Stability Analysis and Stabilization of Fuzzy State Space Models

Vom Fachbereich Mathematik der
Universität Duisburg-Essen
zur Erlangung des akademischen Grades eines
Dr. rer. nat.

genehmigte Dissertation
von
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Tag der mündlichen Prüfung: 12. 07. 2006

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Notation

N	set of natural numbers
$\Re$	field of real numbers
$\Re_+$	set of nonnegative real numbers
C	field of complex numbers
j	unit of imaginary numbers
$\vert . \vert$	2-norm
$\Vert A \Vert$	spectral norm of matrix A
I	unit matrix with appropriate dimension
A^T	transpose of matrix A
A^*	conjugate transpose of matrix A
A^{-1}	inverse of matrix A
$A > 0$	A is a positive definite symmetric matrix
$A \geqslant 0$	A is a semi-positive definite symmetric matrix
$A > B$	$A - B$ is a positive definite symmetric matrix
$diag(A, B)$	diagonal block matrix of A and B
M_i^j	fuzzy sets in the rule base
$\mu_M(.)$	membership function of fuzzy set M
$p_i(t)$	premise variables in fuzzy rules
$\alpha_i(p(t))$	normalized membership functions
r	number of the fuzzy rules
$x(t)$ or $x(k)$	n-dimensional state variable
x_e	equilibrium state
$u(t)$ or $u(k)$	control input
K_i	feedback gains
H_{ij}	$A_i + B_i K_j$
G_{ij}	$(H_{ij} + H_{ji})/2$
$\lambda_{\max}(A)$	maximal eigenvalue of matrix A
$\lambda_{\min}(A)$	minimal eigenvalue of matrix A
$:=$	defined as
$\forall$	any
$\exists$	some
$\nabla f(x)$	gradient of $f(x)$
$V(.)$	Lyapunov candidate function
$\dot{V}(.)$	derivative of V along the system trajectory
$\nabla_x V(x, t)$	partial derivative of V with respect to x

Abstract

Fuzzy control has achieved numerous successful industrial applications. However, stability analysis for fuzzy control systems remains a difficult problem, and most of the critical comments on fuzzy control are due to the lack of a general method for its stability analysis. Although significant research efforts have been made in the literature, appropriate tools for this issue have yet to be found.

This thesis focuses on the problem of stability of fuzzy control systems. Both linguistic fuzzy models and T-S fuzzy models are discussed. The main work of this thesis can be summarized as follows:

(1). A necessary and sufficient condition for the global stability of linguistic fuzzy models is given by means of congruence of fuzzy relational matrices.

(2). A hyperellipsoid-based approach is proposed for stability analysis and control synthesis of a class of T-S (affine) fuzzy models with support-bounded fuzzy sets in the rule base.

(3). Approaches of BMI-based fuzzy controller designs are proposed for the stabilization of T-S fuzzy models.

(4). For the general T-S type fuzzy systems with norm-bounded uncertainties and time-varying delays, sufficient robust stabilization conditions are presented by employing the PDC-based fuzzy state feedback controllers.

On stability analysis of T-S fuzzy models, most reported results based on the method of common quadratic Lyapunov functions require that each subsystem of the fuzzy models be stable in order to guarantee the stability of the overall systems. This restriction is overcome in our results by means of employing the structural information in the fuzzy rules.

Chapter 1

Introduction

The theory of fuzzy logic control stems from Zadeh's pioneering work on fuzzy sets [90]. In 1974 the fuzzy logic technique was first successfully applied to control applications by Mamdani [55]. Since then, fuzzy logic control has achieved numerous industrial applications, and now it has turned out to be one of the most fruitful application areas of the fuzzy set theory. In comparison with the conventional control approaches, fuzzy control has at least two advantages. First, fuzzy control is less sensitive to noise and parameter changes [5]. Moreover, fuzzy control can be applied to a variety of ill-defined processes where the conventional control approaches cannot be applied. As shown in [47], the methodology of fuzzy control appears very useful when the processes are too complex for analysis by conventional quantitative techniques or when the available sources of information are interpreted qualitatively, inexactly or uncertainly.

The wider application of fuzzy control requires a solid and systematic analysis of system performances. Among them, stability is of particular importance. However, due to the non-linearity of fuzzy controllers, stability analysis for fuzzy control is generally quite difficult. We still lack powerful applicable tools for the stability analysis of fuzzy control, and this is also the major drawback of fuzzy control applications.

This thesis is devoted to the stability and stabilization of fuzzy control systems. Before the introduction of the main work of the thesis, we will briefly recall the following related fundamental problems:

1) How to model a fuzzy system?

2) Whether there exists a fuzzy control law to stabilize a given system, in case it can be stabilized?

3) How to design the stabilizing controllers for fuzzy systems?

The first problem deals with fuzzy modeling. For the purpose of analytical stability analysis and model-based controller designs, it is first necessary to have a reliable mathematical model of the plant. In conventional control context, the mathematical model of a system is explicitly described by differential or difference equations. Whereas in fuzzy control context, the mathematical model of a system is implicitly expressed by fuzzy rules. The so-called 'model free' nature of fuzzy control means only 'explicit model free', that is, without the explicit mathematical model of the system a nonlinear controller can also be designed by using the linguistic qualitative knowledge [1]. According to the different output formulations of the fuzzy rules, fuzzy models are generally classified as Mamdani type fuzzy models and T-S (or T-S-K Takagi-Sugeno-Kang) type fuzzy models.

There have been many approaches to fuzzy modeling. Algorithms for the identification of fuzzy models with input-output data of the objective systems are proposed e.g. in [63], [70] and [89]. Approaches to deriving fuzzy models from the given nonlinear systems are presented e.g. in [77], [73] and [44]. Moreover, it has been proved that any nonlinear system can be approximated as accurately as required with some fuzzy rules [45]. That is, fuzzy systems can be taken as universal function approximators.

The second problem is concerned with the so-called universal fuzzy controllers. The problem has been completely solved. As shown in [7] and [6], both the Mamdani type fuzzy controllers and the T-S type fuzzy controllers are universal fuzzy controllers. Thereby, as long as a system is stabilizable, it can be stabilized via fuzzy controllers. Moreover, for any linear time-invariant plant of arbitrary order, a Mamdani type fuzzy controller with only 4 fuzzy rules will always suffice to guarantee the local asymptotic stability [56].

The third topic addresses the design problem. Primarily, the fuzzy controller design methodology involves mainly distilling human expert knowledge about how to control a system into a set of fuzzy rules. This is a heuristic design approach. The major disadvantage of this approach is that the stability of the closed-loop cannot be fully guaranteed. Since experience, intuition and rules of thumb are used in design instead of a firm theory, fuzzy control has been accused of being an unreliable approximate engineering approach. A significant improvement is made when the so-called PDC (parallel distributed compensation) design scheme is proposed for T-S fuzzy models ([81], [74]). The main motivation of this approach is to derive each control rule to compensate each rule of a fuzzy system, then the resulting overall controller is a fuzzy blending of each individual linear controller. The appeal of PDC controller design is that the Lyapunov function based techniques can be directly employed for the stability analysis and control synthesis of T-S fuzzy models. With this design, the state feedback gains of fuzzy controllers can be efficiently solved by numerical methods such as the LMI (Linear Matrix Inequality) tools, and the stability of the closed loop is fully guaranteed, if a common Lyapunov function exists [81].

Besides the PDC scheme, a variety of other different approaches to fuzzy controller designs are proposed in the literature, such as linear stabilizing controller design [91], Lyapunov-based fuzzy controller design [56], variable structure controller design [42], adaptive fuzzy controller design [71], etc.. None of them is appropriate for every application, yet the PDC-based design methodology is employed most frequently in the framework of T-S fuzzy models.

In this thesis, the fuzzy modeling problem is out of our consideration. All the systems under discussion are assumed to have been identified and presented in the form of state space fuzzy models. Moreover, the given systems are further assumed to be stabilizable. Thereby, the existence of stabilizing fuzzy control laws is guaranteed. Our objective is to find the control laws, such that the closed loops are asymptotically stable. In case of Mamdani type fuzzy models, we employ fuzzy relational equations in the form of $u(k) = x(k) \circ R_c$ to represent the fuzzy controllers, where R_c is a fuzzy relational matrix to be determined. In case of T-S fuzzy models, the PDC scheme mentioned above will be applied in design. Thus, the fuzzy controller design problems in this thesis are reduced to determining the feedback gains in the control laws. Within the framework of these assumptions, the stability analysis and control synthesis of the state space fuzzy models will be discussed and some extended stability results will be given. Also, numerical examples will be presented to illustrate the feasibility of the proposed approaches.

The thesis is organized in nine chapters. Chapter 1 gives a brief introduction to the contexts of the work. In Chapter 2 some basic concepts and preliminary results concerning the topic of stability of fuzzy control are listed. Also the design problem is introduced, and as an example, a nonlinear design method for bilinear systems is proposed.

Chapter 3 deals with the stability of linguistic fuzzy models by means of fuzzy relational equations. Due to the fuzzy relational formulations, the general nonlinear methods cannot be applied to the stability analysis of linguistic fuzzy models. Moreover, the stability concept in the sense of Lyapunov is not appropriate for linguistic fuzzy models. We propose first a concept of global stability (see Definition 3.1) with respect to the greatest equilibriums of the given models. More precisely, a linguistic fuzzy model is said to be globally stable, if the trajectory from any normal initial fuzzy state converges to the greatest equilibrium of the model. Also, we propose an algorithm (see Theorem 3.2) for determining the greatest equilibriums of the closed loop linguistic fuzzy models without solving the corresponding fuzzy relational equations. Furthermore, a necessary and sufficient condition (see Theorem 3.4) for the global stability of linguistic fuzzy models is given by means of the congruence of fuzzy relational matrices. Finally, it is to note that the main results of this chapter have been published in [95], [51] and [97].

In Chapter 4 we discuss first the stability of open-loop T-S fuzzy models by the approach of eigenvalue analysis. If a quadratic Lyapunov candidate function $V(x(k)) = x^T(k)Px(k)$ is chosen, the stability of T-S fuzzy models is reduced to the existence of a positive definite matrix P, such that $V(x(k))$ is a common Lyapunov function for all subsystems of the models. Searching for such a matrix has received considerable attention. Although this can be achieved in several ways, such as gradient method, genetic method, LMI method, etc., the necessary and sufficient condition for the existence of such a positive definite matrix is still left open. We prove that if $V(x(k))$ is a common Lyapunov function for all the subsystems (i.e. $A_i^T P A_i - P < 0$), the eigenvalues of the product and average of any number of A_i must be located strictly in the unit circle (see Theorem 4.1). This result improves the necessary condition (Theorem 4.3) for stability in [75]. Next, we present a relaxed eigenvalue constraint (see Theorem 4.2) for the stabilization of T-S fuzzy models using fuzzy state feedback controllers. Solving the eigenvalue constraint can be reduced to the standard BMI (bilinear matrix inequality) feasibility problem, which will be further discussed in Chapter 7 (see Section 7.2).

In Chapter 5 a hyperellipsoid-based method is proposed for the stability analysis of open loop T-S fuzzy affine systems. The motivation of this approach is to overcome the conservativeness of analysis by employing the structural information in the rule base. We provide first an algorithm for constructing minimal hyperellipsoids from the support information of the fuzzy rules. Then, by discussing the maximum of $\dot{V}(x(t))$ on the regions of the constructed minimal hyperellipsoids, we obtain the sufficient stability constraints (see Theorem 5.2, 5.4) for open-loop T-S fuzzy affine models. Our results hold for the common open-loop T-S fuzzy models as well. In this case, the presented result (see Theorem 5.3) is better than the corresponding result (Theorem 4.2) in [75], in the sense that the restriction that all the subsystems must be stable in order to guarantee the stability of the overall system, is removed in our results.

Chapter 6 is focused on the stabilization of a class of T-S fuzzy models with support-bounded fuzzy sets in the fuzzy rules. A fuzzy state feedback controller utilizing the concept of PDC scheme is employed in design, and the proposed hyperellipsoid-based method is applied to derive the sufficient conditions (see Theorem 6.1, 6.2) for stabilization of the models. Then the existence of fuzzy state feedback gains is reduced to the feasibility of a group of bilinear matrix inequalities. Finally, a solution procedure for solving the BMIs is introduced by employing the LMI tools. The presented stability conditions (see Theorem 6.1-6.2, Corollary 6.1-6.3) are less conservative than those LMI-based results in e.g. [46], [81] and [60].

Chapter 7 addresses the problem of BMI-based fuzzy controller designs for T-S fuzzy models. We propose an approach of stability analysis by introducing additional parameters. By this approach, the design of fuzzy output feedback controller and fuzzy observer-based controller is reduced to the BMI feasibility problem (see Theorem7.1 and Corollary 7.1). The introduced parameters can automatically be tuned by the proposed BMI algorithm. Thereby, the chances of finding the desired feedback gains are increased in the procedures of solving the BMIs. Moreover, based on the eigenvalue constraints in Chapter 4, the sufficient conditions for the stabilization of T-S fuzzy models via fuzzy state feedback controllers are also formulated in terms of BMIs (see Theorem 7.2, 7.3).

Chapter 8 is devoted to the stabilization of time delay T-S fuzzy models. An LMI-based stabilization approach using additional parameters as well is developed via the PDC-based fuzzy state feedback controllers (see Theorem 8.1). By applying the improved Razumikhin theorem, a delay-independent sufficient stabilization condition (see Corollary 8.2) is given. Also, delay-dependent results (see Theorem 8.2 and Corollary 8.3, 8.4) for the stabilization of time delay T-S fuzzy models are presented by the Lyapunov functional method.

Chapter 9 deals with the problem of robust stabilization of uncertain nonlinear systems via T-S fuzzy model based approaches. The systems under consideration may have norm-bounded uncertainties and time-varying delays. We propose first a stabilization method for the uncertain models using fuzzy state feedback controller (see Theorem 9.1). Then the H_∞ performance is taken into account additionally, and a stabilization constraint (see Theorem 9.2) for H_∞ control is given. The presented results are formulated in terms of LMIs, thereby, the desired feedback gains can be solved efficiently.

We conclude this thesis with an Appendix, in which the involved LMI problems and the corresponding LMI solvers in Matlab are introduced.

Acknowledgement

I would like to express my sincere gratitude to Prof. Dr. H.H. Gonska and Prof. Dr. Xinlong Zhou for their support and valuable comments during the preparation of this work. In particular, I would like to thank Dr. D. Kacso for her great help on the improvement of the manuscript.

Many thanks to Prof. J. Bauer, Prof. J. Donig, Prof. K.H. Mohn, Prof. W. Schreiber, Dr. P.W. Meyer, Dr. K. Schaefers, Dr. B. Stockenberg, Dr. H. Wegner, Mr. A.W. Bieck, Mr. L. Lorenz, and many others who have provided me with convenience during my study in Duisburg. I have benefited from the contact with them.

I would also like to express my gratitude to my family for their support and encouragement all these years, especially to my wife B.B Wang for proofreading.

Chapter 2

Preliminaries

In this chapter, some basic concepts concerning fuzzy logic are listed. Also the involved preliminary conclusions on the stability issue of control systems are reviewed. Moreover, the controller design problem is introduced and a nonlinear controller design method for bilinear systems is proposed.

2.1 Relevant Terminology in Fuzzy Logic

Definition 2.1 *(Triangular Norms and Triangular Co-Norms, [93]) Let 0 and 1 be the minimal and maximal elements of lattice* $(L, \preceq)$. *Function* $T : L \times L \to L$ *is called a triangular norm if* T *satisfies the following conditions (1)-(4) and* $T(a, 1) = a$ *for all* $a \in L$. *On the other hand, function* $T : L \times L \to L$ *is called a triangular co-norm if* T *satisfies conditions (1)-(4) and* $T(a, 0) = a$ *for all* $a \in L$.

(1) $T(0, 0) = 0; T(1, 1) = 1;$

(2) $T(a, b) = T(b, a)$ *for all* $a, b \in L$;

(3) $a \preceq c, b \preceq d \Rightarrow T(a, b) \preceq T(c, d)$ *for all* $a, b, c, d \in L$;

(4) $T(T(a, b), c) = T(a, T(b, c))$ *for all* $a, b, c \in L$.

In Definition 2.1, $\preceq$ stands for a partial order, and the pair $(L, \preceq)$ is a lattice, which means: $\inf\{a, b\} \in L$, and $\sup\{a, b\} \in L$ for all $a, b \in L$. In the framework of fuzzy control it is enough to choose $L = [0,\ 1]$.

Example 2.1 *Suppose* $L = [0,\ 1]$ *and let:*

$T_0(a, b) := a \wedge b$ *(i.e.* $min(a, b)$*)*, $T_1(a, b) := a \cdot b$,

$T_2(a, b) := a \cdot b/(1+(1-a)\cdot(1-b))$, $T_\infty(a, b) := 0 \vee (a+b-1)$ *(i.e.* $max(0, a+b-1)$*)*,

$S_0(a, b) := a \vee b$, $S_1(a, b) := a + b - a \cdot b$,

$S_2(a, b) := (a + b)/(1 + a \cdot b)$, $S_\infty(a, b) := 1 \wedge (a + b)$.

Obviously $(L, \preceq)$ is a lattice, if $L = [0,\ 1]$ and the partial order $\preceq$ is identical to the ordinary $\leqslant$. It is straightforward to verify that T_0, T_1, T_2, T_∞ are triangular norms on L, and S_0, S_1, S_2, S_∞ are triangular co-norms on L. The operators defined in this example will be used in fuzzy inferences later.

Definition 2.2 *(Fuzzy Set, [47]) A fuzzy set F in the universe of discourse U is characterized by a membership function $\mu_F : U \to [0,\ 1]$. Concisely F can be written as $F = \int_U \mu_F(x)/x$ (or $F = \sum_{i=1}^n \mu_F(x_i)/x_i$ when U is discrete).*

The concept of fuzzy set was first introduced by Zadeh in 1965. Fuzzy set can be viewed as a generalization of the ordinary set, whose membership function takes only two values in {0,1}. Based on fuzzy set theory, the vague concepts in natural language can be described mathematically, which is fundamental in utilizing the human knowledge in fuzzy control.

Definition 2.3 *(T-Complement, [93]) Let N be a function on lattice $(L, \preceq)$ with properties:*

(1) $a \preceq b \Rightarrow N(b) \preceq N(a) \quad \forall a, b \in L$, (2) $N(N(a)) = a \quad \forall a \in L$,

then N is called a T-complement operator on $(L, \preceq)$.

Specially, if $L = [0,\ 1]$, and $N(x) = 1 - x$ for all $x \in [0,\ 1]$, then N is a T-complement operator on $[0,\ 1]$, which is called fuzzy complement and is often denoted as A^c, that is $A^c = \int_U (1 - \mu_A(x))/x$.

Definition 2.4 *($T-$Union and $T-$Intersection, [93]) Let A_1 and A_2 be fuzzy sets in the universe of discourse U. The $T-$ union and $T-$intersection of A_1 and A_2 are defined by:*

$$A_1 \cup A_2 := \int_U (\mu_{A_1}(x) \dotplus \mu_{A_2}(x))/x$$
$$A_1 \cap A_2 := \int_U (\mu_{A_1}(x) * \mu_{A_2}(x))/x$$

respectively, where $\dotplus$ and $*$ are the triangular co-norm and triangular norm operators defined in Definition 2.1.

If the triangular co-norm and triangular norm operators are chosen as S_0 and T_0 defined in Example 2.1, then the $T-$union and $T-$intersection operators $\cup$ and $\cap$ degenerate to the common fuzzy union and fuzzy intersection respectively.

Definition 2.5 *(Fuzzy Relation, [47]) A fuzzy relation is a fuzzy set in $U_1 \times U_2 \times \cdots \times U_m$ and is expressed by:*

$$R_{U_1 \times U_2 \times \cdots \times U_m} = \int_{U_1 \times U_2 \times \cdots \times U_m} \mu_R(u_1, u_2, ..., u_m)/(u_1, u_2, ..., u_m).$$

In the theory of fuzzy control systems, a fuzzy relation is always described by fuzzy rules (If A then B, denoted as fuzzy implication: A→B). The membership functions of fuzzy implications can be inferred from triangular norms and triangular co-norms. There are about 40 distinct fuzzy implication functions proposed in the literature. The following implication functions are frequently employed [47]:

(1) Fuzzy conjunction: $A \rightarrow B = \int_{U\times V} (\mu_A(v) * \mu_B(v))/(u,v)$

(2) Material implication: $A \rightarrow B = \int_{U\times V} (N(\mu_A(u) \dotplus \mu_B(v))/(u,v)$

(3) Propositional calculus: $A \rightarrow B = \int_{U\times V} (N(\mu_A(u) \dotplus (\mu_A(v) * \mu_B(v)))/(u,v)$

(4) Generalization of modus ponens:

$$A \rightarrow B = \int_{U\times V} \sup\{c \in [0,\ 1] : \mu_A(u) * c \leqslant \mu_B(v)\}/(u,v)$$

where $*$, $\dotplus$ and N stand for triangular norm, triangular co-norm and T-complement operators respectively.

Definition 2.6 *(Sup-Star Composition, [47]) If R is a fuzzy relation in $U \times V$, and A is a fuzzy set in U, then the composition of A and R is defined by:*

$$A \circ R = \int_V \sup_{u\in U} (\mu_A(u) * \mu_R(u,v))/v$$

where $*$ is a triangular norm.

According to Definitions 2.5 and 2.6, different triangular norms and triangular co-norms will deduce different fuzzy compositions and fuzzy relations. Due to the diversity of triangular (co)norms, more choices of operators can be provided in applications. In [47] the satisfaction results of various implications are listed under intuitive criteria. Structures of fuzzy controllers with different implications are analyzed in [50]. Generally speaking, no implication is absolutely better than the others. But implications inferred from T_0, T_1 and S_0 defined in Example 2.1 are relatively easy to operate and are commonly used in fuzzy control context. More detailed descriptions of fuzzy inferences can be found e.g. in [47], [93] and [88].

2.2 Basic Configuration of Fuzzy Control Systems

A fuzzy control system is a system with fuzzy controller. The basic configuration of fuzzy control systems is shown in Figure 2.1, in which both the input 'u' and output 'x' of the real controlled systems are non-fuzzy. By executing the Fuzzifier operator, the crisp value 'x' is transformed into a fuzzy set 'X'. The mechanisms of Fuzzy Inference in Figure 2.1 can be formulated in essence as U=X∘R, where '∘' is a fuzzy composition operator and 'R' is a fuzzy relation determined by the fuzzy rules in the rule base. The component Defuzzifier performs a transformation from a fuzzy set 'U' to a crisp value 'u'. The fuzzy rules, which are usually in the form of "If-then-", can be constructed either based on expert knowledge or based on learning

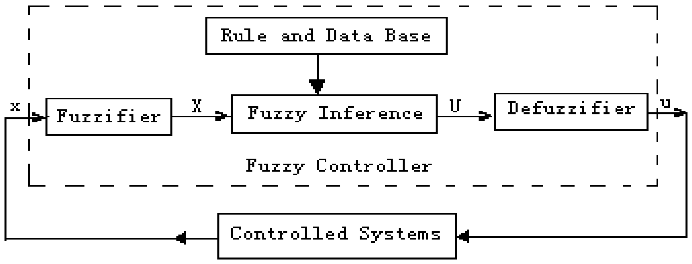

Figure 2.1: Basic configuration of fuzzy control systems

algorithms etc.. The requirement for the rule base is that the properties of consistency and completeness must be satisfied. The completeness property guarantees that every state of the process can infer a proper controller output, and the consistency property gives that there are no contradictory rules in the rule base. More detailed descriptions on the configuration of control systems can be found e.g. in [93] and [47].

Generally fuzzy control systems are classified into Mamdani type fuzzy models and T-S type fuzzy models according to the different consequents of the fuzzy rules. In Mamdani type fuzzy models the consequent of each fuzzy rule is a fuzzy set, whereas in T-S type fuzzy models the consequent of each fuzzy rule is a crisp function of antecedent variables. In the literature (see e.g. [69] and [82]) fuzzy control systems are also classified into three types. The additional type is the so-called singleton type fuzzy models, in which all the fuzzy rules are with singleton consequents. Thereby, this type can be taken as a special case of both Mamdani type fuzzy models and T-S type fuzzy models. For this reason the singleton type fuzzy models are not discussed separately in this thesis, and all our presented results are valid for this type as well.

2.3 Stability Definition and Lyapunov Direct Method

Consider the general form of time varying continuous system:

$$\dot{x}(t) = f(x(t), t), \ x(t_0) = x_0 \tag{2.1}$$

where $x(t) \in \Omega \subseteq \Re^n$ is the state vector, and f is a vector function satisfying conditions for existence and uniqueness of solutions with respect to all initial conditions $x(t_0) = x_0 \in \Omega$. One of the simplest conditions for existence and uniqueness of solutions is the so-called Lipschitz condition: There exists a positive scalar $L > 0$, such that $|f(x_1(t), t) - f(x_2(t), t)| \leqslant L \cdot |x_1(t) - x_2(t)|$ for all $x_1(t), x_2(t) \in \Re^n$. Note that the conditions for existence and uniqueness of solutions are under no circumstances superfluous. It is insignificant to discuss the stability of a solution for some initial condition, if the solution doesn't exist or it exists but is not unique. For brevity, all the systems under discussion in this thesis are also assumed to satisfy the conditions of existence and uniqueness of solutions without specification.

Example 2.2 *([24]) Suppose:*

$$\dot{x}(t) = 2\sqrt{x(t)} \ (x(t) \in [0, +\infty) \subseteq \Re) \tag{2.2}$$

Obviously $x(t) = (t + \sqrt{x_0})^2$ *is the solution of (2.2) for the initial condition* $x(0) = x_0 \geqslant 0$*. But* $x(t) \equiv 0$ *is also a solution of (2.2) for* $x_0 = 0$*, i.e. for the initial condition* $x(0) \equiv 0$*, (2.2) has two different solutions.*

Under the hypothesis of existence and uniqueness of the solutions the stability of the solutions of (2.1) can be defined as follows:

Definition 2.7 *(Stability in the Sense of Lyapunov, [24]) Let the solution* $\overline{x}(t)$ *of (2.1) be well defined for all* $t \geqslant t_0$*, then it is called stable (in the sense of Lyapunov), if for any* $\varepsilon > 0$ *and* $t_1 \geqslant t_0$*, there exists* $\delta(\varepsilon, t_1) > 0$ *such that for any solution* $x(t)$ *with the initial condition* $x(t_1)$ *satisfying* $|x(t_1) - \overline{x}(t_1)| < \delta(\varepsilon, t_1)$*, the inequality* $|x(t) - \overline{x}(t)| < \varepsilon$ *holds for all* $t \geqslant t_1$*. If, in addition,* δ *is independent of* t_1*, then the solution* $\overline{x}(t)$ *is called uniformly stable. If* $\overline{x}(t)$ *is stable and* $|x(t) - \overline{x}(t)| \rightarrow 0$ *as* $t \rightarrow \infty$*, then* $\overline{x}(t)$ *is called asymptotically stable. If* $\overline{x}(t)$ *is asymptotically stable and* $\delta(\varepsilon, t_1)$ *can be arbitrarily large, then* $\overline{x}(t)$ *is called globally asymptotically stable.*

Definition 2.8 *(Exponential Stability, [24]) The solution* $\overline{x}(t)$ *of (2.1) is called exponentially stable if for sufficiently small* $|x(t_0) - \overline{x}(t_0)|$*, there exists* $\alpha, \beta > 0$ *such that* $|x(t) - \overline{x}(t)| \leqslant \beta \cdot e^{-\alpha(t-t_0)} \cdot |x(t_0) - \overline{x}(t_0)|$*. If additionally,* $|x(t_0) - \overline{x}(t_0)|$ *is arbitrary, then* $\overline{x}(t)$ *is called globally exponentially stable.*

The stability in the sense of Lyapunov was originally proposed by Lyapunov based on the concept of energy in the 19th century. Generally speaking, Lyapunov stability may be interpreted as the continuous dependence of the solutions on the initial conditions over an infinite time interval. In fact, in order to verify the stability of solution $\overline{x}(t)$, it is enough to verify the conditions of Definition 2.7 only for some $t_1 \geqslant t_0$ instead of all $t_1 \geqslant t_0$, since on any closed interval $[t_0,\ t_1]$, $|x(t) - \overline{x}(t)|$ can be made arbitrarily small due to the continuous dependence of the solutions on the initial conditions. It is to note, the stability of $\overline{x}(t)$ is not equivalent to the convergence of $|x(t) - \overline{x}(t)|$ as $t \rightarrow \infty$. Even if $|x(t) - \overline{x}(t)| \rightarrow 0$, the solution $\overline{x}(t)$ may be unstable either (see Example 2.3). However, if $|x(t) - \overline{x}(t)|$ is convergent to zero exponentially, the stability of $\overline{x}(t)$ is guaranteed. For general nonlinear systems, the relationship of different stabilities of a solution is shown in Figure 2.2.

A state x_e is called an equilibrium of (2.1), if $x(t)$ reaches x_e and then it will stay at x_e for all the future time, namely: $f(x_e, t) \equiv 0$. Without loss of generality, we can assume that $\overline{x}(t) \equiv 0$ is an equilibrium of (2.1) in Definition 2.7 and Definition 2.8. If it is not the case, let $y(t) = x(t) - \overline{x}(t)$, then $\dot{y}(t) = F(y(t), t)$ and $F(0, t) = 0$, where $F(y(t), t) = f(\overline{x}(t) + y(t), t) - \dot{\overline{x}}(t)$. In this way, the solution $\overline{x}(t)$ of system (2.1) is then transformed into the equilibrium state $y(t) \equiv 0$ of the system $\dot{y}(t) = F(y(t), t)$.

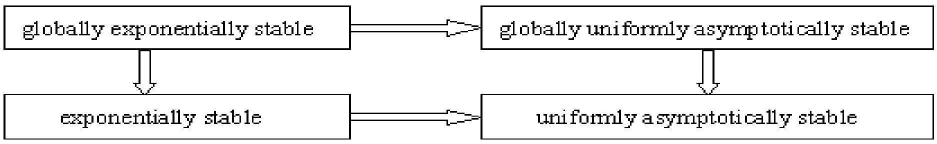

Figure 2.2: Stability and exponential stability

Example 2.3 *Suppose:*

$$\dot{x}(t) = -x(t)^2. \tag{2.3}$$

By separation of variables, solutions of (2.3) can be easily found. For every initial condition $x(t_0) = x_0$*,* $x(t) = x_0/(1 + x_0(t - t_0))$ *is the solution of (2.3). Obviously,* $x(t) \to 0$ *as* $t \to \infty$*. But the trivial solution* $\overline{x}(t) \equiv 0$ *of (2.3) is unstable due to* $x(t) \to \infty$ *as* $t \to t_0 - 1/x_0$ *when* $x_0 \neq 0$*.*

The most frequently employed method for stability analysis of control systems is the so-called Lyapunov direct (or second) method. The idea of this method is to discuss the stability of a solution of the given system through the time-derivatives of a proper definite function (Lyapunov function) along the trajectories of the given system. With this method it is possible to analyze the stability of a solution of the given systems without solving the associated equations, which is very useful for the stability analysis of non-linear systems. However, the problem is that it is always difficult to find a proper Lyapunov function for the given non-linear systems. Some detailed discussions on the construction of Lyapunov functions can be found e.g. in [92]. The main results of Lyapunov direct method are as follows:

Theorem 2.1 *(First Lyapunov Theorem, [24]) Suppose that there exists a continuously differentiable scalar function* $V : \Omega \times [t_0, \infty) \to \Re_+$ *such that* $V(0, t) = 0$*,* $V(x, t) \geqslant \alpha(x)$ *and* $\dot{V}(x, t) \leqslant 0$ *where* $\alpha(x) > 0$ *for* $x \neq 0$*, then the trivial solution* $\overline{x}(t) \equiv 0$ *of system (2.1) is Lyapunov stable. (Note:* $\dot{V}(x, t) = \frac{\partial V}{\partial t}(x, t) + (\nabla_x V(x, t))^T \cdot f(x(t), t)$*)*

Theorem 2.2 *(Second Lyapunov Theorem, [24]) Suppose that there exists a continuously differentiable scalar function* $V : \Omega_0 \times [t_0, \infty) \rightarrow \Re_+ \quad (\Omega_0 \subseteq \Omega)$*, such that* $\alpha_0(|x|) \leqslant V(x,t) \leqslant \alpha_1(|x|)$ *and* $\dot{V}(x,t) \leqslant -\alpha_2(|x|)$*, where* α_0, α_1*and* α_2 *are continuous strictly increasing scalar functions with* $\alpha_0(0) = \alpha_1(0) = \alpha_2(0) = 0$*, then the trivial solution* $\overline{x}(t) \equiv 0$ *of system (2.1) is uniformly asymptotically stable with the domain of attraction* Ω_0*.*

Theorem 2.3 *([24]) Suppose* $\dot{x}(t) = f(x(t), t)$ *where* f *is continuously differentiable. Then the trivial solution* $\overline{x}(t) \equiv 0$ *is globally exponentially stable if and only if there exists a function* $V : \Re^n \times [t_0, \infty) \rightarrow \Re_+$ *and positive scalars* $\alpha_0, \alpha_1, \alpha_2$ *and* α_3 *satisfying:* $\alpha_0 \cdot |x|^2 \leqslant V(x,t) \leqslant \alpha_1 \cdot |x|^2$*, and* $|\nabla_x V(x,t)| \leqslant \alpha_3 |x|$*.*

In this section we have recalled some important concepts and conclusions on the stability of the general non-linear continuous systems. More detailed descriptions and proofs can be found in [24]. Similar concepts and conclusions on discrete systems can be found in [61]. Approaches for the construction of Lyapunov functions are discussed in [92]. Some new results on generalized Lyapunov functions are given e.g. in [42] and [37].

2.4 Stability and Eigenvalues

Consider the time-invariant linear system: $\dot{x}(t) = Ax(t)$ (or $x(k+1) = Ax(k)$). The stability of trivial solution $x(t) \equiv 0$(or $x(k) \equiv 0$) is determined completely by the eigenvalues of matrix A, which can be summarized as the so-called Lyapunov's inequality.

Let $D = \left\{ \lambda \in C : \begin{bmatrix} 1 \\ \lambda \end{bmatrix}^* \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} < 0 \right\}$ be a given open region of the complex plane, where $\begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \in C^{2\times 2}$ has one strictly negative eigenvalue and one strictly positive eigenvalue, and * denotes the transpose conjugate operator. Then Lyapunov's inequality can be formulated as:

Theorem 2.4 *(Lyapunov's Inequality [32]) Matrix* A *has all its eigenvalues in region* D *if and only if there exists a matrix* P *with* $0 < P = P^*$ *such that:*

$$\begin{bmatrix} I \\ A \end{bmatrix}^* \begin{bmatrix} aP & bP \\ b^*P & cP \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0. \tag{2.4}$$

If $a = c = 0$ and $b = 1$, then region D becomes the open left half plane. In this case the matrix inequality (2.4) has the form of $A^*P + PA < 0$ (i.e. $A^TP + PA < 0$

for $A \in \Re^{n\times n}$), which is just the necessary and sufficient condition for the stability of continuous time-invariant systems $\dot{x}(t) = Ax(t)$. On the other hand, if $a = -1$, $b = 0$, and $c = 1$, then D becomes the open region of the unit circle in complex plane. In this circumstance, (2.4) has the form of $A^T PA - P < 0$, which is just the sufficient and necessary condition for the stability of discrete systems $x(k+1) = Ax(k)$.

Related results on the locations of eigenvalues are also discussed in [44] and [18]. A new proof of Lyapunov's inequality is presented in [32]. However, for the linear time-varying systems $\dot{x}(t) = A(t)x(t)$ the stability of trivial solution $x(t) \equiv 0$ is independent of the eigenvalues of matrix $A(t)$, as shown in Example 2.4.

Example 2.4 *([24]) Suppose:*

$$\dot{x}(t) = A_1(t)x(t) \tag{2.5}$$

where

$$A_1(t) = \begin{bmatrix} -1 - 9\cos^2 6t + \sin 6t \cos 6t & 12\cos^2 6t + 9\sin 6t \cos 6t \\ -12\sin^2 6t + 9\sin 6t \cos 6t & -1 - 9\sin^2 6t - 12\sin 6t \cos 6t \end{bmatrix}.$$

The eigenvalues of $A_1(t)$ lie strictly in the left half plane ($\lambda_1 = -1, \lambda_2 = -10$), but the trivial solution of (2.5) is unstable.

On the other hand, suppose

$$\dot{x}(t) = A_2(t)x(t) \tag{2.6}$$

where

$$A_2(t) = \begin{bmatrix} -11 + 15\sin 12t & 15\cos 12t \\ 15\cos 12t & -11 - 15\sin 12t \end{bmatrix}.$$

The trivial solution $x(t) \equiv 0$ of (2.6) is asymptotically stable. However, $A_2(t)$ has an eigenvalue located in the right half plane ($\lambda_1 = 4, \lambda_2 = -26$).

Now we consider the autonomous non-linear systems:

$$\dot{x}(t) = f(x(t)) \tag{2.7}$$

with $f(0) = 0$. If f is twice continuously differentiable in a neighborhood of zero, then (2.7) can be formulated as $\dot{x}(t) = Ax(t) + g(x(t))$, where

$$A = \frac{\partial f}{\partial x}|_{x=0},\ g(x(t)) = (g_1(x(t)), ..., g_n(x(t)))^T,$$
$$g_k(x(t)) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f_k(\theta_k x)}{\partial x_i \partial x_j} x_i x_j,\ 0 < \theta_k < 1.$$

The following result shows that the stability of the trivial solution of (2.7) is to a certain degree dependent on the eigenvalues of matrix A.

Theorem 2.5 *([52]) If all the eigenvalues of matrix A lie strictly in the left half complex plane, then the trivial solution $x(t) \equiv 0$ of (2.7) is asymptotically stable. If A has at least an eigenvalue located in the open right half plane, then the trivial solution $x(t) \equiv 0$ of (2.7) is unstable.*

Finally, it is to note that the stability of fuzzy control systems is also concerned with eigenvalues. Consider the continuous open-loop T-S type fuzzy models:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(x(t))A_i x(t) \tag{2.8}$$

where $\alpha_i(x(t)) \geqslant 0$ for $i = 1, 2, ...r$ and $\sum_{i=1}^{r} \alpha_i(x(t)) = 1$. If all the eigenvalues of $A_i + A_i^T$ are located strictly in the left half plane for all $i = 1, 2, ...r$, then the trivial solution $x(t) \equiv 0$ of (2.8) is asymptotically stable (see Chapter 4 for details). Moreover, if α_i in (2.8) are regarded as completely uncertain parameters independent of $x(t)$, then the stability of solution $x(t) \equiv 0$ can be reduced to whether all the eigenvalues of the polytope matrices $\sum_{i=1}^{r} \alpha_i A_i$ are located strictly in the left half plane. Although $\sum_{i=1}^{r} \alpha_i A_i$ is a convex function with respect to parameters α_i, counterexamples presented in [2] show that the locations of eigenvalues of the polytope matrices cannot be determined by the convex edges. In fact, the general problem for polytope matrices has not yet been completely resolved.

As noted above, eigenvalues play an important role in stability analysis. In the next section, we will show further an application of eigenvectors in controller design.

2.5 On Controller Design

Having recalled some conclusions on the stability of unforced (without control input) systems, we make now some comments on the problem of controller design. The purpose of controller design is to find a proper state or output feedback such that the closed loop systems possess the desired properties. Among them stability is the most important and basic requirement. The presumption of controller design is that the given system must be controllable, i.e. by a proper control the state of the given system can be driven to any final state from any initial condition.

For linear time invariant control systems:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.9}$$

where $x(t) \in \Re^n$,$u(t) \in \Re^m$, the controller design problem is completely resolved. The condition of controllability of (2.9) has been revealed by rank criterion. That is, (2.9) is controllable if and only if $rank[B\ AB\ ...\ A^{n-1}B] = n$. In addition if (2.9) is controllable, it can certainly be stabilized via a linear state feedback controller.

For the general nonlinear systems, the controller design problem is very complicated and it is far from being resolved. Reported techniques for the synthesis of control laws include Jacobian linearization, gain scheduling, feedback linearization, sliding mode control, recursive backstepping, and adaptive control [22]. In addition, the nonlinear systems can be approximated in terms of fuzzy models, then as an alternative, the parallel distributed compensation technique can be employed for the model-based fuzzy controller designs.

Bilinear systems are a class of quite simple nonlinear systems, which are linear in both state and control when considered separately. By Carleman bilinearization a large class of nonlinear systems affine in the input can be described by bilinear systems [21]. In [53] a bang-bang state feedback controller is proposed for bilinear systems with Hurwitz matrix. The design problem of bilinear systems with purely imaginary spectrum is discussed in [20]. We consider a more general class of bilinear systems, and propose a method of nonlinear controller design using the eigenvectors of the system matrix.

Suppose that the bilinear systems under discussion are described by:

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) \tag{2.10}$$

where $x(t) \in \Re^n$ is the n-dimensional state vector and $u(t) = [u_1(t), ..., u_m(t)]^T \in \Re^m$ is the m-dimensional control input, $A \in \Re^{n\times n}$,$B \in \Re^{n\times m}$ and $N_i \in \Re^{n\times n}$ $(i = 1, 2, ..., m)$ are constant matrices. The matrix B can be written as $B = [b_1|b_2|...|b_m]$, where $b_i \in \Re^n$ is the i-th column vector of B for $i = 1, 2, ..., m$. Therefore, (2.10) can be rewritten as:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} (b_i + N_i x(t)) u_i(t). \tag{2.11}$$

Let $\lambda_1, \lambda_2, ..., \lambda_k, \mu_i \pm jv_i$ $(i = 1, 2, ..., q,\ k+2q = n)$ be the eigenvalues of matrix A, and let $\xi_1, \xi_2, ..., \xi_k, \zeta_i \pm j\eta_i$ $(i = 1, 2, ..., q)$ be the corresponding eigenvectors, which implies the following equations:

$$\begin{cases} A\xi_l = \lambda_l \xi_l, \quad l = 1, 2, ..., k. \\ A\zeta_i = \mu_i \zeta_i - v_i \eta_i;\ A\eta_i = \mu_i \eta_i + v_i \zeta_i, \quad i = 1, 2, ..., q. \end{cases} \tag{2.12}$$

Let $T := [\xi_1, ..., \xi_k, \zeta_1, ..., \zeta_q, \eta_1, ..., \eta_q]$ and $P := (T^{-1})^T T^{-1}$ if T is invertible. Then, we can prove:

Theorem 2.6 *Suppose T is invertible, $\lambda_i, \mu_j \leqslant 0$ for $i = 1, 2, ..., k$, $j = 1, 2, ..., q$, and*

$$\left\{x | x^T (A^T P + PA) x = 0\right\} \cap \left(\cap_{i=1}^{m} \left\{x | x^T P (b_i + N_i x) = 0\right\}\right) = \{0\}. \tag{2.13}$$

Then the bilinear system described by (2.10) is globally asymptotically stabilizable via non-linear state feedback control law : $u_i = -x^T P(b_i + N_i x)$ for $i = 1, 2, ..., m$.

Proof. The candidate Lyapunov function is chosen as $V(x) = x^T Px$. Obviously, $V(x) = (T^{-1}x)^T (T^{-1}x) \geqslant 0$, and $V(x) = 0 \Rightarrow x = 0$.

$$\begin{aligned}\dot{V}(x) &= (\dot{x})^T P x + x^T P \dot{x} \\ &= x^T(A^T P + PA)x + 2x^T P \textstyle\sum_{i=1}^m (b_i + N_i x)u_i \\ &= x^T (T^{-1})^T ((T^{-1}AT)^T + T^{-1}AT)T^{-1}x + 2x^T P \sum_{i=1}^m (b_i + N_i x)u_i \end{aligned} \tag{2.14}$$

From the equations of (2.12), it follows:

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_k & & & & & & \\ & & & \mu_1 & & & v_1 & & \\ & & & & \ddots & & & \ddots & \\ & & & & & \mu_q & & & v_q \\ & & & -v_1 & & & \mu_1 & & \\ & & & & \ddots & & & \ddots & \\ & & & & & -v_q & & & \mu_q \end{bmatrix}. \tag{2.15}$$

Substituting (2.15) into (2.14) and replacing u_i for $-x^T P(b_i + N_i x)$, we have:

$$\begin{aligned}\dot{V}(x) &= x^T(T^{-1})^T diag(2\lambda_1, ..., 2\lambda_k, 2\mu_1, ..., 2\mu_q, 2\mu_1, ..., 2\mu_q)T^{-1}x \\ &\quad -2\textstyle\sum_{i=1}^m (x^T P(b_i + N_i x))^2.\end{aligned}$$

Since $\lambda_i, \mu_j \leqslant 0$ for $i = 1, 2, ..., k$, $j = 1, 2, ..., q$, it follows $\dot{V}(x) \leqslant 0$. The condition (2.13) gives that $\dot{V}(x) = 0$ implies $x = 0$. In addition $V(x) \to \infty$ as $|x| \to \infty$. Therefore the closed-loop system under the given control is globally asymptotically stable. ■

If A is a Hurwitz matrix, then for any positive definite symmetric matrix Q, there exists a unique positive definite matrix P satisfying the Lyapunov equation: $A^T P + PA = -Q$. Thereby, the equation $x^T(A^T P + PA)x = 0$ has a unique solution $x = 0$. That is, the condition (2.13) is satisfied. By Theorem 2.6, we obtain immediately the following corollary, which is similar to the result of [53].

Corollary 2.1 *Suppose that all eigenvalues of matrix A have strictly negative real parts, then the bilinear system (2.10) subject to control laws $u_i = -x^T P(b_i + N_i x)$ for $i = 1, 2, ..., m$ is globally asymptotically stable.*

Chapter 3

Stability Analysis of Linguistic Fuzzy Models

Different from the ordinary control systems which are described by differential or difference equations, linguistic fuzzy models are expressed by fuzzy rules and can be formulated by fuzzy relational equations. Based on the relational formulations, a variety of definitions on the stability of linguistic fuzzy models are presented in the literature. In this chapter, some comments on the concept of stability of linguistic fuzzy models are given. Counterexamples are presented to show that it is inappropriate to describe the global stability of linguistic fuzzy models with peak patterns. For the purpose of stability analysis, the closed loop linguistic fuzzy model has to be formulated in the form of iteration. A necessary and sufficient condition is given to reveal the conditions for this transformation. Moreover an algorithm for determining the greatest equilibriums of the closed loop linguistic fuzzy models is proposed. Finally, a necessary and sufficient condition for the global stability of linguistic fuzzy models is presented in terms of the congruence of fuzzy relational matrices.

3.1 Formulation of Linguistic Fuzzy Models

As mentioned in Section 2.2, fuzzy control systems are distinguished into Mamdani type fuzzy models and T-S type fuzzy models according to the different consequents of the fuzzy rules. Mamdani type models are also known as linguistic fuzzy models, in which both the premise and the consequent of the fuzzy rules are described by fuzzy sets. Different from the standard configuration of fuzzy control systems, in this chapter, Fuzzifier and Defuzzifier will be viewed as components of the so-called generalized fuzzy process [14]. Then we have a pure fuzzy system [82] as shown in Figure 3.1. Moreover, we will restrict our consideration to finite discrete linguistic

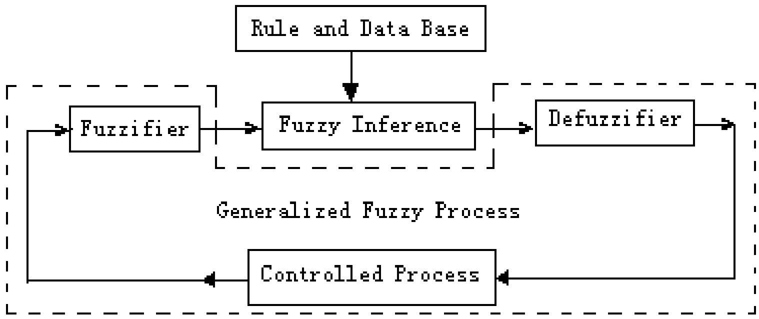

Figure 3.1: Generalized fuzzy process

fuzzy models. As indicated in [79] there are two reasons for the choice of discrete models. First, in all practical situations the power of the fuzzy approach comes from the ability to express process behavior, design goals, and other important system features in linguistic forms. The most natural and simplest representation of such information is in relational terms. Second, any implementation of the ideas must involve a digital computer, which implies both finiteness and discreteness.

Suppose the generalized fuzzy process and fuzzy controller are described by:

Process rules: If $x(k)$ is A_i and $u(k)$ is B_i, then $x(k+1)$ is C_i $(i = 1, 2, ...l\ )$

Controller rules: If $x(k)$ is D_j, then $u(k)$ is E_j $(j = 1, 2, ...s)$

where $x(k)$ and $u(k)$ are state linguistic variable and control linguistic variable with universe of discourse $X = \{a_1, a_2, ..., a_n\}$ and $U = \{b_1, b_2, ..., b_m\}$ respectively. The connective of fuzzy rules is translated as operator '$\vee$', and the connective of fuzzy sets e.g. '$x(k)$ is A_i and $u(k)$ is B_i' is translated as Cartesian product:

$$A_i \times B_i = \big\{((a_p, b_q), \mu_{A_i}(a_p) \wedge \mu_{B_i}(b_q))|a_p \in X, b_q \in U\big\}.$$

Moreover, assume that the fuzzy implication is inferred from fuzzy conjunction with triangular norm T_0 (Example 2.1), and the fuzzy composition is inferred from sup-star composition (Definition 2.6) with respect to triangular norm T_0. Thereby the generalized fuzzy process and fuzzy controller can be formulated as:

$$x(k+1) = (x(k) \times u(k)) \circ \bigvee_{i=1}^{l} (A_i \times B_i \to C_i) = (x(k) \times u(k)) \circ P \qquad (3.1)$$

$$u(k) = x(k) \circ \bigvee_{j=1}^{s} (D_j \to E_j) = x(k) \circ Q \qquad (3.2)$$

where $P = (P_{ij,k})_{nm \times n}$ is a fuzzy relational matrix on $(X \times U) \times X$ and $Q = (Q_{ij})_{n \times m}$ is a fuzzy relational matrix on $X \times U$ with entries:

$$P_{ij,k} = \bigvee_{p=1}^{l} (\mu_{A_p}(a_i) \wedge \mu_{B_p}(b_j)) \wedge \mu_{C_p}(a_k) =: P_{ijk},$$

$$Q_{ij} = \bigvee_{q=1}^{s}(\mu_{D_q}(a_i) \wedge \mu_{E_q}(b_j)).$$

It is to note that our consideration is not restricted to the one input and one output fuzzy linguistic models. With similar arguments as in [80], $x(k)$ and $u(k)$ may be multidimensional linguistic variables too. Under these circumstances, $x(k)$ and $u(k)$ in (3.1) and (3.2) will be replaced by the Cartesian product of state linguistic variables and Cartesian product of control linguistic variables respectively.

3.2 On the Definition of Stability for Linguistic Fuzzy Models

In (3.1) and (3.2), linguistic variables $x(k)$ and $u(k)$ take values of fuzzy sets rather than conventional values. Then, how can the concept of stability be defined? For the sake of convenience we consider first the open loop linguistic fuzzy models:

$$x(k+1) = x(k) \circ R. \tag{3.3}$$

From (3.3) it follows $x(k) = x(0) \circ R^k$. Then in [72] the stability of (3.3) is reduced to the convergence of R^k. If $R^k \to \widetilde{R}$ as $k \to \infty$, the solution $x(0) \circ \widetilde{R}$ of (3.3) is called stable. This definition is too strict, since many real stable systems may not satisfy the condition of convergence. In [80] a relaxed definition is presented with peak patterns. The peak pattern of fuzzy set A on the universe of discourse X is a function $PP : X \to \{0, 1\}$ with $PP(x) = 1$ if $\mu_A(x) = \max\{\mu_A(x)|x \in X\}$ and $PP(x) = 0$ if $\mu_A(x) \neq \max\{\mu_A(x)|x \in X\}$. An equilibrium state x_e (i.e. $x_e = x_e \circ R$) is called stable, if its peak pattern doesn't cover any boundary element of X, and if there exists a K_0 such that $x(k)$ and x_e have the same peak pattern for some initial state $x(0)$ and all $k > K_0$. In the definition the equilibrium state whose membership function takes its maximal value on the boundary of X is not taken into account. For in this situation, the equilibrium may possibly turn infinitely large. In fact, this definition implies that an equilibrium is stable so long as its membership function doesn't take maximal value at the boundary of X (by setting the initial state to equilibrium, this is easy to see). This definition is revised in [79] by defining the degree of stability. The stability degree of (equilibrium) state x_e is defined by $\sigma(x_e) = 1 - I(x_e, X_B)$, where $I(x_e, X_B)$ denotes the degree to which fuzzy set x_e is included in the boundary set X_B of X. But the choice of $I(,)$ is left open. Similarly, the index of stability is given in [30] by a certain measure of fuzziness. That is, if for some $x(0)$ there exists a positive K such that the 'distance' between $x(k)$ and $x(0)$ can be sufficiently small for all $k > K$, then the state $x(0)$ is called stable no matter whether $x(0)$ is an equilibrium or not. The stability definition in [62] is described by the state equivalence, if for some initial state and all sufficiently large k, the membership functions of $x(k)$ and the equilibrium x_e can take maximal value at some point simultaneously, then the equilibrium x_e is defined to be stable. According to the definitions given in [30] and [62], all the equilibriums of (3.3) are also stable. Another definition to note is presented in [43], if the energy of (3.3) decreases monotonically until an equilibrium state is reached, then the equilibrium is stable. But how to measure the energy of (3.3) is difficult. The method given in [43] is mainly based on intuition and physical consideration, so it cannot be applied to general linguistic fuzzy models.

The definitions mentioned above are concerned with local stability of linguistic fuzzy models. In [13] and [14] the stability of (3.3) is so defined that a final state $x(k)$ can approach equilibrium x_e along $x(k+1) = x(k) \circ R$ from any normal initial state $x(0)$. This is in fact a concept of global stability, since the initial state can be an arbitrary normal fuzzy set (A fuzzy set is called normal if the maximal value of its membership function equals 1). But in the definition, what 'approach' means is left open. In the main result of [14], $\beta(x(k), x_e) := 1 - x(k) \circ x_e$ is used to describe how $x(k)$ approaches to x_e. We present two counterexamples to show that it is inappropriate to verify the stability of linguistic fuzzy models by means of $\beta(x(k), x_e)$. That is, even if $\beta(x(k), x_e) = 0$, $x(k)$ may not 'approach' x_e either.

The original main result in [14] is as follows:

Theorem 1 ([14]). Assume that initial state X_k is a normal fuzzy set. Then for any initial state X_k, fuzzy control systems described by $X_{k+1} = X_k \circ R$ are stable and will approach equilibrium state X_e, if and only if, there exists a positive integer N and, when $n \geqslant N$, we have $R^n \circ x_e \neq \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$.

Example 3.1 *Let* $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ *and* $x_e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$*, it follows that x_e is an equilibrium of (3.3) and $R^k \circ x_e = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ for all $k \geqslant 1$. Thereby, according to the sufficient condition of Theorem 1 in [14], the linguistic fuzzy system described by (3.3) is stable for any normal initial state. However, for initial state $x(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, it is easy to see that $x(2k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $x(2k+1) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ for all $k \geqslant 0$. Thereby, $\beta(x(k), x_e) \equiv 0$. Without loss of generality, we suppose that the universe of discourse of $x(k)$ is $X = \{a_1, a_2, a_3\}$. Then fuzzy sets $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ stand for the conventional sets $\{a_1, a_2, a_3\}$, $\{a_1\}$ and $\{a_3\}$ respectively. That is, the non-fuzzy state $x(k)$ will take non-fuzzy values a_1 and a_3 alternatively if the initial state is chosen as $x(0) = a_1$. So (3.3) is unstable for initial state $x(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, which implies the sufficient condition of Theorem 1 in [14] is invalid.*

Example 3.2 *Let* $R = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.3 \end{bmatrix}$, $x_e = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$. *Then for any normal initial state* $x(0)$ *and* $k \geqslant 1$ *we have:* $x(k+1) = x(k) \circ R = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$. *Therefore (3.3) is stable according to the definition in [14], but* $R^k \circ x_e \neq \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ *for any* $k \geqslant 1$. *Thereby, the necessary condition of Theorem 1 in [14] is also invalid. For the same reason Theorem 2 and Theorem 3 in [14] are invalid either.*

It is not difficult to prove that the necessary condition of Theorem 1 in [14] can be revised as follows:

Suppose that R is a maximal relation (i.e. each row of R has at least one element of value 1), and (3.3) is (globally) stable at equilibrium state x_e, then there exists a positive integer K, such that $R^k \circ x_e \neq \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}^T$ for any $k \geqslant K$.

3.3 Condition for the Simplification of Closed-loop Linguistic Fuzzy Models

Consider the closed-loop linguistic fuzzy models described by (3.1) and (3.2). Substituting (3.2) into (3.1) we have:

$$x(k+1) = (x(k) \times (x(k) \circ Q)) \circ P \tag{3.4}$$

Since (3.4) cannot be used in iteration and is difficult to analyze, some simplifications of (3.4) are presented in literature e.g. [79], [14]. Naturally we hope that (3.4) can be simplified as (3.3). For this purpose, (3.4) is formulated in [14] as $x(k+1) = x(k) \circ R(p)$, where $R(p)$ is a relational matrix dependent on p, and p is a positive integer dependent on $x(k)$. Since $R(p)$ varies with $x(k)$, it cannot be used in iteration either. The following result reveals the necessary and sufficient conditions, with which (3.4) can be simplified as (3.3).

Theorem 3.1 *The linguistic fuzzy models described by (3.1) and (3.2) can be formulated as* $x(k+1) = x(k) \circ R$ *if and only if* $R = Q \circ P$ *and:*
$\bigvee_j [(Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr})] = \bigvee_j [(Q_{ij} \wedge P_{ijr}) \vee (Q_{tj} \wedge P_{tjr})] \;\; \forall i, t, r \in \{1, 2, ..., n\}$.

Proof. "⇒" From (3.1) and (3.2) it follows:

$$\mu_{x(k+1)}(a_r) = \bigvee_{i,j} [\mu_{x(k)}(a_i) \wedge \mu_{u(k)}(b_j) \wedge P_{ijr}] \quad \forall a_r \in \{a_1, a_2, ..., a_n\} \tag{3.5}$$

$$\mu_{u(k)}(b_j) = \bigvee_t [\mu_{x(k)}(a_t) \wedge Q_{tj}] \quad \forall b_j \in \{b_1, b_2, ..., b_m\} \tag{3.6}$$

Combining (3.5) and (3.6) we have:

$$\mu_{x(k+1)}(a_r) = \bigvee_{i,j,t} [\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}] \tag{3.7}$$

If (3.1) and (3.2) can be formulated as $x(k+1) = x(k) \circ R$, then:

$$\mu_{x(k+1)}(a_r) = \bigvee_{i} (\mu_{x(k)}(a_i) \wedge R_{ir}) \tag{3.8}$$

Combining (3.7) and (3.8) we obtain:

$$\bigvee_{i} (\mu_{x(k)}(a_i) \wedge R_{ir}) = \bigvee_{i,j,t} [\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}] \quad \forall r \in \{1, 2, ..., n\} \tag{3.9}$$

Note that (3.9) holds for any membership function $\mu_{x(k)}(x)$. Let :

$$\mu_{x(k)}(x) = \begin{cases} 1 & x = a_i \\ 0 & x \neq a_i \end{cases} . \tag{3.10}$$

Combining (3.10) and (3.9), we have:

$$R_{ir} = \bigvee_{j} (Q_{ij} \wedge P_{ijr}). \tag{3.11}$$

Since (3.11) holds for all $i, r \in \{1, 2, ..., n\}$, it follows:

$$R = Q \circ P.$$

Then, by substitution it gives:

$$x(k+1) = x(k) \circ (Q \circ P) = (x(k) \times u(k)) \circ P,$$

which implies:

$$\bigvee_{i} [\mu_{x(k)}(a_i) \wedge (\bigvee_{j} (Q_{ij} \wedge P_{ijr}))] = \bigvee_{i,j} [\mu_{x(k)}(a_i) \wedge \mu_{u(k)}(b_j) \wedge P_{ijr}] = \mu_{x(k+1)}(a_r).$$

Then we have:

$$\bigvee_{i,j} [\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr}] = \bigvee_{i,j,t} [\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}]. \tag{3.12}$$

Choose $\mu_{x(k)}(x)$ as:

$$\mu_{x(k)}(x) = \begin{cases} 1 & x = a_i \ or \ x = a_t \\ 0 & x \neq a_i \ and \ x \neq a_t \end{cases} .$$

Then, from (3.12) it follows:

$$\bigvee_{j} [(Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr})] = \bigvee_{j} [(Q_{ij} \wedge P_{ijr}) \vee (Q_{tj} \wedge P_{tjr})] \ \forall i, t, r \in \{1, 2, ..., n\}$$

"$\Leftarrow$": If it holds:

$$\bigvee_{j} [(Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr})] = \bigvee_{j} [(Q_{ij} \wedge P_{ijr}) \vee (Q_{tj} \wedge P_{tjr})] \, ,$$

then for any $\mu_{x(k)}(a_i), \mu_{x(k)}(a_t) \in [0, \ 1]$ we have:

$\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge [\bigvee_j((Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr}))]$
$= \mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge [\bigvee_j((Q_{ij} \wedge P_{ijr}) \vee (Q_{tj} \wedge P_{tjr}))]$
$= [\bigvee_j(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{ij} \wedge P_{ijr})] \vee [\bigvee_j(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{tjr})]$
$\leqslant [\bigvee_j(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})] \vee [\bigvee_j(\mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{tjr})]$.

That is:

$\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge [\bigvee_j((Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr}))]$

$$\leqslant [\bigvee_j(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})] \vee [\bigvee_j(\mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{tjr})]. \tag{3.13}$$

From (3.13) it follows:

$\bigvee_{i,t}[\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge (\bigvee_j((Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr})))]$

$$\leqslant \bigvee_{i,t}[(\bigvee_j(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})) \vee (\bigvee_j(\mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{tjr}))]. \tag{3.14}$$

For the left side of (3.14), it holds:

$\bigvee_{i,t}[\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge (\bigvee_j((Q_{ij} \wedge P_{tjr}) \vee (Q_{tj} \wedge P_{ijr})))]$
$=[\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{ij} \wedge P_{tjr})] \vee [\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr})]$
$=\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr})$.

Similarly, the right side of (3.14) satisfies:

$\bigvee_{i,t}[(\bigvee_j(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})) \vee (\bigvee_j(\mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{tjr}))] = \bigvee_{i,j}(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})$.

Combining with (3.14) then we have:

$$\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}) \leqslant \bigvee_{i,j}(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr}). \tag{3.15}$$

Note that the right side of (3.15) is a component of the left side as $i = t$. It follows:

$\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}) \geqslant \bigvee_{i,j}(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr})$.

Then we have:

$$\bigvee_{i,j,t}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_t) \wedge Q_{tj} \wedge P_{ijr}) = \bigvee_{i,j}(\mu_{x(k)}(a_i) \wedge Q_{ij} \wedge P_{ijr}). \tag{3.16}$$

Since (3.16) holds for all $r \in \{1, 2, ..., n\}$, it follows:

$x(k+1) = (x(k) \times u(k)) \circ P = x(k) \circ (Q \circ P) = x(k) \circ R$. ■

3.4 Algorithm for Determining the Greatest Equilibrium

Similar to the concept of stability in conventional control systems, the stability of linguistic fuzzy models is also defined with respect to the equilibriums of the systems as mentioned in Section 3.4. In general, linguistic fuzzy models have endless equilibriums. If we find the greatest equilibrium, we can obtain all the equilibriums of a linguistic fuzzy model. For open-loop linguistic fuzzy models described by (3.3), the greatest equilibriums can be solved by the algorithms in [66]. Theorem 3.1 shows that the closed-loop linguistic fuzzy models cannot always be formulated into the form of (3.3). In this section, we propose an algorithm, with which the greatest equilibriums of the closed-loop fuzzy linguistic models can be directly determined without solving the relational equations.

Suppose that the linguistic fuzzy model is described by (3.1) and (3.2). According to (3.7) in the proof of Theorem 3.1, we have

$$x(k+1) = ((x(k) \times x(k)) \circ (Q \circ P).$$

Denote $R := Q \circ P$, then:

$$x(k+1) = (x(k) \times x(k)) \circ R \tag{3.17}$$

where $R = (R_{ijk})_{(n \times n) \times n}$ is a relational matrix with $n \times n$ rows and n columns. The equilibriums of (3.17) are the solutions of $x(k) = (x(k) \times x(k)) \circ R$ according to the definition in [80]. It is easy to find that in general case (3.17) has endless equilibriums. The smallest equilibrium among them is zero fuzzy set, and the greatest equilibrium is the fuzzy union of all its equilibriums. Now we show how the greatest equilibrium can be solved directly.

Let r_i be the maximal value of the i-th column of R for $i = 1, 2, ..., n$, and let x_0 be a fuzzy set with membership function $\mu_{x_0}(a_i) = r_i$ for $i = 1, 2, ..., n$. That is:

$$x_0 = [r_1, r_2, ..., r_n].$$

Lemma 3.1 *If x_e is an equilibrium of (3.17), then $x_e \leqslant x_0$.*

Proof. *Provided that x_e is an equilibrium of (3.17), that is $x_e = (x_e \times x_e) \circ R$, then for all $a_t \in \{a_1, a_2, ..., a_t\}$ it holds:*

$$\mu_{x_e}(a_t) = \bigvee_{i,j} (\mu_{x_e}(a_i) \wedge \mu_{x_e}(a_j) \wedge R_{ijt}). \tag{3.18}$$

Since r_i is the maximal value of the i-th column of R, it follows: $R_{ijt} \leqslant r_t$. Then we have that (3.18) implies $\mu_{x_e}(a_t) \leqslant r_t$. That is: $x_e \leqslant x_0$. ■

Lemma 3.2 *If initial state $x(0)$ is chosen as $x_0 = [r_1, r_2, ..., r_n]$, then $x(k+1) \leqslant x(k)$ by iteration along (3.17) for all $k \geqslant 0$.*

Proof. (Induction) If $k = 0$, then for any $a_t \in \{a_1, a_2, ..., a_n\}$ from (3.17) we have

$$\mu_{x(1)}(a_t) = \bigvee_{i,j}(\mu_{x_0}(a_i) \wedge \mu_{x_0}(a_j) \wedge R_{ijt}) = \bigvee_{i,j}(r_i \wedge r_j \wedge R_{ijt}).$$

Combining with $R_{ijt} \leqslant r_t$ we have $\mu_{x(1)}(a_t) \leqslant r_t$. Therefore $x(1) \leqslant x_0$.

Suppose that $x(k) \leqslant x(k-1)$, then for any $a_t \in \{a_1, a_2, ..., a_n\}$ we have:

$$\begin{aligned}\mu_{x(k+1)}(a_t) &= \bigvee_{i,j}(\mu_{x(k)}(a_i) \wedge \mu_{x(k)}(a_j) \wedge R_{ijt}) \\ &\leqslant \bigvee_{i,j}(\mu_{x(k-1)}(a_i) \wedge \mu_{x(k-1)}(a_j) \wedge R_{ijt}) \\ &= \mu_{x(k)}(a_t).\end{aligned}$$

Thereby $x(k+1) \leqslant x(k)$ holds. By induction it gives $x(k+1) \leqslant x(k)$ for all $k \geqslant 0$. ■

Note that the entries of $x(k)$ come from the entries of R for all $k \geqslant 0$ if the initial state $x_0 = [r_1, r_2, ..., r_n]$, which implies that there are only finite elements in set $\{x(k)\}$. From Lemma 3.2 it follows that $x(k)$ is monotonically decreasing for $k \geqslant 0$. Therefore, the sequence of $x(k)$ for $k \geqslant 0$ must be convergent and there must exist a positive integer N, such that $x(N) = x(N+1) = ... =: X_e$. Then we have $X_e = (X_e \times X_e) \circ R$. We will prove further that X_e is just the greatest equilibrium of (3.17).

Theorem 3.2 *The greatest equilibrium of (3.17) is the limit of sequence $x(k)$ with initial state $x_0 = [r_1, r_2, ..., r_n]$.*

Proof. Suppose that x_e is any equilibrium of (3.17). From Lemma 3.1 we have:

$$x_e \leqslant x_0.$$

Thereby:

$$x_e = (x_e \times x_e) \circ R \leqslant (x_0 \times x_0) \circ R = x(1).$$

Similarly, from $x_e \leqslant x(1)$ we have:

$$x_e = (x_e \times x_e) \circ R \leqslant (x(1) \times x(1)) \circ R = x(2).$$

Repeating the process we have:

$$x_e \leqslant x(k) \text{ for all } k \geqslant 0.$$

Since $x(k) \to X_e$ for initial state $x_0 = [r_1, r_2, ..., r_n]$, then from Lemma 3.2 it follows $x(k) \leqslant X_e$. Thereby, we have:

$$x_e \leqslant x(k) \leqslant X_e,$$

which means that X_e is the greatest equilibrium of (3.17). ■

Example 3.3 *Suppose that the closed-loop linguistic fuzzy model is described by:*

$$x(k+1) = (x(k) \times x(k)) \circ R,$$

where

$$R^T = \begin{bmatrix} 0.4 & 0.5 & 0.8 & 0.5 & 0.8 & 0.6 & 0.3 & 0.5 & 0.1 \\ 1 & 0.7 & 0.4 & 0.3 & 0.2 & 0.4 & 0.9 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.2 & 0.6 & 0.2 & 0.7 & 0.5 & 0.3 & 0.4 \end{bmatrix}.$$

According to Theorem 3.2 we have:

$x_0 = [\ 0.8 \quad 1 \quad 0.7\ ]$, $x(1) = [\ 0.8 \quad 0.8 \quad 0.7\ ]$, $x(2) = [\ 0.8 \quad 0.8 \quad 0.7\ ] = x(1)$. *Thereby, the greatest equilibrium is* $X_e = [\ 0.8 \quad 0.8 \quad 0.7\ ]$.

3.5 Global Stability of Linguistic Fuzzy Models

Definition 3.1 *Suppose that* x_e *is an equilibrium of the linguistic fuzzy model described by (3.3). If* $x(k)$ *converges to* x_e *along (3.3) for any normal initial state* $x(0)$*, then the equilibrium* x_e *(or system (3.3)) is called globally stable.*

It is not difficult to find:

- If x_e is globally stable, then x_e is the greatest equilibrium of (3.3).
- The global stability of a linguistic fuzzy model doesn't imply the uniqueness of equilibriums. In general, (3.3) has infinite equilibriums even if it is globally stable.
- For linguistic fuzzy models, if equilibrium x_e is globally stable, then it is globally asymptotically stable as well. Since the sequence $x(k)$ along (3.3) can take only finite fuzzy sets for any normal initial state $x(0)$, thereby $x(k) \equiv x_e$ holds for all sufficiently large k.
- In the definition, the requirement of normal initial state is natural. If initial state $x(0)$ is not normal, then $x(k)$ along (3.3) converges to equilibrium $x(0) \circ R_e$ (see Lemma 3.3).

Suppose x_e is a fuzzy set described by $x_e = [\ \mu_{x_e}(a_1),\ \ \mu_{x_e}(a_2),\ \ ...,\ \ \mu_{x_e}(a_n)\ ]$ on the universe of discourse $X = \{a_1, a_2, ..., a_n\}$. For the sake of convenience, fuzzy set x_e will be denoted as $x_e = [\ x_1,\ \ x_2,\ \ ...,\ \ x_n\ ]$ and $x \wedge y$ be denoted as xy in the case of no confusion.

Lemma 3.3 *For any normal initial state* $x(0)$*,* $x(k)$ *along (3.3) converges to* x_e*, if and only if* R^k *converges to* R_e *where:*

$$R_e = \begin{bmatrix} x_1 & x_2 & ... & x_n \\ x_1 & x_2 & ... & x_n \\ & & ... & \\ x_1 & x_2 & ... & x_n \end{bmatrix}.$$

Proof. From $x(k+1) = x(k) \circ R$, it follows $x(k) = x(0) \circ R^k$. Since $x(k)$ converges to x_e along (3.3) for any normal initial state $x(0)$, by setting $x(0) = [\ 1 \quad 0 \quad 0 \quad ... \quad 0\ ]$,

$[\ 0\ \ 1\ \ 0\ \ ...\ \ 0\ ], ..., [\ 0\ \ 0\ \ 0\ \ ...\ \ 1\ ]$ respectively we have that R^k converges to R_e. On the other hand, if R^k converges to R_e, obviously the conclusion also holds. ■

In order to deduce the main result, we consider at first a special case of x_e. Suppose that $x_e = [\ x_1,\ \ x_2,\ \ ...,\ \ x_n\ ]$ with $x_1 > x_2 > ... > x_n$. We will prove that the equilibrium x_e of (3.3) is globally stable, if and only if R is a fuzzy relational matrix of the following forms:

$$\begin{bmatrix}
 & & & \overbrace{m} & & \overbrace{m} \\
x_1 & x_2 & & a & & a \\
 & & & x & & x \\
(\geqslant x_1) & (\leqslant x_2) & & = & & \\
 & & \ddots & \underbrace{x_i} & & = \\
(\max & \geqslant & x_1) & (\leqslant x_i) & & \\
 & & & & \ddots & \underbrace{x_n} \\
(\max & & \geqslant & & x_1) & (\leqslant x_n)
\end{bmatrix}. \tag{3.19}$$

It is easy to find that the relational matrix $R = (r_{ij})_{n \times n}$ of the form (3.19) is equivalent to the following conditions:

1° : $r_{ij} \leqslant x_j$ for $i \leqslant j$.

2° : $\bigvee\limits_{1 \leqslant i < j} r_{ij} = x_j$ for $j = 2, 3, ..., n$.

3° : $\bigvee\limits_{1 \leqslant j < i} r_{ij} \geqslant x_1$ for $i = 2, 3, ..., n$.

4° : $r_{11} = x_1$.

Lemma 3.4 *If R satisfies conditions $1^\circ - 4^\circ$, then for any positive integer m, R^m satisfies conditions $1^\circ - 4^\circ$ as well.*

Proof. Let $r_{ij}^{(2)}$ be the element of R^2 located in the i-th row and j-th column. It is enough to verify that R^2 satisfies conditions $1^\circ - 4^\circ$.

1). From $R^2 = R \circ R$ it follows:

$$r_{ij}^{(2)} = \bigvee_{k=1}^{n} (r_{ik} \wedge r_{kj}) = r_{i1}r_{1j} \vee r_{i2}r_{2j} \vee ... \vee r_{ij}r_{jj} \vee ... \vee r_{in}r_{nj} \text{ for } i \leqslant j.$$

Condition 1° gives:

$$r_{1j}, r_{2j}, ..., r_{jj} \leqslant x_j$$
$$r_{i,j+1} \leqslant x_{j+1}, r_{i,j+2} \leqslant x_{j+2}, ..., r_{in} \leqslant x_n.$$

Thereby we have:

$$r_{ij}^{(2)} \leqslant x_j\ (i \leqslant j). \tag{3.20}$$

So R^2 satisfies condition 1°.

2). From 2°, we obtain that there exists an $i_0 < j$, such that:

$$\bigvee_{1 \leqslant i < j} r_{ij} = r_{i_0 j} = x_j \ (i_0 < j). \tag{3.21}$$

Applying 2° again, we have: $\bigvee_{1 \leqslant i < i_0} r_{ii_0} = x_{i_0}$. From $i_0 < j$ it follows $x_{i_0} > x_j$. Combining (3.21) we have that there exists an integer k with $1 \leqslant k < i_0$ such that:

$$r_{ki_0} = x_{i_0} > x_j. \tag{3.22}$$

From (3.21), (3.22) and $r_{kj}^{(2)} = r_{k1}r_{1j} \vee r_{k2}r_{2j} \vee ... \vee r_{ki_0}r_{i_0 j} \vee ... \vee r_{kn}r_{nj}$ we have $r_{kj}^{(2)} \geqslant x_j$. Then from (3.20) it follows $\bigvee_{1 \leqslant i < j} r_{ij}^{(2)} = x_j$ for $j = 2, 3, ..., n$. Therefore R^2 satisfies condition 2°.

3). Condition 3° gives: $\bigvee_{1 \leqslant j < i} r_{ij} \geqslant x_1$. Thereby, for any $i \in \{2, 3, ..., n\}$, there exists $j_0 \in \{1, 2, ..., i-1\}$ such that:

$$r_{ij_0} \geqslant x_1. \tag{3.23}$$

Denote $\bigvee_{1 \leqslant k < j_0} r_{j_0 k} =: r_{j_0 k_0}$ and apply condition 3° we have:

$$r_{j_0 k_0} \geqslant x_1. \tag{3.24}$$

From (3.23), (3.24) and $r_{ik_0}^{(2)} = r_{i1}r_{1k_0} \vee r_{i2}r_{2k_0} \vee ... \vee r_{ij_0}r_{j_0 k_0} \vee ... \vee r_{in}r_{nk_0}$, it follows: $r_{ik_0}^{(2)} \geqslant x_1$, $(k_0 < i)$. Therefore: $\bigvee_{1 \leqslant j < i} r_{ij}^{(2)} \geqslant x_1$ for $i = 2, 3, ..., n$. So R^2 satisfies condition 3°.

4). With direct calculation we have $r_{11}^{(2)} = x_1$. Thereby, R^2 satisfies condition 4°.

Since R^2 satisfies conditions $1° - 4°$, it is easy to show that R^m satisfies conditions $1° - 4°$ for all $m \geqslant 1$ as well. ■

Lemma 3.4 illustrates that the form of (3.19) is invariant under the max-min composition.

Theorem 3.3 *Let* $x_e = [\ x_1 \quad x_2 \quad ... \quad x_n\ ]$ *with* $x_1 > x_2 > ... > x_n$. *State* $x(k)$ *converges to* x_e *along* $x(k+1) = x(k) \circ R$ *for any normal initial state* $x(0)$, *if and only if the relational matrix* R *has the form of (3.19).*

Proof. We prove first that the sufficient condition is valid.

If R satisfies conditions $1° - 4°$, then from condition 4° and condition 2° we have $r_{11} = x_1$ and $r_{12} = x_2$. Therefore R is of the form :

$$R = \begin{bmatrix} x_1 & x_2 & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ & & & \cdots & \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix}.$$

Direct calculation gives $r_{11}^{(2)} = x_1$ and $r_{12}^{(2)} = x_2$. From condition 2° it follows $\max\{r_{13}, r_{23}\} = x_3$. Condition 1° gives: $r_{13} \leqslant x_3, r_{14} \leqslant x_4, ..., r_{1n} \leqslant x_n$. Note that $x_1 > x_2 > ... > x_n$ and $r_{13}^{(2)} = x_1 r_{13} \vee x_2 r_{23} \vee r_{13} r_{33} \vee ... \vee r_{1n} r_{n3}$. Then we have $r_{13}^{(2)} = x_3$.

According to condition 3° we have $r_{21} \geqslant x_1$. Then

$$r_{21}^{(2)} = r_{21} x_1 \vee r_{22} r_{21} \vee ... \vee r_{2n} r_{n1} = x_1 \vee (r_{22} r_{21} \vee ... \vee r_{2n} r_{n1}).$$

From condition 1° it follows $r_{22} r_{21} \vee ... \vee r_{2n} r_{n1} \leqslant x_2 < x_1$. Therefore we get: $r_{21}^{(2)} = x_1$.

Since $r_{22}^{(2)} = r_{21} x_2 \vee r_{22} r_{22} \vee ... \vee r_{2n} r_{n2}$ and $r_{21} \geqslant x_1$, applying condition 1° we have: $r_{22}^{(2)} = x_2$. Therefore R^2 is of the form:

$$R^2 = \begin{bmatrix} x_1 & x_2 & x_3 & r_{14}^{(2)} & ... & r_{1n}^{(2)} \\ x_1 & x_2 & r_{23}^{(2)} & r_{24}^{(2)} & ... & r_{2n}^{(2)} \\ r_{31}^{(2)} & r_{32}^{(2)} & r_{33}^{(2)} & r_{34}^{(2)} & ... & r_{3n}^{(2)} \\ & & & & ... & \\ r_{n1}^{(2)} & r_{n2}^{(2)} & r_{n3}^{(2)} & r_{n4}^{(2)} & ... & r_{nn}^{(2)} \end{bmatrix}.$$

Similarly we can calculate the elements of R^3:

$r_{11}^{(3)} = x_1$, $r_{12}^{(3)} = x_2$, $r_{13}^{(3)} = x_3$, $r_{21}^{(3)} = x_1$, $r_{22}^{(3)} = x_2$, $r_{23}^{(3)} = x_3$, $r_{31}^{(3)} = x_1$, $r_{32}^{(3)} = x_2$.

Since $\max\{r_{14}, r_{24}, r_{34}\} = x_4$ by 2°, and $r_{14}^{(2)} \leqslant x_4, r_{15}^{(2)} \leqslant x_5, ..., r_{1n}^{(2)} \leqslant x_n$ by Lemma 3.4, then we have:

$$\begin{aligned} r_{14}^{(3)} &= \bigvee_{j=1}^{n} (r_{1j}^{(2)} \wedge r_{j4}) = x_1 r_{14} \vee x_2 r_{24} \vee x_3 r_{34} \vee r_{14}^{(2)} r_{44} \vee ... \vee r_{1n}^{(2)} r_{n4} \\ &= x_4 \vee (r_{14}^{(2)} r_{44} \vee ... \vee r_{1n}^{(2)} r_{n4}) \\ &= x_4. \end{aligned}$$

Therefore R^3 is of the form:

$$R^3 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & ... & r_{1n}^{(3)} \\ x_1 & x_2 & x_3 & r_{24}^{(3)} & ... & r_{2n}^{(3)} \\ x_1 & x_2 & r_{33}^{(3)} & r_{34}^{(3)} & ... & r_{3n}^{(3)} \\ r_{41}^{(3)} & r_{42}^{(3)} & r_{43}^{(3)} & r_{44}^{(3)} & ... & r_{4n}^{(3)} \\ & & & & ... & \\ r_{n1}^{(3)} & r_{n2}^{(3)} & r_{n3}^{(3)} & r_{n4}^{(3)} & ... & r_{nn}^{(3)} \end{bmatrix}.$$

Continuing the process we have:

$$R^n = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & ... & x_n \\ & & & & ... & \\ x_1 & x_2 & x_3 & x_4 & ... & r_{n-2,n}^{(n)} \\ x_1 & x_2 & x_3 & r_{n-1,4}^{(n)} & ... & r_{n-1,n}^{(n)} \\ x_1 & x_2 & r_{n3}^{(n)} & r_{n4}^{(n)} & ... & r_{nn}^{(n)} \end{bmatrix}.$$

It is easy to verify that $R^{n+1} = R_e$. Applying Lemma 3.3 we obtain the sufficient condition.

To verify the necessary conditions, we will separate our proofs in the following three steps.

a): If $x(k)$ along $x(k+1) = x(k) \circ R$ converges to x_e, then $x_e = x_e \circ R$. Therefore we have:

$$x_1 r_{11} \vee x_2 r_{21} \vee ... \vee x_n r_{n1} = x_1 \tag{3.25}$$

$$x_1 r_{12} \vee x_2 r_{22} \vee ... \vee x_n r_{n2} = x_2 \tag{3.26}$$

$$...$$

$$x_1 r_{1n} \vee x_2 r_{2n} \vee ... \vee x_n r_{nn} = x_n \tag{3.27}$$

From (3.25) and condition $x_1 > x_2 > ... > x_n$, we obtain $r_{11} \geqslant x_1$. Similarly we have: $r_{12} \leqslant x_2$, $\max\{r_{13}, r_{23}\} \leqslant x_3$,..., $\max\{r_{1n}, r_{2n}, ..., r_{n-1,n}\} \leqslant x_n$. Thereby:

$$r_{ij} \leqslant x_j \ (i \leqslant j). \tag{3.28}$$

In addition, from Lemma 3.3 we have that R^m converges to R_e, which implies: $r_{ii} \leqslant x_i$ for all $i \in \{1, 2, ..., n\}$. Otherwise, if there exists an $i_0 \in \{1, 2, ..., n\}$ such that $r_{i_0 i_0} > x_{i_0}$, then:

$$\begin{aligned} r^{(2)}_{i_0 i_0} &= \textstyle\bigvee_{k=1}^{n} (r_{i_0 k} \wedge r_{k i_0}) \geqslant r_{i_0 i_0} > x_{i_0} \\ r^{(3)}_{i_0 i_0} &= \textstyle\bigvee_{k=1}^{n} (r^{(2)}_{i_0 k} \wedge r_{k i_0}) \geqslant r^{(2)}_{i_0 i_0} \wedge r_{i_0 i_0} > x_{i_0} \\ &... \\ r^{(m)}_{i_0 i_0} &= \textstyle\bigvee_{k=1}^{n} (r^{(m-1)}_{i_0 k} \wedge r_{k i_0}) \geqslant r^{(m-1)}_{i_0 i_0} \wedge r_{i_0 i_0} > x_{i_0}. \end{aligned}$$

Therefore the sequence $r^{(m)}_{i_0 i_0}$ cannot converge to x_{i_0}, which is contrary to the condition of R^m convergent to R_e.

Then, it follows $r_{11} = x_1$ from $r_{11} \geqslant x_1$ and $r_{ii} \leqslant x_i$ for all $i \in \{1, 2, ..., n\}$. Combining with (3.28) we have that conditions 1° and 4° hold if $x(k)$ along (3.3) converges to x_e.

b): According to Lemma 3.3 we have that R^m converges to R_e. Then there exists a positive integer M such that the following equations hold for all $m \geqslant M$.

$$r^{(m+1)}_{21} = \bigvee_{k=1}^{n} (r_{2k} \wedge r^{(m)}_{k1}) = x_1 \tag{3.29}$$

$$r^{(m+1)}_{31} = \bigvee_{k=1}^{n} (r_{3k} \wedge r^{(m)}_{k1}) = x_1 \tag{3.30}$$

$$...$$

$$r^{(m+1)}_{n1} = \bigvee_{k=1}^{n} (r_{nk} \wedge r^{(m)}_{k1}) = x_1 \tag{3.31}$$

Since $r_{2k} \leqslant x_k < x_1$ for $2 \leqslant k \leqslant n$ by (3.28), combining with (3.29) we have: $r_{21} \geqslant x_1$. Similarly we have $\max\{r_{31}, r_{32}\} \geqslant x_1$ from (3.28) and (3.30). Continuing the process, we obtain $\max\{r_{n1}, r_{n2}, ..., r_{n,n-1}\} \geqslant x_1$ from $r_{nn} \leqslant x_n$ and (3.31). Therefore: $\bigvee_{1\leqslant j<i} r_{ij} \geqslant x_1.(i = 2, 3, ..., n)$. So condition 3° also holds.

c): Finally, if R doesn't satisfy condition 2°, then from (3.28) it follows that there exists a positive integer k, such that:

$$\max\{r_{1k}, r_{2k}, ..., r_{k-1,k}\} < x_k. \tag{3.32}$$

Since $r_{ik}^{(2)} = (r_{i1}r_{1k} \vee r_{i2}r_{2k} \vee ... \vee r_{i,k-1}r_{k-1,k}) \vee (r_{ik}r_{kk} \vee ... \vee r_{in}r_{nk})$ for $i < k$, then from (3.28) and (3.32) we have:

$$r_{ik}^{(2)} < x_k \ (i < k). \tag{3.33}$$

Similarly from (3.28), (3.32), (3.33) and

$$r_{ik}^{(3)} = (r_{i1}r_{1k}^{(2)} \vee r_{i2}r_{2k}^{(2)} \vee ... \vee r_{i,k-1}r_{k-1,k}^{(2)}) \vee (r_{ik}r_{kk}^{(2)} \vee ... \vee r_{in}r_{nk}^{(2)}) \quad (i < k)$$

we have: $r_{ik}^{(3)} < x_k$, $(i < k)$. Then, by induction it follows that $r_{ik}^{(m)} < x_k$ $(i < k)$ holds for all positive integer m, which is contrary to the condition of R^m convergent to R_e. Thereby we have: $\bigvee_{1\leqslant i<j} r_{ij} = x_j$ for $j = 2, 3, ..., n$.

So R satisfies conditions $1° - 4°$, which completes the proof. ■

From the above proofs we can see that the sufficient condition of Theorem 3.3 still holds if the entry elements of x_e satisfy $x_1 \geqslant x_2 \geqslant ... \geqslant x_n$ instead of $x_1 > x_2 > ... > x_n$.

Obviously every fuzzy set $\widetilde{x}_e$ can be transformed into the form of x_e by permutation if $\mu_{\widetilde{x}_e}(a_i) \neq \mu_{\widetilde{x}_e}(a_j)$ for all $i \neq j$. In addition, every permutation can be formulated as a matrix composition. Then we can deduce the main result for the general x_e with distinct entries. For the sake of convenience we rewrite the following conventional concepts of algebra in the sense of fuzzy operators.

Definition 3.2 *A matrix is called elementary if it can be obtained through column permutations of the unit matrix.*

It is to note that an elementary matrix here is restricted to the matrix transformed from the unit matrix only with column permutations. It is a little different from the definition in algebra.

Definition 3.3 *Fuzzy relational matrices A and B are called congruent if there exists an elementary matrix P such that: $P^T \circ A \circ P = B$.*

The concept of congruent here is also a little different from the definition in algebra. Relational matrices A and B are congruent means that matrix A can be transformed into matrix B by a series of similar column and row permutations.

Theorem 3.4 *Suppose that the greatest equilibrium of $x(k+1) = x(k) \circ R$ has distinct entry elements, then $x(k+1) = x(k) \circ R$ is globally stable if and only if R is congruent to a matrix of form (3.19).*

Proof. Let $\widetilde{x}_e = [\ \widetilde{x}_1 \quad \widetilde{x}_2 \quad ... \quad \widetilde{x}_n\ ]$ be the greatest equilibrium. Since the entry elements are distinct, then by permutations the entry elements can be rewritten in decreasing order. Denote the permuted vector as $x_e = [\ x_1 \quad x_2 \quad ... \quad x_n\ ]$ where $x_1 > x_2 > ... > x_n$. Note that each permutation of the elements of $\widetilde{x}_e$ is equivalent to a composition of $\widetilde{x}_e$ with a corresponding elementary matrix. Then the relation of $\widetilde{x}_e$ and x_e can be formulated as: $\widetilde{x}_e \circ P = x_e$, where P is a elementary matrix coming from the unit matrix with the same column permutations. Due to the stability definition, $x(k+1) = x(k) \circ R$ is globally stable, if and only if for any given normal initial state $x(0)$, there exists a positive integer M such that $x(0) \circ R^m = \widetilde{x}_e$ for all $m \geqslant M$. Then we have:

$$
\begin{aligned}
& x(0) \circ R^m = \widetilde{x}_e \\
& \Leftrightarrow x(0) \circ I \circ R^m \circ P = \widetilde{x}_e \circ P \\
& \Leftrightarrow (x(0) \circ P) \circ (P^T \circ R^m \circ P) = x_e \\
& \Leftrightarrow (x(0) \circ P) \circ (P^T \circ R \circ P)^m = x_e
\end{aligned}
$$

Denote $P^T \circ R \circ P =: Q$. It follows $R = P \circ Q \circ P^T$. From Theorem 3.3 we have that Q is a matrix of form (3.19). That is, R is congruent to a matrix of form (3.19). ■

Following the proofs of Theorem 3.3, it is easy to see that the sufficiency of Theorem 3.4 also holds, even if the entries of the greatest equilibrium are not distinct.

Example 3.4 *Suppose that the linguistic fuzzy model is described by:*

$$
x(k+1) = x(k) \circ \begin{bmatrix} 0.4 & 1 & 0.5 \\ 0.5 & 1 & 0.2 \\ 1 & 0.3 & 0.1 \end{bmatrix} =: x(k) \circ R
$$

From Theorem 3.2 we have the greatest equilibrium $\widetilde{x}_e = [\ 0.5 \quad 1 \quad 0.5\ ]$ ($\widetilde{x}_e$ has two identical entry elements). By permuting the first and the second entry elements we have: $\widetilde{x}_e \circ P = [\ 1 \quad 0.5 \quad 0.5\ ] =: x_e$, where $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then:

$$
P^T \circ R \circ P = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 1 & 0.4 & 0.5 \\ 0.3 & 1 & 0.1 \end{bmatrix} =: Q.
$$

Obviously Q is of form (3.19), therefore $x(k+1) = x(k) \circ R$ is globally stable according to Theorem 3.4. Since $R^4 = R_e$, the global stability of $\widetilde{x}_e$ is also verified by Lemma 3.3.

Example 3.5 *Suppose that the desired property of a closed loop linguistic fuzzy model is that it is globally stable with respect to the greatest equilibrium* $\widetilde{x}_e = [\ 0.6 \quad 0.4 \quad 0.9\ ]$. *For the purpose of fuzzy controller design, we want to find out all the linguistic fuzzy models satisfying the given property.*

We permute first the entry elements of $\widetilde{x}_e$ *in decreasing order. This can be formulated as:*

$$\widetilde{x}_e \circ P = [\ 0.9 \quad 0.6 \quad 0.4\ ] =: x_e,$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

According to Theorem 3.4, $x(k+1) = x(k) \circ R$ *is globally stable if and only if:*

$$R = P \circ Q \circ P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.6 & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where

$$q_{21} \geqslant 0.9,\ q_{31} \vee q_{32} \geqslant 0.9,\ q_{22} \leqslant 0.6,\ q_{33} \leqslant 0.4,\ q_{13} \vee q_{23} = 0.4.$$

obviously, there are infinite relational matrices satisfying the requirement. Among them the greatest relation matrix is

$$R = \begin{bmatrix} 0.6 & 0.4 & 1 \\ 1 & 0.4 & 1 \\ 0.6 & 0.4 & 0.9 \end{bmatrix}.$$

Now we consider the closed loop linguistic fuzzy models described by (3.1) and (3.2). With the method presented in [15] and [14], (3.1) and (3.2) can be formulated as:

$$x(k+1) = x(k) \circ R(p)$$

where $R(p)$ is a relational matrix dependent on state $x(k)$ and satisfies

$$x(k) \circ R_1 \leqslant x(k) \circ R(p) \leqslant x(k) \circ R_2 \tag{3.34}$$

where

$$R_1 = \bigvee_{i=1}^{l} (A_i \wedge (Q \wedge B_i) \wedge C_i),$$

$$R_2 = \bigvee_{i=1}^{l} ((A_i \vee (Q \wedge B_i)) \wedge C_i).$$

Then, from (3.34) and Theorem 3.4 we obtain directly:

Corollary 3.1 *If there exists an elementary matrix* P*, such that* R_1 *and* R_2 *are congruent to some matrices of form (3.19) respectively, then the closed loop linguistic fuzzy model is globally stable.*

Based on Theorem 3.4, we propose the following fuzzy controller design strategy for the linguistic fuzzy models described by (3.1):

(1) Find out all the feasible matrices R, such that the closed-loop linguistic fuzzy model $x(k+1) = x(k) \circ R$ has the desired stability property (as shown in Example 3.5).

(2) Solve Q from relational equation $Q \circ P = R$ by the method in [93] or [58]. If $Q \circ P = R$ has no solution, replace R by another from (1).

(3) Verify the condition of Theorem 3.1, if it is not satisfied, replace R by another one from (1) and repeat (2) until the condition is satisfied.

Then $u(k) = x(k) \circ Q$ is the desired fuzzy control law.

It is to note if $x(k)$ is multidimensional state or the universes of discourses of $x(k)$ and $u(k)$ have too many elements, the process mentioned above will lead to much computation, and how to simplify the process of design is to be researched further.

Chapter 4

Eigenvalue-based Stability Conditions for T-S Fuzzy Models

Based on Lyapunov's direct method, the stability of T-S fuzzy models can be reduced to finding a common positive definite matrix. We present first a necessary condition for the existence of such a positive definite matrix in terms of the eigenvalues of the system matrices. Then, we give a relaxed eigenvalue constraint for the stabilization of T-S fuzzy models using state feedback controller.

4.1 Formulation of T-S Fuzzy Models

T-S type Fuzzy models were first introduced by Takagi and Sugeno in [73]. Unlike linguistic fuzzy models, the consequent of each fuzzy rule in T-S fuzzy models is a crisp function of the antecedent variables rather than a fuzzy set. The basic idea of fuzzy modeling for T-S fuzzy models is to decompose the input space into a number of fuzzy regions in which the system behavior is approximated by a local linear model. The overall fuzzy model is then a fuzzy blending of the local models interconnected by a set of membership functions. In continuous case, T-S fuzzy models can be described by the following fuzzy rules ([73], [81]):

Plant rules: If $x_1(t)$ is M_1^i and ... and $x_n(t)$ is M_n^i, then:

$$\dot{x}(t) = A_i x(t) + B_i u(t) \ (i = 1, 2, ..., r)$$

where r is the number of fuzzy rules, M_j^i stand for the fuzzy set of the j-th antecedent variable in the i-th fuzzy rule, $u(t) = [u_1(t), u_2(t), ..., u_m(t)]^T$ is the control input, and $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ is the state variable. By the singleton fuzzifier, product inference and the center average defuzzifier, the final outputs of the fuzzy systems can be represented as:

$$\dot{x}(t) = \sum_{i=1}^{r}[\omega_i(x(t))/\sum_{j=1}^{r}\omega_j(x(t))](A_i x(t) + B_i u(t))$$

where $\omega_i(x(t)) = \prod_{j=1}^{n}\mu_{M_j^i}(x(t))$ and $\sum_{i=1}^{r}\omega_i(x(t)) \neq 0$ for all $t \geqslant 0$. Based on the parallel distributed compensation [74], the following control laws are always employed for the stabilization of T-S fuzzy models:

Controller rules: If $x_1(t)$ is M_1^i and ... and $x_n(t)$ is M_n^i, then:

$u(t) = K_i x(t)$ $(i = 1, 2, ..., r)$

where $K_i \in \Re^{m\times n}$ are the feedback gains to be designed. Then, the overall fuzzy state feedback control law can be expressed as:

$$u(t) = \sum_{i=1}^{r}[\omega_i(x(t))/\sum_{j=1}^{r}\omega_j(x(t))]K_i x(t).$$

For the sake of convenience we denote $\omega_i(x(t))/\sum_{j=1}^{r}\omega_j(x(t)) =: \alpha_i(x(t))$. Obviously it holds: $0 \leqslant \alpha_i(x(t)) \leqslant 1$ for all $i = 1, 2, ..., r$ and $\sum_{i=1}^{r}\omega_i(x(t)) = 1$. In general, $\alpha_i(x(t))$ can be regarded as the matching degree between the state variable and the antecedent of the i-th fuzzy rule.

By substituting $u(t)$ we obtain the following formulation of the closed loop models:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x(t))\alpha_j(x(t))(A_i + B_i K_j)x(t). \tag{4.1}$$

Then, the undriven (i.e. $u(t) \equiv 0$) continuous T-S fuzzy models can be formulated as:

$$\dot{x}(t) = \sum_{i=1}^{r}\alpha_i(x(t))A_i x(t). \tag{4.2}$$

Similarly, the discrete T-S fuzzy models and PDC-based fuzzy controllers can be described by the following fuzzy rules respectively:

Plant rules: If $x_1(k)$ is M_1^i and ... and $x_n(k)$ is M_n^i, then

$x(k+1) = A_i x(k) + B_i u(k)$ $(i = 1, 2, ..., r)$.

Controller rules: If $x_1(k)$ is M_1^i and ... and $x_n(k)$ is M_n^i, then

$u(k) = K_i x(k)$ $(i = 1, 2, ..., r)$.

Thus, the closed loop discrete T-S fuzzy models can be formulated as:

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x(k))\alpha_j(x(k))(A_i + B_i K_j)x(k), \tag{4.3}$$

and the open loop discrete T-S fuzzy models can be written as:

$$x(k+1) = \sum_{i=1}^{r}\alpha_i(x(k))A_i x(k). \tag{4.4}$$

The models presented above are most frequently employed in application. In fact, the premise variables in the fuzzy rules needn't be the state variables. The requirement on premise variables is that they must be measurable and independent of the control input.

4.2 Stability Analysis of T-S Fuzzy Models

Since T-S fuzzy models can be finally formulated in terms of differential or difference equations, they can be taken as the conventional nonlinear systems as well. Thereby, most of the stability analysis approaches for nonlinear systems can also be applied to the study of T-S fuzzy models. By Lyapunov's direct method the stability of fuzzy T-S models can be reduced to finding a common positive definite matrix [75]. In order to find the common positive matrix, a lot of numerical approaches have been presented in the literature, such as gradient algorithm [41], genetic approach [28], LMI approach [60], etc.. Moreover, The necessary conditions for the existence of such a common positive matrix are discussed e.g. in [75], [77] and [36]. However, the necessary and sufficient condition remains open. In this section, we present first a necessary eigenvalue constraint for the existence of such a common positive definite matrix. Then we give a relaxed sufficient condition for the stabilization of T-S fuzzy models via state fuzzy feedback controllers.

According to Theorem 4.2 in [75], the open loop model (4.4) is globally asymptotically stable if there is a common positive matrix P such that $A_i^T P A_i - P < 0$ $(i = 1, 2, ..., r)$. If all matrices A_i are non-singular, then the necessary condition for the existence of such a common positive matrix P is that $A_i A_j$ is stable for all $i, j = 1, 2, ..., r$ (Theorem 4.3, [75]). We will show that the non-singular condition of A_i is unnecessary. The result in [75] can be extended as:

Lemma 4.1 *For discrete T-S fuzzy model (4.4), the following sufficient stability conditions are equivalent:*

1. There is a positive symmetric matrix P such that $A_i^T P A_i - P < 0$ $(i = 1, 2, ..., r)$.

2. $A_{i_k}^T A_{i_{k-1}}^T ... A_{i_1}^T P A_{i_1} ... A_{i_{k-1}} A_{i_k} - P < 0$ for all $A_{i_j} \in \{A_1, A_2, ..., A_r\}$.

3. $(\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k})^T P (\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k}) - P < 0$ for all $A_{i_j} \in \{A_1, A_2, ..., A_r\}$ and all $k \in N$.

Proof. (1⇒2) Since $A_i^T P A_i - P < 0$, we have:

$$A_{i_1}^T P A_{i_1} - P =: -Q_1 < 0 \tag{4.5}$$

$$A_{i_2}^T P A_{i_2} - P =: -Q_2 < 0 \tag{4.6}$$

$$A_{i_3}^T P A_{i_3} - P =: -Q_3 < 0 \tag{4.7}$$

$$...$$

$$A_{i_k}^T P A_{i_k} - P =: -Q_k < 0.$$

Multiplying $A_{i_2}^T$ to the left side and A_{i_2} to the right side of (4.5), we have:

$$A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} - A_{i_2}^T P A_{i_2} =: -A_{i_2}^T Q_1 A_{i_2} \leqslant 0. \tag{4.8}$$

Then, from (4.6) and (4.8) it yields:

$$A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} - P = -Q_2 - A_{i_2}^T Q_1 A_{i_2} < 0. \tag{4.9}$$

Again multiplying $A_{i_3}^T$ and A_{i_2} to both sides of (4.9) respectively we obtain:

$$A_{i_3}^T A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} A_{i_3} - A_{i_3}^T P A_{i_3} = -A_{i_3}^T Q_2 A_{i_3} - A_{i_3}^T A_{i_2}^T Q_1 A_{i_2} A_{i_3} \leqslant 0. \tag{4.10}$$

By (4.7) and (4.10) it follows:

$$A_{i_3}^T A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} A_{i_3} - P = -Q_3 - A_{i_3}^T Q_2 A_{i_3} - A_{i_3}^T A_{i_2}^T Q_1 A_{i_2} A_{i_3} < 0.$$

Continue the procedure we obtain: $A_{i_k}^T A_{i_{k-1}}^T ... A_{i_1}^T P A_{i_1} ... A_{i_{k-1}} A_{i_k} < 0$.

(1⇒3) $$(\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k})^T P (\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k}) - P$$
$$= \frac{1}{k^2}(\sum_{j=1}^{k} A_{i_j}^T P A_{i_j} + \sum_{1\leqslant s<t\leqslant k} (A_{i_s}^T P A_{i_t} + A_{i_t}^T P A_{i_s})) - P$$
$$\leqslant \frac{1}{k^2}(\sum_{j=1}^{k} A_{i_j}^T P A_{i_j} + \sum_{1\leqslant s<t\leqslant k} (A_{i_s}^T P A_{i_s} + A_{i_t}^T P A_{i_t})) - P$$
$$= \frac{1}{k^2}(\sum_{j=1}^{k} A_{i_j}^T P A_{i_j} + (k-1)\sum_{j=1}^{k} A_{i_j}^T P A_{i_j}) - P$$
$$= \frac{1}{k}\sum_{j=1}^{k} (A_{i_j}^T P A_{i_j} - P) < 0.$$

(2⇒1) and (3⇒1) are obvious. ■

Theorem 4.1 *If there exists $P > 0$ such that $A_i^T P A_i - P < 0$ for all $i = 1, 2, ..., r$, then*

1). the eigenvalues of the product of any number of A_i ($i = 1, 2, ..., r$) must be located strictly in the unit circle,

2). the eigenvalues of the average of any number of A_i ($i = 1, 2, ..., r$) must be located strictly in the unit circle,

3). the eigenvalues of A_i^l ($i = 1, 2, ..., r$; $l \in N$) must be located strictly in the unit circle.

Proof. Applying Lemma 4.1 and Theorem 2.4, we have the result directly. ■

Obviously, even if all the eigenvalues of each matrix A_i are located strictly in the unit circle, the above three necessary conditions may not be satisfied either. That is, the conditions of all $|\lambda_{\max}(A_i)| < 1$ $(i = 1, 2, ..., r)$ cannot guarantee the stability of (4.4). Note that $||A_i|| < 1$ $(||A_i|| = \sqrt{\lambda_{\max}(A_i^T A_i)})$ implies that $|\lambda_{\max}(A_i)| < 1$. It is easy to see that all the three necessary conditions in Theorem 4.1 are satisfied if all the spectral norms $||A_i|| < 1$. Moreover, due to the equivalence of $||A_i|| < 1$ and $A_i^T I A_i - I < 0$, the existence of a common positive matrix is also guaranteed if all $||A_i|| < 1$. Thereby, we have:

$$\begin{aligned}
&||A_i|| < 1 \; (i = 1, 2, ..., r) \\
&\Rightarrow \exists P > 0, \; s.t. \; A_i^T P A_i - P < 0 \; (i = 1, 2, ..., r) \\
&\Rightarrow |\lambda_{\max}(A_i)| < 1 \; (i = 1, 2, ..., r).
\end{aligned}$$

That is, the region of eigenvalue constraints for the existence of a common $P > 0$ such that $A_i^T P A_i - P < 0$ for all $i = 1, 2, ..., r$ is a region in the unit circle containing $\{\lambda : ||A_i|| < 1\}$.

Now we consider the conditions for the stabilization of T-S models using fuzzy state feedback controller. Concerning this topic, there are numerous researches in the literature. The corresponding results can be found in e.g. [81], [60], [9], [77] and [46]. We denote:

$$\begin{aligned}
H_{ij} &:= A_i + B_i K_j, \\
G_{ij} &:= \tfrac{H_{ij} + H_{ji}}{2}, \\
\lambda_{ij} &:= \lambda_{\max}(G_{ij}^T P G_{ij} - P).
\end{aligned}$$

Then we can prove:

Theorem 4.2 *The closed loop fuzzy T-S systems described by (4.3) is globally asymptotically stable, if there exists a matrix* $P > 0$ *and* $K_i \in \Re^{m \times n}$ *such that* $\lambda_{ii} < 0$ *for* $i = 1, 2, ..., r$ *and* $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1}$ *for* $1 \leqslant i < j \leqslant r$ *excepting the pairs* (i, j) *such that* $\alpha_i(x(k))\alpha_j(x(k)) \equiv 0$.

Proof. Let $V(x(k)) = x(k)^T P x(k)$. It is easy to verify: $V(x(k)) \geqslant 0$ and $V(x(k)) \neq 0$ for $x(k) \neq 0$. Moreover from $\lambda_{\min}(P)|x(k)|^2 \leqslant V(x(k))$ it follows $V(x(k)) \to \infty$ if $|x(k)| \to \infty$.

$$\begin{aligned}
&\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \\
&= (\sum_{i,j} \alpha_i(x(k))\alpha_j(x(k))(A_i + B_i K_j)x(k))^T P (\sum_{i,j} \alpha_i(x(k))\alpha_j(x(k))(A_i + B_i K_j)x(k)) \\
&\quad - x(k)^T P x(k) \\
&= (\sum_{i,j} \alpha_i(x(k))\alpha_j(x(k))x(k)^T H_{ij}^T) P (\sum_{i,j} \alpha_i(x(k))\alpha_j(x(k))H_{ij}x(k)) - x(k)^T P x(k) \\
&= \sum_{i,j,s,t} \alpha_i(x(k))\alpha_j(x(k))\alpha_s(x(k))\alpha_t(x(k))x(k)^T H_{ij}^T P H_{st} x(k) - x(k)^T P x(k) \\
&= \frac{1}{2} \sum_{i,j,s,t} \alpha_i(x(k))\alpha_j(x(k))\alpha_s(x(k))\alpha_t(x(k))x(k)^T (H_{ij}^T + H_{ji}^T) P H_{st} x(k) - x(k)^T P x(k)
\end{aligned}$$

$= \frac{1}{4} \sum\limits_{i,j,s,t} \alpha_i(x(k))\alpha_j(x(k))\alpha_s(x(k))\alpha_t(x(k))x(k)^T(H_{ij}^T + H_{ji}^T)P(H_{st} + H_{ts})x(k)$
$-x(k)^T P x(k)$
$\leqslant \frac{1}{4} \sum\limits_{i,j} \alpha_i(x(k))\alpha_j(x(k))x(k)^T(H_{ij}^T + H_{ji}^T)P(H_{ij} + H_{ji})x(k) - x(k)^T P x(k)$
$= \sum\limits_{i} \alpha_i^2(x(k))x(k)^T H_{ii}^T P H_{ii} x(k) - x(k)^T P x(k) +$
$+\frac{1}{4} \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))x(k)^T(H_{ij}^T + H_{ji}^T)P(H_{ij} + H_{ji})x(k)$
$= \sum\limits_{i} \alpha_i^2(x(k))x(k)^T(H_{ii}^T P H_{ii} - P)x(k) +$
$+ \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))x(k)^T(\frac{(H_{ij}^T + H_{ji}^T)}{2} P \frac{(H_{ij} + H_{ji})}{2} - P)x(k)$
$\leqslant \sum\limits_{i} \alpha_i^2(x(k))x(k)^T \lambda_{ii} x(k) + \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))x(k)^T \lambda_{ij} x(k)$
$= \frac{1}{r-1} \sum\limits_{i} (r-1)\alpha_i^2(x(k))\lambda_{ii}|x(k)|^2 + \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))\lambda_{ij}|x(k)|^2$
$= -\frac{|x(k)|^2}{r-1}[(-\lambda_{11}\alpha_1^2(x(k)) - \lambda_{22}\alpha_1^2(x(k))) + ... + (-\lambda_{r-1,r-1}\alpha_{r-1}^2(x(k)) - \lambda_{rr}\alpha_r^2(x(k)))]$
$+ \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))\lambda_{ij}|x(k)|^2$
$\leqslant -\frac{|x(k)|^2}{r-1}(2\alpha_1(x(k))\alpha_2(x(k))\sqrt{\lambda_{11}\lambda_{22}} + ... + 2\alpha_{r-1}(x(k))\alpha_r(x(k))\sqrt{\lambda_{r-1,r-1}\lambda_{rr}}) +$
$+ \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))\lambda_{ij}|x(k)|^2$
$= \sum\limits_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))(\lambda_{ij} - \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1})|x(k)|^2.$

Thereby we have:

$$\Delta V(x(k)) \leqslant \sum_{i<j} 2\alpha_i(x(k))\alpha_j(x(k))(\lambda_{ij} - \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1})|x(k)|^2. \tag{4.11}$$

For any fixed $x(k)$, if in (4.11) there exists $\alpha_{i_0}(x(k)) \neq 0$ and $\alpha_{j_0}(x(k)) \neq 0$, then from the condition $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1}$ it follows:

$$\Delta V(x(k)) \leqslant 2\alpha_{i_0}(x(k))\alpha_{j_0}(x(k))(\lambda_{i_0 j_0} - \frac{\sqrt{\lambda_{i_0 i_0}\lambda_{j_0 j_0}}}{r-1})|x(k)|^2 < 0 \ (x(k) \neq 0).$$

Otherwise, if there is an $\alpha_{i_0}(x(k)) \neq 0$ but $\alpha_j(x(k)) = 0$ for all $j \neq i_0$, then from the assumption $\lambda_{ii} < 0$ we have:

$$\Delta V(x(k)) = \lambda_{i_0 i_0}\alpha_{i_0}^2(x(k))|x(k)| < 0 \ (x(k) \neq 0).$$

Thereby $\Delta V(x(k)) < 0$ holds for all $x(k) \neq 0$, which complete the proof. ∎

The above result is a generalization of the corresponding result (Theorem 3) presented in [81]. In fact, the stability conditions given in [81] are equivalent to $\lambda_{ii} < 0$ $(i = 1, 2, ..., r)$ and $\lambda_{ij} < 0$ $(i < j)$. The improvement of our result is that λ_{ij} needn't to be negative for $i < j$.

A similar result also holds for the continuous T-S fuzzy models:

Theorem 4.3 *The closed loop continues T-S fuzzy system described by (4.1) is globally asymptotically stable, if there exists a matrix $P > 0$ and $K_i \in \Re^{m \times n}$ such that*

$\lambda_{ii} < 0$ for $i = 1, 2, ..., r$ and $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1}$ for $1 \leqslant i < j \leqslant r$ excepting the pairs (i, j) such that $\alpha_i(x(t))\alpha_j(x(t)) \equiv 0$, where λ_{ij} is the maximum eigenvalue of $G_{ij}^T P + PG_{ij}$.

Proof. It is similar to the proof of Theorem 4.2. ■

It is easy to see that Theorem 4.3 are less conservative than the corresponding result presented in [46].

Based on the stability conditions in Theorem 4.2 and Theorem 4.3, the desired fuzzy state feedback gains can be solved by the following exploratory procedures:

(1). Set $\varepsilon = 0$, $N = 0$.

(2). Find $0 < P \in \Re^{n\times n}$ and $K_i \in \Re^{m\times n}$, such that $\lambda_{ii} + \varepsilon < 0$ for $i = 1, 2, ..., r$.

(3). Verify the inequalities $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{r-1}$ for all $i < j$.

(4). If the inequalities in (3) are not satisfied, set

$$N = N + 1,$$
$$\varepsilon = N \times \varepsilon_0,$$

then go (2).

In the above solution procedures, step (2) can be solved by employing the LMI tools, and ε_0 can be chosen as a sufficiently small positive scalar such that step (2) is always feasible. However, if ε_0 is chosen to be too small, step (3) will involve much computation. In Chapter 7, we will present a BMI-based algorithm for solving the desired feedback gains directly.

4.3 Numerical Example

Example 4.1 *Consider the nonlinear mass-spring-damper system ([46], [60]):*

$$M\ddot{x}(t) + g(x(t), \dot{x}(t)) + f(x(t)) = \phi(\dot{x}(t))u(t)$$

where $M = 1.0$ is the mass, $f(x(t)) = 0.01x(t) + 0.1x(t)^3$ is the spring term, $g(x(t), \dot{x}(t)) = \dot{x}(t)$ is the damper term, $\phi(\dot{x}(t)) = 1 + 0.13\dot{x}(t)^2$ is the input term and $u(t)$ is the force. By applying the PDC-based fuzzy controller designs the above system can be formulated as (see [46] for details):

$$\dot{X}(t) = \sum_{i,j=1}^{4} \alpha_i(X(t))\alpha_j(X(t))(A_i + B_i K_j)X(t)$$

where K_j $(j = 1, 2, 3, 4)$ are the state feedback gains to be designed and:

$$X(t) = (X_1(t), X_2(t))^T = (x(t), \dot{x}(t))^T,$$

$$\alpha_1(X(t)) = (1 - \frac{X_1(t)^2}{2.25})(1 - \frac{X_2(t)^2}{6.75}),\ \alpha_2(X(t)) = (1 - \frac{X_1(t)^2}{2.25})\frac{X_2(t)^2}{6.75},$$

$$\alpha_3(X(t)) = \frac{X_1(t)^2}{2.25}(1 - \frac{X_2(t)^2}{6.75}),\ \ \alpha_4(X(t)) = \frac{X_1(t)^2}{2.25}\frac{X_2(t)^2}{6.75},$$

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix}, A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1.4387 \end{bmatrix}, B_2 = B_4 = \begin{bmatrix} 0 \\ 0.5316 \end{bmatrix}.$$

Following the solution procedures in Section 4.2, we have:

$$P = \begin{bmatrix} 0.016 & 0.0072 \\ 0.0072 & 0.016 \end{bmatrix} > 0,$$

$$K_1 = (-1.0517, -0.1230), K_2 = (-2.6957, -0.3152),$$

$$K_3 = (-0.8953, -0.1230),\ \ K_4 = (-2.2948, -0.3152),$$

$$\lambda_{23} = 7.9853, \sqrt{\lambda_{22}\lambda_{33}} = 68.4744, \lambda_{ij} < 0 (i \neq 2, j \neq 3).$$

Therefore, it holds: $\lambda_{ii} < 0$ $(i = 1, 2, 3, 4)$ *and* $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{4-1}$ $(1 \leqslant i < j \leqslant 4)$. *According to Theorem 4.3 the closed loop fuzzy system is asymptotically stable. Figure 4.1 illustrates the controlled trajectories of the nonlinear mass-spring-damper system by applying fuzzy controller* $u(t) = \sum_{i=1}^{4} \alpha_i(X(t))K_iX(t)$, *where the initial condition is given by* $X(0) = (0.3, 0.5)^T$. *The simulation shows that the proposed approach is feasible.*

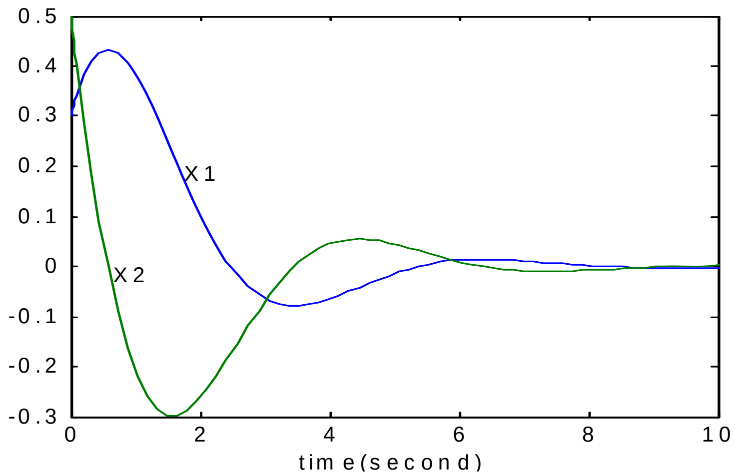

Figure 4.1: Controlled trajectories of the mass-spring-damper system

Chapter 5

Stability Analysis of Fuzzy Affine Systems

In this chapter a hyperellipsoid-based approach is proposed for the stability analysis of fuzzy affine systems. We present first an algorithm for constructing the minimal hyperellipsoids based on the structural information in the fuzzy rules. Then, by discussing the maximum of derivation of the candidate Lyapunov functions in each region of these minimal hyperellipsoids, we obtain the sufficient conditions for the stability of open loop fuzzy affine models in terms of LMIs. Finally, we give two numerical examples (both have some unstable subsystems) to illustrate the feasibility of the proposed approach.

5.1 Constructing the Minimal Hyperellipsoids

We present a lemma to show how to construct the minimal hyperellipsoid containing a given bounded region, where the minimal hyperellipsoid means that it is minimal in volume compared with all the other hyperellipsoids containing the given region.

Lemma 5.1 *Suppose that* $D = \{(x_1, x_2, ..., x_n)^T : a_i < x_i < b_i, i = 1, 2, ..., n\}$ *is a given bounded region in* $\Re^n$*, then the minimal hyperellipsoid containing* D *is:*

$$\frac{(x_1-x_{01})^2}{c_1^2} + \frac{(x_2-x_{02})^2}{c_2^2} + ... + \frac{(x_n-x_{0n})^2}{c_n^2} = 1,$$

where $x_{0i} = \frac{a_i+b_i}{2}$ *and* $c_i^2 = n(\frac{a_i-b_i}{2})^2$ *for all* $i = 1, 2, ..., n$*.*

Proof. For any $(x_1, x_2, ..., x_n)^T \in D$, there must exist $t_i \in (0,\ 1)$ such that: $x_i = a_i(1 - t_i) + b_i$ for all $i = 1, 2, ..., n$. Denote $\frac{(x_1-x_{01})^2}{c_1^2} + \frac{(x_2-x_{02})^2}{c_2^2} + ... + \frac{(x_n-x_{0n})^2}{c_n^2}$ as $F(x_1, x_2, ..., x_n)$, then we have:

$$F(x_1, x_2, ..., x_n) = \frac{(x_1-x_{01})^2}{c_1^2} + \frac{(x_2-x_{02})^2}{c_2^2} + ... + \frac{(x_n-x_{0n})^2}{c_n^2}$$

$$= \frac{[(a_1-b_1)/2-(a_1-b_1)t_1]^2}{n[(a_1-b_1)/2]^2} + ... + \frac{[(a_n-b_n)/2-(a_n-b_n)t_n]^2}{n[(a_n-b_n)/2]^2}$$
$$= \frac{(1/2-t_1)^2}{n/4} + ... + \frac{(1/2-t_n)^2}{n/4}.$$

Since $(1/2 - t_i)^2 < 1/4$ for $t_i \in (0,\ 1)$, then we have $F(x_1, x_2, ..., x_n) < 1$ for all $(x_1, x_2, ..., x_n)^T \in D$. That is, all the points in D are located in the constructed hyperellipsoid. Moreover it is easy to see that all the vertex points of D are located on the surface of the hyperellipsoid. Now we prove that the given hyperellipsoid is minimal in volume. Let

$$\frac{(x_1-\widetilde{x}_{01})^2}{\widetilde{c}_1^2} + \frac{(x_2-\widetilde{x}_{02})^2}{\widetilde{c}_2^2} + ... + \frac{(x_n-\widetilde{x}_{0n})^2}{\widetilde{c}_n^2} = 1$$

be any hyperellipsoid containing the given region. Without loss of generality we can assume $\widetilde{x}_{0i} = x_{0i}$ $(i = 1, 2, ..., n)$, since the volume will not change by the translation of the center point. Following the method presented in [33], we have that the volume of the above hyperellipsoid is:

$$V = \frac{\pi^{n/2}}{\Gamma(1+n/2)}\widetilde{c}_1\widetilde{c}_2...\widetilde{c}_n,$$

where $\Gamma(.)$ is the ordinary Gamma function defined by $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$. Obviously, V is minimal if and only if $\widetilde{c}_1\widetilde{c}_2...\widetilde{c}_n$ is minimal, i.e. $\frac{1}{(\widetilde{c}_1\widetilde{c}_2...\widetilde{c}_n)^2}$ is maximal. Note that all the vertex points of D are located on the surface of the hyperellipsoid, then searching for the minimal hyperellipsoid is reduced to the optimization problem :

$$\begin{cases} maximize \ \ z_1z_2...z_n \\ subject\ to \ \ (\frac{b_1-a_1}{2})^2 z_1 + ... + (\frac{b_n-a_n}{2})^2 z_n = 1, \end{cases}$$

where $z_i = \frac{1}{\widetilde{c}_i^2}$ $(i = 1, 2, ..., n)$.

Let the Lagrange object function be:

$$L(z_1, z_2, ..., z_n) = z_1z_2...z_n + \lambda((\frac{b_1-a_1}{2})^2 z_1 + ... + (\frac{b_n-a_n}{2})^2 z_n - 1).$$

The optimization problem can be solved by the following equalities:

$$L_{z_1} = z_2z_3...z_n + \lambda(\frac{b_1-a_1}{2})^2 = 0$$
$$L_{z_2} = z_1z_3...z_n + \lambda(\frac{b_2-a_2}{2})^2 = 0$$
$$...$$
$$L_{z_n} = z_1z_2...z_{n-1} + \lambda(\frac{b_n-a_n}{2})^2 = 0$$
$$L_\lambda = (\frac{b_1-a_1}{2})^2 z_1 + ... + (\frac{b_n-a_n}{2})^2 z_n - 1 = 0.$$

It is easy to obtain the solutions of the equalities:

$$z_1 = \frac{1}{n}(\frac{2}{b_1-a_1})^2,$$
$$z_2 = \frac{1}{n}(\frac{2}{b_2-a_2})^2,$$
$$...,$$
$$z_n = \frac{1}{n}(\frac{2}{b_n-a_n})^2.$$

Substitute $z_i = \frac{1}{\widetilde{c}_i^2}$ we get $\widetilde{c}_i^2 = n(\frac{a_i-b_i}{2})^2$ for $i = 1, 2, ..., n$. Direct calculation shows that the matrix of d^2L is a semi negative definite matrix, then the Lagrange object function takes maximum at the solution $z_i = \frac{1}{n}(\frac{2}{b_i-a_i})^2$ $(i = 1, 2, ..., n)$. Therefore the given hyperellipsoid in Lemma 5.1 is minimal in volume. ■

5.2 Stability of Continuous Fuzzy Affine Systems

Concerning the stability of T-S fuzzy systems, most results available in the literature require that each subsystem must be stable in order to guarantee the stability of the overall systems. To overcome this restriction, many new approaches have been presented recently by utilizing the structural information in the fuzzy rules. In [9] the information of the number of fired rules is taken into account. In [40] and [69] the information of membership functions in the fuzzy rules are completely utilized in stability analysis of fuzzy systems with singleton consequents. In [37] and [36] the structural information is applied to construct the piecewise quadratic Lyapunov functions. In this chapter, we employ the structural information to construct the minimal hyperellipsoids based on Lemma 5.1. The systems under discussion are described by fuzzy affine T-S models, that is, each subsystem has an additional offset term in the consequent dynamics. In special case, if all the offset terms are zero, the fuzzy affine models are degenerated to the common T-S fuzzy models.

Suppose that the fuzzy affine system is expressed by the following fuzzy rules:

If $x_1(t)$ is M_1^i and ... and $x_n(t)$ is M_n^i, then: $\dot{x}(t) = A_i x(t) + e_i$ $(i = 1, 2, ..., r)$. Similar to the discussion in Section 4.1, the overall system can be deduced:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(x(t))(A_i x(t) + e_i) \tag{5.1}$$

where $0 \leqslant \alpha_i(x(t)) \leqslant 1$ for $i = 1, 2, ..., r$ and $\sum\limits_{i=1}^{r} \alpha_i(x(t)) = 1$. We assume that all the fuzzy sets M_j^i have bounded supports, i.e. there exist $a_{ji}, b_{ji} \in \Re$ $(a_{ji} < b_{ji})$, such that $\{x(t) : \mu_{M_j^i}(x(t)) > 0\} = (a_{ji}, b_{ji})$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$. Then from $\alpha_i(x(t)) = \omega_i(x(t)) / \sum\limits_{j=1}^{r} \omega_j(x(t))$ and $\omega_i(x(t)) = \prod\limits_{j=1}^{n} \mu_{M_j^i}(x(t))$ it follows:

$$\begin{aligned} &\alpha_i(x(t)) > 0 \\ &\Leftrightarrow \mu_{M_j^i}(x(t)) > 0,\ (1 \leqslant j \leqslant n) \\ &\Leftrightarrow x(t) \in (a_{1i}, b_{1i}) \times ... \times (a_{ni}, b_{ni}). \end{aligned}$$

Denote $D_i := (a_{1i}, b_{1i}) \times ... \times (a_{ni}, b_{ni})$. By Lemma 5.1 we have that the minimal hyperellipsoid containing D_i is:

$$\frac{(x_1 - x_{01i})^2}{c_{1i}^2} + \frac{(x_2 - x_{02i})^2}{c_{2i}^2} + ... + \frac{(x_n - x_{0ni})^2}{c_{ni}^2} = 1 \tag{5.2}$$

where $x_{0ji} = \frac{a_{ji}+b_{ji}}{2}$ and $c_{ji}^2 = n(\frac{a_{ji}-b_{ji}}{2})^2$ for all $i = 1, 2, ..., r$ and $j = 1, 2, ..., n$.

Let $x_{0i} := [\ x_{01i}\ \ x_{02i}\ \ ...\ \ x_{0ni}\ ]^T$ and $C_i := \begin{bmatrix} 1/c_{1i}^2 & & & \\ & 1/c_{2i}^2 & & \\ & & \ddots & \\ & & & 1/c_{ni}^2 \end{bmatrix}$.

Then (5.2) can be rewritten as:

$$x^T C_i x + x_{0i}^T C_i x_{0i} - 2x_{0i}^T C_i x = 1. \tag{5.3}$$

For each fuzzy rule, we can construct a minimal hyperellipsoid of form (5.3). Then, according to whether the origin is located in these hyperellipsoids the index set $\{1, 2, ..., r\}$ can be divided into I_0 and I_1, where:

$I_0 = \{i : 1 - x_{0i}^T C_i x_{0i} \geqslant 0, 1 \leqslant i \leqslant r\}$,

$I_1 = \{i : 1 - x_{0i}^T C_i x_{0i} < 0, 1 \leqslant i \leqslant r\}$.

In addition, it is assumed that $e_i = 0$ for $i \in I_0$, which implies that $x(t) \equiv 0$ is a trivial solution of (5.1).

To prove the main results, the following lemmas are required:

Lemma 5.2 *([37]) Let $V(t)$ be a decreasing and piecewise continuous function. If there exist positive scalars α, β, γ such that: $\alpha|x(t)|^2 \leqslant V(t) \leqslant \beta|x(t)|^2$ and $\frac{d}{dt}V(t) \leqslant -\gamma|x(t)|^2$, then $|x(t)|^2 \leqslant \frac{\beta}{\alpha}e^{-\frac{\gamma}{\beta}t}|x(0)|^2$.*

Lemma 5.3 *Suppose that D is a bounded closed set in $\Re^n$, $A^T = A \in \Re^{n\times n}$, $B^T \in \Re^n, C \in \Re$, then $x^T Ax + Bx + C < 0$ $(\forall x \in D)$ if and only if there exists a positive scalar $k \in \Re$, such that $x^T Ax + Bx + C < -kx^T x$ $(\forall x \in D)$.*

Proof. $(\Rightarrow)$ Since D is a bounded closed set and $x^T Ax+Bx+C$ is continuous, then $x^T Ax+Bx+C$ can take the maximum in D. Suppose the maximum is $x_0^T Ax_0+Bx_0+C$. From $x_0 \in D$ it follows: $x_0^T Ax_0 + Bx_0 + C < 0$. Without loss of generality, we can assume $D \neq \{0\}$, namely $\sup\{|x|^2 : x \in D\} \neq 0$. Let $k_0 = -\frac{x_0^T Ax_0+Bx_0+C}{\sup\{|x|^2:x\in D\}}$, then for any $x \in D$, $k \in \Re$ and $0 < k < k_0$, we have:

$x^T Ax + Bx + C$

$\leqslant x_0^T Ax_0 + Bx_0 + C$

$\leqslant (x_0^T Ax_0 + Bx_0 + C)\frac{x^T x}{\sup\{|x|^2:x\in D\}}$

$= -k_0 x^T x$

$< -kx^T x$.

$(\Leftarrow)$ It is obvious. ■

Lemma 5.4 *Suppose that A is a symmetric matrix in $\Re^{n\times n}$ and D is a bounded open set in $\Re^n$ with $0 \in D$, then $x^T Ax < 0$ $(\forall x \in D, x \neq 0)$ if and only if $A < 0$.*

Proof. Since A is a symmetric matrix, then there exists an orthogonal matrix $Q = (q_{ij})_{n\times n} \in \Re^{n\times n}$, such that:

$$Q^T AQ = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

where λ_i $(i = 1, 2, ..., n)$ are the eigenvalues of matrix A. Let $x = kQy$ $(k \in \Re, k \neq 0)$, then:

$$x^T Ax = k^2 y^T Q^T AQy = k^2(\lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2).$$

Obviously $A \neq 0$, since $x^T Ax < 0$ $(\forall x \in D, x \neq 0)$. If A is not a negative definite matrix, then there must exist $\lambda_{i_0} \in \{\lambda_1, \lambda_2, ..., \lambda_n\}$ such that $\lambda_{i_0} \geqslant 0$. Setting $y_{i_o} = 1$ and $y_i = 0$ for all $i \neq i_0$ we have:

$$x = kQy = kQ(0, ..., 0, 1, 0, ..., 0)^T = k(q_{1i}, q_{2i}, ..., q_{ni})^T.$$

Note that $Q = (q_{ij})_{n\times n}$ is an orthogonal matrix, we have $|x|^2 = x^T x = k^2$. From $0 \in D$ it follows $x = kQy \in D$, if k is chosen to be sufficiently small. Hence, we have:

$$x^T Ax = k^2 \lambda_{i_0} y_{i_0}^2 = k^2 \lambda_{i_0} \geqslant 0,$$

which is contrary to the condition $x^T Ax < 0$ $(\forall x \in D, x \neq 0)$.

On the other hand, if $A < 0$, the conclusion is also valid obviously. ■

With the above preparation, we can now present the main results.

Theorem 5.1 *If there exists a symmetric positive definite matrix P and positive scalars τ_i such that:*

$$A_i^T P + PA_i < 0 \quad (i \in I_0) \tag{5.4}$$

$$\begin{bmatrix} \tau_i - \tau_i x_{0i}^T C_i x_{0i} & e_i^T P + \tau_i x_{0i}^T C_i \\ Pe_i + \tau_i C_i x_{0i} & A_i^T P + PA_i - \tau_i C_i \end{bmatrix} < 0 \quad (i \in I_1), \tag{5.5}$$

then every trajectory of (5.1) tends to zero exponentially.

Proof. Choose the candidate Lyapunov function as $V(x) = x(t)^T Px(t)$, then:

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) \\ &= \sum_{i=1}^{r} \alpha_i(x(t))[(x(t)^T A_i^T + e_i^T)Px(t) + x(t)^T P(A_i x(t) + e_i)] \\ &= \sum_{i=1}^{r} \alpha_i(x(t))[(x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t)] \\ &= \sum_{i\in I_1} \alpha_i(x(t))[(x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t)] \\ &\quad + \sum_{i\in I_0} \alpha_i(x(t))x(t)^T(A_i^T P + PA_i)x(t). \end{aligned}$$

Obviously, if $(x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t) < 0$ for all $x(t) \in D_i$ $(i \in I_1)$ and $x(t)^T(A_i^T P + PA_i)x(t) < 0$ for all $x(t) \in D_i$ $(i \in I_0)$, then $\dot{V}(x(t)) < 0$ for all $x(t) \neq 0$. Now we discuss the above two conditions respectively.

Firstly, if $i \in I_0$, then the origin is located in the minimal hyperellipsoid containing D_i. By Lemma 5.4 we have that $x^T(A_i^T P + PA_i)x < 0$ holds for all non-zero $x(t)$

in the minimal hyperellipsoid containing D_i, if and only if condition (5.4) is satisfied. Thereby:

$$x(t)^T(A_i^T P + PA_i)x(t) < 0 \quad (x(t) \in D_i, x(t) \neq 0). \tag{5.6}$$

It follows that there exists a $k_i > 0$, such that:

$$x(t)^T(A_i^T P + PA_i)x(t) < -k_i x(t)^T x(t) \quad (x(t) \in D_i,\ x(t) \neq 0,\ i \in I_0). \tag{5.7}$$

Next, if $i \in I_1$, by Lemma 5.1 we have that the region D_i is included in the minimal hyperellipsoid defined by (5.3). Thereby: $x(t)^T C_i x(t) + x_{0i}^T C_i x_{0i} - 2x_{0i}^T C_i x(t) \leqslant 1$ $(x(t) \in D_i)$. Then for any $\tau_i > 0$ and $x(t) \in D_i$ $(i \in I_1)$, we have:

$$\begin{aligned}
&x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t) \\
&\leqslant x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t) + \tau_i[1 - x^T C_i x - x_{0i}^T C_i x_{0i} + 2x_{0i}^T C_i x] \\
&= x(t)^T(A_i^T P + PA_i - \tau_i C_i)x(t) + 2(e_i^T P + \tau_i x_{0i}^T C_i)x(t) + \tau_i(1 - x_{0i}^T C_i x_{0i}).
\end{aligned}$$

Denote $x^T(A_i^T P + PA_i - \tau_i C_i)x + 2(e_i^T P + \tau_i x_{0i}^T C_i)x + \tau_i(1 - x_{0i}^T C_i x_{0i}) =: f_i(x)$, then the gradient of $f_i(x)$ is given by:

$$\nabla f_i(x) = 2x^T(A_i^T P + PA_i - \tau_i C_i) + 2(e_i^T P + \tau_i x_{0i}^T C_i).$$

Since the condition (5.5) implies that the matrix $A_i^T P + PA_i - \tau_i C_i$ is invertible, then the solution of $\nabla f_i(x) = 0$ can be obtained:

$$x_{Mi}^T := -(e_i^T P + \tau_i x_{0i}^T C_i)(A_i^T P + PA_i - \tau_i C_i)^{-1}.$$

Note that the matrix of $d^2 f_i(x)$ is negative definite $(A_i^T P + PA_i - \tau_i C_i < 0)$, thereby $f_i(x)$ takes maximum at the solution point x_{Mi}. Moreover:

$$f_i(x_{Mi}) = -(e_i^T P + \tau_i x_{0i}^T C_i)(A_i^T P + PA_i - \tau_i C_i)^{-1}(Pe_i + \tau_i C_i x_{0i}) + \tau_i(1 - x_{0i}^T C_i x_{0i}).$$

According to Schur complement, we have:

$$f_i(x_{Mi}) < 0 \Leftrightarrow \begin{bmatrix} \tau_i - \tau_i x_{0i}^T C_i x_{0i} & e_i^T P + \tau_i x_{0i}^T C_i \\ Pe_i + \tau_i C_i x_{0i} & A_i^T P + PA_i - \tau_i C_i \end{bmatrix} < 0.$$

Thus from condition (5.5) it follows:

$$x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t) \leqslant f_i(x) \leqslant f_i(x_{Mi}) < 0. \tag{5.8}$$

Since (5.8) holds for all $x(t)$ satisfying $x(t)^T C_i x(t) + x_{0i}^T C_i x_{0i} - 2x_{0i}^T C_i x(t) \leqslant 1$, applying Lemma 5.3 we have that there exists a positive scalar k_i, such that:

$$x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t) < -k_i x(t)^T x(t) \ (x(t) \in D_i, i \in I_1). \tag{5.9}$$

Combining (5.7) and (5.9), we have that for all $x(t)$ satisfying (5.1), there exists a positive scalar k, such that:

$$\dot{V}(x(t)) = \sum_{i=1}^{r} \alpha_i(x(t))[(x(t)^T(A_i^T P + PA_i)x(t) + 2e_i^T Px(t)] \leqslant -kx(t)^T x(t). \tag{5.10}$$

Setting $\alpha = \lambda_{\min}(P)$, $\beta = \lambda_{\max}(P)$, $\gamma = k$ and applying Lemma 5.2 we have that the trivial solution of (5.1) is exponentially stable. ■

Now we consider a more general case, that is, the premise variables are different from the state variables. In this circumstance, the regions under discussion may be unbounded in the state space, and the constructed minimal hyperellipsoids by Lemma 5.1 may represent unbounded hypercylinders in $\Re^n$.

Suppose that the fuzzy continuous affine systems are described by:

If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then: $\dot{x}(t) = A_i x(t) + e_i$ $(i = 1, 2, ..., r)$. Then the overall fuzzy systems can be written as:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(p(t))(A_i x(t) + e_i) \tag{5.11}$$

where $p(t) = \begin{bmatrix} p_1(t) & p_2(t) & \cdots & p_s(t) \end{bmatrix}^T = Qx(t)$ with $rank(Q) = s$ $(1 \leqslant s \leqslant n)$, $0 \leqslant \alpha_i(p(t)) \leqslant 1$ and $\sum_{i=1}^{r} \alpha_i(p(t)) = 1$. Assume that all the fuzzy sets M_j^i in the first r_1 $(r_1 \leqslant r)$ rules have bounded supports, i.e. $\exists a_{ji}, b_{ji} \in \Re$, $a_{ji} < b_{ji}$ such that:

$\{p_j(t) : \mu_{M_j^i}(p_j(t)) > 0\} = (a_{ji}, b_{ji})$ $(1 \leqslant i \leqslant r_1,\ 1 \leqslant j \leqslant s)$.

But in the other $r - r_1$ fuzzy rules there may be some fuzzy sets with unbounded supports. Now, we consider the support regions of the first r_1 fuzzy rules:

$\{p(t) : 0 < \alpha_i(p(t)), p(t) \in \Re^s\} = (a_{1i}, b_{1i}) \times \cdots \times (a_{si}, b_{si})$ $(1 \leqslant i \leqslant r_1)$.

By Lemma 5.1, the minimal hyperellipsoid in $\Re^s$ containing $(a_{1i}, b_{1i}) \times \cdots \times (a_{si}, b_{si})$ can be formulated as:

$$\frac{(p_1 - p_{01i})^2}{c_{1i}^2} + \frac{(p_2 - p_{02i})^2}{c_{2i}^2} + ... + \frac{(p_s - p_{0si})^2}{c_{si}^2} = 1 \tag{5.12}$$

where $p_{0ji} = \frac{a_{ji}+b_{ji}}{2}$ and $c_{ji}^2 = s(\frac{a_{ji}-b_{ji}}{2})^2$ for all $i = 1, 2, ..., r_1$ and $j = 1, 2, ..., s$.

Let $p_{0i} := \begin{bmatrix} p_{01i} & p_{02i} & ... & p_{0si} \end{bmatrix}^T$ and $C_i := \begin{bmatrix} 1/c_{1i}^2 & & & \\ & 1/c_{2i}^2 & & \\ & & \ddots & \\ & & & 1/c_{si}^2 \end{bmatrix}$, then (5.12) can be rewritten as:

$$p^T C_i p + p_{0i}^T C_i p_{0i} - 2p_{0i}^T C_i p = 1 \ (i = 1, 2, ..., r_1) \tag{5.13}$$

where $p = \begin{bmatrix} p_1 & p_2 & ... & p_s \end{bmatrix}^T \in \Re^s$. The index sets I_0 and I_1 are defined similarly as in Theorem 5.1, namely: $I_1 = \{i : p_{0i}^T C_i p_{0i} > 1, 1 \leqslant i \leqslant r_1\}$, and $I_0 = \{i : i \notin I_1, 1 \leqslant i \leqslant r\}$. Moreover, it is assumed $e_i = 0$ for $i \in I_0$,which shows that $x(t) \equiv 0$ is a trivial solution of (5.11). With these notations in mind, we can prove the following result.

Theorem 5.2 *The trivial solution $x(t) \equiv 0$ of the fuzzy affine system described by (5.11) is asymptotically stable, if there exists a positive definite symmetric matrix P and positive scalars τ_i such that:*

$$A_i^T P + PA_i < 0 \quad (i \in I_0) \tag{5.14}$$

$$\begin{bmatrix} \tau_i - \tau_i p_{0i}^T C_i p_{0i} & e_i^T P + \tau_i p_{0i}^T C_i Q \\ P e_i + \tau_i Q^T C_i p_{0i} & A_i^T P + P A_i - \tau_i Q^T C_i Q \end{bmatrix} < 0 \quad (i \in I_1). \tag{5.15}$$

Proof. Chose the candidate Lyapunov function as $V(x(t)) = x(t)^T P x(t)$, then:

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= \sum_{i=1}^{r} \alpha_i(p(t))[(x(t)^T (A_i^T P + P A_i) x(t) + 2 e_i^T P x(t)] \\ &= \sum_{i \in I_1} \alpha_i(p(t))[(x(t)^T (A_i^T P + P A_i) x(t) + 2 e_i^T P x(t)] \\ &\quad + \sum_{i \in I_0} \alpha_i(p(t)) x(t)^T (A_i^T P + P A_i) x(t). \end{aligned}$$

Applying condition (5.14), we have:

$$\dot{V}(x(t)) \leqslant \sum_{i \in I_1} \alpha_i(p(t))[(x(t)^T (A_i^T P + P A_i) x(t) + 2 e_i^T P x(t)]. \tag{5.16}$$

For all $i \in I_1$ and $p(t) \in \Re^s$ satisfying $0 < \alpha_i(p(t))$, the point $p(t)$ must be located in the hyperellipsoid defined by (5.13). Then we have:

$$p(t)^T C_i p(t) + p_{0i}^T C_i p_{0i} - 2 p_{0i}^T C_i p(t) \leqslant 1. \tag{5.17}$$

Combining (5.16) and (5.17) we have that for any positive scalars τ_i it holds:

$$\begin{aligned} \dot{V}(x(t)) &\leqslant \sum_{i \in I_1} \alpha_i(p(t)) \{[(x(t)^T (A_i^T P + P A_i) x(t) + 2 e_i^T P x(t)] \\ &\quad + \tau_i [1 - p(t)^T C_i p(t) - p_{0i}^T C_i p_{0i} + 2 p_{0i}^T C_i p(t)]\} \\ &= \sum_{i \in I_1} \alpha_i(p(t))[x(t)^T (A_i^T P + P A_i - \tau_i Q^T C_i Q) x(t) + 2(e_i^T P + \tau_i p_{0i}^T C_i Q) x(t)] \\ &\quad + \sum_{i \in I_1} \alpha_i(p(t))(\tau_i - \tau_i p_{0i}^T C_i p_{0i}). \end{aligned}$$

Denote $f_i(x) := x^T (A_i^T P + P A_i - \tau_i Q^T C_i Q) x + 2(e_i^T P + \tau_i p_{0i}^T C_i Q) x + \tau_i - \tau_i p_{0i}^T C_i p_{0i}$. Then it follows:

$$\dot{V}(x(t)) \leqslant \sum_{i \in I_1} \alpha_i(p(t)) f_i(x(t)). \tag{5.18}$$

From condition (5.15) it yields: $A_i^T P + P A_i - \tau_i Q^T C_i Q < 0$. Similarly to the proofs in Theorem 5.1, we can obtain the maximum of $f_i(x)$:

$$\begin{aligned} f_i(x_{Mi}) &:= \max\{f_i(x) : x \in \Re^n\} \\ &= -(e_i^T P + \tau_i p_{0i}^T C_i Q)(A_i^T P + P A_i - \tau_i Q^T C_i Q)^{-1} (P e_i + \tau_i Q^T C_i p_{0i}) \\ &\quad + \tau_i (1 - p_{0i}^T C_i p_{0i}). \end{aligned}$$

Applying Schur complements we have:

$$f_i(x_{Mi}) < 0 \Leftrightarrow \begin{bmatrix} \tau_i - \tau_i p_{0i}^T C_i p_{0i} & e_i^T P + \tau_i p_{0i}^T C_i Q \\ P e_i + \tau_i Q^T C_i p_{0i} & A_i^T P + P A_i - \tau_i Q^T C_i Q \end{bmatrix} < 0 \ (i \in I_1).$$

Thereby, from (5.18) and the condition (5.15) it follows:

$$\dot{V}(x(t)) \leqslant \sum_{i \in I_1} \alpha_i(p(t)) f_i(x(t)) \leqslant \sum_{i \in I_1} \alpha_i(p(t)) f_i(x_{Mi}) < 0,$$

which completes the proof. ■

It is easy to verify that the trivial solution in Theorem 5.2 is exponentially stable, if an extra condition $rank(Q) = n$ holds. However, if $rank(Q) < n$, the hyperellipsoids defined by (5.13) are in fact unbounded hypercylinders in $\Re^n$. In this case, the conditions of Lemma 5.3 are no longer satisfied. Thereby we can only obtain the conclusion of asymptotic stability rather than exponential stability in this circumstance.

5.3 Stability of Discrete Fuzzy Affine Systems

Lemma 5.5 *([61]) If there exist positive scalars α,β, and γ such that: $\alpha|x(k)|^2 \leqslant V(k) \leqslant \beta|x(k)|^2$ and $\triangle V(k) = V(k+1) - V(k) \leqslant -\gamma \cdot |x(k)|^2$ for all $k \in N$, then $x(k)$ tends to zero exponentially as $k \to +\infty$.*

Suppose that the fuzzy discrete system under discussion is expressed by fuzzy rules: If $x_1(k)$ is M_1^i and ... and $x_n(k)$ is M_n^i, then: $x(k+1) = A_i x(k) + e_i$ $(i = 1, 2, ..., r)$. Then the overall system can be deduced:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + e_i). \tag{5.19}$$

Assume that all the fuzzy sets M_j^i have bounded supports. The index sets I_0 and I_1 are defined the same as those in Theorem 5.1. Moreover, it is also assumed that $e_i = 0$ for all $i \in I_0$.

Theorem 5.3 *The trivial solution $x(k) \equiv 0$ of the fuzzy affine system described by (5.19) is exponentially stable, if there exists a symmetric positive definite matrix P and positive scalars τ_i such that:*

$$A_i^T P A_i - P < 0 \quad (i \in I_0) \tag{5.20}$$

$$\begin{bmatrix} e_i^T P e_i + \tau_i - \tau_i x_{0i}^T C_i x_{0i} & e_i^T P A_i + \tau_i x_{0i}^T C_i \\ A_i^T P e_i + \tau_i C_i x_{0i} & A_i^T P A_i - P - \tau_i C_i \end{bmatrix} < 0 \quad (i \in I_1), \tag{5.21}$$

where the notations C_i and x_{0i} are the same as those in Theorem 5.1.

Proof. Let $V(k) = x(k)^T P x(k)$, then:

$$\begin{aligned}\triangle V(k) = &\sum_{i,j=1}^{r} \alpha_i(x(k))\alpha_j(x(k))(x(k)^T A_i^T + e_i^T)P(A_j x(k) + e_j) - x(k)^T P x(k) \\ = &\sum_{i=1}^{r} \alpha_i^2(x(k))[(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)] \\ &+ \sum_{1 \leqslant i < j \leqslant r} \alpha_i(x(k))\alpha_j(x(k))[(x(k)^T A_i^T + e_i^T)P(A_j x(k) + e_j) - x(k)^T P x(k)] \\ &+ \sum_{1 \leqslant i < j \leqslant r} \alpha_i(x(k))\alpha_j(x(k))[(x(k)^T A_j^T + e_j^T)P(A_i x(k) + e_i) - x(k)^T P x(k)]\end{aligned}$$

$$\leqslant \sum_{i=1}^{r} \alpha_i^2(x(k))[(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)]$$
$$+ \sum_{1\leqslant i<j\leqslant r} \alpha_i(x(k))\alpha_j(x(k))[(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)]$$
$$+ \sum_{1\leqslant i<j\leqslant r} \alpha_i(x(k))\alpha_j(x(k))[(x(k)^T A_j^T + e_j^T)P(A_j x(k) + e_j) - x(k)^T P x(k)]$$

Now, we will show that for all $x(k)$ satisfying $x(k) \neq 0$ and $\alpha_i(x(k)) > 0$ it holds:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k) < 0 \ (i = 1, 2, ..., r).$$

Firstly, if $i \in I_0$, it follows $e_i = 0$. By condition (5.20) we have:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)$$
$$= x(k)^T (A_i^T P A_i - P) x(k)$$
$$\leqslant \lambda_{\max}(A_i^T P A_i - P) \cdot |x(k)|^2.$$

That is:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k) \leqslant -\lambda_i \cdot |x(k)|^2 \ (i \in I_0) \tag{5.22}$$

where $\lambda_i = -\lambda_{\max}(A_i^T P A_i - P) > 0$.

On the other hand, if $i \in I_1$ and $\alpha_i(x(k)) > 0$, then $x(k)$ must be located in the minimal hyperellipsoid defined by (5.3). Thereby:

$$x(k)^T C_i x(k) + x_{0i}^T C_i x_{0i} - 2x_{0i}^T C_i x(k) \leqslant 1.$$

Then we have:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)$$
$$\leqslant (x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k)$$
$$+\tau_i(1 - x(k)^T C_i x(k) - x_{0i}^T C_i x_{0i} + 2x_{0i}^T C_i x(k))$$
$$= x(k)^T (A_i^T P A_i - P - \tau_i C_i) x(k) + 2(e_i^T P A_i + \tau_i x_{0i}^T C_i) x(k) + e_i^T P e_i + \tau_i (1 - x_{0i}^T C_i x_{0i}).$$

Denote:

$$f_i(x) := x^T (A_i^T P A_i - P - \tau_i C_i) x + 2(e_i^T P A_i + \tau_i x_{0i}^T C_i) x + e_i^T P e_i + \tau_i (1 - x_{0i}^T C_i x_{0i}).$$

Similar to the process in Theorem 5.1, we can get the maximum of $f_i(x)$:

$$f_i(x_{Mi}) = -(e_i^T P A_i + \tau_i x_{0i}^T C_i)(A_i^T P A_i - P - \tau_i C_i)^{-1}(A_i^T P e_i + \tau_i C_i x_{0i})$$
$$+ e_i^T P e_i + \tau_i (1 - x_{0i}^T C_i x_{0i})$$

By Schur complement, if the condition (5.21) is satisfied, it gives $f_i(x_{Mi}) < 0$. Then, for all $x(k)$ satisfying $\alpha_i(x(k)) > 0$ $(i \in I_1)$ it holds:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k) \leqslant f_i(x(k)) \leqslant f_i(x_{Mi}) < 0 \tag{5.23}$$

Applying Lemma 5.3 we have that there exists $\gamma_i > 0$ such that:

$$(x(k)^T A_i^T + e_i^T)P(A_i x(k) + e_i) - x(k)^T P x(k) \leqslant -\gamma_i \cdot |x(k)|^2 \ (i \in I_1). \tag{5.24}$$

Then from (5.22), (5.24) and Lemma 5.5 we obtain the conclusion. ■

If the premise variables are different from the state variables, then (5.19) is of the form:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(p(k))(A_i x(k) + e_i) \tag{5.25}$$

where $p(k) = Qx(k)$, and Q is a $s \times n$ constant matrix with $rank(Q) = s$. According to the proofs of Theorem 5.2 and Theorem 5.3, it is easy to verify the following result.

Theorem 5.4 *The trivial solution $x(k) \equiv 0$ of the fuzzy affine system described by (5.25) is asymptotically stable, if there exists a symmetric positive definite matrix P and positive scalars τ_i such that:*

$$A_i^T P A_i - P < 0 \quad (i \in I_0) \tag{5.26}$$

$$\begin{bmatrix} e_i^T P e_i + \tau_i - \tau_i p_{0i}^T C_i p_{0i} & e_i^T P A_i + \tau_i p_{0i}^T C_i Q \\ A_i^T P e_i + \tau_i Q^T C_i p_{0i} & A_i^T P A_i - P - \tau_i Q^T C_i Q \end{bmatrix} < 0 \quad (i \in I_1), \tag{5.27}$$

where the notations I_0, I_1, D_i and p_{0i} are the same as those in Theorem 5.2.

Proof. It follows from the proofs of Theorem 5.2 and Theorem 5.3 directly. ■

On the application of the above theorems, it is to note:

- All the matrix inequalities in the above theorems are standard LMIs, so they can be efficiently verified with numerical methods such as LMI control toolbox in Matlab.
- In the above conclusions, the fuzzy sets in the fuzzy rules with index $i \in I_0$ needn't have bounded supports. Moreover, if $I_0 = \{1, 2, ..., r\}$ and all $e_i = 0$, then the presented theorems degenerate to the ordinary conclusions on open loop T-S fuzzy systems in the literature (e.g. Theorem 1 in [81]).
- If in addition $rank(Q) = n$, then under the stability conditions in Theorems 5.2 and 5.4, the trivial solutions of the fuzzy affine systems described by (5.11) and (5.25) are exponentially stable.
- Using the concept of sliding mode in [36], it is easy to show that the above theorems also hold for the piecewise affine systems (as shown in Example 5.1).
- The trivial solution in each of the above conclusions is globally stable, if the domain of definition covers the whole state space $\Re^n$. Otherwise, the trivial solution is only locally stable (as shown in Example 5.2).

5.4 Illustrative Examples with Unstable Subsystems

Example 5.1 *Consider the piecewise continuous affine system described by:*

$$\dot{x}(t) = \begin{cases} A_2 x(t) + B_2 & 4 \leqslant x_2(t) \leqslant 8 \\ A_1 x(t) + B_1 & x_2(t) \notin [4, 8] \end{cases} \tag{5.28}$$

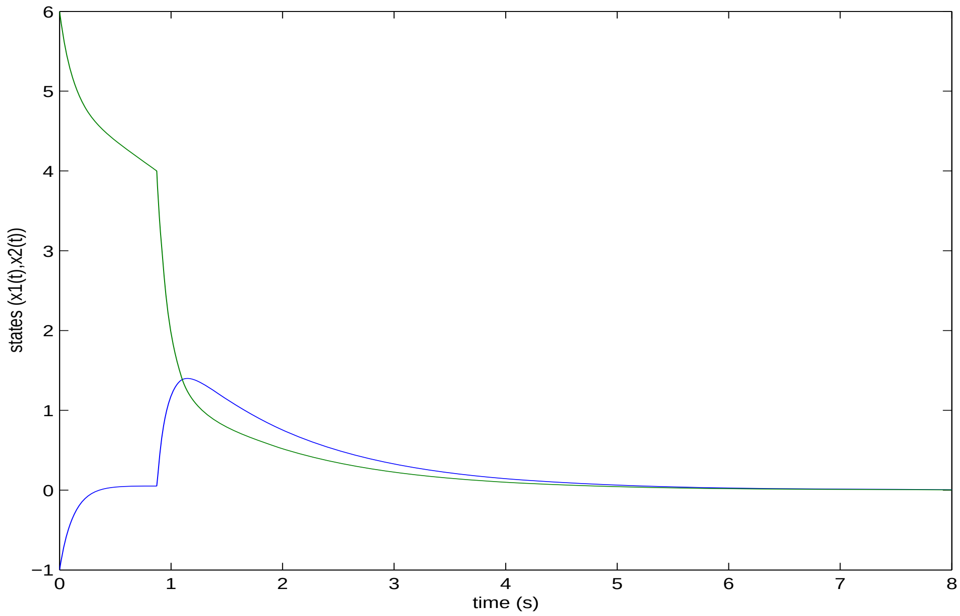

Figure 5.1: Trajectories of $x_1(t)$ and $x_2(t)$ from initial state $[-1, 6]^T$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A_1 = \begin{bmatrix} -3.6837 & 4.15 \\ 4.06 & -6.727 \end{bmatrix}$, $A_2 = \begin{bmatrix} -8 & 0.1 \\ 9 & 0.01 \end{bmatrix}$ $B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $B_2 = \begin{bmatrix} 0 \\ -1.5 \end{bmatrix}$.

Let $p(t) := x_2(t)$, *i.e.* $p(t) = Qx(t)$ *with* $Q = [0\ 1]$. *Then we can rewrite (5.28) as the formulation of standard T-S fuzzy models:*

$$\dot{x}(t) = \alpha_1(p(t))[A_1x(t) + B_1] + \alpha_2(p(t))[A_2x(t) + B_2] \tag{5.29}$$

where $\alpha_1(p(t)) = \begin{cases} 1 & p(t) \notin [4,8] \\ 0 & p(t) \in [4,8] \end{cases}$, $\alpha_2(p(t)) = \begin{cases} 1 & p(t) \in [4,8] \\ 0 & p(t) \notin [4,8] \end{cases}$. *Since* $\alpha_1(0) > 0$ *and* $\alpha_2(0) = 0$, *then* $I_0 = \{1\}$ *and* $I_1 = \{2\}$ *by the denotement in Theorem 5.2. Applying Lemma 5.1 we have that the minimal hyperellipsoid containing the interval* $[4, 8]$ *is:*

$$p(t)^T C_2 p(t) + p_{02}^T C_2 p_{02} - 2p_{02} C_2 p(t) = 1$$

where $p_{02} = 6$ *and* $C_2 = 1/4$. *Applying Theorem 5.2 we have the linear matrix inequalities:*

$$A_1^T P + PA_1 < 0$$

$$\begin{bmatrix} \tau_2 - \tau_2 p_{02}^T C_2 p_{02} & B_2^T P + \tau_2 p_{02}^T C_2 Q \\ PB_2 + \tau_2 Q^T C_2 p_{02} & A_2^T P + PA_2 - \tau_2 Q^T C_2 Q \end{bmatrix} < 0.$$

With the help of LMI control toolbox in Matlab, it is easy to get the feasible solutions of the above inequalities : $P = \begin{bmatrix} 1.0646 & 0.6670 \\ 0.6670 & 0.5875 \end{bmatrix} > 0$ *and* $\tau_2 = 0.8757$. *In this example* A_2 *is a unstable matrix with eigenvalues* $\lambda_1 = 0.1208$ *and* $\lambda_2 = -8.1108$. *However, according to Theorem 5.2, the trivial solution of (5.29) is still asymptotically stable. Moreover, it is globally asymptotically stable, for the domain of attraction is* $\Re^2$. *The stability of (5.28) is illustrated in Figure 5.1, where the initial state is* $[-1,\ 6]^T$.

Example 5.2 *Suppose that the fuzzy discrete affine system is described by the fuzzy rules shown in Table 5.1.*

$x_1(k)$	*PB*	(A_1, e_1)	(A_2, e_2)	(A_3, e_3)	(A_4, e_4)	(A_5, e_5)
	PM	(A_6, e_6)	(A_7, e_7)	(A_8, e_8)	(A_9, e_9)	(A_{10}, e_{10})
	ZO	(A_{11}, e_{11})	(A_{12}, e_{12})	(A_{13}, e_{13})	(A_{14}, e_{14})	(A_{15}, e_{15})
	NM	(A_{16}, e_{16})	(A_{17}, e_{17})	(A_{18}, e_{18})	(A_{19}, e_{19})	(A_{20}, e_{20})
	NB	(A_{21}, e_{21})	(A_{22}, e_{22})	(A_{23}, e_{23})	(A_{24}, e_{24})	(A_{25}, e_{25})
		NB	*NM*	*ZO*	*PM*	*PB*
		$x_2(k)$				

Table 5.1 Rule base for the system under discussion

The table represents 25 fuzzy rules in the rule base. For example, the grid with underline represents the 19-th fuzzy rule:

If $x_1(k)$ *is NM and* $x_2(k)$ *is PM, then* $x(k+1) = A_{19}x(k) + e_{19}$.

The membership functions of the 5 fuzzy sets in Table 5.1 are given by Figure 5.2.

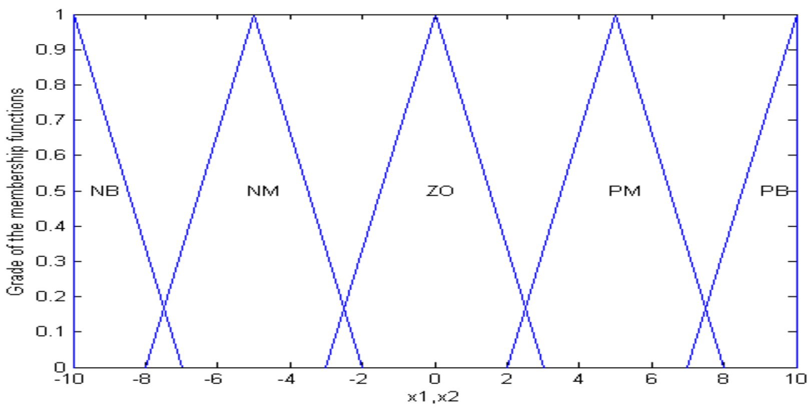

Figure 5.2: Membership functions of the fuzzy sets in rule base

The matrices in Table 5.1 are given as follows:

$$A_3 = A_{23} = \begin{bmatrix} 0.86 & -0.32 \\ 0.25 & -0.72 \end{bmatrix},\ A_{11} = A_{15} = \begin{bmatrix} 1.2 & -0.12 \\ 0.12 & -0.67 \end{bmatrix},$$

$$A_i = \begin{bmatrix} 0.86 & -0.1 \\ 0.12 & -0.89 \end{bmatrix} \ (1 \leqslant i \leqslant 25, i \neq 3, 11, 15, 23),$$
$$e_3 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, e_{23} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, e_i = 0 \ (1 \leqslant i \leqslant 25, \ i \neq 3, 23).$$

Obviously, the two subsystems described by the 11*-th and* 15*-th fuzzy rules are unstable. It is easy to see that the overall fuzzy system can be simplified as:*

$$\begin{aligned} x(k+1) &= [1 - \alpha_3(x(k)) - \alpha_{11}(x(k)) - \alpha_{15}(x(k)) - \alpha_{23}(x(k))]A_1 x(k) \\ &\quad + \alpha_3(x(k))(A_3 x(k) + e_3) + \alpha_{11}(x(k))A_{11}x(k) + \alpha_{15}(x(k))A_{15}x(k) \\ &\quad + \alpha_{23}(x(k))(A_{23}x(k) + e_{23}). \end{aligned} \tag{5.30}$$

where $\alpha_i(x(k))$ *can be computed according to the description of Section 4.1. Applying Theorem 5.3 we can get the feasible solutions of the LMIs (5.20) and (5.21):*

$$P = \begin{bmatrix} 772.1 & 384.9 \\ 384.9 & 2898.4 \end{bmatrix} > 0, \ \tau_3 = 1775.2,$$
$$\tau_{11} = 7779.5, \ \tau_{15} = 7779.5, \ \tau_{23} = 1824.7.$$

So by Theorem 5.3, the trivial solution of (5.30) is exponentially stable with domain of attraction $[-10, 10] \times [-10, 10]$*. The trajectories of* $x_1(k)$ *and* $x_2(k)$ *are illustrated in Figure 5.3, where the initial state is chosen from the unstable region of the* 15*-th subsystem.*

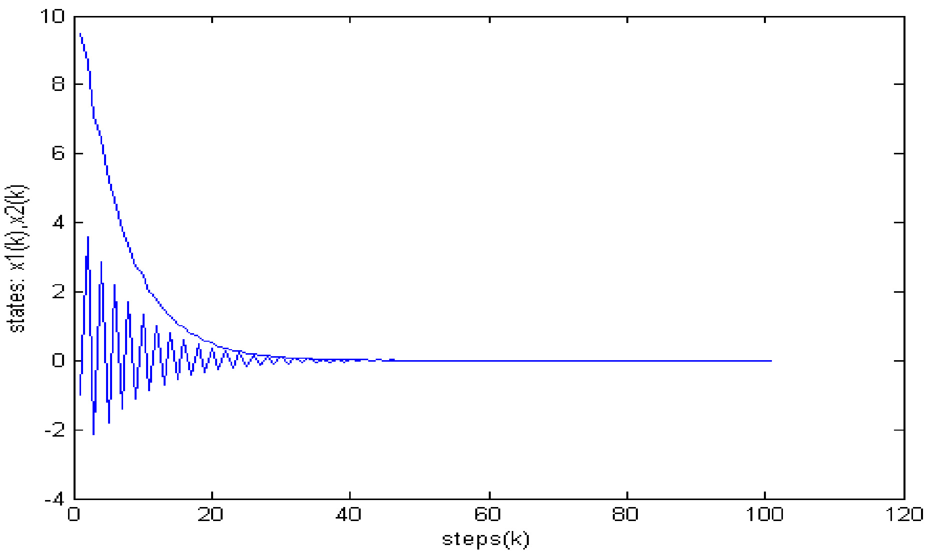

Figure 5.3: Trajectories of $x_1(k)$ and $x_2(k)$ from initial state $[-1 \ 9.8]^T$

Since in (5.30) $x(k)$ *is undefined out of the region* $[-10, 10] \times [-10, 10]$*, the trivial solution is only locally exponentially stable in this case.*

If we define additionally that $\alpha_3(x(k))$, $\alpha_{11}(x(k))$, $\alpha_{15}(x(k))$ and $\alpha_{23}(x(k))$ take the value 0 *for all $x(k) \in \Re^2 - [-10, 10] \times [-10, 10]$, then the domain of definition of (5.30) is whole space $\Re^2$. According to Theorem 5.3, the trivial solution of (5.30) in this case is globally exponentially stable. Figure 5.4 illustrates the trajectory of $x(k)$ from the initial state out of* $[-10, 10] \times [-10, 10]$.

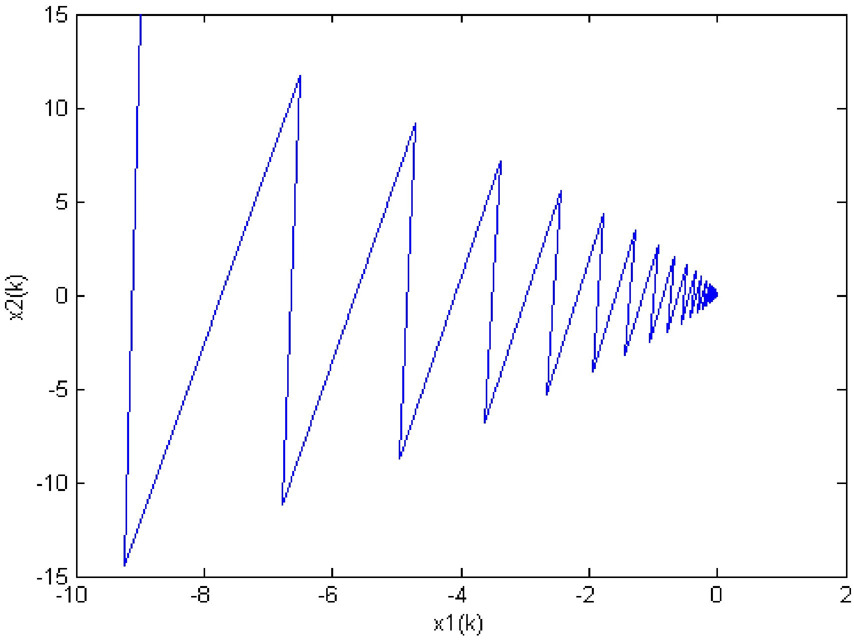

Figure 5.4: Trajectory of $x(k)$ from $[-9\ 15]^T$

In the above two examples, since both have some unstable subsystems, the usual stability conditions (e.g. [75]) are not satisfied, which shows that the conclusions presented in this chapter are less restrictive. However, the proposed approach requires that there must exist some fuzzy sets with bounded supports in the fuzzy rules. Moreover, when the approach is applied to the closed loop fuzzy control systems, the stability conditions can no longer be expressed in terms of LMIs.

Chapter 6

Stabilization of T-S Fuzzy Models with Bounded Supports

In this chapter, the stabilization of a class of T-S fuzzy control systems with support-bounded fuzzy sets in the rule base is discussed via fuzzy state feed back controllers. The stability conditions and fuzzy controller designs are reduced to a series of bilinear matrix inequalities (BMIs) in terms of the minimal hyperellipsoid-based method. Then, based on the LMI tools, the procedures for solving these BMIs are introduced. A simulation example is also given to demonstrate the proposed method.

6.1 Stability Analysis and Design

Suppose that the T-S fuzzy model and the PDC-based fuzzy controller are described by the following fuzzy rules respectively:

Plant rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$\dot{x}(t) = A_i x(t) + B_i u(t) \ (i = 1, 2, ..., r).$$

Controller rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$u(t) = K_j x(t) \ (j = 1, 2, ..., r),$$

where $K_1, K_2, ..., K_r$ are the control gains to be designed. It is assumed that $p(t) := [\ p_1(t),\ \ p_2(t),\ \ ...,\ \ p_s(t)\ ]^T = Qx(t)$ and $rank(Q) = s$. Then the overall system and overall fuzzy controller can be expressed by:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(p(t))(A_i x(t) + B_i u(t)) \tag{6.1}$$

$$u(t) = \sum_{i=1}^{r} \alpha_i(p(t))K_i x(t) \tag{6.2}$$

where $\alpha_i(p(t)) = \prod_{j=1}^{s} \mu_{M_j^i}(p_j(t))/\sum_{i=1}^{r}(\prod_{j=1}^{s} \mu_{M_j^i}(p_j(t)))$ for all $i = 1, 2, ..., r$ according to the description of Section 4.1. Denote $D_i := \{p(t) : \alpha_i(p(t)) > 0\}$ for $i = 1, 2, ..., r$. If D_i is bounded, i.e. $D_i = (a_{1i},\ b_{1i}) \times ... \times (a_{si},\ b_{si})$, then by Lemma 5.1 the minimal hyperellipsoid containing D_i can be constructed:

$$p(t)^T C_i p(t) + p_{0i}^T C_i p_{0i} - 2p_{0i}^T C_i p(t) = 1 \ (i = 1, 2, ..., r)$$

where

$$C_i = \frac{4}{s} diag\{\frac{1}{(a_{1i} - b_{1i})^2}, \frac{1}{(a_{2i} - b_{2i})^2}, ..., \frac{1}{(a_{si} - b_{si})^2}\},$$

$$p_{0i} = [\frac{a_{1i} + b_{1i}}{2}, \frac{a_{2i} + b_{2i}}{2}, ..., \frac{a_{si} + b_{si}}{2}]^T.$$

Moreover, the index set $\{1, 2, ..., r\}$ can be divided into I_0 and I_1 according to the properties of D_i $(i = 1, 2, ..., r)$. If D_i is bounded and the hyperellipsoid containing D_i doesn't contain the original point, then the index i is assigned to I_1. That is:

$$I_1 = \{i : p_{0i}^T C_i p_{0i} > 1, 1 \leqslant i \leqslant r\},\ I_0 = \{1, 2, ..., r\} - I_1.$$

In addition, $A_i + B_i K_j$ is denoted by H_{ij}, and $\frac{H_{ij}+H_{ji}}{2}$ is denoted by G_{ij} for brevity. With these notations in mind, we can present the main result now.

Theorem 6.1 *If there exist $P > 0$, $\tau_i \geqslant 0, \tau_{ii} \geqslant 0$ and $K_i \in \Re^{m\times n}$ for $i = 1, 2, ..., r$, such that:*

$$G_{ij}^T P + PG_{ij} < 0 \quad (i, j \in I_0 \ and \ 1 \leqslant i \leqslant j \leqslant r), \tag{6.3}$$

$$\begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q \\ \tau_{ii} Q^T C_i p_{0i} & H_{ii}^T P + PH_{ii} - \tau_{ii} Q^T C_i Q \end{bmatrix} < 0 \quad (i \in I_1), \tag{6.4}$$

and for the pairs $(i, j) \in \{(i, j) : 1 \leqslant i < j \leqslant r,\ i \in I_1 \ or \ j \in I_1\}$:

$$\begin{bmatrix} \tau_i(1 - p_{0i}^T C_i p_{0i}) + \tau_j(1 - p_{0j}^T C_j p_{0j}) & \tau_i p_{0i}^T C_i Q + \tau_j p_{0j}^T C_j Q \\ \tau_i Q^T C_i p_{0i} + \tau_j Q^T C_j p_{0j} & G_{ij}^T P + PG_{ij} - \tau_i Q^T C_i Q - \tau_j Q^T C_j Q \end{bmatrix} < 0, \tag{6.5}$$

where $\tau_i = 0$ for $i \in I_0$ in (6.5), then the fuzzy system described by (6.1) is asymptotically stabilized via fuzzy state feedback controller (6.2).

Proof. Let the Lyapunov candidate function $V(x(t)) = x(t)^T Px(t)$, then

$$\dot{V}(x(t)) = \sum_{i,j=1}^{r} \alpha_i(p(t))\alpha_j(p(t))x(t)^T (H_{ij}^T P + PH_{ij})x(t)$$
$$= \sum_{i\in I_0} \alpha_i^2(p(t))x(t)^T (H_{ii}^T P + PH_{ii})x(t)$$

$$+ \sum_{\substack{1\leqslant i<j\leqslant r\\ i\in I_0, j\in I_0}} 2\alpha_i(p(t))\alpha_j(p(t))x(t)^T(G_{ij}^T P + PG_{ij})x(t)$$
$$+ \sum_{i\in I_1} \alpha_i^2(p(t))x(t)^T(H_{ii}^T P + PH_{ii})x(t)$$
$$+ \sum_{\substack{1\leqslant i<j\leqslant r\\ i\in I_1 \ or \ j\in I_1}} 2\alpha_i(p(t))\alpha_j(p(t))x(t)^T(G_{ij}^T P + PG_{ij})x(t)$$
$$\overset{(6.3)}{\leqslant} \sum_{i\in I_1} \alpha_i^2(p(t))x(t)^T(H_{ii}^T P + PH_{ii})x(t)$$
$$+ \sum_{\substack{1\leqslant i<j\leqslant r\\ i\in I_1 \ or \ j\in I_1}} 2\alpha_i(p(t))\alpha_j(p(t))x(t)^T(G_{ij}^T P + PG_{ij})x(t).$$

Note that $\alpha_i(p(t)) > 0$ implies $p(t)^T C_i p(t) + p_{0i}^T C_i p_{0i} - 2p_{0i}^T C_i p(t) < 1$ for $i \in I_1$ Thereby, we have:

$$\dot{V}(x(t)) \leqslant \sum_{i\in I_1} \alpha_i^2(p(t))\{x(t)^T(H_{ii}^T P + PH_{ii})x(t)$$
$$+\tau_{ii}[1 - p(t)^T C_i p(t) - p_{0i}^T C_i p_{0i} + 2p_{0i}^T C_i p(t)]\}$$
$$+ \sum_{\substack{1\leqslant i<j\leqslant r\\ i\in I_1 \ or \ j\in I_1}} 2\alpha_i(p(t))\alpha_j(p(t))\{x(t)^T(G_{ij}^T P + PG_{ij})x(t)$$
$$+\tau_i[1 - p(t)^T C_i p(t) - p_{0i}^T C_i p_{0i} + 2p_{0i}^T C_i p(t)]$$
$$+\tau_j[1 - p(t)^T C_j p(t) - p_{0j}^T C_i p_{0j} + 2p_{0j}^T C_j p(t)]\}$$

where $\tau_i = 0$ for $i \in I_0$.

Then, similar to the proofs of Theorem 5.2, it is easy to show that $\dot{V}(x(t)) \leqslant 0$ if conditions (6.4) and (6.5) are satisfied, which completes the proof. ■

The matrix inequality constraints in the above theorem have been formulated in the form of BMIs with respect to the parameters τ_i, τ_{ii}, K_i and P $(i = 1, 2, ..., r)$. The method for solving these BMIs will be discussed in the next section.

In the proof of Theorem 6.1, each fuzzy rule is considered separately. Thereby, only $|I_1|$ minimal hyperellipsoids are constructed. If consider the supports intersection of two fuzzy rules simultaneously, then the conditions of Theorem 6.1 can be further improved, since the hyperellipsoid regions under discussion are reduced in this case.

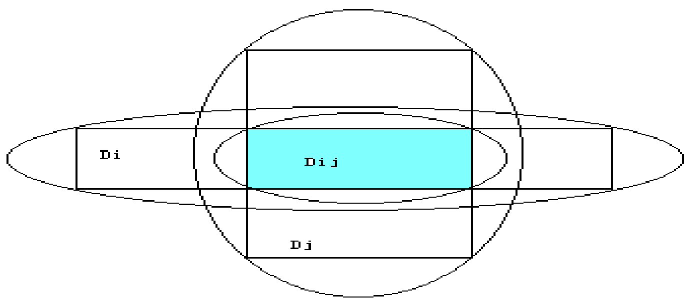

Figure 6.1: Hyperellipsoid for the intersection of two fuzzy rules

If $D_{ij} := \{p(t) : \alpha_i(p(t))\alpha_j(p(t)) > 0\}$ is bounded, by Lemma 5.1 the minimal hyperellipsoid containing D_{ij} can be constructed (as shown in Figure 6.1):

$$p(t)^T C_{ij} p(t) + p_{0ij}^T C_{ij} p_{0ij} - 2p_{0ij}^T C_{ij} p(t) = 1,$$

where

$$p(t) = Qx(t), rank(Q) = s,$$
$$C_{ij} = \frac{4}{s} diag\{\frac{1}{(a_{1ij} - b_{1ij})^2}, \frac{1}{(a_{2ij} - b_{2ij})^2}, ..., \frac{1}{(a_{sij} - b_{sij})^2}\},$$
$$p_{0i} = [\frac{a_{1ij} + b_{1ij}}{2}, \frac{a_{2ij} + b_{2ij}}{2}, ..., \frac{a_{sij} + b_{sij}}{2}]^T,$$
$$a_{kij} = \max\{\inf\{p_k(t) : \alpha_i(p(t)) > 0\}, \ \inf\{p_k(t) : \alpha_j(p(t)) > 0\}\},$$
$$b_{kij} = \min\{\sup\{p_k(t) : \alpha_i(p(t)) > 0\}, \ \sup\{p_k(t) : \alpha_j(p(t)) > 0\}\}$$

for $i, j = 1, 2, ..., r$ and $k = 1, 2, ..., s$.

In this case, we denote

$$I_1 := \{(i, j) : p_{0ij}^T C_{ij} p_{0ij} > 1, 1 \leqslant i \leqslant j \leqslant r\}, \ I_0 := \{(i, j) : 1 \leqslant i \leqslant j \leqslant r\} - I_1.$$

Then, by substituting D_i and D_j for D_{ij} in the proofs of Theorem 6.1 we have the following improved result.

Corollary 6.1 *If there exist $P > 0$, $\tau_{ij} \geqslant 0$ and $K_i \in \Re^{m \times n}$ for $1 \leqslant i \leqslant j \leqslant r$, such that:*

$$G_{ij}^T P + PG_{ij} < 0 \quad (i, j) \in I_0,$$

$$\begin{bmatrix} \tau_{ij}(1 - p_{0ij}^T C_{ij} p_{0ij}) & \tau_{ij} p_{0ij}^T C_{ij} Q \\ \tau_{ij} Q^T C_{ij} p_{0ij} & G_{ij}^T P + PG_{ij} - \tau_{ij} Q^T C_{ij} Q \end{bmatrix} < 0 \quad (i, j) \in I_1,$$

then the fuzzy system described by (6.1) is asymptotically stabilized via fuzzy state feedback controller (6.2).

Proof. It is similar to the proofs of Theorem 6.1. ■

In comparison with Theorem 6.1, the improvement of Corollary 6.1 results from:

(i) The regions D_{ij} may be bounded, even if both D_i and D_j are unbounded.

(ii) The regions under discussion are further reduced, since $D_{ij} \subseteq D_i \cup D_j$.

However, in applications, Corollary 6.1 may involve more computations than Theorem 6.1, since more minimal hyperellipsoids (at most $\frac{r(r+1)}{2}$ hyperellipsoids) have to be constructed.

For the discrete T-S fuzzy control systems, the overall process and overall fuzzy controller can be formulated as:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(p(k))(A_i x(k) + B_i u(k)), \tag{6.6}$$

$$u(k) = \sum_{i=1}^{r} \alpha_i(p(k))K_i x(k). \tag{6.7}$$

Similarly we can prove:

Theorem 6.2 *If there exist $P > 0$, $\tau_i \geqslant 0, \tau_{ii} \geqslant 0$ and $K_i \in \Re^{m\times n}$ for $i = 1, 2, ..., r$, such that:*

$$\begin{bmatrix} -P & G_{ij}^T P \\ PG_{ij} & -P \end{bmatrix} < 0 \quad (i, j \in I_0, 1 \leqslant i \leqslant j \leqslant r), \tag{6.8}$$

$$\begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q & 0 \\ \tau_{ii} Q^T C_i p_{0i} & -P - \tau_{ii} Q^T C_i Q & H_{ii}^T P \\ 0 & PH_{ii} & -P \end{bmatrix} < 0 \quad (i \in I_1), \tag{6.9}$$

and for the pairs $(i, j) \in \{(i, j) : 1 \leqslant i < j \leqslant r, i \in I_1 \text{ or } j \in I_1\}$:

$$\begin{bmatrix} \tau_i(1 - p_{0i}^T C_i p_{0i}) + \tau_j(1 - p_{0j}^T C_j p_{0j}) & \tau_i p_{0i}^T C_i Q + \tau_j p_{0j}^T C_j Q & 0 \\ \tau_i Q^T C_i p_{0i} + \tau_j Q^T C_j p_{0j} & -P - \tau_i Q^T C_i Q - \tau_j Q^T C_j Q & G_{ij}^T P \\ 0 & PG_{ij} & -P \end{bmatrix} < 0, \tag{6.10}$$

where all notations are the same as those in Theorem 6.1 and $\tau_i = 0$ for $i \in I_0$ in (6.10), then the fuzzy system described by (6.6) is asymptotically stabilized via fuzzy state feedback controller (6.7).

Proof. Similar to the proofs of Theorem 6.1, it is easy to show that (6.6) is asymptotically stabilizable via (6.7), if the following conditions are satisfied:

$$G_{ij}^T PG_{ij} - P < 0 \quad (i, j \in I_0 \text{ and } 1 \leqslant i \leqslant j \leqslant r), \tag{6.11}$$

$$\begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q \\ \tau_{ii} Q^T C_i p_{0i} & H_{ii}^T PH_{ii} - P - \tau_{ii} Q^T C_i Q \end{bmatrix} < 0 \quad (i \in I_1) \tag{6.12}$$

and for the pairs $(i, j) \in \{(i, j) : 1 \leqslant i < j \leqslant r, i \in I_1 \text{ or } j \in I_1\}$:

$$\begin{bmatrix} \tau_i(1 - p_{0i}^T C_i p_{0i}) + \tau_j(1 - p_{0j}^T C_j p_{0j}) & \tau_i p_{0i}^T C_i Q + \tau_j p_{0j}^T C_j Q \\ \tau_i Q^T C_i p_{0i} + \tau_j Q^T C_j p_{0j} & G_{ij}^T PG_{ij} - P - \tau_i Q^T C_i Q - \tau_j Q^T C_j Q \end{bmatrix} < 0. \tag{6.13}$$

The matrix inequality constraints (6.11)-(6.13) are neither LMIs nor BMIs with respect to the parameters τ_i, τ_{ii}, K_i and P. For the sake of computation, we will rewrite them in terms of BMIs. We prove that (6.11)-(6.13) are equivalent to the BMI constraints (6.8)-(6.10) respectively.

(I) $G_{ij}^T PG_{ij} - P < 0$

$\Leftrightarrow (G_{ij}^T P)P^{-1}(PG_{ij}) - P < 0$

$$\Leftrightarrow \begin{bmatrix} -P & G_{ij}^T P \\ PG_{ij} & -P \end{bmatrix} < 0 \text{ by Schur complement.}$$

(II) $\begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q \\ \tau_{ii} Q^T C_i p_{0i} & H_{ii}^T P H_{ii} - P - \tau_{ii} Q^T C_i Q \end{bmatrix} < 0$

$$\Leftrightarrow \begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q \\ \tau_{ii} Q^T C_i p_{0i} & -P - \tau_{ii} Q^T C_i Q \end{bmatrix} + \begin{bmatrix} 0 \\ H_{ii}^T P \end{bmatrix} P^{-1} \begin{bmatrix} 0 & PH_{ii} \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} \tau_{ii}(1 - p_{0i}^T C_i p_{0i}) & \tau_{ii} p_{0i}^T C_i Q & 0 \\ \tau_{ii} Q^T C_i p_{0i} & -P - \tau_{ii} Q^T C_i Q & H_{ii}^T P \\ 0 & PH_{ii} & -P \end{bmatrix} < 0 \text{ by Schur complement.}$$

(III) Similar to the process (II), it gives (6.10)⇔(6.13). ■

Corollary 6.2 *If there exist $P > 0$, $\tau_{ij} \geqslant 0$ and $K_i \in \Re^{m \times n}$ for all $1 \leqslant i \leqslant j \leqslant r$, such that:*

$$\begin{bmatrix} -P & G_{ij}^T P \\ PG_{ij} & -P \end{bmatrix} < 0 \quad (i, j) \in I_0,$$

$$\begin{bmatrix} \tau_{ij}(1 - p_{0ij}^T C_{ij} p_{0ij}) & \tau_{ij} p_{0ij}^T C_{ij} Q & 0 \\ \tau_{ij} Q^T C_{ij} p_{0ij} & -P - \tau_{ij} Q^T C_{ij} Q & G_{ij}^T P \\ 0 & PG_{ij} & -P \end{bmatrix} < 0 \quad (i, j) \in I_1,$$

where all the notations are the same as those in Corollary 6.1, then the fuzzy system described by (6.6) is asymptotically stabilized via fuzzy state feedback controller (6.7).

Proof. It is similar to the proof of Theorem 6.2. ■

As shown above, the structural information is utilized to release the conservatism of analysis. But, in order to construct the minimal hyperellipsoids, it is required that there must exist some fuzzy sets with bounded supports in the fuzzy rules. In fuzzy control context, three kinds of fuzzy sets are most frequently employed, namely, triangular-shaped fuzzy sets, trapezoid-shaped fuzzy sets and bell-shaped fuzzy sets. The proposed approach is appropriate for the fuzzy systems with triangular-shaped fuzzy sets and trapezoid-shaped fuzzy sets. But for the fuzzy systems with bell-shaped fuzzy sets, no minimal hyperellipsoids can be constructed, so all the above conclusions degenerate to the usual ones in literature e.g. [60], [81].

Another way to overcome the conservatism of analysis is to employ the generalized Lyapunov functions instead of the common global quadratic Lyapunov functions. In [37] and [36] a method for constructing the piecewise quadratic Lyapunov functions is proposed for the stability analysis of open loop fuzzy models. In [42] the piecewise quadratic candidate Lyapunov function is given by $V(x(t)) = max\{x(t)^T P_i x(t) : 1 \leqslant i \leqslant N\}$ disregard of the structural information in the fuzzy rules. Based on the method in [42], we can further improve our result of Corollary 6.1 just by substituting the candidate Lyapunov functions in the proofs.

Lemma 6.1 *(Generalized Lyapunov Function, [42]) Let* $\{\Omega_i : i = 1, 2, ..., N\}$ *be a partition of* $\Re^n$ *(i.e.* $0 \in \Omega_i$ *for* $i = 1, 2, ..., N$, $\bigcup_{i=1}^{N} \Omega_i = \Re^n$, $\overset{\circ}{\Omega}_i \cap \overset{\circ}{\Omega}_j = \emptyset$ *for* $i \neq j$*). The continuous function* $V : \Re^n \to \Re$ *is a generalized Lyapunov function for (6.1), if:*

(1) V *is proper on each* Ω_i, *i.e.* $\{x \in \Omega_i : V(x) \leqslant a\}$ *is compact for all* $a > 0$.

(2) V *is positive definite on each* Ω_i, *i.e.* $V(0) = 0$, $V(x) > 0$ *for all* $0 \neq x \in \Omega_i$.

(3) For each $0 \neq x \in \overset{\circ}{\Omega}_i$, *there exists some* u *such that along the trajectory of (6.1) it holds:* $\dot{V} < 0$, *where* $\overset{\circ}{\Omega}_i$ *stands for the interior of* Ω_i.

Corollary 6.3 *The fuzzy system described by (6.1) is asymptotically stabilized via fuzzy state feedback controller (6.2), if there exist* $P_l > 0$, *scalars* $\tau_{ij}, \sigma_{ijlm} \geqslant 0$ *and matrices* $K_i \in \Re^{m\times n}$ *such that:*

$$G_{ij}^T P_l + P_l G_{ij} + \sum_{m=1}^{N} \sigma_{ijlm}(P_l - P_m) < 0 \ for \ (i,j) \in I_0 \ and \ 1 \leqslant l \leqslant N, \tag{6.14}$$

$$\begin{bmatrix} \tau_{ij}(1 - p_{0ij}^T C_{ij} p_{0ij}) & \tau_{ij} p_{0ij}^T C_{ij} Q \\ \tau_{ij} Q^T C_{ij} p_{0ij} & G_{ij}^T P_l + P_l G_{ij} - \tau_{ij} Q^T C_{ij} Q + \sum_{m=1}^{N} \sigma_{ijlm}(P_l - P_m) \end{bmatrix} < 0 \tag{6.15}$$

for $(i,j) \in I_1$ *and* $1 \leqslant l \leqslant N$, *where the notations* $I_0, I_1, C_{ij}, p_{0ij}$ *and* Q *are the same as those in Corollary 6.1.*

Proof. Choose the Lyapunov candidate function as

$$V(x(t)) = max\{x(t)^T P_i x(t) : 1 \leqslant i \leqslant N\},$$

then we obtain a partition $\{\Omega_i : i = 1, 2, ..., N\}$ of $\Re^n$, where

$$\Omega_i = \{x \in \Re^n : x^T P_i x \geqslant x^T P_j x, j \neq i\} \text{ for } i = 1, 2, ..., N.$$

Thereby, $V(x(t)) = x(t)^T P_i x(t)$ when V is restricted to the region $\overset{\circ}{\Omega}_i$. The conditions (1) and (2) in Lemma 6.1 are satisfied obviously. Following the proofs of Theorem 6.1 we can prove that condition (3) is also satisfied on each region $\overset{\circ}{\Omega}_i$ under the BMI constraints (6.14) and (6.15). So, we obtain the result according to Lemma 6.1. ■

It is easy to show that Corollary 6.3 and Corollary 6.1 are equivalent, if N is set to 1. However, if a greater N is chosen, both the number of BMIs and the number of parameters will be greatly increased in Corollary 6.3. The main drawback of this generalized Lyapunov function method is that an appropriate N is difficult to give in advance. If N is set too small, the related BMIs may be infeasible, and if N is chosen too large, the related BMI constraints will turn out to be very complicated.

6.2 Solution Procedure of BMIs

Definition 6.1 *(BMI Feasibility Problem, [65]) Suppose that* $F : \Re^{n_1} \times \Re^{n_2} \rightarrow \Re^{m\times m}$ *is a bilinear function defined by:*

$$F(x,y) = \sum_{i=1}^{n_1}\sum_{i=1}^{n_2} x_i y_j F_{ij}$$

where $x = [x_1, x_2, ..., x_n]^T \in \Re^{n_1}$, $y = [y_1, y_2, ..., y_n]^T \in \Re^{n_2}$ *and* $F_{ij} = F_{ji}^T \in \Re^{m\times m}$. *To find the solutions* x *and* y, *if they exist, such that* $F(x,y) < 0$ *is called the bilinear matrix inequality feasibility problem. If additionally the constraints* $x_1 = 1$ *and* $y_1 = 1$ *are imposed, then this is called the biaffine matrix inequality feasibility problem.*

Both biaffine matrix inequalities and bilinear matrix inequalities will be simply referred to as BMIs, since every biaffine matrix inequality can be equivalently formulated as a bilinear matrix inequality [65]. Moreover, when some parameters are fixed, BMIs turn out to be LMIs with respect to the other parameters, and vice versa. Due to the non-convexity of BMIs, the BMI feasibility problem is in general difficult to solve [26]. As shown in [78], the global optimization for BMIs is NP-hard, i.e. it is unlikely to find a polynomial time algorithm for the optimal solution of BMIs. However, in some special cases, such as full-order control and full-state feedback, the BMI feasibility problem may be reduced to the LMI feasibility problem equivalently [34]. In [26] a heuristic LMI-based approach for solving BMIs is proposed. That is, we try to find the feasible solutions of BMIs by solving the related double LMIs alternatively. Based on the algorithm in [26], we introduce the following procedures for solving the BMIs presented in Section 6.1.

Since $A < 0$ and $B < 0$ is equivalent to $diag(A, B) < 0$, we can rewrite the BMI constraints in each conclusion of Section 6.1 as one BMI $F(P, \tau, K) < 0$, where τ and K stand for the sets of parameters $\tau_i, \tau_{ii}, \tau_{ij}$ and K_i respectively. Fix the parameters τ, K and P alternatively, we can solve the BMIs by means of the LMI approaches. The solution procedures are given in Table 6.1.

In the procedures, the so called generalized eigenvalue problem:

$$\begin{cases} \min\text{imizing } \lambda \\ \text{subject to } A(x) < \lambda B(x) \end{cases}$$

is involved. The problem can be efficiently solved by the LMI solver *gevp*( ) in LMI Control Toolbox in Matlab. However, this LMI-based method cannot guarantee that we can find a feasible solution necessarily, even if it exists. This depends on the choice of the initial conditions.

It is to note, if all the parameters $\tau_i, \tau_{ii}, \tau_{ij}$ are set to be identical (e.g. $\tau_i = \tau_{ii} = \tau_{ij} =: \overline{\tau}$), then the BMI constraints in each conclusion in Section 6.1 can be equivalently formulated as LMIs with respect to P^{-1},$\overline{\tau}P^{-1}$ and K_iP^{-1}. In this case, the feasible solutions can be solved directly by the LMI tools in Matlab. As a result, the possibility of finding the feasible solutions will be also reduced in this case.

More detailed discussions on the BMI feasibility problem can be found in [27], and the so-called barrier approach for BMIs is also presented therein.

Table 6.1:Solution Procedures of BMIs

$\widetilde{P} = I$	
Repeat	{
	Find $\widetilde{\tau}, \widetilde{K}$ and λ_1 by $\begin{cases} \min\text{imizing } \lambda_1 \\ \text{subject to } F(\widetilde{P}, \tau, K) < \lambda_1 I \end{cases}$.
	If $\lambda_1 < 0$, exit
	Find P and λ_2 by $\begin{cases} \min\text{imize } \lambda_2 \\ \text{subject to } F(P, \widetilde{\tau}, \widetilde{K}) < \lambda_2 I \end{cases}$. Let $\widetilde{P} = P$
	If $\lambda_2 < 0$, exit.
	}

6.3 Simulation Example

Example 6.1 *Consider the T-S fuzzy system described by (6.1), where:*

$$A_1 = \begin{bmatrix} 2 & -7 \\ -3 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 5 & 1 \\ -4 & -3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ B_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

(all A_1, A_2 and A_3 are unstable matrices) the premise variable $p(t) = Qx(t)$ with $Q = [\ 1 \quad 0\ ]$, and the fuzzy sets M_i (i=1,2,3) in the rule base are given in Figure 6.2. We employ the fuzzy state feed back fuzzy controller (6.2) for the stabilization of (6.1). By Corollary 6.1 we have the following BMI-type stability constraints:

$$H_{22}^T P + P H_{22} < 0,$$

$$H_{33}^T P + P H_{33} < 0,$$

and for $(i, j) \in \{(1, 1), (1, 2), (1, 3)\}$:

$$\begin{bmatrix} \tau_{ij}(1 - p_{ij}^T C_{ij} p_{ij}) & \tau_{ij} p_{ij}^T C_{ij} Q \\ \tau_{ij} Q^T C_{ij} p_{ij} & G_{ij}^T P + P G_{ij} - \tau_{ij} Q^T C_{ij} Q \end{bmatrix} < 0,$$

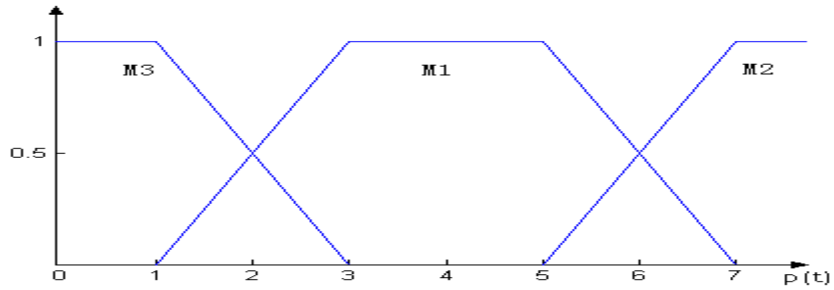

Figure 6.2: Membership functions in the rule base

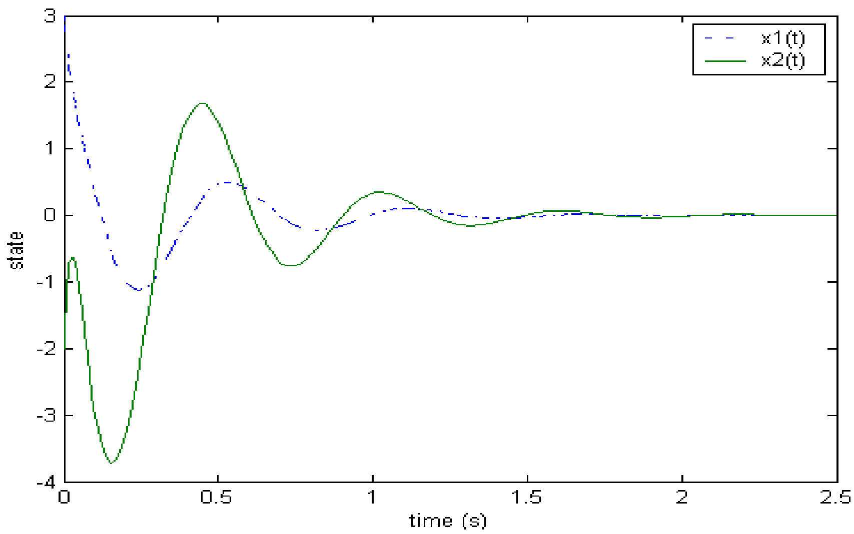

Figure 6.3: Controlled trajectories of $x_1(t)$ and $x_2(t)$ under initial condition [3, -2]^T

where $p_{11} = 4$, $p_{12} = 6$, $p_{13} = 2,$, $C_{11} = \frac{1}{9}$, $C_{12} = C_{13} = 1$. *By executing the procedures in Section 6.2, the following feasible solutions are obtained:*

$$\tau_{11} = 7.6586, \tau_{12} = 24.5955, \tau_{13} = 7.0268,$$

$$K_1 = \begin{bmatrix} -21.7973 & 5.2037 \end{bmatrix}, K_2 = \begin{bmatrix} -51.1813 & 11.4937 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -15.6981 & 4.1080 \end{bmatrix}, P = \begin{bmatrix} 0.1618 & -0.0295 \\ -0.0295 & 0.0251 \end{bmatrix}.$$

So, the conditions of Corollary 6.1 are satisfied. The stability of the closed loop system is demonstrated via the simulation result shown in Figure 6.3.

Chapter 7

BMI-based Fuzzy Controller Design for T-S Fuzzy Models

In this chapter, we present some relaxed sufficient conditions for the stabilization of T-S fuzzy models via state feedback, output feedback and observer-based fuzzy controllers respectively. We introduce a block parameter matrix in analysis and formulate the stabilization conditions in terms of BMIs. The design of fuzzy controllers is reduced to the BMI feasibility problem, so the state feedback gains, output feedback gains and observer gains can be solved by the BMI solution procedures. The proposed design methods are finally illustrated by the control simulations on the chaotic Lorenz system.

7.1 Output Feedback Controller Design

Suppose that the continuous T-S fuzzy models are described by the following fuzzy rules:

Plant rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y_i(t) = C_i x(t) \end{cases} \quad (i = 1, 2, ..., r). \tag{7.1}$$

where $y_i(t)$ is the output of the i-th subsystem. Based on the PDC technique, the output feedback fuzzy controllers can be expressed by:

Controller rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$u(t) = K_i y_i(t) \quad (i = 1, 2, ..., r),$$

where K_i $(i = 1, 2, ..., r)$ are the output feedback gains to be designed. Then, similar to the discussions in Section 4.1 the overall formulations of the continuous T-S fuzzy models can be inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(p(t))(A_i x(t) + B_i u(t)), \tag{7.2}$$

$$y(t) = \sum_{i=1}^{r} \alpha_i(p(t)) y_i(t)) = \sum_{i=1}^{r} \alpha_i(p(t)) C_i x(t), \tag{7.3}$$

$$u(t) = \sum_{i=1}^{r} \alpha_i(p(t)) K_i y_i(t)) = \sum_{i=1}^{r} \alpha_i(p(t)) K_i C_i x(t), \tag{7.4}$$

where all the notations are the same as those in Section 4.1. Then, the design of output feedback fuzzy controllers is reduced to determining the output feedback gains, such that the closed loop system (7.2) can be asymptotically stabilized via the output feedback controller (7.4).

Similarly, the overall outputs of discrete T-S fuzzy models can be formulated as:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(p(k))(A_i x(k) + B_i u(k)), \tag{7.5}$$

$$y(k) = \sum_{i=1}^{r} \alpha_i(p(k)) y_i(k)) = \sum_{i=1}^{r} \alpha_i(p(k)) C_i x(k), \tag{7.6}$$

$$u(k) = \sum_{i=1}^{r} \alpha_i(p(k)) K_i y_i(k)) = \sum_{i=1}^{r} \alpha_i(p(k)) K_i C_i x(k). \tag{7.7}$$

Based on the stability results in [60] and [81], we have that the continuous fuzzy system described by (7.2) is stabilized via the output feedback fuzzy controller (7.4), if there exists $Q > 0$ and matrices K_i such that:

$$QG_{ij}^T + G_{ij} Q < 0 \tag{7.8}$$

for all $1 \leqslant i \leqslant j \leqslant r$ except $\alpha_i(p(t))\alpha_j(p(t)) \equiv 0$, where $G_{ij} = \frac{1}{2}(A_i + B_i K_j C_j + A_j + B_j K_i C_i)$. Similarly, if there exists $Q > 0$ and matrices K_i such that:

$$\begin{bmatrix} -Q & QG_{ij}^T \\ G_{ij} Q & -Q \end{bmatrix} < 0 \tag{7.9}$$

for all $1 \leqslant i \leqslant j \leqslant r$ except $\alpha_i(p(k))\alpha_j(p(k)) \equiv 0$, then the discrete system described by (7.5) is stabilized via the output feedback controller (7.7).

The BMI constraints (7.9) are less conservative than the corresponding result presented in [11], since an additional constraint $Q^{-1} = \sum_{t=0}^{\infty} (G_{ii}^T)^t C_i^T C_i G_{ii}^t$ is imposed in [11] for the sake of computation. The following result is theoretically parallel to the BMI constraints of (7.8).

Theorem 7.1 *The continuous T-S fuzzy system described by (7.2) is globally exponentially stabilized via the output feedback fuzzy controller described by (7.4), if there exists a matrix $Q > 0$, scalar parameters $\tau_{ij} > 0$ $(1 \leqslant i < j \leqslant r)$ and matrices K_i $(1 \leqslant i \leqslant r)$, such that:*

$$\begin{bmatrix} R_{11} & R_{12} + \tau_{12} I & \cdots & R_{1r} + \tau_{1r} I \\ R_{12} + \tau_{12} I & R_{22} & \cdots & R_{2r} + \tau_{2r} I \\ \dots & \dots & \dots & \dots \\ R_{1r} + \tau_{1r} I & R_{2r} + \tau_{2r} I & \dots & R_{rr} \end{bmatrix} < 0, \tag{7.10}$$

where $R_{ij} = QG_{ij}^T + G_{ij}Q$ and $G_{ij} = \frac{1}{2}(A_i + B_iK_jC_j + A_j + B_jK_iC_i)$ for all $i \leqslant j$.

Proof. Choose the candidate Lyapunov function as $V(x(t)) = x(t)^T Q^{-1} x(t)$, then we have: $\lambda_{\min}(Q^{-1})|x(t)|^2 \leqslant V(x(t)) \leqslant \lambda_{\max}(Q^{-1})|x(t)|^2$.

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}(t)^T Q^{-1} x(t) + x(t)^T Q^{-1} \dot{x}(t) \\
&= \sum_{i,j=1}^{r} \alpha_i(p(t))\alpha_j(p(t)) x(t)^T [(A_i + B_iK_jC_j)^T Q^{-1} + Q^{-1}(A_i + B_iK_jC_j)] x(t) \\
&= \sum_{i=1}^{r} \alpha_i^2(p(t)) x(t)^T (G_{ii}^T Q^{-1} + Q^{-1} G_{ii}) x(t) \\
&\quad + \sum_{1 \leqslant i < j \leqslant r} 2\alpha_i(p(t))\alpha_j(p(t)) x(t)^T (G_{ij}^T Q^{-1} + Q^{-1} G_{ij}) x(t) \\
&= \sum_{i=1}^{r} \alpha_i^2(p(t)) x(t)^T Q^{-1} R_{ii} Q^{-1} x(t) \\
&\quad + \sum_{1 \leqslant i < j \leqslant r} 2\alpha_i(p(t))\alpha_j(p(t)) x(t)^T Q^{-1} R_{ij} Q^{-1} x(t) \\
&= x(t)^T Q^{-1} \begin{bmatrix} \alpha_1(p(t))I \\ \alpha_2(p(t))I \\ \cdots \\ \alpha_r(p(t))I \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1r} \\ R_{12} & R_{22} & \cdots & R_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1r} & R_{2r} & \dots & R_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1(p(t))I \\ \alpha_2(p(t))I \\ \cdots \\ \alpha_r(p(t))I \end{bmatrix} Q^{-1} x(t).
\end{aligned}$$

For the sake of brevity, we denote :

$$\alpha_i := \alpha_i(p(t)) \ (1 \leqslant i \leqslant r)$$

$$(R_{ij})_{r\times r} := \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1r} \\ R_{12} & R_{22} & \cdots & R_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1r} & R_{2r} & \dots & R_{rr} \end{bmatrix}$$

$$(\tau_{ij} I)_{r\times r} := \begin{bmatrix} 0 & \tau_{12} I & \cdots & \tau_{1r} I \\ \tau_{12} I & 0 & \cdots & \tau_{2r} I \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{1r} I & \tau_{2r} I & \dots & 0 \end{bmatrix}.$$

Since:

$$(Q^{-1}x(t))^T [\ \alpha_1 I \quad \alpha_2 I \quad \dots \quad \alpha_r I\ ] (\tau_{ij} I)_{r\times r} [\ \alpha_1 I \quad \alpha_2 I \quad \dots \quad \alpha_r I\ ]^T (Q^{-1}x(t))$$

$= (Q^{-1}x(t))^T(\sum_{1\leqslant i<j\leqslant r} 2\alpha_i\alpha_j\tau_{ij}I)(Q^{-1}x(t))$
$= (\sum_{1\leqslant i<j\leqslant r} 2\alpha_i\alpha_j\tau_{ij})|Q^{-1}x(t)|^2 \geqslant 0,$
then it follows:
$\dot{V}(x(t))$
$= (Q^{-1}x(t))^T[\ \alpha_1 I \quad \alpha_2 I \quad \ldots \quad \alpha_r I\ ](R_{ij})_{r\times r}[\ \alpha_1 I \quad \alpha_2 I \quad \ldots \quad \alpha_r I\ ]^T(Q^{-1}x(t))$
$\leqslant (Q^{-1}x(t))^T[\ \alpha_1 I \quad \alpha_2 I \quad \ldots \quad \alpha_r I\ ](R_{ij}+\tau_{ij}I)_{r\times r}[\ \alpha_1 I \quad \alpha_2 I \quad \ldots \quad \alpha_r I\ ]^T(Q^{-1}x(t))$
$\leqslant \lambda_{\max}((R_{ij}+\tau_{ij}I)_{r\times r})(\alpha_1^2+\alpha_2^2+...+\alpha_r^2)|Q^{-1}x(t)|^2$
$\leqslant \lambda_{\max}((R_{ij}+\tau_{ij}I)_{r\times r})\lambda_{\min}^2(Q)(\alpha_1^2+\alpha_2^2+...+\alpha_r^2)|x(t)|^2.$
From condition (7.10) it yields $\lambda_{\max}((R_{ij}+\tau_{ij}I)_{r\times r}) < 0$. Then, we have the result by Lemma (5.2). ■

From the proofs of Theorem 7.1, it is easy to see:

1). All the blocks (i, j) in (7.10) can be set to zero if $\alpha_i(p(t))\alpha_j(p(t)) \equiv 0$.

2). If all $C_i = I$ (i.e. the case of state feedback control), the BMI constraint (7.10) degenerates to a standard LMI with respect to parameters τ_{ij}, Q and N_i, where $N_i = K_iQ$. In this case, K_i can be directly solved by the LMI tools.

3). The BMI constraint (7.10) doesn't require $R_{ij} = QG_{ij}^T + G_{ij}Q < 0$ for the pairs $(i,\ j)$ such that $\alpha_i(p(t))\alpha_j(p(t)) \neq 0$, whereas this is necessary in the stability conditions described by (7.8).

4). It is easy to see, $(R_{ij})_{r\times r} < 0$ implies that there exist $\tau_{ij} > 0$ such that $(R_{ij} + \tau_{ij}I)_{r\times r} < 0$, but the inverse of this statement doesn't hold since $(\tau_{ij}I)_{r\times r}$ is a non-positive symmetric definite matrix. That is, by introducing the additional parameters τ_{ij}, the chances of finding the feasible solutions of (7.8) will be increased.

5). Theorem 7.1 also holds for the discrete T-S fuzzy model described by (7.5), if R_{ij} is replaced by $G_{ij}^TQG_{ij} - Q$. In this case, the constraint (7.10) is no longer a bilinear matrix inequality.

7.2 State Feedback Controller Design

In this section, all the matrices C_i $(i = 1, 2, ..., r)$ in (7.4) and (7.7) are restricted to the unit matrix. That is, we discuss the BMI-based state feedback fuzzy controller designs for the stabilization of T-S fuzzy models. The main results in this section are based on the eigenvalue constraints presented in Chapter 4.

Lemma 7.1 *Suppose* $\lambda_{ii}, \lambda_{jj}, \lambda_{ij} \in \Re$ *and* $\lambda_{ii} < 0, \lambda_{jj} < 0$. *Then:*

$$\lambda_{ij} < \sqrt{\lambda_{ii}\lambda_{jj}} \Leftrightarrow \exists \tau < 0,\ s.t.\ \left\{ \begin{array}{l} \lambda_{ij} < \tau\lambda_{ii} \\ \lambda_{jj} < \tau\lambda_{ij} \end{array} \right. .$$

Proof. If $\lambda_{ij} > 0$, then:

$$\begin{aligned}
&\lambda_{ij} < \sqrt{\lambda_{ii}\lambda_{jj}} \\
&\Leftrightarrow \lambda_{ij}^2 < \lambda_{ii}\lambda_{jj} \\
&\Leftrightarrow \frac{\lambda_{ij}}{\lambda_{ii}} > \frac{\lambda_{jj}}{\lambda_{ij}} \\
&\Leftrightarrow \exists\tau < 0,\ s.t.\ \frac{\lambda_{ij}}{\lambda_{ii}} > \tau > \frac{\lambda_{jj}}{\lambda_{ij}} \\
&\Leftrightarrow \exists\tau < 0,\ s.t.\ \left\{ \begin{array}{l} \lambda_{ij} < \tau\lambda_{ii} \\ \lambda_{jj} < \tau\lambda_{ij} \end{array} \right. .
\end{aligned}$$

If $\lambda_{ij} \leqslant 0$, it is obvious. ■

Lemma 7.2 *The discrete T-S fuzzy system described by (7.5) is globally asymptotically stabilized via the state feedback fuzzy controller (7.7), if there exists a matrix* $Q > 0$, *and matrices* $K_i \in \Re^{m\times n}$ $(1 \leqslant i \leqslant r)$, *such that:*

$$\left\{ \begin{array}{l} \lambda_{ii} < 0 \quad (i = 1, 2, ..., r) \\ (r-1)\lambda_{ij} < \sqrt{\lambda_{ii}\lambda_{jj}} \quad (1 \leqslant i < j \leqslant r) \end{array} \right.$$

where λ_{ij} *is the maximal eigenvalue of* $QG_{ij}^T Q^{-1} G_{ij} Q - Q$ *for all* $1 \leqslant i \leqslant j \leqslant r$.

Proof. Substitute P for Q^{-1} and $x(k)$ for $Qz(k)$ in the proofs of Theorem 4.3, then we have the result. ■

Lemma 7.3 *The continuous T-S fuzzy system described by (7.2) is globally asymptotically stabilized via the state feedback fuzzy controller (7.4), if there exists a matrix* $Q > 0$, *and matrices* $K_i \in \Re^{m\times n}$ $(1 \leqslant i \leqslant r)$, *such that:*

$$\left\{ \begin{array}{l} \lambda_{ii} < 0 \quad (i = 1, 2, ..., r) \\ (r-1)\lambda_{ij} < \sqrt{\lambda_{ii}\lambda_{jj}} \quad (1 \leqslant i < j \leqslant r) \end{array} \right.$$

where λ_{ij} *is the maximal eigenvalue of* $QG_{ij}^T + G_{ij}Q$ *for all* $1 \leqslant i \leqslant j \leqslant r$.

Proof. It is similar to the proofs of Lemma 7.2. ■

Theorem 7.2 *The discrete T-S fuzzy system described by (7.5) is globally asymptotically stabilized via the state feedback fuzzy controller (7.7), if there exists a matrix* $Q > 0$, τ_{ij}, $\delta_{ij}^{(1)}$, $\delta_{ij}^{(2)} \in \Re$, $\tau_{ij} < 0$, *and matrices* $M_i \in \Re^{m\times n}$ $(1 \leqslant i \leqslant r)$, *such that:*

$$\left[\begin{array}{cc} -Q & QG_{ii}^T \\ G_{ii}Q & -Q \end{array} \right] < 0 \quad (1 \leqslant i \leqslant r) \tag{7.11}$$

$$\left[\begin{array}{cc} -Q - \frac{1}{r-1}\delta_{ij}^{(1)} I & QG_{ij}^T \\ G_{ij}Q & -Q \end{array} \right] < 0 \quad (1 \leqslant i < j \leqslant r) \tag{7.12}$$

$$\left[\begin{array}{cc} -Q - \tau_{ij}\delta_{ij}^{(1)} I & QG_{ii}^T \\ G_{ii}Q & -Q \end{array} \right] < 0 \quad (1 \leqslant i < j \leqslant r) \tag{7.13}$$

$$\left[\begin{array}{cc} -Q - \delta_{ij}^{(2)} I & QG_{jj}^T \\ G_{jj}Q & -Q \end{array} \right] < 0 \quad (1 \leqslant i < j \leqslant r) \tag{7.14}$$

$$\begin{bmatrix} -Q - \frac{1}{r-1}\tau_{ij}\delta_{ij}^{(2)}I & QG_{ij}^T \\ G_{ij}Q & -Q \end{bmatrix} < 0 \quad (1 \leqslant i < j \leqslant r) \tag{7.15}$$

where $G_{ij}Q = \frac{1}{2}(A_iQ + A_jQ + B_iM_j + B_jM_i)$ *for all* $1 \leqslant i \leqslant j \leqslant r$. *Then, the state feedback gains can be calculated by* $K_i = M_iQ^{-1}$.

Proof. By Schur complement, the BMI constraints (7.11)-(7.15) are equivalent to the following matrix inequalities (7.16)-(7.20) respectively:

$$QG_{ii}^TQ^{-1}G_{ii}Q - Q < 0 \quad (1 \leqslant i \leqslant r) \tag{7.16}$$

$$QG_{ij}^TQ^{-1}G_{ij}Q - Q < \frac{1}{r-1}\delta_{ij}^{(1)}I \quad (1 \leqslant i < j \leqslant r) \tag{7.17}$$

$$QG_{ii}^TQ^{-1}G_{ii}Q - Q < \tau_{ij}\delta_{ij}^{(1)}I \quad (1 \leqslant i < j \leqslant r) \tag{7.18}$$

$$QG_{jj}^TQ^{-1}G_{jj}Q - Q < \delta_{ij}^{(2)}I \quad (1 \leqslant i < j \leqslant r) \tag{7.19}$$

$$QG_{ij}^TQ^{-1}G_{ij}Q - Q < \frac{1}{r-1}\tau_{ij}\delta_{ij}^{(2)}I \quad (1 \leqslant i < j \leqslant r). \tag{7.20}$$

Then, it yields from (7.16)-(7.20) respectively:

$$\begin{aligned} \lambda_{ii} &< 0 \quad (i = 1, 2, ..., r), \\ \lambda_{ij} &< \frac{1}{r-1}\delta_{ij}^{(1)} \quad (1 \leqslant i < j \leqslant r), \\ \delta_{ij}^{(1)} &< \frac{1}{\tau_{ij}}\lambda_{ii} \quad (1 \leqslant i < j \leqslant r), \\ \lambda_{jj} &< \delta_{ij}^{(2)} \quad (1 \leqslant i < j \leqslant r), \\ \delta_{ij}^{(2)} &< \frac{r-1}{\tau_{ij}}\lambda_{ij} \quad (1 \leqslant i < j \leqslant r), \end{aligned}$$

where λ_{ij} is the maximal eigenvalue of $QG_{ij}^TQ^{-1}G_{ij}Q - Q$ for all $1 \leqslant i \leqslant j \leqslant r$.

Thereby, we have:

$$\begin{aligned} &\lambda_{ii} < 0 \quad (i = 1, 2, ..., r) \\ &(r-1)\lambda_{ij} < \tfrac{1}{\tau_{ij}}\lambda_{ii} \quad (1 \leqslant i < j \leqslant r) \\ &\lambda_{jj} < \tfrac{1}{\tau_{ij}}(r-1)\lambda_{ij} \quad (1 \leqslant i < j \leqslant r). \end{aligned}$$

By applying Lemma 7.1 and Lemma 7.2, it gives the result. ∎

Theorem 7.3 *The continuous T-S fuzzy system described by (7.2) is globally asymptotically stabilized via the state feedback fuzzy controller (7.4), if there exists a* $Q > 0$, τ_{ij}, $\delta_{ij}^{(1)}$, $\delta_{ij}^{(2)} \in \Re$, $\tau_{ij} < 0$, *and matrices* $M_i \in \Re^{m\times n}$ $(1 \leqslant i \leqslant r)$, *such that:*

$$QG_{ii}^T + G_{ii}Q < 0 \quad (1 \leqslant i \leqslant r) \tag{7.21}$$

$$(r-1)(QG_{ij}^T + G_{ij}Q) < \delta_{ij}^{(1)} I < \tau_{ij}(QG_{ii}^T + G_{ii}Q) \quad (1 \leqslant i < j \leqslant r) \tag{7.22}$$

$$(QG_{jj}^T + G_{jj}Q) < \delta_{ij}^{(2)} I < (r-1)\tau_{ij}(QG_{ij}^T + G_{ij}Q) \quad (1 \leqslant i < j \leqslant r) \tag{7.23}$$

where $G_{ij}Q = \frac{1}{2}(A_iQ + A_jQ + B_iM_j + B_jM_i)$ *for all* $1 \leqslant i \leqslant j \leqslant r$*. Then, the state feedback gains can be calculated by* $K_i = M_iQ^{-1}$*.*

Proof. The BMI constraints (7.21)-(7.23) imply respectively:

$$\begin{aligned}
&\lambda_{ii} < 0 \quad (1 \leqslant i \leqslant r) \\
&(r-1)\lambda_{ij} < \tau_{ij}\lambda_{ii} \quad (1 \leqslant i < j \leqslant r) \\
&\lambda_{jj} < (r-1)\tau_{ij}\lambda_{ij} \quad (1 \leqslant i < j \leqslant r).
\end{aligned}$$

where λ_{ij} is the maximal eigenvalue of $QG_{ij}^T + G_{ij}Q$ for all $1 \leqslant i \leqslant j \leqslant r$.

Then we have the result by Lemma 7.1 and Lemma 7.3. ■

Obviously, the BMI constraints (7.8) and (7.9) are special cases of the BMI constraints of Theorem 7.3 and Theorem 7.2 respectively (e.g. $\delta_{ij}^{(1)} = \delta_{ij}^{(2)} = 0$ and $\tau_{ij} = -1$). That is, the above results are less restrictive than the related LMI-based results in the literature e.g. [60], [81].

Remark 7.1 *The conditions given in Theorem 7.2 and Theorem 7.3 are standard BMI constraints with respect to the parameters* $Q, M_i, \delta_{ij}^{(1)}, \delta_{ij}^{(2)}$ *and* τ_{ij}*. They can be solved by the solution procedures presented in Section 6.2 directly. Moreover, for the pairs* (i, j) *with* $\alpha_i(p(k))\alpha_j(p(k)) \equiv 0$ *(or* $\alpha_i(p(t))\alpha_j(p(t)) \equiv 0$*), the related BMI constraints in Theorem 7.2 (or Theorem 7.3) don't have to be satisfied. Thereby, these BMIs needn't be solved in executing the BMI solution procedures.*

7.3 Observer-based Controller Design

Suppose that the plant rules are described by (7.1). An observer-based fuzzy controller is to be designed. Based on the PDC technique, the regulator rules and controller rules can be expressed as follows respectively:

Regulator rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$\begin{cases} \dot{\hat{x}}(t) = A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t)) \\ \hat{y}_i(t) = C_i\hat{x}(t) \end{cases} \quad (i = 1, 2, ..., r),$$

Controller rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$u(t) = K_i\hat{x}(t) \quad (i = 1, 2, ..., r).$$

where L_i and K_i $(i = 1, 2, ..., r)$ are the observer gains and controller gains to be designed.

Then, the overall regulator output and controller output can be inferred:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} \alpha_i(p(t))[A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))] \tag{7.24}$$

$$\hat{y}(t) = \sum_{i=1}^{r} \alpha_i(p(t))\hat{y}_i(t) = \sum_{i=1}^{r} \alpha_i(p(t))C_i\hat{x}(t) \tag{7.25}$$

$$u(t) = \sum_{i=1}^{r} \alpha_i(p(t))K_i\hat{x}(t) \tag{7.26}$$

Combining (7.24)-(7.26) and (7.1) we have:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(p(t))\alpha_j(p(t))[A_i x(t) + B_i K_j \hat{x}(t)] \\ &= \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(p(t))\alpha_j(p(t))[(A_i + B_i K_j)x(t) - B_i K_j e(t)], \\ \dot{e}(t) &= \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(p(t))\alpha_j(p(t))[A_i - L_i C_j]e(t),\end{aligned}$$

where $e(t) = x(t) - \hat{x}(t)$. Then, it follows:

$$\dot{\tilde{x}}(t) = \sum_{i,j} \alpha_i(p(t))\alpha_j(p(t))\widetilde{H}_{ij}\tilde{x}(t), \tag{7.27}$$

where $\tilde{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ and $\widetilde{H}_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_i - L_i C_j \end{bmatrix}$ $(1 \leqslant i, j \leqslant r)$. It is easy to show that (7.28) is asymptotically stable, if there exists a matrix $P > 0$ such that:

$$\widetilde{H}_{ij}^T P + P\widetilde{H}_{ij} < 0 \quad (1 \leqslant i,\ j \leqslant r). \tag{7.28}$$

In [23] it has been proved that the existence of a matrix $P > 0$ is equivalent to the existence of a diagonal block matrix $P > 0$ in (7.28). the variable matrix P can be chosen as a diagonal block matrix . Then, the constraints in (7.28) can be formulated into LMIs equivalently by restricting P to a diagonal block matrix ([23], [9]). Based on the results of Theorem 7.1, and 7.3 we can give the further relaxed stability constraints in terms of BMIs.

Note that:

$$\begin{aligned}\widetilde{H}_{ij} &= \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_i - L_i C_j \end{bmatrix} \\ &= \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} K_j [\ I \quad -I\ ] - \begin{bmatrix} 0 \\ I \end{bmatrix} L_i [\ 0 \quad C_j\ ] \\ &=: \widetilde{A}_i + \widetilde{B}_i K_j [\ I \quad -I\ ] - \begin{bmatrix} 0 \\ I \end{bmatrix} L_i \widetilde{C}_j\end{aligned}$$

where I and 0 are unit matrix and zero matrix with appropriate dimensions and

$$\widetilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix}, \widetilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \widetilde{C}_j = [\ 0 \quad C_j\ ].$$

Applying the similar proof procedures of Theorem 7.1 to (7.27) we have:

Corollary 7.1 *The fuzzy system described by (7.1) is globally exponentially stabilized via the observer-based fuzzy controller (7.26), if there exists a positive definite matrix* P*, matrices* K_i, L_i $(i = 1, 2, ..., r)$ *and scalars* $\tau_{ij} > 0$ $(1 \leqslant i < j \leqslant r)$ *such that:*

$$\begin{bmatrix} \widetilde{R}_{11} & \widetilde{R}_{12} + \tau_{12} I & \cdots & \widetilde{R}_{1r} + \tau_{1r} I \\ * & \widetilde{R}_{22} & \cdots & \widetilde{R}_{2r} + \tau_{2r} I \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & \widetilde{R}_{rr} \end{bmatrix} < 0$$

where $*$ *stands for the transposed element in the symmetric position,* $\widetilde{R}_{ii} = \widetilde{H}_{ii}^T P + P \widetilde{H}_{ii}$ $(i = 1, 2, ..., r)$*,* $\widetilde{R}_{ij} = P(\widetilde{H}_{ij} + \widetilde{H}_{ji})$ $(1 \leqslant i < j \leqslant r)$*.*

Proof. It is similar to the proofs of Theorem 7.1. ■

Corollary 7.2 *The fuzzy system described by (7.1) is globally asymptotically stabilized via the observer-based fuzzy controller (7.26), if there exists a positive definite matrix* Q*,* τ_{ij}*,* $\delta_{ij}^{(1)}$*,* $\delta_{ij}^{(2)} \in \Re$*,* $\tau_{ij} < 0$*, and matrices* K_i*,* L_i $(1 \leqslant i \leqslant r)$*, such that:*

$$Q\widetilde{H}_{ii}^T + \widetilde{H}_{ii} Q < 0 \quad (1 \leqslant i \leqslant r)$$

$$(r-1)(Q\widetilde{G}_{ij}^T + \widetilde{G}_{ii} Q) < \delta_{ij}^{(1)} I \quad (1 \leqslant i < j \leqslant r)$$

$$Q\widetilde{G}_{ii}^T + \widetilde{G}_{ii} Q < \tau_{ij} \delta_{ij}^{(1)} I \quad (1 \leqslant i < j \leqslant r)$$

$$Q\widetilde{G}_{jj}^T + \widetilde{G}_{jj} Q < \delta_{ij}^{(2)} I \quad (1 \leqslant i < j \leqslant r)$$

$$(r-1)(Q\widetilde{G}_{ij}^T + \widetilde{G}_{ij} Q) < \tau_{ij} \delta_{ij}^{(2)} I \quad (1 \leqslant i < j \leqslant r),$$

where $\widetilde{G}_{ij} = \frac{1}{2}(\widetilde{H}_{ij} + \widetilde{H}_{ji})$ *for all* $1 \leqslant i \leqslant j \leqslant r$*.*

Proof. It is similar to the proofs of Theorem 7.3. ■

It is easy to see that the similar results of (7.1) and (7.2) also hold for the discrete T-S fuzzy models.

7.4 Simulation

Consider the design problem of the chaotic Lorenz system [48]:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -10x_1(t) + 10x_2(t) \\ 28x_1(t) - x_2(t) - x_1(t)x_3(t) \\ x_1(t)x_2(t) - \frac{8}{3}x_3(t) \end{bmatrix}. \tag{7.29}$$

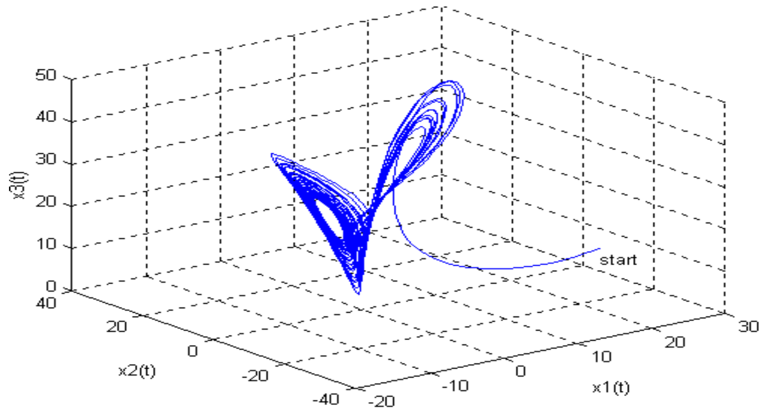

Figure 7.1: Trajectory of the chaotic Lorenz system

The trajectory of (7.29) is shown in Figure 7.1. Our objective is to design the BMI-based fuzzy controllers to stabilize the trajectory of (7.29). Since

$$x_1(t)x_2(t) = (M_1 g_1(x) + M_2 g_2(x))x_2(t),$$

where

$$g_1(x) = \frac{-x_1(t)+M_2}{M_2-M_1},\ g_2(x) = \frac{x_1(t)-M_1}{M_2-M_1},$$

then (7.29) can be rewritten as:

$$\dot{x}(t) = \sum_{i=1}^{2} g_i(x) A_i x(t), \tag{7.30}$$

where

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -M_1 \\ 0 & M_1 & -\frac{8}{3} \end{bmatrix}, A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -M_2 \\ 0 & M_2 & -\frac{8}{3} \end{bmatrix}.$$

As shown in Figure 7.1, $x_1(t)$ is likely to be bounded in $[-20,\ 30]$. Thereby M_1 and M_2 can be set to -20 and 30 respectively. Let $\alpha_1(x)$ and $\alpha_2(x)$ be the membership functions of the fuzzy sets 'about M_1' and 'about M_2' as shown in Figure 7.2.

Then (7.30) can be expressed by the following fuzzy rules:

If $x_1(t)$ is about M_1, then $\dot{x}(t) = A_1 x(t)$

If $x_1(t)$ is about M_2, then $\dot{x}(t) = A_2 x(t)$.

(I) State feedback fuzzy controller design

Suppose that the state feedback fuzzy controller are described by (7.4), where B_1, B_2, C_1, C_2 are given as in [48], i.e. $B_1 = B_2 = [\ 1\ \ 0\ \ 0\ ]^T$, $C_1 = C_2 = I$. Calculation shows that they are all feasible, when we apply the BMI solution procedures to the LMI constraints in (7.8), the BMI constraints in Theorem 7.1 and Theorem 7.3. For example, by executing the BMI solution procedures the following feasible solutions of

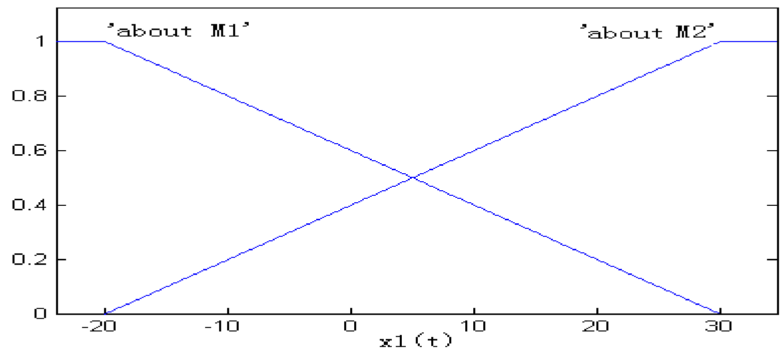

Figure 7.2: Membership functions: $\alpha_1(x)$ and $\alpha_2(x)$

the BMI constraints in Theorem 7.3 are obtained:

$$Q = 10^4 \begin{bmatrix} 9.5199 & -0.0644 & -0.0006 \\ -0.0006 & 1.6775 & 0.0067 \\ -0.0006 & 0.0067 & 1.6699 \end{bmatrix},$$

$$K_1 = [\ 8.5526 \quad -168.9841 \quad -0.1388\ ],$$

$$K_2 = [\ 8.5543 \quad -169.0094 \quad 1.7908\ ].$$

The controlled trajectory of the chaotic Lorenz system is shown in Figure 7.3, where the initial condition is given by $x(0) = [10,\ 20,\ -10]^T$.

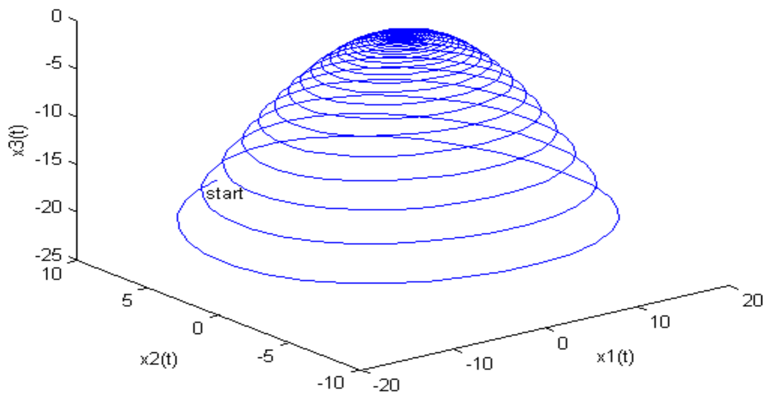

Figure 7.3: Simulation of state feedback control based on Theorem 7.3

For the sake of comparison, we set now $B_1 = [1, 0, 0]^T$, $B_2 = [-1, 0, 1]^T$ and $C_1 = C_2 = I$. Then, we can solve the usual LMI constraints in (7.8), the BMI constraints in Theorem 7.1 and Theorem 7.3 again. Calculation shows that the constraints of (7.8) and Theorem 7.3 are infeasible in this case. But the BMIs in Theorem 7.1 are feasible, and the obtained feasible solutions are as follows:

$$Q = 10^{-3}\begin{bmatrix} 0.2926 & -0.1432 & 0.0587 \\ -0.1432 & 0.1042 & 0.0136 \\ 0.0587 & 0.0136 & 0.1401 \end{bmatrix},$$

$$K_1 = 10^3[\ -1.1277 \quad -1.8100 \quad -0.6441\ ],$$

$$K_2 = [\ 400.2084 \quad 527.8397 \quad -284.7570\ ].$$

The controlled trajectories via state feedback fuzzy controller (7.4) are shown in Figure 7.4, where the initial condition is given by $x(0) = [25,\ -15,\ 10]^T$.

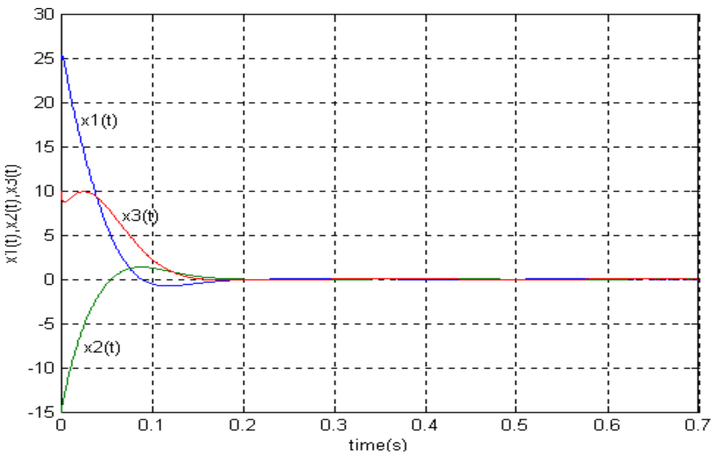

Figure 7.4: Simulation of state feedback control based on Theorem 7.1

(II) Output feedback fuzzy controller design

Suppose that B_1, B_2, C_1, and C_2 are given as follows:

$$.\ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, C_1 = C_2^T = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

where both C_1 and C_2 are singular matrices. The output feedback fuzzy controller to be designed are expressed by (7.4). By applying the BMI solution procedures to the BMI constraints (7.10) the following feasible solutions are obtained:

$$K_1 = [\ -8.7416 \quad -5.5883 \quad -8.9783\ ],$$

$$K_2 = [\ -25.4751 \quad 26.5558 \quad 14.0102\ ],$$

$$Q = 10^5\begin{bmatrix} 1.4628 & -1.1877 & 0.3581 \\ -1.1877 & 1.8522 & 0.4836 \\ 0.3581 & 0.4836 & 1.6630 \end{bmatrix}.$$

Thereby, the conditions of Theorem 7.1 are satisfied. The controlled trajectories via the output feedback fuzzy controller (7.4) are illustrated in Figure 7.5, where the initial condition is given by $x(0) = [25,\ -15,\ 10]^T$.

Similarly, we can design the observer-based fuzzy controller for the stabilization of the chaotic Lorenz system.

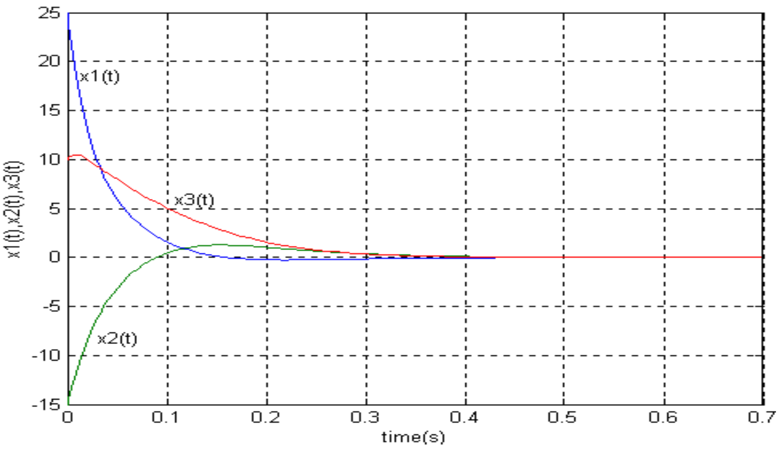

Figure 7.5: Simulation of output feedback control based on Theorem 7.1

Chapter 8

Stabilization of Time-Delay T-S Fuzzy Models

In this chapter, the stabilization of nonlinear time-delay systems is discussed in terms of T-S fuzzy models. We present first a stability result independent of the delays by the improved Razumikhin theorem. Then, we give the delay-dependent stability conditions via the Lyapunov functional method. Based on the presented results, the state feedback gains can be solved via the LMI tools directly. The presented results are finally illustrated by a simulation example of truck-trailer.

8.1 Introduction to the Time-Delay Systems

Razumikhin type theorems and the Lyapunov functional method are the main approaches to deal with the stabilities of the time delay systems. First, we introduce some basic results on the stability of retarded functional differential equations.

The general time delay systems are described by the following retarded functional differential equations ([31], [68]) :

$$\dot{x}(t) = f(t, x_t) \tag{8.1}$$

where $t \in J = [\delta,\ \infty)$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,\ 0]$, and f is a given function from $J \times C([-\tau,\ 0],\ \Re^n)$ to $\Re^n$, where $C([-\tau,\ 0],\ \Re^n)$ stands for the Banach space of continuous functions mapping the interval $[-\tau,\ 0]$ into $\Re^n$. The norm in $C([-\tau,\ 0],\ \Re^n)$ is defined by:

$$\|\phi\| = \sup_{\theta \in [-\tau,\ 0]} |\phi(\theta)|_p \qquad (\phi \in C([-\tau,\ 0],\ \Re^n)).$$

where $|\cdot|_p$ stands for any kind of p-norms, such as the ordinary $1, 2$ and ∞ norms.

It is assumed that $f(t, 0) = 0$ for all $t \in J$, i.e. $x(t) = 0$ $(\forall t \in J)$ is a trivial solution of (8.1) (if it is not the case, by setting $z(t) = x(t) - y(t)$, where $y(t)$ is a solution of

(8.1) for the given initial condition, then $\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t)$ has $z(t) = 0$ as a trivial solution). In addition, for any given $t_0 \in J$ and any initial condition $\phi \in C([-\tau,\ 0],\ \Re^n)$, the retarded functional differential equation (8.1) is assumed to have a unique solution $x(t_0, \phi)$ which can be described by:

$$\begin{cases} x_{t_0}(\theta) = \phi(\theta) & \theta \in [-\tau,\ 0] \\ x(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s)ds & t \geqslant t_0 \end{cases}.$$

Retarded functional differential equations can be viewed as an extension of the ordinary differential equations ($\tau = 0$). With slight modification the ordinary concepts of stability can be extended to the retarded functional differential equations.

Definition 8.1 *(Asymptotic Stability, [86]) The trivial solution of (8.1) is called stable, if for any $t_0 \in J$, $\varepsilon > 0$, there exists a $\sigma(\varepsilon, t_0) > 0$ such that $\|\phi\| < \sigma(\varepsilon, t_0)$ implies $\|x_t(t_0, \phi)\| < \varepsilon$ for all $t \geqslant t_0$.*

In addition, if $\sigma(\varepsilon, t_0)$ is independent of t_0, then the trivial solution is called uniformly stable.

If the trivial solution is stable, and for any $t_0 \in J$, there is a $\eta(t_0) > 0$ such that $\|\phi\| < \eta(t_0)$ implies $x(t_0, \phi) \to 0$ as $t \to \infty$, then the trivial solution is called asymptotically stable.

The following results will be used in the proofs of our main results.

Lemma 8.1 *(Razumikhin Theorem, [31]) Suppose $u, v, w : \Re_+ \to \Re_+$ are strictly monotonically increasing continuous functions with $u(0) = v(0) = 0$ and $w(0) \geqslant 0$. If there is a continuous function $V : J \times \Re^n \to \Re_+$ such that:*

(i) $u(|x|) \leqslant V(t, x) \leqslant v(|x|) \qquad t \in J, x \in \Re^n$,

(ii) there is a continuous non-decreasing function $p(s) > s$ for $s > 0$ and for any $t_0 \in J$, $\dot{V}(t, x) \leqslant -w(|x|)$, if $V(t+\theta, x(t+\theta)) < p(V(t, x))$ for $\theta \in [-\tau,\ 0]$ and $t \geqslant t_0$, then the trivial solution of (8.1) is uniformly asymptotically stable. If additionally $\lim\limits_{s\to\infty} u(s) = \infty$, then the trivial solution is uniformly asymptotically stable in the large.

Lemma 8.2 *(Improved Razumikhin Theorem, [86]) Suppose $u, v, w : \Re_+ \to \Re_+$ are strictly monotonically increasing continuous functions with $u(0) = v(0) = 0$ and $w(0) \geqslant 0$. If there is a continuous function $V : J \times \Re^n \to \Re_+$ such that:*

(I) $u(|x|) \leqslant V(t, x) \leqslant v(|x|) \qquad t \in J, x \in \Re^n$

(II) there is a positive $q > 1$ and for any $t_0 \in J$, $\dot{V}(t, x) \leqslant -w(|x|)$, if $|x(t+\theta)| < q\,|x|$ for $\theta \in [-\tau,\ 0]$ and $t \geqslant t_0$,
then the trivial solution of (8.1) is uniformly asymptotically stable. If additionally $\lim\limits_{s\to\infty} u(s) = \infty$, then the trivial solution is uniformly asymptotically stable in the large.

Lemma 8.3 *(Lyapunov-Krasovskii Theorem,[29]) Suppose that f in (8.1) maps $\Re \times$ (bounded sets in $C([-\tau,\ 0],\ \Re^n)$) into bounded sets in $\Re^n$, and $u, v, w : \Re_+ \to \Re_+$ are continuous non-decreasing functions satisfying $u(0) = v(0) = 0$, $u(s), v(s), w(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = \infty$. If there exists a continuous differentiable functional $V : \Re \times C([-\tau,\ 0],\ \Re^n) \to \Re$, such that*

$$u(|\phi(0)| \leqslant V(t,\phi) \leqslant v(\|\phi\|),$$
$$\dot{V}(t,\phi) \leqslant -w(\|\phi(0)\|),$$

then the trivial solution of (8.1) is globally uniformly asymptotically stable.

More stability results and detailed descriptions on the retarded functional differential equations can be found e.g. in [29], [31], [68].

8.2 Delay-independent Stability Conditions

There have been a lot of studies (e.g. [9], [16], [49], [35]) on the stability of the retarded nonlinear control systems by means of T-S fuzzy model approaches. In this section, some new stability conditions independent of delays will be deduced via the (improved) Razumikhin theorems. The time delay control systems under discussion are described by the following fuzzy rules as in [9]:

Plant rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then:

$$\begin{cases} \dot{x}(t) = A_i x(t) + A_{di} x(t - d_i(t)) + B_i u(t) \\ x(t) = \phi(t) \quad t \in [-\tau,\ 0],\ d_i(t) \in [0,\ \tau] \end{cases} \quad (i = 1, 2, ..., r).$$

We employ the PDC based design for the stabilization of the above model. The controller rules can be expressed by:

Controller rules: If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$u(t) = K_i x(t) \quad (i = 1, 2, ..., r).$$

where K_i $(i = 1, 2, ..., r)$ are the feedback gains to be designed. Similarly as discussed in Section 4.1, the overall outputs can be inferred:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(p(t))[A_i x(t) + A_{di} x(t - d_i(t)) + B_i u(t)], \tag{8.2}$$

$$u(t) = \sum_{i=1}^{r} \alpha_i(p(t)) K_i x(t). \tag{8.3}$$

Then the closed loop time delay systems can be formulated as:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} \sum\limits_{j=1}^{r} \alpha_i(p(t)) \alpha_j(p(t)) [(A_i + B_i K_j) x(t) + A_{di} x(t - d_i(t))] \\ x(t) = \phi(t) \quad t \in [-\tau,\ 0], d_i(t) \in [0,\ \tau] \end{cases}. \tag{8.4}$$

Before presenting the main results, we give first a required conclusion.

Lemma 8.4 *For any given matrices $A < 0$ and $B^T = B$, there exists a scalar $\varepsilon > 0$ such that $A + \varepsilon B < 0$.*

Proof. If $\lambda_{\max}(B) > 0$, then $A+\varepsilon B < 0$ holds for all $0 < \varepsilon < -\lambda_{\max}(A)/\lambda_{\max}(B)$. On the other hand, if $\lambda_{\max}(B) \leqslant 0$, then $A + \varepsilon B < 0$ holds for all $\varepsilon > 0$. ■

Theorem 8.1 *If there exists a $P > 0$, scalars $\tau_{ij} > 0$, and matrices M_i such that*

$$\begin{bmatrix} R_{11} & R_{12}+\tau_{12}I & \cdots & R_{1r}+\tau_{1r}I \\ * & R_{22} & \cdots & R_{2r}+\tau_{2r}I \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & R_{rr} \end{bmatrix} < 0 \tag{8.5}$$

where $*$ stands for the transposed element in the symmetric position,

$R_{ij} = G_{ij}P + PG_{ij}^T + \frac{1}{2}(A_{di}PA_{di}^T + A_{dj}PA_{dj}^T) + P$ $(1 \leqslant i \leqslant j \leqslant r)$,

$G_{ij} = \frac{1}{2}(A_i + A_j + B_iM_jP^{-1} + B_jM_iP^{-1})$ $(1 \leqslant i \leqslant j \leqslant r)$,

then the time delay system (8.2) is globally asymptotically stabilized via the fuzzy controller described by (8.3) and the feedback gains can be calculated by $K_i = M_iP^{-1}$ for all $i = 1, 2, ..., r$.

Proof. Choose the candidate Lyapunov function as $V(x(t)) = x(t)^TP^{-1}x(t)$.

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}(t)^TP^{-1}x(t) + x(t)^TP^{-1}\dot{x}(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))\{x(t)^T[(A_i+B_iK_j)^TP^{-1}+P^{-1}(A_i+B_iK_j)]x(t) \\ &\quad +x(t)^TP^{-1}A_{di}x(t-d_i(t)) + x^T(t-d_i(t))A_{di}^TP^{-1}x(t)\} \\ &\leqslant \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))\{x(t)^T[(A_i+B_iK_j)^TP^{-1}+P^{-1}(A_i+B_iK_j)]x(t) \\ &\quad +x(t)^TP^{-1}A_{di}PA_{di}^TP^{-1}x(t) + x^T(t-d_i(t))P^{-1}x(t-d_i(t))\}.\end{aligned}$$

Substituting $V(x(t-d_i(t)) \leqslant \gamma V(x(t))$ into the above inequality we have:

$$\begin{aligned}\dot{V}(x(t)) &\leqslant \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))x(t)^T[(A_i+B_iK_j)^TP^{-1} + P^{-1}(A_i+B_iK_j) \\ &\quad +P^{-1}A_{di}PA_{di}^TP^{-1} + \gamma P^{-1}]x(t) \\ &= \sum_{i=1}^{r}\alpha_i^2(p(t))(P^{-1}x(t))^T[R_{ii} + (\gamma-1)P](P^{-1}x(t)) \\ &\quad + \sum_{i<j}2\alpha_i(p(t))\alpha_j(p(t))(P^{-1}x(t))^T[R_{ij} + (\gamma-1)P](P^{-1}x(t)) \\ &= (P^{-1}x(t))^T\begin{bmatrix}\alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I\end{bmatrix}^T (\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1r} \\ * & R_{22} & \cdots & R_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & R_{rr}\end{bmatrix}\end{aligned}$$

$$+(\gamma-1)\begin{bmatrix} P & P & \cdots & P \\ * & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & P \end{bmatrix})\begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}(P^{-1}x(t)),$$

where $\alpha_i := \alpha_i(p(t))$. Since $\sum_{i<j} 2\alpha_i\alpha_j\tau_{ij}\left|P^{-1}x(t)\right|^2 \geqslant 0$ for all $x(t) \in \Re$ and $\tau_{ij} > 0$, then we have:

$$\dot{V}(x(t)) \leqslant (P^{-1}x(t))^T \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T (\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1r} \\ * & R_{22} & \cdots & R_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & R_{rr} \end{bmatrix}$$

$$+(\gamma-1)\begin{bmatrix} P & P & \cdots & P \\ * & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & P \end{bmatrix})\begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}(P^{-1}x(t))$$

$$+(P^{-1}x(t))^T \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} 0 & \tau_{12}I & \cdots & \tau_{1r}I \\ * & 0 & \cdots & \tau_{2r}I \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}(P^{-1}x(t))$$

$$= (P^{-1}x(t))^T \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T (\begin{bmatrix} R_{11} & R_{12}+\tau_{12}I & \cdots & R_{1r}+\tau_{1r}I \\ * & R_{22} & \cdots & R_{2r}+\tau_{2r}I \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & R_{rr} \end{bmatrix}$$

$$+(\gamma-1)\begin{bmatrix} P & P & \cdots & P \\ * & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & P \end{bmatrix})\begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}(P^{-1}x(t)).$$

If the condition (8.5) is satisfied, by Lemma 8.4 we have that there must exist a $\gamma_0 > 1$ and $w > 0$ such that $\dot{V}(x(t)) \leqslant -w\left|x(t)\right|^2$. Then the proof is completed by applying Lemma 8.1. ■

Corollary 8.1 *If there exists a matrix $P > 0$ and matrices M_i such that*

$$\begin{array}{l}\frac{1}{2}[(A_i+A_j)P + P(A_i+A_j)^T + (B_iM_j+B_jM_i)+ \\ \quad (B_iM_j+B_jM_i)^T + (A_{di}PA_{di}^T + A_{dj}PA_{dj}^T)] + P < 0\end{array} \quad (1 \leqslant i \leqslant j \leqslant r) \qquad (8.6)$$

then the time delay system (8.2) is globally asymptotically stabilized via the fuzzy controller described by (8.3). Then, the state feedback gains can be calculated by $K_i = M_iP^{-1}$ $(i = 1, 2, ..., r)$.

Proof. It follows from the proof of Theorem 8.1 directly. ■

It is easy to verify that Corollary 8.1 is equivalent to the Theorem 2 of [9], where the conditions are given by:

$$\begin{cases} S_i \geqslant P \quad (i = 1, 2, ..., r) \\ A_i P + P A_i{}^T + B_i M_i + M_i^T B_i^T + A_{di} S_i A_{di}^T + P < 0 \quad (i = 1, 2, ..., r) \\ \begin{array}{l} (A_i + A_j)P + P(A_i + A_j)^T + B_i M_j + B_j M_i + M_i^T B_j^T \\ \quad + M_j^T B_i^T + A_{di} S_i A_{di}^T + A_{dj} S_j A_{dj}^T + 2P < 0 \end{array} \quad (1 \leqslant i < j \leqslant r) \end{cases}. \tag{8.7}$$

Since $S_i \geqslant P$ implies $A_{di} S_i A_{di}^T \geqslant A_{di} P A_{di}^T$, then we have that the constraints in (8.6) hold if the conditions in (8.7) are satisfied. On the other hand, by Lemma 8.4 it follows that (8.6) also implies (8.7). The improvement of Corollary 8.1 is that the number of parameters and the number of LMIs are reduced compared with the result of [9]. Moreover, the constraints in (8.6) require $R_{ij} < 0$ for all $1 \leqslant i < j \leqslant r$ except the pairs (i, j) such that $\alpha_i(p(t))\alpha_j(p(t)) \equiv 0$, whereas this restriction is removed in Theorem 8.1 by introducing additional parameters.

By applying the improved Razumikhin Theorem, the following result can be obtained.

Corollary 8.2 *If there exists a matrix $P > 0$, scalars $\gamma_i > 0$ and matrices M_i such that*

$$\begin{bmatrix} \widetilde{A}_{ii} & P \\ P & -\gamma_i I \end{bmatrix} < 0 \quad (i = 1, 2, ..., r) \tag{8.8}$$

$$\begin{bmatrix} \widetilde{A}_{ij} + \widetilde{A}_{ji} & P & P \\ P & -\gamma_i I & 0 \\ P & 0 & -\gamma_j I \end{bmatrix} < 0 \quad (1 \leqslant i < j \leqslant r) \tag{8.9}$$

where $\widetilde{A}_{ij} = P A_i^T + A_i P + B_i M_j + M_j^T B_i^T + \gamma_i A_{di} A_{di}^T$ for all $1 \leqslant i \leqslant j \leqslant r$, then the time delay system (8.2) is globally asymptotically stabilized via the fuzzy controller described by (8.3). Then, the state feedback gains can be obtained by $K_i = M_i P^{-1}$ for all $i = 1, 2, ..., r$.

Proof. Choose the Lyapunov candidate function as $V(x(t)) = x(t)^T P^{-1} x(t)$. Then:

$$\begin{aligned} \dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r} & \alpha_i(p(t))\alpha_j(p(t))\{x(t)^T[(A_i + B_i K_j)^T P^{-1} + P^{-1}(A_i + B_i K_j)]x(t) \\ & + x(t)^T P^{-1} A_{di} x(t - d_i(t)) + x^T(t - d_i(t)) A_{di}^T P^{-1} x(t)\} \\ \leqslant \sum_{i=1}^{r}\sum_{j=1}^{r} & \alpha_i(p(t))\alpha_j(p(t))\{x(t)^T[(A_i + B_i K_j)^T P^{-1} + P^{-1}(A_i + B_i K_j)]x(t) \\ & + x(t)^T P^{-1} A_{di} \gamma_i A_{di}^T P^{-1} x(t) + x^T(t - d_i(t)) \gamma_i^{-1} x(t - d_i(t))\}. \end{aligned}$$

Substituting $|x(t - d_i(t)| \leqslant \gamma\, |x(t)|$ into the above inequality we have:

$$\dot{V}(x(t)) \leqslant \sum_{i=1}^{r} \alpha_i^2(p(t))(P^{-1}x(t))^T[\widetilde{A}_{ii} + \gamma^2\gamma_i^{-1}P^2](P^{-1}x(t))$$
$$+\sum_{i<j} \alpha_i(p(t))\alpha_j(p(t))(P^{-1}x(t))^T[\widetilde{A}_{ij} + \widetilde{A}_{ji} + \gamma^2(\gamma_i^{-1}P^2 + \gamma_j^{-1}P^2)](P^{-1}x(t)).$$

If the LMI constraints (8.8)-(8.9) are satisfied, by the Schur complement we have:

$$\begin{cases} \widetilde{A}_{ii} + \gamma_i^{-1}P^2 < 0 \quad (i = 1, 2, ..., r) \\ \widetilde{A}_{ij} + \widetilde{A}_{ji} + \gamma_i^{-1}P^2 + \gamma_j^{-1}P^2 < 0 \quad (1 \leqslant i < j \leqslant r) \end{cases}. \tag{8.10}$$

Applying Lemma 8.4 to (8.10) it follows that there exists $\gamma > 1$ such that

$$\begin{cases} \widetilde{A}_{ii} + \gamma^2\gamma_i^{-1}P^2 < 0 \quad (i = 1, 2, ..., r) \\ \widetilde{A}_{ij} + \widetilde{A}_{ji} + \gamma^2(\gamma_i^{-1}P^2 + \gamma_j^{-1}P^2) < 0 \quad (1 \leqslant i < j \leqslant r) \end{cases}.$$

Thereby, there must exist $w > 0$ such that $\dot{V}(x(t)) \leqslant -w\left|x(t)\right|^2$. Then, the proof is completed by applying Lemma 8.2. ■

8.3 Delay-dependent Stability Conditions

Based on the Lyapunov functional method, we present some delay dependent stability conditions for the time delay control systems described by (8.2) and (8.3), where $d_i(t)$ are assumed to satisfy $\dot{d}_i(t) \leqslant \varepsilon_i < 1$ for all $i = 1, 2, ...r$ additionally.

Theorem 8.2 *If there exist symmetric matrices $P > 0$, $R > 0$, scalars $\tau_{ij} > 0$, and matrices M_i such that*

$$\begin{bmatrix} \widetilde{R}_{11} & \widetilde{R}_{12} + \tau_{12}I & \cdots & \widetilde{R}_{1r} + \tau_{1r}I & P \\ * & \widetilde{R}_{22} & \cdots & \widetilde{R}_{2r} + \tau_{2r}I & P \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & \widetilde{R}_{rr} & P \\ * & * & \cdots & * & -R \end{bmatrix} < 0 \tag{8.11}$$

where

$\widetilde{R}_{ii} = A_iP + PA_i^T + B_iM_i + M_i^TB_i^T + \frac{r}{1-\varepsilon_i}A_{di}RA_{di}^T$ *$(i = 1, 2, ..., r)$,*

$\widetilde{R}_{ij} = \frac{1}{2}[(A_i + A_j)P + P(A_i + A_j)^T + (B_iM_j + B_jM_i) + (B_iM_j + B_jM_i)^T]$ $(i < j)$,

then the time delay system described by (8.2) is asymptotically stabilized via the fuzzy controller described by (8.3) with the feedback gains $K_i = M_iP^{-1}$ $(i = 1, 2, ..., r)$.

Proof. Choose the candidate Lyapunov-Krasovskii functional as

$$V(t, \phi) = \phi^T(0)P^{-1}\phi(0) + \frac{1}{r}\sum_{i=1}^{r}\int_{-d_i(t)}^{0}\phi^T(s)R^{-1}\phi(s)ds.$$

That is:

$$V(t, x_t) = x^T(t)P^{-1}x(t) + \frac{1}{r}\sum_{i=1}^{r}\int_{t-d_i(t)}^{t}x^T(s)R^{-1}x(s)ds.$$

Then, there must exist $\delta_1, \delta_2 > 0$ such that

$$\delta_1 |x(t)|^2 \leqslant V(t,x_t) \leqslant \delta_2 \sup_{\theta\in[-\tau,0]} |x(t+\theta)|^2 .$$

The derivative of $V(t,x_t)$ along the trajectory of (8.4) gives:

$$\begin{aligned}\dot{V}(t,x_t) = &\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))\{x^T(t)[(A_i+B_iK_j)^TP^{-1}+P^{-1}(A_i+B_iK_j)]x(t)\\ &+x^T(t)P^{-1}A_{di}x(t-d_i(t))+x^T(t-d_i(t))A_{di}^TP^{-1}x(t)\}\\ &+\tfrac{1}{r}\sum_{i=1}^{r}[x^T(t)R^{-1}x(t)-(1-\dot{d}_i(t))x^T(t-d_i(t))R^{-1}x(t-d_i(t))]\\ \leqslant &\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))\{x^T(t)[(A_i+B_iK_j)^TP^{-1}+P^{-1}(A_i+B_iK_j)]x(t)\\ &+\sum_{i=1}^{r}\{x^T(t-d_i(t))[\alpha_i(p(t))A_{di}^TP^{-1}x(t)]+[\alpha_i(p(t))A_{di}^TP^{-1}x(t)]^Tx(t-d_i(t))\}\\ &+x^T(t)R^{-1}x(t)-\sum_{i=1}^{r}x^T(t-d_i(t))(\tfrac{1-\varepsilon_i}{r}R^{-1})x(t-d_i(t)).\end{aligned}$$

Applying the inequality $X^TY+Y^TX\leqslant X^TQX+Y^TQ^{-1}Y$ $(Q>0)$, we have

$$\begin{aligned}\dot{V}(t,x_t) \leqslant &\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(p(t))\alpha_j(p(t))x^T(t)[(A_i+B_iK_j)^TP^{-1}+P^{-1}(A_i+B_iK_j)]x(t)\\ &+\sum_{i=1}^{r}[x^T(t-d_i(t))(\tfrac{1-\varepsilon_i}{r}R^{-1})x(t-d_i(t))+\alpha_i^2(p(t))x^T(t)P^{-1}A_{di}(\tfrac{r}{1-\varepsilon_i}R)A_{di}^TP^{-1}x(t)]\\ &+x^T(t)R^{-1}x(t)-\sum_{i=1}^{r}x^T(t-d_i(t))(\tfrac{1-\varepsilon_i}{r}R^{-1})x(t-d_i(t))\\ = &\sum_{i=1}^{r}\alpha_i^2(p(t))(P^{-1}x(t))^T(\widetilde{R}_{ii}+PR^{-1}P)(P^{-1}x(t))\\ &+\sum_{i<j}2\alpha_i(p(t))\alpha_j(p(t))(P^{-1}x(t))^T(\widetilde{R}_{ij}+PR^{-1}P)(P^{-1}x(t))\\ = &(P^{-1}x(t))^T\begin{bmatrix}\alpha_1I & \alpha_2I & \cdots & \alpha_rI\end{bmatrix}(\begin{bmatrix}\widetilde{R}_{11} & \widetilde{R}_{12} & \cdots & \widetilde{R}_{1r}\\ * & \widetilde{R}_{22} & \cdots & \widetilde{R}_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \dots & \widetilde{R}_{rr}\end{bmatrix}\\ &+\begin{bmatrix}PR^{-1}P & PR^{-1}P & \cdots & PR^{-1}P\\ * & PR^{-1}P & \cdots & PR^{-1}P\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \dots & PR^{-1}P\end{bmatrix})\begin{bmatrix}\alpha_1I\\ \alpha_2I\\ \vdots\\ \alpha_rI\end{bmatrix}(P^{-1}x(t))\end{aligned}$$

where $\alpha_i := \alpha_i(p(t))$. Since $\sum_{i<j}2\alpha_i\alpha_j\tau_{ij}\left|P^{-1}x(t)\right|^2 \geqslant 0$ for all $x(t)\in\Re$ and $\tau_{ij}>0$, then it follows:

$$\dot{V}(t,x_t)\leqslant (P^{-1}x(t))^T\begin{bmatrix}\alpha_1I\\ \alpha_2I\\ \vdots\\ \alpha_rI\end{bmatrix}^T(\begin{bmatrix}\widetilde{R}_{11} & \widetilde{R}_{12}+\tau_{12}I & \cdots & \widetilde{R}_{1r}+\tau_{1r}I\\ * & \widetilde{R}_{22} & \cdots & \widetilde{R}_{2r}+\tau_{2r}I\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \dots & \widetilde{R}_{rr}\end{bmatrix}$$

$$+\begin{bmatrix} P \\ P \\ \vdots \\ P \end{bmatrix} R^{-1} \begin{bmatrix} P & P & \cdots & P \end{bmatrix}) \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} (P^{-1}x(t)).$$

Then, by the Schur complement we have $\dot{V}(t, x_t) \leqslant -w\,|x(t)|^2$ for some $w > 0$, if the LMI constraint (8.11) is satisfied. By Lemma 8.3 it follows that the closed loop time delay system (8.4) is globally asymptotically stable, which completes the proof. ■

Corollary 8.3 *If there exist symmetric matrices $P > 0$, $R > 0$ and matrices M_i such that*

$$\begin{bmatrix} \widetilde{R}_{ii} & P \\ P & -R \end{bmatrix} < 0 \quad (i = 1, 2, ..., r)$$

$$\begin{bmatrix} \widetilde{R}_{ij} & P \\ P & -R \end{bmatrix} < 0 \quad (1 \leqslant i < j \leqslant r)$$

where $\widetilde{R}_{ij}$ $(1 \leqslant i \leqslant j \leqslant r)$ are the same as in Theorem 8.2, then the time delay system (8.2) can be asymptotically stabilized via the fuzzy controller described by (8.3) with the feedback gains $K_i = M_i P^{-1}$ $(i = 1, 2, ..., r)$.

Proof. It follows from the proof of Theorem 8.2. ■

If the candidate Lyapunov-Krasovskii functional is chosen as

$$V(t, x_t) = x^T(t)P^{-1}x(t) + \frac{1}{r}\sum_{i=1}^{r}\int_{t-d_i(t)}^{t} x^T(s)R_i^{-1}x(s)ds \tag{8.12}$$

where $P^{-1}, R_i^{-1} > 0$, applying the proof procedure of Theorem 8.2, we obtain the following result, which is an extension of Corollary 8.3.

Corollary 8.4 *If there exist symmetric matrices $P > 0$, $R_i > 0$ and matrices M_i such that*

$$\begin{bmatrix} \widehat{R}_{ii} & P & P & \cdots & P \\ * & -rR_1 & 0 & \cdots & 0 \\ * & * & -rR_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -rR_r \end{bmatrix} < 0 \quad (i = 1, 2, ..., r) \tag{8.13}$$

$$\begin{bmatrix} \widehat{R}_{ij} & P & P & \cdots & P \\ * & -rR_1 & 0 & \cdots & 0 \\ * & * & -rR_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -rR_r \end{bmatrix} < 0 \quad (1 \leqslant i < j \leqslant r) \tag{8.14}$$

where $*$ stands for the transposed element in the symmetric position,

$\widehat{R}_{ii} = A_iP + PA_i^T + B_iM_i + M_i^TB_i^T + \frac{r}{1-\varepsilon_i}A_{di}R_iA_{di}^T$ $(i = 1, 2, ..., r)$,

$\widehat{R}_{ij} = \frac{1}{2}[(A_i + A_j)P + P(A_i + A_j)^T + (B_iM_j + B_jM_i) + (B_iM_j + B_jM_i)^T]$ $(i < j)$,

then the time delay system (8.2) can be asymptotically stabilized via the fuzzy controller described by (8.3) with the feedback gains $K_i = M_iP^{-1}$ $(i = 1, 2, ..., r)$.

Proof. Substituting (8.12) into the proof of Theorem (8.2), we have

$$\dot{V}(t, x_t) \leqslant \sum_{i=1}^{r} \alpha_i^2(p(t))(P^{-1}x(t))^T(\widehat{R}_{ii} + \frac{1}{r}\sum_{i=1}^{r} PR_i^{-1}P)(P^{-1}x(t))$$
$$+ \sum_{i<j} 2\alpha_i(p(t))\alpha_j(p(t))(P^{-1}x(t))^T(\widehat{R}_{ij} + \frac{1}{r}\sum_{i=1}^{r} PR_i^{-1}P)(P^{-1}x(t)).$$

From the LMI constraints (8.13) and (8.14) it follows

$$\widehat{R}_{ii} + \frac{1}{r}\sum_{i=1}^{r} PR_i^{-1}P < 0 \quad (i = 1, 2, ..., r)$$
$$\widehat{R}_{ij} + \frac{1}{r}\sum_{i=1}^{r} PR_i^{-1}P \quad (1 \leqslant i < j \leqslant r).$$

Thereby, there exists scalar $w > 0$ such that $\dot{V}(t, x_t) \leqslant -w\,|x(t)|^2$. Applying Lemma 8.4, we obtain the result. ■

Theoretically, all the conclusions presented in this chapter are parallel, except that Corollary (8.4) is a generalized result of Corollary (8.3) (i.e. $R_1 = R_2 = ... = R_r$). The conservativeness of the these conditions will be compared via the simulation results in Section 8.4.

8.4 Numerical Example

Suppose that the delay truck-trailer system is given by the following fuzzy rules [9]:

If $p(t)$ is F_i, then: $\dot{x}(t) = A_ix(t) + A_{di}x(t-\tau) + B_iu(t)$ $(i = 1, 2)$

where

$$p(t) = \begin{bmatrix} a\frac{v\bar{t}}{2L} & 1 & 0 \end{bmatrix} x(t) + (1-a)\begin{bmatrix} \frac{v\bar{t}}{2L} & 0 & 0 \end{bmatrix} x(t-\tau),$$

$$A_1 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, A_{d1} = \begin{bmatrix} (a-1)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{5v^2\bar{t}^2}{L\pi} & \frac{10v\bar{t}}{\pi} & 0 \end{bmatrix}, A_{d1} = \begin{bmatrix} (a-1)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{5v^2\bar{t}^2}{L\pi} & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix}.$$

The membership functions of fuzzy sets F_1 and F_2 are given by:

$$\mu_{F_1}(p(t)) = \frac{1}{1 + \exp(-3p(t) - 1.5\pi)}(1 - \frac{1}{1 + \exp(-3p(t) + 1.5\pi)}),$$

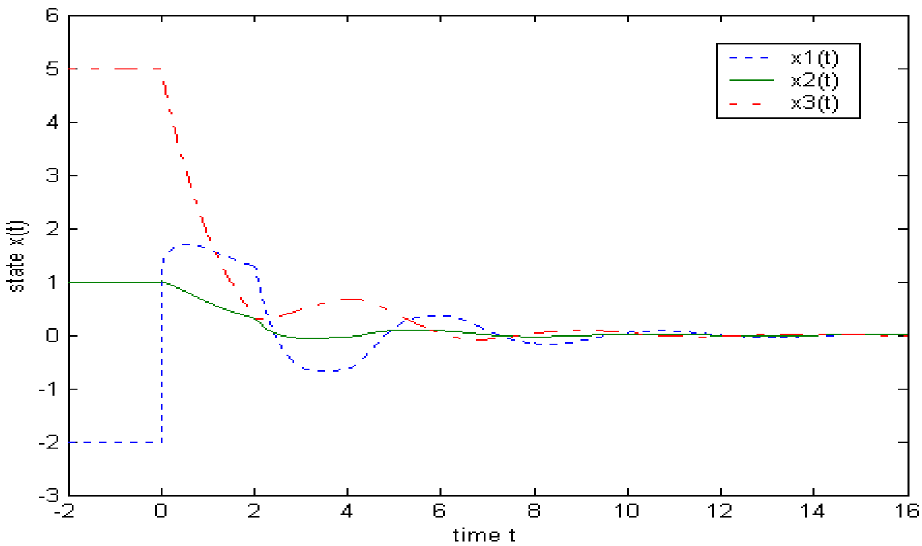

Figure 8.1: Simulation result by applying the controller designed by Theorem 8.1

$$\mu_{F_2}(p(t)) = 1 - \mu_{F_1}(p(t)).$$

The parameter $a \in [0,\ 1]$ stands for the retarded coefficient. A smaller value of parameter a means conversely a greater value of the delay terms. The other model parameters are given by $L = 5.5$, $l = 2.8$, $v = -1$, $\bar{t} = 2$, $t_0 = 0.5$. More detailed descriptions of the model can be found e.g. in [74] and [9].

Based on the LMI tools, the intervals of parameter a can be found, in which the LMI constraints of the presented conclusions are feasible. That is

Theorem 8.1 feasible for $a \geqslant 0.672$	Theorem 8.2 feasible for $a \geqslant 0.586$
Corollary 8.1 feasible for $a \geqslant 0.617$	Corollary 8.2 feasible for $a \geqslant 0.501$
Corollary 8.3 feasible for $a \geqslant 0.586$	Corollary 8.4 feasible for $a \geqslant 0.586$

which means, that the conditions of Theorem 8.1 are most conservative, whereas the conditions of Corollary 8.2 are most relaxed for this model.

If a is set to 0.7 as in [9], the feasible solutions of the LMIs in Theorem 8.1 are:

$$P = \begin{bmatrix} 2.9429 & 0.7055 & 0.6382 \\ 0.7055 & 0.3016 & 1.0166 \\ 0.6382 & 1.0166 & 5.9609 \end{bmatrix}, K_1 = \begin{bmatrix} 80.3551 \\ -378.1108 \\ 55.7143 \end{bmatrix}^T, K_2 = \begin{bmatrix} 47.3605 \\ -217.8820 \\ 32.0560 \end{bmatrix}^T,$$

and $\tau_{12} = 0.0340$. The controlled trajectory is shown in Figure 8.1 with $\tau = -2$ and the initial conditions $x(t) = \begin{bmatrix} -2 & 1 & 5 \end{bmatrix}^T$ for $t \in [-2,\ 0]$.

Applying Corollary 8.2 to the model ($a = 0.7$), we obtain the following feasible solutions:

$$P = \begin{bmatrix} 1389.3 & 208.5 & -297.3 \\ 208.5 & 66.4 & 87.8 \\ -297.3 & 87.8 & 766.7 \end{bmatrix}, K_1 = \begin{bmatrix} 11.3042 \\ -54.4413 \\ 10.1249 \end{bmatrix}^T, K_2 = \begin{bmatrix} 10.9549 \\ -52.5159 \\ 9.7005 \end{bmatrix}^T,$$

$$\gamma_1 = 1.7886 \times 10^3, \gamma_2 = 1.6044 \times 10^3.$$

The system response is illustrated in Figure 8.2 , where $\tau = -2$ and the initial conditions $x(t) = \begin{bmatrix} -2 & 1 & 5 \end{bmatrix}^T$ for $t \in [-2,\ 0]$.

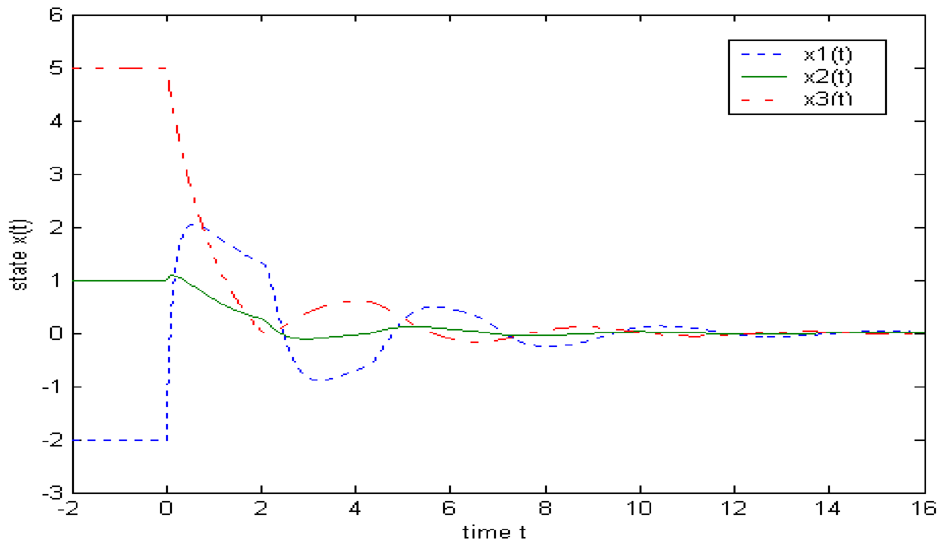

Figure 8.2: Simulation result by applying the controller designed by Corollary 8.2

Chapter 9

Robust Stabilization of Uncertain Delay T-S Fuzzy Models

In this Chapter, the problem of robust stabilization of T-S fuzzy models with time varying delays and norm bounded uncertainties is discussed by employing the PDC based state feedback fuzzy controllers. Sufficient robust stability conditions are presented in terms of Lyapunov functional method and Razumikhin type theorems respectively. In the same framework the design of H_∞ fuzzy controllers is also considered. The results are formulated in the form of LMIs and the synthesis procedures are finally illustrated by a numerical example.

9.1 Robust Stability Conditions

Robust stability problem is an important subject in control research, which is concerned with the systems containing uncertainties. To treat the robust control problem, two time domain approaches are often adopted, namely the Riccati equation approach and the LMI approach [64]. Recently, the interests are focused on the latter, since the LMI constraints can be efficiently solved by the interior point algorithms, and all the parameters in LMIs don't need to be tuned manually.

The uncertainty of a plant may stem from internal structure and external disturbance. The maximum uncertainty that can be dealt with by feedback is discussed in [83]. In the literature, the system uncertainties are often assumed to satisfy matching conditions e.g. [54], [57], rank-one conditions e.g. [19], [67] and norm bounded conditions e.g. [49], [48]. It is shown in [64], that the matching conditions are not appropriate constraints for the system uncertainties. In this chapter, it is assumed that all the involved uncertainties satisfy the norm bounded conditions.

The model under discussion has both time varying delays and norm bounded uncertainties. Moreover, different subsystems in the model may have different time delays. Few researches on such a model are reported in the literature, but many special cases are discussed e.g. [16], [60], [49]. Assume that the nonlinear uncertain delay systems are expressed by the following fuzzy rules:

If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$\begin{cases} \dot{x}(t) = (A_i + \Delta A_i)x(t) + A_{di}x(t - d_i(t)) + (B_i + \Delta B_i)u(t) \\ x(t) = \phi(t) \quad t \in [-\tau,\ 0], d_i(t) \in [0,\ \tau] \end{cases} \quad (i = 1, 2, ..., r) \tag{9.1}$$

where $\dot{d}_i(t) \leqslant \tau_i < 1$ and $\Delta A_i, \Delta B_i$ are the system uncertainties satisfying the norm bounded conditions:

$$\begin{cases} \Delta A_i = \widetilde{H}_{ai}\widetilde{F}_{ai}(t)\widetilde{L}_{ai} \quad \widetilde{F}_{ai}^T(t)\widetilde{F}_{ai}(t) \leqslant I \\ \Delta B_i = \widetilde{H}_{bi}\widetilde{F}_{bi}(t)\widetilde{L}_{bi} \quad \widetilde{F}_{bi}^T(t)\widetilde{F}_{bi}(t) \leqslant I \end{cases} \quad (i = 1, 2, ..., r). \tag{9.2}$$

By setting $H_i := [\widetilde{H}_{ai} \quad \widetilde{H}_{bi}]$, $F_i := diag(\widetilde{F}_{ai}(t), \widetilde{F}_{bi}(t))$, $L_{ai} =: [\widetilde{L}_{ai}^T \quad 0]^T$ and $L_{bi} =: [0 \quad \widetilde{L}_{bi}^T]^T$, (9.2) can be rewritten as:

$$[\Delta A_i \quad \Delta B_i] = H_i F_i [L_{ai} \quad L_{bi}] \quad (i = 1, 2, ..., r). \tag{9.3}$$

Based on the PDC technique, the fuzzy controller for (9.1) can be described by:

If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$u(t) = K_i x(t) \quad (i = 1, 2, ..., r) \tag{9.4}$$

where K_i are the state feedback gains to be designed.

Then, the closed loop system can be inferred:

$$\dot{x}(t) = \sum_{i,j} \alpha_i(p(t))\alpha_j(p(t))\{[(A_i + \Delta A_i) + (B_i + \Delta B_i)K_j]x(t) + A_{di}x(t - d_i(t))\}. \tag{9.5}$$

For the stability analysis of (9.5), the following result is required.

Lemma 9.1 *([84]) Given matrices Q, H, E, R of appropriate dimensions with $Q = Q^T$, $R = R^T$ and $R > 0$, then*

$$Q + HFE + E^T F^T H^T < 0$$

for all F satisfying $F^T F \leqslant R$, if and only if there exists some $\varepsilon > 0$, such that

$$Q + \varepsilon H H^T + \varepsilon^{-1} E^T R E < 0.$$

Theorem 9.1 *The closed loop system (9.5) is globally uniformly asymptotically stable, if there exist symmetric positive matrices $P > 0$, $R_i > 0$ $(i = 1, 2, ..., r)$, scalars $\varepsilon_{ij} > 0$ $(i \leqslant j)$ and matrices M_i $(i = 1, 2, ..., r)$, such that*

$$\begin{bmatrix} \widetilde{G}_{ii} + \varepsilon_{ii} H_i H_i^T & * & * \\ L_{ai}P + L_{bi}M_i & -\varepsilon_{ii} I & * \\ P A_{di}^T & 0 & -\frac{1-\tau_i}{r} R_i \end{bmatrix} < 0 \quad (i = 1, 2, ..., r) \tag{9.6}$$

$$\begin{bmatrix} \widetilde{G}_{ij} + \widetilde{G}_{ji} + \varepsilon_{ij}(H_i H_i^T + H_j H_j^T) & * & * \\ L_{ai}P + L_{bi}M_j & -\varepsilon_{ij} I & * \\ L_{aj}P + L_{bj}M_i & 0 & -\varepsilon_{ij} I \end{bmatrix} < 0 \quad (i < j) \tag{9.7}$$

where $\widetilde{G}_{ij} = P A_i^T + A_i P + B_i M_j + M_j^T B_i^T + \frac{1}{r} \sum_{s=1}^{r} R_s \quad (i \leqslant j)$.
Then, the state feedback gains can be calculated by $K_i = M_i P^{-1}$ for all $i = 1, 2, ..., r$.

Proof. Choose the candidate Lyapunov-Krasovskii functional as

$$V(t, x_t) = x^T(t) P^{-1} x(t) + \frac{1}{r} \sum_{i=1}^{r} \int_{t-d_i(t)}^{t} x^T(s) P^{-1} R_i P^{-1} x(s) ds.$$

Then the derivative of $V(t, x_t)$ along the trajectory of (9.5) gives:

$$\begin{aligned} \dot{V}(t, x_t) = & \sum_{i,j} \alpha_i(p(t)) \alpha_j(p(t)) x^T(t) [P^{-1}(A_i + \Delta A_i + B_i K_j + \Delta B_i K_j) \\ & + (A_i + \Delta A_i + B_i K_j + \Delta B_i K_j)^T P^{-1}] x(t) \\ & + \sum_{i=1}^{r} \alpha_i(p(t)) [x^T(t) P^{-1} A_{di} x(t - d_i(t)) + x^T(t - d_i(t)) A_{di}^T P^{-1} x(t)] \\ & - \frac{1}{r} \sum_{i=1}^{r} (1 - \dot{d}_i(t)) x^T(t - d_i(t)) P^{-1} R_i P^{-1} x(t - d_i(t)) \\ & + \frac{1}{r} \sum_{i=1}^{r} x^T(t) P^{-1} R_i P^{-1} x(t). \end{aligned}$$

Substitute $\dot{d}_i(t)$ for τ_i, and then apply the inequality

$$X^T Y + Y^T X \leqslant X^T Q^{-1} X + Y^T Q Y \quad (Q > 0)$$

to the term

$$[\alpha_i(p(t)) A_{di}^T P^{-1} x(t)]^T x(t - d_i(t)) + x^T(t - d_i(t)) [\alpha_i(p(t)) A_{di}^T P^{-1} x(t)],$$

where Q is set to $\frac{1-\tau_i}{r} P^{-1} R_i P^{-1}$,
we have:

$$\begin{aligned} \dot{V}(t, x_t) \leqslant & \sum_{i=1}^{r} \alpha_i^2(p(t)) (P^{-1} x(t))^T [(A_i + \Delta A_i + B_i K_i + \Delta B_i K_i) P \\ & + P(A_i + \Delta A_i + B_i K_i + \Delta B_i K_i)^T + A_{di}^T \frac{r}{1-\tau_i} P R_i^{-1} P A_{di}^T + \frac{1}{r} \sum_{i=1}^{r} R_i] (P^{-1} x(t)) \\ & + \sum_{i,j} \alpha_i(p(t)) \alpha_j(p(t)) (P^{-1} x(t))^T [\frac{1}{r} \sum_{i=1}^{r} R_i \\ & + (A_i + \Delta A_i + B_i K_j + \Delta B_i K_j + A_j + \Delta A_j + B_j K_i + \Delta B_j K_i) P \\ & + P(A_i + \Delta A_i + B_i K_j + \Delta B_i K_j + A_j + \Delta A_j + B_j K_i + \Delta B_j K_i)^T] (P^{-1} x(t)). \end{aligned}$$

Obviously, there exists some $\varpi > 0$ such that $\dot{V}(t, x_t) \leqslant -\varpi |x(t)|^2$, if the following conditions are satisfied:

$$\begin{aligned}&(A_i+\Delta A_i+B_iK_i+\Delta B_iK_i)P+P(A_i+\Delta A_i+B_iK_i+\Delta B_iK_i)^T\\&+A_{di}^T\tfrac{r}{1-\tau_i}PR_i^{-1}PA_{di}^T+\tfrac{1}{r}\sum_{i=1}^{r}R_i<0\quad(i=1,2,...,r),\end{aligned}\tag{9.8}$$

$$\begin{aligned}&\tfrac{2}{r}\sum_{i=1}^{r}R_i+(A_i+\Delta A_i+B_iK_j+\Delta B_iK_j+A_j+\Delta A_j+B_jK_i+\Delta B_jK_i)P\\&+P(A_i+\Delta A_i+B_iK_j+\Delta B_iK_j+A_j+\Delta A_j+B_jK_i+\Delta B_jK_i)^T<0\quad(i<j).\end{aligned}\tag{9.9}$$

Now we prove that (9.8) and (9.9) are equivalent to the LMI constraints (9.6) and (9.7) respectively. Substituting the norm bounded conditions (9.3) into left side of (9.8), we have:

$$\begin{aligned}&\widetilde{G}_{ii}+(H_iF_iL_{ai}P+H_iF_iL_{bi}K_iP)+(H_iF_iL_{ai}P+H_iF_iL_{bi}K_iP)^T\\&\quad+A_{di}^T\tfrac{r}{1-\tau_i}PR_i^{-1}PA_{di}^T<0.\\&\Leftrightarrow[\widetilde{G}_{ii}+A_{di}^T\tfrac{r}{1-\tau_i}PR_i^{-1}PA_{di}^T]+H_iF_i(L_{ai}P+L_{bi}K_iP)\\&\qquad+(L_{ai}P+L_{bi}K_iP)^TF_i^TH_i^T<0.\\&\Leftrightarrow\widetilde{G}_{ii}+A_{di}^T\tfrac{r}{1-\tau_i}PR_i^{-1}PA_{di}^T+\varepsilon_{ii}H_iH_i^T\\&\qquad+\varepsilon_{ii}^{-1}(L_{ai}P+L_{bi}K_iP)^T(L_{ai}P+L_{bi}K_iP)<0\text{ by Lemma 9.1.}\\&\Leftrightarrow\widetilde{G}_{ii}+\varepsilon_{ii}H_iH_i^T\\&\qquad+\begin{bmatrix}L_{ai}P+L_{bi}K_iP\\PA_{di}\end{bmatrix}^T\begin{bmatrix}\varepsilon_{ii}^{-1}I&\\&\tfrac{r}{1-\tau_i}R_i^{-1}\end{bmatrix}\begin{bmatrix}L_{ai}P+L_{bi}K_iP\\PA_{di}\end{bmatrix}<0.\\&\Leftrightarrow\begin{bmatrix}\widetilde{G}_{ii}+\varepsilon_{ii}H_iH_i^T&*&*\\L_{ai}P+L_{bi}M_i&-\varepsilon_{ii}I&*\\PA_{di}^T&0&-\tfrac{1-\tau_i}{r}R_i\end{bmatrix}<0\text{ by Schur complement.}\end{aligned}$$

Similarly, we can prove that (9.9) is equivalent to (9.7). Then, by applying the Lyapunov-Krasovskii theorem the proof is completed. ■

Different from most of the reported results in the literature, in Theorem 9.1 the delay terms A_{di} don't appear in the LMI constraints for $i<j$. Moreover, it is easy to see:

1). If $\Delta A_i=0$, $\Delta B_i=0$ for $i=1,2,...,r$, then Theorem 9.1 is equivalent to the result of Corollary 8.4.

2). Theorem 9.1 is also an extension of the main result in [48], where $A_{di}=0$ for all $i=1,2,...,r$.

3). If the delay terms in (9.5) have also uncertainties, i.e. the closed loop system is described by

$$\begin{aligned}\dot{x}(t)=&\textstyle\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))\{[(A_i+\Delta A_i)+(B_i+\Delta B_i)K_j]x(t)\\&+(A_{di}+\Delta A_{di})x(t-d_i(t))\},\end{aligned}$$

where $\Delta A_{di}=H_{di}F_{di}L_{di}$ and $F_{di}^TF_{di}\leqslant I$ for $i=1,2,...,r$, then Theorem 9.1 still holds if the LMI constraints (9.6) are replaced by:

$$\begin{bmatrix} \widetilde{G}_{ii}+\varepsilon_{ii}(H_iH_i^T+H_{di}H_{di}^T) & * & * & * \\ PA_{di}^T & -\frac{1-\tau_i}{r}R_i & * & * \\ 0 & L_{di}P & -\varepsilon_{ii}I & * \\ L_{ai}P+L_{bi}M_i & 0 & 0 & -\varepsilon_{ii}I \end{bmatrix} < 0 \quad (i=1,2,...,r).$$

If the candidate Lyapunov function is replaced by $V(x(t)) = x^T(t)P^{-1}x(t)$, by the Razumikhin type theorem we obtain the following delay independent robust stability conditions:

Corollary 9.1 *The closed loop system (9.5) is globally uniformly asymptotically stable, if there exists a symmetric positive matrix* $P > 0$, *scalars* $\gamma_i > 0$ $(i = 1, 2, ..., r)$, $\varepsilon_{ij} > 0$ $(i \leqslant j)$ *and matrices* M_i $(i = 1, 2, ..., r)$, *such that*

$$\begin{bmatrix} \widetilde{A}_{ii}+\varepsilon_{ii}H_iH_i^T & * & * \\ L_{ai}P+L_{bi}M_i & -\varepsilon_{ii}I & * \\ P & 0 & -\gamma_iI \end{bmatrix} < 0 \quad (i=1,2,...,r),$$

$$\begin{bmatrix} \widetilde{A}_{ij}+\widetilde{A}_{ji}+\varepsilon_{ij}(H_iH_i^T+H_jH_j^T) & * & * & * & * \\ L_{ai}P+L_{bi}M_j & -\varepsilon_{ij}I & & & \\ L_{aj}P+L_{bj}M_i & & -\varepsilon_{ij}I & & \\ P & & & -\gamma_iI & \\ P & & & & -\gamma_jI \end{bmatrix} < 0 \quad (i<j),$$

where $\widetilde{A}_{ij} = PA_i^T + A_iP + B_iM_j + M_j^TB_i^T + \gamma_iA_{di}A_{di}^T \quad (i \leqslant j)$.
Then, the state feedback gains can be calculated by $K_i = M_iP^{-1}$ *for all* $i = 1, 2, ..., r$.

Proof. Similar to the proof procedure of Theorem 9.1, the result follows by applying Corollary 8.2 and Lemma 9.1. ■

Corollary 9.2 *The closed loop system (9.5) is globally uniformly asymptotically stable, if there exists a symmetric positive matrix* $P > 0$, *scalars* $\varepsilon_{ij} > 0$ $(i \leqslant j)$ *and matrices* M_i $(i = 1, 2, ..., r)$, *such that*

$$\begin{bmatrix} \overline{A}_{ii}+\varepsilon_{ii}H_iH_i^T & * \\ L_{ai}P+L_{bi}M_i & -\varepsilon_{ii}I \end{bmatrix} < 0 \quad (i=1,2,...,r),$$

$$\begin{bmatrix} \overline{A}_{ij}+\overline{A}_{ji}+\varepsilon_{ij}(H_iH_i^T+H_jH_j^T) & * & * \\ L_{ai}P+L_{bi}M_j & -\varepsilon_{ij}I & \\ L_{aj}P+L_{bj}M_i & & -\varepsilon_{ij}I \end{bmatrix} < 0 \quad (i<j),$$

where $\overline{A}_{ij} = PA_i^T + A_iP + B_iM_j + M_j^TB_i^T + A_{di}A_{di}^T + P \ (i \leqslant j)$.
Then, the state feedback gains can be calculated by $K_i = M_iP^{-1}$ *for all* $i = 1, 2, ..., r$.

Proof. It follows from Corollary 8.1 and Lemma 9.1. ■

9.2 H_∞ Controller Design

The H_∞ control problem is concerned with the controller design which stabilizes a system, while an H_∞ norm bound constraint on disturbance attenuation is satisfied. It is shown in [38], that the H_∞ control for linear systems can be solved by solving an algebraic Riccati equation, whereas the result in [39] shows that the H_∞ control problem is essentially a certain type of quadratic stabilization problem. The result of [39] is further extended to the linear systems with uncertainties in all the system matrices e.g. [85], [84]. Recently, based on the quadratic stabilization approach, the H_∞ control for nonlinear systems is investigated e.g. in [49], [12] via T-S fuzzy models. In this section, we will discuss the problem of H_∞ controller design of the following fuzzy models:

If $p_1(t)$ is M_1^i and ... and $p_s(t)$ is M_s^i, then

$$\begin{cases} \dot{x}(t) = (A_i + \Delta A_i)x(t) + A_{di}x(t - d_i(t)) + (B_i + \Delta B_i)u(t) + E_i w(t) \\ z(t) = C_i x(t) + D_i u(t) \quad (i = 1, 2, ..., r) \end{cases} \tag{9.10}$$

where $w(t)$ is the square integrable disturbance, $z(t)$ is the controlled output, $d_i(t)$ is the state time varying delay satisfying $0 \leqslant d_i(t) < \infty$ and $\dot{d}_i(t) \leqslant \tau_i < 1$. Moreover, the system uncertainties ΔA_i and ΔB_i are assumed to satisfy the norm bounded conditions (9.3) for $i = 1, 2, ..., r$. The objective of state feedback H_∞ controller design is to construct control law $u(t) = K(t)x(t)$, such that for all the admissible system uncertainties and time delays:

(1) the closed loop system with $w(t) = 0$ is asymptotically stable,

(2) subject to the zero initial condition, it holds $\int_0^\infty |z(t)|^2 dt \leqslant \gamma^2 \int_0^\infty |w(t)|^2 dt$, where γ is a prescribed level of disturbance attenuation. If such a control law exists, then the nonlinear system described by (9.10) is said to be stabilizable with H_∞ norm bound γ. For linear systems, only linear controller is needed to achieve the robust performance. Moreover, it is shown in [38], that the linear dynamic state feedback offers no advantage over the linear static state feedback, concerning the minimization of the H_∞ norm of the closed loop system. However, these properties don't hold for nonlinear systems. We employ the PDC based fuzzy controller of form (9.4) for the H_∞ control of (9.10), that is:

$$u(t) = \sum_{i=1}^{r} \alpha_i(p(t))K_i x(t). \tag{9.11}$$

Theorem 9.2 *The uncertain delay system described by (9.10) is stabilizable with H_∞ norm bound γ via fuzzy controller (9.11), if there exist symmetric positive definite matrices $P > 0$, $R_i > 0$, scalars $\varepsilon_{ij} > 0$ and matrices M_i, such that the following LMI* *constraints are satisfied:*

$$\begin{bmatrix} \widetilde{G}_{ii} + \varepsilon_{ii} H_i H_i^T + \gamma^{-2} E_i E_i^T & * & * & * \\ C_i P + D_i M_i & -I & & \\ L_{ai} P + L_{bi} M_i & & -\varepsilon_{ii} I & \\ P A_{di}^T & & & -\frac{1-\tau_i}{r} R_i \end{bmatrix} < 0 \quad (i = 1, 2, ..., r), \quad (9.12)$$

$$\begin{bmatrix} \widetilde{G}_{ij} + \widetilde{G}_{ji} + \varepsilon_{ij}(H_i H_i^T + H_j H_j^T) + \gamma^{-2}(E_i E_i^T + E_j E_j^T) & * & * & * \\ (C_i + C_j)P + D_i M_j + D_j M_i & -2I & & \\ L_{ai} P + L_{bi} M_j & & -\varepsilon_{ij} I & \\ L_{aj} P + L_{bj} M_i & & & -\varepsilon_{ij} I \end{bmatrix} < 0 \quad (9.13)$$

for all $i < j$, where

$$\widetilde{G}_{ij} = P A_i^T + A_i P + B_i M_j + M_j^T B_i^T + \frac{1}{r} \sum_{s=1}^{r} R_s \quad (i \leqslant j).$$

Then, the state feedback gains can be obtained by $K_i = M_i P^{-1} \quad (i = 1, 2, ..., r)$.

Proof. Note that the LMI constraints (9.6) and (9.7) are implied by the LMI constraints (9.12) and (9.13) respectively. Then it follows by Theorem 9.1, that the closed loop system with $w(t) = 0$ is asymptotically stable, if the conditions (9.12) and (9.13) are satisfied. Now, we show that the H_∞ performance for the prescribed constant γ is also guaranteed as subject to the zero initial condition and the LMI constraints (9.12),(9.13).

Let $J := \int_0^\infty (|z(t)|^2 - \gamma^2 |w(t)|^2) dt$, we have:

$$\begin{aligned} J &= \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)] dt \\ &= \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \dot{V}(t, x_t)] dt \\ &\quad - \lim_{t \to \infty} [x(t)^T P^{-1} x(t) + \frac{1}{r} \sum_{i=1}^{r} \int_{t-d_i(t)}^{t} x^T(s) P^{-1} R_i P^{-1} x(s) ds] \end{aligned}$$

where $V(t, x_t)$ is the candidate Lyapunov-Krasovskii functional defined by:

$$V(t, x_t) = x^T(t) P^{-1} x(t) + \frac{1}{r} \sum_{i=1}^{r} \int_{t-d_i(t)}^{t} x^T(s) P^{-1} R_i P^{-1} x(s) ds.$$

Since $P > 0$ and $R_i > 0$, then we have

$$\begin{aligned} J &\leqslant \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \dot{V}(t, x_t)] dt \\ &= \int_0^\infty \{[\sum_{i,j} \alpha_i(p(t)) \alpha_j(p(t)) (C_i + D_i K_j) x(t)]^T [\sum_{i,j} \alpha_i(p(t)) \alpha_j(p(t)) (C_i + D_i K_j) x(t)] \\ &\quad - \gamma^2 w(t)^T w(t) + \sum_{i=1}^{r} 2\alpha_i(p(t)) x^T(t) P^{-1} A_{di} x(t - d_i(t)) \\ &\quad + \sum_{i,j} 2\alpha_i(p(t)) \alpha_j(p(t)) x^T(t) P^{-1} [(A_i + \Delta A_i) + (B_i + \Delta B_i) K_j] x(t) \\ &\quad + \sum_{i=1}^{r} 2\alpha_i(p(t)) x^T(t) P^{-1} E_i w(t) + \frac{1}{r} \sum_{i=1}^{r} x^T(t) P^{-1} R_i P^{-1} x(t) \\ &\quad - \frac{1}{r} \sum_{i=1}^{r} (1 - \dot{d}_i(t)) x^T(t - d_i(t)) P^{-1} R_i P^{-1} x(t - d_i(t))\} dt. \end{aligned}$$

By applying the inequalities:

$$2x^T(t)P^{-1}E_iw(t) \leqslant \gamma^{-2}x^T(t)P^{-1}E_iE_i^TP^{-1}x(t) + \gamma^2w^T(t)w(t),$$

and

$$\begin{aligned}&2\alpha_i(p(t))x^T(t)P^{-1}A_{di}x(t-d_i(t))\\ &\leqslant \alpha_i^2(p(t))x^T(t)P^{-1}A_{di}(\tfrac{r}{1-\tau_i}PR_i^{-1}P)A_{di}^TP^{-1}x(t)\\ &\quad +x^T(t-d_i(t))(\tfrac{1-\tau_i}{r}P^{-1}R_iP^{-1})x(t-d_i(t)),\end{aligned}$$

it follows:

$$\begin{aligned}J \leqslant \textstyle\int_0^\infty &(P^{-1}x(t))^T\{[\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))P(C_i+D_iK_j)^T]\\ &\cdot[\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))(C_i+D_iK_j)P]\\ &+\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))[P((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)^T\\ &+((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)P+\gamma^{-2}E_iE_i^T]+\tfrac{1}{r}\sum_{i=1}^r R_i\\ &+\sum_{i=1}^r\alpha_i^2(p(t))\tfrac{r}{1-\tau_i}A_{di}PR_i^{-1}PA_{di}^T\}(P^{-1}x(t))dt.\end{aligned}$$

Then, we have $J \leqslant 0$ if the following inequality is satisfied:

$$\begin{cases}[\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))P(C_i+D_iK_j)^T][\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))(C_i+D_iK_j)P]\\ +\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))[P((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)^T\\ +((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)P+\gamma^{-2}E_iE_i^T+\tfrac{1}{r}\sum_{s=1}^r R_s\\ +\sum_{i=1}^r\alpha_i^2(p(t))\tfrac{r}{1-\tau_i}A_{di}PR_i^{-1}PA_{di}^T<0.\end{cases} \tag{9.14}$$

Applying the Schur complement to (9.14), we have:

$$\begin{bmatrix}\sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))\widetilde{\Omega}_{ij}+\sum_{i=1}^r\alpha_i^2(p(t))\frac{r}{1-\tau_i}A_{di}PR_i^{-1}PA_{di}^T & * \\ \sum_{i,j}\alpha_i(p(t))\alpha_j(p(t))(C_i+D_iK_j)P & -I\end{bmatrix}<0 \tag{9.15}$$

where

$$\begin{aligned}\widetilde{\Omega}_{ij} = &P((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)^T+((A_i+\Delta A_i)+(B_i+\Delta B_i)K_j)P\\ &+\gamma^{-2}E_iE_i^T+\tfrac{1}{r}\sum_{s=1}^r R_s.\end{aligned}$$

Rewrite (9.15) as:

$$\begin{aligned}&\sum_{i=1}^r\alpha_i^2(p(t))\begin{bmatrix}\widetilde{\Omega}_{ii}+\frac{r}{1-\tau_i}A_{di}PR_i^{-1}PA_{di}^T & * \\ (C_i+D_iK_i)P & -I\end{bmatrix}\\ &+\sum_{i<j}\alpha_i(p(t))\alpha_j(p(t))\begin{bmatrix}\widetilde{\Omega}_{ij}+\widetilde{\Omega}_{ji} & * \\ (C_i+C_j+D_iK_j+D_jK_i)P & -2I\end{bmatrix}<0.\end{aligned}$$

It follows $J \leqslant 0$, if the following conditions are satisfied:

$$\begin{bmatrix}\widetilde{\Omega}_{ii}+\frac{r}{1-\tau_i}A_{di}PR_i^{-1}PA_{di}^T & * \\ (C_i+D_iK_i)P & -I\end{bmatrix}<0 \quad (i=1,2,...,r), \tag{9.16}$$

$$\begin{bmatrix} \widetilde{\Omega}_{ij}+\widetilde{\Omega}_{ji} & * \\ (C_i+C_j+D_iK_j+D_jK_i)P & -2I \end{bmatrix} < 0 \ \ (i<j). \qquad (9.17)$$

By Lemma 9.1 and the norm bounded conditions (9.3), it is easy to show, that (9.16) and (9.17) are equivalent to the LMI constraints (9.12) and (9.13) respectively. This completes the proof. ■

Obviously, if the LMI constraints (9.12) and (9.13) are feasible for the prescribed attenuation level γ, then they are also feasible for all the attenuation levels $\widetilde{\gamma} > \gamma$. Substituting different attenuation levels into the LMI constraints, we can obtain the minimal value of disturbance attenuation such that the LMI constraints (9.12) and (9.13) are feasible. In this case, the feasible solution of (9.12) and (9.13) can be taken as a suboptimal solution to the H_∞ optimal control problem [38]:

$$\begin{cases} \min\ \gamma \\ \textit{subject to } \int_0^\infty |z(t)|^2 dt \leqslant \gamma^2 \int_0^\infty |w(t)|^2 dt \end{cases}.$$

The procedure will be shown in the illustrative example. Similar work can also be found in [12], where a suboptimal H_2 control design is proposed by means of EVP (eigenvalue problem) optimization.

9.3 Illustrative Example

To illustrate the proposed approach, we consider the revised chaotic Lorenz system (see [48] or Section 7.4). Assume that the model is described by:

$$\text{If } x_1(t) \text{ is } \widetilde{M}_1, \text{ then } \begin{cases} \dot{x}(t) = \widetilde{A}_1 x(t) + A_{d1}x(t-d_1(t)) + B_1u(t) + E_1w(t) \\ z(t) = C_1x(t) + D_1u(t) \end{cases},$$

$$\text{If } x_1(t) \text{ is } \widetilde{M}_2, \text{ then } \begin{cases} \dot{x}(t) = \widetilde{A}_2 x(t) + A_{d2}x(t-d_2(t)) + B_2u(t) + E_2w(t) \\ z(t) = C_2x(t) + D_2u(t) \end{cases}$$

where $D_1 = D_2 = 1$, $d_1(t) = d_2(t) = 0.5(1-\sin(0.04t))$, $\gamma = 2$,

$$\widetilde{A}_1 = \begin{bmatrix} -\delta_1 & \delta_1 & 0 \\ \delta_2 & -1 & 20 \\ 0 & -20 & -\delta_3 \end{bmatrix}, \widetilde{A}_2 = \begin{bmatrix} -\delta_1 & \delta_1 & 0 \\ \delta_2 & -1 & -30 \\ 0 & 30 & -\delta_3 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T,$$

$$A_{d1} = A_{d2} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The uncertain parameters δ_1, δ_2 and δ_3 can take values randomly on intervals

$[10(1-40\%), 10(1+40\%)]$, $[28(1-20\%), 28(1+20\%)]$, $[\frac{8}{3}(1-30\%), \frac{8}{3}(1+30\%)]$

respectively, and the membership functions of the fuzzy sets $\widetilde{M}_1$ and $\widetilde{M}_2$ are given by:

$$\mu_{\widetilde{M}_1}(x(t)) = \begin{cases} 1 & \textit{if } x_1(t) < -20 \\ 0.6 - 0.02x_1(t) & \textit{if } -20 \leqslant x_1(t) < 30 \\ 0 & \textit{if } x_1(t) \geqslant 30 \end{cases},$$

$$\mu_{\widetilde{M}_2}(x(t)) = 1 - \mu_{\widetilde{M}_1}(x(t)).$$

Rewrite $\widetilde{A}_1$ as

$$\widetilde{A}_1 = \begin{bmatrix} -10(0.4\xi_1+1) & 10(0.4\xi_1+1) & 0 \\ 28(0.2\xi_2+1) & -1 & 20 \\ 0 & -20 & -\frac{8}{3}(0.3\xi_3+1) \end{bmatrix} = A_1 + H_1 F_1 L_1$$

where ξ_1, ξ_2, ξ_3 are random numbers on interval $[-1, 1]$, $F_1 = diag(\xi_1, \xi_2, \xi_3)$,

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 20 \\ 0 & -20 & -\frac{8}{3} \end{bmatrix}, H_1 = \begin{bmatrix} 0.4 & & \\ & 0.2 & \\ & & 0.3 \end{bmatrix}, L_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & 0 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}.$$

Similarly, $\widetilde{A}_2$ can be formulated as $\widetilde{A}_2 = A_2 + H_2 F_2 L_2$,

where $A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -30 \\ 0 & 30 & -\frac{8}{3} \end{bmatrix}$, $H_2 = H_1$, $F_2 = F_1$, $L_2 = L_1$.

By executing the LMI algorithm to the stability constraints of Theorem 9.2, we obtain the following feasible solution:

$$P = \begin{bmatrix} 0.0022 & -0.0040 & -0.0020 \\ -0.0040 & 0.0170 & -0.0002 \\ -0.0020 & -0.0002 & 0.0181 \end{bmatrix}, R_1 = \begin{bmatrix} 0.0574 & -0.0231 & -0.0028 \\ -0.0231 & 0.0633 & -0.0007 \\ -0.0028 & -0.0007 & 0.0173 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.0556 & -0.0162 & -0.0020 \\ -0.0162 & 0.0378 & -0.0001 \\ -0.0020 & -0.0001 & 0.0050 \end{bmatrix}, M_1 = [-0.5767, -0.1434, -0.0351],$$

$$M_2 = [-0.6539, -0.3067, -0.0190], \varepsilon_{11} = 0.4817, \varepsilon_{12} = 0.5714, \varepsilon_{22} = 0.4523.$$

Then the desired feedback gains are obtained by $K_i = M_i P^{-1}$:

$$K_1 = [-638.3665, -160.8648, -74.9890], K_2 = [-757.0267, -198.7586, -87.7825].$$

Figure 9.1 shows the controlled trajectories of the closed loop system:

$$\dot{x}(t) = \sum_{i,j=1}^{2} \mu_{\widetilde{M}_1}(x(t))\mu_{\widetilde{M}_2}(x(t))[(A_i + H_i F_i L_i + B_i K_j)x(t) + A_{di}x(t - d_i(t)) + E_i w(t)]$$

where the disturbance signal $w(t)$ is set to $w(t) = \exp(-t + \cos t)$, and the initial state is given by $x(t) = [-2, 5, 3]^T$ for $t \leqslant 0$.

Substitute the attenuation level for a smaller one, and solve the related LMI constraints (9.12) and (9.13) until they are infeasible, we obtain the minimal admissible attenuation level $\gamma = 1.5323$. The feasible solutions with respect to this minimal attenuation level are as follows:

$$P = \begin{bmatrix} 0.0029 & -0.0025 & -0.0019 \\ -0.0025 & 0.0113 & -0.0002 \\ -0.0019 & -0.0002 & 0.0125 \end{bmatrix}, R_1 = \begin{bmatrix} 0.0024 & -0.0088 & -0.0002 \\ -0.0088 & 0.0351 & 0.0004 \\ -0.0002 & 0.0004 & 0.0104 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.0010 & -0.0028 & 0.0006 \\ -0.0028 & 0.0188 & 0.0002 \\ 0.0006 & 0.0002 & 0.0023 \end{bmatrix}, M_1 = [-0.2658, -0.1787, -0.0647],$$

$M_2 = [-0.2725, -0.0491, 0.0721]$, $\varepsilon_{11} = 0.3273$, $\varepsilon_{12} = 0.1808$, $\varepsilon_{22} = 0.3818$. That is $K_1 = [-156.5104, -51.4328, -29.8173]$, $K_2 = [-134.7061, -34.8254, -15.3390]$. Simulation for the minimal attenuation level ($\gamma = 1.5323$) is shown in Figure 9.2, where the disturbance signal $w(t)$ and the initial condition are the same as in Figure 9.1.

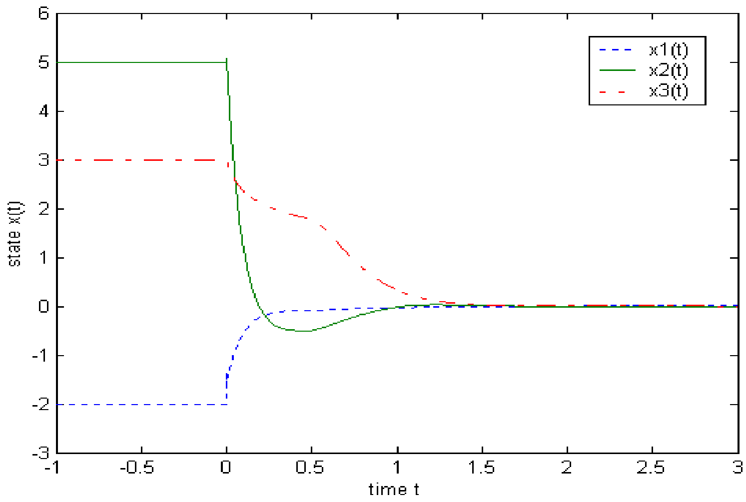

Figure 9.1: Controlled trajectories with disturbance attenuation level $\gamma = 2$

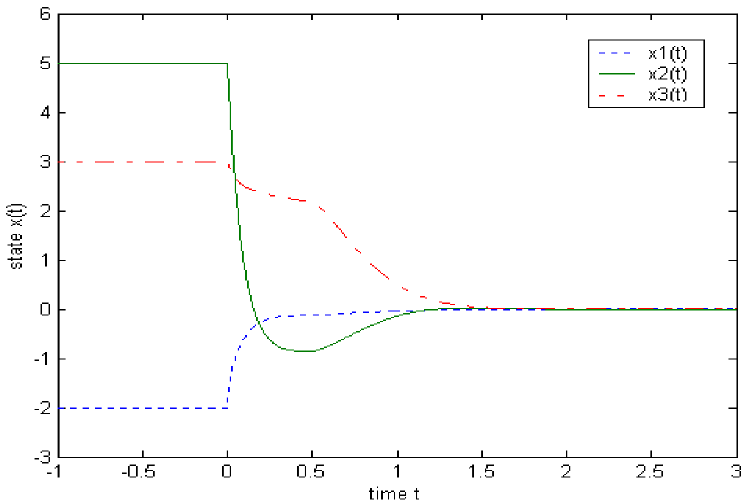

Figure 9.2: Controlled trajectories with the minimal attenuation level $\gamma = 1.5323$

Appendix

Introduction to LMI Problems

The history of linear matrix inequality techniques can be traced back to 100 years ago, when the first linear matrix inequality (i.e. $A^TP + PA < 0$) appeared in about 1890. Since then, a variety of approaches for solving linear matrix inequalities are proposed in the literature, such as graphical method, algebraic Riccati equation method, convex programming and interior-point algorithms [3]. Now, the linear matrix inequality technique is widely utilized in control context. The following introduction is based on the LMI control toolbox in Matlab.

A linear matrix inequality (LMI) is a constraint of the form:

$$A(x) := A_0 + x_1A_1 + ... + x_NA_N < 0 \tag{18}$$

where $x = [x_1, x_2, ..., x_N]^T$ is a vector of scalar variables, and $A_0, A_1, ..., A_N$ are the given symmetric matrices. Note that $A(x) < 0$ and $A(y) < 0$ imply $A(\frac{x+y}{2}) < 0$, i.e. (18) is a convex constraint with respect to variable x, thereby, finding the feasible solutions of (18) is essentially a convex optimization problem.

In most control applications, the resulted LMIs often have the form:

$$F_i(X_1, X_2, ..., X_M) < 0 \quad (i = 1, 2, ..., r) \tag{19}$$

where $F_i(.)$ $(i = 1, 2, ..., r)$ are affine functions of the structured matrix variables $X_1, X_2, ..., X_M$. It is easy to see that (19) can be formulated into the standard form of (18) equivalently by defining the scalar variables $x_1, x_2, ..., x_N$ as the independent entries of $X_1, X_2, ..., X_M$. In fact, the LMI solvers in LMI control tool box are so designed as to be based on this structured form of (19) rather than the form of (18).

There are the following three types of standard LMI problems. The corresponding LMI solvers in Matlab are designed by means of Nesterov and Nemirovski's Projective Method described in *Interior Point Polynomial Methods in Convex Programming: Theory and Applications,* SIAM, Philadelphia, 1994.

1). LMI feasibility problem

That is, to find a solution x, if it exists, satisfying the LMI constraint:

$$A(x) < 0.$$

The LMI solver for LMI feasibility problem is $feasp(lmis, options, target)$, where $lmis$ stand for the LMI constraints, $options$ is a optional five-entry vector of control parameters (Default=$[-,\ 10^2, 10^9, 10, 0]$), and $target$ is an optional objective value for termination (Default=0).

Specially, for the linear matrix inequalities of type $M + P^T XQ + Q^T X^T P < 0$, the feasible solution of X can also be directly solved by *basiclmi* (M, P, Q).

2). Linear objective minization problem

$$\begin{cases} \min imize & c^T x \\ subject\ to & A(x) < 0 \end{cases}.$$

This problem can be solved by the LMI solver $mincx(lmis, c, options, xinit, target)$, where *xinit* is a optional guess for x, the default value of *options* is $[10^{-2}, 10^2, 10^9, 10, 0]$ and the default *target* is -10^{20}.

3). Generalized eigenvalue minization problem

$$\begin{array}{ll} \min imize & \lambda \\ subject\ to & \begin{cases} A(x) < \lambda B(x), \\ B(x) > 0, \\ C(x) < 0 \end{cases} \end{array}.$$

The corresponding LMI solver is $gevp(lmis, nlfc, options, \lambda init, xinit, target)$, where $nlfc$ stands for the number of LMIs involving λ, the entries $\lambda init$ and *xinit* are optional initial guesses for λ and x, the default *options* is $[10^{-2}, 10^2, 10^8, 5, 0]$ and the default *target* is -10^5.

Both the LMI feasibility problem and the linear objective minization problem are convex problems. But the generalized eigenvalue minization problem is no longer a convex problem, it is quasi convex. It is to note that the LMI feasibility problem can be reduced to the generalized eigenvalue minization problem. Moreover, $A(x) \leqslant 0$ type matrix inequalities can also be solved via the LMI solver of the generalized eigenvalue minization problem, since $A(x) \leqslant 0$ is feasible if and only if the minimum $\lambda_{\min} \leqslant 0$, where $\lambda_{\min}$ is the solution of λ such that: $\begin{cases} \min imize & \lambda \\ subject\ to & A(x) < \lambda I \end{cases}.$

More detailed descriptions on LMI problems can be found in e.g. [4], [3] and [27].

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