Stock Markets as Evolving Complex Systems.
Simulations and Statistical Inferences.

Dissertation

zur Erlangung des akademischen Grades eines Doktors der
Wirtschaftswissenschaften Dr.rer.pol
durch den Fachbereich Wirtschaftswissenschaften der Universität
Duisburg-Essen, Standort Essen

vorgelegt von
Dipl-Volkswirt Hans-Jürgen Holtrup aus Dorsten

Tag der Prüfung: 18. Januar 2006

Erstgutachter: Prof. Dr. W. Gaab
Zweitgutachter: Prof. Dr. Assenmacher

Contents

Chapter 1

Introduction

Financial markets have always been playing an important part in economic research. Many economists see financial markets, whether stock, foreign exchange or future markets, as prime examples of complete markets. Information that might be important for the value of the traded asset is quickly propagated through the media. There is no personal affection for specific equities assumed, and the focus is only to buy cheap and to sell high. Furthermore, shares can be traded without personal contacts by placing buy and sell orders. This in turn should reduce transaction costs by a considerable amount. In fact, financial markets can claim to posses a very efficient mechanism of finding trading partners. In short, financial markets among all markets should be the place where prices come nearest to fully reflect the opinions of the participants. These are moreover supposed to have perfect knowledge about the intrinsic values of each equity. The *Efficient Market Hypothesis* is a logical consequence of these circumstances.

However, the importance of financial markets does not only come from this more theoretical statement. Financial markets are also important in financial intermediation. For example, stock markets allow an efficient risk sharing as stressed by Diamond (1967). They also provide incentives to gather information, which drives stock prices more closely to its true values. These market prices then provide signals for an efficient allocation of financial capital (see e.g. Diamond and Verrecchia (1981)). A more practical point is that financial markets offer a chance to make profits. If it would be possible to forecast future price developments, short or long positions should yield high profits. However, this possibility collides with the theoretical assessments. If financial markets really reflect the whole information, then prices cannot be forecasted because only new information alters the prices.[1] Recent years witnessed a lively debate about models of financial markets that are somewhat in-between these conflicting positions. Empirical and theoretical challenges to the efficient view have come up

[1] i.e. future prices are unknown. Principally, one can make a forecast based e.g. on pure intuition. But this is not the meaning of unforcastable above.

with competing views about traders who do not act in the fully rational way assumed by the protagonists of the efficient view. On the other side, evidence for a possibility to employ technical analysis (charts) in order to forecast prices is - at least - very sparse. So the search to find a realistic picture of the processes in financial markets is still ongoing.

One promising advance is made by the introduction of psychological explanation for indiduals' action in financial markets. Here, peoples motivation is analysed and experimental as well as theoretical results are then transformed to explain the dynamcis within e.g. stock markets. Other lines of finance theory focus on the specific microstructure of markets, or try to rationalise common behaviour by introducing some form of private information that only some market participants posess. However, while these directions are able to explain some specific empirical features of financial markets, they cannot account for the more general behaviour of asset prices.

The following work is inspired by the ideas of several theoretical physicists. They developed propositions about financial markets so as to interpret them as examples of complex self-organising buildings, similar to many other natural systems. They stress the fact that some systems, physical, social or financial, display similar statistical properties, which cannot completely explained by exogenous factors. Historical events like the famous Tulipmania bubble or the south see bubble feed the assumption of an endogenous reason for large price fluctuations, because they show no signs of fundamental exogenous reasons. To be more precise, physicist claim to have found some universal statistical features that prevail in every system that consists of a large number of interacting members. For financial markets, the members are the people who trade assets and the interaction is usually interpreted as the communication that takes place between them. These members, so the hypothesis, build a network that in few cases work so as to align all traders to behave in the same manner, thus creating a herd that produces bubbles and crashes.

This work in divided into three parts. The first shortly summarises the Efficient Market Hypothesis and its principal empirical shortcomings as well as the competing theoretical line called Behavioural Finance Theory. The presentation of a new idea based on the theory of complex systems completes part one. The second part analyses the main statistical facts of financial markets. Because these empirical characteristics are the yardstick with which proposed new models have to be compared, it is essential to have a precise picture of what should be targeted. The last part presents a new framework to model stock markets. It is based on the idea that these markets consist of many heterogenous interacting traders. These traders determine through their actions the price dynamics. The simulations of part three will try to use this concept in order to convert it into numerical models that reproduce the facts of part two. It must be stressed that the provision of some new empirical estimations and two new variants of simulations are not the sole contribution of this work. It also aims to give an overview of the whole concept of statistical physics and its application to economic problems. There are by now some notetably introducory books published (Mantegna and Stanley (2000), Bouchaud and Potters (2000) and Lévy, Lévy and Solomon (2000) among others), but none of these tries to give a complete picture that connects the empirical facts with the numerical simulations. They focus either on the statistical featurs of financial markets or its simulation.

Part I

Efficient Markets and other Concepts

Chapter 2

The Efficient Market Hypothesis and its Challenges

The notion of efficiency in financial markets has a long tradition. The idea behind the term, originally coined by Harry Roberts (1967), goes back to Gibson (1889) and Bachelier (1900) gives a first mathematical treatment of the subject.[1] The concept of the *Efficient Market Hypothesis* (henceforth EMH) in its most general form claims that prices of financial assets reflect all relevant information, or as Mandelbrot (1971, p. 225) explains: "Roughly speaking, a competitive market of securities, commodities or bonds may be considered efficient if every price already reflects all the relevant information that is available. The arrival of new information causes imperfection, but *it is assumed that every such imperfection is promptly arbitraged away.*" As this efficiency concept involves the modellation of information, the expression of information efficiency is also frequently used to characterise efficient financial markets (as opposed to other familiar notions of economic efficiency like, for example, Pareto-efficiency).

In its strongest interpretation, individuals do not have different comparative advantages in information acquisition. All people trade on the same complete information set that even includes inside information. Because this reading demands an ability of information gathering that is rarely met in reality Fama (1970) divides the EMH into three categories depending on the information set:

[1] See Shiller (1998). In fact, Bacheliers thesis *Theorie de la Speculation* already included the idea of a martingale measure for the evalutation of assets. He explicitely modelled the markets prices as a continous Markov process. Bachelier was also the first who developed many of the mathematical properties of Brownian Motion - five years prior to Einstein's famous work on the same subject (1905). For a short review of Bacheliers work see Courtault et al. (2000). Other early works on the topic include Williams (1938) and Graham and Dodd (1934, 1996).

(i) *Strong form of market efficiency*
There is no public or even private information that will allow an investor to earn abnormal returns based on that information. It is assumed that *all* information is available to everyone at the same time cost-free, i.e., a *perfect* market exists.

(ii) *Semistrong form of market efficiency*
There is no public information that will allow an investor to earn abnormal returns based on that information. Public information includes all stock market information *plus* all publicly available financial, economic, or other type of information on the specific company, the national economy, the world, etc. Security prices react immediately to all new information.

(iii) *Weak form of market efficiency*
There is no information in past stock prices (of a particular asset) which will allow an investor to earn abnormal returns (from that particular asset) based on that information. Stock market information includes stock prices as well as relevant macroeconomic and firm specific data.[2]

The concept of efficient markets as stated above is an appealing idea since it is difficult for an economist to sustain the case that agents in financial markets do not behave rationally and maximise their profits by processing all available information. On the other side is it hard to imagine that traders live in a perfect rational world where psychology plays no role at all. Actually, even strong supporters of the rational behaviour paradigm would accept some irrational beliefs as a factor of influence at least to some of the market participants. But the problem with these often called *noise traders* is that they are buying overpriced while selling underpriced assets. As a consequence, their profits are lower than those of smart traders who make greater profits by exploiting arbitrage deals. As Friedman (1953) noticed, this is not a situation that can last forever, because noise traders will eventually leave the market because of permanently losing against the rational actors. Through this process, the EMH should be restored at least in the middle-run.

The concept of informational efficient markets is closely associated with a probabilistic handling of the subject. Economists use the concept of martingale theory to formalise the idea of an informational efficient market in an elegant and compact manner.[3]

Definition 2.1 (Martingale):

Let $\Omega = (\Omega_t)_{t \in T}$ be a family of information subsets of T up to the time index $t < T$, and let $E[x_t|\Omega_s]$ be the expectation of x_t

[2] In this form, informational efficient market requires that the costs for gathering information and trading are zero (Grossmann and Stieglitz (1980)).

[3] A rigorous treatment can be found in Doob (1953) or Billingsley (1976). The first who used martingales as a description of asset prices where Samuelson (1965) and Mandelbrot (1966).

conditional on the information available up to time index $s < t$, where $E[.]$ is taken with respect to a probability measure P. Then a stochastic process $x = (x_t)_{t \in T}$ is called a martingale, if $E[x_t] < \infty$ and $E[x_t|\Omega_s] = x_s;\ s < t\ \forall\ s, t$. x is called a supermartingale if $E[x_t|\Omega_s] \leqq x_s$, and a submartingale if $E[x_t|\Omega_s] \geqq x_s$.[4]

A martingale states that the best forecast in t of the price of an asset in $t+1$ using all available information up to t is simply its value in t. This is exactly the notion of EMH as explained above. There might be some forecast errors at any time, i.e. if the price in t, P_t, replaces the variable x_t, the difference $P_t - E_{t-1}[P_t|\Omega_t]$ is non-zero. But these errors are not correlated and have an expectation value of zero. This implies only unanticipated impulses to cause the actual value of the asset to differ from its value one time step before. For super- and submartingales this assumption must not to hold. Therefore martingale theory is able to account for the empirical observed upward trend in asset prices. In this case a submartingale would characterise the process of P_t. It is important to note that martingales say little about the probability density function which the random variables have to obey. The next chapter offers the three main assumptions about the process that fits the martingale definition from above.

2.1 The Random Walk Hypothesis

The concept of a random walk is nowadays a solid integral part of finance theory. It is closely related to martingale theory and constitutes another formal way to express the EMH. According to the random walk hypothesis (RWH henceforth), the dynamics of the price process are given by

$$P_{t+1} = \mu + P_t + \varepsilon_t, \tag{2.1}$$

where $\mu > 0$ is a drift parameter. The crucial point lies in the assumption about the term ε_t. ε is a disturbance term that represents the news. Now, the most common assumption about those news demands that ε is identically and independently distributed (*iid*), $\varepsilon \overset{iid}{\sim} (0, \sigma^2)$ with mean 0 and variance σ^2. Campbell et al. (1997) call this the RWH I. It is in fact not unreasonable to suppose that news coming from the whole spectrum of relevant events (like the invention of a new procedure for the production, the announcement of an expansive monetary policy or the release of the latest figures on unemployment) do not depend on each on other. There is a another more pragmatic reason for the widespread use of this assumption. Considering the sum S_n of n *iid* random variables,

$$S_n = \sum_{i=1}^{n=\infty} \varepsilon_i. \tag{2.2}$$

[4]See Billlingsley (1976, p. 407).

In the context of asset prices, S_n can be interpreted as the accumulated value of incoming news up to time index n. By successive backward substitution in (2.1), S_n can be replaced by the current price P_t as the cumulation of all past news plus drift term:

$$P_t = \sum_{i=1}^{n=\infty} \varepsilon_i + \mu. \tag{2.3}$$

From the *iid* assumption of ε it follows that its mean and variance is given by

$$E\left[\varepsilon\left(n\Delta t\right)\right] = \sum_{i=1}^{n} E\left[\varepsilon_i\right] = 0 \tag{2.4}$$

and

$$E\left[\varepsilon^2\left(n\Delta t\right)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\varepsilon_i\varepsilon_j\right] = E\left[\varepsilon_i^2\right] = n\sigma^2 \tag{2.5}$$

respectively. Setting $t \equiv n\Delta t$, the mean and variance of (2.1) is expressed through $E[P_t] = \mu t + P_0$, for an arbitrary initial value of the price in $t = 0$ and $E[P_t]^2 = Var[P_t] = n\sigma^2 = \frac{\sigma^2}{\Delta t}t$. In the continuous limit for $n \to \infty$ or $\Delta t \to \infty$ (so that t is finite) and by defining $\sigma^2 \equiv D\Delta t$ one has finally $Var[P_t] = D\Delta t$, where D is called the diffusion constant. If the (in the limit infinitely many) contributions to S_n are indeed independent and identically distributed, statements about the form of the resulting probability density function (*pdf* henceforth) can be derived by the *Central Limit Theorem*. It states that, after normalising S_n by its standard deviation$\sqrt{\sigma^2} = \sigma_n$,

$$\widetilde{S_n} = \frac{S_n}{\sigma_n}, \tag{2.6}$$

$\widetilde{S_n}$ results in a Gaussian *pdf* of the form

$$P\left(\widetilde{S_n}\right) = \frac{1}{2\pi} e^{\left(\frac{-\widetilde{S_n^2}}{2}\right)}. \tag{2.7}$$

This implies for the price process with

$$\Delta P_t = P_t - P_{t-1} = \varepsilon_t + \mu \tag{2.8}$$

that the *pdf* of ΔP_t for $t \to \infty$ also follows a Gaussian distribution.[5]

The distributional outcome underlying the RWH I simplifies many calculations, but has one inevitable disadvantage: if ΔP_t obeys a Gaussian distribution, then there is always a positive probability that prices become negative. To avoid such impossible outcomes, one often encounters the assumption that the natural

[5] One should be aware of the fact that this normal distribution affords an infinite support which is not given by real data! An answer to the question of how fast this random walk convergences to the Gaussian can be found in Bérry (1941) and Esséen (1945).

logarithm of the price, ln $P_t = p_t$, follows the above described random walk, hence

$$p_{t+1} = \mu + p_t + \varepsilon_t, \; \varepsilon \overset{iid}{\sim} \left(0, \sigma^2\right).^6 \qquad (2.9)$$

Equation (2.1) and its logarithmic equivalent (2.9) has to be understood as a theoretical model, a hypothesis about evolving asset prices. The question is whether the assumption of independent *and* identical news is realistic? Considering a time span of more than one hundred years of stock price history, one has to concede the arrival of many new events during that period: technology-pushes, new information techniques etc. It is hard to imagine that the disturbances ε during that century all came from a never changing environment.

In order to meet this point, a broadening of the RWH I assumption is adequate. Following Campbell et al. (1997) a new proposition of a still independent but no longer identically distributed news process is introduced under the name of the random RWH II. This opens up the set for other important distributions like the class of Lévy-stable distributions (which includes the Gaussian as a special case). Although this version of the RWH is weaker than the former, prices are still not forecastable (besides the drift parameter μ as in the RWH I). The RWH III constitutes a further generalisation. But here the disturbances are no longer independent. Instead, ε is allowed to show a non-zero correlation in their squared value, i.e. $Cor[\varepsilon_t^2, \varepsilon_{t-j}^2] \neq 0$. Because temporal correlation between the disturbance terms is still zero, the RWH III retains the martingale property and so still does not contradict the EMH. Important models that display such behaviour are the ARCH (**a**utoreg**r**essive **c**onditional **h**eteroscedasticity.) and GARCH (**g**eneral **a**utoreg**r**essive **c**onditional **h**eteroscedasticity.) models introduced by Engle (1982) and Bollerslev (1986).

2.2 Theoretical and Empirical Challenges to the EMH

Although the EMH needs only a fraction of all traders to be rational in order to drive prices to its fundamental value, even this more flexible idea can be challenged. There are three major directions of criticism in the economic literature. The first tackles one of the cornerstones of EMH, i.e. agents act fully rational (at least those who eventually bring the market back to the fundamental value). In this sense, it is a rather principal criticism in that it raises general doubts about the possibility of a purely rational behaviour. A second line asserts that

[6] However, the argument that the log-normal distribution should be preferred because of the possibility of eventually negative prices for the Gaussian is not particular convincing according to Bouchaud and Potters (2000). They stress the fact that both statistics are just approximations to reality, and so both can claim validity only for a certain range of values. Furthermore, for volatilities of up to 20%, the two distributions look very similar (see Bouchaud and Potters (2000, p. 9ff.)).

it is feasible to have non-rational agents permanently participating or even dominating the market. In fact, this branch of finance theory helps economists to explain some empirical facts that contradict the EMH. These facts, also called anomalies or puzzles, constitute the third line of criticism.

2.2.1 Bounded Rationality

Extreme models of individual's unbounded rationality assume that economic decision markers

(i) have knowledge about all possible alternatives including their outcomes and the strategies needed to achieve or realise them;

(ii) their preference ordering is

- · complete, which implies the ability to rank each alternative,
- · monotone, which means that more is better, and
- · transitive, which means that if alternative A is as good as B and B is as good as C, then A must also be as good as C;

(iii) they are able to decide which alternative is their most preferred.[7]

Despite being the by far most frequently applied concept of modelling decisions, there have always been critics. Their main objection is that the axiomatic presentation of people's behaviour does not meet the real situation, where humans regularly miss their theoretical optimum because of imperfections.

Simon (1957) is one of the first economists who argues that bounded rationality is the rule rather than an exception. His assertion rests upon an alleged imperfectly known set of decision possibilities. This is due either to the costs of collecting and storing information and/or to a natural limit in the processing of this information into decision proposals. Simon stresses that in several situations the number of strategies is too high to evaluate every single outcome, which he illustrates with many examples including chess. Financial markets can serve as an additional example. Considering the task of a trader who has to evaluate all incoming news (supposing he really gets them without any costs) on their exact impact on a portfolio of assets. Is it e.g. a realistic view to suppose that anybody is able to predict the future net value of a new invention that has not yet entered the market? Practitioners would hardly agree with that perfect view.

Baumol and Quandt (1964) consider the consequences of costs in the process of acquiring, calculating and refining the necessary information for making decisions in a changing environment. They conclude that it is not rational to consider every available factor that is potentially able to influence the choice of action. Instead, it might be sensible to concentrate on the major points if the

[7] See for example Tisdell (1996).

acquisition of additional information is connected with high cost. This of course is just an application of a well known economic rule: engage in a costly activity as long as it produces an outcome that gives more utility or profit. In the limit, the marginal cost should equal the marginal benefit. Given the rational incorporation of information costs, it is a simple but realistic assumption that in most decisions a lot of other useful information is left apart. In this situation, rules of thumb and habits might serve as a guide through a not entirely known set of decision possibilities.

There is a fundamental objection to the EMH that is based on the so-called *Neo-Austrian theory of market processes* which fits the Simon critique quite well. According to this theory a competitive market is a systematical set of forces that evolves through the entrepreneurial alertness (i.e. the eagerness to make profits). These dynamics tend to reduce mistakes from market participants but never eliminate them completely since this would end in a stagnant market. The crux of the story is that investors might detect the market failure but hesitate to exploit this knowledge unless they realise the nature of the inefficiency. Confronted with this situation investors may use rules of thumb to beat the market. This is a situation very similar to financial markets.[8]

A weaker version of rationality accepts the impossibility of possessing perfect information, yet assumes an unbounded rational decision process. These models are based on the *Expected Utility Maximisation Theory* (EUT).[9] According to this theory, people face alternatives that are not sure in their outcome and so bring a form of uncertainty into the decision process. The individuals now chose the alternative that leads to the highest expected utility. Although being a universally accepted decision rule, some problems still remain. One of the most serious caveats concerns the way in which probabilities are built. The EUT takes probabilities for the uncertain events as given and interprets this as a case of risk involved in the decision making process. It does however, not say anything about how these probabilities are calculated. Of course, people can build probabilities by sequentially updating them according to Bayes' rule, but this behaviour is strictly taken not included in EUT. So personal and logical (i.e. objective) expectation need not coincide. This leaves the EUT with some coign of attack.

The so-called subjective EUT (SEUT) focusses on the building of the beliefs. This concept roots in the mathematical and philosophical tradition of subjective probabilities (see Ramsey (1931) and de Finetti (1937); see also Fine (1973) for a survey). It has the following problem: individuals are usually able to give reasons for their decisions, although the basis of the decision may not be rational when seen under full information. Taking the SEUT literally, every bubble and crash can be justified as long as market developments are in accordance with the

[8] Uncertainly over the economic nature of the mispricing impedes arbitrage (see Merton (1987)).

[9] See von Neumann and Morgenstern (1944) and Savage (1954).

subjective beliefs of the traders. However, this view is outside the possibility of an empirical falsification.[10]

2.2.2 Market Anomalies

After the seventies of the last century, some disturbing anomalies have been discovered and over the years the list of unusual empirical facts got considerable longer. Just to mention a few, over- and underreaction to news releases, excess volatility, the small firm effect etc.[11] These peculiarities are hard to reconcile with market efficiency, since they allow for systematic predictions in asset prices and returns. The next sections present some of the well known puzzles.

2.2.2.1 Time Patterns in Security Prices

Intraday and day-of-the-week patterns are nowadays widely accepted empirical features of stock markets. Both refer to the same phenomenon, a relatively regular up and down in the prices. For example, especially Mondays appear to have negative returns. Cross (1973) and French (1980) found Monday price indices on the Standard and Poors composite index to be significantly lower than their Friday closing values. Using the Dow Jones index, Gibbons and Hess (1981) are able to confirm this result.

Concerning monthly pattern, it is noticed that the daily returns on common stocks of the first five to seven days of January are considerably higher than their performance during the rest of the year, in particular for small firms (see Kinney and Rozeff (1976), Keane (1983) and Fama (1991)). There are explanations for the January effect like the usual year-end selling of stocks in order to realise capital losses thus reducing tax payments. But, if the phenomenon would be stable over time (as it actually is), there should be at least some smart investors who notice this fact and accordingly buy stocks in December when they are cheap and sell them in the first week of the new year. This would give them a clear edge over the market as long as their actions do not equate prices and eliminate the effect, which must ultimately happen according to theory because of a growing number of traders who will capitalise on the regularity. The same argument could be applied to the Monday effect, so the question remains why these anomalies still exist.

2.2.2.2 Excess Volatility

In 1981 Robert Shiller published a work on volatilities of the Standard and Poors (S&P) Composite Stock Price Index from 1871-1979. His concern was to compare historical volatility with what would be expected by a simple efficient market model. The basic idea of the test is as follows: Shiller employs a version

[10] Besides this, unbounded rationality relies heavily on the completeness of individuals preference ordering. Unfortunately experiments have shown that this is not always given in reality. For a detailed exploitation of bounded rationality see Tisdell (1996) and Gigerenzer and Selten (2002).

[11] See Siegel (1998) for a review of those market anomalies.

of efficient market model that claims asset prices to equal all discounted expected dividends, i.e.

$$P_t^* = \sum \rho^{k+1} E_t[D_{k+t} \mid \Omega_t], 0 < \rho < 1, \tag{2.10}$$

where P_t^* is the fundamental price of the asset, ρ denotes the discounting factor, D stands for dividends and Ω_t is the complete information set available at time t. Because $E_t[D_{k+t}]$ is not known at time t, it has to be forecasted. The EMH now asserts that the price is equal to the best forecast of the sum of future dividends. Since all information about these future dividends are already incorporated in the price, i.e. $P_t = E[P_t^*]$. At each subsequent time step new information on dividends enters the market so that $P_t^* = P_t + \varepsilon_t$, where ε is supposed to be *iid*. Because the variance of $(P_t + \varepsilon_t)$ is the sum of the variance of P_t plus the variance of ε_t , it follows that variance of P_t^* cannot be greater than the variance of P_t. Shiller is able to calculate a time series for P_t^* with the help of historic data for dividends from 108 years ago onwards. By comparing the variance of P_t^* and P_t respectively, he claims that stock prices exhibit a volatility, too high to be explained by the fluctuations in dividends. Due to the nature of (fundamental) prices as being the sum of a moving average, changes in dividends would have to be huge in order to alter P_t substantially. However, such large dividend changes are missing in the data. In order to meet critics who object the negligence of time-varying real discount rates, Shiller (2003) sets ρ equal to historical interest rates. However, conclusions remained the same. Prices are still too volatile to be in accordance with the dividend model.[12]

2.2.2.3 The Equity Premium Puzzle

The equity premium puzzle, first introduced in the seminal paper of Mehra and Prescott (1985), is one of the most cited examples for the failure of full rationality in financial markets. There are indeed only few investors left who have not recognised that stocks have outperformed bonds by a considerable margin if measured over long horizons. Picking up this empirical fact, Mehra and Prescott argue that the equity premium, the amount by which the (average) returns to stocks exceed the riskless returns is inconsistently high to be explained by the riskiness of common stocks. Employing a variation of Lucas' (1978) exchange economy with two assets, Mehra and Prescott calculate risk premia by varying the values of risk aversion and the discount factor. Their maximum simulated value is 0.35%, which is remarkably low compared with the historical premium of 6.18% calculated from relatively risk-less short-term securities (0.8%) and the average real return on the Standard and Poors 500 Composite Stock Index in the period between 1889-1978 (6.98%). Since 1985 a lot of work has been devoted to solve this puzzle. Suggested explanations go in several directions. Epstein and Zin (1989) and Weil (1989) consider a so-called recurse utility function and

[12] This result is bolstered by LeRoy and Porter (1981) who come virtually to the same conclusion. Moreover, employing different models of asset prices, like the consumption discount model of Breeden (1979) does not alter the general impression (see Mankiw et al. (1985)). For an analysis of the German Stock Market from 1876-1990 see DeLong and Becht (1992).

assume that intertemporal utility fits into this set-up. By further assuming that risk preferences exhibit first-order risk aversion they are able to reach a premium of 2%, still less than the Mehra-Prescott outcome. Constantinides (1990) uses a habit formation model to resolve the puzzle. Campbell and Cochrane (1999), employing this habit persistence formulation are able to show that by calibrating their model it is possible to fit the high historical equity premium. Unfortunately, by doing so new puzzles arise namely the long-run predictability of stock and bond returns and high volatility of asset prices despite tranquil and unpredictable dividends.

Other lines of reasoning allow transaction costs to enter the considerations (see Heaton and Lucas, 1995). But Kocherlakota (1996) shows that this explanation needs a very improbable 600% price of trading costs in order to solve the puzzle. Further attempts are made by Brown, Goetzmann and Ross (1995), where survivorship bias plays an important role, and by Constantinides, Donaldson and Mehra (2001), who consider a three stage life-cycle model where the young underlie a borrowing constrain, the middle-aged save and the old dissave. Unfortunately, none of these works remedies the still existing problem (see Kocherlakota (1996) Siegel and Thaler (1997) and Mehra (2001)).

2.2.2.4 Other Anomalies

There are several other anomalies that have not been mentioned so far. For example, Jegadeesh and Titman (1993) and Chan, Jegadeesh and Lakonishok (1996) find evidence that in the medium run (i.e. 3-12 months) prices are positively autocorrelated. This is called a Momentum strategy. A related phenomenon is the Contrarian strategy (see De Bondt and Thaler (1985) who found that portfolios with a bad performance (looser-portfolios) in the past are able to become winner-portfolios with a better than average performance in the future). In a more detailed analyses Lakonishok, Shleifer and Vishny (1994) examine the strategy to buy stocks that have low prices relative to earnings, dividends, historical prices or book assets. Liu, Strong and Xu (1999) can show that the most profitable strategy lies in a ranking on the basis of the last 12 months. Investing in a winner-loser portfolio for months yields an annualised return of 19.5% on UK stock prices for a data set from 1977-1996.

In an early paper on anomalies, Basu (1977) identifies Price/Earnings (P/E) ratios as predictors of subsequent performance. In particular high P/E firms underperformed while low P/E firms overperformed if compared with averaged values.[13] Underreaction and overreaction in security returns is another stylised fact encountered on several occasions. The literature comprises empirical works on Initial Public Offerings (Loughran and Ritter (2001)), Seasoned Equity Offerings (Loughram and Ritter (1995)), equity repurchases (Ikenberry, Lakonishok and Vermaelen (1995)) and stock splits (Ikenberry, Rankine and Stice (1996)). Daniel et al. (1998) and Fama (1998) provide summaries of a large number of

[13] See also Basu (1983) who finds a P/E-effect after controlling for firm size.

related event studies. Common to all these works is a delayed reaction to news. When new information comes into the market, theory predicts an instantaneous and correct response of the investors. Empirical research has come up with evidence against this view. In reality, people sometimes overreact to news, i.e. the price overshoots its new fundamental value. Reversion takes a time span during which the price fluctuations are negatively correlated with the first increment after the news event. Underreaction occurs when prices need some time to reach their new level. In this case, price changes would be positively correlated.

2.2.3 Big Crashes

The common picture of a crash is that of a sudden massive drop in asset prices, such as the fall in stocks of about one-third of its value between early to mid-october 1987. Cases like the *Black Monday* have always been a point of concern. Critics of the EMH argue that these instances cannot be explained by rational behaviour. Considering the RWH I with news distributed according to the Gaussian, the empirical distribution of $\Delta P_t = \varepsilon_t$ should mimic the normal distribution. Commensurate with this is a probability of 10^{-160} that a drop as large as the October 1987 event happens.[14] Although this might be astonishing low, it by no means constitutes a definite falsification of the EMH. Simply generalising the assumptions about the distribution of news gives rise to distributions that are able to capture even the largest drops found in history within a realistic probability value. But the problem with such extraordinary events is not only the very low probability of its occurrence. Far more important is the lack of an economic story. The EMH states that the price increments are only caused by unpredictable news. Therefore, big crashes and huge upward shifts should be accompanied by either a series of bad or good news coming in very quickly, or by one single important new incident. Critics object exactly the non existence of such economic relevant news. This is in fact the crucial point. Both parts, the supporters as well as the critics of the EMH, have to show either the news that lead to the crash or bubble or its missing. The next paragraphs give a short overview of three of the most severe examples of alleged irrational behaviour.

2.2.3.1 The Tulipmania

The arguably most famous bubble in the history of asset price recordings is the so-called tulipmania, happening in the 17-th century Holland. In his book *Extraordinary Popular Delusions and the Madness of Crowds,* Mackay (1852) gives an early description of the course of events leading to a climax in 1636 where a single tulip bulb had the value of twelve acres of building land. There he describes the attitude of even ordinary dutchmen to trade on tulips as "insensibly attached". Galbraith adds that "by 1636, a bulb of no previously apparent worth might be exchanged for a new carriage, two grey horses and a complete

[14] See Rubinstein (1988).

harness."[15] Although this incident seems to be an obvious example of a severe mispricing, Garber (1990, 2000) argues that the roar of tulip bulb prices was actually driven by fundamentals - or, to be more precise, by beliefs about fundamentals. In order to support his claim he gives several explanations that ought to show that prices were not that irrational as mostly believed.[16] The gist of Garber's story is that traders expected tulip prices to rise because of their inherent fundamental value. In fact, tulips became a luxury good soon after their arrival in Holland (in 1593), and those goods are always subject to the shaky taste of wealthy buyers. Garber's conclusion is that people assumed prices of rare bulbs to rise because affection to the beautiful patterns on the flowers grew rapidly. It might be true that these expectations turned out to be wrong in the end, but after Garber, they were based on sound facts.

This line of defence roots in the SEUT where probabilities can be built on subjective beliefs. In the context of the Tulipmania, prices where thought to have a high probability of rising. However, it is always able to find reasons for even the most extreme behaviour. And it is one thing to defend the idea of rational actions, but another to show that the tulipmania was indeed an example of such behaviour. Garbers logical arguments do not obscure the lack of new information for the period 1634-1637. Garber himself holds the view that the data basis on the 17-th century market situation and its related news is too sparse to judge by how far prices exceeded fundamental values. Moreover, he admits that in 1637 "common bulbs became objects of speculation among the lower classes in a future market (...)"[17]. Although, as Garber outlines, serious traders ignored this market, it is then an example of mispricing from amateur traders. But this raises the question why the professionals have not succeeded in bringing prices down by exploiting the severe mispricing. Furthermore, Garber cannot provide an explanation for what fundamentals actually were. While he nevertheless tries to make a point for rational explanations of the Tulipmania, Galbraith (1994) instead claims that a "mass mania" was the true cause. All ries of the dutch society borrowed money to feed the speculative bubble. When in 1637 some of the more serious and some of the very nervous began to detach, panic brought out and prices quickly began to tumble, leading to numerous

[15] See Galbraith (1994, p. 29f).

[16] These explanations include the Bubonic plague as one possible explanation for speculation. He cites Van Damme (1976, p. 38) in order to establish a link between the desease and the high prices: "In the midst of all this misery (the plague) that made our city suffer, people were caught by a special fever, by a particular anxiety to get rich in a very short period of time. The means to this were thought to be found in the tulip trade. This treade, so well konwn in the history of our country, and so well developed in our city should be taught to our fellow citizens as a proof of forefatherly folly." A not very convincing example of a rational behaviour. Garber (2000, p. 78) furthermore argues that the fast declines were typical for newly developed bulb variates: "Single bulbs in the eighteenth century commanded prices as high as 1000 guilders. In this context, the 1000-2000 guilder price of Semper Ausgustus from 1623 to 1625 or even its 5500 guilder price in 1637 do not appear obviously overvalued." But this is not a proove of a fundamental value but merely the documentation of other high prices. Interpreting this different: he has discovered other bubbles.

[17] Garber (1990, p. 39).

bankruptcies. Shiller (2000) shares this view when he expresses his doubts on the Garber view. He stresses the fact that Garber is unable to offer a satisfactory story based on fundamental reasons.

2.2.3.2 The Crash of 1929

On October 28, 1929, the Dow Jones Industrial Average realised a drop of 12.8% in one day. This was followed by a further 11.7% decline on the next day. Until 1987 there has never been a more dramatic daily fall in asset prices as on those two days. Up to now a lot of stories have been told to explain the 1929 crash and the events leading to it, but none has solely been based on fundamental reasons. Shiller (2000) passes through the newspapers of the critical days in order to find the news that might have been responsible for the big drop - without success. He concludes that no current news were important enough to account for the big decline, but maybe an event occurring a few days before: a temporary drop of the Dow of 12.9%. This was not a daily record since the market experienced a roaring last hour before closing, thereby reducing the loss to just 2%. Shiller argues that the sudden decline stuck in the memories of all traders. So albeit the fact that the newspapers reported sound market conditions, a phase of falling prices on the October 28 causes fears of a drop similar in strength to the one happening four days before. As a result, everybody tried to get rid of his assets as soon as possible, thereby initiating the crash.

An extensive review of the 1929-crash is provided by Kenneth Galbraith (1997). There is no doubt that he blames the exaggerated mood of the speculations that was not based on fundamental developments. Dismissing external factors like the weakening in industrial production and other economic indices, Galbraith sees the development of asset values in the late twenties as driven by expectations about rising prices. There had been of course supportive circumstances like the huge savings, the easy to get credits and the low interest rates, but "(f)ar more important ... is the mood. Speculation on a large scale requires a pervasive sense of confidence and optimism and conviction that ordinary people were meant to be rich. People must also have faith in the good intentions and even in the benevolence of others, for it is by the agency of others that they will get rich. In 1929 Professor Dice observed: 'The common folks believe in their leaders. We no longer look upon the captains of industry as magnified crooks. Have we not heard their voices over the radio? Are we not familiar with their thoughts, ambitions, and ideals as they have expressed them to us almost as a man talks to his friend?' Such a feeling of trust is essential for a boom. When people are cautious, questioning, misanthropic, suspicious, or mean, they are immune to speculative enthusiasms."[18]

[18] Galbraith (1994, p. 169f). Irving Fischer, an advocat of sound fundamental values, later explained the crash also as being initiated by a psychology of panic. However, there are economists who are suspicious about the bubble hypothesis. They claim that the market was not over- but even undervalued in the autumn of 1929 (see McGrattan and Prescott (2001)). Though, up to now no convincing news have been found that can explain the sharp

2.2.3.3 The Crash of 1987

Whereas the 1929 event was soon followed by the Great Depression, the aftermath of 1987-crash did not see a substantial decrease in real economic variables. Yet, the October 19 of that year witnessed the largest percentage decline on a single day in the whole history of the US stock exchange. The 20% loss of the Standard and Poors (S&P) index was even surpassed by the 26% downward movement of the S&P Futures index. Such dramatic events naturally initiate a great search for the reasons. However, very little is found in favour of an explanation based on fundamentals. For instance, Shiller (1987) took up the opportunity to issue questionnaires to individual and institutional investors in order to find out what it was that made them acting in this extraordinary way. It comprised questions about the situation, observations and reaction of the trader on that particular day and about incidents some days before that may had been responsible for the decline in prices. Shiller additionally includes a list of news that were claimed to be possible determinants of the crash and asked the investors to tell him which of these were important. The investors had to rank the news from 1 indicating complete unimportance to 7 indicating the highest importance. They could also fill in other news which the investor felt to be important. It turned out that most news got a moderate 4. But the most astonishing was the fact that the 200-point drop in the Dow on 19-th of October was rated to be the most important news.

Shiller also asked the traders to give a specific theory about the reason for the Black Friday. The most frequently given answer (about 1/3 of all respondents) was the overpricing of the market - a correct observation but no theory at all. Another compelling example is the question: "Which of the following better describes your theory about the declines: a theory about investor psychology [or] a theory about fundamentals such as profits or interest rates?"[19] Approximately 2/3 of the respondents opted for the investor psychology. A last interesting point to mention is the 96,7% of all individual investors who reported of roumers that a crash was about to start. Apparently a kind of nervous tension prevailed among the traders on that day. In fact, also institutional investors experienced "a contagion of fear from other investors".[20] Conclusions drawn from these answers should not be seen as definite signs of irrationality, but they also do not include any support for the EMH!

Although not as devastating to the real economy as the 1929-crash, the incident caused major concerns about the reason for the price decline. The then president R. Reagan set up the Brady commission to investigate the circumstances leading to this crash. In their summary of an explanation they wrote:

discontinouity in investors behaviour. See Cutler et al. (1989) for another failure to uncover the news that lead to the crash.

[19] Shiller (2000, p. 90).

[20] Shiller (1989, p. 388).

> *The precipitous market decline of mid-October was "triggered" by specific events: an unexpectedly high merchandise trade deficit which pushed interest rates to new high levels, and proposed tax legislation which led to the collapse of the stocks of a number of takeover candidates. This initial decline ignited mechanical, price insensitive selling by a number of institutions employing portfolio insurance strategies and a small number of mutual fund groups reacting to redemptions. The selling by these investors, and the prospect of further selling by them, encouraged a number of aggressive trading-oriented institutions included, in addition to hedge funds, a small number of pension and endowment funds, money management firms and investment banking houses. This selling, in turn, stimulated further reactive selling by portfolio insures and mutual funds.*[21]

This is not too different from the Shiller results since non-fundamental actions were supposed to have enforced the decline of asset prices. Contrary to Shiller, the high balance of payment deficit is named as one of the causes why the crash initially started. But this reasoning is very hard to sustain because these bad news should have been reflected in prices much earlier on.[22] In fact, up to now no convincing story exclusively based on economic factors have been found to explain the black Friday. To cite Mark Rubinstein (2000, p. 10), one of the leading financial experts who worked intensively on the subject: "I think we can say with considerable certitude that the crash was not caused by fundamental news about the economy, either in the US or in other countries." Instead, two other causes are frequently proposed as possible explanations: the already above mentioned actions of portfolio insurancers and a changed perception of risk. In the first case trades are made automatically in a given situation, i.e. assets are sold after market declines and brought when the market rises, thereby backing an already overall existing trend (see for instance Rubinstein (1988) for a link between portfolio insurance and the 1987-crash). The other view is particularly brought forward by Fischer Black (1988). According to his theory, the risk of equity investing rose in October without being compensated by the necessary amount of the cash flows to firms. In this situation, people tried to get assets sold so to buy bonds. It is hard to figure out the true responsible mechanisms of the crash and this is not the place to decide which the explanations is correct - it was probably a mixture of many factors. But it should in any case be clear that a fundamental reason cannot be provided.

2.2.4 Behavioural Finance

Behavioural Finance is a relatively new branch of finance theory. It started with the works of Paul Slovic (1972) and in particular Daniel Kahneman and Amos Tversky (1974, 1979). Considered to be a rather curious idea in the

[21] Presidential Task Force on Market Mechanisms, Report of the Presidential Task Force on Market Mechanisms (Brady Commision, 1988, p. v).

[22] See Leland and Rubinstein (1988)

beginning, it has by now gained full reputation as a useful path towards a more realistic modelling of financial markets and their acting members within. Behavioural Finance Theory (BFT henceforth) is defined by Lintner (1998, p. 7) as "the study of how humans interpret and act on information to make informed investment decisions".

A common feature of Behavioural Finance is their psychological and sociological background which already presupposes an economic behaviour that is not entirely driven by rationality. The aim of Behavioural Finance is foremost to give a better picture of people's motivation to act as they do. As a by-product it can be used to reconcile the various empirical puzzles with explanations of human action taking their (inherent) fallibility into account. Although literature on that topic is immense, there is still no unifying theory in sight. Almost every model designed for financial markets focuses on a special case, trying to understand a particular anomaly. Hence some theories do well in a distinct set-up while failing to explain other features of empirical research. However they all draw from the same pillars of psychological reasoning, namely prospect theory, regret theory, anchoring and over- and underreaction.[23]

2.2.4.1 Limited Arbitrage

A central argument of BFT is that arbitrage is limited. This contrasts the idea of Friedman that smart money assures fundamentally priced stocks. One of the key tenants of financial economics is the concept of arbitrage. It requires in its purest form no own capital and is furthermore risk free. The assertion of limited arbitrage has two different sources. The first is fundamental in nature: if the arbitrageur holds a short position because prices now are above their fundamental values, he is always exposed to the danger of incoming good news that in the end might justify the high price. The second problem arises because noise-traders, who are erroneously convinced to have superior information about the future price, can create risk through their actions.[24] Even if prices are too high, trend followers can push up the value even further. So if arbitrageurs have finite time horizons, they have to bear the risk that assets become even more overpriced than before (DeLong, Shleifer, Summers and Waldmann (1990)).[25] Furthermore, Friedman's argument that irrational traders loose money over time and as a result will eventually quit the market is far from being self-evident. Noise-traders, since not totally orientated on fundamentals, bear more risks and so may earn even higher profits than arbitrageurs.

[23] Shleifer (1999) gives a review of some key models of BFT.

[24] See Palomino (1996).

[25] The assumption of non-infinitive time horizons for arbitrageurs can be justified by considering the fact that they have to borrow cash in order to undertake their actions and therefore must pay fees to their lender each period. Besides the problem of riskiness two other sources hamper the ability of arbitrageurs to ensure correctly priced assets: transaction and holding costs.

2.2.4.2 Prospect Theory

The term Prospect Theory originates from an article published by the psychologists Kahneman and Tversky (1979). It suggests that economic agents act differently in situations depending on whether they have to face an event leading to a gain or a loss. Kahneman and Tversky (1979) contains several experiments and empirical findings that are inconsistent with the standard economic expected utility theory. For example, people who were asked to give a choice between getting \$ 1,000 with certainty or \$ 2,500 with a probability of $\frac{1}{2}$ mainly opted for the sure incident, perfectly in accordance with risk-aversion. But the same people when facing the choice between a 50% chance of loosing \$ 2,500 and a sure loss of \$ 1,000 converted in majority to the more risky first outcome which manifests a severe contradiction. Kahneman and Tversky's heuristic conjecture posits that individuals become more distresses when facing a loss than getting happy by a gain of the same size. This kind of *loss aversion* leads people to accept more risks in order to avoid losses than to realise gains.

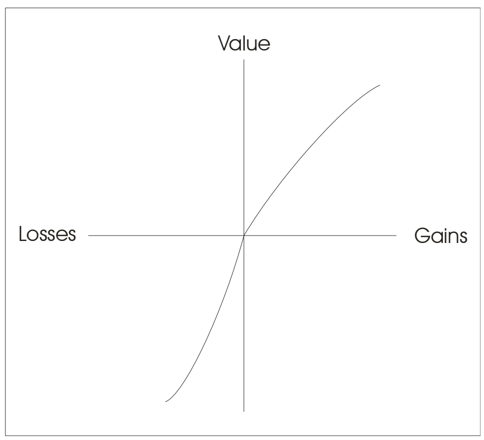

Figure 2.1: Typical value function as proposed by Kahneman and Tversky (1979).

In order to remedy the failure of EUT to explain the experimental outcomes, Kahneman and Tversky offer a new concept by which the decisions under uncertainty are described. They replace the utility function by a so-called *value function* which evaluates outcomes according to a reference point. This is a point all outcomes are compared with. Figure 2.1 displays a typical value function. It is concave for gains and convex for losses when the reference point is zero. This makes individuals risk averse in gain situations but risk loving in loss situations, just as the experiments reveal.

A notable application of prospect theory is given in Benartzi and Thaler (1995). They claim that the Kahneman-Tversky approach is able to resolve the equity premium puzzle. In their work, a relatively short investment horizon of about one year leads to a loss aversion explaining the high equity premium (see also Siegel and Thaler (1997)).

2.2.4.3 Regret Theory

Regret theory tries to take account of people's attitude towards having made a mistake, whether it be the selling of stocks shortly before they have gone up or the buying of stocks in the dawn of a decline. According to this psychological attitude, some investors may orientate their behaviour in order to avoid such unpleasant feelings, although judged from a purely rational point of view this can lead to a sub-optimal outcome. Shefrin and Statman (1985) is an early example of using the concept of regret theory to explain the attitude of individuals to defer the selling of stocks that have already gone down for quite a considerable time: they want to avoid the regret of having made a bad investment. This behaviour is empirically documented by Ferris, Haugen and Makhija (1988) and Odean (1999). Another consequence of regret theory leads to a more common behaviour among investors. People often invest in well-situated, respected companies because they carry an implicit insurance against regret (see Koening (1999)). Regret theory is closely related to what psychologists call cognitive dissonance or denial (Festinger (1957)). Humans get into a mental conflict when faced with their own wrong decisions. As in the case of regret theory, in order to avoid such feelings, people refrain from collecting new information or even reject to notice the arrival of bad news that are contrary to their beliefs and preconceived ideas.[26]

2.2.4.4 Anchoring

Anchoring theory suggests that in the absence of better information people assume current prices to be correct. EMH-protagonists would emphatically stress this as a sign of validation but this would presuppose that the prevailing price level reflects the true intrinsic value of the assets. Kahneman and Tversky give the following heuristic description of anchoring: ”In many (uncertain) situations, people make estimates by starting from an initial value that is adjusted to yield the final answer. The initial value, or starting point, may be suggested by the formulation on the problem, or it may be the result of a partial computation. In either case, adjustments are typically insufficient. That is, different starting points yield estimates, which are biased toward the initial values. We call this phenomenon anchoring”.[27]

By now, anchoring is a well-established psychological pattern. One of the most telling examples of its economic application comes from Northcraft and Neale (1987). There, professional real estate agents and undergraduate business school students were given exhaustive information about a house currently for sale. Included was all information deemed to be necessary to evaluate a piece

[26] The concept of regret and disappointment was proposed by Bell (1982, 1983), Loomes and Sugden (1982, 1987) and Fishburn (1988).

[27] Kahneman and Tversky (1974, p. 1228).

of residential property.[28] The data was correct with the exception of the sellers asking price. This exception was planed to act as the anchor. As it turned out, probands (both amateurs *and* professionals), when asked to provide estimates for the value of the house, took the given value as an anchor and stayed close to it.

As Shiller (2000) points out, there is no agreed-upon theory of what a stock index like the DAX or the Dow Jones should be worth. Of course, concepts like the Captial Asset Pricing Model (CAPM) or the Arbitrage Pricing Theory (APT) exist, but so far none of these have come up with a totally convincing empirical confirmation. The latest record can therefore serve as a first proxy for prognoses. This enforces the similarity of day-to-day pattern in stock markets.[29] It must be added that not only recent prices are conceivable anchors but also remembered historical values. This might explain the reversal of trends in individual stocks (and gives thus chartists a point when applying technical analysis such as *supporting* and *resistance lines*). Another possible anchor are price changes of related individual firms. In fact those stocks often show a remarkable co-movement.[30]

2.2.4.5 Overconfidence and Over- and Underreaction

Overconfidence stands for people's tendency to overestimate and exaggerate their own knowledge, talents and understanding of the processes of financial markets. On the other side they underestimate the consequences for themselves resulting from bad events. As Goldberg and von Nitsch (2000) show, this is particularly true in situations where people are not well-informed. Under– and overreaction is a related phenomenon. It basically claims that the market does not react to news in the correct way but often shows an inclination to either over- or underestimate releases of new information.[31]

There are currently two lines of explaining why people behave (in some situations) in this way. The first is proposed by Kahneman and Tversky (1974) who call their heuristic explication representativeness. According to this, people tend to categorise news into already familiar classes. Hence they assign the importance of a typical member of that class to the event without realising this specific nature. A phenomenon captured by the representative assumption is the (false) believe to recognise pattern in truly random sequences. The second explanation is based on Edwards' (1968) idea of conservatism. Here people are awkward about news and value them less than prior events. Edwards concludes

[28] The clues about which of the information has a significant importance were given by local real estate agents.

[29] Anchoring has also been applied to the phenomenon that forward discounts do properly explain subsequent exchange rate movements. Gruen and Gizychi (1993) show that the stickiness in variables can be explained by the fact that past prices are used as proxies for new prices.

[30] See Shiller (1989).

[31] See DeBondt and Thaler (1985) for an early empirical work on the subject.

that it takes two to five observations to do one observation's worth in inducing a subject to change his opinions. Barberis, Shleifer and Vishny (1998) offer a switching model that comprises both phenomena where the investors are in either of two regimes: the representative or the conservative regime.

2.2.5 Herd Behaviour

Common behaviour of people is a widespread observation in the human society. Several reasons can be brought forward to explain this phenomenon. One is the every day communication between people. Information, believes and attitudes spread out and so often let people to think similar. Shiller (2000) attributes this to be the main source of herd behaviour. Conversation and in particular the distribution of information through mass media play an important part in forming believes and attitudes. Another reason for herding is the fact that humans often feel a pressure, possibly due to a loyalty induced psychological motivation, to conform to their social environment (Jost (1995)). Forms of this behaviour includes fashion and fads.

Complementary to these psychological based explanations, other models try to rationalise herd behaviour on the basis of differently informed traders. Here, contrary to the assumptions of classic asset theory, the existence of private information is permitted. If other traders have superior information, then in the absence of own private knowledge it is rational to figure out who the better informed are and then imitate their actions. Explanations for such rational herding is contributed by the information cascade models of Banarjee (1992), Bikchandani et al. (1992) and others.[32] Their main idea is presented by a sequential trade process in which the first mover receives a private signal, sometimes indicating the true nature, sometimes misleading the trader. Now, either the surrounding environment (that must have the chance to observe the trade) or the first partner tries to draw conclusions from the action of the first mover. For example, if the trade offer is a buying of a large amount, this might be interpreted as a signal of good private news. If the subsequent movers really assume this, they will place a buy-offer. The rest of the market has by then observed two or more buy-offers, and the probability (updated by a Bayesian rule) of a good private news has become greater and so initiates further buys. In general, such a buyer-cascade starts when the number of investors who bought is greater by two or more than the number of those who sold assets. Theoretically these processes last until the whole information dissipates, i.e. until no further information can be deducted from the action. In reality, a cascade stops whenever a new private signal comes into the market that is contrary to the process.

One might argue that this is an example of how well the market works since information once owned by one single trader gradually spreads out into the market. But this is only true if all non-informed interpret the sign correctly.

[32] For a compact survey of herding models see Bikhchandani and Sharma (2000).

Because they have to deal with uncertainties, this won't be the case in every instant, whether qualitatively (differentiating between good and bad news) and quantitatively (evaluating the correct measure of the impact on the price of a share). Furthermore, already the first sign could have been misleading, so information cascades may start with false or even no information at all! Moreover, the process takes some time to proliferate throughout the whole market. In the meantime, trades and thus price changes occur without the arrival of news.

Information cascades are not the only kind of herd models. Reputation models are based on asymmetric information. The employees (principal) is uncertain about the ability of his portfolio manager (who knows about his own capabilities). Now, if this manager has a low ability, he might try to hide this by imitating other managers, a common behaviour can result by the aim to maintain his reputation.[33] A last point may be that it can be risky to hold positions contrary to the trend. DeLong et al. (1990) show that for traders with risk aversion, trend following is sometimes superior to acting against a trend, despite own fundamental knowledge which would advise to do so. The crucial point of herd behaviour is that agents do not act homogeneously, they differ in their beliefs and private information sets and, even more important, they act not independently of each other.

There is still up to now no clear cut opinion about the value of BFT, the robustness of the empirical anomalies or the soundness of models with bounded rationality. Burton Malkiel (2003), one of the supporters of the EMH, points out that "(i)t is far from clear that any stock price patterns are useful for investors in fashioning an investment strategy that will dependably earn excess returns"; Odean (1999) raises doubts about the excess profits coming from momentum strategies, and Fama tries to rebut empirical anomalies in several papers (Fama (1991, 1998)). Furthermore, Fama and French (1993) offer a possible explanation for firm specific anomalies with their common risk factor model. A witty defense is Fama's notion that "(m)arket efficiency per se is not testable".[34] Indeed, market efficiency has to be tested jointly with a model of the true asset price evaluation. Because there are various conceivable candidates, Hawawini and Keim (1998) conclude that there is no such a test as to distinguish apparent inefficiencies from wrong asset price models.

Notwithstanding these objections, the complete ignorance of anomalies that stand against the EMH and the various psychologically based explanations is hard to justify. People are no identical infinite-capacity calculating machines differing only in their utility function. As psychology explores the human judgement and its behaviour, it can teach economics how investors differ in reality from the conventional way of modelling them. One point of concern, though, is quite serious. BFT does not offer a coherent theory that is able to account for all anomalies. For example, BFT tend to be mutually inconsistent, so it cannot

[33] See Scharfstein and Stein (1990).
[34] Fama (1991, p. 1575).

explain why people show in some cases signs of overreaction, but in other cases signs of underreaction.[35] Moreover, and this is even more crucial, BFT is unable to reproduce the overall behaviour of asset prices, which is characterised by long periods of smooth and tranquil patterns, disturbed only occasionally by some short turbulent phases. Time series on asset prices are not dominated by everlasting cascades or by a switching regime where people just over- and underestimate news. Therefore, another, more general hypothesis about the mechanisms of financial markets is still warranted. The next chapter tries to outlay a possible candidate for such a general hypothesis.

[35] Brav and Heaton (2001) give a critical exposition of BFT and its shortcomings.

Chapter 3

The Theory of Complex Systems

A good starting point for a new theory of financial markets may be the summary of what the critics of the EMH have pointed out. First, it is unlikely that traders are homogenous. Financial markets comprise different types of traders with different expectations, information sets and restrictions: chartists who extrapolate past trends, amateur traders who believe in the expertise of famous pundits, liquidity traders who act according to special constraints, fundamentalists looking only for the true value in the information of news and market makers who have to take care of open positions.[1] Secondly, markets consist of many members. This is indeed a triviality but will become important when combined with thirdly: market participants interact with each other. They do not base their actions solely on individual and rational reasons but are influenced by what others do, too. This is shown in the various papers on information cascades and herding. In the presence of private information it is quite logical to try to infer these news from the buying or selling signals of other traders. Apart from pure economical reasons, social scientists and psychologists have raised serious doubts on the assumptions that every single trader just optimises his portfolio with respect to his utility function. These points can be presented in an even more compact manner by asserting that financial markets are open systems (because of incoming news) that consist of many interacting heterogenous agents.

Provided this very sparse description of financial markets would suffice, then surely lots of other cases exist for which this rough characterisation is also adequate. Almost every social and ecological system, in fact the whole economy itself could be characterised in this way. And the list of examples is by no means

[1] Questionnaires have shown that in reality traders do not exclusively employ one pure strategy. Instead the behaviour is determined by the appliance of a mixture of strategies, see e.g. Menkhoff (1998).

restricted to systems including living beings. The universe is a combination of many different objects (stars, planets, black wholes) that clearly do interact (via Newton's law of gravity). A natural question arises of the use of a theory that describes so many social and natural phenomena? How can one employ a theory that is not able to discriminate between so many systems?

The answer to these questions is not found in the description of the constituent parts of the system (the traders, the planets, the animals etc.), but in the behaviour of the system as a whole. The interesting systems all share an empirical feature which the other systems do not possess: the logarithm of fluctuations in the macrovariable of the system (the asset prices in the case of stock markets) can be assembled on a straight line when plotted against the logarithm of the number of its occurrences. In mathematics, such lines are associated with power-law functions.[2] A function, $N(x)$, is a power law if the quantity N of a variable x (here, the size of the fluctuation) can be expressed by x raised to an exponent -α, i.e.

$$N(x) \sim x^{-\alpha}.^{3} \tag{3.1}$$

The logarithmic form is then given by

$$\log N(x) = -\alpha \log x. \tag{3.2}$$

Thus, plotting the log of the quantity of some fluctuations versus the log of its different sizes results in a straight line with a slope $-\alpha$. This regularity is by no means a natural or obvious outcome. It shows a missing of any kinks or bumps that would indicate a special importance of a particular scale, i.e. a scale at which almost all events happen, which is typical for many other systems.[4]

The power-law feature leads physicists like Per Bak to distinguish such systems from other systems by claiming that they are complex in a specific way, while the others are not. Besides the appearance of power-laws, there are a couple of other important points which complex systems have in common. The next section gives an overview of the characteristics of complex systems. These points are a loose collection of the typical features of complex systems. Nevertheless, this section aims to incorporate all essential ingredients.[5]

[2] Power-law distributions are said to be scale-free or scale-invariant because the relative probability to observe an event of a given size and another size, say four times larger, does not depend on the reference scale. To be correct, in reality this *scaling* property does not hold over the whole range of fluctuations. Shroeder (1991) gives an overview of natural systems showing power-law behaviour. Another feature always encountered in the complex system is their fractal nature. The empirical part of this work deals extensively with these points.

[3] $\sim$ means asymptotic equal not approximate or proportional. I.e., $N(x)$ becomes a power-law as $x \to \infty$.

[4] For example, distributions for fluctuations of Brownian Motion do not follow a power-law, but exhibit an exponential run.

[5] There exist a couple of views about which ingredients are indispensible for making up a complex system. The following list is a summary of the most important points. It should make clear the peculiarities of the kind of systems employed in this work as compared with

3.1 Characteristics of Complex Systems

(i) Complexity: Only systems with a large number of mutually interacting members (large many-body systems) are candidates for complex systems. The connection inside the system can cause highly non-linear dynamics that cannot be described by linear equations.[6] As a consequence complex systems are not amenable to a mathematical treatment but have to be explored by numerical methods. Simulations are so far the only possible way to analyse the behaviour of such systems.

(ii) Irreducibility: Complex systems only appear as a whole. They cannot be analysed by decomposing the system into its single components. The isolated observation of the behaviour of just one member does not unveil the bahaviour of the whole system. Instead, the dynamics are governed by the interaction of the constituent members and every decomposition destroys the aspects that gives the system its individual character. The outcome of a system is to a large degree independent of the microscopic details. This is sometimes disturbing especially for economists, since they are usually focussed on the correct and thorough modelling of the behaviour of agents. Although this is by no means unimportant, the kind of interactions between the parts play a more crucial role.

(iii) Emergence: The behaviour of complex systems is surprising; the appearance of a property or feature like huge catastrophic events cannot be previously observed (or derived) as (from) a functional characteristic of the system. This implicates the impossibility of forecasting. It is presumably this characteristic which distinguishes complex systems from other systems most prominently.

(iv) Adaption: Complex systems are constituted of intelligent agents. They often act on the basis of some information about the whole system. Intelligence should not be taken literally but viewed as the ability to change the behaviour in the face of a different environment. In this generality biological, chemical and even physical units can adopt to changing circumstances. This is of course even more true for human beings.

(v) Imbalance: Complex systems are not in balance, i.e. even minor disturbances can cause major fluctuations. The reason for this bahaviour lies in the collective mechanism of the constituent microunits, where a small impulse can trigger off a kind of chain-reaction that ends in a catastrophic outcome. Such systems are also described as operating *out-of-equilibrium.* They tend to exhibit long periods of smooth and tranquil behaviour where the system seems to be in a kind of equilibrium state. But regularly

other systems which do not have all of the enlisted features and therefore are no candidates for models of financial markets.

[6] The quantitative measure of the complexity of a system is usually related to information theory. It means that a figure of complexity is related to the amount of information that is needed in order to describe the system (see Bar-Yam, 1997, chapter 0.5.2).

(shorter) periods of turbulent behaviour interrupts this temporary calm movement and so give rise to the emergence of the above mentioned power-law. This feature is also called *intermittency*. Systems operating in a stable equilibrium are not intermittent. On the contrary, normal equilibrium states are characterised by a finite response to small perturbations, i.e. these responses decay *exponentially* (sometimes in an oscillatory manner), and there is no such a mechanism as to start from an avalanche-like process.

(vi) Openness: This point is not an indispensable necessity but all real-life systems are open to influences from outside. These disturbances hit the system regularly thereby causing reactions of the members, but are themselves not the cause of the large dramatic events (in some cases only their initiators). The exception to open systems are closed systems undergoing phase transitions (e.g. water evaporating to gas; at this point the systems evolves from a disordered state into an ordered state).[7]

(vii) No chaos: Chaotic systems do not show a power-law behaviour as complex systems do. Instead they release signals that are very hard to discern from white noise. Moreover, chaotic dynamics stem in general from deterministic non-linear functions, whereas in complex systems the stochastic element is far more important. There is but one interesting point to mention. Chaotic systems have a point at which the so far relatively tranquil signal transforms itself into chaos. At precisely this point, the system displays a power-spectrum. This is why complex systems are said to behave *on the edge of chaos*.

(viii) Critical points: This is another important characteristic of complex systems. Large changes occur as the result of the accumulation of small events. The small stimuli aggregate and by reaching a particular threshold value (the critical point) they cause huge fluctuations in the system, just as small grains of snow can precipitate large avalanches. Large fluctuations are thus endogenous and the search for specific external causes of big-scale events is therefore often futile.

Some words are in order to get these features connected to financial markets. Financial markets are open to the influence of external events like news; in fact this is a central point of EMH and unpredictability is a crucial feature of efficient markets, too. However, point (iii) is different from the notion of unpredictability as understood by the EMH. Emergence includes huge changes, catastrophes as the Black Friday. It is not so much the impossibility of every day forecasts but more the surprising emergence of completely different behaviours. On the other side, adaption and learning should be unknown to the supporters of EMH

[7] For a detailed exposition of phase transitions see in particular Stanley (1973). Deducing the properties of water vapour already becomes a computationally intractable problem because of the exponentially large number of molecules involved.

because rational expectations imply an already existing perfect knowledge. If evolutionary learning would be accepted - as the neo-austrian market theory strongly suggests - then longer phases of mispricing cannot be ruled out. Complexity itself is not a feature of efficient markets. Though also incorporating many traders, these are not diverse in the sense of complex systems and, even more important, their actions have rarely an impact on other traders. A crucial and related point is the irreducibility of complex systems. Financial markets if comprised of individual profit maximising actors are reducible. The observation (modelling) of just one fully rational investor would suffice to determine asset prices since everybody is trading on the same information set and according to the one and only asset price model. Also, rational markets are neither instable nor on the edge of chaos. The RWH poses severe restrictions on the evolving fluctuations of prices. These cannot be described by a set of non-linear (deterministic) functions but are the result of random changes in the news process. Furthermore, every impulse stemming from these news is instantaneously incorporated. Lagged actions that can aggravate to an avalanche of sells (or buys) are ruled out by the assumption of rational agents, so point (viii) cannot describe financial markets according to the EMH.

However, it is exactly the similarity between complex systems and financial markets that initiated physicists to offer a new explanation for the mechanism by which the observed price process is driven. As these *econophysicists*[8] aim to show, stock markets behave principally like any other complex system in the natural science and therefore exhibit not only power-law fluctuations but also share all the other characteristics listed above, with a particular emphasis on the last point. To give an imagination why such parallels are drawn, the next chapter provides some of the well known examples of complex systems in the natural science.

3.2 Examples of Complex Systems

Figure 3.1 displays the (spatial) distribution of earthquake magnitudes in the New Madrid zone in the southeastern United States during the period 1973-1983. The right picture shows a clustering around some critical areas, while the left picture gives the relationship between the magnitude of the earthquakes and its associated frequency, both measured on a logarithmic scale.

The figure shows an expected outcome: small earthquakes occur much more frequently than heavy ones. In fact, earthquakes of magnitude 4 on the Richter scale happen about 1000 times every day while the large and catastrophic events show up only rarely. But the astonishing is that the distribution of their strength can be assembled on a straight line if plotted on a log-log scale against the number of occurrences. This is a manifestation of the Gutenberg-Richter law,

[8] The term *econophysics,* a hybrid of economy and physics, was coined by Eugene Stanley in order to describe the application of methods of statistical physics to economy in general.

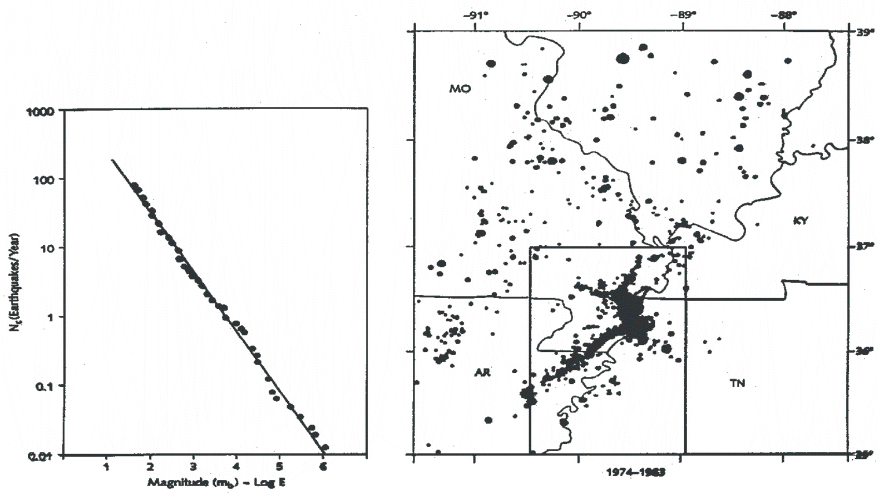

Figure 3.1: Left figure: Number of earthquakes with a magnitude larger than a given magnitude m. Right figure: Location of the earthquakes from the left figure. See Bak (1996, p. 13).

which says that the number of earthquakes with magnitude M greater than m is given by

$$\log_{10} N(M > m) \sim -bm, \ b > 0.^{9}$$

As Bak (1996, p. 14) emphatically explains: "The importance of the Gutenberg Richter law cannot be exaggerated. It is precisely the observation of such simple empirical laws in nature that motivates us to search for a theory of complexity." In order to fully grasp why earthquakes are taken as an introductory example, it is necessary to explain their emergence.

Most earthquakes are caused by plate tectonics. The earth's surface is divided into some large and several minor crustal plates. They all together constitute the lithosphere, a rigid layer floating on top of the earth's hot mantle. This is the by now overall accepted theory of plate tectonics, first introduced by Alfred Wegener in 1912.[10] The exact mechanism by which plates are driven is not

[9] On the Richter scale, the magnitude of an earthquake is proportional to the $\log_{10}$ of the maximum amplitude of the earths motion. Because of this logarithmic scale, an earthquake with magnitude 8 moves the earth's crust 10000 times heavier than an earthquake of magnitude 4. Nota bene: the differences in released energy are even greater. Here, the base factor 10 has to replaced by a factor of 32, so in the example, the bigger earthquake releases energy that is 1000000 times greater than in the smaller case.

[10] To be more precise, the plates are swimming on the asthenosphere, the outer partially molten part of the mantle. There are about six to eight large plates; the exact number is still not determined (see Anderson, 2001).

fully understood so far but it is known that the movement is principally caused by convection within the upper mantle. Hot liquid (heated by the outer core of the earth) rises to the surface where it is cooled and then spreads laterally downwards, thereby moving the rigid but brittle plates that are interconnected by ridges, transform faults, and trenches. Plate tectonics display four types of different activities: the sea floor spreading which is caused by convection currents that creates new oceanic crusts (this is also called a *divergent boundary*); the continental slide where the oceanic plate slides under the lighter continental plate (subduction) and bends downwards to the inner earth (*convergent boundary*); the continental crush (*collisional boundary*) that happens whenever two continental crusts meet; and the sliding (*transform boundary*), where two plates move against each other, thus building up tension.[11]

Plates move with a velocity of some centimeters per year,[12] and the stress that is caused by the sliding gradually builds up during a long time span (of hundreds or even thousands of years). If tension is above some threshold, a sudden release leads to the earthquake rupture which lasts usually just for a few seconds. This mechanism works both for small and large earthquakes so the natural question arises about what is responsible for the difference in the magnitude? It is not the naive, though natural expectation from classical mechanics that it is the existence of a single main rupture fault on which the deformation occurs. The empirical evidence is strongly against this conjecture: it is the successive rupture events that give birth to complex and self-similar fault patterns. I.e. the catastrophic shock waves are not the result of a single mighty rupture but of repetitive smaller ruptures. This means that small earthquakes cause other instabilities to unlock faults which themselves are of a negligible magnitude, but by enforcing each other and so building up a cascade of subsequent ruptures, can finally end in the large catastrophic events. Thus, even great ruptures can be seen merely as culminations of progressive nucleation, growth and fusion between microcracks which themselves would not cause much damage. Most chain-like incidents, however, stop before ruptures become extremely large. Only a few of them do not stop, but these are the ones that release huge amounts of mechanical energy.[13] This tentative explanation is corroborated by the observed smooth transition from small to large earthquakes. A characteristic size of earthquakes, which would show up as horizontal line on a log-log plot cannot be detected. This is why earthquakes are called *scale-free*.

Although this is a crude description of what is going on in the lithosphere, the main point should be understood. The set of different plates comprises a complex system in which the members do not behave independently but are

[11] It is this activity that is responsible for most continental earthquakes because the tension is released in a sudden jerk. This is why most earthquakes are found to happen along the margins of plates. The San Andreas Fault is a prime example where two plates are dragged sideways against each other.

[12] See for example Hamilton (1995).

[13] See Main (1995) and Grasso and Sornette (1998).

connected with each other and so move slowly like a conveyor belt. Based on this inside, several models have been proposed that see the lithosphere as a system which organises itself into a *critical state* (see point (viii) in the last section). Self-organisation just means the existence of some order of patterns in systems that emerge from the interaction of its members and does not come from outside, so there is no parameter that controls the development of the system entirely. The concept of critical states or critical points involves all systems that operate on the limit between an everlasting smooth development and chaos. At this point, long periods with only minor fluctuations are irregularly interrupted by large and middle-sized events. The important thing is that this point is reached without the interference of an external control parameter. No single factor from outside the system is responsible for this state. This is the so-called self-organised critically (SOC henceforth) hypothesis.[14] The main idea is presented by a very crude simulation of the earthquake fault. Figure 3.2 illustrates the construction of the Bak-Tang spring-slider block model.

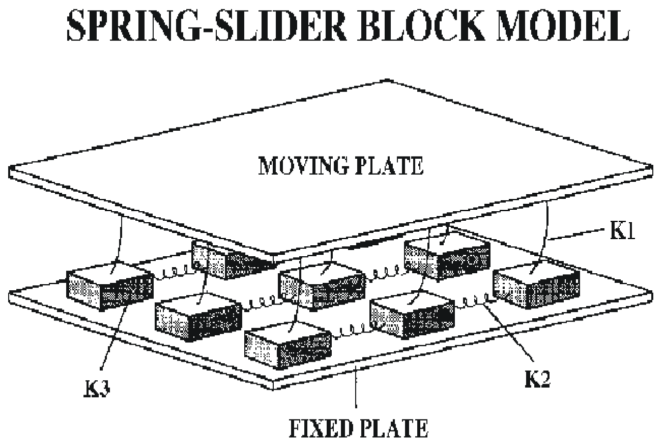

Figure 3.2: Representation of the earthquake model of Bak and Tang (1989). See Bak (1996, p. 90).

In this model, the hot asthenosphere is represented by a moving plate that keeps the blocks (the plates) in action through the leaf springs. Interaction is simulated by the other coil springs. The system exhibits a smooth and regular motion (the blocks stick to the surface) as long as the sum of the forces of the moving plates working on the blocks is under a particular threshold. If it is above this value, the block slips upward towards the moving plate, thereby enlarging the force on the other (four) neighbouring blocks (plates). This might

[14] See Bak et al. (1989) as the initial paper on SOC; see Sornette and Sornette (1989), Bak and Tang (1989) and Ito and Matsuzaki (1990) for other early SOC-models of earthquake faults.

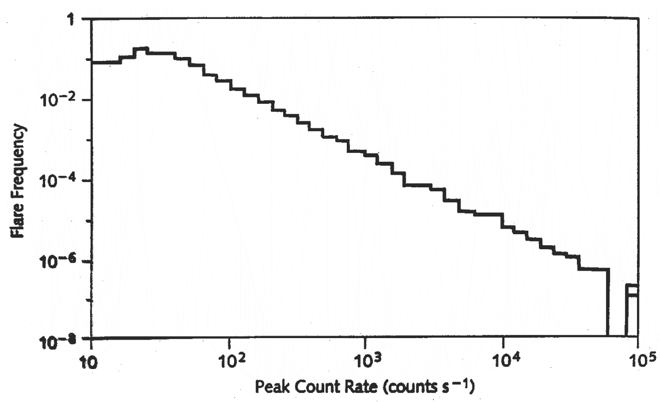

Figure 3.3: X-ray intensity from solar flares. See Bak (1990, p. 103).

start a chain reaction, leading to very large earthquakes while most of the time the transition of energy between the elements stops before a large event occurs.[15] Results obtained from this and related simulations show a distribution of earthquake sizes that is similar to the one observed in empirical studies. It confirms the hypothesis that the system, after setting initial parameter values, is able to organise itself into a state which is neither stable nor totally instable. Therefore, physicists often call this a metastable behaviour. Reality confirms the simulated results in that it shows a more or less tranquil motion in most areas of the world, most of the time. But at some stage the catastrophic events take place without major external events happening. A very important feature of self-organised complex systems is the fact that even after large fluctuations, the systems comes back to its normal behaviour without the interference of an external regulator!

The second example comes from astrophysics. Figure 3.3 shows the X-ray intensity of solar flares.[16] Solar flares are massive explosions on the surface of the sun (the chromosphere). They show up as sudden, rapid variations in brightness. Solar flares are distinguished from other similar phenomena like protuberances by the fact that they release energy up to the order of 10^{36} erg in just a few minutes.[17]

X-ray intensity is a measure of how big the flare is. There is an obvious similarity to figure 3.1. Most flares are small, but some are so large that they cause disruptions of radio communication within the earth's atmosphere. However, all of them can be again assembled on a straight line if measured on a

[15] See Bak and Tang (1989) and Bak (1996).

[16] The release of energy in flares takes place in several forms: as Gamma and X-rays, as protons and electrons (loaded particles) and as an outflow of mass.

[17] An erg is the unit of work or energy in C.G.S. system. One erg is equivalent to 10^{-17} joule. Solar flares are in fact the most energetic explosions in the solar system.

double logarithmic scale. The interesting question concerns the cause of the flares. Although a precise explanation is still warranted, it is today an accepted theory that flares are associated with instabilities in the magnetic field around sunspots. The mechanism by which a flare takes place can be described in a similar way as in the earthquake example: energy builds up and if tension is too high it will be released. Supposing that there are two magnetic regions of opposite polarities which are divided by a so-called neutral line. Regions where the field lines are perpendicular to the magnetic neutral line are called potential configurations. If, by successively adding new charged material from the innermost layer of the sun, the core, these magnetic regions slide and the field lines will drift away from perpendicular towards the neutral line, creating a so-called sheared configuration. In an extreme situation, the field lines are almost paralleled to the neutral line. If the regions are twisted in this fashion, the magnetic field then has more energy than the corresponding potential configuration.[18] In this situation instabilities occur: the magnetic configuration breaks down and a sudden reconnection of the field lines takes place in order to reinstall the configuration before the shearing.[19] During this process the additional energy is released and radiation is emitted. However, shearing, although necessary, is not the only reason for flares. In addition, the complexity of active regions around sunspots is also crucial. In reality more than two regions are involved in the creation of a flare and it is found that major flares are associated with the complexity of active regions. The idea is that the random loadings from the inner part of the sun do not cause many instabilities during the quite phases. But if a local threshold criterion for stored free energy is fulfilled, the magnetic field becomes relaxed, thereby triggering some of its neighbouring fields to relaxation and so on. And the more regions are involved, the higher the radiation emittance becomes.

Lu and Hamilton (1991) and Charbonneau et al. (2001) develop an avalanche-like theory to explain large flares. In these models, small energy releases cause subsequent instabilities thereby leading to a chain of events. The dynamics stop when having reached a stage of a new stable magnetic configuration. Most chains end up in relatively small flares but sometimes a large avalanche starts emitting the very large amounts of energy - just as in the earthquake example.[20] And just like there, the system consists of interacting members (the magnetic fields), which lead through their mutual influence to highly non-linear dynamics.

[18] This extra energy is called free magnetic energy.

[19] The magnetic shear is defined as the angular difference between the azimuth of the potential transverse field and the observed transvers magnetic field (see Hagyard et al., 1984). Given this definition, a flare takes place where the magnetic field departs most from a potential field.

[20] The main idea about what happens before and during a flare was first proposed by Parker (1983, 1988). In his model mechanical energy is first stored as magnetic energy within the photosphere of the sun and is then being transferred to the plasma. A heating process sets in and leads in combination with the high electronical conductinty preraling in the coronal plasma to magnetic reconnections. This leads to the local release of magnetic energy and so to a reconfiguration of the magnetic field.

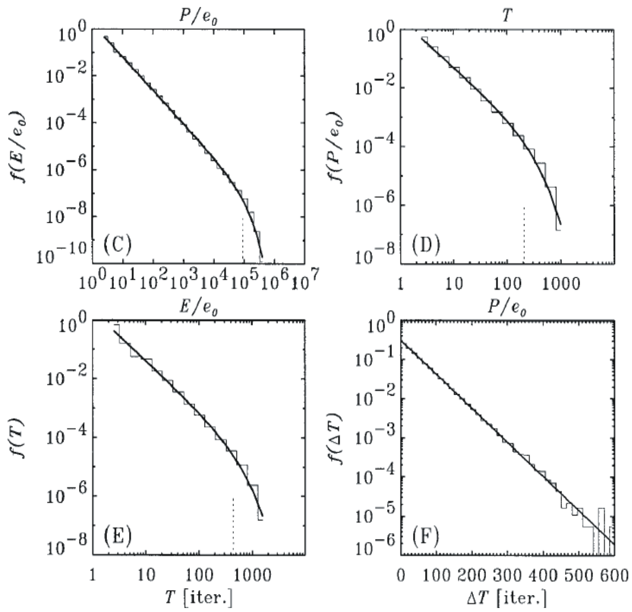

Figure 3.4: Simulation results of the solar flare model of Charbonneau et al. (2001). The pictures (labeled (C), (D), (E) and (F) respectively) show normalised frequency distribution for energy releases (E), peak energy (P), duration (T) and waiting time (ΔT). Al variables follow a power-law. See Charbonneau et al. (2001, p. 335).

This gives rise to the assumption that a similar avalanche-type mechanism is responsible for the coexistence of small and large flares.

The next example is taken from evolutionary biology. Figure 3.5 shows the extinction of species in the last 600 million years. Extinction is the process in which groups of organisms (species) die out. It is calculated as a net value and occurs whenever the birth-rate of new species is less than the death-rate of existing species. Because of natural inabilities of some species to adapt to changes in the environment, extinction is a common process taking place in each area of earth's history. However, some periods are characterised by mass extinctions where up to 50% of all species became extinct.[21]

As one can see, extinction is a common phenomena but mostly encountered in small net die out rates. Catastrophic events occur rarely as it should be expected. The surprising fact lies in the power-law behaviour of this curve if

[21] The fire largest mass extinctions occurred in the (i) late Ordovician period (438 million years ago); (ii) the late Devonian (360 million years ago) with a 30% of animal extinction rate; (iii) the late Permian (245 million years ago) with 50% of animal foucilies and 95% of marine species died out; (iv) the late Triassic (208 million years ago) and the (v) Cretaceous-Tertiary boundary - the K-T-Event - (65 million years ago), where about 50% of all life forms extincted – including the dinosaurs.

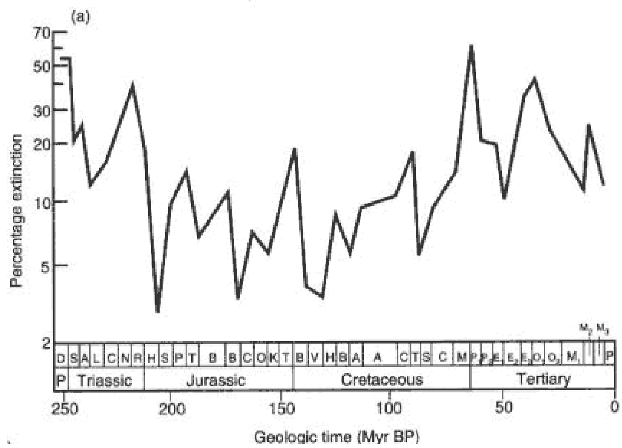

Figure 3.5: Initial results of Raup and Sepkoski (1984) for 567 marine families. See Ridley (1993, p. 609).

measured on a logarithmic scale. Here again a straight line appears to represent a simple relationship between the quantity and the size of an event, in this case biological extinction.

A popular explanation of the large extinctions is the occurrence of similar rare and catastrophic events like the hitting of meteorites, climatic changes especially cooling and volcanic activity. In fact, research supports this idea by confirming that dinosaurs became extinct because of an external event (see in particular Alvarez et al. (1980, 1990) who discovered an anomalously large concentration of iridium at the Cretaceous-Tertiary boundary, an element only found after volcanoes or asteroid hits). Moreover, some periods of earth cooling coincides with mass extinctions. This line of reasoning is prominently advocated by Raup(1991) in his book *Extinction: Bad Luck or Bad Genes?* in which he makes several points on his theory of external events.[22] However, there are some severe shortcomings. Bak (1996) argues that the existence of a power-law indicates a common mechanism for both small and large extinctions "because otherwise the size and frequency of these large events would have no correlations with the smaller extinction events."[23] Indeed, if only external forces were responsible for the large events, what causes the medium sized and small ones? Extraterrestrial impacts seem not to be a good answer for every-time extinctions. Furthermore, cooling periods do not coincide with every large extinction but just with one for sure (the late Ordovician) and maybe with a second (the late Devonian, see Ridley (1993, p. 612 f.)).

[22] Another line blames the imperfect fossil records for the appearence of mass extinctions. See Gould (1989) on that point.

[23] Bak (1996, p. 152).

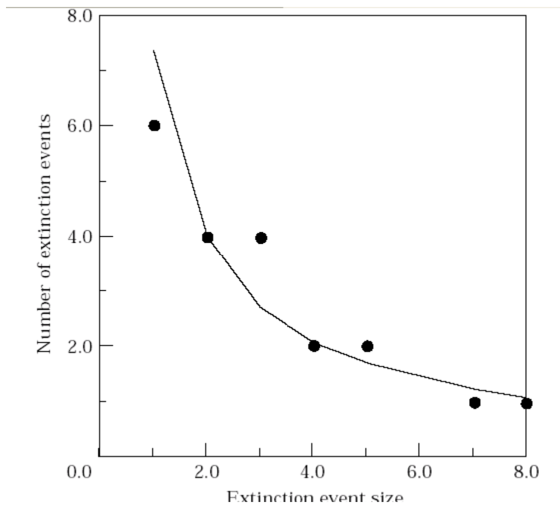

Figure 3.6: Empirical distribution of extinction events for the Hawaiian avifauna. See Keitt and Marquet (1996, p. 164).

An alternative to the external view interprets the extinctions as the result of internal forces. Each species depends on the existence of other species serving as food. Viewed this way, all life-forms can be ordered within a hierarchical system, where the species at the end of the food chain depend on the subordinate species. Such a system is build up like a house of cards. It is possible that the extinction of just one species has no significant effects on the upper levels of the food chain if other species can fill in the gap as nutrition. In fact this might be the case for the majority of species. However, some may be crucial for the stability of the whole system. The extinction of these species initiates further extinctions and at some point in time an avalanche starts drawing down a massive number of life-forms. In this respect one has described a complex system with many interacting members. The idea that extinction rates are the outcome of a self-organising process of competing species is brought forward by Kaufman (1993) and later refined by others. All of these models are based on the argument that species in an ecosystem do not develop themselves independently. Together with random mutations and genetic variation, interconnections play a crucial role and their mechanics can lead to the empirically observed disturbances.[24] This view is strengthened by Newman (1997), who is able to simulate coevolutionary avalanches that display the same distribution of the sizes of extinctions as the empirical data.

An interesting example that confirms the intrinsic view is given by Keitt and Marquet (1996). They study bird species introduced to the Hawaiian Islands over the last century. The isolated position of the islands has the advantage of

[24] See Bascompte and Solé (1996), Bak and Paczuski (1993) and Newman (1997).

excluding large external effects. Figure 3.6 displays the relationship between the number and the size of extinction events. The picture fits well into the pattern of the mass extinction over the last 600 million years. Keitt and Marquet (1996, p. 161) argue that competition among species is a candidate mechanism for extinction: "As the number of potential interactions between species increases proportional to the number of species squared, species richness that competition is forcing some species to extinction."

The three examples of earthquakes, solar flares and mass extinctions are by no means the only complex systems that show a power-law bahaviour of their fluctuations. In the field of geophysics volcanoes, erosion and landscapes and in particular the climate and the weather are suspected to behave principally in the way complex systems do. So is protein folding and the related question about the origin of life. The sequence of examples where simulations based on the idea of complex systems showed good approximations to empirical data can easily be extended to social areas. Andreev et al. (1996) for example try to mimic the strikes of industrial workers in Russia, and majority voting is treated in Galam (1990, 1997). Another famous example of power-law distributions in systems with many interacting parts in Zipf's law. Zipf (1947) noticed a linear relationship between the size of cities as measured by its inhabitants and its ranking. Although external components like rivers and mines play an important part, other driving forces for the development of cities are social and economic interactions between people, the competition between near-by cities, infrastructure and spatial decisions about land development.

So far, all presented examples display the same power-law pattern. As above already mentioned, systems that are in balance do not show such a behaviour. A gas of atoms or a bowl of water are both systems with many interacting members (the atoms and molecules), too. But a small pertubation does not have huge consequences for the behaviour of the system. Instead, only large external events can cause large fluctuations. This is the way most economists picture not only financial but also nearly every other commodity market. Perfect rationality and frictionless markets are assumed to secure stable equilibria. In this world, the systems response linearly to shocks from outside. Not so in complex systems. There, small events enhance each other through the connections inside the system and hence are able to cause an avalanche process. This would also be in total agreement with the scale-free nature of their fluctuations: small and large events follow the same power-law, thus indicating a common source. In fact this is the working hypothesis of a complex theory for financial markets in general and stock markets in particular.

3.3 The Explanatory Range of a Theory of Complex Systems

Because of the inherent impossibility of forecasting complex systems, models trying to rebuild financial markets on that basis can hardly be taken for improvements in forecasting. Even a good, i.e. a one-to-one reproduction of historic data cannot be performed. It is impossible to rebuild exact time series of empirical data with a simulation of complex financial systems. What can be done is the reconstruction of the statistical features of complex systems. It will never be the case that the artificial stock market is able to produce two crashes (1929, 1987) at precisely the same time points but it should be able to reproduce two crashes within a time span proportional to the real observed data. It should also reproduce the more frequent less dramatic events. In short, the statistical distribution of the macrovariable is the yardstick of each simulation.

Besides this quantitative task, the modelling of complex systems aim to come up with crucial qualitative answers to questions after the mechanism that drives the systems. Or whether it is possible to control the system in order to prevent at least the biggest catastrophes. It is maybe untenable to understand all of what the complex systems involves, but it is a task to exclude constellations that do not lead to the emergent behaviour observed in nature. On the other hand finding out constraints that lead to desirable outcomes may serve as a crude guide to political decisions, though this is surely the ultimate future goal of simulating complex systems. All these thoughts can be readily be applied to stock markets.

Karl Popper's notion of falsification is the common watershed that divides empirical sciences from meta-science.[25] The question is whether a theory based on complex system has the potential to be rejected. Usually the criterion of falsification is applied to experiments. Given a predefined environment, does the experiment deliver the result forecasted by theory? Unfortunately not every object under scrutiny can be transformed into an experiment. This is obvious for almost all examples described so far: solar flares, earthquakes, mass extinction and the distribution of the size of cities. The same is true for stock markets. No astrophysicist can change the parameters of the sun in order to see what happens. And no economist has any influence on the traders to alter their behaviour in order to see what the consequence will be. Econometricians might argue that a properly designed, i.e. testable model will reveal the truth about the new theory. But this overlooks a severe problem: models of complex systems cannot be put into a closed mathematical form. Therefore usual econometric tools are useless as there are no equations that can be estimated.

Furthermore, every scientist has to undergo something like a trial and error process in order to arrive at the desired outcome. On this way he usually

[25] Popper (1959, 1979).

changes parameter values and sometimes even the model itself. At this point confirmation theory might argue that this is not science. The researcher uses his influence to drive the system to its critical state and therefore leaves the simulation with no other chance but to give sensible results (however, this critique is principally applicable to every simulation). So what makes it sure that the real world works with the same parameters and constellations as proposed? The unpleasant answer is: nothing! Computer simulations on complex systems can only offer a new hypothesis about financial markets that is consistent with reality. And this hypothesis is limited in that it can only try to explain the statistical pattern, but as Per Bak claims: "To predict the statistics of actual phenomena rather than the specific outcome is a quite legitimate and ordinary way of confronting theory with observations."[26]

[26] Bak (1996, p.11).

Part II

Stylised Facts of Financial Markets

The empirical facts presented in part one are mostly a collection of special incidents challenging the statistical content of the EMH. Their existence is a serious thread to the idea of rational markets, but besides these peculiarities one has to address the question to the more general behaviour of asset prices in order to give a realistic picture of stock markets. This is captured in what empirical researchers call stylised facts. However, after Bachelier's early groundbreaking work on the random character of asset prices, statistical analysis of stock markets languished for a long time. The 1950's then witnessed a body of quantitative work that formed the basis for the EMH and were soon collected in the classic anthology of Cootner (1964). Shortly afterwards, a first complete study on daily returns was provided by Fama (1965). In his later survey (Fama (1970)) few studies can be found that challenge the idea of efficient markets, and in 1978 Jensen concludes that "there is no other proposition in economic which has more solid empirical evidence supporting it than the EMH."[27] However, the arrival of high frequency data unleashed another boost of empirical research. These studies comprise data sets up to fourty million data points and so constitute a huge step towards a more precise description of the fundamental statistical properties that those markets exhibit. How are returns distributed, what is the pattern of volatility or is there any kind of autocorrelation are typical questions of the researchers. As an introduction, the following lines give a short summary of the arguably most important stylised facts of financial markets.[28] Any hypothesis about the mechanism of asset markets has to meet these statistical features.

(i) *Fat tails and aggregational normality:* The distribution of returns do not show the exponential decay of a Gaussian distribution but a power-law behaviour that resembles Pareto-like distributions, including the class of stable distributions. These statistical models are much closer to the empirical data than the Gaussian, because they exhibit a tail behaviour that allows for more extreme values. This feature, termed fat tails, does, however, not prevail for every time scale. It is noticed that the distribution of returns over larger periods of time (a month, half a year, a year) departs remarkably from those of high-frequency data. For example, quarterly returns are distributed like a Gaussian.

(ii) *Multiscaling and multifractality:* Brownian Motion and its generalisation fractional Brownian Motion have special scaling abilities. These manifest themselves as a direct relation between averaged absolute price changes and the corresponding time intervals, irrespective of whether first, second or higher moments are taken. Real financial time series show a more complicated pattern called multiscaling. Closely related to this topic is the question of the fractal, respectively multifractal dimension of the time series.

[27] Jensen (1978, p. 95).

[28] For other compact reviews of stylised facts of financial markets see in particular Cont (2001) and Bouchaud and Potters (2001). See also Pagan (1996) for a more extensive treatment of some aspects.

(iii) *Short and long term autocorrelation of returns and volatility clustering:* Linear autocorrelation is found to show statistical significant results only for very short time scales. Although this seems to violate the weak form of EMH, correlations are so small that, considering transaction costs, profits are hard to realise. For longer time scales, empirical research seems to support the weak form of market efficiency. This is totally different for absolute returns. They display a very slow decay in their autocorrelation function. Volatility, whether measured as the variance or as the absolute value of price changes occurs in cluster, i.e. events of high variability tend to be followed also by high variability events.

(iv) *Other stylised facts:* There are some other statistical features that are common among financial time series but are not pinpointed by simulations to such an extent as the above mentioned. These are namely:

a) *Volume-volatility correlations*: trading volume is positively correlated with the volatility.

b) *Leverage effect:* past returns and future volatility is negatively correlated. This feature holds both for individual assets and indices.

c) *Cross-correlations in periods of high volatility:* Cross-correlations between individual stocks fluctuate substantially in time, but a common finding is that it increases in times of high volatility.

d) *Gain-Loss asymmetry*: Drawdowns show a different pattern compared with the course of big upward shifts. Especially big drawdowns happen in a sharp and sudden manner recovering in the same way, while upward shifts occur in a more smooth kind. Moreover, negative maximal price changing are larger in a crash than they are for positive values in a bubble.

The subsequent chapters will only deal with the first three points in more detail, because these are the facts that are targeted by the simulations almost exclusively. All other points, though important, have not attracted the same interest. This may be due either to the less general character of some of them (e.g. the cross-correlation between individual stocks) or to the problems of the simulations to reproduce them. In any case, own simulations will also focus on the "main" facts, i.e. the fat tails, the multifractality and the autocorrelation structure.

All of the following chapters are organised in the same way: the first section explains the necessary mathematical background of the subject. This is indispensable for a proper understanding of the enclosed statistical subsection where the empirical methods are presented. Because this work draws heavily from previous empirical studies, the third section comprises the facts found so far. The last section then provides own empirical analysis of some of the previously not studied financial markets.

Chapter 4

Heavy tails

To mention before one of the main findings of empirical research in finance, stock returns have been found not to be well approximated by the Gaussian normal distribution. This is clearly pointed out by the seminal papers of Mandelbrot (1963a,b) and Fama (1965), who were able to show that *stable-Pareto distributions* with infinite second moment yield a better empirical fit. Since then many researchers have contributed new facts about the distribution of asset returns, but the overall consensus is that it has fatter tails than the normal.[1] Intuitively, fat-tailedness means that the distribution has more large observations in its (right and left) tail than a reference distribution, which in the case of returns was for a very long time the normal. Moreover, the Gaussian displays an exponential decay of its values, separating it from the power-law decay of (many) heavier distribution. So far, neither longer time series nor higher frequencies have altered this finding. Knowing the exact form of the distribution plays beside its scientific content a vital role for the risk management of financial institutions.

Despite being a seemingly undisputed stylised fact, the question of what exactly fatter tails are is still open. The literature does not have a general accepted definition under which a tail ranking is possible, and so a variety of attempts have been proposed. For example, some focus on the 4-th moment of the distribution (its kurtosis). According to this definition a random variable x is fat-tailed if

$$E\left[\frac{(x-\mu)^4}{\sigma}\right] > 3.$$

Here, $E[x] = \mu$ and σ is the standard deviation from this mean. The Gaussian has as its 4-th central moment a value of 3, so in case the above inequality applies for x, excess kurtosis is present, and the distribution is said to be leptokurtic, i.e. it has a high peak, thin midrange, and the fat tails. This definition, however,

[1] The literature has numerous denotations for this fact. It is alternatively called heavy, fat, long or thick tales. All terms will be used synonymously throughout this work.

hinges crucially on the existence of the 4-th moment, which is not given for some important distributions. In this case, no discrimination can be achieved by considering only the kurtosis. A similar approach demands an infinite second moment in order to be classified as a heavy tailed distribution distributions. But as Bryson (1982) claims, the attempts that focus on a single moment are too crude and should be replaced by other concepts which consider the tail behaviour of the distribution more explicitly. Accordingly, distributions are said to be fat tailed if their tails are heavier than exponential. This criterion is the crucial point of almost every recent classification. Additionally, the aim is often to provide a definition as general as possible in order to account for a wide range of known distributions. A first step was made by using distributions with regularly varying tails. However, this class turned out to be not complete (for instance they do not include the Weibull) and so the class of heavy-tailed distributions are often represented by the so-called sub-exponentiell distributions. In the following, a brief classification is provided that aims to give a ranking from the most general (even more general than the sub-exponentiell) to more specific examples of distributions showing heavy-tailed forms.[2]

4.1 Heavy tailed distributions

4.1.1 The Class L

Let $F(x)$ be a distribution function in the domain $(0, \infty)$ and let $\overline{F}(x)$ denote its tail, $\overline{F}(x) = 1 - F(x) = P(X > x)$, where the last term is the probability that the variable X exceeds some value x. A density function (*df*), or equivalently a random variable (*rv*) X is called fat-tailed if for $\overline{F}(x) > 0$, $x \geq 0$ and any other *rv* $y \geq 0$, the following property holds:

$$\lim_{x \to \infty} P(X > x + 1/X > x) = \lim_{x \to \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1. \tag{4.1}$$

Defining $a(x) \sim b(x)$ as meaning $a(x)/b(x) \to 1$ as $x \to \infty$, then (4.1) can also be expressed through

$$\overline{F}(x+y) \sim \overline{F}(x), \quad \forall\, y \geq 0. \tag{4.2}$$

All distributions that satisfy the above given properties are summarised as class $L-$distributions, $F \in L$ (or $X \in L$). Expressions (4.1) and (4.2) have the following intuitive interpretation: if it is known that a *rv* X exceeds a large value x, then it is likely that it also will exceed any larger value. It can be shown that this class of distributions is characterised by the nonexistence of its exponential moments, i.e.

$$E\left(e^{\varepsilon X}\right) = \infty,\ \varepsilon > 0, \tag{4.3}$$

[2] The classification used in the next sections draws heavily from Embrechts, Klüppelberg and Mikosch (1997). See also Bamberg and Dorfleitner (2001).

and furthermore

$$\lim_{x\to\infty} F(x)\exp(\lambda x) = \infty,\ \forall\ \lambda > 0. \tag{4.4}$$

Class L distributions comprise a great deal of different distributions and are therefore a reasonable candidate for defining heavy-tailedness if the broadest possible criterion is aimed for. However, this class suffers from the possibility of infinite variances and even means. As a consequence, some distributions $F \in L$ are of neglectable use in the empirical analysis.

4.1.2 The Class S

In 1964 Chistyakov introduced the class S of the sub-exponentiell distributions. This class is at present the favourite candidate for heavy tailed distributions - in part due to its widely use in insurance, stochastic networks, communication and finance. It can be characterised in two different ways, where the first is concerned with the convolution property of the tails of the *df* of $F(x)$. This property asserts that the tail of a distribution $\overline{F}(x)$ retains its shape after the summation of identical and independent copies of the *rv* x, or equivalently described, the tail of the sum of n *iid rvs* displays the same form as the tail of the *rv* itself.

If $\overline{F^{*n}}$ denotes the n-fold convolution of $\overline{F}$, i.e. $\overline{F^{*n}}$ is the sum of n summations of $\overline{F}$, $\overline{F^{*n}} = \overline{F} + \overline{F} + ... + \overline{F}$, the convolution property can be defined as follows:

> Definition 4.1 (Sub-exponentiell distributions; Embrechts, Klüppelberg and Mikosch (1997), p. 39 f.):
>
> The *df* $F(x)$ in the domain $(0,\infty)$ is called sub-exponentiell if for $\overline{F}(x) > 0$, $x \geq 0$ and *iid*, and all $n \geq 2$,
>
> $$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n(x) \tag{4.5}$$
>
> holds.

This is denoted by $F \in S$.[3]

The definition of subexponential distributions can also be expressed by considering the probability of the sum of *rvs*

$$P(X_1 + X_2 + ... + X_n > x) = P(S_n > x).$$

Relation (4.5) can thus be restated by

$$P(S_n > x) \sim nP(X > x), \tag{4.6}$$

[3] A sufficient condition for sub-expontiality is given in Embrechts, Klüppelberg and Mikosch (EKM henceforth, 1997, p. 40): $F \in S$ whenever $\lim_{x\to\infty} \sup \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} \leq 2$. For a proof see EKM (1997, p. 40).

which says that the sum of *rvs* that is larger than some x has (in the limit $x \to \infty$) the same *df* as the n-th convolution of the probability that X exceeds the same value x, or

$$\lim_{x\to\infty} \frac{P(S_n > x)}{P(M_n > x)} = 1, \tag{4.7}$$

where M_n is the maximal value of $S_n = \sum_{i=1}^{n} X_i$.

In this case, the intuition behind (4.7) is that the sum S_n is dominated by its largest value or, similarly, S_n is likely to become large because one of the *rvs* becomes large. Although there exist some distributions for which $F \in S$ but $F \notin L$, class S distributions satisfy properties (4.5) and (4.7) and may also have infinite first and second moments. Therefore, the same reservations that were brought forward against the class L does also hold for this class. However, since class S is still quite broad, not all of these distributions feature the problem of infinite moments. Klüppelberg (1988) introduces a subclass of S, the class S^* which contains only distributions with a finite mean μ.

Definition 4.2 (Goldie and Klüppelberg (1998, p. 444)):

Let F be a *df* defined on the domain $(0, \infty)$ such that

$$F(x) < 1, \ \forall \ x \geq 0.$$

Then $F \in S^*$ if F has finite mean μ and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\, dy = 2\mu. \tag{4.8}$$

Conditions for $F \in S^*$ can be found in Goldie and Klüppelberg (1998). Examples of distributions $F \in S^*$ include the Burr, the Weibull, the lognormal and the Benktander type I and II distribution. Two other examples that belong to the class S^* are the Pareto and the Lévy-stable distribution. Because they play an important role in the empirical analysis of fat tails, both are treated separately in more detail. Another example, the student's t-distribution, is treated in chapter 4.2.

4.1.3 Power-law distributions

This subset of class S distributions is characterised by a tail which approximately follows an inverse power-function, i.e. it decays as $x^{-\alpha}$. All *df*s that exhibit such tail behaviour are usually collected under the name of *regularly varying functions*.

Definition 4.3 (regular varying functions):

A positive Lebesgue measurable function $f(x)$ on $(0, \infty)$ is regularly varying with index α, if

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^{\alpha}, \ t > 0$$

$$\overline{F} = x^{-\alpha} f(x); \quad x > 0$$

in order to have a regularly varying tail. For detailed accounts of proofs and properties of regular varying function see Bingham, Goldie and Teugels (1989).

Equivalently, the behaviour of the tails can be described by

$$\lim_{x \to \infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\alpha}, t > 0. \tag{4.10}$$

This is also a widespread used definition of fat tailedness.[4] It says that the tail of a distribution, when measured in its far out region ($x \to \infty$) behaves like an inverse power-law.

4.1.4 The Pareto distribution

In contrast to many members of the last class, this class only comprises distributions which have an exact Pareto tail. The Pareto distributions were originally introduced by Vilfredo Pareto (1897) who tried to model the distribution of wealth. Its cumulative distribution function is given by

$$F(x) = 1 - u^{\alpha} x^{-\alpha}, \ x \geq u \text{ and } u > 0. \tag{4.11}$$

The correspondent tail is therefore

$$\overline{F}(x) = u^{\alpha} x^{-\alpha}. \tag{4.12}$$

Since the Pareto distribution is a subclass of the power-law distribution, all regular varying functions also possess tails of the form

$$\overline{F}(x) \sim K x^{-\alpha}, \ x \to \infty, \ K, \alpha > 0. \tag{4.13}$$

The density of the exact Pareto distribution is given by $\alpha u^{\alpha} x^{-\alpha-1}$ and is therefore also called a power-function distribution. Its most interesting parameter α can be related to the moments by

$$E\left[x^k\right] = \alpha u^{\alpha} \int_u^{\infty} x^{k-\alpha-1} dx. \tag{4.14}$$

[4] See for instance Shiryaev (1999, p. 334).

Therefore, only the first $k < \alpha$ moments exist. For example, the first moment is in the case of the Pareto given by $E[x] = \frac{\alpha}{\alpha-1}$ for $\alpha > 1$, but does not exist for $\alpha \leq 1$, so $E[x] = \infty$. The same is true for $\alpha > 2$ where $Var[x] = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}$, but $Var[x] = \infty$ for $\alpha \leq 2$.[5]

Having a tail index above or below 2 is a crucial point for Pareto-like distributions. If $\alpha > 2$, the *df* features fat tails but its variance exists (and so enables the use of many standard statistical tools). If $\alpha < 2$ tails are so long that the integral of $F(x)$ does not converge to a finite value. This case gives rise to the stable distributions treated in the following paragraph.

An extension to the Pareto distribution is the generalised Pareto distribution (GPD). Its *df* is defined by

$$F(x, \sigma, k) = \begin{cases} 1 - (1 - kx\sigma^{-1})^{\frac{1}{k}}, \text{if } k \neq 0 \\ 1 - \exp(-x\sigma^{-1}), \quad \text{if } k = 0. \end{cases} \tag{4.15}$$

Here, k is called the shape parameter and σ the scale parameter. The GPD was introduced to model exceedances over some prespecified threshold value X (see Pickands (1975) as the original source). It will play a role in the treatment of extreme events.

4.1.5 The Lévy Stable distribution

This is the distribution Mandelbrot (1963a,b) initially suggested as a statistical model for asset prices.[6] The choice of stable distributions for modelling asset prices can be tributed to two convenient facts. The first is that they constitute a domain of attraction for the sums of *iid rv*s, i.e. the sum of such *rv*s eventually converges to a limit distribution, which belongs to the stable family. This *Generalised Central Limit Theorem* also includes the Gaussian distribution, but the class of (Lévy) stable distributions (henceforth LSD) represents a generalisation of the normal in the sense that it allows for both asymmetries and fat tails. The second reason is the property of being stable, which implicates shape-invariance under scale transformations. This has a desirable consequence: the same distribution can be applied to daily, weekly and monthly data.

Before the characteristics of stable distributions will be laid out in more detail, a closer look is taken of the meaning of the word *stable*. A first heuristic explanation follows the above reasoning: Let $P(x(n\Delta t))$ be the *pdf* of a variable x that is a function of time t. How does this *pdf* change with time? Loosely speaking, a stable distribution may vary in scale - it becomes wider with increasing time index $t(n\Delta t)$ - but not in form, or the distribution conserves its

[5] For a more thorough treatment of the Pareto distribution see Johnson, Kotz and Balakrishnan (1994). For an extensive analysis of the close relationship between Pareto and exponential distributions see in particular Galambos and Kotz (1978).

[6] He used the term stable Pareto distribution to characterise and separate it from the Pareto distribution with $\alpha \geq 2$.

shape under temporal convolution. A more mathematical treatment gives the following

Definition 4.4a (stable distributions):

A distribution function $F(x)$ is said to be stable if there exist constants $c > 0$ and γ that satisfy

$$F(ax + \alpha) \oplus F(bx + \beta) = F(cx + \gamma) \tag{4.16}$$

for any $a, b > 0$ and any a, b.

Definition 4.4a asserts that for every n, the sum $x_1 + x_2 + ... + x_n$ of independent random variables with common distribution F has a distribution function of the form $F(c_n x + \gamma_n)$ so that

$$\frac{x_1 + x_2 + ... + x_n}{c_n} - \gamma_n$$

has the same distribution F as the x_i.[7]
An equivalent definition that rests upon the random variables itself is given by

Definition 4.4b (stable distributions):

A non-degenerative random variable X is said to be stable or stable in the broad sense if for x_1 and x_2 independent copies of X and some positive constants a and b exists in the form of

$$ax_1 + bx_2 \overset{d}{=} cX + d, \tag{4.17}$$

for some c and $d \in R$. If $d = 0$, then X is said to be strictly stable or stable in the narrow sense. If furthermore $X \overset{d}{=} -X$, then the random variable is termed symmetric stable.[8]

Both definitions give a broad meaning but do not provide a functional form for the distribution of X. This problem was solved by Lévy (1925) and Khintchine and Lévy (1936). They found a way to describe all possible stable distributions by its characteristic function (CF) or Fourier transform. For a continuous random variable x with density $F(x)$, the CF is defined by

$$\phi(t) = E\{\exp(itx)\} = \int \exp(itx)F(x)dx.$$

This function $\phi(t)$ does not only describe the distribution completely, but has the advantage of a closed form presentation. The parametrisation of the following definition is the currently prevailing one (see Samorodnitsky and Taqqu (1994)):

[7] For a proof see Ibragimov and Linnik (1971). There are several other monographs on stable distributions. The more recent are Samorodnitsky and Taqqu (1994), Janicki and Weron (1994) and Uchaikin and Zolotarev (1999). All of them treat stable distributions in a comprehensive mathematical manner, which cannot be outlined in this extensive form here.

[8] These are not the only definitions of stable random variables, see e.g. Samarodnitsky and Taqqu (1994, p. 2f.).

Definition 4.4c (stable distributions):

A random variable x has a stable density function F if for parameter values of $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in R$, such that the CF of F can be expressed by

$$E\left\{\exp\left(itx\right)\right\} = \int e^{itx} dF(x)$$

$$= \begin{cases} \exp\left\{-\sigma^{\alpha}\left|t\right|^{\alpha}\left(1 - i\beta\left(\operatorname{sign} t\right)\tan\frac{\pi\alpha}{2}\right) + i\mu t\right\}, \alpha \neq 1 \\ \exp\left\{-\sigma\left|t\right|\left(1 - i\beta\frac{2}{\pi}\left(\operatorname{sign} t\right)\ln\left|t\right|\right) + i\mu t\right\}, \quad \alpha = 1 \end{cases} \quad (4.18)$$

with the sign function defined by

$$\operatorname{sign} t = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

For $\alpha = 1$ the factor $\ln|u|$ appears in the imaginary part of (4.18). This is the source of many numerical problems in computing the CF. Therefore, $\alpha = 1$ is usually treated as a separate case. The CF can also be written as

$$E\left\{\exp\left(itx\right)\right\} = \exp\left\{\sigma^{\alpha}\left(-\left|t\right|^{\alpha} + itw\left(t, \alpha, \beta\right)\right) + i\mu t\right\}, \quad (4.19)$$

where

$$w\left(t, \alpha, \beta\right) = \begin{cases} \beta\left|t\right|^{\alpha - 1}\tan\frac{\pi\alpha}{2}, & \text{if } \alpha \neq 1 \\ -\beta\frac{2}{\pi}\ln\left|t\right|, & \text{if } \alpha = 1. \end{cases}$$

Although the functional form of these CFs are far from being obvious, definitions 4.4a and 4.4b already imply definition 4.4c.[9] A proof of the relation between the two expressions can be found for example in Gnedenko and Kolmogorov (1954, section 34) or Ibragimov and Linnik (1971, chapter 1-2); Samorodnitzky and Taqqu (1994) offer a sketch of the proof.

The class of stable distributions has four parameters to fully characterise its form:

1. The scale factor σ: This parameter can be interpreted as a measure of the width of the distribution.
2. The symmetry factor β: With $\beta = 0$, LSD are symmetric around the mean irrespective of the other parameters. $\beta > 0$ gives a skewed distribution to the right while $\beta < 0$ results in a left-skewed form.

[9] The form of (4.19) is additionaly presented because one estimation technique is based on it. It should also be noted that (4.19) becomes $E\left\{\exp\left(iux\right)\right\} = \exp\left\{-\delta^2 u^2 + i\mu w\right\}$ in the case of $\alpha = 2$, which is the characteristic function of the Gaussian with mean μ and variance $2\delta^2$.

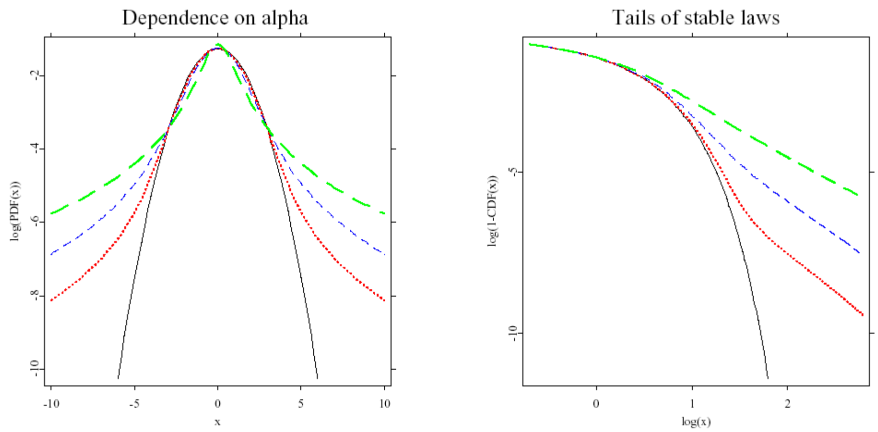

Figure 4.1: Left picture: pdf of four symmetric variants (all with $\beta = \mu = 0$). The solid line is the usual Gaussian distribution. The green long-dashed has $\alpha = 1.0$, the blue dashed $\alpha = 1.5$, the red dotted line $\alpha = 1.8$ and the black solid line represents $\alpha = 2.0$ (i.e. the Gaussian). Only the centered part is shown because the tails are too havy to be plotted with a normal in the same figure. The right panel shows the associated (right) tails. Borak, Härdle and Weron (2005, p. 3).

3. The shift parameter μ: μ represents the mode of the distribution, and is also frequently called the parameter of location. μ indicates the position of the mode. With $\mu = 0$, the distribution is centred around 0 for $\alpha \geq 1$. $\mu > 0$ indicates a positive, $\mu < 0$ a negative mode.

4. The index parameter α: This is the most important parameter of LSD, as it is the index of stability. It determines the shape of the tails.

Figure 4.1 shows the pdf for four different distributions. The parameters σ, β and μ are held constant with $\beta = \mu = 0$ and $\sigma = 1$ respectively. α takes on values of $1.0, 1.5, 1.8$ and 2.0. As can be seen, tails becoming less pronounced the closer α gets to the Gaussian ($\alpha = 2$). However, the picture also gives an idea of how slowly the LSD approaches the normal as $\alpha \uparrow 2$.

Lévy (1925) was able to show that stable distributions feature a Pareto-like tail behaviour. Specifically, if x is a standardised random variable with skewness parameter β that belongs to the family of $\mathrm{LSD}_{\alpha,\beta}(x; 1, 0)$ with $\alpha < 2$, the tail $\bar{F}(x)$ behaves like

$$\bar{F}(x) \sim (1 + \beta) C_\alpha x^{-\alpha}, \tag{4.20}$$

with

$$C_\alpha = (2 \int x^{-\alpha} \sin x dx)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin(\frac{\alpha\pi}{2}). \tag{4.21}$$

as $x \to \infty$.[10] It is thus a possible model for the distribution of empirical data if heavy tails are observed in the variables.

The difficulty with stable distributions is the missing of closed form solutions. Integral representation only exist for the Lévy-Smirnov distribution ($\alpha = 0.5$, $\beta = 1$), the Cauchy distribution ($\alpha = 1$, $\beta = 0$) and the Gaussian distribution ($\alpha = 2$). By now some good numerical calculations have been performed, but the problem of the bad applicability for statistical tests still remains. The Gaussian distribution is of course the most familiar member of LSD and its tractability has made it a routinely assumption for the error terms in a linear regression. But since many samples contain outliers that are improbably large for the normal distribution, the stable distribution is a logical extension from a theoretical point. Though, LSD with $\alpha \neq 2$ have for a long time been given little attention in the field of statistical inference. Because of its infinite variance, non-normal LSDs preclude many statistical applications. However, this pragmatic view cannot rule out the possibility of an infinite-variance distribution.

There is a notable new proposal called Truncated Lévy flights (TLF) which allows for finite variances but still preserves the shape of the body of a LSD with $\alpha < 2$. Its distribution $P(x)$ is given by

$$P(x) \equiv \begin{cases} 0 \;\; x > l \\ cP_l(x) - l \leq x \leq l \\ 0 \;\; x < -l \end{cases} \tag{4.22}$$

where c is a normalising constant and P_l is a $LSD_\alpha(x; \alpha, \beta, \delta, \mu)$. The idea is that a LSD might be a good statistical approximation if the data set is considered for $-l \leq x \leq l$, but fails to model the distribution for very large values. I.e. the truncated Lévy distribution is "Lévy-like" in the central part of the distribution, but in the far out tails it decays faster than a pure LSD. It is this feature that ensures finite moments and an exponential decay as $x \to \infty$. Another way to think of the TLF is by observing its behaviour under temporal convolution. Stable distributions conserve their form, so that their *pdf* is the same for daily ($\Delta t = n$ =1 day) and monthly ($\Delta t = 30n \approx 1$ month) data. Because of (4.22), the TLF is not stable: different time scales n yield also different distributions, which means that daily returns can be statistically distinguished from monthly returns. Hence, there is a crossover value for the used time index $\Delta t = n$, n_x, so that for $n < n_x$ $P(x_n)$ behaves very much like a pure LSD, in particular, it features a power-law tail with exponent α. As soon as $n > n_x$ x progressively converges to a Gaussian distribution $P_G(x_n)$. This can be expressed formally by

[10] See property 1.2.15 of Samorodnitsky and Taqqu (1994); see also Bergström (1952). Here, $\Gamma(\alpha)$ represents the Gamma function. Because of this tail behaviour, LSDs are also frequently called stable Paretian distributions. This is the term Mandelbrot (1963a) used originally in his work.

$$P(x_n) \approx \begin{cases} P_L(x_n), & \text{if } n < n_x \\ P_G(x_n), & \text{if } n > n_x. \end{cases} \quad (4.23)$$

n_x serves as a cut off point that divides the two regimes. Mantegna and Stanley (1994) show that the convergence to the Gaussian regime is generally very slow and can take up to $n \geq 10^4$ data points to ensure the crossover.

The process of a TLF (4.23) has a sharp truncation at n_x which might not be very sensible for empirical data, because it would indicate a LSD with e.g. $\alpha = 1.5$ for a particular range $\Delta t = n$. But as soon as this range is passed, the behaviour predicted by the CLT sets in. To account for a more gradual transformation, Koponen (1995) introduced the smooth TLF (STLF). Its distribution is defined by

$$P(x_n) = \begin{cases} Ca\,|x_n|^{-1-\alpha}\, e^{-\lambda\left|\bar{x}\right|}, & \text{for } x < 0 \\ Cb{x_n}^{-1-\alpha} e^{-\lambda \bar{x}}, & \text{for } x > 0, \end{cases} \quad (4.24)$$

with Ca and Cb being constants.[11] Here, the process does not feature only two abruptly divided regimes but allows for a more flexible approach to the normal distribution. In any case, the most distinguishing factor between the ordinary LSD and its two variants is the finiteness of the moments. In contrary to the ordinary Lévy distribution, TLF and STLF finally converge to a Gaussian process. This however, limits the estimation of the tail-index to some extend. Even finding values that would indicate an infinite second moment does not rule out the possibility of normal distributions. It may be the case that the true process reveals itself only after considering returns for progressively larger time differences.

4.2 Alternatives to the stable distribution

There are some other statistical models that are used in the literature to describe financial data sets. This section briefly cites the most important alternatives to the LSD assumption and its variants. The class of hyperbolic distributions introduced by Barndorff-Nielsen (1977) is a notable recent alternative to the LSD when it comes to model the statistical features of price returns. A property that distinguishes it from the Pareto like distributions is the semi-fatness of their tails, which means that it interpolates between a Gaussian body and exponential tails. The general one dimensional hyperbolic distribution is defined by

[11] These values are defined by the symmetry parameter through $\beta = \frac{Ca-Cb}{Ca+Cb} = \frac{x_1^\alpha - x_2^\alpha}{x_1^\alpha + x_2^\alpha}$; see Koponen (1995).

$$P_{hyp}(x) = \frac{(\alpha^2 \ - \ \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\sigma^{\lambda}K_{\lambda}(\sigma\sqrt{\alpha^2-\beta^2})}\left[\sigma^2+(x-\mu)^2\right]^{(\lambda-1/2)/2}\times \quad (4.25)$$

$$\times K_{\lambda-1/2}\left(\alpha\sqrt{\sigma^2+(x-\mu)^2}\exp(\beta(x-y))\right).$$

K is the modified Bessel function of the third kind with index λ. The parameters α and β, $\alpha > 0$ and $0 \leq |\beta| < \alpha$, determine the shape of the distribution. μ is the location parameter and σ is the scale of $P_{hyp}(x)$. For $\sigma \to \infty$ and $\sigma/\alpha \to \sigma^2$ $P_{hyp}(x)$ goes over to the Gaussian (in the limit of n $\to \infty$). However, if $\sigma < \infty$ the hyperbolic distribution behaves for large values of x like $\exp(-\alpha\,|x|)$ and therefore has fatter tails than the Gaussian, although not as fat as those of power-law distributions. The name of this class of distributions comes from the hyperbolic shape of their log-density. The task of the researcher is to find the values of the parameters that does replicate the empirical data with the best fit.[12]

In 1974 Blattberg and Gonedes questioned the LSD and instead proposed the Student's t-distribution as the most appropriate statistical model of price changes. Its distribution is given by

$$P(x) = \frac{C_n}{(1+x^2/n)^{(n+1)/2}}, \quad (4.26)$$

where

$$C_n = \frac{\Gamma\left[(n+1)/2\right]}{\sqrt{\pi n}\Gamma(n/2)}. \quad (4.27)$$

For $n = 1$, $P(x)$ coincides with the Cauchy distribution, and $n \to \infty$ leads to the Gaussian. So for $1 < n < \infty$, the Student's t-distribution can be varied freely to approximate the distribution of price change. Because the distribution is a member to the class S, it falls under the most popular definition of fat-tailedness. Its tails in fact follow a power-law and so the t-distribution is a reasonable candidate for a statistical model.

Clark (1973) introduced the mixture of Gaussian distributions. The basis of this model uses the concept of subordinated processes.[13] The idea is that a random process $X\left[\Omega(t)\right]$ is formed out of another random process $x(t)$. $X\left[\Omega(t)\right]$ is then subordinate to $x(t)$, since this process governs X. This concept is adopted to financial markets by noticing that market activity is not equally distributed over the whole time span, but fluctuates. For this reason the sequence of equally spaced time points $t_1, t_2, ..., t_n$ is not a good candidate for the underlying driving

[12] See e.g. Barndorff-Nielsen and Prause (2001) who are able to achieve a good fit for the (hourly) log-returns for the US-Dollar/Deutsche Mark exchange rate.
[13] See Feller (1971).

force of price changes. Instead Clark (1973) proposes the trading volume as a measure of the dynamics of ΔP. The cumulative volume between two time points t_1 and t_2 represents the directing process $\Omega(t)$. The price changes are then governed by $\Omega(t)$ and the distribution is simply $P\left[S\ \left[\Omega(t)\right]\right]$ where S is here the price. This process is subordinate to $P\left[S(t)\right]$ which is the empirical distribution with usual time indices, but is directed by $P\left[\Omega\right]$. Clark assumes a Gaussian distribution for $P\left[S(t)\right]$. By claiming $P\left[\Omega\right]$ to have all moments finite, he could show that $P\left[S\ \left[\Omega(t)\right]\right]$ has a leptokurtic form with finite moments.

4.3 Discussion

Student's t and the hyperbolic distributions are analytically more convenient if compared with the LSD because they can be presented in a closed mathematical form. The LSD in turn is only numerically to compute. It furthermore restricts the tail index to be lower than 2, producing an infinite variance, while the other may have finite moments.[14] This consequence often raises serious concerns about the LSD. The problem lies in the routinely use of the variance $Var\left(x\right)$ as a measure to describe the distribution of price increments and the volatility – often identified as the risk of the market under scrutiny. With infinite second moment, $Var\left(x\right)$ does not have a reasonable meaning. But the parameters μ and σ can take the place of mean and variance, because they determine the localisation of the mode and the width of the distribution. A more flexible approach is the introduction of TLF and STLF. It combines LSD and finite moments, but it is just as the hyperbolic and student's t-distribution not stable under temporal aggregation. The question is whether this constitutes really a severe disadvantage given the empirical fact of the aggregational gaussianity mentioned in the introduction to part two. In this case, stability would not be desirable.

In the remaining course of this work, the focus will be laid solely on the LSD distribution and its variants. The families of hyperbolic and mixture distributions are neglected. This restriction has three reasons. First, the majority of work on statistical features uses the LSD and its variants. Then, these distributions seem to be flexible enough to model real data. They are able to account not only for fat tails but also for a possible convergence towards the normal distribution. A last point is that the comparison of all statistical candidates for modelling return distributions is beyond the scope of this work, since it is primarily concerned with the problems of simulating stock markets.

4.4 Extreme Value Theory

In many applications of heavy tailed distributions, cumulative sums of *iid rvs* are the sole target of interest. This is justified in financial applications by

[14] Finiteness of moments is not secured for all parameter values of the student's t-distribution.

the study of $P(t)$ which should according to the EMH reflect the accumulated informational contents of the news that came into the market during a particular time span. However, analysing the n largest values of a time series has a right on its own. The methodological background for this is provided by a branch of statistical mathematics called *Extreme Value Theory* (EVT). It is concerned with the asymptotic distribution of standardised maxima from a series of *rvs* with a common distribution F. To be more precise, considering a stationary sequence $x_1, x_2, ...$ of *iid rvs* with common *pdf* F. Let

$$M_1 = x_1,\ M_n = \max\left(x_1, x_2, ..., x_n\right),\ n \geq 2 \tag{4.28}$$

denote the sample maximum of the sequence. M_n is an upper order statistic where n is generally arbitrary but often taken to be in the range of 1% - 5% of the whole sample size.

The CLT states that the sum S_n of *iid rvs* converges in limit to the normal distribution. Upon replacing S_n by M_n, one can ask the same question as to what distribution M_n will go. The following theorem, the basis of EVT, gives a summarised version of the answer.

Fisher-Tippett theorem for limit laws of maxima:[15]

Let x_n denote a sequence of *iid rvs* with *pdf* F. If there exists a norming constant $c_n > 0$, $d_n \in R$ so that

$$c_n^{-1}\left(M_n - d_n\right) \xrightarrow{d} H,\ n \to \infty, \tag{4.29}$$

where H is a *pdf*, then H belongs to one of the following three types of EV- *pdfs*:

Gumbel (Type I):

$$\Lambda_\alpha\left(x\right) = \exp\left\{-e^{-x}\right\}, x \in R, \tag{4.30}$$

Fréchet (Type II):

$$\Phi_\alpha\left(x\right) = \begin{cases} 0 & \text{if } x \leq 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x > 0 \end{cases} \qquad \text{for } \alpha > 0, \tag{4.31}$$

Weibull (Type III):

$$\Psi_\alpha\left(x\right) = \begin{cases} \exp\left\{-\left(-x\right)^\alpha\right\} & \text{if } x \leq 0, \\ 1 & \text{if } x > 0 \end{cases} \qquad \text{for } \alpha > 0. \tag{4.32}$$

[15] See EKM (1997, p. 121).

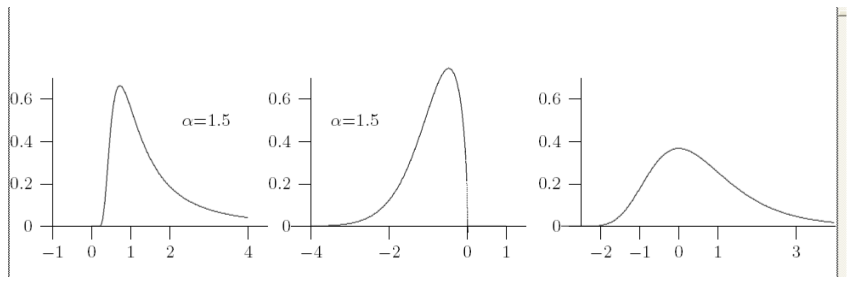

Figure 4.2: Probability densities for the Fréchet, the Weibull and the Gumbel. α values for the first two are both fixed at 1.5. See Gilli and Kellezi (2003, p. 5).

This theorem states simply spoken, that the order statistic M_n, if being appropriately scaled by some normalising constants (c_n, d_n), converges in probability, $P\left(c_n^{-1}\left(M_n - d_n\right) \leq x\right)$, weakly to one of the three types of *pdf*, also frequently termed as the three max-stable distributions (Leadbetter et al. (1983)).[16] Figure 4.2 shows the densities of all three types.

The main task that arises in this context is to determine what process belongs to the MDA of which of the above given max-stable distributions? In other words, the question is from which *pdf* F come variables whose n-largest values take the form of either $\Psi_\alpha(x)$, Λ_α or $\Phi_\alpha(x)$? Because Type III has short tails, it is of no interest for financial applications.[17] The MDA $\Lambda_\alpha(x)$ is much more important since it compromises the normal, the exponential and the log-normal distribution as special cases. However, in the face of heavy tails in empirical data sets, the Fréchet distribution plays the key role in the discussion of an appropriate statistical model. A very important fact is that the set of distributions which are attracted by the Fréchet is confined to those with regular varying tails, which exactly corresponds to the class of power-law distributions. This can be motivated by a Taylor expansion of Φ_α, since for $\alpha > 0$ one has

$$1 - \Phi_\alpha(x) = 1 - \exp\left\{-x^{-\alpha}\right\} \sim x^{-\alpha}, x \to \infty. \tag{4.33}$$

Thus, the tail of Φ_α decreases like a power-law. The following theorem restates this finding more accurately:

> Theorem for the maximum domain of attraction for Φ_α :[18]
>
> The *pdf* of a sequence (x_n) of *iid* non-degenerate rvs belongs to the $MDA(\Phi_\alpha)$ with $\alpha > 0$, if and only if
>
> $$1 - F(x) = x^{-\alpha} L(x), \ x > 0 \tag{4.34}$$

[16] A thourough derivation of the max-stable distributions and its subsequent proof is presented in Resnick (1987).

[17] An example of a distribution which is attracted towards the Weibull is the uniform distribution.

[18] See EKM (1997, p. 131).

with $L(x)$ being a slowly varying function as defined in (4.9). If $F \in MDA\left(\Phi_\alpha\right)$, then

$$c_n^{-1} M_n \xrightarrow{\alpha} \Phi_\alpha. \qquad (4.35)$$

Thus, the class of distribution that satisfy this theorem contains the Pareto, the Cauchy, the Burr and the LSD with $0 < \alpha < 2$. They all have tails of the form $1 - F(x) \sim K x^{-\alpha}$, $x \to \infty$, for some constant K. It is again important to note that the Gaussian does not belong to this group.

It is possible to comprise all three types of extreme value distributions in a one-parameter representation, ξ, the so called *Generalised Extreme Value* (GEV) distribution.[19]

Definition 4.5 (Jenkinson - von Mises representation of extreme value distributions):

The family of GEV distributions is given by

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{\frac{-1}{\xi}}) & \text{for } \xi \neq 0 \\ \exp(-e^{-x}) & \text{for } \xi = 0, \end{cases} \qquad (4.36)$$

where x is chosen so that $1 + \xi x > 0$.

The three different types can easily be regained by the replacement of ξ through α in the following way:
$\xi = \alpha^{-1} > 0$ yields the Fréchet,
$\xi = 0$ yields the Gumbel, and
$\xi = -\alpha^{-1} < 0$ yields the Weibull distribution.
An equivalent definition asserts that there exists a function $a(u)$ so that

$$\lim_{u \uparrow x_F} \frac{\overline{F}\left(u + x a(u)\right)}{\overline{F}\left(u\right)} = \begin{cases} (1+\xi x)^{\frac{-1}{\xi}} & \text{for } \xi \neq 0 \\ e^{-x} & \text{for } \xi = 0, \end{cases} \qquad (4.37)$$

and $1+\xi x > 0$.[20] If x belongs to a *df* with $MDA\left(H_\xi\right)$, then the above definition can be reformulated as

$$\lim_{u \uparrow x_F 0} P\left(\frac{x-u}{a\left(u\right)} > x | x > u\right) = \begin{cases} (1+\xi x)^{\frac{-1}{\xi}} & \text{for } \xi \neq 0 \\ e^{-x} & \text{for } \xi = 0, \end{cases} \qquad (4.38)$$

In this form (4.38) presents an approximation to the MDA for normalised excess over some threshold u (chosen to be sufficiently high). This gives rise to the important

[19] The adjective generalised does not imply that GEV is more general than the above given three expressions, but is merely a useful reparametrisation.

[20] u is a sufficiently high threshold and x_F its right endpoint. See EKM (1997, p. 159).

Balkema-de Haan theorem for maximal losses in excess of a threshold:[21]

A df F which has a MDA (H_ξ) possesses a distribution of excess losses that can be approximated by a generalised Pareto distribution of the form (4.15), provided the threshold is sufficiently large.

By replacing x through $\widetilde{x} = \frac{x-\mu}{\sigma}$, where μ and σ are location and scale parameter respectively, the GDP is enlarged by $GDP_{\xi;\mu,\beta}$.

Summarising the results so far, a subsample consisting of the n-largest values of a time series leads, provided that n is sufficiently large, to one of the three types of max-stable distributions. Moreover, if the sample is drawn from a distribution with regular varying tails, then its MDA is the Fréchet whose tails behave like a power-law. In the case the variable of interest is the maximal loss for example of a portfolio, the relevant limit distribution is the GDP or the $GDP_{\xi;\mu,\beta}$.

4.5 Empirical Methods for Heavy Tailed Distributions

4.5.1 Quantile Plots

Before applying more sophisticated estimation methods, a close look at the data itself can give a first hint about the underlying *df*. The use of graphical techniques is e.g. strongly advised by Gumbel in his book *Statistics of Extremes* (1958). A powerful graphical tool for testing whether two distinct data sets come from the same distribution is by applying the so-called qq- (or quantile-) plot as a goodnes-of-fit procedure. It begins with the specification of a reference distribution $F(x, \theta)$. For the returns this is usually the Gaussian as the standard EMH assumption. The quantiles of the empirical data and the reference distributions are then plotted against each other. If the data coincide in distribution, the Glivenko-Cantelli theorem implies a straight line for the plot. Deviations show up as bended functions.

4.5.2 Estimation Methods for Heavy Tailed Distributions

Statistical methods that are concerned with heavy tailed data are quite large in number. The upcoming application of EVT and the usage of stable heavy-tailed distributions in many fields like insurance, traffic and economics has given rise to the development of various new techniques of estimation. Some of them hinges on an a priory determined *pdf* to which the empirical data is supposed to belong to. Even the qq-plots are compared with a prespecified distribution. Estimation methods fall into one of the four following categories: tail estimators, quantile methods, maximum likelihood methods and estimation based on the

[21] See EKM (1997, p. 152).

characteristic function of LSD. The next chapters explain the way in which these different techniques work by picking up their most representative estimators. It is important to note that the performance of each method (especially its small sample behaviour) is not reported in the section where it is introduced. Instead, the last paragraph summarises some of the recent studies on this topic.

4.5.3 Tail Estimators

There are many estimators that follow the advice of Du Mouchel (1983) that when it comes to estimate the *fatness* of a distribution, the tails should speak for themselves, i.e. only the tails of the empirical distribution should be taken for the estimation. All of these estimators are based on the n-largest (or lowest) observations in a sample. The results of chapter 4.4 have demonstrated that the upper (or lower) tail of a fat-tailed distribution function F behaves asymptotically as the tail of the (generalised) Pareto distribution, and it is this fact that makes all following methods rely on the order statistics of extreme events. One of the first and most popular tail-estimators is the Hill-estimator (Hill, 1975). It is still widely in use and takes on the following form

$$\widehat{\alpha}_{n,m}^{(H)} = \left\{ \frac{1}{n} \sum_{j=1}^{n} \left[\ln X_{j,n} - \ln X_{m,n} \right] \right\}^{-1}, \ m > 1. \tag{4.39}$$

$X_{1,n},\ X_{2,n}, ..., X_{m,n}$ are the order statistics in decreasing order and the bracketed term $[\ln X_{j,n} - \ \ln X_{m,n}]$ is the logarithmic difference between them. The integer n determines where the tail area begins, i.e. the threshold after which values are taken for the order statistic. However, determining n involves a trade-off between the number of variables in the order statistic that can be used in the estimation and the question how far one is going out in the tails. The further this is, the less data survives for (4.39), but the more it concentrates on the high values and so avoids possible flaws because of too many smaller values that may indicate finite variances.

The reasoning behind the Hill estimator follows from the fact that

$$F \in MDA\left[\Phi_\alpha\left(x\right)\right]$$

only if the tail of a *df* $F(x)$ is regularly varying. Reformulating (4.39) through integrating by parts yields

$$\int_t^\infty \left(\ln x - \ln t\right) dF\left(x\right) = \int_t^\infty \frac{\overline{F}\left(x\right)}{x} dx. \tag{4.40}$$

By Karamata's theorem this leads to[22]

$$\frac{1}{\overline{F}(x)}\int_{t}^{\infty}(\ln x-\ln t)\,dF(x)\rightarrow\frac{1}{\alpha},\ t\rightarrow\infty. \tag{4.41}$$

An estimator of $\xi=\frac{1}{\alpha}$ can be found by replacing t with a sufficient high level. In the case of (4.39), $t=X_{n,m}$ is taken for some $m=m(n)$. This replacement yields the estimator

$$\frac{1}{\overline{F}_n(X_{m,n})}\int_{X_{m,n}}^{\infty}(\ln x-\ln X_{m,n})\,dF_n(x)=\frac{1}{n}\sum_{j=1}^{n}\ln X_{j,n}-\ln X_{m,n}.$$[23]

The Hill-estimator has the convenient feature of being weakly consistent under fairly general conditions:

> Theorem for the consistency of the Hill estimator (EKM, 1997, p. 336f.):
>
> The Hill-estimator $\widehat{\alpha}^{(H)}_{n,m}$ is weakly consistent i.e. $\widehat{\alpha}^{(H)}\stackrel{d}{\rightarrow}\alpha$ if one of the following three conditions is satisfied:[24]

(i) (x_n) is *iid* (Mason (1982)),

(ii) (x_n) is weakly dependent, i.e. the temporal dependence between the rvs becomes weaker as the time separation increases (Roótzen, Leadbetter and de Haan (1992)), or

(iii) (x_n) is a linear process (Resnick and Stărică (1995, 1996)).[25]

Econophysicists often use the simplest and most straightforward method for estimating α, a plot of the right (or left) tail of the empirical cumulative density function $1-F(x)$ against $\log(x)$. The method uses the fact that

[22] Karamata's theorem says that for $L\in R_0$ in the open intervall $[x_{0,}\ \infty]$, and $x\geq 0$ one has for $x\rightarrow 0$:
(i) $\int_{x_0}^{\infty}t^{\alpha}L(t)dt\sim(\alpha+1)^{-1}x^{\alpha+1}L(x)$, if $\alpha>-1$, and
(ii) $\int_{x}^{\infty}t^{\alpha}L(t)dt\sim(\alpha+1)^{-1}x^{\alpha+1}L(x)$, $if\ \alpha<-1$.
See Karamata (1933).

[24] $\stackrel{d}{\rightarrow}$ means asymptotic in distribution.

[25] Conditions for strong consistency are derived by Deheuvels, Häusler and Mason (1988). There are a couple of variants of the Hill estimator which are all derived by the same considerations as above.These variants include proposals by Pickands (1975), Dekkers et al. (1990) and De Haan and Resnick (1979). Comparisions between the Hill-estimator and other similar estimators is given in de Haan and Peng (1998).

$$\log \bar{F}(x) \sim -\alpha \log(x).$$[26]

It is therefore often referred to as the cumulative or log-log method. The procedure starts with selecting the n largest values of a sample. Then the log-probability of each rare event is taken. Summing up these numbers in decreasing order gives $\log \overline{F}(x)$. In a second step logarithms are also taken of the values itself. Plotting both series against each other on a x, y-diagram and estimating the slope by OLS gives an estimated value $\widehat{\alpha}$. Compare the linear relationship above to equation (3.2). It makes the connection to power-laws and complex systems obvious, because the equation simply says that the (n) highest values of price fluctuations appear as a straight line if plotted against their probability of occurrence. This is nothing else than the universal law Bak found in all complex systems.

4.5.4 Sample Quantiles Methods

The idea to use quantiles as a basis for estimation was first proposed by Fama and Roll (1968, 1971), but their method is restricted to symmetric *pdfs*. McCulloch (1986) proposes a generalised approach that is able to provide consistent estimators for all possible values of β $[-1, 1]$, μ and σ, and for α in the range $[0.6, 2]$.[27] The arguably most attractive feature of the McCulloch estimator is its computational simplicity.

Let x_i, $i, ..., n$, be n independent drawings from a numerically computed $LSD(x; \alpha, \beta, \mu, \sigma)$. If $p = f(LSD(\alpha, \beta, \mu, \sigma))$ is the probability of finding a value that exceeds $\overline{x}$, then x_p is the corresponding p-th population quantile. $\widehat{x}_P$ is the equivalent p-th quantile of the empirical distribution, which is supposed to take the form of a LSD but with unknown parameters. In order to estimate α and β explicitly, McCulloch defines the function

$$\nu_\alpha = \frac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}}. \tag{4.42}$$

Its empirical counterpart is the statistic

$$\widehat{\nu}_\alpha = \frac{\widehat{x}_{0.95} - \widehat{x}_{0.05}}{\widehat{x}_{0.75} - \widehat{x}_{0.25}}. \tag{4.43}$$

This statistic is a consistent estimator of ν_α, since ν_α is strictly decreasing in α.

In a similar fashion ν_β and $\widehat{\nu}_\beta$ are defined by

$$\nu_\beta = \frac{x_{0.95} - x_{0.05} - 2x_{0.5}}{x_{0.95} - x_{0.05}}, \tag{4.44}$$

[26] See (4.34).

[27] However, this poses no severe restriction since $\alpha < 1$ yields distributions with no finite first moment, a situation highly unsuitable for financial data.

$$\widehat{\nu}_\beta = \frac{\widehat{x}_{0.95} - \widehat{x}_{0.05} - 2\widehat{x}_{0.5}}{\widehat{x}_{0.95} - \widehat{x}_{0.05}}. \tag{4.45}$$

This function is for given values of α strictly increasing in β and is therefore a consistent estimator of ν_β as well. Because ν_α and ν_β do not depend on σ and μ, but only on α and β one may write

$$\widehat{\nu}_\alpha = \phi_1(\alpha, \beta) \quad \text{and} \quad \widehat{\nu}_\beta = \phi_2(\alpha, \beta).$$

These relationships can be inverted to infer the desired estimated parameters by:

$$\widehat{\alpha} = \varphi_1(\widehat{\nu}_\alpha, \widehat{\nu}_\beta) \quad \text{and} \quad \widehat{\beta} = \varphi_2(\widehat{\nu}_\alpha, \widehat{\nu}_\beta).$$

McCulloch (1986) provides tables with values for α and β depending on ν_α and ν_β.

The method of estimating the scale parameter is done in the same manner by defining

$$\nu_\sigma = \frac{x_{0.75} - x_{0.25}}{\sigma}. \tag{4.46}$$

and relating the behaviour of ν_σ to α and β :

$$\widehat{\nu}_\sigma = \phi_3(\alpha, \beta).$$[28]

The empirical values for $\widehat{x}_{0.75}$ and $\widehat{x}_{0.25}$ and the already estimated parameters $\widehat{\alpha}$ and $\widehat{\beta}$ can now be used to derive a consistent estimator for σ by

$$\widehat{\sigma} = \frac{\widehat{x}_{0.75} - \widehat{x}_{0.25}}{\phi_3\left(\hat{\alpha}, \hat{\beta}\right)}.$$[29]

4.5.5 Maximum Likelihood Estimation

The first Maximum Likelihood Estimation (MLE) has been conducted in a remarkably early stage of using LSDs to fit empirical data. Du Mouchel (1971) is able to compute log-likelihood functions by first grouping data into bins and then using a combination of means. Later approaches to MLE are undertaken by Brant (1984) who approximated the likelihood function with the help of characteristic functions, and Brorsen and Yang (1990) and McCulloch (1998). In this work, the MLE-approach of Nolan (1997) will be applied. His program STABLE gives not only accurate calculations of stable distributions, which is in any case an indispensable condition for afterwards used estimation techniques. It also contains all ingredients needed to numerically estimate the shape of a

[29] μ is calculated in a similar fashion. For details see McCulloch (1986).

given data set. The following representation is a short summary of how the Nolan-MLE procedure works.[30]

The variables of the sample are assumed to come from a stable distribution with parameter space $\theta = (\alpha, \beta, \sigma, \mu)$. The parameter space is restricted to $\alpha\,(0,2]\,,\ \beta\,[-1,1]\,,\ \sigma\ (0,\infty)\,,\ \mu\,(\infty,-\infty)$. First, sample values $x_1, ..., x_n$ are normalised by

$$\tilde{x} = \frac{x-\mu}{\sigma}.$$

The values for μ and σ are taken from some prior estimation, which in the case of the Nolan program is done with the quantile method of McCulloch (1986). The standardised *pdf* of the variables is therefore given by

$$F(\tilde{x}; \alpha, \beta, \sigma, \mu) \equiv f(\tilde{x}; \theta).$$

The MLE of θ is then obtained by maximising

$$L(\theta, x) = \sum_{i=1}^{N} \log f(x_i \mid \theta)\,. \tag{4.47}$$

with respect to the unknown parameter vector θ.

The main difficulty is the absence of closed formulas for stable densities. Since a numerical approximation is computer-intensive, this obstacle prevented a frequent use of MLE. As an initial starting point for the search of $\hat{\theta}$, the estimation results of the quantile method of McCulloch are taken. Then a quasi–Newton method is used to maximise $L(\theta, x)$ numerically. The goal of this procedure is to find that set of parameter values θ which, given the sample variables, achieves a best fit to one of the set of all calculated LSDs. For $\widehat{\theta}$ in the interior of the parameter space, Du Mouchel (1973) is able to show that MLE of θ leads to consistent and asymptotically normal distributed under certain regularity conditions, i. e.

$$\sqrt{N}(\hat{\theta} - \theta_0) \underset{\rightarrow}{d} N(0, I^{-1}(\theta_0))$$

where I^{-1} is the inverse of the Fisher information matrix $I:\ I_{ij} = \int\limits_{-\infty}^{\infty} \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} dx$.

4.5.6 Estimators based on the Characteristic Function of LSD

The idea to estimate the four parameters of a LSD by fitting the characteristic function of a sample to the stable distribution model was first proposed by

[30] There exsists no paper that explicitely describes the routines used in the program. A Matlab routine is available; the version used here is the DOS-version that is freely downloadable from John Nolan's homepage http://academic2.american.edu/~jpnolan/.

Press (1972). Following his example Paulsen, Holcomb and Leitch (1975), Arad (1980), Koutrouvelis (1980, 1981) and Feuerverger and McDunnough (1981) also employ approaches based on the sample characteristic function. From all these methods it is shown that the Koutrouvelis estimation has the best performance (Akgiray and Lamoureux (1989)). It is therefore the natural choice for selecting it as the representative of this method. Koutrouvelis presents a regression-type method which starts from the characteristic function of LSD as defined by

$$E\left\{\exp\left(itx\right)\right\} = \exp\left\{-\left|\sigma t\right|^{\alpha}\left(1 - i\beta f\left(t,\alpha,\sigma\right)\right) + i\mu t\right\}, \tag{4.48}$$

where $f\left(t,\alpha,\sigma\right)$ is given by.

$$f\left(t,\alpha,\sigma\right) = \begin{cases} \frac{t}{|t|}\tan\frac{\overline{u}\alpha}{2}, & \text{if } \alpha \neq 1 \\ -\frac{2t}{\pi|t|}\log\left|\sigma t\right|, & \text{if } \alpha = 1.^{31} \end{cases}$$

For the derivation of his estimator Koutrouvelis takes the logarithm of (4.48):

$$\log\phi(t) = -|\sigma t|^{\alpha} + i\left(\mu t + |\sigma t|^{\alpha}\beta\frac{t}{|t|}\tan\frac{\alpha\pi}{2}\right), \tag{4.49}$$

for the case of $\alpha \neq 1$. He then extracts the real part of (4.49),

$$\mathrm{Re}[\log\phi(t)] = -\left|\sigma t\right|^{\alpha} = -\sigma^{\alpha}\left|t\right|^{\alpha}, \tag{4.50}$$

and takes its logarithm in order to arrive at the linear function

$$\log\mathrm{Re}[\log\phi(t)] = \alpha\log|t| - \alpha\log\sigma, \tag{4.51}$$

which constitutes a linear relationship between $\log|t|$ and $\log\left[\mathrm{Re}\log\left[\phi\left(t\right)\right]\right]$. In a simple linear regression α could then be interpreted as the slope and σ is determined by the intercept. The imaginary part of (4.49) is given by

$$\mathrm{Im}(\log\phi(t)) = \mu t + \beta\left|\sigma t\right|^{\alpha} f\left(t,\alpha,\sigma\right). \tag{4.52}$$

Without considering principal values, the last two equations lead to

$$\arctan = \frac{\mathrm{Im}(\phi(t))}{\mathrm{Re}(\phi(t))} = \mu t + \beta\left|\sigma t\right|^{\alpha}\frac{t}{|t|}\tan\frac{\alpha\pi}{2}. \tag{4.53}$$

Having established the relationships between parameters and the log-CF, the question remains how to get the desired estimates $\widehat{\alpha}$ and $\widehat{\sigma}$. Before the regression takes place, the sample data is normalised by the following reparametrisation

$$\bar{x} = \frac{x - \widehat{\mu}_0}{\widehat{\sigma}_0}.$$

$\widehat{\mu}_0$ and $\widehat{\sigma}_0$ are taken from prior estimations of spread and location parameters. These are done by a quantile method.[32] The sample characteristic function is then given by

[32] Koutrouvelis used the quantile method of Fama and Roll (1971).

$$\tilde{E}\exp\left(it\tilde{x}\right) = \frac{1}{N}\sum_{N=1}^{N}\exp\left[it\widetilde{x}\left(t\right)\right], \tag{4.54}$$

with N being the sample size. Now, $\widehat{\alpha}$ and $\widehat{\sigma}$ are obtained by the regression

$$\log\left[-\log\phi(t_k)\right] = \left(\widehat{\alpha}\log|t_k| + \alpha\log\widehat{\sigma}\right) + e_k,\ e_k \sim iid\ (0,\sigma^2), \tag{4.55}$$

where (t_k) is a set of real numbers, dependent on the sample size and α.[33] Once $\widehat{\alpha}$ and $\widehat{\sigma}$ are determined, the remaining two parameters $\widehat{\beta}$ and $\widehat{\mu}$ are obtained with the help of (4.52):

$$\operatorname{Im}\log\phi\left(t_k\right) = \left(\widehat{\mu}t_k + \beta\left|\widehat{\sigma}t_k\right|^{\widehat{\alpha}} f\left(t_k,\widehat{\alpha},\widehat{\sigma}\right)\right) + \eta_k,\ \eta \sim iid\ (0,\sigma^2). \tag{4.56}$$

The regressions can now be performed by using OLS. Having obtained $\widehat{\alpha}$ and $\widehat{\sigma}$ from the first regression these values can be used as new estimates to normalise x for a second round. The procedure now consists of the following steps:

1. Normalise the empirical data
2. Use the normalised data in order to calculate $\log\left[-\log\phi(t_k)\right]$.
3. Perform regression (4.55) and (4.56).
4. Take the estimated values for a new parameterisation of x and repeat steps 2. and 3.
5. Stop when a predefined criterion for improvements is met.

4.5.7 The performance of the estimators

This chapter summarises some recent studies that analyse the possibilities and limitations, especially the small sample behaviour of the estimators presented so far. Their performance is usually measured by the root mean-square error, RMSE, of the parameter estimate

$$RMSE\ \left(\hat{\theta}\right) = \left[\frac{1}{N}\sum_{N=1}^{N}\left(\theta - \widehat{\theta}\right)^2\right]^{\frac{1}{2}},$$

where N is the the number of realisations used in the simulation, θ is the true estimator and $\widehat{\theta}$ is the estimated parameter value here the tail exponent α.

[33] The need of a set of real numbers t_k comes from the fact that $\operatorname{Re}\phi(t) = \exp(-|\sigma t|^{\alpha})\cos\left[\mu t - |\sigma t|^{\alpha}\beta\ sgn(t)\tan\frac{\pi\alpha}{2}\right]$. In order to calculate the empirical counterpart $\operatorname{Re}\hat{\phi}(t)$, one must account for the periodicity of cos [.]. Koutrouvelis proposes to use $t_k = \frac{2k\pi}{25}, k = 1, 2, ..., 134$.

(i) Weron (1995) compares the McCulloch and the Koutrouvelis estimator by generating Lévy-stable distributed random variables with varying parameter combinations. For each combination, r samples of size $N = 500$ are calculated. This is indeed a very low value compared to available data sets even for a daily frequency. If the estimators perform reasonable for $N = 500$, then ordinary data sets of about three to five thousand points should not cause any problems. The analysis reveals biases ranging from about 0.001 to 0.01 standard deviations for the index of stability. Thus, they do not show a severe missing of the true parameter and this impression gets even stronger for sizes of $N = 2000$, where the RMSE is reduced to values not higher than 0.003. Moreover, one might notice that both methods perform similar in the parameter space $\alpha \in [0.6, 2.0]$, $\sigma \in [0.1, 10]$. A disadvantage of McCulloch's method is its restriction to α-values above 0.6, and its rather poor performance for wide distributions ($\sigma > 10$). But for financial data, the index of stability can be supposed to lie somewhat above 1 and all tests on logarithmic data show that the width parameter hardly reaches values of around 10.

(ii) Kogon and Williams (1998) take the McCulloch, the Koutrouvelis and an estimator based on EVT for their study.[34] They use sample sizes of $N = 200, 500, 1000, 2000$ and 5000 points of simulated LSDs to compare the behaviour of the selected methods for estimating all four parameters, where the index of stability takes on the values $\alpha = 1.0, 1.2, 1.5$ and 1.95. As it turns out, the Koutrouvelis method is slightly better than McCullochs quantile estimator, both becoming more reliable as $\alpha \uparrow 2$. The RMSE of α is 0.15 for $N = 200$, but quickly approaches values under 0.04 for sample sizes of 2000 and more data points. In all cases, the EVT-estimator performance is the worst. This picture does not change dramatically for the symmetry, spread and location parameter.[35] The method based on the CF (Koutrouvelis) slightly outperforms the McCulloch estimator. For example $N < 1000$ results in a RMSE, which is above 0.1, but soon approaches errors well under 0.1 ($N = 2000$). The general impression is that the estimators perform better the closer α is to 2. In general, the impression of the Weron study is approximately reproduced. The biases are considerable low, even for small samples. However, only LSDs are considered, so there is no test of the usefulness of the estimators for other distributions.

(iii) Nolan (1997) calculates stable random variables in order to infer the small sample behaviour of his estimator. As in all other studies, the performance considerable rises with growing sample size. For $N = 1000$ the standard error is below 0.1. This might cause some problem for α-values near the boundary, but in the more interior of the parameter space, a validation of the stable hypothesis is fairly accurate. However, the MLE of Nolan faces the same problem as the Koutrouvelis and McCulloch estimator when non-stable distributions with tails longer than the Gaussian are considered. Nolan simulates Pareto-distributed

[34] They use a tail estimator similar to the Hill-estimator.

[35] Because the EVT-estimator only targets the tail, it cannot be used to calculate β, μ or σ and is therefore omitted in the comparision.

variables with $\alpha = 1.5$. As it turns out, the (averaged) estimated values yield 1.23 and thus give a rather poor result. In this case, Nolan suggests to use further tests such as qq-plots in order to infer the non-stability of the underlying data set.

(iv) McCulloch (1997) performs a Monte Carlo simulation with symmetric stable distributions for various different values of α ($0.2 \leq \alpha \leq 2$). He chooses a sample size of only 3000 points and uses the pooled 5% upper and lower values of the simulated set.[36] Table 4.1 shows the minimum, the median and maximum values obtained by applying the Hill estimator.

Table 4.1: Results of McCulloch's (1997) Monte Carlo Simulation

true α	Min.	Median	Max.
2.00	5.117	6.728	8.838
1.99	4.203	6.095	8.313
1.95	2.963	4.609	7.164
1.90	2.211	3.160	5.365
1.80	1.559	2.211	3.988
1.70	1.308	1.827	3.159
1.60	1.195	1.645	2.935
1.50	1.111	1.485	2.687
1.40	1.097	1.328	2.542
1.30	1.014	1.214	2.130
1.20	0.886	1.111	1.744
1.10	0.799	1.029	1.617
1.00	0.706	0.931	1.484
0.80	0.573	0.739	1.202
0.60	0.431	0.552	0.840
0.40	0.294	0.365	0.548
0.20	0.158	0.185	0.282

As can be seen from the table, even for true values $\alpha < 2$, the estimated tail indices are well above 2. The biases aggravate the closer the stable gets to the normal distribution. For true values of $\alpha \leq 1.6$, the Hill gives at least approximately reliable results for rejecting a Gaussian.

(v) Höpfner and Rüschendorf (1999) is a further analysis of the Hill-estimator. They come to the conclusion that it is only reliable for $\alpha \leq 1.5$. Above this value, the Hill-estimator cannot discriminate between a Gaussian and a stable non-Gaussian distribution and thus Höpfer and Rüschendorf are able to confirm similar results of previous studies like the one above from McCulloch (1997) or Resnick (1997) and Drees and Kaufman (1998). As a consequence, they suggest for values of α in the neighbourhood of 2 to use a whole sequence of estimates for varying N, which gives a path of $\widehat{\alpha}(N)$. Only if one is able to detect flat

[36] This was because he wants to replicate the situation of a former paper by Loretan and Phillips (1994) whose daily records were about 3000 points.

parts in this curve of $\widehat{\alpha}(n)$, then "there is strong indication that the level of this flat is a good estimator of α."[37] The study furthermore corroborates the finding that even in the case of a LSD with appropriate low index of stability, only extremely large data sets assure reliable estimates.[38]

(vi) Weron (2001) analyses the Hill and the cumulative method. Like the Hill, the cumulative method is very sensitive to the sample size and the number of extreme events chosen to estimate $\overline{F}(x)$. Weron simulates α-stable distributions with parametrisation ($\alpha = 1.95$, 1.8, $\beta = \mu = 0$, $\sigma = 1$) with a sample size of $N = 10^4$, 10^6. By taking only the 0.5 % of each sample, his conclusion is that inferences on tail exponents are strongly biased upwards, when regarding the smaller sample size, but are more reliable for the 10^6 data set. However, even in these cases α-estimates are slightly above the true parameter.

To summarise the results, inferences that are based only on the maximum values of a data set are crucially depended on the size of the order statistic. The smaller M_n, the less extreme values are available and the estimation becomes less reliable. Financial records with 5000 data points have the shortcoming that for example their 1%-highest values yield only an order statistic of $n \approx 50$. But this incorporates the danger of a considerable upward bias for the index of stability. Thus, estimated values of the Hill estimator and the cumulative method have to be taken with some caution. However, own empirical results for both tail estimators are presented even for the daily records because they are the only ones that do not hinge on a presupposed distribution. Exactly this is the drawback of the parametric estimators, since they work well only when the sample under consideration comes indeed from a LSD. The Nolan, the Koutrouvelis and the McCulloch estimator can cope with relatively small data sets much better, but have no opportunity to reject the stable hypothesis in favour of other fat-tailed, though not stable distributions. Consequently, as suggested by many authors, only very large data sets offer a trustworthy decision about the appropriateness of the LSD-assumption.

4.6 Empirical Results in the Literature

Since Mandelbrot's seminal papers (1963a,b) many economists analysed the returns of various asset prices.[39] Instead of going through all contributions,

[37] Höpfer and Rüschendorf (1999, p. 147).

[38] Efforts have been undertaken to derive the optimal size of the max order statistic. See e. g. Danielsson and de Vries. (1997). However, it has turned out that even these suggestions fail to deliver accurate estimates, especially for values of α below 1.5 (see Höpfner and Rüschendorf (1999)).

[39] Surprisingly these efforts have not started immediately after Mandelbrots proposal of what he called *stable Paretians*. Some might argue the poor estimation techniques available at that time were responsible for the lack of interest in fat tails, others (like Mirowski (1995)) blame economists for not pursuing paths apart from mainstream econometric thought. It may be true that the lack of proper statistical tools hindered a quick application of LSD, but one of the important contribution to LSD came in 1971 with the Ph.d. thesis of DuMouchel.

a more interesting way is to observe how results are related to a particular estimation method. As could be expected, contributions from studies using the parametric approaches usually confirm the Mandelbrot hypothesis of returns with an infinite variance. For example Koutrouvelis applies his estimator to four corporations for the period between 1956 and 1975. The monthly data yield values of around 1.78-1.83 for all series. Nolan (1999) uses stock price data from the Center for Research in Security Prices (CRSP) to apply his MLE-estimator. He reaches a good fit with $\hat{\alpha} = 1.855$. More can be found in Rachev and Mittnick (2000) who give an extensive review of the use of stable distributions in finance.

Loretan and Phillips (1994) is an early example of examining a broad range of series including stock returns and exchange rates by applying the cumulative method. Their results show α-values in the range from $\alpha = 2.5$ to 3.2 for the monthly stock returns, and from 3.1 to 3.8 for daily stock returns. The problem with these values is the relatively small sample from which they are derived. The associated maximum order statistics is highest with $n = 250$ for the daily US stock markets, but only $n = 100$ for all other data sets. The studies reported in chapter 4.5.7 show that reliable estimates afford longer time series.

Lux (1996) takes the 30 German stocks form the DAX and applies the Hill-estimator to their daily returns from 1988-1994. By varying the upper and lower bounds for maximum returns from 15 % to 10%, 5 % and 2.5 %, Lux is able to derive values for α depending on the size of the order statistic. His estimates confirm the impression of Loretan and Phillips (1994), but still suffer from the same problem.

An early study of high frequency data (the Standard&Poor's 500 from 1984-1989) is provided by Mantegna and Stanley (1995, 1997). They use time intervals from 1 min. (493,545 data points) up to 1,000 min. (with only 562 data points remaining) to construct fluctuations defined by $\Delta R(t) = \ln P(t + \Delta t) - \ln P(t)$, where $P(t)$ denotes the value of the S&P index at time t and Δt symbolises the different time intervals for which $R(t)$ is computed. By using the scaling abilities of the LSD, they are able to estimate a value for α of 1.4. This would indicate a LSD with $\alpha < 2$, but only if the process is well described by a Lévy distribution over the whole range of Δt.

Gopikrishnan et al. (1998) are the first who get values for α above 2 with the help of the cumulative probability distribution $P(\Delta R)$. They study the three major US stock markets, the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX), and the National Association of Securities Dealers Automated Quotation (NASDAQ). The data is given on a tick-by-tick frequency comprising a total of $10^7 \times 4$ points. The fluctuation is defined by

Shortly afterwards Hill (1975) and Pickands (1975) introduced estimators for calculating the tails of distributions, so improvements on the estimation site have probably not hampered economists in employing LSD instead of a Gaussian distribution as a statistical description of asset returns.

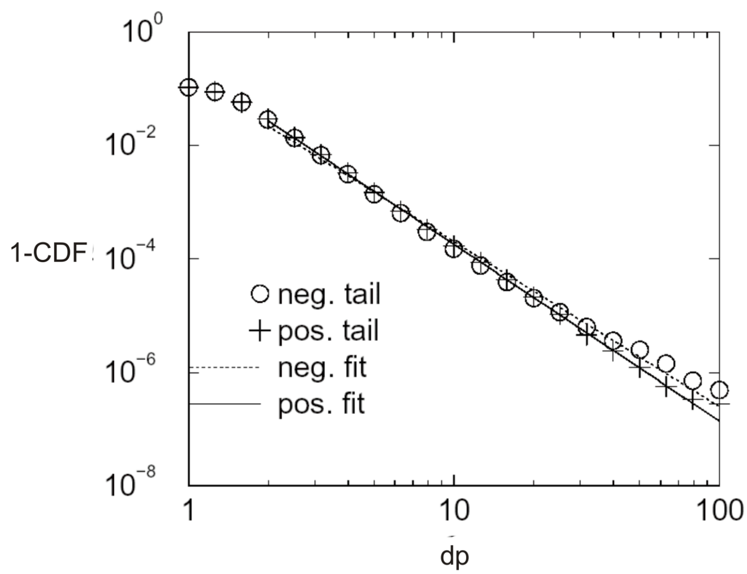

Figure 4.3: Cumulative probability distribution of the normalised price increments from the S&P 500 for the range $2 < g < 100$. The solid and the dashed line symbolise the power-law fit. See Gopikrishnan et al. (1998, p. 140).

$g_i = \ln P_i(t + \Delta t) - \ln P_i(t)$, where Δt is of the order of 5 minutes. Then they calculate the cumulative probability distribution for increments larger than a threshold g as given by $P(g) \equiv P(g_i(t) \geq \Delta g)$. The choice of g plays a crucial role in this procedure since the higher the threshold value, the further one goes out in the tails and the less data points remain. Therefore Gopikrishnan et al. apply $P(g)$ to values of g in the range $2 < g < 100$. Figure 4.3 shows this cumulative distribution.

Estimating the slope gives a fit according to $log\ P(g) \sim -\alpha \log\, \Delta g$ with $\alpha = 3.1 \pm 0.03$ for the positive increments (right tail) and $\alpha = 2.84 \pm 0.12$ for the negative increments (left tail). In order to test the robustness of $\alpha > 2$, Δt is increased from 5 to 120 minutes, but none of these changes in the time scale has a significant influence on the results which are still at $\hat{\alpha} \approx 3$. The next table reports the main results of recent works on the question of how heavy the tails really are.

Table 4.2: Some recent contributions on the literature on fat tails

Author	Data	Results
Gopikrishnan et al. (1999, 2000)	In both cases the S&P 500 and the Nikkei	Both studies yield tail parameters above 2 for both indices
Wang and Hui (2001)	Hang Seng (1994-1997) 1 min frequency	$\hat{\alpha}$ is about 4, indicating a possible finite fourth moment
Huang (2002)	Using the same data as in Wang and Hui (2001)	The cum. method yields $\hat{\alpha} = 4$ for the left tail and $\hat{\alpha} = 5$ for the right tail
Matia et al. (2002)	Daily returns for various commodity prices	$\hat{\alpha}$ is found to exceed 2 for each time series
Coronel-Brizio and Hernandez-Montoya (2004)	Daily records for the Dow Jones and the Mexican ICP (1990-2004)	Both indices yield a value of $\hat{\alpha}$ around three

All of the studies above use the cumulative method and as one can see, estimated tail exponents are above 2 in every case. This is in agreement with the assumption of a finite second moment. Hence there is a considerable confirmation of the conjucture that by using semi-parametric estimators, the LSD can be ruled out in favour of distribution like the TLF or the STLF. The following sections provide some own estimations on the tails of stock price increments.

4.7 Own Empirical Tests on the Tail parameter

4.7.1 The Data Sets

This work uses different types of data for the empirical investigation. They can be broadly put into two classes. The first consists of daily and therefore aggregate price records of various national stock indices and indivdual stocks from the Deutsche Aktienindex (DAX). Daily data is chosen deliberately in order to facilitate comparisons with the literature because there it is the prevailing frequency. Another motivation is the convenience in acquisition as daily records are provided by many free sources. The second group includes the so-called high-frequency data. These sets feature data points that are separated by only a few seconds. One set (the DAX) is provided on a tick-by-tick frequency, i.e. every new price is recorded at the time it is built. Therefore, these time series are not homogeneously spaced in the time domain. In order to circumvent problems that arise with the application of estimation techniques that are in need of homogenous series, the following (linear) interpolation method is used: Take a time interval that is large enough to have at least one tick in it and then take always the most recent value. So if there are three points in the interval

12:00:00 and 12:00:30, for example at 12:00:05, 12:00:12 and 12:00:28, the last one will take the place of the price at 12:00:30.[40] The other two sets (NEMAX and FDAX) already provide a small amount of aggregation with time intervals of 15, 30 and 60 seconds respectively so that there is no need to manipulate them. To be concrete, the data sets with a daily frequency comprises the following time series

- Standard and Poors 500 Index (S&P),
- Sydney All Ordinaries Index,
- Tokyo Nikkei 225 Index,
- London FTSE-30 Index
- Mexican IPC Index,
- Hong Kong Hang Seng Index,
- Singapore Straits Index,
- Paris CAC-40 Index,
- Toronto Composite index,
- Varta AG,
- BASF AG,
- BMW AG,
- Buderus AG,
- Bewag AG,
- Continental AG,
- Bayer AG,
- Kugelfischer AG,
- Phoenix AG and
- Harpen AG.[41]

[40] See Dacorogna et al. (2001) for other interpolation techniques.

[41] The data for the S&P and the FTSE-30 are provided by Terence Mills on http://www.lboro.ac.uk/departments/ec/cup/data.html. The FTSE is a daily record from 1935-1994, the S&P 500 is from 1928-1991. The time periods for the other daily records are: Singapore (28.12.1987 - 26.11.2003), Sydney (3.8.84 -26.11.03), Toronto (15.8.84 - 26.11.03), Hong Kong (31.12.86 - 26.11.03), Paris (31.12.87 - 26.11.03), Mexiko (19.4.1990 - 3.4.2003); all of these data sets are freely downloadable from freelunch.com. The data for the German companies, listed at the DAX is kindly provided by the Karlsruher Kapitalmarktdatenbank. The records span a period from the 12-th of march 1973 to the 28-th of december 2001.

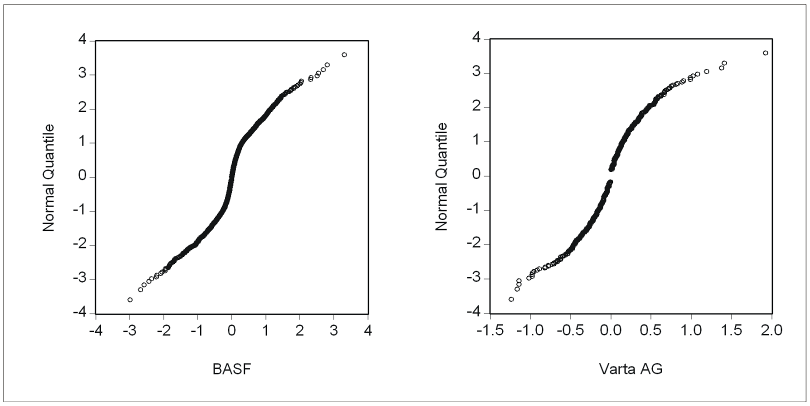

Figure 4.4: QQ-plots of of price changes for two daily price records, the BASF and Varta. The bent curvatures indicates non-normality.

The high-frequency data consists of the

- Deutscher Aktienindex (DAX),
- Nemax All Share and the
- Future Dax (FDAX).[42]

4.7.2 QQ-plots

Below, two of the above mentioned daily time series are tested for normality by applying qq-plots. These are the Varta AG and the BASF AG. As can be seen, these daily records show significant differences to the normal quantils. This is not a singular outcome. In fact, all of the daily time series show very similar qq-plots that all indicate a departure from the normal distribution. The comparison of the quantiles leads to the assumption that real data has much heavier tails (both left and right) than the normal.

4.7.3 Estimation Results for daily Price Records

In this section only samples with a daily frequency are considered in order to infer the distributional characteristics of returns. Here, especially the number and values of very large events, i.e. the points that are assembled in the tails of the distribution, are of interest. Although the overall distribution form of price changes can have important implications for e.g. portfolio selection, it is in particular the large events (both positive and negative) that decides upon

[42] The whole data sets amount to millions of quotes, kindly provided by the Karlsruher Kapitalmarktdatenbank.

the question whether empirical data sets are better modelled by an LSD with infinite variance or another statistical model for which there are finite moments at least up to an order of 2. Besides this rather principal point, a precise knowledge about the size of the tail gives the simulation models of part three a numerical goal to target. Thus, in order to capture the tail mass of a given empirical distribution most precisely, it is mandatory to estimate the so-called tail index (α) with the help of the instruments introduced in the subsections of chapter 4.5.2. At the start, only the parametric estimators are applied. They are all based on the CF of $LSD(x; \alpha, \beta, \sigma, \mu)$ and therefore the only question is whether $\alpha < 2$ or not. Table 4.3 displays values of α as the result of each estimation technique. The time series taken for the estimation are in each case the simple increments of the original price indices $P(t)$, i.e.

$$P\left(t+\Delta t\right)-P\left(t\right),$$

where Δt is one day.

Table 4.3: Parametric tail estimation for the daily records

	Estimators		
Series	Nolan's MLE	Koutrouvelis	McCulloch
S&P 500 Index	1.4454	1.5286	1.4452
Sydney All Ordinaries Index	1.7253	1.8287	1.7249
Tokyo Nikkei Index	1.2194	1.2709	1.226
London FTSE-30 Index	1.4457	1.5294	1.4451
Mexican IPC Index	1.6227	1.7299	1.6232
Hong Kong Hang Seng Index	1.4965	1.625	1.4981
Singapore Straits Index	1.529	1.6229	1.5301
Paris CAC-40 Index	1.642	1.7848	1.6429
Toronto Composite Index	1.5288	1.6289	1.53
Varta	1.3149	1.3993	1.3191
BASF	1.5539	1.7598	1.554
BMW	1.453	1.5929	1.4527
Buderus	1.2827	1.3666	1.2795
Bewag	1.2405	1.2787	1.243
Continental	1.515	1.692	1.517
Bayer	1.571	1.7507	1.571
FAG Kugelfischer	1.454	1.575	1.454
Phoenix	1.4069	1.5788	1.4077
Harpen	1.2038	1.107	1.2133

All estimated values are comparable to the results reported in the literature. There is almost no time series that shows a normal distribution. Even values of α close to two (say 1.9) are not detectable. Notable is the striking similarity between the Nolan-MLE and the McCulloch estimator for most of the time series, where the Koutrouvelis values regularly exceed the Nolan-MLE results. Anyway, the overall impression seems to validate the assumption of an underlying

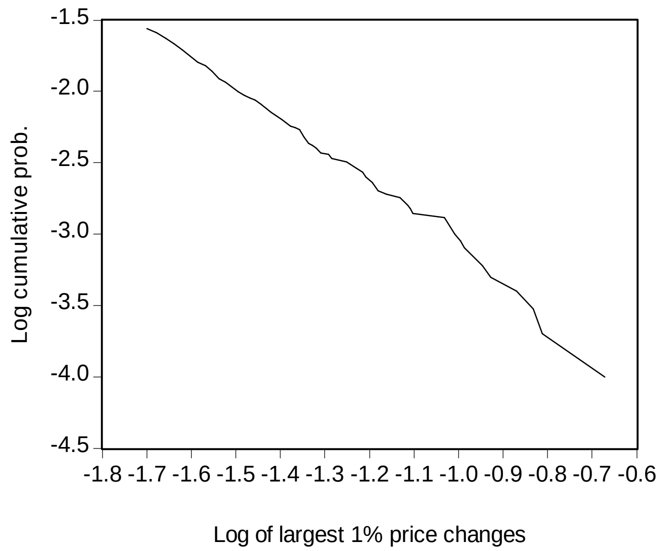

Figure 4.5: Cumulative distribution for the 1% highest negative price changes of the Sydney stock index. The function is plotted on a log-log scale, thus probabilities on the y-axe are negative.

stable process. However, the parametric estimators consider the whole distribution and the tail index is just one of four estimated parameters.[43] Moreover, these values are, because of the underlying LSD assumption, bound to yield values of $\alpha \leq 2$. But for the purposes of calculating the largest price fluctuations, it is only important to look at the tails. Hence, the semi-parametric tail estimations are next performed. Since they are not based on a specific a priori assumption about the data generating process, they are much more flexible than there parametric counterparts. Like former studies, estimated α-values now become much higher and regularly exceed the critical number two, rejecting the LSD-hypothesis in favour of a distribution with finite variance.

Figure 4.5 is an example of how the (log-) tail behaves when plotted against the highest (log) values. The behaviour of the cumulative probability is - in this case - consistent with a value of $\alpha \approx 3$. Recall that the method uses the fact that $\log \bar{F}(\Delta P) \sim -\alpha \log(\Delta P)$.

The remaining estimation results of the cumulative method are given in the next table. The left tail comprises the 1% highest negative (log) price changes

[43] The other parameter values, β, γ and δ are not reported in this work. However, they all have a very similar pattern. β is always very near to zero, so distributions do not show strong indications for asymetry. The location or shift parameter is also close to zero (i.e. μ ranges from -0.115 to 0.0633). The width parameter has a large fluctuation but is in most cases around 10.

while the estimation for the right uses the 1% highest positive values.

Table 4.4: Tail estimation by the cumulative method

Name of time series	left tail	right tail
S&P 500	3.364	3.382
Sydney stock index	3.511	3.117
Nikkei 225	3.427	3.309
FTSE	3.74	3.127
Mexican ICP	3.206	3.600
Hang-Seng	2.078	3.159
Singapur	3.254	3.089
CAC 40	3.996	3.929
Toronto stock index	2.515	3.217
Varta	3.225	3.301
BASF	3.802	4.252
BMW	2.764	3.397
Buderus	3.513	3.900
Bewag	3.71	3.466
Continental	3.322	3.811
Bayer	2.877	3.926
Kugelfischer	2.42	2.537
Phoenix	3.31	4.363
Harpen	3.614	3.118

Table 4.5 gives the estimated values for the Hill method. To account for the dependence of $\hat{\alpha}$ on the size of the set of largest (log) $\Delta P-$values for which the order statistic is calculated, the estimation is performed with varying percentage values (p). p is given in decimal numbers, i.e. p=0.01 is equal to the 1% (absolute) highest values of ΔP.The other estimations are carried out with the 2.5%, 5% and 10% of the highest price changes. This classification will be used throughout the whole work, both for the empirical data as well as for the simulated in order to facilitate the comparibility between them.

Table 4.5: Hill estimation of the tails for the daily records

Name of time series	p=0.01	p=0.025	p=0.05	p=0.1
S&P 500	2.7446	2.4825	2.3502	2.0522
Sydney stock index	4.7917	3.8436	3.1496	2.6245
Nikkei 225	3.2698	2.8546	2.4889	2.0233
FTSE	2.7446	2.8546	2.4889	2.0522
Mexican ICP	3.8586	3.5442	2.9058	2.4169
Hang-Seng	3.109	3.1976	2.7593	2.4725
Singapur	3.0588	3.0727	2.4372	2.1652
CAC 40	3.9098	4.385	3.688	2.9322
Toronto stock index	3.1603	2.9856	2.8368	2.4355
Varta	3.3217	3.0664	2.7321	1.9135
BASF	4.817	4.231	3.476	2.743
BMW	3.442	3.183	3.033	2.2887
Buderus	4.736	3.48	2.869	2.026
Bewag	3.497	2.9332	2.385	1.923
Continental	4.572	3.939	3.174	2.632
Bayer	3.999	3.367	3.2144	2.785
Kugelfischer	3.1498	3.11	2.692	2.133
Phoenix	4.335	4.4615	2.9163	2.2557
Harpen	3.065	2.99	2.593	1.905

The estimated parameter α displays a considerable amount of variability for different values of p. For some series the data seems to have finite third or even finite fourth moments if only the 1% highest values are taken but using a 10% cut off yields much smaller values, which for three cases indicate a stable regime. However, the McCulloch-study (1997) shows that only the largest values should be used to calculate the tail mass. The estimated values with $p = 0.01$ should be considered the most reliable, because there tail estimation is concentrated on the very large events.

What can be said about the outcomes of the different estimations so far? In general, they are in good agreement with the findings in the literature. As there, all parametric estimators yield $\alpha-$values well below 2 but the semi-parametric alternatives confirm the findings that distributions of price changes have finite moments up to order 2 at least, probably 3 and possibly even 4. This implicates that the LSD assumption of Mandelbrot proposed in his early contributions is not tenable. The empirical distribution has nevertheless much heavier tails that the Gaussian and is therefore, at a daily frequency, surely non-normal. However, does a drastic increase of frequency change the results? In order to answer that question, the three high-frequent data sets are investigated in the same way.

4.7.4 Own Estimation with high-frequency Price Records

Now, the three German high-frequency data sets are investigated. As above, the Koutrouvelis, the McCulloch and the Nolan estimator are first applied.

Table 4.6: Parametric tail estimation of the high-frequency records

Series	Nolan's MLE	Koutrouvelis	McCulloch
DAX	1.683	1.499	1.68
NEMAX	1.729	1.662	1.735
FDAX	1.501	1.37	1.455

As could be expected, estimated values of the index of stability are all in the range of the LSD. But, just as for the daily data, the application of the Hill and the cumulative method shows values above two.

Table 4.7: Tail Parameters of the high-frequency records (cum. method)

Series	right tail	left tail
DAX	2.945	3.081
NEMAX	3.49	3.55
FDAX	3.102	3.214

Table 4.8: Hill estimation of the tails of the high-frequency data

Series	p=0.01	p=0.025	p=0.05	p=0.1
DAX	3.072	2.522	2.36	2.22
NEMAX	3.953	3.01	2.514	2.13
FDAX	3.84	2.97	2.462	2.258

It should be noted that the two methods based on the EVT (the cumulative and the Hill estimator) can now be applied with much more confidence than for the daily records because of the large data sets that have enough points even if only the 1% largest values are used. Like the empirical studies in Table 4.2, the tail exponents are now all above the threshold value of 2. This confirms the view that another distribution, distinct from the LSD has to be chosen. Moreover, the empirical literature (see e.g. Eberlein and Keller (1999), Mantegna and Stanley (2000) and Bouchaud and Potters (2001)) is unambiguous in finding aggregational Gaussianity, i.e. the lower the frequency of the data, the more it approaches the normal distribution. Hence empirical data sets are not stable under aggregation which is a distingtive feature of the LSD. In the light of this empirical fact, the TLF and the STLF becomes more appropriate models than the pure LSD.

Chapter 5

Fractal Dimensions and Scaling Laws for Financial Time Series

The concepts of fractal geometry and scaling are nowadays important tools in the analysis of objects and time series in many disciplines of the natural and social sciences. Although the mathematical roots can be traced back to Hausdorff (1918), the real development started with Mandelbrot's (1977, 1982) extensive work on the subject. Applications proved first to be fruitful in physics, especially for chaotic systems, but soon also biologists, astronomers, geologists and economists used fractals extensively in their work.

A fractal is an irregular geometric object that cannot be characterised by an integer dimension. This criterion distinguishes it from Euclidean objects like a solid cube which has the inter dimension three. The second criterion is the infinite nesting of its structure at all scales, called self-similarity. Roughly speaking, self-similarity means that the geometric object is composed of sub-units and sub-sub-units on various levels which all look like the whole object itself. Closely associated to self-similarity is the almost synonymously used expression of *scale-invariance*. This concept is often employed to describe time series and it says that the appearance of the data is invariant to transformations of different time scales, for example transforming daily into weekly data. Both criteria, fractals and self-similarity, can be quantitatively characterised by using metrics which are the topics of the next paragraphs. But before these are described in more detail, a simple example of such an irregular object should serve as a first introduction. Figures 5.1 and 5.2 show the construction of the triadic Koch-curve.[1]

The construction begins with the line segment of unit length $L(1) = 1$ called

[1] The Koch-curve was introduced by the Swedish mathematician Helge von Koch in 1904.

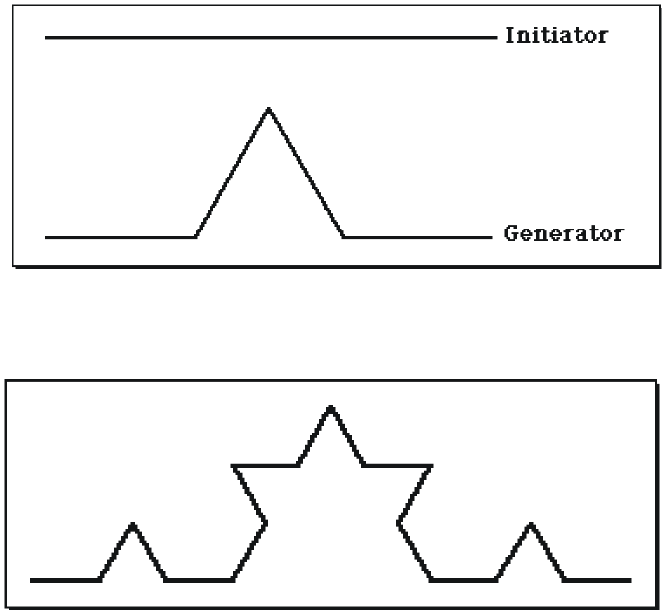

Figure 5.1: Construction of the Koch-curve. The figures show the first two steps.

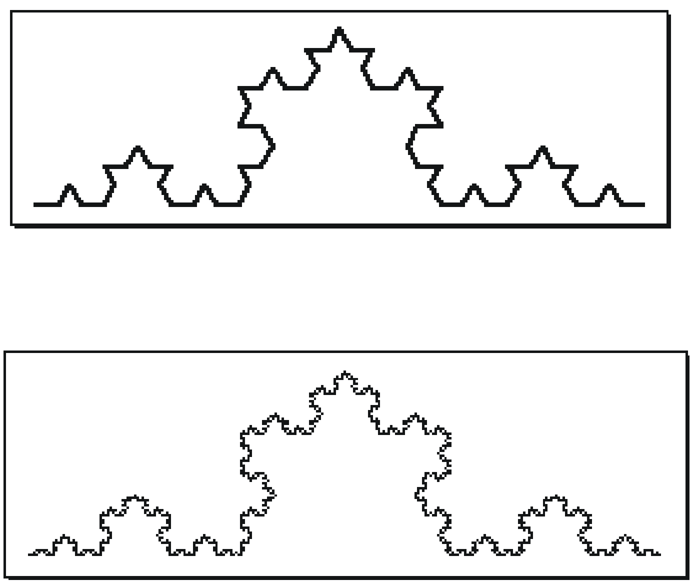

Figure 5.2: Construction of the Koch-curve continued. Two further steps are displayed. However, no matter how far the steps go, each segment always looks like the curve after step one.

the initiator. In the second step, the middle third is replaced by an equilateral triangle and the baseline is removed, so that the geometrical figure has now four segments each with a length of $\frac{1}{3}$. The new figure is called the generator. After this first step the total length of the curve has grown to a value of $L(1/3) = \frac{4}{3}$. The procedure is repeatedly applied to all lines. For example, at stage $n = 3$ the length has already risen to $L(1/9) = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$. After the n-th transformation one finds $L\left(3^{-n}\right) = \left(\frac{4}{3}\right)^n$. Since $n = -\frac{\ln(4/3)}{\ln 3}$, L can be obtained by

$$L(\delta) = L\left(\frac{4}{3}\right)^n = \exp\left[-\frac{\ln\delta(\ln 4 - \ln 3)}{\ln 3}\right]. \tag{5.1}$$

Looking at the small sub-units after the last transformation reveals the same identical appearance as the geometrical figure at stage 1. In this example the object is scaled by a factor of 3 because at each consecutive step every line segment is replaced by a smaller version of the original figure which themselves divides the line into 3 sections, therefore $\delta = \frac{1}{3}$.[2]

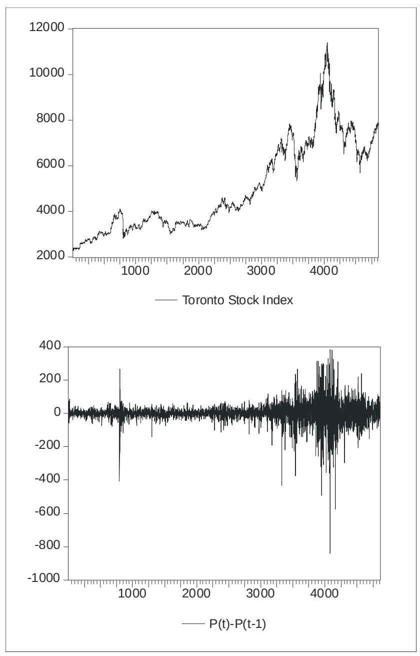

Figure 5.3: The picture above schows the price chart for the Toronto Stock Index. The picture beneath shows the first difference.

Deterministic fractals like the Koch-curve are artificially constructed and

[2] In general, a scaling factor δ transforms a point in a two-dimensional Euclidean space, given by the Cartesian coordinates $p = (x_1, x_2)$ in a point $\delta(p) = (\delta x_1, \delta x_2)$ (see Voss (1988, p. 59)): $P = (x_1, x_2) \rightarrow \delta(P) = (\delta x_1, \delta x_2)$.

therefore preserve their shape exactly at each step. However, nature does not produce objects that are exactly self similar. Real physical objects like coastlines, rivers and clouds have smaller copies that look like the whole but only with some random variations. These natural shapes are irregular but still obey scaling laws. This is also true for the time series of financial markets. Figure 5.3 and 5.4 show the Toronto stock index on a daily frequency from August, 15-th 1984 to November, 26-th 2003. The index started with a value of 2360 and grew to 7822,34 for the last day of the data set. In between, major fluctuations occurred as can be seen from the four graphs.

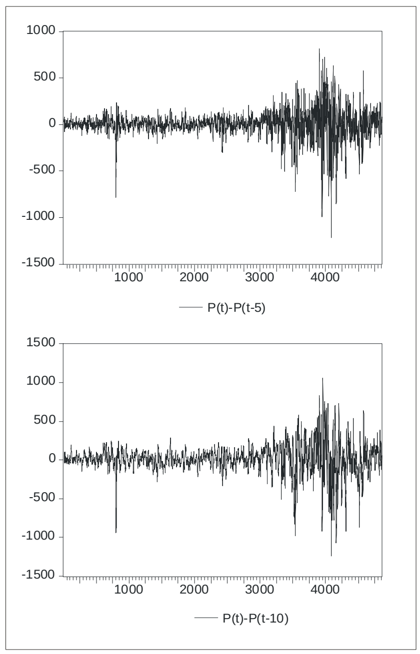

Figure 5.4: Price changes of the Toronto Stock Index for the 5-th and 10-th difference respectively.

The pictures of the price differences for the first, fith and 10-th lag look similar, but they are obviously not exactly self-similar. To capture such (natural) circumstances Mandelbrot (1967, p. 636) coined the term of *statistical self-similarity*. It requires that the statistical characteristics of the distribution are invariant to transformations.

The following chapters introduce methods that deal with the quantitative treatment of such complicated objects. The first section is concerned with the determination of the dimension of these objects. As it will turn out, integer numbers are not appropriate to characterise them. The next chapters explain fractals and the scaling laws of time series and ways how to attribute a measure to them.

5.1 Fractal Dimensions

The most intuitive idea of a dimension is the usual Euclidean dimension, denoted by D_E. D_E is defined to be 0 for a single point, 1 for a line, 2 for a plane, and 3 for a cube, because it needs one coordinate to describe all points of the line, two coordinates for the plane and three for the cube. Another standard concept of dimension is the topological dimension, D_T.[3] The topological dimension of a space is defined as one plus the maximum of its local dimension, where the local dimension is defined to be the lowest dimensional object needed to separate any neighbourhood of the space into two parts. A line therefore has $D_T = 1$ because it can be separated by a point with dimension $D_T = 0$, and a plane is divided by a line and so has dimension $D_T = 1 + 1 = 2$. The problem with the Koch-curve is that it has an Euclidean dimension of two (because of the need of two coordinates to describe it), but a topological dimension of one (because only one point is needed to separate it in any segment). It may therefore be a enticing to assume the dimension of the Koch-curve to lie somewhat between one and two. By looking at figures 5.5 and 5.6 it seems straightforward to assume the same of the price process for the Toronto stock index. Indeed, the graph has $D_E = 2$ but $D_T = 1$ just as the Koch-curve. It might therefore be probable to use the same tools for describing both geometrical objects although the Koch-curve shows an exact self-similarity, whereas the graph of the time series for the price index is only self-similar in a statistical sense.

The problem of determining non-integer, i.e. fractal dimensions of geometric objects was first solved by Felix Hausdorff (1918). His concept, though, contains some severe problems in the practical usage. Therefore the next paragraphs introduce some closely related concepts that are heuristically more accessible, and can furthermore readily be applied to self-similar objects.

5.1.1 The self-similar Dimension

Examples like the Koch-curve are drawn from a long list of artificially constructed objects, which all have the advantage of being amenable to an exact mathematical treatment.[4] It is this feature which makes all of them such an attractive geometrical object to start with. Self-similarity in geometrical terms means that by resolving a figure step by step one encounters a constant relationship between the scaling factor δ and the number of parts the figure can be divided into without changing their appearance. This is easiest to see on a one-dimensional line with unit length. By choosing $\delta = \frac{1}{2}$, the line is separated into two lines, each with half the length of the original line. The next step sees four segments with length $\frac{1}{4} \times L$. A simple generalisation to the N-th consecutive step gives $\delta = N^{-1}$, or $N = N^{-1}$ respectively.

[3] Topology is concerned with geometrical objects that are invariant to transformations, translations and rotations (so-called homeomorphisms).

[4] For an introduction into other deterministic fractals see for example Peitgen, Jürgens and Saupe (1998).

In the introduction, the scaling factor was chosen to be $\delta = \frac{1}{3}$ for the Koch-curve, i.e. each straight line is subdivided into three segments of which the middle part consists of two lines yielding a triangular like appearance. This was done because of the prescribed Koch algorithm that resulted in the above shown self-similar object.[5] Table 5.1 presents the relationships between the scaling factor δ and the number of parts N for the line and the Koch-curve.

Table 5.1: Relationship between the number of parts and δ

Object	Dimension	Number of parts	Scaling factor δ
Line	1	3	1/3
Line	1	6	1/6
Line	1	173	1/173
Koch-curve	D_s[6]	4	1/3
Koch-curve	D_s	4^2	$1/3^2$
Koch-curve	D_s	4^k	$1/3^k$

There seems to be a universal relation between δ and N for each object. In fact, dimension, scale and number of copies follow the pattern of

$$N = \delta^{-D}. \tag{5.2}$$

Assuming the same relation for the Koch-curve, its self-similar dimension can be derived by transforming $N = \delta^{-D_s}$ to $\log N = -D_s \times \log \delta$, which gives

$$D_s = \frac{\log N}{\log \frac{1}{\delta}} = \frac{\log 4}{\log 3} = 1.2619. \tag{5.3}$$

Of course, D_s can be computed in the same way for any multiplicative k of $\frac{1}{3}$, so in general

$$D_s = \frac{\log N^k}{\log \left(\frac{1}{\delta}\right)^k} .$$

It must be emphasised that this simple method is not applicable to geometrical objects which are not exactly self-similar. The following section shows a way how to cope with that problem.

5.1.2 The Box Dimension

The box dimension, D_b, is a concept designed to cope with every geometrical structure that has $D_E = 1, 2, ..., n$. Since the Koch-curve has $D_E = 2$, the method is represented in the usual two-dimensional coordinate system. In a

[5] Does this outcome hinge on the value of $\delta = \frac{1}{3}$, or is self-similarity in this example independent of the scaling factor? The line, the square or the cube can be scaled by every arbitrary δ to get the same object. For fractal objects, this is not the case. There, just one specific factor δ or any multiplicative of δ yields a self-similar structure, and therefore $\delta = \frac{1}{3}$ is not arbitrary for the Koch-curve.

[6] D_s denotes the self-similar dimension.

first step the object is laid on a regular lattice with square meshes of arbitrary size s; see figures 5.5 and 5.6. Now, the crucial question is by how many boxes the Koch-curve is fully covered. Let $N(s)$ be the number of boxes of size s needed to cover the Koch-curve.

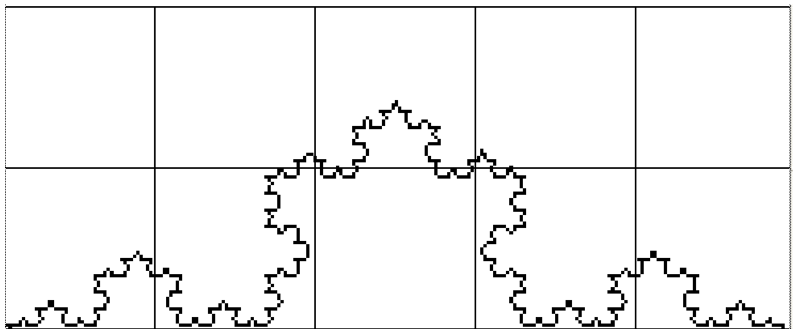

Figure 5.5: Koch-curve coverd by boxes of size s.

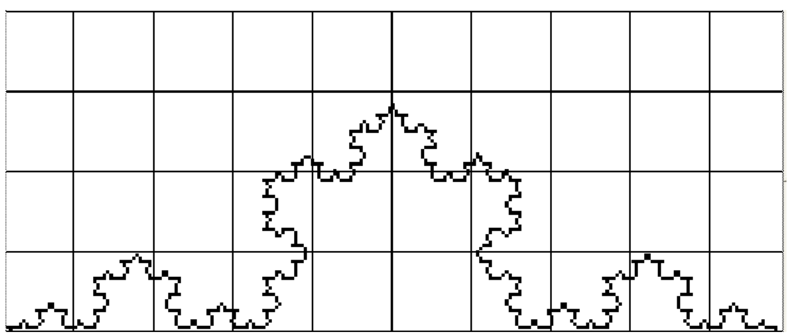

Figure 5.6: Koch curve covered by boxes of size s/2.

The pictures show two steps of the procedure. The two figures differ only in the size of the boxes, where the right figure has boxes with $\frac{1}{2}$ of the size used in the left one.[7] In this case, each box of the right part is subdivided into four boxes with half the wide of the former one. $N(s)$ is counted for both cases and because of the smaller meshes in the right picture, more boxes are needed to cover the Koch-curve. Doing this for varying values of s, one obtains a series of values $N(s)$ for each corresponding size. Table 5.2 gives the values for $\log N(s)$ against $\log\left(\frac{1}{s}\right)$ for various s.[8] Using these values for an OLS estimation (which corresponds to the relation $\log N(s) = const. - D_b \log(\frac{1}{s})$) gives a good

[7] The reduction of s by a factor of $\frac{1}{2}$ is a common choice.

[8] The choice of the size of the largest meshes is arbitrary. Note that the number in the table do not coincide with the pictures above.

approximation of the fractal dimension of the Koch-curve. In the example, the slope is being estimated as $D_b = 1.22$.

Table 5.2: Number of cover-boxes versus size s

s	log(1/s)	N(s)	log(N(s))
2	-0.7731	2983	8.00075
4	-1.38629	1234	7.11802
6	-1.79176	798	6.68211
8	-2.07944	597	6.39192
12	-2.48491	358	5.88053
16	-2.77259	251	5.52545
24	-3.17805	159	5.06890
32	-3.46574	115	4.74493
50	-3.91202	54	3.98898
64	-4.15888	49	3.89182
100	-4.60517	25	3.21888

Compared with the exactly calculated self-similar dimension in chapter 5.1.1, D_b gives a fairly accurate value. However, this procedure seems a bit curious and one may ask whether the box-dimension is also applicable to the usual geometrical objects. Using D_b to measure the unit line segment it takes one box of diameter 1, two boxes of diameter $\frac{1}{2}$, four boxes of $\frac{1}{4}$ etc. to cover it completely. Here, as in all examples with regular geometrical objects, the box-dimension agrees with the topological and Euclidean dimension.[9] It might be enticing to apply this method in the same way to the charts of stock markets in order to calculate the fractal dimension of the Toronto stock index. Unfortunately, a simple transfer of the box-counting procedure to time series is not that easy. Considering lattices with different box-sizes is impossible because one coordinate is measured on a time scale while the other is measured in units of the analysed variable, in this case the price.[10] However, it is feasible to cope with this problem. Peitgen, Jürgens and Saupe (1998) are able to show that the box-dimension of a BM is given by

$$D_b = 2 - H,$$

where H is a scaling factor that has the value $\frac{1}{2}$. This scaling factor is often called *Hurst or Hölder exponent*. The meaning of this parameter will be explained in more detail in chapter 5.2.

The result of $D_b = 2 - H$ is, however, only valid for the box-counting procedure. There are similar methods for calculating non-integer dimensions. For

[9] For the fractal cases, D_b is often said to be embedded in an Euclidean space, i.e. D_E is the next integer dimension, $[D_b] > D_E$ where [ ] are the Gaussian brackets. For $D_b = 1.2$ the embedding or Euclidean dimension is obviously 2, since the Koch-curve is completely covered by a plane.

[10] See Mandelbrot (1982). See also Peitgen, Jürgens and Saupe (1998) for explaining the problems with applications of fractal dimension concepts to time series.

example, the so-called circle-dimension uses circles to cover the object instead of boxes. However, these geometrical techniques do not always yield the same dimensions. Furthermore, time is treated as a geometrical object which is not unproblematic. Therefore, other concepts of dealing with stochastic time dependent processes must be applied. This will be done in chapter 5.2. However, continuing with the analysis of fractal dimension will yield an intuitive way of introducing the methods that are used to cope with the complicated time series of real financial data sets.

5.1.3 The Pointwise Dimension

Another concept of dimension and an indispensable preliminary to multifractal measures is the pointwise dimension, denoted by D_p. So far geometrical objects have been treated as being continuous on the whole domain. This is true for the Koch-curve and many other fractals, but generally a figure consists of many discrete points. An empirical time series is an ample example of such an object since it consists strictly taken only of a set of disjunctive data points, defined on a discrete time scale, albeit the fact that those are regularly connected by line segments for visual reasons. The pointwise dimension is a concept that is explicitly based on the points of a sample set S whose dimension is to be estimated. Let R be the embedding space of S, so that S is a subset of R. In the case of a time series, R is the two-dimensional plane. Because D_p is sensitive to the behaviour of a set in the vicinity of a specific point $x \in S$, the method starts by defining a region around that point. Let $B(r, x)$ be the set of all points $\in R$ whose distance from x is less than r. For $R \in R^2$, $B(r, x)$ is a disk or a circle, in the general case $R \in R^n$ it is a n-dimensional sphere with radius r, centred around the point $x \in S$.

By letting $Card\ R \equiv N(r, x)$ be the number of points in $B(r, x)$, and $Card\ S \equiv \|S\|$ represents the number of elements in the whole set S, a probability measure can be defined by

$$\mu(r, x) = \frac{N(r, x)}{\|S\|}. \tag{5.4}$$

Here, μ must be interpreted as an estimate of the probability of finding a point of S in $B(r, x)$. Given these preliminaries, the pointwise dimension is defined by

$$D_p(x) = \lim_{r \to 0} \frac{\ln \mu(B(r, x))}{\ln r}\ .^{11} \tag{5.5}$$

The pointwise dimension assumes that D_p is the same for all n points, $x_i, i = 1, 2, ..., n$, in the set R. Since r is the same for every x, $\mu(Br, x)$ has also to be equal. The last condition constitutes the main feature of the pointwise dimension: every n-dimensional ball with radius $r \to 0$ must contain the same number of points.

[11] See Young (1982).

5.1.4 The Multifractal Spectrum

5.1.4.1 The Spectrum of D$_q$-dimensions

The box dimension uses boxes with embedding dimension $D_E > D_b$ to cover an irregular structure. Now, these boxes are investigated more closely. As can be thought of, there are cases in which all boxes cover the same number of points. This would be in consistency with the assumption of the pointwise dimension. However, there might be some objects for which this is not true. Therefore, a slight modification of (5.4) is introduced by defining a new probability measure that depends on the number of points in box i:

$$\mu_i = \lim_{N\to\infty} \frac{\mu\left(B_i, N\right)}{\|S\|}, \tag{5.6}$$

where B_i is box i and $\mu\left(B_i, N\right)$ is the number of points lying in this box. $\|S\|$ is as before the total number of points. The measure μ_i is also called the natural measure of box i.

If one finds $\mu_i = \mu_j$, for all $j = 1, ..., n$, then x_i is the typical point of the structure. In fact, for the Koch-curve and some other well-known artificially constructed fractals only one point $x \in S$ is needed to characterise the fractal structure of the whole object. However, for more complicated objects the vast majority of points may fall into a smaller subset of all boxes, and therefore a hierarchy of different measures arises. In order to account for such cases, Grassberger (1983), Grassberger and Procaccia (1983) and Hentschel and Procaccia (1983) provide a concept of dimensions that depends on a continuous index q:

$$\begin{aligned} D_q &= \frac{1}{1-q} \lim_{r\to 0} \frac{\ln I\left(q, r\right)}{\ln\left(\frac{1}{r}\right)}, \text{ where} \qquad (5.7) \\ I\left(q, r\right) &= \sum_{i=1}^{N(r)} \mu_i^q. \end{aligned}$$

For boxes with a large fraction of points, $q > 0$ ensures heavier weights in the calculation of D_q. Setting $q = 0$ leads to $I(0, r) = N(r)$ and so

$$D_q = \lim_{r\to 0} \frac{\ln N(r)}{\ln\left(\frac{1}{r}\right)} = D_b, \tag{5.8}$$

and the box dimension is recovered. In the case of $\mu_i = \mu_j \vee i,j = 1, ..., n$ one finds that $\mu_i = N(r)^{-1}$ and ln $I\left(q, r\right) = \left(1-q\right) N\left(r\right)$, resulting in a measure independent of q. The next paragraph outlines the treatment of different fractal measures depending on q more explicitly. There are other ways in determining a relationship between fractal dimensions and the exponent q, for example by introducing binomial multiplicative processes as done by Falconer (1990) or Feder (1988). However, using the concept of box and pointwise dimension seems to be straightforward given the above preliminaries.

5.1.4.2 The Singularity Spectrum

The starting point for the derivation of a multifractal measure is actually the same as in the box-counting method: the invariant set S is covered with boxes of size r, where $\mu_i = \mu(B_i)$ is as above a probability measure on S. Now, let r be very small so that the number of boxes $N(r)$ needed to cover the object is very large. Furthermore, each box B_i becomes associated with a special number α_i, called the *singularity index*. This is expressed by $\mu_i = r^{\alpha_i}$. Then, by counting the number of boxes with α_i in the range α and $\alpha + \Delta\alpha$, one is able to derive a functional form that relates the number of boxes with the same singularity index $N(\alpha)$, to α and the size of the boxes r. This is done by the following considerations. First, for r being very small the discrete range of α_i between α and $\alpha + \Delta\alpha$ can be replaced by a continuum of α values, so that $d\alpha$ can henceforth be used instead of $\alpha + \Delta\alpha$. Then, D_p as defined by (5.5) is calculated for all points, $x \in S$. Collecting all points with the same pointwise dimension gives a set of points with same measure. This set can now be characterised by its box-counting dimension, $D_b = f(\alpha)$. Thus, interpreting $\mu_i = r^{\alpha_i}$ so as to assign the value α_i as the pointwise dimension of x, $f(\alpha)$ is a measure of the (box) dimension of the set of points with equal D_p. Let $n(\alpha)$ $d\alpha$ be the number of sets with α_i between α and $\alpha + \Delta\alpha$, the number of boxes $N(\alpha)$ with the same α_i is given by

$$N(\alpha, r) = n(\alpha) d\alpha r^{-f(\alpha)}.\text{[12]} \tag{5.9}$$

The integral over different α_i is then

$$\begin{aligned} I(q,r) &= \int d\alpha \xi(\alpha) r^{-f(\alpha) r^{q\alpha}} \\ &= \int d\alpha \xi(\alpha) \exp\left[(f(\alpha) - q\alpha) \ln\left(\tfrac{1}{r}\right) \right]. \end{aligned} \tag{5.10}$$

In the case of $r \to 0$, $\ln(1/r)$ is a very large number so that $(f(\alpha) - q\alpha)$ dominates the integral over the range α. $I(q,r)$ can then be approximated by only considering the maximum of the function $f(\alpha) - q\alpha$. For $f(\alpha)$ being twice differentiable, the maximum at a point $\alpha' = \alpha(q)$ is given by

$$\frac{d}{d\alpha'} \left[f(\alpha') - \alpha' q \right]_{\alpha' = \alpha(q)} = 0, \tag{5.11}$$

provided that

$$\frac{d^2}{d(\alpha')^2} \left[f(\alpha') - \alpha' q \right]_{\alpha' = \alpha(q)} < 0. \tag{5.12}$$

Formulating (5.11) and (5.12) slightly different yields:

$$\begin{aligned} f'(\alpha(q)) &= q, \\ f''(\alpha(q)) &< 0 . \end{aligned}$$

[12] This is easiest to see for a monofractal by noting that $N(\alpha, r) r^{-f(\alpha)} = n(\alpha) d\alpha$. If $\mu_i = \mu_j$ for all points, then $n(\alpha) d\alpha = \|S\|$ and it needs $N(\alpha)$ boxes with size r to cover the object entirely, thereby yielding the fractal dimension $f(\alpha)$.

The integral in equation (5.10) is now approximately given by

$$\begin{aligned} I(q,r) &\approx \exp\left([f(\alpha(q) - q)] \ln \tfrac{1}{r}\right) \int d\alpha \xi(\alpha) \, r^{\frac{1}{2} f''[\alpha - \alpha(q)]^2} \\ &\approx \exp\left([f(\alpha(q)) - q\alpha(q)] \ln \tfrac{1}{r}\right) \ . \end{aligned} \tag{5.13}$$

Thus, D_q yields

$$D_q = \frac{1}{q-1} \left[q\alpha(q) - f(\alpha(q)) \right]. \tag{5.14}$$

A relationship between D_q and $\alpha(q)$ can be established by multiplying (5.14) with $(q-1)$ and differentiating this with respect to q:

$$\begin{aligned} \frac{d[(q-1)D_q]}{dq} &= \zeta'(q) = \alpha(q) \ \text{, with} \\ \zeta(q) &= (q-1)D_q, \ \text{and} \ \zeta'(q) \equiv \frac{d\zeta}{dq}. \end{aligned} \tag{5.15}$$

Thus, having determined $D_q, d(q)$ and $f(\alpha)$ are computed via

$$\begin{aligned} f(\alpha) &= q \times \tfrac{d[(q-1)D_q] - (q-1)D_q}{dq} \\ &= q\zeta'(q) - \zeta(q). \end{aligned} \tag{5.16}$$

The last two equations establish a parametric representation of the behaviour of the fractal dimension of a geometric object when its measure is raised to varying powers. (5.16) constitutes a so-called Legendre transformation.

At this stage, the use of fractal and multifractal measures to describe time series may not seen to be an obviously appropriate tool. However, by noting that time is generally treated as a space parameter, a geometrical treatment can be rationalised.

5.2 The Scaling Properties of Fractional Brownian Motion

5.2.1 The scaling of Brownian Motion

The second important feature of fractal objects is the ability to project their structures into smaller scales, thereby preserving its geometrical appearance. Such projections are manifested in what is called the scaling-laws. A special case of scaling, the exact self-similarity, was encountered in chapter 5.1.1. But, as already mentioned in the introduction, natural phenomena and real time series are not exactly self-similar. They show a kind of statistical self-similarity, which in the case of time series is furthermore complicated by the fact that it is not sufficient to use only one scaling factor.

Although Brownian Motion (BM henceforth) is not the most general stochastic process consistent with the EMH, it is a sensible starting point for the derivation of the scaling laws of real financial time series.[13] Let $|\varepsilon|$ represent the (absolute) step-length of a particle moving in two directions (up and down). In accordance with chapter 2.1, this can be interpreted as a news process, where e.g. an upward jump would symbolise good news. A sequence of n such independent jumps results in the usual random walk and gives the position of the particle at every n-th step,

$$x\left\{t = n\triangle t\right\} = \sum_{i=1}^{n} \varepsilon_i,$$

with $\triangle t$ being the time unit interval during which the jump happens. Supposing ε appears only at each second time step, i.e. $b\Delta t$ with $b = 2$ and so $x\left\{t = n2\triangle t\right\} = \sum\limits_{i=1}^{n} \varepsilon_i,$. The appropriate scaling of $x\left(t\right)$ can be derived by the following consideration.

The increment ε is the sum of two independent jumps, ε_1 and ε_2, both occurring during $\triangle t' = 2\triangle t$. Given the fact that the joint probability of the two jumps, i.e. the probability that ε_1 occurs in the interval $(t, \Delta t)$ and ε_2 in the interval $(t + \Delta t, t + 2\Delta t)$, is the product of the probability distributions of both jumps taken from the same Gaussian distribution

$$P\left(\varepsilon, \triangle t\right) = \frac{1}{\sqrt{2\pi D\triangle t}} e^{\left(-\frac{\varepsilon^2}{2D\Delta t}\right)}, \tag{5.17}$$

where the variance σ^2 is here explicitly expressed by $D\Delta t$, D being the drift parameter. Thus

$$P(\varepsilon_1, \varepsilon_2, \Delta t') = P(\varepsilon_1, \Delta t) P(\varepsilon_2, \Delta t). \tag{5.18}$$

Taking the integral over all possible combination of ε_1 and ε_2 that sum up to ε leads to the probability distribution of ε:

$$P\left(\varepsilon, 2\triangle t\right) = \int\limits_{-\infty}^{\infty} d\varepsilon_1 P\left(\varepsilon - \varepsilon_1, \triangle t\right) P\left(\varepsilon_1, \triangle t\right) = \frac{1}{\sqrt{2\pi D 2\triangle t}} e^{\left(-\frac{\varepsilon^2}{2D2\Delta t}\right)}. \tag{5.19}$$

This is a Gaussian distribution similar to the one imputed to the two εs. It has the same expected position of $x, E[x] = 0$, but a variance of

$$\sigma^2 = Db\Delta t = 2D\Delta t,$$

i. e. twice the value of σ^2 for the process underlying the individual ε_1- and ε_2-jumps.

[13] For the derivation of the scaling characteristics of Brownian Motion in the form presented here, see in particular Feder (1988).

Now, both processes $x\left\{(t), t=n\triangle t\right\}$ and $x\left\{(t), t=n2\triangle t\right\}$ are statistically indistinguishable if the vertical scale is replaced by $\varepsilon' = \sqrt{b}\varepsilon$ and by setting $\triangle t' = b\triangle t$ for the horizontal axis.[14] Then, the scaling relation for the probability densities is expressed by

$$P(\varepsilon' = b^{1/2}, \Delta t' = b\Delta t) = b^{1/2}(P\varepsilon, \Delta t).$$

Thus

$$P(b^{1/2}\left[x(bt) - x(_0)bt\right] = b^{1/2}P\left[x(bt) - x(_0)bt\right] \qquad (5.20)$$

gives the scaling relation for the probability distribution of the particle position $x(t)$. From (5.20) the first and second moments, $E[x(t) - x(t_0)]$ and $E[x(t) - x(t_0)]^2$, are calculated by

$$\langle[x(t) - x(t_0)]\rangle = 0 \qquad (5.21)$$

and

$$\left\langle[x(t) - x(t_0)]^2\right\rangle = 2D\left|t - t_0\right|, \qquad (5.22)$$

respectively. Since ε is the difference between two positions, $x\left(t+\triangle t\right)$ and $x\left(t\right)$, one can also write the scaling relation in the form of

$$\triangle x = b^{\frac{1}{2}}\varepsilon. \qquad (5.23)$$

Because of $b = |t - t_0|$, (5.23) can be written as

$$x\left(t\right) - x\left(t_0\right) \sim \varepsilon\left|t - t_0\right|^H \quad , \qquad (5.24)$$

where H is the (local) *Hölder exponent* that symbolises the scaling factor. In this case, $H = \frac{1}{2}$ is associated with the BM. This is a scaling relation that connects the differences in x to the differences in time.

5.2.2 The Scaling of fractional Brownian Motion

Setting $H = \frac{1}{2}$ in (5.24) results in the well-known properties of a Gaussian process, namely the absence of autocorrelation, a finite variance and a stable mean. A generalisation of this special case of independent increments was introduced by Mandelbrot and Ness (1968) with the concept of fractional Brownian motion (fBM henceforth). They allow H to take on values between, $0 \leq H \leq 1$, thereby creating a random process which is no longer independent. This process, denoted by $B_H\left(t\right)$, is defined by the function

$$B_H\left(t\right) = \frac{1}{\Gamma\left(H + 0.5\right)}\int_{-\infty}^{t}\left(t - s\right)^{H-0.5}dB\left(s\right), \, t > s, \qquad (5.25)$$

[14] See Feder (1988, p. 167f).

where $\Gamma(x)$ represents the Gamma function and $t > s$.[15] Its increments are given by

$$B_H(t) - B_H(s) = dB(t). \tag{5.26}$$

The process $B_H(t)$ is scale invariant since by changing the time scale by a factor b, one obtains

$$B_H(bt) - B_H(0) = \frac{1}{\Gamma(H+0.5)} \int_{-\infty}^{bt} (bt - s)\, dB(s). \tag{5.27}$$

$$= \frac{1}{\Gamma(H+0.5)} \int_{-\infty}^{0} \{(bt - bs')^{H-0.5} - \tag{5.28}$$

$$-(-bs')^{H-0.5} dB(s) + \int_{0}^{bt} (bt - bs')^{H-0.5}\, dB(bs')\},$$

Substituting $s = bs'$ yields:

$$\begin{aligned} B_H(bt) - B_H(0) \quad &= \quad \frac{1}{\Gamma(H+0.5)} \int_{-\infty}^{0} \{(bt - bs')^{H-0.5} - (-bs')^{H-0.5} dB(s) + \\ &\quad + \int_{0}^{bt} (bt - bs')^{H-0.5}\, dB(bs')\}, \end{aligned} \tag{5.29}$$

This is equivalent to

$$\triangleq \quad \frac{1}{\Gamma(H+0.5)} \int_{-\infty}^{0} \{b^{H-0.5}[(t - s')^{H-0.5} - (-s')^{H-0.5}] b^{0.5} dB(bs') +$$

$$+ \int_{0}^{bt} b^{H-0.5}\,(t - s')^{H-0.5}\, b^{0.5} dB(s')\},$$

[15] The functional form of (5.25) is more easily understood by approximating the integral by a summation of the increments of a fBM, $dB(t) = B_H(t) - B_H(s)$. Dividing each time interval into n small subintervals enables a re-statement of the integration $s = \frac{j}{n}$ by the time steps $j = -\infty, ..., -\frac{2}{n}, -\frac{1}{n}, 0, \frac{1}{n}, ..., \frac{t}{n}$. $dB(t)$ is then replaced by $jn^{-\frac{1}{2}}$, where nj is a discrete Gaussian random variable with mean zero and unit variance, and $n^{-\frac{1}{2}}$ is the above derived rescaling factor that takes account of the decreasing time steps. Now it is possible to approximate (5.25) by

$B_H(t) \approx \frac{1}{\Gamma(H+0.5)} \sum_{j=-\infty}^{nt} \left(t - \frac{j}{n}\right)^{H-0.5} n^{-0.5} y_j$. See Feder (1988).

$$\triangleq \quad \frac{b^{0.5}}{\Gamma\left(H+0.5\right)} \int\limits_{-\infty}^{0} [(t-s')^{H-0.5} - (-s')^{H-0.5}] dB\left(bs'\right) +$$

$$+ \int\limits_{0}^{bt} \left(t-s'\right)^{H-0.5} dB\left(s'\right)\},$$

and so

$$B_H\left(bt\right) - B_H\left(0\right) = b^H \left\{B_H\left(t\right) - B_H\left(0\right)\right\} \ . \tag{5.30}$$

Especially by choosing $t = 1$ and $\Delta t = bt$

$$B_H\left(\Delta t\right) - B_H\left(0\right) = \left|\Delta t\right|^H \left\{B_H\left(1\right) - B_H\left(0\right)\right\} \sim \left|\Delta t\right|^H \ \ , \tag{5.31}$$

or in the notation $x_t = B_H(t)$

$$x(\Delta t) - x(0) = \left|\Delta t\right|^H \left(x\left(1\right) - x\left(0\right)\right) \sim \left|\Delta t\right|^H .$$

where $\sim$ means that the increment of x is proportional to $\left|\Delta t\right|^H$ in distribution. Compared with (5.24), the only difference lies in allowing H to take on any value between 0 and 1. The fBM might be viewed as a biased random walk, where the probability that a step upwards is followed by another step in the same direction is not 0.5 as for the ordinary BM but smaller for values $H < 0.5$ and greater for $H > 0.5$. The characteristics of the associated autocorrelation functions are described in chapter 6.1.1.2, where problems of long range dependencies between increments are discussed in more detail.

5.3 Multiscaling and Multifractality

The scaling laws of fBM can be extended to expressions for their q-th moments, i.e.

$$[x(t+\Delta t) - x(t)]^q \sim |(t+\Delta t) - t|^{qH} \, \forall q < \infty, \tag{5.32}$$

or more compact

$$[\Delta x(\Delta t)]^q \sim |(\Delta t)|^{qH} .\text{[16]} \tag{5.33}$$

It is important to notice the fact that the exponent H remains the same irrespective of q so qH is a linear function of q. Thus, the scaling property of (5.32) implies a uniform oscillatory behaviour for fBM and $\Delta x(\Delta t)$ simply behaves as a power-law. Unfortunately, real world signals do not always follow this simple pattern. They often possess a changing scaling exponent, which limits fBM as a

[16] The scaling symmetry can also be expressed by $[x(t)-x(t-1)]^q \sim |x(t) - x(t-T)| \left(\frac{t}{T}\right)^{qH}$, where T is the longest time for which the scaling holds.

good approximation to a considerable extend. Therefore, monofractals have to be replaced by multifractals in order to describe the self-similarity of these time series more properly. The main difference between mono- and multifractuals is that the latter needs a whole spectrum of exponents to define its scaling characteristics completely (thus the name multiscaling) while the first only requires one (Hölder) exponent.[17] In the multifractal case a hierarchy of dimensions (the singularity spectrum) is required.

As a consequence, multiscaling is a relaxation of the linear relation indicated by qH. In order to account for a possible non-linear relationship, a so-called zeta-function $\zeta(q)$ is introduced. Formally (5.32) now reads as

$$[x(t+\Delta t) - x(t)]^q \sim \Delta t^{\zeta(q)}]. \tag{5.34}$$

In this form, $\zeta(q)$ is held explicitly general and still allows for a linear function like Hq. However, former considerations already lead to the derivation of a functional form that expresses a relation between the moments and the appearance of $\zeta(q)$ in case of multifractality. Reformulating (5.16) slightly different gives

$$\zeta(q) = q\tau'(q) - f(\alpha). \tag{5.35}$$

Now, $\zeta(q)$ could principally be determined by successively calculating the Box dimension of the price changes up to various moment orders. However, determining $\zeta(q)$ this way is rather tedious. Fortunately, a relative simple algorithm exists that starts from the scaling relation (5.34) and as it will turn out, the estimation technique only requires the use of OLS regression despite the highly complex behaviour of the signal itself.

5.3.1 Estimation of the Zeta-(q)-function

Let $x(t)$, $t = 1, 2, \ldots, n$, be the time series under scrutiny. As has been seen in chapter 5.2.2, scaling laws give a direct relation between time intervals Δt and the moments of the average differences for the same intervals. The procedure to calculate the zeta-(q)-function starts with the calculation of the averaged absolute increments of a variable x, here usually the price:

$$\langle |x(t+\Delta t) - x(t)| \rangle \tag{5.36}$$

for different Δt.[18] The use of averaged absolute values for financial time series can be explained by noting that the variable of interest is the fluctuation of prices, and the absolute value of the differences serve as a proxy for it.

[17] The two expressions multiscaling and multifractality will be used interchangeably throughout this work.

[18] $\langle\ \rangle$ denotes averaged values.

In a second step, (5.36) is risen to the powers of q from 0.1 to 8.4.[19] Raising (5.36) to other powers than 1 or 2 might then be interpreted as studying *generalised* average volatilities for different time scales.[20] The relation between the averaged price differences at the time differences is visualised by plotting log $\langle|x(t+\Delta t)-x(t)|^q\rangle$ versus log Δt for every q (see figure 5.7). Repeating this procedure for various time differences gives a set of time series that are all used for an estimation of log $\langle|x(t+\Delta t)-x(t)|^q\rangle$ on Δt. The slope of each pair gives then an estimate of the relation between the scaling exponent and the moment q.

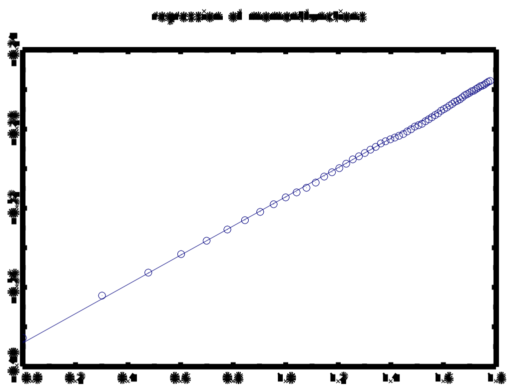

Figure 5.7: Example of a regression of the log of equation (5.36) from a simulated fBM (H=0.8) for the first moment on log Δt. Because this is a simulated series, the points, representing the log $\langle|x(t+\Delta t)-x(t)|\rangle^{q=1}$ for different lags, lie very near to the straight line, so R^2 is near 1. This shows a very stable scaling relationship (for the first moment). This regression is progressively repeated with higher moments. The collection of all slope estimates gives the function zeta-(q).

Collecting all individual exponents gives an estimate of $\zeta(q)$. In the case of monoscaling, this must result in a linear function. The procedure has thus the

[19] The steps do not need to be integers. In order to get a sufficient number for the regression later analysis uses 0.1 steps. The choice of 8.4 is arbitrary. However, the number of different moments should ensure a good regression. With 0.1 steps, a maximum of q=8.4 fives 84 values for the estimation which is sufficient.

[20] Because of

$$E[x(t)-x(t_0)]^2 = \left\langle [x(t)-x(t_0)]^2 \right\rangle = 2D\,|t-t_0|$$

the standard deviation is given by

$$\sigma = (2D\,|t-t_0|)^{1/2}$$

and scales as

$$\sigma = E([x(t)-x(t_0)^2])^{1/2} \sim \varepsilon\,|t-t_0|^H\,.$$

following steps:

- compute (5.36) for different time lags;
- then raise each of the time series computed in step one to different powers (q);
- select all values with the same power (this series has of course different time lags);
- regress the series with the same power to Δt^q;
- repeat the last step for each power;
- collecting the results for all q gives an estimate of $\zeta(q)$.

5.3.2 Empirical Evidence of Multiscaling (Multifractality)

Because of the inherent scaling abilities of time series, there is no privileged time interval at which real data sets should be investigated.[21] It is thus important to analyse how different (time-) measures relate to each other. Since Müller et al. (1990) have found the existence of scaling laws for the mean volatility of returns in foreign exchange rates, much work has been undertaken to verify the existence of more generalised laws for various financial markets. In fact, a large volume of papers have confirmed the findings of Müller et al (1990). Those papers are mainly concerned with the detection of multiscaling processes and most of them conclude that proposing simple stochastic processes like the fBM do not describe real financial time series well enough.[22] Although it is surely a worthwhile endeavor attempting to find out the true scaling process, the consequences for an economic explanation of financial markets are even more crucial. However, a simple interpretation is not easy. What should be interesting to economists is the fact that those multiscaling or multifractal processes originate from high-dimensional systems that are - because of their complex structure - totally different to the picture of the EMH. On the other side, mechanisms that are found to be responsible for multifractal structures in physics do not necessarily correspond to financial processes.[23] The most promising idea is based on information cascades, where traders with different investment horizons are heterogeneously affected by incoming news (this point will be picked up in more detail at the end of the next section). The next lines briefly summarise the main results of previous empirical studies on this topic.

[21] For practical reasons, the most common time scales in former studies have been daily, weekly and monthly records because higher frequency were unavailable. The upcoming of high-frequency data made it possible to analyse intra-day data. However, all choices of time variables hinges purely on the analysed topic. For example, studying the behaviour of market makers in the foreign exhchange markets affords the use of very short time intervals.

[22] See Brock (1999) for an extensive review of scaling laws in economics and finance.

[23] See Bunde and Havlin (1991) for an introduction into multifractality in different physical systems.

Multifractal analysis is a relatively new tool of looking at financial markets. The maybe earliest contribution on the topic is the analysis of Vassilicos, Demos and Tata (1993). There, two time series of foreign exchange rates ($/DM and $/SF with a tick-by-tick frequency) and the daily returns from the New York Stock Exchange (in the period 1885 – 1988) are put under scrutiny. The original intention was to test them for signs of chaos. Chaotic systems produce strange attractors that can be characterised by a fractal dimension. Vassilicos, Demos and Tata (1993) employ a method in the sense of equation (5.7) in order to discriminate between homogenous mono- and non-homogeneous multifractality. They cover the time axis with a grid of intervals of size s. Each interval is labeled by a series of integers i. The number of data points (here, the price increments) that lie within these intervals are denoted by λ_i. Here, the Box dimension concept in used despite its practical inconvenience. The authors then define the quantity

$$N_q(s) = \sum \lambda_i^q. \tag{5.37}$$

They apply (5.37) with $q = 0, 1, ..., 4$. If $q = 0$, $N_q(s)$ is just the number of boxes needed to cover all points, so that

$$N_0(s) \sim s^{-D_0} \tag{5.38}$$

determines the fractal dimension in the sense of the box-counting method. The generalised dimension

$$D_q = \frac{D_0}{1-q} \tag{5.39}$$

should be independent of q, i.e. $D_0 = D_1 = D_2 = ...$ for a homogeneous fractal. Vassilicos et al. find that this is not the case for the analysed time series.

As can be seen from figure 5.8 D_q is strictly decreasing in q as it should be for a multifractal. The authors thus conclude that a simple BM is inadequate to describe the data. Furthermore they suggest to interpret the result as "a reflection of very high frequency oscillatory changes of prices from one local equilibrium to another following private, or perhaps even public, news."[24]

One of the early works that uses the algorithm described in the last section is the contribution of Schmitt, Schertzer and Lovejoy (1999). They analyse daily foreign exchange rates for five major countries (Switzerland (CHF), Germany (DEM), USA (USD), GB (GBP) and Japan (JPY), all evaluated against the French Franc). Figure 5.9 shows the $\zeta(q)$ functions for the currencies. The points refer to the estimates of $\log(\Delta t)$ on $\log\langle|e(t + \Delta t) - e(t)|\rangle$, with e as the exchange rate, for moments up to $q = 4$. The non-linearity is an indication of multiscaling

[24] Vassilicos, Demos and Tata (1993, p. 263).

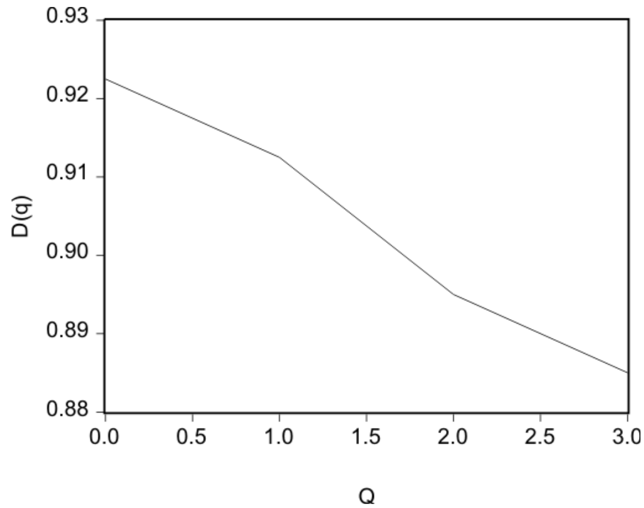

Figure 5.8: Devlopment of the generalised dimension $D_0 - D_3$. D_q is strictly decreasing with increasing q. Monofractality would show up as a horizontal line. See Vassillicos et al. (1993, p. 260).

which catches an important aspect of price patterns that statistical models like the ARCH- and GARCH-processes cannot. In fact, the authors extend their analysis by comparing different other statistical proposals (the LSD, the TLF, the ARCH- and GARCH-models) with the results of $\zeta(q)$ and conclude that none of them is able to produce the outcome of the empirical data.

Table 5.3 summarises the literature on multifractals for stock and foreign exchange markets.

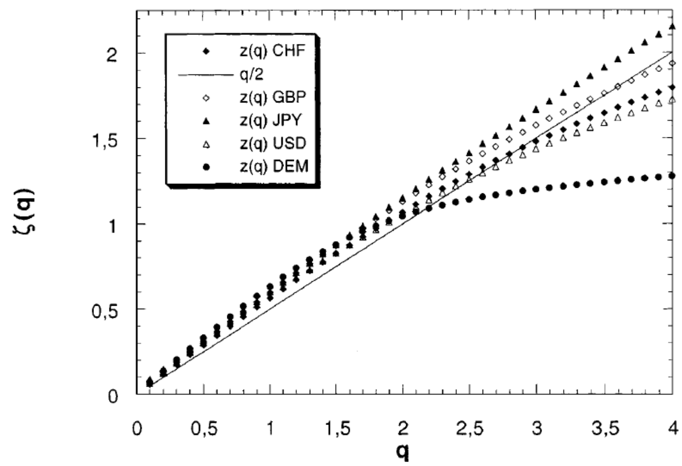

Figure 5.9: Zeta-(q)-functions of five foreign exchange rates. See Schmitt, Schertzer and Lovejoy (1999, p. 35).

Table 5.3: Recent literature on multiscaling in financial markets

Author	Data	Results
Vandewalle and Ausloos (1998)	$\frac{\$}{DM}$− and $\frac{JNY}{\$}$−rates daily records (Jan. 1980 to Dec. 1996)	Both time series show signs of multisacaling. zeta (q) is non-linear.
Ivanova and Ausloos (1999)	Dow Jones Indust. average on a daily frequency from Feb. 1991 to May 1997	For low moments up to 3, the series shows a variety of Hölder exponents, thus indicating mulitscaling
Bershadskii (1999)	Uses the same data as in Vandewalle and Ausloos (1998)	Confirms the results of Vandewalle and Ausloos (1998)
Górski et al. (2002)	High frequency returns for the DAX 1997-1999 (10^6 data points)	The scaling exponents vary for different values of q
Matia et al. (2002)	Daily prices of 2449 indiv. stocks for a 15-year period	All analysed series show non-linear sacling behaviour
Fillol (2003)	Paris stock index (daily frequency)	The zeta(q)-function is non-linear for moments up to 5
Balcilar (2003)	Istanbul and Moscow stock markets (daily frequency)	Both markets display multifractal spectra

The above table comprises some of the recent contributions to the multiscaling debate in financial markets. Not all of them are exclusively focussed on stock markets. Foreign exchange markets in particular are another source of large high-frequency data sets and thus are used regularly in the analysis. Some authors even take commodity or gold prices on a daily basis as the test procedure is not in need of such huge data sets as in the case of e.g. the Hill-estimator. It is hence fair to say that the empirical literature covers a wide range of financial markets and its overall picture is unequivocal: there is no single study that rejects the early multiscaling findings of Vassilicos, Demos and Tata (1993). For example, Matia et al. (2002) analyse daily prices of 29 commodities and 2449 individual stocks and find that the latter mentioned have a significantly narrower multifractal spectrum than for the commodity prices, i.e. the curvature for $\zeta(q)$ is less pronounced for stocks than for commodities. But in any case they detect multiscaling for each single time series! Pasquini and Serra (2000) is a supplement paper that studies the volatility rather than the prices itself. However, the multiscale phenomenon remains. By analysing daily returns on the NYSE they find that their volatilities exhibit a spectrum of different exponents. The striking parallels across markets and national boundaries lead to the conclusion that - up to now - the hypothesis of multiscaling is based on a reasonable amount of empirical validation. Despite this fact, none of the above cited papers tries to unveil the economic dynamics that produce the data. All confine themselves to stress the fact of the failure of the simple fBM to account for this phenomenon.

5.3.2.1 Own Tests on Multiscaling

This chapter investigates the same indices as in chapter 4.7.1 for their non-linear scaling behaviour. Likewise, daily records are first tested. Let

$$\langle |P(t+\Delta t) - P(t)| \rangle \tag{5.40}$$

be the averaged difference of the stock indices. The moments range from 0.1 to 8.4 in 0.1-steps so that subsequent regressions are performed with 84 points. In order to detect non-linearity, this range turned out to be sufficiently large.[25] The scaling range with which these series are tested has in any case a minimum of $\Delta t = 60$ days. This amounts of taking differences up to almost three months. The reason for this choice can be understood by the following reasoning. Figure 5.10 shows the scaling of the absolut averaged price changes of the Paris CAC 40 for the first moment up to time differences of $\Delta t = 80$.[26] The points are quite close to the line which indicates a good fit.

However, figure 5.11 shows the log differences for the Toronto stock index as plotted against $\Delta t = 100$ and moment $q = 8$. As can be seen, the fit becomes poorer the larger Δt gets (and the higher the moments). The estimation has a

[25] If q is chosen too small, multiscaling that appears only late cannot be detected. Thus q must be large enough to account for this.

[26] The choice of the Paris stock index was not arbitrary as not all indices shows such a good scaling.

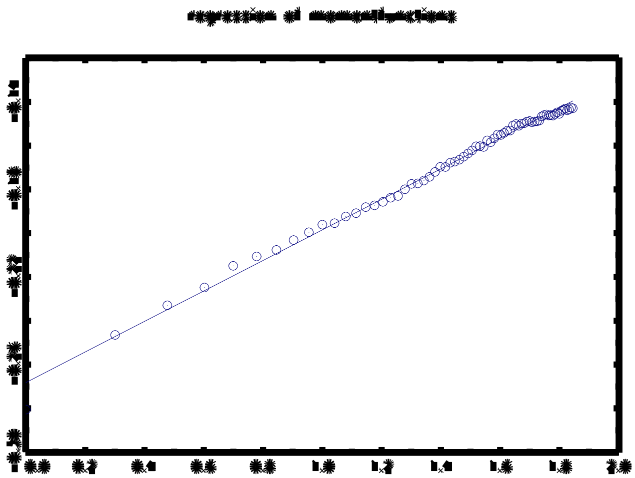

Figure 5.10: Regression of log $\langle |x(t + \Delta t) - x(t)| \rangle$ for lags $\Delta t = 1 - 80$ on Δt for the CAC 40. x is here the value of the index.

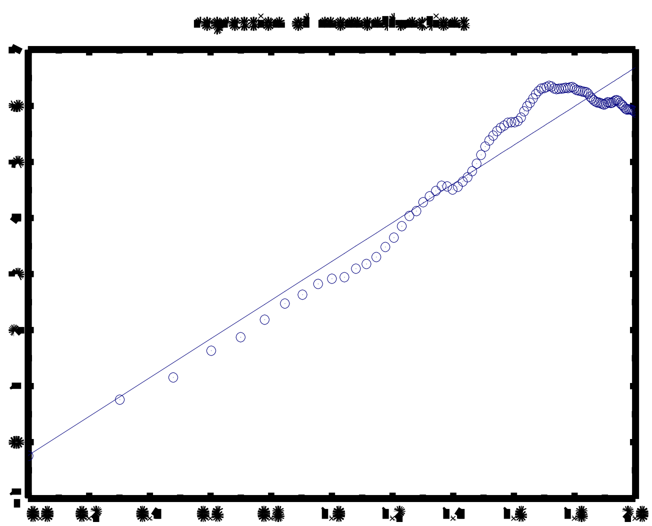

Figure 5.11: Regression of log $\langle |x(t + \Delta t) - x(t)| \rangle^8$ for lags $\Delta t = 1 - 100$ on Δt for the Toronto Stock Index. The points depart substantially from the regression line which indicates the failure of a proper scaling relation. For such series, lower lag length have to be taken in order to yield relieable estimates.

lower R^2 and this indicates that scaling is no longer prevailing. The choice of $\Delta t = 60$ for some indices is due to those time series for which scaling breaks down if Δt is higher than 60 days.[27] Not all of the used time series have this problem. However, for some of them, the scaling extends to a quarterly frequency, although this is only rarely encountered. In these cases $\Delta t \approx 80$ is the limit in order to obtain a regression with at least $R^2 \approx 0.9$. A proper scaling with $\Delta t = 100$ days is not detected in any series.

The next figures (5.12-5.15) display the zeta-(q)-function of all daily time series. For all of the indices, the regression (of $\log \langle |P(t + \Delta t) - P(t)| \rangle^q$ on $\log \Delta t$ for different q's) exhibits "good" regression coefficients for the vast majority of points, i.e. R^2 is always near or above 0.9, so the fit validates the scaling conjecture. The functions itself show a non-linear behaviour that is distinctively different from the linear function qH. For a better comparability, the straight lines show $\zeta(q)$ for the ordinary BM with a slope of 1/2. This is the zeta-(q)-function that would emerge if the real price records follow a simple random walk. If the data generating process would have been a fBM, still a straight line should be the result, but with different slope, i.e. steeper than 1/2 for a fBM with H>0.5 or flatter for a fBM with H<0.5.

It is clear from the figures that the different indices show different behaviours in their moment-functions, where it is important to note that the more bent the curvatures are, the wilder is the price fluctuation. Some of them display an early departure from linearity, others have only a weak form of non-linearity, i.e. their curves do not bend as much as others. For example, the non-linearity of the curve for the Bewag AG in figure 5.16 is rather mild compared with the result from the Buderus AG. In fact, the Bewag AG has a function that is almost linear. Though even the series for which the non-linearity becomes clear only for high moments (like the CAC 40 or Continental) are distinctively different to the BM.

Figure 5.16 shows the thre zeta-(q)-functions for the high-frequency data. There is no qualitative difference to the functions for the daily records. The non-linear behaviour again indicates the existence of a whole spectrum of scaling moments. Another point worth mentioning is the scaling that now extends to $\Delta t = 100$. Naturally, this comes from the fact that the 100-th difference just covers 100 minutes (for a frequency of 1 min, which is taken to be the basis frequency of all three series) and the transmission to a normal distribution occurs only at a lag of 40 to 50 days. Noteworthy is the stark bend of the function compared to some of the daily counterparts. This indicates that series with higher frequency show wilder fluctuations.

As in chapters 4.7.3 and 4.7.4, own empirical tests confirm the stylised fact of a multiscaling feature in the stock data. All series show a significant non-linear

[27] Note that the transmission from a LSD to a Gaussian distribution can be noticed for quarterly data, which coincides with the above findings quite well. This however, comes as no surprise since different distributions for different time scales must show up as a breaking down of scalin (see e.g. Bouchaud and Potters (2000)).

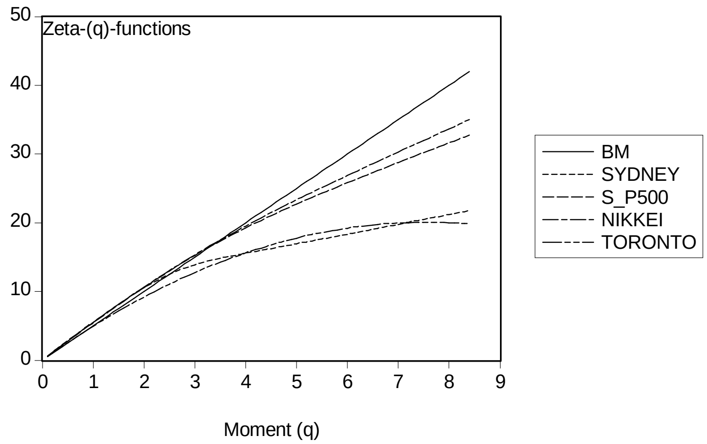

Figure 5.12: Zeta-(q)-functions for four national daily stock indices. The straight line is added in order to compare it with the functions for the real data. This linear function would prevail if the indices were following a simple random walk.

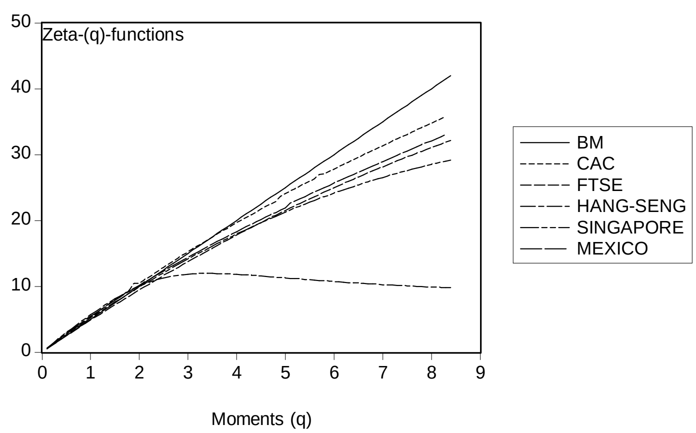

Figure 5.13: Zeta-(q)-functions for five nationl daily stock indices.

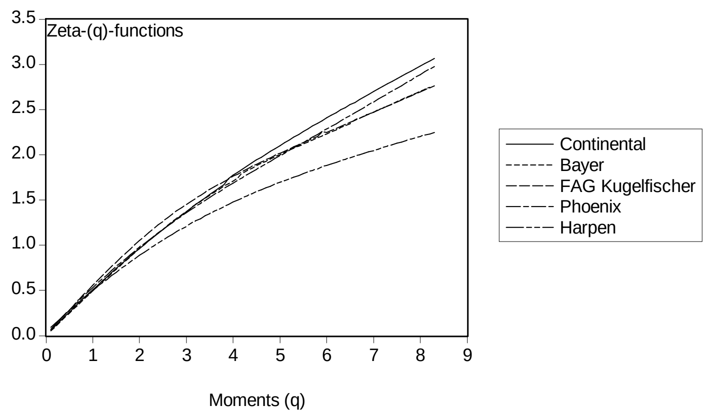

Figure 5.14: Zeta-(q)-functions for five daily price records. Although different in shape, all examples show the typical non-linear function as a sign of multifractality.

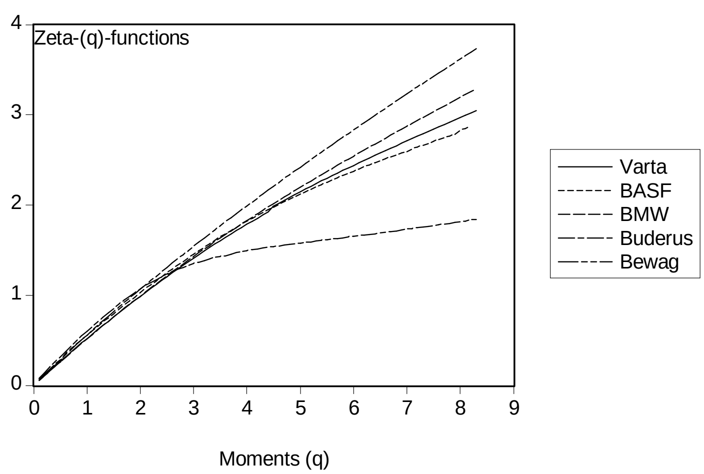

Figure 5.15: Zeta-(q)-functions for five daily price records. The functions are all similar to the above examples.

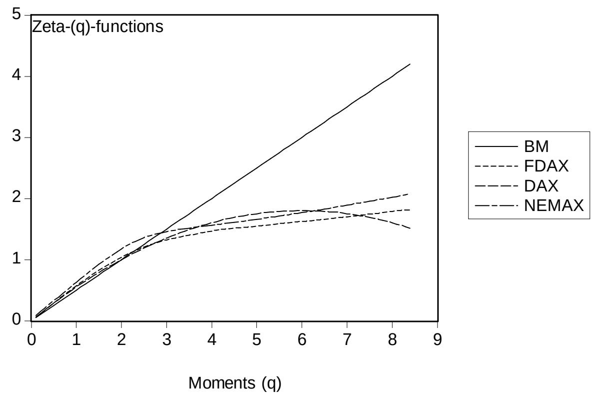

Figure 5.16: Zeta-(q)-functions for the three high-frequency price records. Easy to see is the huge similarity of the functions for the DAX and the FDAX, while the NEMAX shows are pronounced hyperbolic behaviour.

zeta-(q)-function that is distinct from a BM or a fBM. However, establishing the multiscaling feature of financial markets gives no hint about its possible economic origin. The problem with finding an explanation lies in the fact that even for well studied physical systems with the same feature, the precise mechanism by which systems produce such complex patterns is sometimes unclear. Moreover, in the case one knows the mechanism, physical reasons are not easily transformed into an economic context. However, the last years have produced some fruitful concepts of interpreting multifractal price records economically. A recent contribution by Mandelbrot, Fisher and Calvet (1997) explains the observed multifractality by a compound stochastic process that generates asset returns. There, multifractal returns are proposed to originate from a process in which multifractal cascades serve as a tool for time transformations. The return according to Mandelbrot, Fisher and Calvet (1997) can be written as

$$r(t) = B_H \left[Q(t) \right], \tag{5.41}$$

where B_H is the fBM with index H. $Q(t) = M_D\,[0, t]$ is the trading time with M_D as a multifractal cascade. An intuitive explanation for $Q(t)$ may come from the fact that trading time, i.e. the time interval in which trades take place, is not equally distributed. For example, early and late trading hours witness a stronger trading than the rest of the day, so much of the whole fluctuation is concentrated on a relatively small time interval. The conjecture is that trading volume has a multifractal distribution and so directs the return process.

The most promising advance in the search for an economic underpinning of multifractality in financial time series is the *heterogeneous trader assumption*, first introduced by Müller et al. (1993), Müller et al. (1997) and Dacorogna et al. (1998). They propose to see a typical market as an aggregation of many investors with different time horizons. This is indeed a very realistic picture. There are the day-traders who try to capitalise on short-run trends and thus have a high trading frequency; there are the pension and hedge funds investors with a low dealing frequency and there are also banks or other commercial organisations which restructure their portfolios once a month. Müller et al. (1997) and Dacorogna et al. (1998) now assume that the differences in dealing frequencies reflect differences in the reaction to news. These news are common to all market participants but are interpreted differently, or rather *used* differently. Investors with long horizons are only interested in the intrinsic value of the information for the new price. Therefore their behaviour can roughly be identified as those of fundamentally orientated traders. Investors with a medium planning horizons are not primarily focussed on the long lasting effects but try to figure out what the consequences may be for the next few months. On the other side, day-traders have to pay attention to what the market thinks about these news, i.e. how other traders evaluate the news, then act accordingly and thus create a momentum which is most important to those with a high dealing frequency. In a homogenous market where all act in the same way, such considerations would be fruitless. And it is the interaction or rather the sum of the components with different time scales produces the multifractal effect.

Another, though in essence very similar way of interpreting multifractal phenomena in stock markets is put forward by Ghashghaie et al. (1996), Arneodo et al. (1998) Muzy et al. (2000) and Breymann et al. (2000). Their proposal is inspired by an analogy with turbulent cascades in hydrodynamics. For the case of turbulent flows, successive multiplicative cascade steps define the transmission of energy from larger to smaller eddies. These steps form a hierachical order where the energy from the largest scale is transmitted to smaller scales in a stochastic manner that produces non-uniform distributions. Figure 5.17 demonstates such a cascade process.

Because stock markets consist of interacting traders, it is assumed that a similar transmission takes place with economic information playing the role of energy. The idea is that information dissipates from higher to lower levels just as in figure 5.17. This process can be described in a way corresponding to the information cascades from herding models mentioned in chapter 2.2.5. Higher levels are occupied by those traders with larger investment horizons. They convert the information as proposed by the EMH. Traders at lower levels try to infer the economic content from the actions of the investors from higher levels. Additionally they can also be allowed to observe the information but do not interpret it in the same way as those fundamental traders from the first level. This mixture of misinterpretation and probabilistic inference is able to account for a stochastic cascade process that build multifractal stock price series.

Up to now, several authors have succeeded in generating multifractal time series with models that use multiplicative cascades and so the analogy between

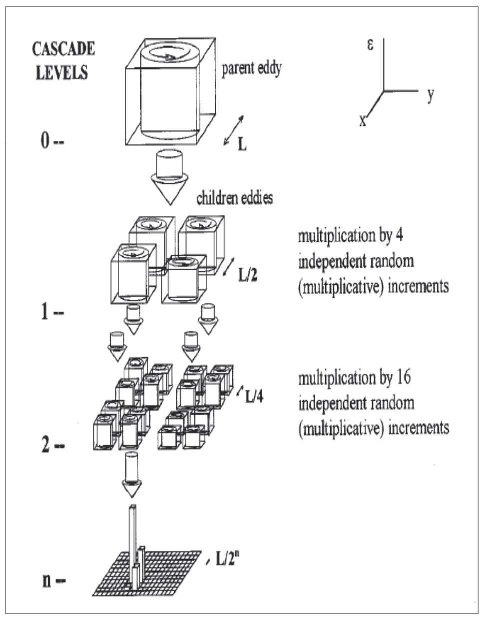

Figure 5.17: Schematic presentation of a multiplicative cascade, where the energy at the highest level (the parent eddy) is transformed into the lower levels. Important to note is the changing distribution of energy at the different levels. This particular example was chosen to explain the multifractal distribution of phytoplankton. See Seuront (1999, p. 887).

price fluctuations and turbulent flow models is backed up by those examples. However, this does not ensure that the true nature of the market process is identified. Although the heterogenous-trader-assumption and the cascade model are both convincing (and complementary) proposals, there is no direct way to use these models for an empirical test. It is impossible to observe the actions of all involved traders which would be a prerequisite. Hence, one is confined to simulations. As part three will reveal, many of those simulations that aim to reproduce the main stylised facts often incorporate two or more groups of differently acting agents. These agents are often interpreted as fundamentalists and noise-traders respectively. Now, these trader types can be put into the heterogenous-trader-framework and the cascade models by identifying fundamentalists as investors with the longest time horizon and noise-traders as technical orientated day-traders who follow short-run price trends. In this respect, the simulation can be seen as experiments that are in accordance with the theoretical propositions.

Chapter 6

Autocorrelations and Volatility Clustering in the Stock Markets

The last of the analysed stylised facts of financial markets touches the arguably most highlighted point of the literature on technical trading. Although practical trading rules comprise a wide range of different advises, the search for linear sample autocorrelations in the price increment (or return) process of stock data is still in the foreground. However, interest in the temporal dependence of successive price changes or returns is not confined to those who are eager to make personal profits. The topic is also a crucial issue for the scientific community as the EMH claims that the correlation between any two values of ΔP_t and ΔP_{t-k} should be zero. Providing evidence for a significant autocorrelation structure different from zero would indicate a severe flaw in the EMH. Although recently the EMH assumption of a totally random behaviour of prices has been weakened in favour of concepts that permit some mild form of positive correlations in order to account for the holding of risky assets,[1] systematic patterns still imply predictive power and the possibility to beat the market. It is therefore of little surprise that practitioners are particularly interested in finding some - typically short-term - patterns in the price process or its associated volatility in order to capitalise on them.[2]

Another major topic of this section will be the question of possible existing long-range autocorrelations in returns and volatility as opposed to the above mentioned short-run dependencies. The presence of long run temporal dependencies in stock data has in principal the same implication for the EMH. If stock returns or price changes really exhibit a non-zero autocorrelation between realisations widely separated in time, past values can help to predict future prices.

[1] See Campbell et al. (1997).

[2] The importance of volatility is mostly due to its interpretation as a measure of risk.

This again would stand in sharp contrast to the notion of weakly efficient market as stated by Fama (1970). Other fields effected by long range dependencies in return series are the portfolio optimisation or the optimal consumption/saving ratio as they become dependent on the time horizon. Furthermore, linear models of asset prices would have to be replaced by non-linear models, because they are the only ones capable of producing such long range dependencies. It is hence quite natural to see econometricians eargerly searching to find an answer to the question whether long memory is present in financial time series or not.

The next chapters review the main findings of the literature on both topics, short- and long-term autocorrelation. A thorough and detailed treatment would afford more room, for the last two decades have seen a massive bulk of research examining the time dependencies of price changes or returns and its volatility. As the following sections do not only provide a general overview of existing results but do also explain the main techniques needed to deal with the phenomena empirically, this chapter should be seen as a supplement to the time-series based findings on short- and long-run dependencies in the existing literature.

6.1 First-order short run Correlations

Linear dependencies between the current and past values of a variable x are usually captured by the autocorrelation function

$$\rho(k) = \frac{\sum[x_{t+k} - \langle x \rangle][x_t - \langle x \rangle]}{[\sum[x_{t+k} - \langle x \rangle]^2 \sum[x_t - \langle x \rangle]^2]^{\frac{1}{2}}}.^3 \tag{6.1}$$

This function is applied to many time series of price changes or returns and it is by now a common knowledge that the value of $\rho(k)$ is extremely sensitive to the time lag k. For k between some seconds and at most a few minutes, empirical research finds $\rho(k)$ to be significantly different from zero. By investigating high-frequency data, several studies show a negative correlation between successive returns that fades out very quickly, i.e. after only two to three lags. This is documented in the early studies of MacKinlay and Ramaswamy (1988) for the S&P 500 index for the traded futures on the Chicago Mercantile Exchange, Yodav and Pope (1990) for the FTSE 100 and Lim (1990) who analyses the Nikkei 225 index.[4] Recently, Engle and Russel (2002) find a negative autocorrelation in

[4] Explanations for the negative autocorrelations go typically in two directions. The first proposes that noise, defined as typing errors or rounding errors, is responsible. Wang (2003) gives an example how such errors may produce the stylised fact: supposing the following five successive prices are observed: $1.4, 1.5, 1.6, 1.7$ and 1.7. The first-order autocorrelation of the returns is then given by $\rho(k = 1) = 0.87$. Now, if the third price is accidently recorded as 2.5 $\rho(k)$ changes abruptly to -0.42 and so this noise effect produces a negative instead of positive autocorrelation. The second possible source for the autocorrelation is Roll's (1984) bid-ask bounce. Let B indicate a bid price and A an ask price. Supposing that at time t a sale to the market maker takes place at this price B. For $t + 1$, the prices can only either be a further sale or a buy from the market maker. The last possibility would result in a price

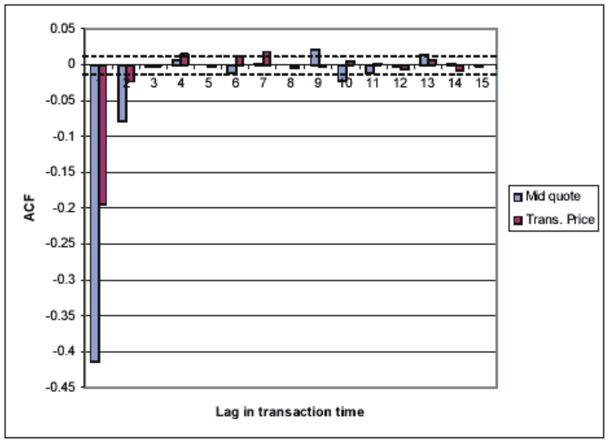

Figure 6.1: Autocorrelation function for the first difference of transaction price and mid quote (the average of bid and ask prices). Although there is a quantitative difference in the values between both measures, the course is similar: after two lags, autocorrelation is insignificant for almost all lags. See Engle and Russel (2002, p. 5).

the US Airgas stock. They use a tick-frequency for their series, i.e. a new data point occurs whenever a transaction takes place. Figure 6.1 adapted from their publication shows the typical strong dependence for the first lag which extends in less strong manner up to the second transaction. Afterwards, no significant values for $\rho(k)$ can be detected.

There are some other studies that find positive instead of negative autocorrelations for the first lags. E.g. Bouchaud and Potters (2000), analysing the S&P 500 with a frequency of 5 minutes find significant positive values for price changes up to 20 minutes.[5] Furthermore, short-run correlations in high frequency data are not confined to developed markets. Early analysis has focussed on stock indices from the major stock markets, but in the meantime, intra-day prices for other markets are also easily obtained. By studying the log-price changes of the MOL (Hungarian Oil Company), the OTP (National Savings Bank) and the TVK (TVK Chemical Company) on a tick-basis, Palá-

$A > B$. Assuming further that sales and buys are equally probable, the expected price of time period $t+1$ is $A + B/2 > B$. Supposing that a buy happens then price is at A. By the same considerations, expected price in $t+2$ is $A + B/2 < A$. Therefore, a lower price (B in t) is followed by an expected higher price (A+B/2) and a higher price A (in $t+1$) is succeded by an expected lower price (A+B/2). According to Roll, this mean reverting phenomenon is responsible for the anticorrelation.

[5] Positive lagged autocorrelations can be explained by the lagged adjustment model of Holden and Subrahamamyam (1992). They claim the existence of so-called leading indices that react very quickly to new information. The other stocks adjust to new values with a greater delay by a news transmission similar to the information cascade models of Bikchandani et al. (1992).

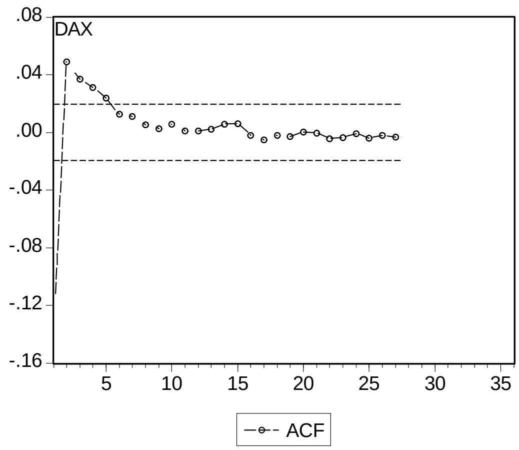

Figure 6.2: Autocorrelation function for the DAX. The dashed lines represent the 95% (Bartlett) confidence intervals of the null hypothesis of no autocorrelation at lag k (1-36). The critical values are computed by $\pm 1.96T^{1/2}$, where T is the total number of observations. See Bartlett (1946).

gyi and Mantegna (1999) are able to confirm that "the autocorrelation function of log-price difference shows a significant anticorrelation in successive transactions...".[6] However, similar to the results above, the correlation vanishes after only two successive transactions, indicating an extremely short memory.

The fact of a fast decaying autocorrelation seems to be well settled. However, the question is still open whether it is positive or negative. While it is unambiguous to have a very short-lived negative dependency in foreign exchange markets (see Dacorogna et al. (2001)), stock markets do not feature such a clear behaviour. In many studies, negative, in some other cases positive correlations are found.

In the following, high-frequency data for the three German indices are investigated.[7] The original series have different frequencies and therefore, the series are manipulated by the linear interpolation as described in chapter 4.7, in order to ensure the comparibility of the results. As can be seen, they are principally in agreement with the literature. Figures 6.2 and 6.3 display the autocorrelation functions for the DAX and the Future DAX.

[6] Palágyi and Mantegna (1999, p. 134).

[7] To compare the daily records of the last chapters with high-frequency data is a senseless effort, because autocorrelations always fade out long before daily intervalls and thus will never show the same correlation structure. Anyway, the daily records do not posses significant autocorrelations.

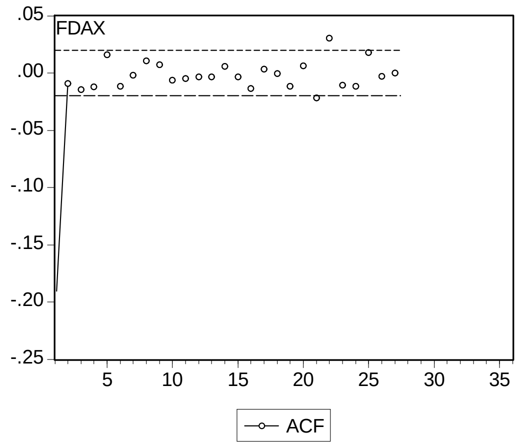

Figure 6.3: Autocorrelation for the FDAX. The ACF is similar to the above result. Here, only the first lag displays a significant values.

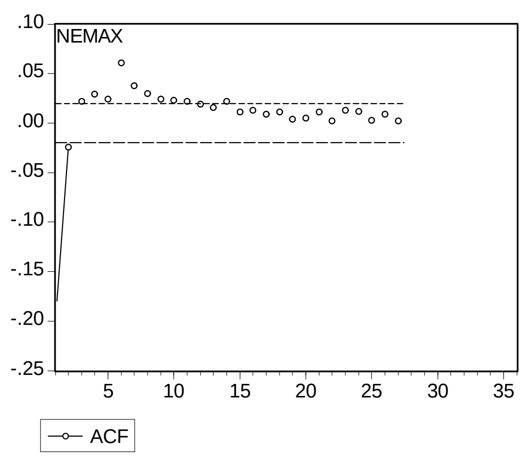

Figure 6.4: Autocorrelation function for the NEMAX.

The most evident result for both series is their strong negative first order autocorrelation that quickly fades out. The DAX has more significant lags, but generally correlation breaks down just after some 5 lags. Figure 6.4 presents the ACF of the minutely price changes for the NEMAX. Quite interestingly, significant values for $\rho(k)$ appear up to 11 minutes.

All in all, the outcome is in good accordance to the weak form of the EMH as it is questionable whether even large institutional banks are able to exploit the moderate correlations for high frequency data given transaction costs.[8]

6.1.1 First-order long-run Correlations

Long memory models have been playing a major role in economic literature, at least since Granger's (1966) article on *The typical spectral shape of an economic variable*. However, the origin of interest in time series that exhibit long-range dependencies has come from observations in fields as diverse as climatology or geology. The perhaps most well-known example is the study of reservoir control for the river Nile by the hydrologist Hurst (1951). He was the first who extensively studied what was later come to be known as biased random walks or fractional Brownian motions. Their specific characteristic is the presence of long range dependencies that can be roughly defined in terms of a persistence in observed autocorrelations over very long time records. Later on, Mandelbrot took up the subject and popularised it in many papers (Mandelbrot and Wallis (1968), Mandelbrot (1971, 1972)). This effort encouraged others to further empirical studies. Greene and Fielitz (1977) is an early example that analyse the daily returns of securities of the NYSE and found long-memory properties in many of them. Chapter 6.1.2 will provide an overview of the more recent literature.

6.1.1.1 Long Memory versus short Memory

Existing literature provides a couple of formal definitions of long-range dependency in order to separate it from the short memory. The first intuitive way is obtained by considering the average of a discrete time series process $x(t)$. If the sequence of random variables $x(1), ..., x(n)$ is *iid* and has a finite variance, then $E[x] = \overline{x}$ is asymptotically normal and

[8] One might argue that the reduction in transaction costs due to electronic commerce is of such an order as to preserve some profits from a strategy that employs empirical autocorrelations. In fact anyone now can use web browsers for their trades thus reducing the full service brokerage rates by a huge factor. With online brokerage, greater proliferation of online financial information should also effect the market. This however has two different implications. For the most part of the literature on the effects of a decrease in transaction costs on the performance of markets, it is suggested that such a decline is beneficial, i.e. markets should become more efficient (see e.g. Malone and Rockart (1991)). On the other hand empirical research shows that investors with access to online trading act more frequently but with lower total returns than those who do not place their orders online (Barber and Odean (2001)). It is thus thinkable that lower transaction costs attract more uninformed noise-traders rather than fundamentally orientated investors. This would stand in contrast to the tendency of a more efficient market. However, no definite result is obtained so far.

$$var(\overline{x}) = n^{-1}\sigma^2. \tag{6.2}$$

This result, however, hinges on the assumption of uncorrelated variables. If the time series is allowed to have correlated members, (6.2) has to be modified to

$$var(\overline{x}) = n^{-1}\sigma^2 \sum_{i,j=1}^{n} \rho(k), \tag{6.3}$$

where $\sum \rho(k)$ captures the autocorrelations for x between time indices i and j, i.e. a lag of length $k = |i - j|$.[9] For $\sum\limits_{i,j=1}^{n} \rho(k) \neq 0$, (6.3) can also be written as

$$var(\overline{x}) = \sigma^2(1 + \delta_n(\rho)n^{-1}, \text{with} \tag{6.4}$$

$$\delta_n(\rho) = \sum_{i,j=1}^{n} n^{-1}\rho(k).^{10} \tag{6.5}$$

Upon replacing $1+\delta_n(\rho)$ by a constant $c(\rho)$ one obtains a generalisation of (6.4) with

$$var(\overline{x}) \sim \sigma^2 c(\rho) n^{-1}.^{11} \tag{6.6}$$

This form is valid for many different kinds of time series namely the Markov and the ARMA processes. The behaviour of $var(\overline{x})$ in (6.4) is similar to the normal process since it is still proportional to n^{-1}. With the constant added (6.6) is able to account for even strong temporal dependencies such as an AR(1) process like $x_t = 0.9x_{t-1} + \varepsilon_t$. I.e. even with a high positive autocorrelation, the average variance decays with a rate that is proportional to the inverse of n.

However, the inclusion of a constant does not account for every kind of temporal connections between lagged variables. There are processes which cannot be modelled by just including terms for a possible short-term autocorrelation. In fact, some data sets show a lower convergence than n^{-1}. A simple way to cope with such phenomena while maintaining the functional form of (6.6) is by introducing a new parameter α so that

$$var(\overline{x}) \sim \sigma^2 c(\rho) n^{-\alpha}, \text{ where} \tag{6.7}$$

$\alpha \in (0, 1)$ and $c(\rho)$ is defined by

$$\lim_{n\to\infty} n^{\alpha-2} \sum_{i\neq j} \rho(k). \tag{6.8}$$

[9] $|i - j|$ is taken in absolute values because of an assumed symmetrical behaviour of $\rho(k)$.

[10] See Beran (1994, p. 3).

[11] The replacement of $1 + \delta_n(\rho)$ by $\delta(\rho)$ is possible because $\delta_n(\rho)$ tends to a finite constant for $n \to \infty$, i.e. $\lim\limits_{n\longrightarrow\infty} \delta_n(\rho) = \sum\limits_{i,j=1}^{n} n^{-1}\rho(k) = \delta(\rho) < \infty$.

By considering the correlations that only depend on the lag $|i-j|$ it is clear that the sum of all these correlations is now proportional to $n^{1-\alpha}$, i.e.,

$$\sum_{k=-(n-1)}^{n-1} \rho(k) \sim cn^{1-\alpha}, \tag{6.9}$$

where $c \neq c(\rho)$ is a constant. By the fact that α is bound to lie between 0 and 1,

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty. \tag{6.10}$$

This is exactly the definition of long-memory given by McLeod and Hippel (1978). According to them a given discrete time series $x(t)$ with autocorrelation function $\rho(k)$ possesses a long-range dependency if the sum of all correlations decay to zero so slowly that they are not summable. In particular, (6.9) holds for

$$\rho(k) \sim c\,|k|^{-\alpha}, \tag{6.11}$$

if $k \to \infty$ and c being a positive constant.

An equivalent condition can be obtained by considering the spectral density $f(\omega)$ of the process, as defined by

$$f(\omega) = \frac{\sigma^2}{2\pi} \sum_{k=-\infty}^{\infty} \rho(k) e^{ik\omega}.^{12} \tag{6.12}$$

(6.11) then implies

$$f(\omega) \sim c\,|k|^{\alpha-1}, \tag{6.13}$$

and again for $\alpha < 1$, f_ω tends to infinite for $\omega \to 0$.[13]

An alternative way of describing the behaviour of a time series process is provided by using the partial sum

$$S_T = \sum_{t=1}^{T} x_t.$$

Rosenblatt (1956) defines short-memory dependency as the property of satisfying so-called strong-mixing conditions. Loosely speaking, these conditions demand that the correlations between any two points of the process becomes

[12] See e.g. Priestley (1981).

[13] Markov and ARMA processes have autocorrelations with an asymptotic exponential decay that is bounded by $|\rho(k)| \leq bm^{-k}$, for large k and $0 < m < 1$, and hence feature a short memory.

trivially small as the distance $|i-j|$ gets larger. In order to be more precise, the time series $x(t)$ has short memory if

$$\sigma^2 = \lim_{n\to\infty} E[n^{-1}\left(\sum x_t\right)^2] = [n^{-1}S_T^2] < \infty, \text{ and} \tag{6.14}$$

$$[\sigma^{-1}T^{1/2}]S_{[rT]} \overset{asym.}{\to} BM(r), \tag{6.15}$$

where $[rT]$ is the integer of rT and $BM(r)$ is the standard BM. Otherwise the process has a long memory characteristic.[14] Although the two definitions above differ in detail, they both incorporate the idea of a non-vanishing dependence of any two data points, no matter how far the temporal distance between them becomes. The next two sections give some examples of processes with long-range dependencies in order to illustrate the behaviour of the corresponding autocorrelation.

6.1.1.2 Fractional Brownian Motion

In chapter 5.2.2 fBM was introduced as a generalisation of ordinary BM. With Hölder exponents different from $\frac{1}{2}$, the fractal nature of financial time series should be modelled more realistically as compared with the unbiased random walk hypothesis of the EMH. Though, as it turned out, a pure fBM cannot account for the multiscaling feature of empirical data sets and thus is a suggestion too narrow to deliver a satisfactory agreement with the theoretical predictions. Despite this shortcoming, fBM still serves as a theoretical basis for the long memory characteristics of financial time series. Let $x(t)$ be a self-similar process with stationary increments and with self-similarity or Hölder exponent H so that

$$x_k - x_0 = |k|^H (x_1 - x_0) \sim |k|^H ,$$

just as in chapter 5.2.2. Let furthermore be assumed that $E[x_t] = 0$ and $E[x_t^2] = \sigma^2$ as the variance of the increment process $\Delta x_t = x_t - x_s$, $s < t$. Then

$$E[(x_t - x_s)^2] = E[(x_{t-s} - x_0)^2] = \sigma^2(t-s)^{2H}. \tag{6.16}$$

For $H = 0.5$ one recovers the well known relation

$$E[(x_t - x_s)^2] = \sigma^2(t-s),$$

which says that the variance is growing in $t-s$. From the fact that

$$E[(x_t - x_s)^2] = E[x_t^2] - 2E[x_t x_s] + E[x_s^2] = \sigma^2 t^{2H} - 2\gamma(t,s) + \sigma^2 s^{2H}$$

it follows that the covariance γ is given by

[14] These conditions can be extended (see e.g. Lo and MacKinlay (1995)) but in any case they require the finiteness of moments up to a specific order. For further details of mixing conditions see in particular Eberlein and Taqqu (1986). An even wider definition of long-memory that seperates it from the short-memory is given in Resnik (1997).

$$\gamma(t,s) = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}\sigma^2, \qquad (6.17)$$

where σ^2 is supposed to be finite. With $H = 0.5$, $\gamma(t,s) = 0$ follows.

A similar consideration leads to the covariance and correlation function of the increments of a fBM process. The covariance $\gamma(k)$ between Δx_t and Δx_{t+k} is given by

$$\gamma(k) = 0.5\sigma^2\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right].^{15} \qquad (6.18)$$

The autocorrelation of Δx is thus

$$\frac{\gamma(k)}{\sigma^2} = \rho(k) = 0.5\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right]. \qquad (6.19)$$

The asymptotic behaviour of $\rho(k)$ for $k \to \infty$ can be obtained by a Taylor expansion and is found to be of the form

$$\rho(k) = H(2H-1)\,k^{2H-2}.^{16} \qquad (6.20)$$

Again, $H = 0.5$ yields $\rho(k) = 0$ which has to be expected for a usual BM, but $0.5 < H < 1$ results in positive values for $\rho(k)$ no matter how large k becomes and the correlation decays so slowly that

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty, \qquad (6.21)$$

which is the definition of long memory as given by (6.10). If $0 < H < 0.5$, the process has short-range dependence and the correlations sum up to zero. This behaviour is termed *antipersistence* by Mandelbrot. The increment process of x can also be expressed in the frequency domain by

$$f(\omega) \sim c_f\,|\omega|^{1-2H}, \qquad (6.22)$$

for $\omega \to 0$ and c_f as a positive constant.[17] Thus, for $0.5 < H < 1$, the process has an unbounded spectral density at zero frequency,

$$f(0) = \infty.^{18}$$

The fBM is chronologically the first theoretical process with a long memory property. However, the next section deals with another example that has been used more frequently in the recent econometric literature.

[15]See Beran (1994, p. 51).
[16]See Beran (1994, p. 52).
[17]See Sinai (1976) for a derivation and mathematical proove. See also Beran (1994, p. 3).
[18]The formulation in the frequency domain will proove to be vital for the comprehension of two of the empirical tests.

6.1.1.3 Fractional White Noise

The fBM or fractional white noise process is a continuous stochastic process. Its discrete time equivalent is the fractionally differenced white noise. According to Granger and Joyeux (1980) and Hosking (1981) who first developed this process, it is defined by the following difference equation:

$$(1-L)^d(x_t-\mu)=\varepsilon_t,\ \varepsilon_t\sim(0,\sigma^2), \tag{6.23}$$

where L is the lag operator, $\mu=E[\bar{x}]$ and d is a difference parameter that can take on any value in the open interval $(0,1)$. (6.23) can thus be interpreted as an extension to an ARIMA (Autoregressive integrated moving average) $(0,d,0)$ process where d is not confined to integer values. However, having $d=0,1$ or 2 is an usual assumption in economic time series, but a value of $d=0.3$ may seem puzzling at first sight. Allowing non-integer values of d in (6.23) can be achieved by expanding $(1-L)^d$ with the help of the binomial development, i.e.

$$(1-L)^d=\sum_{k=0}^{\infty}(-1)^k\binom{d}{k}L^k. \tag{6.24}$$

Applying this expansion to (6.23) results in

$$(1-L)^d x_t=\sum_{k=0}^{\infty}(-1)^k\binom{d}{k}L^k x_t=\sum_{k=0}^{\infty}\pi_k x_{t-k}=\varepsilon_t, \tag{6.25}$$

where the π_k represents infinite-order autoregressive weights. With $\pi_0=1$, the coefficients π_k can be expressed through

$$\pi_k=(-1)^k\binom{d}{k}=\frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)}=\prod_{k=1}^{k}\frac{k-1-d}{k}. \tag{6.26}$$

For $-0.5\leq d\leq 0.5$, $x(t)$ is invertible and can also be expressed as an infinite-order MA process

$$x_t=(-L)^d\varepsilon_t=B(L)\varepsilon_t, \tag{6.27}$$

$$x_t=\sum_{k=0}^{\infty}\frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)}\varepsilon_{t-k}, \tag{6.28}$$

where $B(L)$ describes the coefficients of the lagged terms ε_{t-k}, $k=1,2,...$and μ is assumed to be zero. The covariance and correlation function is then given by

$$\gamma(k)=\frac{(-1)^k\Gamma(1-2d)}{\Gamma(k+1-d)\Gamma(-k+1-d)}\sigma_\varepsilon^2\sim ck^{2d-1}, \tag{6.29}$$

$$\rho(k)=\frac{\Gamma(k+d)\Gamma(1-d)}{\Gamma(k+1-d)\Gamma(d)}.^{19} \tag{6.30}$$

[19] See e.g. Baillie et al. (1996).

Thus for $d > 0$, the autocorrelation decays so slowly that their sum diverges to infinity. To be more concrete

- for $0 < d < 0.5$, the process is stationary and has a long memory in the sense of condition (6.10);
- for $-0.5 < d < 0$, the process has a sum of absolute values for $\rho(k)$ that is finite. Hence it has a short-memory and is antipersistent in the sense of Mandelbrot and Wallis (1969);
- for $d = 0$, the process is white noise with no autocorrelations;
- and for $d = \pm 0.5$ the process is $1/f$-noise.[20]

While both processes, the fBM and the fractional white noise are able to cope with the autocorrelation structure found in financial time series, attention in recent years has focused on a more flexible approach that was also independently introduced by Granger and Joyeux (1980) and Hosking (1981), the ARFIMA (ρ, d, q) process

$$\varphi(L(1-L)^d)(x_t - \mu) = \psi(L)\varepsilon_t, \tag{6.31}$$

where ε is again the usual white noise term. All roots of $\varphi(L)$ and $\psi(L)$ are assumed to lie outside the unit circle. As an extension to the ARIMA class, the functional form of (6.31) allows for values of d different from integers. It has similar properties as the fractional white noise. Its Wold decomposition has autocorrelation functions which decay at a hyperbolic rate and so represent a long memory characteristic. The process is covariance stationary for d in the open interval $(-0.5, 0.5)$.[21]

The spectral density of an ARFIMA process has the form

$$f(\omega) = \left|1 - e^{i\omega}\right|^{-2d} \frac{\sigma^2}{2\pi} \left|\frac{\psi(e^{i\omega})}{\varphi(e^{i\omega})}\right|^2 .^{22} \tag{6.32}$$

or simply

$$f(\omega) \sim c_f \left|1 - e^{i\omega}\right|^{-2d} . \tag{6.33}$$

Recalling the functional form of $f(\omega)$ for the fBM, it is easy to see the relationship

$$d = H - 0.5.$$

[20] In the time domain, noise signals can be seen as superpositions of periodic signals at different frequencies. Supposing the noise has a spectral density that is proportional to $1/f^{\alpha}$. Then if $\alpha = 0$, one has white noise, if $\alpha = 1$, it is called $1/f$-noise or flicker noise. This noise is a widespread phenomenom in many physical phenomena like electronic devices, astronomy etc.

[21] See Baillie et al. (1996) for further details.

[22] Beran (1994, p. 63).

Thus $d > 0$ indicates a long memory.

The above introduced examples of stochastic processes are not the only ones with a long memory property. Other suitable models are the Gegenbauer process which is considered by Gray, Zhang and Woddward (1989), the seasonal fractionally differenced process introduced by Porter-Hudak (1990) or the fractional cointegration (Granger 1981, 1983). Some models are especially developed in order to reproduce a long memory in the volatility of the process. These comprise the fractionally integrated GARCH (FIGARCH) model (see Baillie et al. (1993)), the fractionally integrated exponential GARCH (FIEGARCH) model of Bollerslev and Mikkelsen (1996) or the long memory ARCH (LM-ARCH) model of Ding and Granger (1996), which all possesses the characteristic of a hyperbolic rate of decay for the autocorrelation functions.

In any case, none of these theoretical propositions give any clue about the source of the long-run dependencies. Thus, detecting a similarity or accordance of real time series to the theoretical models says nothing about the economic reasons for that phenomena. It must therefore be emphasised that the processes presented above are just but statistical models which obey the envisaged long-run dependency. Nevertheless, many of the methods that are designed to detect the long memory characteristics build upon these theoretical time series concepts. The next section discusses two of the most frequently used tests in the literature. The use of the simple autocorrelation function is not a good method for detecting long-range temporal dependencies. For example, the autocorrelation function affords a well defined variance function in order to be applied. However, this prerequisite is not fulfilled for some stochastic processes like the LSD with index of stability less than 2.[23] Nevertheless, in the following $\rho(k)$ will be plotted against the lags in order to give a first impression.

6.1.1.4 Hurst's Rescaling Method

This arguably best known method was developed by Hurst in order to study the cyclical behaviour of the River Nile. Let $y_j = \sum_{i=1}^{j} x_i$ be the sum of a sequence of random variables $x(i) = x_1, x_2, \ldots$from time i to j.[24] Then, the rescaled adjusted range, or simply the R/S statistic is calculated by

$$\begin{aligned} Q_k = \frac{R_k}{S_k} \quad = \quad & \frac{1}{\sqrt{S_k^2}} \left[\max_{0 \leq i \leq k} \left(y_{t+i} - y_t - \frac{i}{k}\left(y_{t+k} - y_t\right)\right) \right] \\ & - \frac{1}{\sqrt{S_k^2}} \left[\min_{0 \leq i \leq k} \left(y_{t+i} - y_t - \frac{i}{k}\left(y_{t+k} - y_t\right)\right) \right].^{25} \end{aligned} \tag{6.34}$$

[23] See Mandelbrodt (1972, 1975) for a study of the deficiencies of ordinary autocorrelation functions in detecting long memory characteristics.

[24] Where the variable x_i in the Nile-example represents the inflow at time i; this variable can here be thought of as price increments

[25] See Beran (1994, p. 33).

The bracketed term is the range, i.e. the maximum minus the minimum over k partial sums of the first k deviations of y_j from its mean and

$$S_k = \sqrt{S_k^2} = \sqrt{k^{-1} \sum_{i=t+1}^{t+k} x_i^2 - k^{-2} \langle x \rangle_k^2}$$

is the standard deviation. Its inclusion can be regarded as a standardisation of R_k which makes the statistic independent of the scale. Hurst (1951), Mandelbrot and Wallis (1969) and Taqqu (1975, 1977) is able to show that

$$p \lim \left\{ k^{-H} (R_k / S_k) \right\} = \text{ constant.}$$

The idea is to calculate $\log(R_k(/S_k)$ and regress the values for different lag lengths k on a constant and $H(\log(k))$. More precisely, the R/S statistic is constructed first by subdividing the time series of length N into K blocks of size N/K each. Then, starting at points $k_i = i \ N/(K+1)$, $i = 1, 2, ...,$ $R(k_i, k/S(k_i, k))$ must be computed for each k, matching the condition $k_i + n \leq N$. For values of n smaller than N/K, one has K different estimates of R_k/S_k. By plotting the logarithm of R/S[26] against various values of k, Hurst noted that the values were scattered around a straight line with a slope that was above $1/2$. This was in sharp contrast to the then usual view of a stationary time series. For those processes log R/S results in a line with slope $1/2$, i.e. the regression

$$\log E[R/S] = c + \hat{H} \log k + \varepsilon_t, \ \varepsilon_t \overset{iid}{\sim} (0, 1) \tag{6.35}$$

should yield a value of $\hat{H} = 1/2$. The outcome with values of $\hat{H} > 1/2$ for the River Nile data was called the *Hurst effect*.

However, some severe problems arise in the application of the R/S statistic. First, the regression is valid only for a large enough k; thus deciding how far one goes back in history is crucial for $\hat{H}$. For example, choosing k too small may lead to misinterpretations because some series reveal their characteristic only after extreme long lags. The second problem concerns the estimation of H. For finite samples, Q is neither normally nor symmetrically distributed and hence the use of a simple OLS technique is questionable.[27]

Another deficiency of the R/S-statistic is its sensitivity to the presence of short-memory in the time series. I.e. values of $\widehat{H} > 1/2$ may come merely from the presence of some short-range dependencies in the data. To overcome this shortcoming, Lo (1991) proposes a modified R/S statistic. He replaces the denominator S_n by a consistent estimator of $\sqrt{var(\ y_n)}$. The reason for implementing a new denominator is the fact that, in the presence of short-run correlations, the variance of the partial sum is not simply the sum of the variance of the individual $x_i's$. In addition, their autocovariance have also to be included. Lo (1991) calculates the new denominator by

[26] The subscript k is from now on skipped for conveniance.

[27] The exact distribution of Q is rather difficult to calculate. For details see Mandelbrot and Wallis (1969).

$$
\begin{aligned}
\hat{S}_n^2(q) &= n^{-1}\sum_{j=1}^{n}(x_j-\bar{x}_n)^2 + 2n^{-1}\sum_{j=1}^{q}\omega_j(q)\left[\sum_{i=j+1}^{n}(x_i-\bar{x}_n)(x_{i-j}-\bar{x}_n)\right] \\
&= \hat{S}_x^2 + 2\sum_{j=1}^{q}\omega_j(q)\hat{\gamma}_j, \text{ with}
\end{aligned}
\tag{6.36}
$$

$$
\omega_j(q) \equiv 1 - \frac{j}{q+1} \text{ for } q < n. \tag{6.37}
$$

$\hat{S}_n^2$and $\hat{\gamma}_j$ are the sample variance and autocovariance estimators respectively. The weights $\omega_j(q)$ are taken from Newey and West (1987) and q is determined via Monte Carlo simulations. The modified R/S-statistic is then given by

$$
\widetilde{Q}_n = \frac{R_n}{\hat{S}_n}. \tag{6.38}
$$

Lo is able to show that given the correct choice of q, the distribution of $n^{-1/2}\widetilde{Q}_n$ is asymptotic to a Brownian Bridge. He constructs an interval [0.809, 1.862] which he uses as the 95% (asymptotic) acceptance region for testing the null hypothesis of $H_0 = H = 0.5$, i.e. no long-range dependence.

This new method is a significant improvement as shown by Hauser, Kunst and Reschenhofer (1994). Short-run correlations do not longer influence the estimation and values of $\hat{H} > \frac{1}{2}$ become more trustworthy. Nevertheless, the method faces similar problems as the classical R/S statistic. Lo's results are asymptotic since they afford N and $q = q(N)$ both to go to infinity. In reality N is finite and q is dependent on the choice of the researcher. Naturally, the question is how to choose q correctly and the Monte Carlo studies of Andrews (1991) and Lo and MacKinlay (1989) give only a vague guidance for selecting a right truncation lag. For q being large compared to the sample size n, the finite-sample distribution may become different from its asymptotic limit and for q being small the danger exist to neglect substantial autocovariance in the weighted sum. This is in fact a crucial point as Teverovsky, Taqqu and Willinger (1998) are able to show. Neither a small nor a large q ensures the detection of the true nature of the process automatically. For example, time series with high short-run correlations generally result in a too frequent rejection of the null hypothesis $H_0 = 0.5$, despite no long memory. On the other hand, for q being large, H_0 is almost never rejected although a long-memory characteristic existed.

To sum up, the results of the $R/S-$ statistic and its modification by Lo has to be taken with some care. However, it is irrespective of its deficiencies a useful approach to obtain a first picture about the possibility of a long-term behaviour in the empirical process.

6.1.1.5 Least Square Regression using the Periodogramm

This method exploits the fact that the spectral density of an increment process x for a fBM or ARFIMA process is given by (6.22) or (6.33), thus the spectral density at the origin is

$$f(\omega) \sim c_f \, |\omega|^{-2d}, \text{ for } |\omega| \to 0.$$

Therefore, a regression of

$$\log f(\omega) = \log c_f - 2d \log |\omega| + \log \xi \tag{6.39}$$

should yield an estimator for the slope $2d$ and thus also $\widehat{d}$. Because the periodogramm $I(\omega)$ is an asymptotically unbiased estimator of $f(\omega)$ for each fixed $\omega \neq 0$, one has $E[I(\omega)] = f(\omega)$. Thus once $I(\omega)$ is obtained for different frequencies ω, an estimation of d is achieved by using OLS. Note that $E\left[\log \xi\right] = -0.577$ which is known as the Euler constant. The periodogramm $I(\omega)$is calculated by

$$I(\omega) = \frac{1}{2\pi T} \mid \sum_{j=1}^{T} x_j e^{ij\omega} \mid^2 , \tag{6.40}$$

where T is the number of observations for the empirical price changes, returns or the volatility.

Geweke and Porter-Hudak (1983, GPH henceforth) in particular regress a number of logarithmic periodogramms on a constant and a nonlinear function of the frequencies ω according to

$$\log\left[I(\omega_j)\right] = c + \beta \log\left[\sin^2(\frac{\omega_j}{2})\right] + \eta_{j,} \tag{6.41}$$

where $j = 1, ..., n < T$ is the number of periodogramm ordinates used in the regression, $\omega_j = (2\pi j)/T$ and $\eta_j = \log \xi_j$. The estimation of the fractional difference parameter d in (6.39) is $\frac{1}{2}$ times the negative of β. One problem remains. It is the choice of $n = T^{\mu}$ as the number of periodogramm ordinates. A common choice is to set $n = T^{0.5}$ though this may not be the best value. Therefore empirical tests often use different values of μ to account for the sensitivity of the regression. Robinson (1992) proposes an alternative way of regression by using weighted least squares. This is usually the second variant that is used for the empirical tests.

Like the former method, regressing the periodogramm has its caveats. First, the procedure assumes a proportionality of $f(\omega) = E[I(\omega)]$ and $-2d$ for the whole range for which it is computed, i.e. $(-\pi, \pi)$. However, this might be wrong if the spectral density is proportional only in a small neighbourhood around zero. This problem can be solved by using only low frequencies in the estimation, as done in Geweke and Porter-Hudak (1983) and Robinson (1992). Unfortunately, this causes a lower precision of the regression. Moreover, the distribution of the error term in (6.39), is highly skewed which makes the OLS-estimates inferior to estimators that account for this property. Despite these problems, the methods using periodogramms have been shown to perform better than the R/S-statistics. Karagiannis, Faloutsos and Riedi (2002) testing a wide range of methods conclude that the periodogramm gives satisfying estimations, after being applied to a simulated fractional white noise. This is corroborated by de Peretti and Marimoutou (2001) who also perform tests for different methods. There, AR-processes of various orders and two ARFIMA processes are generated for the purposes of testing the effectiveness of the estimators. Again, the method that uses periodogramms (in this case Robinson's method) gives more robust results than the R/S-statistic of Lo (1991).[28]

6.1.2 Empirical Evidence of long Memory in Raw Returns

Contrary to the fat tails and the multiscaling phenomenon which both proved to be well settled stylised facts, recent studies on the presence of long memory in stock returns is rather mixed. Indeed, some contributions find evidence for long range predictability in stock returns, see Fama and French (1988), Poterba and Summers (1988) and Mills (1993) inter alia. On the other hand, studies of Lo (1991), Goetzman and Jorion (1993), and Nelson and Kim (1993) do not show significant results for existing positive autocorrelations on longer time horizons. Whereas some of the positive findings may be attributed to the (wrong) use of the classical R/S-statistic later studies employ the modified version of Lo and the approaches of Geweke and Porter-Hudak (1983) and Robinson (1992).

A recent study with a wide range of tested time series is the one of Cheung and Lai (1995). They extend the Lo (1991) study to 17 stock markets. In addition, they also use the GPH method as a second estimation technique besides the modified R/S-statistic. However, results are still more supportive for the Lo findings of no long range dependencies than for the alternative hypothesis of a long memory in raw returns. Table 6.1 gives the outcome of the Cheung and Lai (1995) study.

[28] It must be mentioned that both studies have more estimators tested than the two explained here. However, these are the most prevailing ones in the economic literature.

Table 6.1: Cheung and Lai (1995) estimation results on raw returns[29]

Country	$\hat{d}$ $(n = T^{0.5})$[30]	$\hat{d}$ $(n = T^{0.55})$	R/S
Australia	0.214	0.027	1.158
Austria	0.330	0.386*	1.696
Belgium	0.383	0.303	1.456
Canada	-0.043	-0.100	0.871
Denmark	0.069	0.123	1.28
France	0.032	0.070	1.013
Germany	0.086	0.103	1.129
Hong Kong	0.019	-0.018	1.351
Italy	0.560*	0.400*	1.635
Japan	0.468*	0.410*	1.285
Netherlands	0.311	0.067	1.217
Norway	-0.051	0.037	1.946
Singap./Malays.	0.031	0.095	1.049
Spain	0.522*	0.418*	1.745
Sweden	0.159	0.173	1.301
Switzerland	0.168	0.149	0.948
United Kingdom	-0.109	-0.044	1.228
United States	-0.001	-0.148	1.118

The table shows that only a few countries feature return data in favour of the long memory hypothesis. Although all series have a value of $\hat{d} \neq 0$, which would indicate long memory, they are all statistically insignificant. Italy, Spain and Japan are the three series that can claim to yield two significant estimation values for the GPH method while Austria is significant only for the second choice of n. In particular the R/S statistic displays poor outcomes for the long memory proposition. Notetably is that even slight differences in n can lead to substantial differences in the estimated values. Significance, however, is only different in one case (Austria).

Another study is the one of Barkoulas and Baum (1996). They take two aggregate stock indices with a daily frequency (the S&P 500 index and the Nasdaq Index) and 7 series with a monthly frequency. By applying the residuals regression, the fractional differencing parameter d is estimated for each time series, the authors could not find "any consistent, convincing evidence supporting the long memory (biased random walk) hypothesis for the returns series of any of the aggregate or sectorial stock indices"[31] A more mixed picture emerges when individual series of the Dow Jones Industrial companies are considered, though evidence of long memory in this data package remains sparse. Only two out of 24 time series indicate the presence of long range dependencies. In a subsequent paper Barkoulas, Baum and Travlos (2000) focus their attention on the stock market represented by the Athens Stock Exchange (the ASE). By using the

[29] Astericks indicate significance at the 5% significance value.
[30] Recall that n is the number of periodogram ordinates used in the regression (6.41).
[31] Barkoulas and Baum (1996, p. 123).

spectral regression method of GPH, the authors are able to find evidence for the presence of long memory in the weekly data from January 1981 to December 1990.

Tolvi (2003) analyses the Finnish stock market with the help of the GPH and the Robinson method. The data consists of daily returns for six indices from the Helsinki stock exchange (the HEX 20). Tolvi reports of 35% of the series to "have statistically significant long memory at the 10% level, and 26% (12 out of 46) at 5% level; when applying the GPH-estimation."[32] This is at least not a total rebuttal of long range dependencies as found in the papers referred before.

Henry (2002) is another recent study that uses the GPH method. Here, only monthly data for 9 national stock indices are considered. Similar to the Tolvi results, the continuously compounded returns, $R_{i,t} = \log(P_{i,t}/P_{i,t-1})$, show a rather mixed picture.

Table 6.2: Henry's (2002) estimation results on raw returns

Country	$n = T^{0.475}$	$n = T^{0.5}$	$n = T^{0.525}$
US	-0.232 (0.512)	0.326 (0.424)	-0.400 (0.363)
Japan	0.481 (0.214)	0.405 (0.202)	0.157 (0.227)
Germany	0.318 (0.159)	0.258 (0.137)	0.215 (0.120)
UK	0.055 (0.180)	-0.058 (0.163)	-0.194 (0.161)
Hongkong	0.180 (0.217)	0.330 (0.217)	0.236 (0.211)
Taiwan	0.646 (0.175)	0.462 (0.182)	0.293 (0.185)
South Korea	0.701 (0.199)	0.696 (0.163)	0.589 (0.160)
Singapore	-0.0205 (0.180)	0.031 (0.234)	-0.094 (0.220)
Australia	0.068 (0.283)	-0.073 (0.247)	-0.106 (0.213)

As can be seen, the South Korean index shows the strongest evidence for long memory. Even with standard errors considered, the estimated values stays above 0.5. For Germany, Japan and Taiwan the evidence is less convincing and for the five remaining time series the long memory hypothesis should be rejected.

To sum up, the literature shows mixed evidence for the existence of long memory in the time series for raw returns in stock markets. In most cases, test results are not significantly different from the no long memory hypothesis and for the small group of supporting examples, values are close to the theoretical value for short memory correlations. Considering the still existing problems with the statistical methods, this is not a strong indication for the presence of long-run dependencies. Hence, the overall impression of past studies on the raw returns of stock price data, whether indices or individual equities, is more in accordance to the EMH than against it.

[32] Tolvi (2003), p. 5.

6.1.3 Own Estimations for Raw Returns

In order to check the findings of the literature, the empirical time series for the stock indices and individual assets of chapter 4.7 are taken for an additional testing. The applied estimation techniques will be the modified R/S-statistic of Lo (1991) and the spectral regression method of GPH (1983). Applying the GPH spectral procedure needs the specification of the number of low frequency ordinates used in the spectral regression because in the literature there is no agreed upon best number. The following tests all use the proposition of Hurvich and Beltrao (1994), which found $n = T^{0.8}$ to be the best choice. The lag length q in (7.36) for the R/S-statistic is chosen according to Lo (1991).[33] Returns are computed by taking the (natural) logarithms of price differences, i.e. $R_t = \ln(P_t/P_{t-1})$. Because the number of data points exceed 3000 in any case, the tests do not suffer from problems of a set too small to deliver trustworthy estimates. Estimated values of H from the R/S-statistic are given below in table 6.3 together with the estimations for d of the GPH test procedure. Like above, astericks indicate significance at the 5% level. The numbers in brackets for the GPH are the t-values.[34]

[33] These settings are used in all own estimations.

[34] This notation is kept throughout the whole work, including part three.

Table 6.3: Long memory tests of the raw returns of the daily price records[35]

Series	R/S-statistic	GPH (with n=$T^{0.8}$)
S&P 500	0.611	0.008 (1.229)
Sydney Stock Index	0.56	0.012 (1.44)
Nikkei 225	0.51	-0.0089 (0.739)
FTSE 30	0.54	0.1088 (0.953)
Mexican IPC	0.486	-0.2337 (1.841)
Hang-Seng	0.442	-0.098 (1.552)
Singapore	0.613	0.1322 (1.298)
CAC 40	0.577	0.1761 (1.842)
Toronto Stock Index	0.483	-0.013 (1.19)
Varta	0.59*	0.058 (1.9543)
BASF	0.58	0.006 (1.003)
BMW	0.42	-0.1 (1.348)
Buderus	0.521	0.0002 (1.048)
Bewag	0.44	0.152 (0.995)
Continental	0.506	0.065 (1.901)
Bayer	0.46	-0.036 (1.578)
Kugelfischer	0.467	-0.03 (1.451)
Phoenix	0.571	0.061 (1.7794)
Harpen	0.526*	0.0571 (2.488)

The analysis suggests an absence of long memory in the indices. Almost no time series is able to reject the hypothesis of no long-term dependence. The only exceptions are Varta and Harpen for the R/S-statistic and Harpen for the GPH estimator. This is not a very convincing evidence in favour of the long memory hypothesis. It is much more in line with the empirical literature, where only a few series with a long temporal dependency in the price changes are found.

This first round test for the presence of a long memory characteristic is completed by examining the high-frequency data. Table 6.4 shows the estimated values for the three German indices of the DAX, the FDAX and the NEMAX.

Table 6.4: Long memory tests of the raw retruns of the high-frequency data

Time series	R/S statistic	GPH
DAX	0.551	0.03 (0.985)
FDAX	0.892	0.211 (1.442)
NEMAX	0.701	0.174 (1.233)

There is no evidence in favour of the long memory hypothesis for the R/S-statistic and the GPH estimator. None of the estimations reject H_0 despite the high values of $\hat{H}$ and $\hat{d}$ for the NEMAX and the FDAX. Thus, own tests mirror the literature in its mixed findings with the majority of series in contradiction to the long memory hypothesis. This is an important outcome as the existence

[35] Astericks indicate significance at the 5% significance value as calculated by Lo (1991). Values in brackets are the t-values for the null hypothesis $H_0 = d = 0$.

of a long-memory process would indicate the possibility of using the remote past in creating more reliable forecasts of future prices. In this case, consistent speculative profits could be made from the observation of past prices which contradicts the weak form of market efficiency. As the tests show, such a claim cannot be made.

6.2 Second-order Correlations

The fact of a rapid decay of the autocorrelation function of returns is supportive for the EMH, but it does not automatically imply an independent random character of the return. The presence of $E[x_t, x_{t-i}] = 0$ still allows for dependencies in higher moments, especially the second as a measure for the volatility of prices. The phenomenon of volatility clustering was already recognised in Mandelbrots early contribution to economics. It roughly says that large price movements are followed by large fluctuations and vice-versa. Because volatility enters as an important factor in many financial applications like option pricing, risk evaluation or portfolio optimisation, it is natural to see both practitioners and researchers aiming to provide a statistical model for the volatility process as close to real data as possible.

Approaches that attempt to model volatility for short horizons often focus on the cyclical behaviour of intra-day data. For example, stock markets - unlike foreign exchange markets - do not operate 24 hours but have closing times. One of the consequences is a seasonality of the volatility that is different across the trading hours: the opening usually has a large volatility followed by a decrease. Shortly before the closing time volatility again rises. This course is called the U-shape of volatility and is empirically found e.g. by Ghysels and Jasiak (1995), Andersen and Bollerslev (1997) and Hasbrouck (1999).

Although deepening the insight of intra-day behaviour is of special interest to practitioners and researchers operating in the field of high frequency data, it seems to be less interesting to the simulators because so far non of the simulations have tried to tackle this feature. One reason for this neglect may be the fact that much of the cyclical behaviour can be explained by the working hours of stock markets and this is not a general component of complex financial markets. Therefore, a summary of empirical results is exclusively devoted to the behaviour of volatility on longer horizons.

6.2.1 Empirical evidence of long memory in the volatility process

Contrary to the rather mixed results for raw returns, the presence of long range dependencies in the volatility process has been documented across many financial time series. Taylor (1986) is an early example of an extensive study. A more recent paper by Lobato and Savin (1997) finds evidence in favour of the long memory property in several international stock indices, confirming earlier conclusions in Ding and Granger (1993).[36] This is independent to whether squared return data or just the absolute values are used.

More recent work on other time series corroborates this picture. Elekdag (2001) utilises the log-periodogram regression of GPH to measure the long memory properties of a large set of markets. His results show that almost all examined indices feature the long persistence characteristic of their associated volatility process. The only exceptions, Argentina, the Netherlands, Switzerland and Venezuela, indicate anti-persistence. Table 6.5 provides the point estimates of d from Elekdag.

Table 6.5[37]: Elekdag's (2001) estimation results on absolute returns

Argentina	-0.0578	India	0.4462	Philippines	0.2796
Australia	0.0995	Indonesia	0.4426	Poland	0.3133
Austria	0.2636	Ireland	0.1833	Portugal	0.3995
Belgium	0.1342	Israel	0.5929	Russia	0.3378
Brazil	0.0451	Italy	0.3381	Singapore	0.4228
Canada	0.2992	Japan	0.4555	South Africa	0.2104
Chile	0.3581	Jordan	0.3687	Spain	0.1906
Colombia	0.7264	Korea	0.2817	Sri Lanka	0.1238
Czech Rep.	0.3388	Luxemburg	0.6391	Sweden	0.495
Denmark	0.3728	Malaysia	0.4565	Switzerland	-0.0619
Egypt	0.3874	Mexico	0.3758	Taiwan	0.6422
Finland	0.5975	Marocco	0.2224	Thailand	0.1387
France	0.4368	Netherlands	-0.0027	Turkey	0.5159
Germany	0.3239	New Zealand	0.2455	UK	0.0919
Greece	0.3546	Norway	0.133	US	0.3633
Hong Kong	0.2617	Pakistan	0.5712	Venezuela	-0.414
Hungary	0.221	Peru	0.2425		

One should notice the huge differences between the values. Argentina has a fractional difference parameter of $\hat{d} = -0.0578$ while Finland has $\hat{d} = 0.5975$. This indicates either substantial differences in the behaviour of the markets or the sensitivity of estimates on small differences in the data sets (which are, though, not detectable from the description of the data in the study). However, the most important result remains the significance of all values.

[36] Interestingly Lobato and Savin (1997) cannot find the long memory characteristics in the raw returns and so the study is in good agreement with the overall picture in the literature. Many of the new studies follow the proposition of Baillie et al. (1996) to use the concept of the above already mentioned FIGARCH models in which the conditional variance is allowed to behave according to a fractional integrated process. These models are not treated here, but in principal they are almost all in line with the other findings of an existing long memory process.

[37] All values reported are statistically significant.

Dark (2004), studying the Australian All Ordinaries Index and the Share Price Index futures on a daily and a high frequency basis confirms the presence of long memory. Using Lo's modified R/S statistic, both data frequencies show long range correlations for the Index as well as the SPI futures.

6.2.2 Own Estimations for long Memory in the Volatility Process

Because of the overwhelming evidence in favour of the long memory hypothesis for the volatility of asset time series, it ought to be expected that this complementary analysis will confirm the literature. Volatility is calculated by the absolute price changes, i.e.

$$|P_t - P_{t-1}|.$$

The autocorrelation function for the absolute value of the price change for the Mexican ICP is shown in figure 6.5. As can be seen, the usual long sequence of positive autocorrelations is present in the data. Table 6.6 reports the results of the estimation for all daily records.

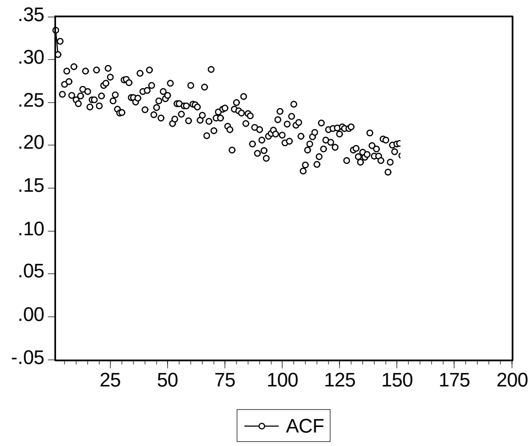

Figure 6.5: Autocorrelation of the absolute (first) price changes of the Mexican ICP for the first 200 lags. The ACF displays significant values for all lags. This prelimenary result in a first indication of the presence of long memory.

Table 6.6: Long memory tests of absolute returns of the daily records

Series	R/S	GPH
S&P 500	0.594*	0.04 (2.954)
Sydney Stock Index	0.6*	0.116 (4.05)
Nikkei 225	0.697*	0.258 (3.33)
FTSE 30	0.584*	0.113 (4.099)
Mexican IPC	0.723*	0.263 (6.954)
Hang-Seng	0.731*	0.259 (4.665)
Singapore	0.69*	0.221 (4.461)
CAC 40	0.689*	0.237 (3.447)
Toronto Stock Index	0.658*	0.189 (7.031)
Varta	0.766*	0.2 (5.518)
BASF	0.564*	0.103 (6.213)
BMW	0.483*	-0.009 (2.11)
Buderus	0.611*	0.142 (6.892)
Bewag	0.59*	0.118 (4.571)
Continental	0.592*	0.061 (7.997)
Bayer	0.593*	0.088 (6.361)
Kugelfischer	0.706*	0.197 (5.638)
Phoenix	0.726*	0.21 (6.113)

For the R/S -statistic, now all series show a significant value different from a zero long range correlation, contrary to the results for the raw returns. The GPH estimator has also all its t-values clearly above 2, thus rejecting H_0. Again, the testing extends to the high-frequency data. However, changing the frequency does not alter the results. All three time series indicate the presence of the long memory property.

Table 6.7: Long memory tests for the high-frequency data

Series	R/S statistic	GPH
DAX	0.641^*	0.109 (7.373)
FDAX	0.677^*	0.136 (5.881)
NEMAX	0.81^*	0.273 (6.175)

This section has provided an overall confirmation of the volatility clustering already mentioned in Mandelbrot's early studies. It is thus an unambiguous target that any theory or simulation of financial markets has to hit. However, as in the case of multifractal time series, finding empirical support for the long-memory hypothesis does not say anything about the reasons for such a phenomenon. The models of chapter 6.1.1.2 and 6.1.1.3 just offer theoretical statistical processes that produce time series with exactly this fact, but they do not present any clue about the underlying economic process that leads to it. Suggestions that have a more economic content are put forward by Müller et al. (1997) and Andersen and Bollerslev (1997). They see long memory in the volatility of a time series as coming from the aggregation of multiple volatility components caused either by heterogenous traders or heterogenous information flows. The first mentioned contribution is the one already encountered in the chapter about multifractals. In Müller et al. (1995), a heterogenous ARCH-process (HARCH-process) was introduced by allowing traders to follow different planning horizons. The authors are able to show that different dependent variances lead to a hyperbolic decay of autocorrelation of volatilities, just like the long-memory processes. It is interesting to see that the same ideas can be responsible for both multifractality and long-memory. Although this is by no means a definite clue that the explanation hits the true process, it at least corroborates the hypothesis that financial markets consists of many different agents. Seemingly, only a kind of heterogeneity within the market can produce the complicated price fluctuations observed in stock, future and foreign exchange markets. Homogeneous models are not capable to give a realistic picture of the true market mechanism.

Part III

The Simulation of Financial Markets

One of the most striking features of complex systems - and the working hypothesis as proposed in chapter 3 is still that financial markets belong to this class of systems - is their inherent inability of being put into a closed mathematical form. As a consequence, the variable of interest, the result of the dynamic system, cannot be calculated analytically. In order to overcome the problem of tractability, two ways are generally conceivable. One approach to capturing the mechanisms of the market is the preparation of a well defined experiment. For example, 1000 probands, endowed with a starting capital could be asked to trade under similar circumstances as in real life. The results of such experiments could then serve as examples of the way in which financial markets work. However, apart from the problem of construction, a point of fundamental criticism would always be the fact that the probands do not deal with their own money. The second alternative is a simulation. This can be thought of as a kind of an experiment, though an artificial one. In this case everything which makes up a stock market has to be specified by the simulator. From the news generating process, the formation of demand and supply, the determination of the decision process for all agents including the kind of interaction between them, to details like the distribution of the starting capital, the price formation and the different trading strategies of the noise trader. The outcome of this simulation gives then a time series that can be compared with the empirical data.

Historically, these kinds of microsimulations had one of its earliest application in the search of controlling nuclear fission. By now, the use of numerical methods has extended to almost every area of the natural sciences. To name just a few examples: protein folding, immunology, meteorology, solar physics and many more. All of these systems are complex in the sense outlined in chapter 3.1: they consist of a large number of interwoven parts which produce highly non-linear outcomes due to their collective behaviour. In some of the examples from the natural science, one has detailed and in a few cases even well-tested knowledge of the basic laws for the parts that build the system. This foundation of established knowledge makes it possible to put simulations on the basis of experimental natural sciences.

In social systems exact knowledge about the behaviour of individuals is much harder to infer. Thinking about societies in general, the constituent elements are the human beings and their interactions are characterised by communication, competition and cooperation. It is thus not only a matter of detailed knowledge about people's behaviour but also about their connecting entities.[38] Within economics the new branch of experimental economics tries to find out what individuals really do in a given situation, but so far no definite results have been obtained. The main difficulty lies in the fact that the formation of opinions

[38] Microsimulations for the social scienes were first introduced by Orcutt in the 1950s (see Orcutt, Caldwell and Wertheimer (1976)).

or convictions is unobservable.[39] Even if actions are observable in all possible situations, the intrinsic motivation is not. What is left is the usual way of modelling individuals by means of utility functions. This is the economists method to determine demand and supply.

Although the concept of utility functions is derived from an axiomatic setting and not through experimental results it is a reasonable way to represent the individuals decision problem, despite their deficiencies. Moreover, this kind of theoretical based modelling allows for the incorporation of many psychological factors that were introduced by Behavioural Finance. As will be seen later, some researchers use even simpler elements like profit functions in order to determine traders actions. In any case, the decisive point is the reference to microeconomic considerations. With such a setting, the researcher determines the reaction of each individual agent to its changing surroundings by the choice of the exact functional form of his utility function. And so, after having devised an artificial economic environment, the agents are left "to their own devises" and the desired macrovariable evolves through the interactions within the system. The picture of a typical approach that roots in such a theoretical-economical based simulation can thus be described in the following manner:

(i) device a method according to which people decide upon their actions; this should include determining the functional form of the utility function, the values of all involved parameters and the parameter accounting for the risk aversion;

(ii) arranging the economic environment with all its ingredients like, e.g. incoming news, the dynamics of the dividends, the total number of market participants, the way in which equilibrium prices are determined etc.;

(iii) after setting reasonable starting values, let the environment evolve according to the set up without further interventions and let the agents react to it so that the market dynamics can unfold.

Because all ingredients are fixed by the researcher, such simulations can be called *deterministic simulations*. There is, however, another direction of simulations that is much more based on stochastic considerations and therefore is not in need of such a detailed modelling. Here, the decisions of agents are not derived through an optimisation process but by means of a probability function. Chapters 7 and 8 are devoted to the explanation of both concepts. The first contains what here is called *stochastic simulations*, the second is entitled with *deterministic simulations*. So far none of these adjectives are chosen to describe and partition the various microsimulation models, thus the classification in this work is new to the literature. However, the striking difference of both concepts in deciding upon the demand and supply of the microagents justifies the chosen partition.

[39] In fact, the field that deals with the problem of an empirical determination of this formation is neurology. It's topic is the brain which is - because of the interconnected neurons - a complex system itself.

Chapter 7

Stochastic Simulations

In deterministic simulation models of financial markets, a concrete rule often derived from the findings of utility theory is devised for every agent. It tells him what to do in a particular situation. No random factors influence his decision; it is the researcher who sets the frame in which the agents act according to these rules. Not so in the stochastic models, where the agents are considered as micro-units whose decision process cannot be observed. As a consequence, traders actions in a given situation are treated as unknown variables. It is therefore impossible for the researcher to fall back on empirical data about traders actions in the past in order to model future decisions. For the same reason, the approach also rejects utility theory as a means of modelling agents behaviour, because this would presuppose an acquaintance with the true preference structure. Instead, a probability distribution takes over the role as the determinant for the specific actions. This probability distribution says how probable a particular choice is given the economic situation of the agent, e.g. his financial constraints, the composition of his portfolio or any other factors that are deemed to influence his choice. In any case, the choice is not directly determined by the environment.

To view agents as stochastic micro-units[1] has recently been put forward by Aoki (1996, 2002), Brock (1993) and Brock and Durlauf (1995). The general goal is to explain the emergence of multiple equilibria in dynamic economic systems that consist of a large number of microeconomic agents. While it is difficult for many deterministic models to explain multiple equilibria, stochastic formulations have in principal less conceptual hassle with it.[2] The essence of a stochastic approach is the fact that a given macroeconomic variable, for example the gross domestic product or a price index, is compatible with many different combinations of microeconomic states, or actions.

To give a first idea about how these kind of models work, consider an artificial

[1] Throughout this work, the terms individuals, people, (micro-) agents, investors or traders are all used synonomously as the constituent elements of the system at the lowest (micro-) level, micro-units.

[2] There are, however, exceptions like Krugman (1996).

market that consists of a collection of a large, but finite number of traders. Each of whom faces a set, S, of possible M choices. This set is usually not dependant on any other feature of the model, i.e. the researcher provides a collection of possible choices from which the microagents select their actions. A simple toy-model of a stock market would have only one asset that can be traded. A description of the decision set S would be to restrict the alternatives to three choices: buying one unit of that asset, selling one unit and holding. The function of a probability distribution can also be kept extremely simple by assigning an equal probability of $1/3$ to each alternative. Given these components, a first round is then undertaken by letting a random generator attribute the individual choices of the agents according to the uniform distribution. Since the goal is to analyse the evolution of the price of that asset, and because price fluctuations can be determined by the differences between supply and demand, aggregating the different actions of all traders would result in the desired variable of interest, the change of prices.

However, a simulation that consists of these elements only would be a caricature of a real market. Of course, since each decision is equally probable, such a simulated time series of price changes would end up as random fluctuations around zero which is realistic. But the crude uniform distribution of actions from above neglects any kind of economic or psychological factors and is thus insufficient for a realistic modelling of traders. The task is now to provide a way that relates the concrete choice of the agents as determined by probabilistic considerations to some comprehensible economic reasoning. The range of possibilities that influence the agents to favour a specific decision is generally open to many elements. Most models have a form of *local* influence which simply means that agents orientate themselves on the actions of other traders they have contact to. The idea is borrowed from the fact that agents do not decide upon their actions in an isolated world; they are influenced by their environment. Often, some kind of technical analysis is also incorporated. Traders may look back at the price history and try to detect hidden patterns. Another indispensable element should be a fundamental influence that is exogenous. Systems that feature such an element are called open systems. Financial markets are open systems because every day lots of new information come into the market. It cannot seriously be asserted that all information flow stems from endogenous elements. A further important ingredient is the kind of interaction between the micoragents. Do they just observe the actions of others and try to figure out supposed hidden private information, or do they maintain a more intimate exchange of opinions? No matter which factors are taken, they do not mechanically decide upon the exact choice. It is only that these elements increase the tendency towards a particular action. And so the design of an appropriate probability distribution for the description of financial traders becomes a crucial point of all stochastic simulations.

One way of deriving such an expression would start from economic considerations and would then attribute some a priori probabilities for each action dependant on the economic environment. However, such a description would contradict the unobservability postulate. A second way, and this will be the one pursued in this work, starts from the macrovariable and derives a suitable distribution with the help of the concept of *entropy.* Furthermore, since the actual variable of interest is the fluctuation of prices, it is important to describe how agents change their behaviour. Normally, a set of transition rates are specified to model the evolution of one microunit. These rates are also based on probabilistic considerations, i.e. the transition from one choice to another does not follow a prescribed road but occurs randomly. The derivation of such probabilities is the main topic of the next sections.

7.1 The Basis of Stochastic Modellings of Economic Systems

7.1.1 The Multiplicity of Microstates

The modelling of a large set of interacting agents and the examination of aggregate dynamics together with their associated stochastic fluctuations stands at the hard of this approach. It is foremost the last aspect, the random combinatorial element that is often unfamiliar to economists. They are used to view economic actions as the result of a deterministic process in which people decide upon their best available choices given a particular situation. Instead, in the approach outlined in the following pages, economic processes are modelled by means of probabilistic theory where decisions are governed by random factors. A common feature of all these models is the use of a finite but large number of agents, each of whom is assumed to have an identical choice set, S, from which he can chose. As a matter of fact, situations where agents are almost uniformly distributed over the choice set change with situations where one or two choices are prevailing. Such combinatorial patterns are called *configurations* or *states* and form, when analysed dynamically, aggregate emergent patterns of the behaviour of large collections of agents. The evolution of the macroeconomic variable is then deduced from this collective behaviour. The next preliminary considerations will try to aid in following the more formal presentation of the following section. They hopefully give a first impression about the way in which stochastic approaches describe financial markets.

Let Y_t denote the macrovariable of an economic system at a special time index t. And let there be N agents who all face the same set of M different choices. $S = (S_1, S_2, ..., S_N)$ is then a complete description of all N microunits, where each of the N S_i is able to chose among a set of M possibilities $A = (a_1, a_2, ..., a_M)$. For example $S_{1,t} = a_2$ would indicate that agent 1 at time index t chooses the possibility a_2 out of the set of actions available to him. Then Y_t can be interpreted as the result of a particular combination of microstates S_i; this is formally expressed by

$$S_t = \{S_{1,t} = a_j, S_{2,t} = a_i, ..., S_{N,t} = a_M, \} \rightarrow Y_t, \tag{7.1}$$

where the arrow indicates the mapping of S into Y.[3] The probability of encountering this particular configuration out of the many possible if all of them are assumed to have the same probability is given by the multinominal distribution

$$P(N_1, N_2, ..., N_M) = \frac{N_j}{\prod\limits_j N_j} M^{-N},^4 \tag{7.2}$$

where N_1 represents the number of individuals that took the possibility 1. The total number of combinations in the example, denoted by $W(N)$, is then given by

$$W(N) = \sum_j \frac{N!}{\prod\limits_j N_j} M^{-N} = M^N. \tag{7.3}$$

Limiting the number of choices to two yields $W(N) = 2^N$.

In order to measure Y quantitatively, the parameters a_i must also have a quantitative nature. To give a simple example that can later be used to describe an artificial financial market, let $M = 3$ be the number of different choices each trader can take on. Let furthermore a_i denote the value of the choices from the set A which is equal for all traders. Thus

$$A = \{a_1 = -1,\ a_2 = 0,\ a_3 = +1\}. \tag{7.4}$$

Then the desired macrovariable, for example the price of the traded asset, may be calculated by

$$Y_t = \sum_{i=1}^{N} S_{i,t}. \tag{7.5}$$

There is a problem with this interpretation because Y_t may become negative which is impossible for asset prices. Two ways are conceivable to solve this problem. The first is to add a constant to Y_t so that it almost never remains under the critical value of zero. Although this seems to be a rather crude method, it is not that problematic because the variable of interest of all simulations concerning financial markets is actually the *fluctuation of the price* and not its level. Hence, the analysis is focussed on the difference of the macrovariable, i.e. $\Delta Y_t = Y_t - Y_{t-1}$, and so the constant drops out. The second interpretation

[3] Just to avoid misunderstandings: the number of agents usually does not coincide with the number of choices, thus $N \neq M$ (and $N > M$ in almost all cases).

[4] This is known as the Maxwell-Boltzmann distribution for distinguishable microunits. See Feller (1971). There, it is treated as an occupancy problem for balls and urns. However, it is not necessary for the calculation to identify each agent's decision. It suffices to know how many of the N agents took up choice 1, how many choice 2 and so on. This description is much less detailed but nonetheless represents the composition of the total population of agents.

views Y_t as the change in prices caused by an excess demand as calculated by the difference of buying orders and sell orders. As will be seen in the review of stochastic simulations, both possibilities can be encountered in the literature.[5]

Independent of the interpretation of Y_t, it is important to note that each particular value of Y can also be achieved by another combination of the microstates, e.g. $S_t = (S_{1,t} = a_k, S_{2,t} = a_M, ..., S_{N,t} = a_v)$. In fact, the concrete value of Y_t can be the outcome of many different configurations of S_t and therefore, observing the macrovariable without simultaneously observing the microstates does not lead to an identification of the actions of the N agents. Consequently, a constant value for Y over some time span does by no means indicate a stagnant system. It is well possible to have a great deal of fluctuation at the microlevel while maintaining the same value for the macrovariable. This non-uniqueness of configurations is called the *multiplicity of microstates*. As a quantitative measure, it calculates the number of different ways that can result in the same macrovariable. Intuitively, the more agents attain the system and the more choices they have, the higher will be this expression. Furthermore, it is possible to derive the most probable configuration of microstates that is compatible with a specific value of Y_t. Considering the case of $Y_t = 1$. It is obviously much more probable to achieve such a value with a diversified distribution of S_i, i.e. some agents hold, many sell and the same amount of agents plus one buy, than with a situation of all $S_i = 0$ and only one $S_i = +1$, although the last configuration is possible as well. Many of the stochastic simulations presented in chapter 7.3 rely on that considerations and it is the main task of the next paragraphs to derive a relation between the multiplicity of states, the most probable combination of micro-units and the way traders form their decisions. In this connection, the entropy which is intimately related to the information content of a system will play an important role. It must also be stressed that the derivation will be undertaken without an a priori knowledge about the probabilities with which the agents select between their choices.

The main concern is thus to obtain a way that makes it possible to describe a stochastic economic system without having a priori knowledge about the underlying probability distribution. The goal is to start from the observations of the macrovariable and derive an appropriate probability for each agent, $\pi(S_i)$, with as few assumptions as possible. Once $\pi(S_i)$ is established, the microunits of the system evolve according to it and so determine the dynamics of Y_t. Let $W(N, k)$ again denote the number of different combinations of S_i to realise the same Y_k. k denotes the number of agents that opt for $+1$ while the remaining $N - k$ chose -1. Then a low value of W indicates a macrovariable that is only rarely observed. The more configurations are compatible with a specific Y_t, the more often this value will show up, regardless of the probabilities for S_i. Thus, it is natural to ask which $W(N, k)$ is the highest of all configurations. Having

[5] It must be said that stochastic simulations usually do not feature an explicit price determination process as it is e.g. provided by the auctioneer model (see e.g. Mas Colell, Whinston and Green (1995)).

determined max $W(N,k)$, then the only remaining task left over is to derive the probabilities for S_i that are compatible with this W. These values for $\pi(S_i)$ will then form the basis of a stochastic prescription for the buy, sell and hold decision. Fortunately, all of this can be achieved under fairly general conditions and the following lines give a short derivation.

7.1.2 Entropy and the Gibbs-distribution

First of all it is important to have a proper understanding of the relationship between entropy and the information of a system. Given a set of M discrete possible observable decisions for trader i and an expectation value for the macrovariable Y, $E[Y]$. What is the probability distribution which describes the information we have about the observable in the least biased way, while still considering that the motivation for a particular decision are unobservable? Assuming one has to compare the information content of two different situations. In the first one, the observable agent can decide only between $M = 2$ possibilities, i.e. $S_i = a_1$ and $S_i = a_2$. The second one has a wider set of $M = N > 2$ different actions to choose from. The observer has no idea which of the actions will be taken but can only see the outcome after agent has chosen i. Information theory asserts that the gain in information is much higher in the second case because the amount of uncertainty before the actual outcome is observed is higher for $M = N > 2$ possibilities than for $M = 2$. In the first case, one has a fifty-fifty chance to predict the true result which is not as bad as the N^{-1}chance from the second experiment. The (information theoretic) entropy, F, for such an experiment with 2 and M outcomes respectively is defined by

$$F(2) = \log 2 \text{ and } F(M) = \log M, \tag{7.6}$$

thus entropy F is higher the more possibilities exist.[6] From (7.6) and the assumption that all outcomes are equally probable it follows that the probability and entropy may be written as

$$F = -\log \pi(S) \text{ and } \pi(S) = \exp(-F). \tag{7.7}$$

However, a market consists of many traders and not only one. To take this into account, a new formulation is devised. Now, the set of outcomes has again M different choices but there are N agents. The phase space is completely portioned into a set $C = (c_1, ..., c_M)$ of mutually exclusive substates, also called microstates.[7] c_1 then represents for example the first choice out of the set. If the

[6] At this point some words are in order to avoid confusions that typically arise because of the use of the term entropy as it is understood in information theory. In physical systems entropy is largest when the system is in equilibrium. This may lead to the assumption that complex systems are in equilibrium which is counter to the usual understanding of complex systems. The solution is that the above definition is used to specify the microstate of the system rather than the macrostate. Generally, entropy is a measure of the randomnes of a system.

[7] The parameter c_i has replaced the term a_i from above because the last represents individual choices while the former stands for a state that can be occupied by many agents.

state c_i contains N_i agents that took the same decision, where the probability for each individual to chose a particular state is equal for all M and thus still N^{-1}, then $\pi(c_i)$, the probability to find a particular agent in c_i is given by

$$\pi(c_i) = \frac{N_i}{N}. \tag{7.8}$$

The entropy according to (7.6) is given as

$$F(c_i) = \log N_i. \tag{7.9}$$

If one observes c_i with its members, then the formally existing ignorance is reduced by $\log N - \log N_i = -\log \pi(c_i)$. Because c_i happens with probability $\pi(c_i)$, the (average) gain in knowledge is measured by the entropy

$$F = -\pi(c_i) \log \pi(c_i). \tag{7.10}$$

For all M states that can be observed one has then

$$F = -\sum_{i=1}^{M} \pi(c_i) \log \pi(c_i). \tag{7.11}$$

Up to now, states are only differentiated by the different choices but do not have a numerical value assigned to. But since c_i indicates a particular and measurable decision, the expected value of the macrovariable Y can be calculated. Let x_i be the numerical value of state c_i. Then $E[Y]$ is calculated by

$$E[Y] = Y^e = \sum_{i=1}^{M} \pi(c_i) x_i.^{8} \tag{7.12}$$

And because of $\pi(c_i) \geq 0$ for each state, the relationship

$$1 = \sum_{i=1}^{M} \pi(c_i) \tag{7.13}$$

must also be satisfied. In order to obtain the highest possible information gain of the system, the entropy in (7.11) has to be maximised with respect to (7.12) and (7.13). The problem thus becomes

$$\max \; F = -\sum_{i=1}^{M} \pi(c_i) \log \pi(c_i) + \beta(Y^e - \sum_{i=1}^{M} \pi(c_i) x_i + \alpha(1 - \sum_{i=1}^{M} \pi(c_i)), \tag{7.14}$$

where α and β are Lagrange multipliers. The stationary conditions $\partial F / \partial \pi(c_i) = 0$ lead to

[8] Note the coincidence with $Y = \sum S_i$. The probabilities which occur in (7.12) indicate the ex ante character of the consideration whereas $Y = \sum S_i$ is an ex post value.

$$\log \pi(c_i) = -\beta x_i - (1+\alpha) \text{ or} \tag{7.15}$$

$$\pi(c_i) = \frac{\exp(-\beta x_i)}{\exp(1+\alpha)}. \tag{7.16}$$

Substituting into (7.13) and noting the independence of α and i gives

$$\pi(c_i) = \frac{\exp(-\beta x_i)}{Z(\beta)}, \text{ with} \tag{7.17}$$

$$Z(\beta) = \sum_{i=1}^{M} \exp(-\beta x_i) \tag{7.18}$$

as the so-called partition function or *Zustandssumme*.[9] This is the desired probability function for agent's actions: the Gibbs-distribution. It determines the probability of each agent to find him in state i.[10] Thus, for the individual perspective, the probability for agent j to chose i is also

$$\pi(S_j = a_i) = \exp(-\beta x_i)/(Z(\beta)).$$

So far, the considerations are limited to a static situation. However, the final goal of each simulation is to produce realistic time series of the macrovariable, and so the procedure cannot stop here. A dynamic element has to be added so that the stochastic concept is indeed an (alternative) way to describe the mechanism of financial markets. Thus, the task left over is to provide a probability distribution that is valid in every round. Fortunately, the above derived Gibbs distribution is a good candidate because it satisfies certain conditions that ensure a stationary distribution of $\pi(S_i)$. These conditions are known as *detailed balance conditions.*

7.1.3 Detailed Balance

The dynamics of a financial system as described by the stochastic concept is exclusively governed by the transition of choices, undertaken by the agents. Because the fluctuation of prices is the most important variable to observe, and this in turn is determined by the changes in the decisions of the traders, it is decisive to model a probability distribution for those changes. Let c_i be the state of any arbitrary chosen trader. Let furthermore c_j be another state of the whole set of choices. Then $R(c_i \to c_j)$ represents the transition from decision c_i

[9] For a more thourough treatment of the derivation of the Gibbs-distribution see Haken (1984).

[10] An interesting and important point to note is that (7.14) can be used to highlight an interpretation of the parameter β. If the entropy is differentiated with respect to Y, one obtains $\frac{\partial H}{\partial Y} = \beta$. Thus, the Lagrange multiplier β measures the amount of increase in the entropy the system would achieve if the macrovariable is slightly increased.

to c_j.[11] Although not knowing the concrete appearance, let ω_{c_j} symbolise the probability that this change happens during a short time interval Δt. If these probabilities are allowed to vary over time, then a master equation would give the evolution of $\omega_{c_j,t}$ in terms of rates $R(c_i \to c_j)$ as

$$\frac{\Delta\omega_{c_j}}{\Delta t} = \sum_i [\omega_{c_j,t} R(c_i \to c_j) - \omega_{c_i,t} R(c_i \to c_j)]. \tag{7.19}$$

The first term of the sum is the rate at which the agent (and thus the system since every change in agent's i decision also changes the macrovariable) undergoes a transition into state c_i, while the second term represents transition rates out of state c_j.[12]

However, stationary distributions demand the values for ω_{c_i} to stay at the same value for all time indices. This is indispensable for a time consistent description of agent's behaviour. Hence (7.19) must be zero and consequently

$$\sum_{c_i} \pi_j R(c_j - c_i) = \sum_{c_i} \pi_i R(c_i - c_j). \tag{7.20}$$

This is nothing else than a discrete-time version of (7.19) set to zero. Equation (7.20) says that the transition rates of the system to get into state c_i and to get out of it must be equal. Applying the sum rule, (7.20) reduces to

$$\pi_j = \sum_i \pi_i R(c_i \to c_j). \tag{7.21}$$

Unfortunately, even this simplification does not ensure that the probability distribution really tends to π_i, in this case the Gibbs-distribution. There is always the possibility of a dynamic equilibrium in the sense of a rotation of the probability distribution around a number of different values.[13] To avoid such unpleasant circumstances one can impose a stronger version of (7.21):

$$\pi_i R(c_i \to c_j) = \pi_j R(c_j \to c_i),$$

or

$$\frac{R(c_i \to c_j)}{R(c_j \to c_i)} = \frac{\pi_j}{\pi_i}. \tag{7.22}$$

This is the announced detailed balance condition. Its intuitive meaning is that the balance between the probability of leaving a state c_i and arriving in it from another state holds *for any pair of states* and thus for the overall system and its macrovariable as well. An economic analogy might be the price evolution that is determined by the aggregation of the states of all agents. Detailed balance then

[11] In terms of using S_i as above, a change from +1 to -1 is simply written as $R(S_i = +1 \to -1)$.

[12] A natural constraint is that the probabilities must sum up to 1, i.e. $\sum_i \omega_{C_j}(t) = 1$.

[13] This is called a limit cycle.

says that the transitions cancel each other out, leaving the macrovariable unaffected. (7.22) can thus be interpreted as a condition for the dynamic equilibria of the system. It is however important to notice that the detailed balance conditions does not imply a stagnant system where the macrovariable cannot change over time. The condition (7.22) just says that, on average, the macrovariable, consistent with the highest entropy, stays at its value, the value with the highest probability. Thus, huge changes are far less probable than minor ones, but they are certainly possible. (Otherwise, detailed balance conditions would not make much sense for dynamic systems). The main reason for this rather long introduction to the technical aspects of the stochastic approach lies in the fact that physicists often apply the well-known Ising magnetisation model as a basis for their models of financial markets. And in Ising models Gibbs-distributions serve as the probability distribution that decides upon the value of the micro-units.

The considerations so far show that one may take *any set of transition probabilities* that satisfies equation (7.22) in order to ensure a stationary probability distribution. This is of no help as it does not prescribe an explicit form for π. However, given the fact that the goal is to maintain the Gibbs-distribution, values for π_i and π_j should exactly follow this distribution. The detailed balance condition then demands that transition probabilities have to obey

$$\frac{\pi_j}{\pi_i} = e^{-\beta(E_{c_j}+E_{c_i})}.$$

Now the transition probability for changing the state from i to j can be written as

$$\pi(c_i \rightharpoonup c_j) = e^{-\beta(E_{c_j}+E_{c_i})}, \qquad (7.23)$$

since the detailed balance conditions allows for many different functional forms of (7.23). Having determined these rates still leaves the problem of deciding whether transitions take place or not. The solution is to compare the values of $\pi(c_i \rightharpoonup c_j)$ with a uniformly distributed random variable in the open interval (0,1). If the changing probability is higher than this random number, a change takes place. Otherwise, the agent stays in state c_i.

At this point there are some words in order to summarise and comment the results. Starting from the assumption that unobservable agents have an equal probability to chose one of the M possible states, the entropy of the system with N agents and M possible choices is maximised. This represents, loosely speaking, the situation with the highest randomness, or, in other words a situation that is compatible with the expected value of the macrovariable Y. From there on, a probability distribution for the individual choices is derived - the Gibbs distribution. Of course, a severe objection of the approach outlined so far is the missing of any economic reasonings. The crude assignation of equal probabilities for taking a specific decision is surely insufficient. However, this point is valid only up to this preliminary stage. The approach itself is open to a huge variety of psychological, institutional and economic factors that alter the uniform probability distribution. Yet none of these factors have entered the model. Moreover, the important feature of interconnections between agents, which is an essential feature of complex systems still has to be considered.

As will be seen shortly, the Ising magnetisation model offers a way in which different factors like technical trading and foremost the interconnections of traders can be incorporated.[14] It is based on the same stochastic considerations as those described above.[15] The next section aims to provide an introduction into the main structure of the model. Once the general principal of the mechanism is grasped, variations should be fairly easy to understand.

7.2 Ising related Models for Financial Markets

7.2.1 The General Structure

Empirical studies suggest that large price fluctuations are similar to phase transitions in physical systems, where the height of the changes follow a power scaling law. It thus comes as no surprise that econophysicists are inclined to use well known examples from the natural sciences in order to model the stylised facts of financial markets. Considering crashes as a kind of a phase transition in a system of many interacting traders, physicists employ simulations where this transition leads to an ordering of the actions of all agents, analogous to the paramagnetic-ferromagnetic transition in the Ising model or to percolation models.[16] The key assumption here is that bubbles and crashes are to a large extent caused by self-reinforcing imitation between investors who are not exclusively focussed on the fundamental value of an asset. These investors, often called noise-traders, make their decisions dependent on the nearest neighbours as in the Ising model, or on other traders they meet randomly as in the percolation models. In both cases, if this tendency of imitation increases up to a certain point, the *critical point*, then almost all noise-traders place the same orders at the same time, hence building up a bubble or causing a crash. This is the brief story of econophysics. They see financial markets as systems which are, at least occasionally, driven by cooperative effects that cause herding behaviour. And this in turn leads to *critical points* in the sense of point (viii) of chapter 3.1 that can principally be explained by means of statistical physics.

[14] However, the modelling of the human behaviour will never a one-to-one shot of reality. It is always a huge abstraction or simplification (if not oversimplification) of the true processes that are going on during the search for the best decision. But it should be noted that this is the best simulations can aim for because their task is to unveil the essential reasons for the behaviour of the macrovariable and not to offer a picture of the microunits as precise as possible.

[15] The appendix provides a short introduction into the mechanism of this important model of statistical physics. It is a purely physical explanation and thus does not entail any economic ingredients. It is nevertheless important, for it eases the understanding of the economic applications.

[16] See Stauffer and Aharony (1994) for an introduction.

The term *Ising-related* modelling stands for a variety of simulation models that all draw their base from the original work on spontaneous magnetisation. Common to all are some characteristic components:

(i) decisions of agents are based on a special form of the Gibbs probability distribution; this includes the formulation of transition rates for the decision based on that probability distribution;

(ii) agents are distributed over an n-(usually two- or three-) dimensional lattice; the lattice just serves as a kind of platform to posit the traders on. The dimension determines the number of surrounding agents;

(iii) interaction between agents is usually achieved through the influence of their neighbours; this means that locally connected traders exchange some kind of information, thereby creating a common tendency in behaviour; alternatively, long range connections can also be considered;

(iv) the variable of interest is computed by the aggregated buy and sell actions of all agents each simulation run.

Within this broad structure, room remains for the concrete modelling.

7.2.2 The Mechanics of the System

In Ising models agents are treated as spins who are put on a d-dimensional, regular lattice. Each site of the lattice has attached a number S_i that represents his decision and which usually can take on only three values: $+1$ (the spin is up; this indicates a buy order), -1 (the spin is down; sell order) and 0 (holding). Figure 7.1 illustrates a typical situation.

The special feature of this model is the way in which traders are connected. Like in the original magnetisation model, agents are influenced by their local colleagues. I.e. their actions determine - at least in part - the decisions of agent i. The shaded trader in figure 1 has as his nearest neighbours four traders who all sell, i.e the next traders left and right and up and down. These traders work in such a way as to get him in line with his surrounding environment. If, as in this case here, all four neighbours sell, then the probability of the shaded trader to join the decision rises substantially.[17] This local influence itself can be thought of in several ways: as a herding process in the sense of modern microstructure models, as a pure psychological behaviour or simply as a kind of roomers about fundamental factors. The herding component is usually motivated by the models of Bannerjee (1992), Bikchandani et al. (1992) and Scharfstein and Stein (1990). Agents communicate with their surrounding environment, thereby convincing or merely persuading other agents to act in the same manner, or they try to detect supposed hidden private information

[17] However, even in this case a selling decision is not a sure outcome due to the probabilistic nature of the decision process.

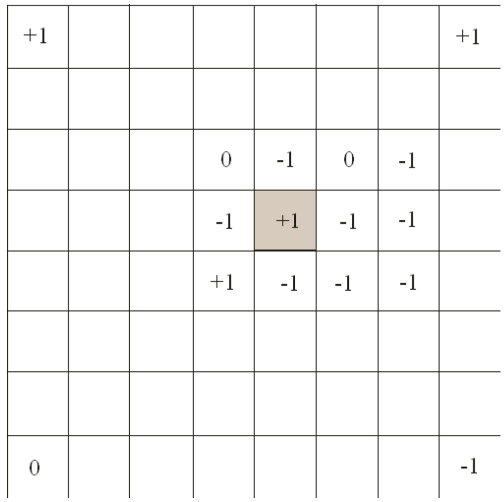

Figure 7.1: Illustration of an Ising model. The above situation shows the typical structure of a two dimensional lattice. Traders are represented by either +1, -1 or 0. Not all aquares are filled with values because of convinience. The own simulation will have all squares occupied.

from the observed actions of other traders. The modelling itself is, because of its simplicity, open to several interpretations.

Regardless of the specific interpretation, this element parallels the mutually magnetising effect of the Ising model and it is especially this feature which makes the Ising model so attractive to econophysicists: the easy way to mimic networks. The local influence is formally expressed in accordance with the original magnetisation model by a so-called Hamiltonian function H for each agent i

$$H_i = \sum_{i \neq j} J_{ij} S_i, \tag{7.24}$$

where J_{ij} is the parameter that measures the strength of influence of trader j on trader i and $j \neq i$ means that agent i is excluded. Usually the four nearest neighbours, also known as the von Neumann neighbours, are taken as a proxy for the local influence. If $J_{ij} > 0$, then influence is designed towards a uniform behaviour of local cluster. In the case of $J_{ij} < 0$, trader i would instead try to make superior profits by acting against the local trend.

For economic purposes H_i can be thought of as an indicator function for the opinion of agent i. Values of H above zero result from positive orders by the neighbours, and a positive J_{ij}. Hence $H_i > 0$ means that trader i shares the optimistic opinion of his local surrounding. According to the energy minimisation of the ferromagnetic model, a function E is defined for every agent i that has to be minimised:

$$E_i = -S_i H_i = -S_i \sum_{j \neq i} J_{ij} S_j. \tag{7.25}$$

The logic of this formulation can be demonstrated by assuming $\sum J_{ij} S_j < 0$, i.e. the local environment has the majority of neighbours selling. In this case $S_i > 0$ would result in a positive value for E_i whereas $S_i < 0$ leads to a negative E_i. Hence, with $J_{ij} > 0$ assumed, the trader i when trying to minimise E_i should share the opinion of his nearby colleagues and is hence ceteris paribus inclined to sell. One may call this a policy of avoiding to stay alone. Self evident, it can also be termed a majority following strategy. It should be noted that this cooperative behaviour does not only prevail in situations like the one in figure 7.1 where trader i was surrounded by four uniformly acting neighbours. Two sellers, a holder and one buyer may be sufficient to induce a selling inclination if connections are strong enough. Thus even unclear environments are theoretically able to force neighbours to follow local majorities. Below, conditions for such strong interconnections will be discussed in more detail. However, given the case they exist, the situation of trader i occurs to all sides, and so the equilibrium would eventually see the whole market selling. For obvious reasons, this outcome is one that every researcher wants to evade.

This is understandable, for building models exclusively on the local influence component without any other economic features would be a very poor picture of the real world. Simulations are thus always amended by other elements. For example, a stochastic disturbance term that is unrelated to the local trends, or an external factor similar to the external field in an ferromagnetic Ising model can be incorporated. For example

$$H_i = \sum_{j \neq i} J_{ij} S_i + A \tag{7.26}$$

has a factor A added that can potentially account for all exogenous elements. If (7.26) is furthermore enhanced by A_i, then each trader becomes an individual factor attached to. A sensible way to interpret such a component is to see A_i as a private information that is different across the traders.

Another factor is the global development of the market. If technical traders are able to capture the overall mood within the market, then they will use it as a source of information about the future evolution of the price. In this case the Hamiltonian may have a form like

$$H_i = \sum_{j \neq i} J_{ij} S_i - B \sum_{i=1}^{N} S_i. \tag{7.27}$$

B is here a parameter indicating the influence of the whole market on trader i. Some models also include the possibility of a deliberate trading against the mood of other market participants.

Independent of the particular modelling, this is the crucial part of each simulation model. It determines by which factors traders are influenced when forming their opinions. Having fixed the functional form of H_i, the probability to find a particular trader in a specific state is then given by the Boltzmann distribution which is a special variant of the Gibbs distribution

$$\pi(S_i = +1) = \frac{e^{E_i/\beta}}{\sum e^{E_i/\beta}}, \tag{7.28}$$

where $\beta \equiv k_B T$, with T being the temperature and k_B as the Boltzmann constant.[18]

The dynamics of traders actions, i.e. their willingness to change from a sell to a buy position is governed by

$$e^{-\Delta E_i/\beta} \lesseqgtr rn, \tag{7.29}$$

where rn is a uniformly distributed random number in the interval (0,1), and $\Delta E_i = E_i(S_i = \textit{new state}) - E_i(S_i = \textit{old state})$. In the case of a lower random number, the trader does not change his decision and remains in his former state. For the reverse case, spin i flips. The two crucial parameters in (7.29) are ΔE_i and β (or T as k_B is a constant). The first one reflects the difference in the disagreement function calculated for the current decision and the possible new one. If this difference is small, reflecting a disagreement that would not change much, the probability of a transition towards selling is also small. Compared with the uniformly distributed random number, the smaller this probability is, the less frequent becomes a flip. On the other hand, if ΔE_i is large, then $\exp(-\Delta E_i/\beta) > rn$ becomes far more probable. In order to illustrate this, consider a situation where agent i has $S_i = +1$ and all his neighbours have -1, so $J \sum S_i = +4$. Then

$$\Delta E_i = E_i(S_i = +1) - E_i(S_i = -1) = 0.$$

In this case $e^{-\Delta E_i/\beta} = 1$ (regardless of β) and the flip is a sure outcome. On the other hand, if this agent has $S_i = -1$, then $\Delta E_i = 8$ and the probability with β fixed at 1 is $e^{-\Delta E_i/\beta} \approx 0.000355$ which makes a flip a rather unlikely event.

However, β, and with it T, is even more crucial to the dynamics of the system. In analogy with the Ising model, this is the parameter that ultimately brings the system to its crucial point. There, it decides upon the strength of the connections. For T below the crucial temperature T_c, spontaneous ordering leads to a uniform orientation of the spins. Above T_c, the local connections are not strong enough to prevent a random regime where the spins flip without clear direction. As will be seen later, own simulation results do not have that

[18] It is important not to confuse the use of β in (7.28) with the Lagrange multiplier in (7.14).

severe dependence on T. Nevertheless, the close connection between T and the behaviour of the system on this parameter will still limit the range of reasonable values for T.

Because temperature plays such an important role, an economic interpretation is strongly warranted. First, a decreasing T also decreases β. Supposing agent i faces the following situation

$$\Delta E_i < 0,$$

i.e. a flip would reduce the disagreement of this particular trader. Given the value of ΔE_i, the smaller β is, the greater is $\exp(-(\Delta E_i)\beta^{-1})$ and the more likely becomes a change. Hence reducing T increases the sensitivity of the traders for their neighbourhood and thus the local influence factor is stronger the smaller T is.

To gain some order, the next two paragraphs separate the Ising–related models into two subsets: the Cont–Bouchaud models which are basically percolation models and the more genuine Ising–models. The principal difference between the two subclasses is that while Ising-models use fixed local connections, percolation models are ruled by a stochastic exchange of information between agents, where the connections change randomly and are often stretched far more out on the lattice.

7.3 Previous Simulations

7.3.1 Chowdury and Stauffer (1999)

7.3.1.1 The Model

The Chowdury-Stauffer model is formulated as a classical super-spin simulation where N spins, each denoted by S_i, are distributed over a two-dimensional lattice. The spins, representing the traders, are characterised by three different states. A positive S_i^+ with probability π, a negative S_i^- with the same probability and a state $S_i = 0$ with probability $1-2\pi$. Contrary to the magnetisation model where the values of the S_i are restricted to ± 1, the magnitude of $|S_i|$ is distributed according to

$$P\left(|S|\right) \sim |S|^{-(1+\alpha)}. \tag{7.30}$$

This distribution follows a so-called Levy flight. Levy flights are characterised by a random walk with jumps of a size l with probability $p(l) \sim l^{-(1-\alpha)}$, $0 < \alpha < 2$.[19] The remainder is canonical. The Hamilton function is formulated as

$$H_i = \sum_{j \neq i} J_{ij} S_j. \tag{7.31}$$

[19] This formulation does not affect the distribution of the price changes. See Chowdury and Stauffer (1999, p. 478 ff).

H_i is the opinion of trader i, which is in case no other components totally dominated by the behaviour of the nearest neighbours S_j. J_{ij} is the parameter that measures the strength of influence of trader j on i vice versa. In order to keep the simulation as simple as possible, the authors set $J_{ij} = J > 0$, so opinion now becomes

$$H_i = J \sum S_j. \tag{7.32}$$

For $H_i = 0$, trader i has as many optimistic (buying) neighbours as pessimistic (selling) ones and is therefore indifferent. If H_i is positive (negative) the majority opinion of his neighbours makes the agent more inclined to buy (sell). So far, all traders are modelled as pure noise traders. They form their opinion only on the basis of local trends. Chowdury and Stauffer now introduce a so-called disagreement-function E_i that traders want to minimise

$$E_i = -S_i H_i = -J \sum_{j \neq i} S_i S_j. \tag{7.33}$$

The system built without any additional ingredients will eventually end up in a situation where all spins are aligned, either with all selling or all buying. This is the same outcome that would prevail in the original Ising-model. Because this is not a realistic situation for financial markets the authors use a fictitious temperature T that ensures a random fluctuation around a mean of zero.

An important point of this model is the incorporation of what the authors call a fundamental influence. It is denoted by h_i and represents an individual bias. Chowdury and Stauffer interpret $h_i > 0$ ($h_i < 0$) as an optimistic (pessimistic) factor that may be different to the local trend. The value itself is not fixed over the complete simulation but varies in magnitude.[20] However, the value for h_i must be limited in both directions. It must not be too large, because then h_i would always determine S_i^- or S_i^+ irrespective of the local connections. On the other hand, choosing a value too low is also wrong since in this case the individual bias is always dominated by the neighbour influence and thus pointless.

The disagreement function for these traders now becomes

$$E_i = -S_i \left(H_i + h_i \right). \tag{7.34}$$

It should be clear that if $|h_i| > |H_i|$ but with different signs, the individual bias overcompensates the local trend and trader i is more inclined to act against his surrounding environment. Of course, terming such an element as fundamental is quite daring. It has nothing to do with the kind of fundamental analysis economists are used to. There is nothing fundamental in h_i, since it merely constitutes a term that can be opposite to the local trend, without any resort

[20] Unfortunately, there is no information provided by the authors about the exact mechanism for the changes in h_i or about its precise magnitude.

to the intrinsic value of e.g. incoming news about the asset. Furthermore, the subscript i indicates a different value for each agent. This, however, is inappropriate for a fundamentally based modelling. If h_i really should symbolise a fundamental influence, then h_i must be equal for all i because every agent trades on the same information. It makes much more sense to view h_i merely as a factor that introduces an element (to some agents) besides the neighbour influence. This factor then has the potential to let bubbles burst or bring the market up from a crash. Anyhow, equation (7.34) is the disagreement function for those with a fundamental orientation. Chowdury and Stauffer simulate the model with varying portions of fundamentalists. Finally the authors calculate the price changes by the difference of supply and demand

$$\Delta P = \sum_i S_i. \tag{7.35}$$

It should be noted that this is equivalent to the determination of M in the magnetisation model, there a level variable.

7.3.1.2 Results

The authors first chose a fundamentalist/noise-trader ratio of 50%. They furthermore impose the following restriction to the behaviour of the fundamentalist. If ΔP exceeds or falls below an arbitrary chosen value of 0.4, then fundamentalists change their biases by flipping h_i. Again, this kind of modelling cannot be easily related to a fundamentalist like behaviour. One might think of traders who get afraid of too much fluctuation in the market and thus reverse their bias. On the other side, the flipping occurs irrespective of whether h_i is positive or negative. In the case of $\Delta P > 0.4$ and $h_i > 0$, a fundamentalist may become more distressed about the enormous rise in the price and suspect that a bubble must be going on. Hence, he tries to act against it (i.e. sells the equity) in order to avoid the losses when the burst occurs. But if $\Delta P > 0.4$ and $h_i < 0$, the same interpretation cannot be used. Instead, a revision of h_i towards positive values might be justified by picking up an argument from DeLong et al. (1990): if fundamentalists observe a persistent rise in prices, they may change their pessimistic view and join the overall trend because this strategy promises more profits. However, both interpretations are contradictory, and the authors are silent on that point.

As it turns out, the outcome of the variant where the $h_i s$ flip whenever $|\Delta P| > 0.4$, does not produce realistic time series. Price fluctuations show a price variation too periodic to be a satisfactory result. Without this component, the fluctuations are non-periodic, unpredictable and produce price changes with a distribution similar to the empirical ones.

Figure 7.3 shows the dependence of the distribution on the number of traders, while figure 7.4 displays the cumulative distribution for the price changes.

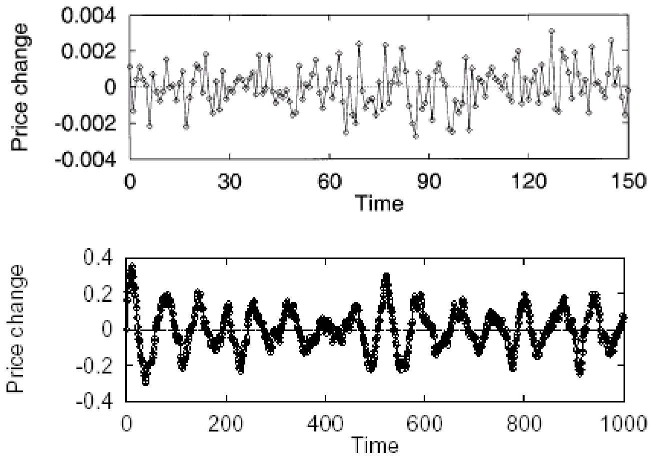

Figure 7.2: Fluctuations of price changes in the Chowdury-Stauffer model. The lower picture shows the price changes for the simulation with 50% traders flipping whenever $-0.4 \leq M \leq 0.4$. See Chowdury and Stauffer (1999, p. 481).

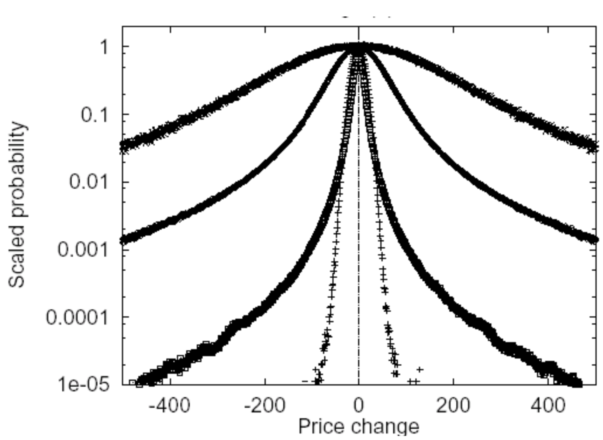

Figure 7.3: Distribution of the price changes for different numbers of traders. The widest distribution is obtained with N=5000, the second widest with N=10^3, and the third widest with N=10^2 traders respectively. For all three $\alpha = 3/2$. The most narrow distribution is obtained with N=10^3 and $\alpha = 7/2$. The picture shows the distribution on a semi-log scale. See Chowdury and Stauffer (1999, p. 478).

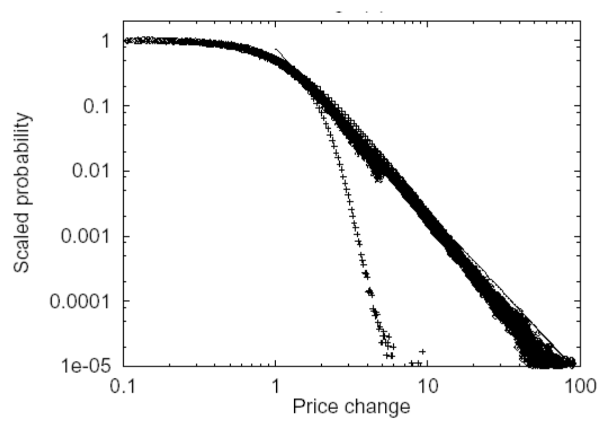

Figure 7.4: Log-log plot of price changes versus probability as obtained from the distributions in the figure above. Notetable is that only the variations that produces the thinnest distribution departs from the other, which are in turn undistinguishable in the picture. See Chowdury and Stauffer (1999, p. 478).

The power-law from ΔP is clearly visible and thus simulation results are qualitatively in accordance to the empirical data. The problem with these results is the missing of any quantitative values in the publication. There is for example no estimated tail index for the displayed distributions. Moreover, the possibility of a long memory behaviour or a possible multiscaling phenomena are not examined.

7.3.2 Kaizoji (2000)

7.3.2.1 The Model

Kaizoji's model of traders who only have the possibility to buy or sell, denoted by $S_i = +1$ and $S_i = -1$ respectively, is another attempt to reproduce realistic price time series with the help of an Ising-variation. The question of how trader i will act in a given economic environment hinges on two different factors. The first can be called the fundamental one as it captures the influence of future profits on the firm's value. In the model this is done by taking the ratio of ordinary profits to total capital and interpreting changes in this ratio as changes in the fundamental value. The author defines an investment environment B according to

$$B \equiv ratio\ of\ ordinary\ profits\ to\ capital - long\text{-}term\ interest\ rate. \quad (7.36)$$

The last term represents the return for the alternative asset with fixed returns (e.g. a bond). Now, changes in B (e.g. a decrease) influence the opinion of the fundamentalists (here, it would induce the trader to favour a sell). The second factor is the noisy one: it is the desire to act in line with the majority of the other traders. This can be interpreted as a risk minimising strategy; traders simply want to stay with the market trend and don't dare to act against it. Thus the term fundamental is still quite ambitious although much better motivated than above. Kaizoji uses the formal concept of the Ising model to define the disagreement function E_i:

$$E_i(S) = -\frac{1}{2}\sum_{j=1}^{N} J_{ij} S_i S_j - b_i B S_i, \tag{7.37}$$

where J_{ij} represents the strength of influence of trader j and i, and b_i is the parameter that measures the influence of the investment environment B. Consequently individual investment attitude is determined by the minimisation of a so-called *disagreement* function E_i quite similar to the formulation in Chowdury and Stauffer (1999):

$$\min E_i\left(S\right) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij} S_i S_j - \sum_{i=1}^{N} b_i B S_i.^{21} \tag{7.38}$$

This equation captures the two different elements of influences of traders decisions. Apart from the first sum which is the cornerstone of every Ising model, the second part reflects the attitude of each trader towards the fundamental factor. It should be noted that the author does not divide the agents into two groups. Here, all agents take the two factors into account when forming their opinion.[22]

As it is the case for all Ising-models, decisions upon buying or selling are subject to the Boltzmann distribution. Temperature is interpreted as a "market temperature. According to Kazoji it represents the degree of randomness in the actions of the traders. This is similiar to the explanations of the introducing paragraph 7.2.2. If T is too high, connections are cut off and the changes in decisions occur randomly, and so preventing any trends. Excess demand N^D is given by

$$N^D = q\sum_{i=1}^{N} S_i\ , \tag{7.39}$$

where q is the fixed amount of stock that each agent trades. Thus, price increments are obtained by

$$\Delta P_{t+1} = \lambda q \sum_{i=1}^{N} S_{it}\ . \tag{7.40}$$

The parameter λ measures the speed of price adjustment, i.e. with a high λ small differences in supply and demand lead to a large ΔP.

[22] The formulation of course allows for the appereance of pure noise trader by setting $b_i = 0$.

7.3.2.2 Results

Contrary to the above contributions, Kaizoji obtains his results not by a simulation but from an analytical method called mean field approximation.[23] The calculations show the outcome as being dependent on the strength of the two elements, neighbourhood influence and investment environment.

In a first round, investment environment is set to zero, $B = 0$. By changing the parameter of trend influence, the author obtains a two phase system, one which is totally dominated by sell orders, the other a pure bull market. Prices are thus either extremely low (near zero) or extremely high, in both cases creating almost no fluctuations. This unrealistic result changes when the fundamental element comes into play. Kaizoji calculates a situation where the parameter for the trend has only a moderate value. B can be positive (good fundamental environment) as well as negative (bad fundamental environment). As it turns out, $B > 0$ starts a bear regime. The conclusion seems to be that a weak noisy influence does not effect a (more a less) correct reaction of the market. The fundamental news dominate the technical trading.

A turbulent and thus more realistic regime is achieved by enlarging the trend factor. However, the model still needs a finer tuning since without restricting the parameter values, the system has three different equilibria. Kaizoji is able to show that a situation where $|B|$ is above a specific critical point B^* induces an equilibrium where strong bandwagon effect change with quieter phases, and bubbles and crashes occur. As $|B|$ rises, the system alternates between a fundamental, a bear and a bull market. When $|B|$ reaches B^*, only the bull market survives and results in a bubble. A similar behaviour is obtained by letting $|B|$ start from a value far above B^*. As soon as $|B| = B^*$, the bull and the fundamental equilibrium vanishes leaving only the bear market and hence a crash. The problem with these results is the lack of any (calculated) time series. It is therefore not able to provide a real comparison. Nonetheless, the mechanics of the model show the typical constitutes.

7.3.3 Bornholdt (2001)

7.3.3.1 The Model

The Bornholdt attempt to simulate market dynamics with the help of the Ising magnetisation model is also closely related to the classical set-up. Traders are arranged around a two dimensional lattice with only two possibilities, $S_i = +1$ and $S_i = -1$. Their actions are again determined by two opposite effects. The first is termed the *herding component*. It is characterised by the influence of the nearest neighbours of a trader. If all neighbours buy, the trader, in absence of any other economic influence, will also be inclined to act accordingly. This is

[23] Mean-field approximation is a technique that obtains his results from a differential equation modelling of a discrete time algorithm (the spin flips). It is a valid approximation as long as the individual changes at each step of the algorithm can be considered small which is true for a one spin flip out on N^2 possible flips.

the by now familiar local trend component. The second is an attitude towards trading against or in accordance with the market opinion, which is formulated by

$$-\alpha C_{i,t}\left|N^{-1}\sum_{j=1}^{N}S_{j,t}\right|,\ \alpha > 0. \tag{7.41}$$

$N^{-1}\sum_{j=1}^{N}S_{j,t}$ is simply the averaged decision of all agents (including agent i). C_i is the parameter that represents the attitude of each trader towards this overall trend, and α measures the strength of this reaction. The sign of C_i is decisive for the direction of the influence. Bornholdt restricts the values of C_i to $+1$ and -1. In the case of $C_i = +1$, traders are inclined to act against the majority of all agents. The author calls this a minority strategy following the literature on minority games. For $C_i = -1$, traders are positively coupled not only to the local environment but also to the global development.

Bornholdt calls this second factor *fundamental*. He justifies the termination by assuming that traders with such a component have a knowledge about the fundamental value of the asset (without telling the reader the nature of that knowledge). However, the careless notion does not come up with the way economists would model a fundamental trader. First of all, fundamentalists do not principally act against a trend. They would follow the market if it acts correct, and correct is what the fundamental news entail. I.e., if good news come in, prices *should* jump upwards. Thus, news would be the true fundamental factor of each simulation. Because the Bornholdt model has no such a factor implemented, it is not correct to relate this element to a fundamental evaluation. Despite this critique, the Bornholdt model has introduced an interesting element with (7.41). The factor that is responsible for actions against a trend is related to an endogenous development within the system: the global trend. With this, traders react to situations where either all buy or sell, i.e. in trend periods. Hence, the dynamics enforced by this element come from an endogenous motivation and not from an exogenously imposed factor. However, this above addressed fundamental element introduces a kind of split opinion into H_i. On one side, local couplings lead to a uniform behaviour of traders while on the other side these same traders "have a desire to join the global minority, for example in order to invest in possible future gains."[24] A mixed strategy, consistent of technical and fundamental orientated trading is well conceivable. But the imagination of a person who wants to get in line with his local surrounding and nevertheless also tries to counter the whole market trend seems to be quite unmotivated.[25]

[24] Bornholdt (2001, p. 669).

[25] However, the idea is borrowed from the literture on so-called minority games. Minority games are repeated games with an odd number of players who must decide to take one out of two possible choices. As it turns out, those who decide to take the alternative which attracted

A possible way to justify such a modelling is by assuming that the two elements come from different sources. The local influence is due to personal connections with other traders who exchange their individual convictions or decisions. On the other hand, the overall market development, manifested in the price, is easy to observe for everybody and thus may serve as a second source of information. And so, traders are confronted with two, sometimes contradictory signs.

If the neighbourhood influence is always strong enough to overcompensate the minority term, the picture is not different to the one that would prevail for all agents being pure noise trader: the market eventually reaches an equilibrium with either all spins $+1$ or -1. But with (7.41) sufficiently large, all traders reverse their opinion and the situation changes completely. However, such an evolution of decisions is totally unrealistic as it would imply a permanent switch from one extreme situation to another. A more reasonable description should restrict the number of traders with $C_i = +1$. It is because of this consideration that the model does not have a fixed ratio of agents with $C_i = +1$ or $C_i = -1$. Instead, Bornholdt suggests to change C_i according to

$$C_{i,t+1} = -C_{i,t}, \text{ if } \alpha S_i\left(t\right) C_{i,t} \sum_{j=1}^{N} S_{j,t} < 0 \ . \tag{7.42}$$

α represents the strength of the global coupling of the spins. He justifies his choice by the fact that each agent can receive a penalty for his strategy which rises with absolute value of price. In order to illustrate the logic behind (7.42) one may think about a situation with $S_i < 0$, $C_{i,t}$ and $\sum S_{i,t} > 0$. In this case, the opinion H_i misses to join the overall market trend. This is the risk of following a minority strategy: it carries the risk of heavy losses when the market trend is strong and stable.

The remaining set-up is canonical. Decisions to buy ($S_i = +1$) or sell ($S_i = -1$) are determined according to

$$S_{i,t} = +1 \text{ with probability } p = \frac{1}{\left[1 + \exp\left(-2\beta H_{i,t}\right)\right]} \ , \tag{7.43}$$

$$S_{i,t} = -1 \text{ with probability } 1 - p, \tag{7.44}$$

less players, win. The structure of these models stems from the "El-Faroll's bar" problem. The El-Faroll bar is a bar near the Santa-Fe institute of Technology. Once a week, the bar offers an attractive program (life music, cheap beer etc.). However, if too many people decide to go to the bar, the room is packed and the whole atmosphere turns out to be rather bad. But if only a small group of interested goes to the bar, they get the full advantages of the program. Here, the minority wins. The model is full of interactions as it is a repeated game, where people update their decisions based on the last experiences. The set of possibilities is here "go to the bar" or "do not go to the bar", and in the case of stock markets it is "buy" or "sell". For an introduction to minority games see Zhang (1998).

$$\text{with } H_{i,t} = \sum_{j=1}^{N} J_{ij} S_j - \alpha \left(C_{i,t} \left| \frac{1}{N} \sum_{j=1}^{N} S_{j,t} \right| \right). \tag{7.45}$$

$H_{i,t}$ contains the interactions of spins and the fundamental parameter as well.[26] The fact of a unique Hamiltonian for all indicates the missing of a devision of traders into fundamentalists and noise traders. Instead each agent has potentially both components. J_{ij} represents the influence of the four nearest neighbours and is set equal to all i, i.e. $J_{ij} = J$.

Price changes are calculated by taking the difference of the magnetisation

$$M = \sum_{i=1}^{N} S_i \tag{7.46}$$

of two consecutive runs, i.e.

$$\Delta P_t = M_t - M_{t-1}. \tag{7.47}$$

This is in line with the Ising magnetisation model since there M is the level variable and fluctuations are measured by the first difference. As explained in section 7.1.1, this interpretation has the inconvenient side effect of producing negative prices, but the author does not offer a solution.

7.3.3.2 Results

Fixing β in (7.43) at the critical value of 2.2, Bornholdt gets the desired intermittent phases as can be depicted from figure 7.5.

Although the similarity with real data sets is visible, the appearance alone does not guarantee that the outcome belongs to one of the statistical models proposed in part two. In order to assert an accordance with e.g. a truncated LSD, the author would have to provide estimates about the distribution of the returns or price changes. Unfortunately he does not give these results and thus cannot claim to reproduce a time series with approximately the same tail thickness as the empirical records. However, the power-law scaling of the cumulative distribution of the absolute returns is clearly visible in figure 7.6.

Another outcome that is in agreement with the empirical data is the autocorrelation function of absolute returns, which decay in a very slow fashion (and vanishes just after some 1000 runs).

7.3.4 Kaizoji, Bornholdt and Fujiwara (2002)

7.3.4.1 The Model

This simulation is an extension of the Bornholdt model. The set-up is principally the same, but some new features are incorporated. Fundamentalists are now

[26] The functional form of (7.43) is merely a variation of the usual Boltzmann distribution.

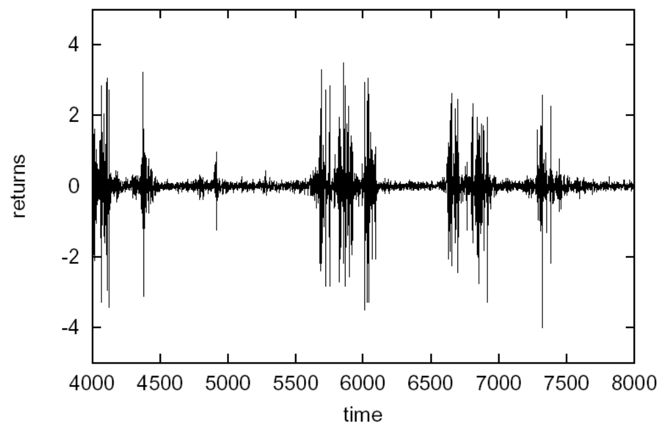

Figure 7.5: Simulated returns of the Bornholdt simulation. The picture shows the logarithmic relative change of the magnetisation $M = \frac{1}{N} \sum S_i$. This result is obtained with parameter values $T = 1.0$ and $\alpha = 8$. See Bornholdt (2001, p. 669).

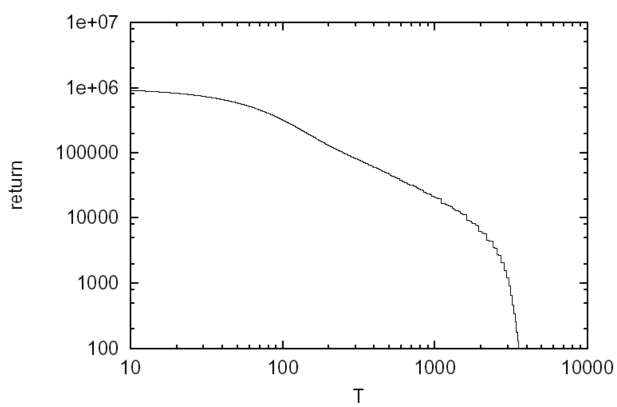

Figure 7.6: Cummulative distribution of absolute returns.
See Bornholdt (2001, p. 669).

modelled as traders who always act when the actual price does not coincide with the fundamental value, denoted by P_t^*. Kaizoji et al. model this fundamental price as a simple random walk, thus following idea of efficient markets. Total fundamentalists demand is given by

$$N_{F,t}^D = bm \left[\ln P_t^* - \ln P_t \right] \ , \tag{7.48}$$

where m is the number of fundamentalists in the market and b represents the strength of the reaction to the difference of P_t^* and P_t. P_t is the price prevailing at time t. The interpretation is straightforward. If $\ln P_t^* - \ln P_t > 0$, actual prices are below their fundamental value, hence $N_F^D > 0$. With a negative difference, the fundamentalists sell their assets because of an overvaluing. Furthermore, the more severe this mispricing becomes the stronger the fundamentalists reaction.

The second group of participants are termed interacting traders. These traders are similar to the ones modelled in the Bornholdt simulation above. Again, they simultaneously have a tendency to join the majority of traders but on the other hand see possibilities in making capital gains by trading against them. The Hamiltonian is thus very similar to the one of the last model

$$H_i = \sum_{j=1}^{N} J_{ij} S_j - \alpha S_{j,t} |M_t|. \tag{7.49}$$

Interestingly, this counter-reaction term is no longer attributed to a fundamental strategy. Instead the interpretation now is that "an interacting trader in the majority group would expect that the larger the value of $|M_t|$ is, the more difficult a further increase in size of the majority group would be. Therefore, interacting traders in the majority group tend to switch to the minority group in order to avert capital losses, e.g. to escape a large crash, as the size of the majority group increases."[27] The decision about traders' choice is ruled by the same probabilistic formulation as in (7.43), where H_i is here determined by (7.49). Total demand of the interacting traders is determined by

$$N_{I,t}^D = anM_t, \ M_t = \frac{1}{N} \sum_{i=1}^{N} S_i \ . \tag{7.50}$$

n is the number of interacting traders and a measures again the strength of the reaction on M_t.

In this model a market maker ensures an equilibrium by adjusting the price to its market clearing value.[28] This value is determined through

$$N_{F,t}^D + N_{I,t}^D = bm \left[\ln P_t^* - \ln P_t \right] + anM_t = 0, \tag{7.51}$$

and thus

[27] Kaizojy, Bornholdt and Fujiwara (2002, p. 4).
[28] However, the precise process remains unexplained in the publication.

$$\ln P_t = \ln P_t^* + \lambda M_t,\ \lambda = \frac{an}{bm}. \tag{7.52}$$

The volume is then given by

$$an\frac{1+|M_t|}{2}. \tag{7.53}$$

As can be seen from (7.52), if $M_t = 0$, i.e. the demand and supply of interacting noise traders cancel each other out, $\ln P_t$ is equal to its fundamental value $\ln P_t^*$. In this case, the price follows a pure random walk and the density of the price changes end up in a normal distribution. When interacting traders come in, prices can substantially deviate from $\ln P_t^*$. For example, $M_t > 0$ indicates a bull market and the actual price exceeds its fundamental value.

Kaizoji et al. use a 101×101 quadratic lattice for their simulation. The transition probabilities between buying and selling are again given by the Boltzmann distribution. Simulations are carried out with $T = 1/\beta = 0.5$. The remaining exogenous parameters were set to $J = J_{ij} = 1$ and $\alpha = 20$. The resulting artificial time series is a considerably good approximation of real data. Figure 7.6 shows intermittent phases where quite periods alternate with crashes and bubbles.

An interesting point is the correlation of periods with high volatility and high trading volume. Most of the time, price fluctuations are moderate. But the appearance of turbulent phases where the relatively stable behaviour of ΔP is interrupted by huge up's and down's in the price index, can be seen periodically (though not regularly spaced in time). These phases are accompanied by a large trading volume. Moreover, the authors stress the fact that the volume has to exceed some critical value in order to produce periods with high volatility.

The authors provide statistical analysis of their simulation. First, they estimate the tails of the density for the log-return r_t. The tail exponent as reported in the paper does not quite match the desired value (approximately 4), but with 2.3 it is clearly outside the range of LSD.[29] Kaizoji et al. do not say anything about the estimation technique, but it seems to be clear that the cumulative procedure was applied.

They then perform the multiscaling analysis. Different to many empirical investigations, this is done with volatilities rather than returns. With volatility as defined by $|r_t|$, Kaizoji et al. find a range of scaling moments that confirm the empirical findings about the non-linearity of the zeta-function. Finally, the autocorrelation function for the absolute changes shows the typical long lasting decay of temporal dependencies. Even after a 1000 runs the value of the correlation function is still around 0.1.

[29] Desired because of the results in Gopikrishnan et al. (1998), where a tail exponent α of around 4 is estimated. See chapter 5.3.2.

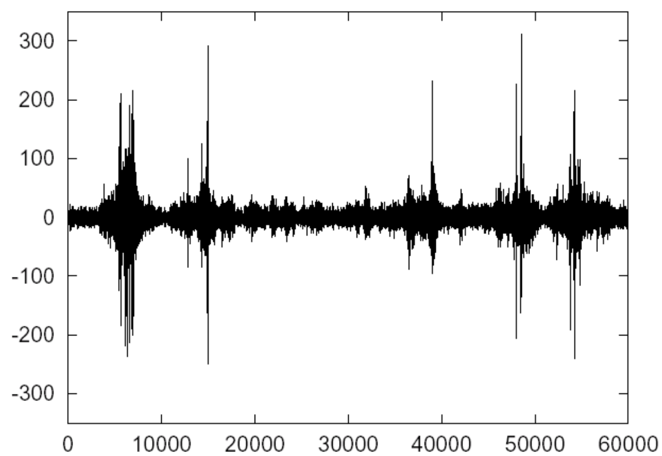

Figure 7.7: Price changes in the model of Kaizoji et al. (2002). See Kaizoji et al. (2002, p. 446)

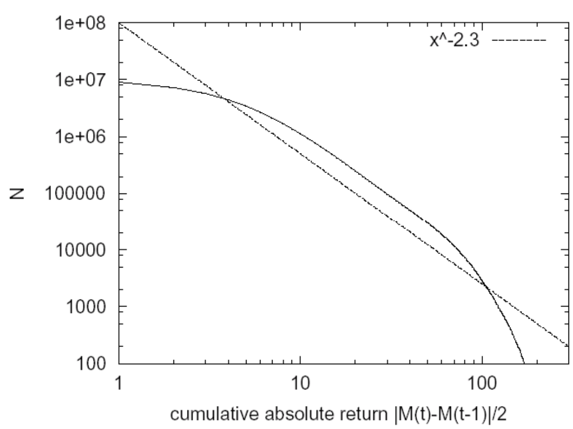

Figure 7.8: Cumulative probability function for absolute returns. See Kaizoji et al. (2002, p. 448)

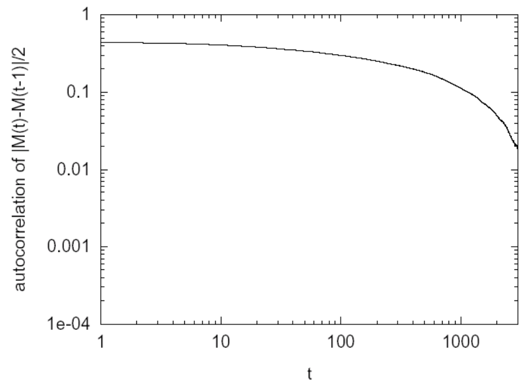

Figure 7.9: Autocorrelation function of magnetisation increments. See Kaizoji et al. (2002, p. 449).

7.3.5 Iori (2002)

7.3.5.1 The Model

In the Iori model traders who are as usual distributed on a square lattice are now endowed with a particular amount of capital. This (financial) capital is the sum of two assets: cash M and N units of a single stock. Because of the finiteness of capital, agents are bounded in their actions: if it happens that the individual stock is reduced to zero, selling is forbidden. This is a new, though not essential feature of this model. Initially all agents start with the same endowment and are only allowed to buy and sell one unit of the stock or being inactive.

Traders base their decision on two different signals, both received at the beginning of each run. The first is the usual signal of the four nearest neighbours. The second is an individual signal $\eta_i(t)$, uniformly distributed over $[-1;+1]$, which symbolises shocks to the personal preferences of trader i. This element is obviously not a fundamental ingredient otherwise each fundamentalist would receive the same signal. It should though be seen as an idiosyncratic change in the willingness to buy or sell. Probably the best economic interpretation is an altered risk aversion. Then $\eta_i < 0$ indicates a more careful attitude towards the risky asset.[30] The aggregated signal H_i is written as

$$H_{i,t} = \sum J_{ij} S_{j,t} + A\eta_{i,t} \ . \tag{7.54}$$

A measures the strength of η_i. As in the majority of simulations, J_{ij} is set to be equal for each site thus, $J_{ij} = J = 1$. Iori introduces a flexible element for the neighbourhood influence by taking $J_{ij} = 1$ with probability π and $J_{ij} = 0$ with probability $1 - \pi$. Of course, $\pi = 0$ would lead to uncorrelated decisions for

[30] But then it would be surprising to see frequently occuring parameter changes.

all agents while $\pi = 1$ leads to the usual Ising structure. In the following, the author mainly deals with values of $\pi = 1$.[31] Another new feature is introduced in form of a trade friction χ that does prevent traders from acting too frequently. This trade friction is not equal to all but is drawn from a Gaussian distribution for each trader individually, $\chi_i\,({}_t) \overset{iid}{\sim} (0, \sigma^2)$.[32] With these variations, decision possibilities are formulated by

$$S_{i,t} = \begin{cases} +1, \text{ if } H_{it} \geq \chi_{i,t} \\ \quad 0, \text{ if } -\chi_{i,t} < H_{i,t} < \chi_{i,t} \\ -1, \text{ if } H_{i,t} \leq -\chi_{i,t}. \end{cases} \tag{7.55}$$

Clearly, in the absence of neighbour influence, each trader would act only in accordance to its idiosyncratic shocks, resulting in an aggregate price that follows a random walk.

In the model, a market maker sets prices according to a non-linear function of excess demand. Let N^D denote the aggregate demand and N^S the aggregate supply:

$$N_t^D = \sum_{i:S_{i,t}>0} S_{i,t},\ N_t^S = -\sum_{i:S_{i,t}<0} S_{i,t} \tag{7.56}$$

and $Vo_t = N_t^S + N_t^D$ is the trading volume. Prices are adjusted by the following rule:

$$P_{t+1} = P_t \left(\frac{N_t^D}{N_t^S}\right)^{\alpha},\ \text{where } \alpha = a\frac{Vo_t}{L^2}\ . \tag{7.57}$$

with $a > 0$ as an arbitrary parameter, and L^2 as the number of traders (which is equal to the number of sites of the lattice). In order to let the threshold evolve over time, Iori formulates the following dynamics for the individual friction

$$\chi_{i,t+1} = \chi_{i,t}\frac{P_t}{P_{t-1}}. \tag{7.58}$$

According to (7.58), χ_i is positively correlated to the market. If prices rise, individual thresholds do also shift up. This evolution of χ_i is added in order to preserve the symmetry in the probability of buying and selling. Returns are defined by

$$r_t = \log\frac{P_t}{P_{t-1}}, \tag{7.59}$$

and

[31] Values of $0 < \pi < 1$ lead to percolation models which will be considered in the next chapters. Therefore results for such cases are not referred here. They play a very moderate rule in model, anyway.

[32] The term is identified as a proxy for transactions costs. However, transaction costs are no individual costs but should be the same for any trader.

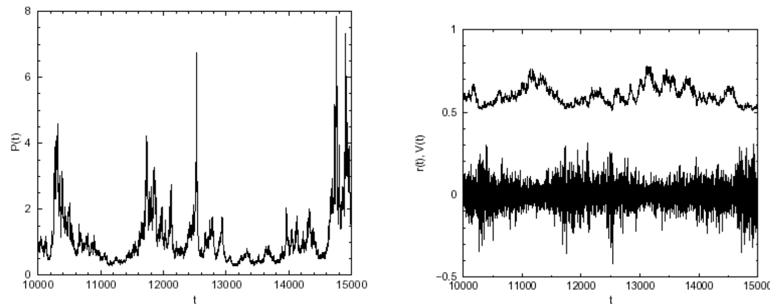

Figure 7.10: Left panel: Price development in the Iori simulation. Right panel: The associated return (lower series) and trading volume (upper series). See Iori (2002, p. 85).

$$\sigma_t = |r_t| \tag{7.60}$$

denotes the market volatility. Simulations are carried out with starting values of $N_i(0) = 100$ and $M_i = 100$, $P(0) = 1$. Initially $\sigma_\chi(0) = 1$ and a was fixed at 0.2.

7.3.5.2 Results

Iori concentrates on three stylised facts of stock markets, the volatility clustering and long memory property of absolute returns, the power law scaling of returns, and the multiscaling feature. The three points are discussed in length in the paper. The main results are the following:

Volatility clustering is achieved, especially when all factors of price determination are present, i.e. imitation, adjusted thresholds χ_i and a variable price determination that is dependent on the volume traded in that period. Important insight into the model is obtained by letting the agents act independently of each other. For $J_{ij} = 1$ and $\pi = 0$, the system is unable to build up networks and the price fluctuations, although driven by heterogenous agents, follow a random walk. Nonetheless, Vo_t is occasionally very small, so even minor differences in demand and supply lead to pronounced price changes. Another parameter combination that is able to produce intermittent time series for P_t is found by $\pi = 0$ and $\alpha = 1$. Figure 7.10 shows the charts forP_t, r_t and Vo_t.

Iori explains her results with the speed of price adjustment that takes over the responsibility as the crucial mechanism. Employing the adjusting thresholds, a positive $\Delta P(t)$ leads to a reduction of active traders, thus thinning the market. But this can cause major fluctuations despite $\pi = 0$ as already mentioned above.

The author also controls for the long-range dependence of absolute returns by employing Lo's modified R/S-statistic. She finds a Hurst exponent of 0.85 for the whole range of the simulation.

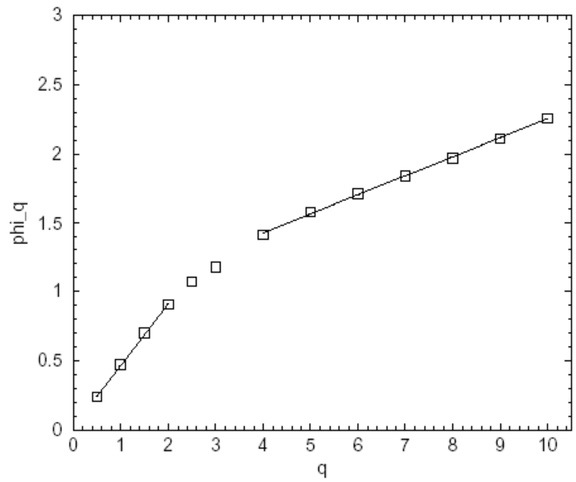

Figure 7.11: The zeta-(q)-function for the simulation, here called phi_q. The course is non-linear, but piecewise linear which is different to the bended functions of chapter 5. See Iori (2002, p. 86).

Testing for the multiscaling feature of the simulated time series, it shows up that it produces two different phases, each with a different scaling parameter. Figure 7.11 displays the case of the zeta-function. Although being non-linear because of the break at $q \approx 3$, it is distinctively different from the results obtained for the empirical data, because the Iori simulation produces a time series with only two scaling exponents. Indeed, the zeta-(q)-function can be divided into two distinct sections each with a different slope. The first from $0 \leq q = 3$ features a slope of 0.47 which is close to the value that would be obtained by a simple BM. Passing $q = 3$, the slope decreases to 0.14.

The cumulative distribution of returns is plotted in figure 11. It shows the by now familiar power-law and estimations of the slope as indicated by the dotted line yields a value of approximately -3, thus $P(r) \sim r^{-3}$. This result is consistent with the empirical literature in its finding of an exponent above 2, indicating a distribution with finite second moment. In general, the results are quite satisfactory.

7.3.6 The Cont-Bouchaud Percolation Simulation (2000)

The Cont-Bouchaud (C-B)-model is one of the earlier efforts by econophysicists to explain the fluctuations in stock markets by borrowing ideas from the natural science. It has by now become a pillar-stone of computational modelling and is widely used as a base for other models. Its main contribution is the different communication structure as compared to classical Ising models. Like

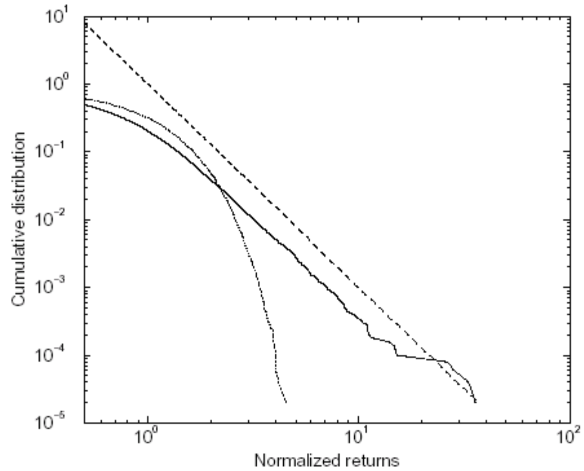

Figure 7.12: Cumulative probability of normalised returns. See Iori (2002, p. 90).

all percolation-type models traders are now randomly distributed on a lattice with a certain concentration, i.e. the number of occupied sites is no longer equal to the total number of sites. The percolation clusters are formed through the random connections between two or more agents. These clusters are then interpreted as herding groups or companies where all connected sites behave in the same manner.

The market consists of N agents who can trade only one unit of a risky asset. The individual decision is denoted by

$$S_i = \begin{cases} +1, & \text{if agent } i \text{ buys one asset} \\ 0, & \text{if agent } i \text{ is inactive} \\ -1, & \text{if agent } i \text{ sells one asset.} \end{cases} \tag{7.61}$$

The probabilities with which decisions are made are given by

$$\pi\left(S_i = +1\right) = \pi\left(S_i = -1\right) = a, \; \pi\left(S_i = 0\right) = 1 - 2a. \tag{7.62}$$

With (7.61) and (7.62), the description so far is reminiscent of the classical Ising structure. However, the exogenous character of π should be noted. There is no Gibbs-distribution which accounts for the transition of decisions. This is possible because there is no fixed number of agents and there are no individual decisions. Moreover, percolation models are principally not related to thermodynamic considerations.[33] The price increment in period t is calculated via

[33] See Stauffer and Aharony (1994).

$$\Delta P_t = P_t - P_{t-1} = \frac{1}{\lambda}\sum_{i=1}^{N} S_{i,t-1} \ , \qquad (7.63)$$

where λ is a parameter that represents the market depth. It should be interpreted as a measure of market sensitivity: a low value of λ means that only slight differences in demand and offer lead to strong-price movements. This is typically encountered in thin markets with low liquidity.

Cont and Bouchaud assume that agents align themselves towards a uniform action once a group is established. This assumption restricts traders to a collective behaviour without the slightest opportunity to deviate. Moreover, the question of which value S_i takes on hinges solely on the given probabilities. This is also true for the Ising models, but there economic factors have a significant influence on the probabilities. Here, no other element enters the determination of $\pi(S_i)$ and thus people decision's are completely exogenous and cannot be traced back to any economic reasoning whatsoever. The authors justify this rather crude modelling with the intension "to focus on the effect of herding".[34] The model does not have any economic modules built in to explain the decision of a group. It is hence a random analog to the pure Ising model.

The special feature of the C-B model is the way in which clusters are formed. The authors use binary links between traders in order to represent connections that lead to a common behaviour. These links are not, however, established automatically but only with a probability $\pi < 1$, i.e. two agents are mutually connected with a particular probability $\pi_{i,j}$. For reasons of simplification, $\pi_{i,j}$ is chosen to be equal among all traders and thus $\pi_{i,j} = \pi$. This makes up an average of $(N-1)\pi$ links for each agent. Thus, the C-B model is not restricted to the local environment of a prespecified number of traders with influence. This is the main difference between the C-B and Ising-models which uses nearest–neighbour connections only. There, it is always a fixed group of four or eight traders that have an influence of trader i. Here, influences may span over a wide range within the market. And the more traders are connected, i.e. the larger the size of the cluster, the more influence it has on the price dynamics. A realistic projection incorporating these links pictures traders as subjects who meet each other randomly on the trading floor, exchanging opinions and information. Cont and Bouchaud set π equal to c/N, i.e. proportional to N^{-1}. Thus c becomes the crucial parameter that decides upon the distribution of the clusters. This parameter can be interpreted as a measure of the "willingness of agents to align their actions".[35] With c being low, $\pi_{i,j}$ is also small and this will end in small clusters. Figure 7.13 gives an example of a typical situation.

Structures as presented by figure 7.13 are known as random graphs. It is a part of topology and delivers a range of solutions. Because of this, Cont

[34] Cont and Bouchaud (1998, p. 175).
[35] Cont and Bouchaud (2000, p. 175).

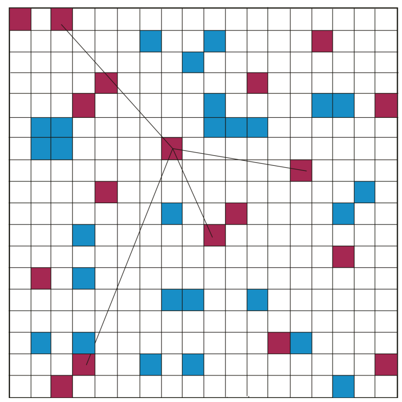

Figure 7.13: Illustration of a typical percolation model. The 18 x 18 square lattice represents an arbitrary situation. Red squares indicate buy orders, blue sell orders and white is a waiting position, here the prevailing one. The lines symbolise connections between agents. These connected traders then form a uniformly acting cluster, in the case above, they all buy.

and Bouchaud do not have to perform a simulation but can solve the system analytically. The authors use a method called random graph theory and propose this mechanism as being responsible for the statistical properties of price fluctuations in the artificial stock market.[36]

Since the techniques of random graph theory are not a topic of this work, the next lines summarise only the main results. Cont and Bouchaud set the parameter of coordination c close to 1, which amounts to having each trader linked with only one other agent. However, despite this rather low communication activity, large cluster can nevertheless occur through successive binary links. Though the choice of c is also motivated by the results of Bollobas (1985) who shows that for $c = 1$ the probability density for the cluster size W behaves as a power law. Using this results, the authors are now able to show that for $c = 0.9$ and and average number of actively trading agents of around 1000, $P(\Delta P_t)$ has the form of a power law with exponential parameter $\alpha = 3/2$, at that time believed to be the correct value, thus

$$P(\Delta P_t) \sim \Delta P_t^{-3/2}.$$

Despite this outcome, the model can be criticised in many ways. Its economic structure is indeed very minimal. It is so sparse as to seem too unrealistic. Furthermore, the grade of precision and the range of the results is admittingly not very high. Its only concern is the power-law decay of price changes. Autocorrelation and multiscaling features are not reported. However, this model

[36]See Diestel (1996) for a good introduction into random graph theory and Ioannides (1996) for a review of economic applications.

was originally the first that was built with the help of percolation theory, a discipline also very familiar to physicists. The model is thus reviewed because many econophysicists have been working along similar lines. Some of these, here in particularly the works of Stauffer and his collaborators, are presented beneath. Their primary focus lies on a good correspondence of artificial and real price records with a special emphasis on the behaviour of large price changes or returns. Contrary to the C-B model, the analysis are undertaken with the help of a simulation.

7.3.7 Stauffer and Penna (1998)

The Stauffer and Penna simulation is more closely related to the original Ising set-up than the C-B model, because it uses nearest-neighbour influences to capture the connection between traders. Different to the Ising-models, not all sites are occupied by an agent. Instead, a site represents an agent only with probability π, but with $(1-\pi)$ it stays empty. As usual traders can either buy or sell one unit of equity if they are active, or they abstain from trading. Again, the decision to which of those possibilities an agent belongs is based on probabilities. I.e. with a probability of π, occupied sites represent active trading, where the buy or sell decision is equally probable. Then, $(1 - \pi)$ is the probability of holding the asset. The crucial point is that, very much like the C-B approach, connected agents build a cluster (denoted by W_i) and perform in a uniform manner. Every trader who belongs to a cluster either sells or buys, or holds. This description is somewhat different to the classical Ising simulations because here agents form a homogenous group. In the classical Ising model, nearest neighbours influence the trader to act as they do, but at last the decision is determined by a probability function and thus the outcome is uncertain. The Stauffer-Penna concept instead introduces a unitary behaviour for all agents who belong to a connected area, irrespective of any other influences.

Stauffer and Penna now construct the following trading mechanism: at a first step π determines the number of occupied sites and the resulting cluster sizes are computed. Then it is randomly chosen whether these cluster are active or not. Subsequently, in the case of an active state, a decision is made about the demand (positive or negative). These steps are repeated several times. Stauffer and Penna take the most simple functional form for the price dynamics assuming that increments are proportional to the sum of the $W_i s$:

$$\Delta P \sim \sum_{i=1} N_i W_i, \tag{7.64}$$

where N_i is the number of traders in cluster i. Of course, the higher N_i, the more influence a cluster has on the price development. Thus, clustering is an important quantity. π determines the structure of the market. As known from percolation theory, for π above some critical value π_c traders form an infinite cluster connecting top and end points of the lattice.[37] For π close to zero,

[37] Infinite because of an endless flow of information from top to down and vice versa.

traders are isolated and there is little information flow. For π below π_c but near to it, large but finite clusters build up. Figures 7.14 and 7.15 show the distribution of price changes and the behaviour of the associated cumulative probability function respectively.

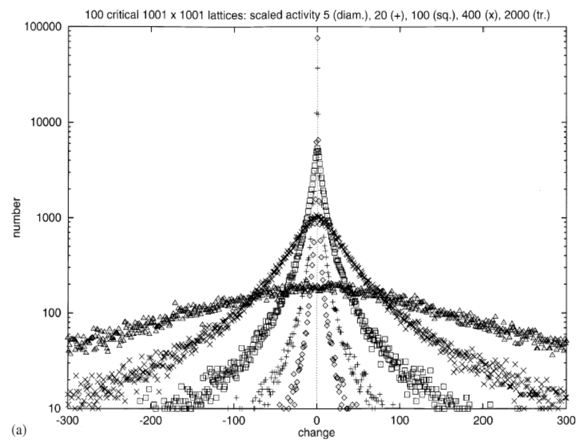

Figure 7.14: Probability distribution for price changes with different parametrisations. The different symbols represent different numbers of actively trading agents during one time step. Here, active sides are 5 (diameters), 20 (crosses), 100 (squares), 400 (x) and 2000 (triangles). The higher the activity, the heavier the tails. See Stauffer and Penna (1998, p. 286).

The diagram on the right site displays the dependence of the *pdf* for ΔP on the trading activity a. Price changes are more pronounced when trading activity is small. Stauffer and Penna offer the following interpretation of this result: if rising activity can be identified with increasing traded time (at fixed activity), then the phenomenon of aggregational gaussianity is reproduced. Indeed, for rising a, the *pdf* converges from a power law to a bell-shaped form.

7.3.8 Stauffer and Sornette (1999)

There are just some minor differences between this simulation and its predecessor the Stauffer-Penna-model. The probability parameter π is now chosen randomly instead of being fixed by the authors. This is motivated by an assumed development in the process of information exchange: communication changes over time and thus also alters the strength of the connections between traders. Because the only way in which the model can account for these differences is the number of traders in each cluster, π becomes again the crucial parameter to vary. Of course, one might argue about the purely random character of the changes in π, but this option is chosen for simplicity as the authors

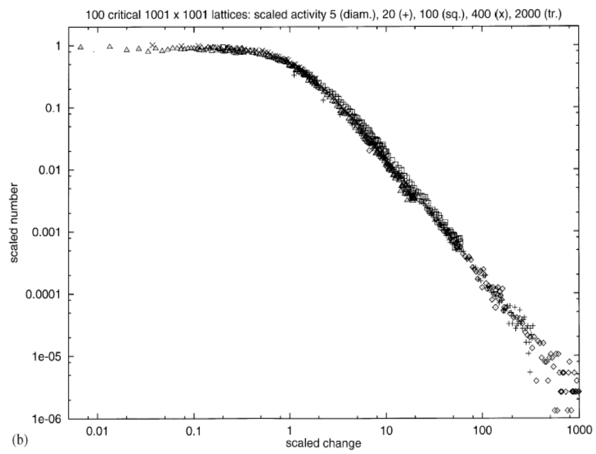

Figure 7.15: Cumulative probability distribution of the simulated price change. Surprisingly, the different activities have no big influence on the form of the cumulative distribution. They are almost identical, hence the points are clustered so tight tht differences are not visible. See Stauffer and Penna (1998, p. 286).

argue. Consequently, ΔP is now computed over a spectrum of different probabilities. This affords a multiple repetition of the simulation with randomly varying πs. Stauffer and Sornette then take an average over each time series of ΔP. As above, samples that contain values of π near or identical to π_c are the only candidates for the generation of intermittent price records. In fact, the samples are dominated by those probabilities because they contribute the runs with the largest clusters, thus generating the biggest price fluctuations.

A first round of simulations produces a tail parameter of around 2.5 which the authors find to be inconsistent with the empirical data. Their goal is to reach a value that is compatible with the results obtained by Gopikrishnan et al. (1998) where the tail exponent is approximately 4. Stauffer and Sornette alter their model by assuming a dependency between the parameter of activity a and cluster size s in the form of:

$$a = \frac{0.5}{\sqrt{s}}. \tag{8.65}$$

This specification implies a reciprocal relation, i.e. if s rises a becomes lower and less active agents trade in the market. The authors justify this approach by noting that "big investors, such as the mutual and/or retirement funds with their prudent approach, their emphasis on low risk, and their enormous inertia due to the fact that large positions move the market unfavorably, have to and do trade less often than small professional investors who have to generate their income from active trading rather than from sheer mass".[38] With this configuration, large clusters trade less frequently than smaller ones and one may expect a lesser influence on the average of the sample. The modification leads to values of 3.5 which is much closer to 4 than before.

7.3.9 Chang and Stauffer (1999)

Again, this simulation is a direct successor of Stauffer and Penna (1998). Here, a is set to 0.05 in order to have only a small fraction of active traders in each run. This is the lesson learned from the previous attempts: small trading activity favours large fluctuations. Furthermore, 1% of all agents (occupied sites) move randomly to one of the nearest-neighbour sites (in the case they are free). This introduces another random element into the model in that it constitutes new cluster sizes by removing 1% of the agents from existing blocks and adding those to other cluster. As in the Stauffer-Sornette simulation, the probabilities to buy or sell are no longer equal. Different to it, the probability of selling is no longer equal to the probability of buying but instead hinges on the following expressions:

$$\pi\left(S_i = -1\right) = \left(1 + \varepsilon\phi\right) a, \ \varepsilon > 0, \tag{7.65}$$

$$\pi\left(S_i = +1\right) = \left(1 - \varepsilon\phi\right) a, \tag{7.66}$$

where ϕ is defined by

$$\phi = \log\left(\frac{P_t}{P_0}\right). \tag{7.67}$$

P_t is the actual prevailing price while P_0 is the starting price that is set to 100. For example $\phi = -0.004$ means a ratio of 99/100 and thus indicates a lower price. The interpretation is that ϕ is the relative change of prices and if it happens to be large, more clusters want to sell, thereby initiating a prompt mean-reverting process. According to (7.67), this action is stronger the higher the relative change is.

An economic reasoning for the tendency of the price to fall after a huge roar may be explained by the existence of fundamentalists, who immediately detect an increase in P that is not backed by a fundamental change and act against it. Of course this scenario is consistent with the EMH, but in the context of the simulation it is just one possible interpretation. The other is that of a bursting bubble. Even noise-traders have a feeling about the long time true (fundamental) value of an asset. If the actual price is too far away from that value, the risk of engaging in further buying activities rises to values which might induce even the chartists to reduce their risk of big loss by selling before the crash happens. Both lines of argumentation are possible, but one should

[38] Stauffer and Sornette (1999, p. 500).

not drive the economic interpretation too far. The set-up does not reveal any kind of different motives for the actions and the physicists Chang and Stauffer do not offer any. It is a very crude image of a real financial market. Simulated time series for ϕ hinge critically on the value of ε. The authors detect that for $\varepsilon = 0$, the price has a large deviation from its equilibrium value of zero, while $\varepsilon = 10$ let ϕ_t stay close to zero. The distribution of price changes in this simulation is quite similar to the ones in the former simulation. Unfortunately, the authors do not provide a detailed description of the used parameter values. A further result is the very slow decay of the autocorrelation in the volatility process. This is similar to real data.

7.3.10 Summary

There are naturally more simulations in the literature as those reviewed above. For example, a one dimensional Ising structure is provided by Sznajd-Weron and Weron (2002) which is able to reproduce the fat tailedness of return distributions as well as long memory for the raw returns as tested by the R/S statistic and GPH-estimator. Ponzi and Aizawa (2000) use the Ising structure and add an evolutionary component that decides upon the strength of the connections between agents. Here, price returns follow a power-law and volatility, measured by the absolute value of price changes, confirms the long memory hypothesis. A simulation more related to the percolation models of Cont and Bouchaud is the one of Eguíluz and Zimmermann (2000), where the only parameter of the model accounts for rate of information passed to connected traders. By fixing this parameter to a critical value the authors achieve fairly good approximations to real data. Stauffer and Jan (2000) building on the paper of Chang and Stauffer (1999) are able to tackle the up/down-asymmetry of stock returns, a stylised fact only rarely targeted by the literature. This feature is obtained by introducing a feedback rule that prescribes a changing of trading activity that is proportional to the last price changes. Other stochastic simulations are put forward by Arifovic (1996), Krawiecki, Holyst and Helbing (2002), Yang et al. (2004). However, this list is also just but a small selection of papers that use the general structure of Ising models to perform financial market simulations. Common to all is the same simple modellation of traders motivations to buy, sell or hold. But even these simplified versions of real stock markets are all able to reproduce the most striking empirical features of financial time series.

7.4 A new Ising Model with heterogenous Traders and Information Inflow

The structure of this new Ising-variation is deliberately chosen to stay close to the ones described in chapter 7.3. As the following simulation aims to work out the essential features, it is important not to depart too far from the established models.[39] By keeping the number of different models moderate, it is much easier to isolate the crucial points. Other set-ups taken from other fields of physics may also be able to reproduce intermittent price records. But it is then much harder to compare the effects of the different elements, because it might become unclear whether results can be attributed to the structure itself (i.e. a particular structure of a model that is known to generate these outcomes) or the other economic factors as e.g. the description of the interaction between fundamental and noise trader. Therefore this simulation a mixture of the previous models with one new element added: a regular injection of new information from outside the market.

7.4.1 The Model

As usual, traders are ordered on a two-dimensional (32 x 32) lattice where each site is occupied by one agent, who has the choice of selling or buying a unit of a risky asset indicated as usual by $S_i = -1$ and $S_i = +1$ or just waiting, $S_i = 0$. Although decisions ultimately hinge on a probability distribution, there are two major elements that have a strong influence on the behaviour of the agents. The first is the division of all traders into two groups, chartists and fundamentalists. Chartists do not care about any extra information from outside the system while fundamentalists are influenced by what is taken to represent news about the traded asset. This difference will constitute different behaviour.

7.4.1.1 The Behaviour of Chartists

Chartists are generally viewed as trend chasers who believe in technical analysis as a superior tool for making excess profits. However, as the questionnaires of Menkhoff (1998) show, real acting agents base their decision on a diversified spectrum of information sources. They try to exploit price trend but they do not neglect the opinions of others and, interestingly, they also have a good command of fundamental analysis. This non-uniform behaviour will be considered when it comes to model the fundamentalists. Though, in order to keep a clear distinction of both trader groups, the noise traders are restricted to their basic approach. In this simulation, people of the chartist group take care of two signals: one coming from the local environment, the other is the overall market or global trend.

The global trend is the same for all agents as it is simply the averaged aggregated decisions of all members. It is calculated by the averaged sum over all traders,

$$\frac{1}{N}\sum_{i}^{N} S_i.^{40} \tag{7.68}$$

[39] The following simulation is programmed with Perl, a script language, whose interpreter is freely downloadable on http://www.perl.com/language/misc/Artistic.html

[40] In order to keep the proportion to the influence of the neighbours, this global trend is

The local component is expressed through the by now familiar form of the $j = 4$ nearest neighbours of agent i, i.e.

$$\sum_{j \neq i}^{4} J_{ij} S_j. \tag{7.69}$$

To keep things as simple as possible, J_{ij}, the parameter of connection, is equal among all four neighbouring traders and thus $J_{ij} = J \; \forall \; j$. Hence, there is no big player or pundit in the market that has a stronger impact on neighbour agents than others.

Since both factors are the sole sources of information for the technical traders, it is only a matter of weighting which of the two dominates. The opinion composed of both technical components is then written as

$$H_i^c = aJ \sum_{i \neq j} S_j + (1-a) \frac{1}{N} \sum_{i}^{N} S_i, \tag{7.70}$$

where a and $(1-a)$ are the weights and the superscript c in the Hamiltonian H_i^c indicates the presence of a chartist. In accordance with the other Ising models, a function is then defined that agents minimise:

$$E_i^c = -S_i H_i^c = -S_i \left[aJ \sum_{i \neq j} S_j + (1-a) \frac{1}{N} \sum_{i}^{N} S_i \right]. \tag{7.71}$$

The interpretation is straightforward: no one wants to be isolated in his view of the market. If global as well as local sources indicate an optimistic attitude towards the equity of interest, it may be a good advise to follow the overall mood instead of trying to lean against the wind. However, a market with all traders using the same technical analysis only is not a realistic picture. First of all, technical trading is not characterised by a uniform rule but provides many different advises how to act. Moreover, history rarely witnessed even small time periods with all market participants sharing the same opinion. And this would, moreover, produce situations with no trading volume at all. It is hence a necessity to introduce traders who use other information as a guide for their decisions.

7.4.1.2 The Behaviour of Fundamentalists

Picking up the point that traders use different sources of information, members of this group are not stuck to a single factor when building their opinions. Hence, fundamentalist's strategy will have technical components too. Yet, to

averaged. Otherwise, situations where the sum over all spins is much larger than the neighbour or fundamental influence would occur too frequently, thereby reducing the influence of the other factors to almost zero.

justify the term fundamental in an economic sense, a component must be introduced that has a relation to the value of the traded asset. Within this framework, the fundamental component is chosen to be a random element ε that represents incoming new fundamental information about the traded asset. ε_t is an unpredictable element that regularly comes into the market but is only noticed by the fundamentalists. This news component is modelled as an *iid* process with zero mean and unit variance. With such a news term, results that do not show a normal distribution cannot be traced back to the introduction of this external component. The Hamiltonian for the fundamentalists has this news elements incorporated; the noise trading component is denoted by $H_i^{C,F}$ in order to indicate the similarity to the opinion-building block of the chartists:

$$H_i^F = b(\varepsilon_t) + (1-b)H_i^{C,F}, \tag{7.72}$$

where the weights of $H_i^{C,F}$ according to (7.70) are set to $a = (1-a) = \frac{1}{2}$ for simplicity. It is easy to see that ε_t can work against an overall trend. E.g., if the market is pessimistic ($H_i^{C,F} < 0$), good news, $\varepsilon_t > 0$, may be even larger and hence drive the trader to buy and not to sell like the noise-traders. This information component can also be seen as a private signal that is only received by some of the market participants. Then through the nearest neighbour interaction, the buying of fundamentalists may influence others to buy so that the informational content of ε spreads out into the market. It is possible to reduce the factors of influence for fundamentalists to only the news component. In this case, they would be totally uncoupled from the market trend. The simulation allows for this by setting b equal to 1. Determining weights for both components in (7.72) lies in the hands of the researcher. However, the incorporation of $H_i^{C,F}$ even for the fundamentalists is a more flexible approach and is in line with reality.

There is a last component added to the description of the fundamentalist behaviour which is a more tender point to justify. Similar to the Bornholdt model of paragraph 7.3.3, the opinion function incorporates a minority term that indicates a non-conformity with the overall trend:

$$\xi_i \left| N^{-1} \sum_i^N S_{i,t} \right|, \tag{7.73}$$

where $\xi_i = \xi < 0$ for all i measures the strength of the reaction. It is worth noting how different (7.73) works compared with the news term. The release of important new information might be in contrast to the local and/or global trend, but it is equally probable to point in the same direction and so enforcing the cooperative behaviour. A counter-reaction to the latest market development is something different because it deliberately acts against the majority opinion despite their direction.

One may dismiss this kind of anti-behaviour as not being compatible with a fundamental view.[41] This is not an easy problem to deal with since theory demands a unique interest in the intrinsic evaluation of an equity and nothing else. However, there are some arguments for the inclusion of the term. The first is the one taken on by a special branch of econophysics, the minority games. This point was already mentioned in the review of the Bornholdt simulation. Its essential message is that those who happen to be in the minority will win. Thus, if the majority opts for a buy order, the winning strategy is selling. It is doubtful that such an austere modelling suffices to describe investors behaviour, but the number of papers applying minority games has grown immensely in the last years. In the present simulation it is only an additional factor and thus does not determine the behaviour completely. However, as the results will show, it has an immense influence.

The second and more economical based way of explaining (7.73) draws from the literature on rational-bubbles. Here, assets prices are determined by two elements: the fundamental value and the rational bubble. The first is defined by an exponentially weighted sum of present and expected future values of all important fundamental elements. One may call this the true fundamental or intrinsic value of the asset. The second term is a weighted sum of all expected future asset price increases and is regularly ruled out by assuming that the sum converges to zero, which is referred to as a transversality condition. However, this is often just an assumption without economic justification. If it does not hold, the asset price has an infinite number of different possible values. Some bubbles have the unpleasant implication of an asset price that diverges from the intrinsic value forever and so create an everlasting rise of prices. In order to circumvent such implausible behaviour, Blanchard (1979) and Blanchard and Watson (1982) introduced the so-called stochastic bubbles. These bubbles have the more realistic feature of having a positive probability to burst. Very interestingly, the probabilities can be linked to factors such as the length of time the bubble has lasted or the amount of mispricing, i.e. *the difference between the actual asset price and its fundamental value.* Therefore, (7.73) should be amended by a term that can account for such considerations. With a slight modification, (7.73) now becomes

$$\xi|\Delta P_t - \varepsilon_t|, \tag{7.74}$$

where the price changes, ΔP_t, are assumed to be proportional to $\sum S_i$, that is

$$\Delta P_t = N^{-1}\left(\sum_i^N S_{i,t} - \sum_i^N S_{i,t-1}\right).^{42} \tag{7.75}$$

[41] It should be remembered that in the Bornholdt simulation such an element of anti-reaction suffices to call them fundamentalists.

[42] There is a priori no clear cut way how to define the price in this simulation as mentioned in section 7.1.1. The most important requirement is surely that prices should go up when demand exeeds supply, which is met by (7.75). The normalisation is added in order to leave both factors in (7.75) with approximately the same strength as $\Delta P \in [-1, 1]$ can be interpreted as a probability just like ε.

The idea behind this anti-reaction term is to use the information about the difference between markets price development and fundamental news. But in the case of $\Delta P_t - \varepsilon_t$ being large, the market must have mispriced the information and so induces fundamentalists to trade against it. If this difference is large, fundamentalists, although being also influenced by others, are more inclined to act against the trend. It should be noted that the news term alone cannot account for this behaviour.[43] H_i^F for fundamentalists with this additional feature now looks like

$$H_i^F = b(\varepsilon_t) + (1-b)(H_i^{C,F}) + \xi \left|\Delta P_t - \varepsilon_t\right|, \text{ or} \tag{7.76}$$

$$H_i^F = b(\varepsilon_t) + (1-b)(H_i^{C,F}) + \xi \left| N^{-1} \sum_i^N S_{i,t} \right|. \tag{7.77}$$

The disagreements functions are then given by

$$E_i^F = -S_i H_i^F = -S_i \left[b(\varepsilon_t) + (1-b)(H_i^{C,F}) + \xi \left|\Delta P_t - \varepsilon_t\right| \right], \tag{7.78}$$

or in the other case

$$E_i^F = -S_i H_i^F = -S_i [b(\varepsilon_t) + (1-b)(H_i^{C,F}) + \xi \left| N^{-1} \sum_i^N S_{i,t} \right|].^{44} \tag{7.79}$$

7.4.1.3 Strategy Changes

The second crucial point for the behaviour of each agent is the determination of strategy-changes. In the first version, ratios for noise and fundamental traders will be fixed by the researcher. The more endogenously designed versions will allow for transitions between the two groups. As this model does not feature a portfolio that can be maximised (and whose development can be taken as a sign of a good strategy), another proxy for evaluating the success of a particular strategy has to be found.

Considering a time period of say 30 runs. During this time each agent will find himself, at least sometimes, in a situation where he regrets his decision. For example, he might have sold assets for 4 consecutive runs (because of neighbour influence), but the market price has constantly gone up during that time. Surely, everyone would be disappointed not to have waited for the first downward shift in P_t (to be more precise, the selling should have occurred shortly before the

[43] One may object the double consideration of the news term in (7.76). However, this counter-reaction explicitely takes the difference of the price development and the news into account. The buy and sell decision is therefore not only motivated by ε_t alone but also by the latest size of the mispricing.

[44] The simulation is performed with both variants.

drop). In order to account for such feelings, a simple mechanism is designed to mirror this regret. Each agent has an account that keeps track of the following process

$$S_{i,t}\Delta P_t + S_{i,t+1}\Delta P_{t+1} + ... + S_{i,_{t+n}}\Delta P_{t+n}. \tag{7.80}$$

A maximum is achieved whenever S_i and ΔP coincide in sign for all t. Because ΔP is calculated after all agents have decided upon their S_i, (7.80) can be interpreted as a process that captures the forecasting ability of a strategy. For example, if $S_i = -1$ and $-\Delta P$ is large, agent i has avoided a huge drop in the value of his (imaginary) portfolio. On the other hand, for ΔP being positive and large and $S_i = +1$, the same agent bought the asset before its value has risen substantially, which would have been, ex post, the right decision. It should also be noted that minor errors cannot account for major regrets, i.e. a situation with $S_i = +1$ and $-\Delta P$ very small is close to being almost ignored by the trader. With this argumentation, the strategy with the highest value signals the most trustworthy concept. As a consequence, traders whose account is significantly lower than those of traders with the rivalry strategy cease from following their own believes, convictions and considerations and transpose e.g. from a fundamentalist to a noise-trader. The general idea behind this concept is easiest revealed by considering extreme market situations. For example, in a market situation that is prone to trending, technical trading may generate more profits than a fundamental trading; at least as long as the bubble does not burst. In these situations, the noise-trader account should deliver relatively high positive values compared to the fundamentalist's one. It is thus natural to assume a transition from a fundamental strategy to a technical one during such market conditions. The economic background can be explained with the help of several chartists-fundamentalists models like Frankel and Froot (1990) and DeLong et al. (1990). There, it is suggested that even fundamentally orientated traders will engage in technical analysis in times where noise trading dominates the price dynamics.

The remaining question is with whom the individual agents should be compared. Candidates could be the next (right) neighbour, an average over the nearest neighbours or a randomly selected trader. The problem with the second proposition is that those agents must a) all have the same strategy and b) this cannot be the same strategy as that of the agent itself. In principal, the same caveat appears to be valid for the first and third possibility. For example, in a case of an extremely uniform distribution of traders, say noise-traders, it is in some cases impossible to find a fundamentalist to which portfolios can be compared. Therefore, and for the sake of simplicity, the artificial market has two "blind" agents added. These are termed *virtual fundamentalist* and *virtual noise-trader*, because they do not appear as individual spins on the lattice, but as a kind of *shadow traders* with all other features being equal to the "real" traders. Hence they act in the same way as those and will consequently have an account that is similar to them. This account serves as the criterion for strategy changes.

The dynamics of the system are governed by the same principals as in all Ising modells, i.e. when it comes to choosing between the three possible actions, the Gibbs distribution (in particular the Boltzmann distribution of (7.28)) is again the decisive mechanism for determining the value of an individual trader. Consequently a change of the individual state occurs whenever

$$e^{-\Delta E_i/\beta} > rn.$$[45]

For convenience, $\beta = k_B T$, where the Boltzman constant is fixed at $k_B = 1$, so that $T = \beta$.[46]

7.4.2 Results Variation A

In this variation, ratios of noise traders and fundamentalists are given exogenously, so there is no opportunity for traders to change their strategy. The interconnections are thus reduced to the global and local couplings. On a first stage, no fundamentalists are allowed to enter the market. The market consists of noise traders only and therefore price dynamics are exclusively driven by intrinsic factors. It is clear that the complete exclusion of exogenous elements is a severe neglect of an essential factor. However, the outcome of this rather unrealistic situation may be seen as a first benchmark. Figures 7.17 and 7.16 show two of the three typical equilibria that emerge with only noise-traders in the market.

The first shows an outcome that would pertain in a situation where all agents have minimised their disagreement, i.e. all of the investors make decisions only depending on what the others are doing. In these cases, the market ends up in one of two possible states, where all traders either buy or sell. Here, prices cannot go up or down any further because no trade happens.[47] Hence price fluctuations are, after a transition time until the final equilibrium stage is reached, zero. The third equilibrium is obtained by increasing T so much, that the connections are detached and become investors interdependent of each other. In this situation, price fluctuations are mild around zero and no clear market trend emerges. As a consequence, technical elements cannot display its influence. This situations produces a random series for the price changes that is very similar to a BM as it does not posses fat tails nor any kind of autocorrelation (see figure 7.18).

The next step includes fundamentalists at low ratios of 5%−15% of all agents. Figure 7.19 shows more realistic looking price changes. This series is obtained

[46] T represens the temperature in physical terms. Because of the three states, a preselection in the comparision of two states (the old and the new) has to be undertaken. First, one of the alternative states is chosen randomly with each having the same probability to be selected. The chosen state value is then taken to calculated the difference of old and new disagreement, ΔE_i, which in turn is used to calculate the probability to change.

[47] This is an interpretation for real markets. In such unrealistic situations, there are no counterparts that want to sell any assets. However, this interpretation is strictly taken not adequate for the simulation because there, no explicite trading process is modelled.

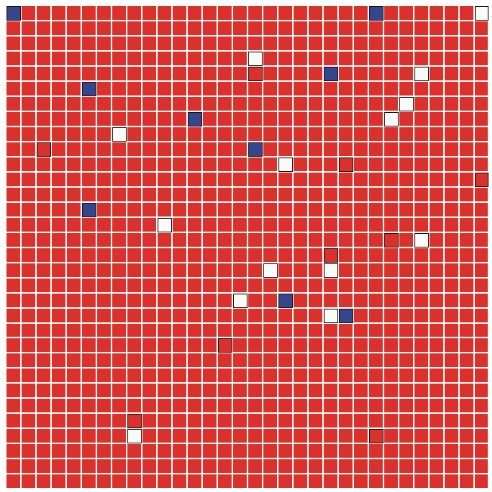

Figure 7.16: Situation of an extremely bullish market. Only few (fundamentalistic orientated) traders do not buy. The blue squares represent sell orders, red squares buy orders and white squares stand for holding. The framed squares mark the fundamentalists. All noise-traders are aligned to the buying strategy.

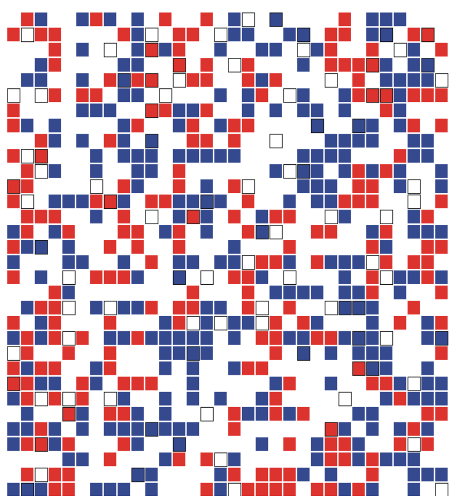

Figure 7.17: Snapshot of the simulation during a complete run.

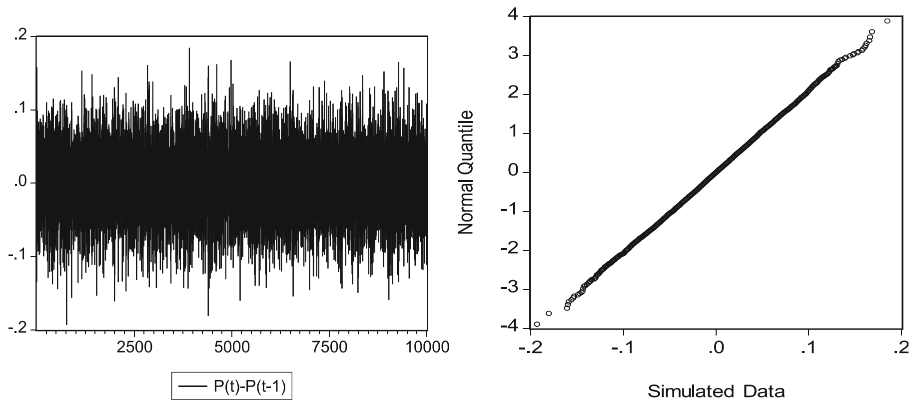

Figure 7.18: Price differences for the case of T being high. The qq-plot indicates almost normality.

with $T = 0.8$, $\xi = 3$, and a counter-reaction term for the fundamentalists as in (7.76).[48] All of the following results are obtained with this set-up, but the outcome does not depend on the choice of (7.76) or (7.77), as the second variant produces qualitatively the same artificial price record.

The difference to the first outcomes is obvious: periods of mild fluctuations are irregularly interrupted by huge price changes and the similarity to real time series is striking. Compared with the figures of the time series in part two, it is hard to discern the simulated price record from real data sets. Interestingly, the outcome does not depend on the functional form of the counter-reaction term, i.e. both (7.76) and (7.77) result in the (qualitatively) same time series. The presented series is simulated with (7.76). Of course, visual inspection does not suffice. So, in order to prove the correspondence of real and simulated data, the statistical tools of part two must be applied. The working hypothesis is that the data has the same statistical features as the time series of real financial markets. The associated qq-plot in the right of figure 7.19 confirms the inappropriateness of a Gaussian distribution. It can be seen that the tails of the data are too heavy to be in the range of a normal distribution.

But, how heavy are they exactly? Table 7.1 displays the values of interest as estimated by three parametric methods, namely the Nolan, the Koutrouvelis and the McCulloch estimator. In order to shed some light on the sensitivity of the results to variations, estimation values for different ratios and parameter settings are also displayed. Additionally, the result for a high counter-reaction term is given. Here, estimated values are close to the value for a Gaussian

[48] The higherthe values of ξ are, the quicker is the counterreaction of the fundamentalists. If the parameter is too high, no bubbles should emerge because the slightest deviations from a pure random behaviour of ΔP induces a trading against this trend for all fundamentalists. The statistical tests will show the correctnes of this conjecture.

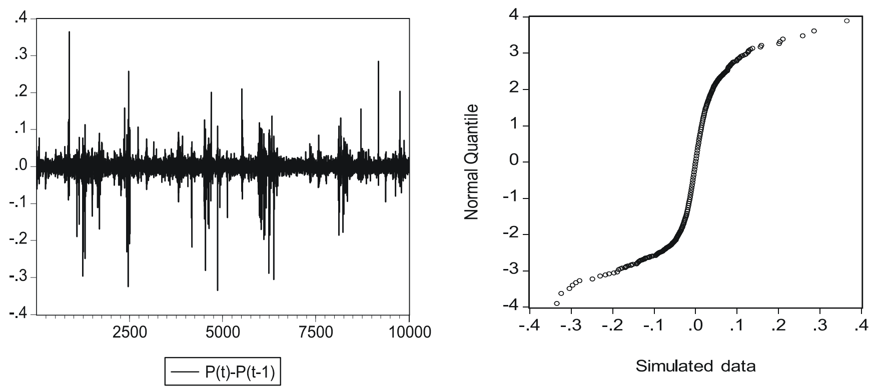

Figure 7.19: Left panel: Price changes obtained from variation A. Right panel: The associated qq-plot.

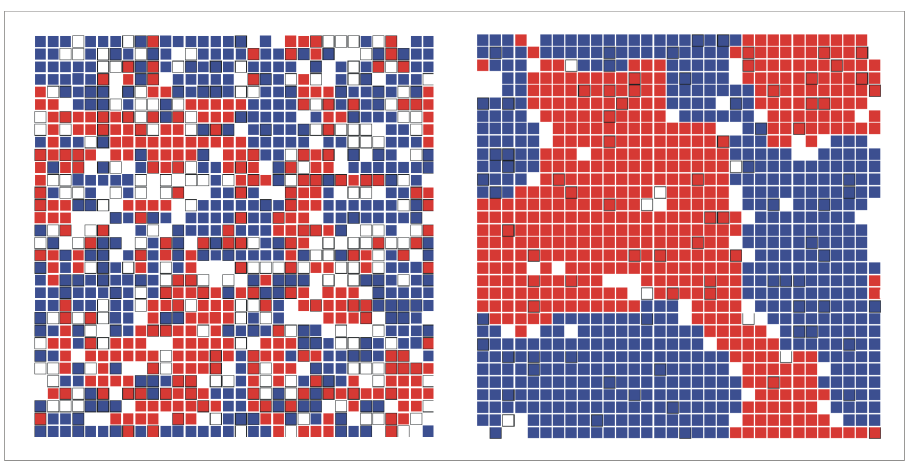

Figure 7.20: Ilustration of a typical development during a simulation. The first picture shows an almost random distribution with only small regional clusters. The second is achieved from a situations a few runs later. Now large cluster have built, but still no side is dominating.

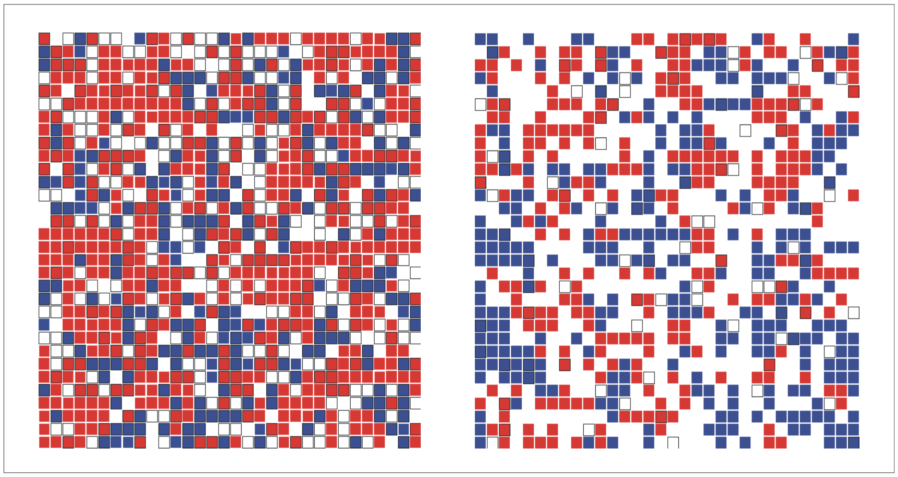

Figure 7.21: In the left picture, buyers have gained the majority. With a sufficiently high ξ this situation induces counter reactions by the fundamentalists. The right pictures shows the situation a couple of runs after the right configuration. The bubble is brought down and former buyers have become sellers by now.

($\alpha = 2$). This exemplifies that a high value of ξ prevents the emergence of large bubbles, which is easy to explain: with ξ being high, even slight deviations from minor fluctuations induce fundamentalists to counter-react, thus suppressing any forms of excessive price movements.

Table 7.1: Parametric estimates for the tail parameter

Series simulated with	Koutrouvelis	McCulloch	Nolan
5% Fund. and $\xi = 3$	1.3	1.365	1.363
5% Fund. and $\xi = 5$	1.411	1.362	1.359
10% Fund. and $\xi = 3$	1.38	1.443	1.437
10% Fund. and $\xi = 5$	1.399	1.471	1.477
15% Fund. and $\xi = 3$	1.594	1.601	1.64
15% Fund. and $\xi = 5$	1.53	1.6	1.607
15% Fund. and $\xi = 35$	1.94	1.911	1.932

All estimators (except the one for $\xi = 35$) show a value for the index of stability well below 2. This would indicate the presence of a stable distribution with infinite second moment. However, all of the estimators are bound to the stable hypothesis and thus cannot detect whether the distributions may have fatter tails than a Gaussian but nevertheless also have finite second moments. Because data limitations do not play a role for simulations, the method of cumulative probabilities can now be used with more confidence than for the daily records.

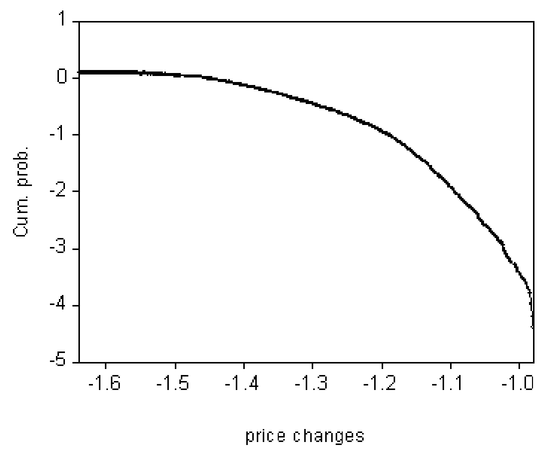

Figure 7.22: Cumulative probability for all (absolute) values of ΔP as obtained by the simulation that leads to figure 7.19. Estimation is performed only with the 1% highest values, i.e the part of the function to the right, where the (negative) slope is steepest. As can be seen, this range is consistent with a power-law, where the slope gives the estimation for the exponent.

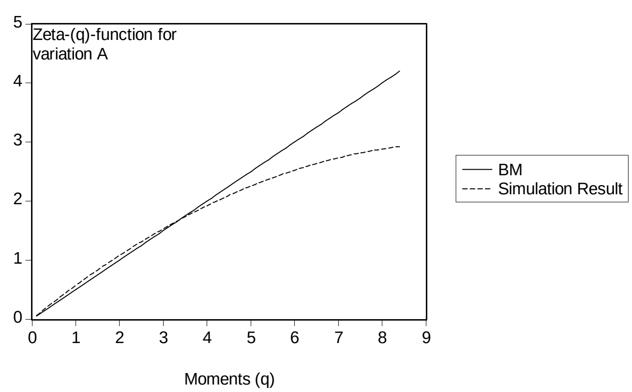

Figure 7.23: Zeta-(q)-function of variation A, obtained with 10% Fundamentalists and $\xi = 5$.

The estimation of the (negative) slope for the 1% highest price changes from the data underlying figure 7.22 gives a value of -3.24, which is properly in the range of the values estimated in the own empirical analysis. Price change distributions thus follow a power law with exponents around 3 for all cases. Even the simulation with $\xi = 35$ has an estimated exponent above 2. In addition, the Hill estimator is also applied. Its results indicate finite moments up to the fourth order, again confirming the literature and the own estimations of part two.

Table 7.3: Tail estimation by the cum. method

Series simulated with	left	right
5% Fund. and $\xi = 3$	3.24	3.29
5% Fund. and $\xi = 5$	3.19	2.99
10% Fund. and $\xi = 3$	3.51	3.4
10% Fund. and $\xi = 5$	3.02	2.84
15% Fund. and $\xi = 3$	2.94	2.7
15% Fund. and $\xi = 5$	2.3	2.16

Table 7.4: Results for the Hill estimator

Series simulated with	p=0.01	p=0.025	p=0.05	p=0.1
5% Fund. and $\xi = 3$	4.21	3.817	3.001	2.68
5% Fund. and $\xi = 3$	4.06	3.903	2.88	2.281
10% Fund. and $\xi = 3$	3.89	3.5	3.093	2.48
10% Fund. and $\xi = 3$	3.8	3.57	3.28	2.42
15% Fund. and $\xi = 3$	3.73	3.33	3.1	2.673
15% Fund. and $\xi = 3$	2.13	2.21	2.4	2.075

The next point concerns the multiscaling characteristics of the simulated time series. This is a crucial point for many pure statistical approaches to financial data. Often, fat tails and long-memory phenomena can be reproduced. But the additional multiscaling feature is much harder to achieve. Figure 7.23 shows the zeta-(q)-function of the simulated data.

The function is non-linear and it thus rejects a simple BM hypothesis as well as a fBM, whereas it fits quite well into the examples of part two. The simulation thus reproduces this specific feature of real price records well and the accordance with the zeta-(q)-functions for their data is clearly visible. This outcome confirms the assumption that the simulation has a mechanism built in that cannot be too different from the one working in real stock markets.

Next, the characteristics of the temporal dependencies between successive price changes are investigated. Empirical investigations have shown a significant value for the autocorrelation at very short time scales but with a rapid decay towards zero, too. Figure 7.24 displays the corresponding changes of the simulation for the first 36 lags.

The concordance with the results for the high-frequency data is obvious: except of the first lag who has a relatively large non-zero value for $\rho(k)$, no longer lasting dependencies can be found. Hence, this artificial stock market produces

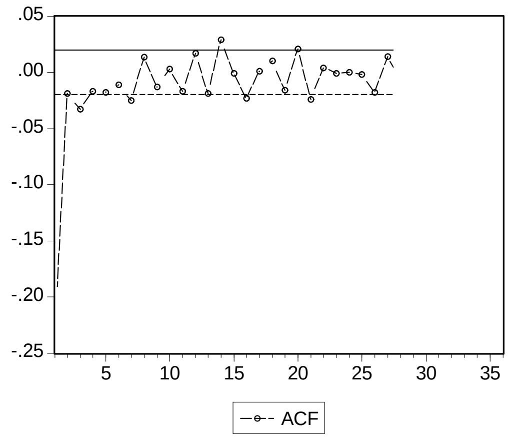

Figure 7.24: Autocorrelation function of the simulated series of price differences. Remarkable is the similarity to the empirical findings of a relatively large negative value for the first lag.

independent price developments although connections between neighbours can last for longer periods. Especially in phases when bubbles start to build and small regional clusters grow, traders act fairly in the same way. Nevertheless, ΔP is (with the exception of the first lag) not autocorrelated.[49]

The last point concerns the long-run characteristics of the simulated time series. While long time correlations are expected to be present in the volatility, they should not be found in the raw returns. However, the last point is not that clear cut when considering the mixed results the literature. Taking first raw returns as calculated by the logarithm of the first price difference, table 7.5 displays the results of the modified R/S statistic, and the GPH estimator.

[49] This is far from clear when only the model set-up is considered. However, physicists know that Ising models produce independent magnetisation increments.

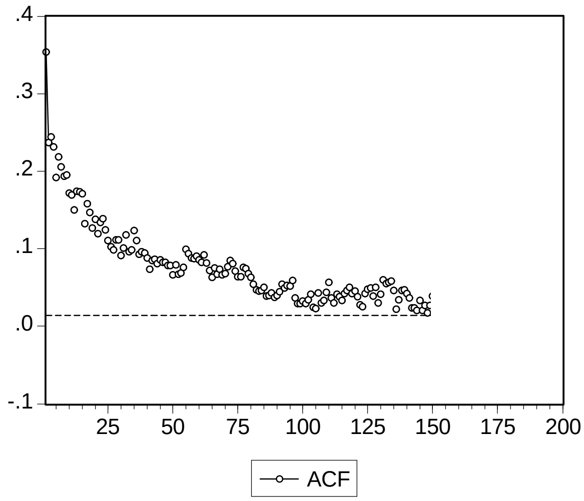

Figure 7.25: Autocorrelation for the absolut price changes $|\Delta P|$.

Table 7.5: Results of R/S analysis and GDP estimates for raw returns[50]

Series simulated with	R/S	GPH
5% Fund. and $\xi = 3$	0.701	0.1552 (1.551)
5% Fund. and $\xi = 5$	0.683	0.1433 (1.079)
10% Fund. and $\xi = 3$	0.71	0.187 (1.32)
10% Fund. and $\xi = 5$	0.69	0.179 (1.66)
15% Fund. and $\xi = 3$	0.68	0.16 (1.395)
15% Fund. and $\xi = 5$	0.7	0.162 (0.99)
15% Fund. and $\xi = 35$	0.489	-0.03 (0.682)

As for the empirical data, the results show no signs of long memory. This picture changes when absolute returns as a measure of volatility are considered. As a first indication, the autocorrelation for the first 200 runs is plotted below.

The simple autocorrelation is not a tool designed to detect long memory, but the two more sophisticated statistical methods from chapter 6.1 do not contradict this first impression. Both estimators reject the null hypothesis of no long memory.

[50] Asteriks indicate significance at the 5% significance value. Values in brackets are again the t-values for the null hypothesis $H_0 = d = 0$.

Table 7.6: Results of R/S analysis and GDP estimates for absolute returns

Series simulated with	R/S	GPH
5% Fund. and $\xi = 3$	0.619^*	0.153 (5.237)
5% Fund. and $\xi = 5$	0.701^*	0.2 (7.241)
10% Fund. and $\xi = 3$	0.673^*	0.15 (4.23)
10% Fund. and $\xi = 5$	0.682^*	0.192 (5.09)
15% Fund. and $\xi = 3$	0.721^*	0.179 (3.77)
15% Fund. and $\xi = 5$	0.708^*	0.182 (3.92)
15% Fund. and $\xi = 35$	0.55	0.02 (1.229)

With the exception of the variation with $\xi = 35$, all other set-ups show significant values of $\hat{H}$ other than 0.5 and $\hat{d}$ other than zero. The estimated values itself are quite close to the results for the empirical data sets.

In order to demonstrate the role of ratios a simulation is also performed with 70% of fundamentalists and $\xi = 5$. The results are close to the situation where the traders became detached because of a high value in the parameter T. Tail estimators are in all cases near two and the autocorrelation structure is also close to white noise. Only the estimated zeta-(q)-function displays a very mild non-linearity. However, the outcome in this variation does not show severe departures from a simple random walk. The reason for this is easy to explain: it is the high percentage of fundamentalists that is responsible for the non-existence of larger and longer lasting price trends, because they act against the majority whenever price differences differ substantially from incoming news (ε). However, the exact ratio which produces such results is hard to determine because the counter-reaction parameter ξ plays also a crucial role, but generally 40% fundamentalists suffice to suppress major price fluctuations irrespectable of ξ, though, not independent of T. As in all cases, T above the critical value produces normally distributed price changes.

There is a last, but very crucial point to mention. In order to test the importance of the counter-reaction term, a simulation is also performed with $\xi = 0$, so that fundamentalists now do only act against a trend when the private news signal exceeds his technical component by a wide enough margin. Theoretically, this alone should suffice to ensure that bubbles burst. However, as it turns out, this is not the case. With a low ratio of fundamentalists, large clusters remain large for very long time horizons. On the other hand, with a high percentage of fundamentalists, market situations change randomly without dominating price trends. This result confirms the conclusion from above (many fundamentalists prevent bubbles and crashes), but the exclusion of a minority term leads to non-realistic time series that do not show large intermittent phases with high turbulences. It seems to be the case that this component is vital for the system, because it ensures that at one point - which depends on the strength of the counter-reaction and the number of fundamentalists - an overall market trend can suddenly be reversed. Only a fraction of all traders is needed to let a bubble burst. The actions of those traders influence their environments to follow them. In the beginning, just a few may share the fundamentalists opinion, but soon more traders join the sceptical view. And if the trend once is broken, noise-trader may change their former succesful strategy and now also become sellers.

7.4.3 Results Variation B

This variation allows for a wide range of endogenously determined parameters. First, there is no fixed ratio of chartists and fundamentalists. All traders regularly check their strategy as explained above, i.e. they calculate the success of their past decisions according to (7.80). And if it happens to be the case that another strategy performed better in the latest n runs, they will adopt that strategy.[51] Furthermore, the influence of the nearest neighbours and the market trend is not fixed anymore. The idea now is to let the traders, who observe local and global trends with the same accurateness, weight these technical signals according to their "definiteness". It seems to be quite natural to suppose that people who are surrounded only by sellers will still hesitate to follow them if at the same time the overall market unmistakably points towards a buy. This factor works already in variation A, but here its influence depends on the strength of that signal; hence, the longer and more clearly the market trend is, the higher the weight gets in (7.70) and (7.76). Though, in order to maintain the structure of the Ising model, the weight of the influence of the nearest neighbours cannot be reduced to values below 0.5. In a similar fashion, the weights of the fundamental opinion are made flexible. The weights in H_i^F are now dependent on the strength of ε_t and the noisy component. If the fundamental news are high, then local and global trends will play a less important role in building the opinion. But if both factors together are high and coincide so that they point in the same action, then the technical elements is fostered.

It is important to note that the endogenity of the weights produce a strengthening of the prevailing situation: periods with a clear direction, where almost every agent chooses the same option, are now designed to enforce opinions to share this view since exactly these weights are driven up. Moreover, the possibility of a transition from a fundamental to a technical strategy reduces the number of traders that are able to bring the market trend down. This is because in times of a high $N^{-1}\sum S_i$, fundamental trading with a changing news component misses to be highly correlated with ΔP_t. Noisy traders instead have this high concordance and thus also a higher account. So if the global price development does not change its direction for a longer period, agents abandon their fundamental attitude in favour of a technical one which in turn induces them to enforce the trend even more.

In general, these tendencies do not play a major role because trends are eventually brought down by the remaining fundamentalists, as long as there are enough of them. However, occasionally the simulation ends up in an unrealistic scenario where almost all agents have adopted the noise-trader strategy

[51] n has to be fixed exogenously which is in this case $n = 50$ runs.

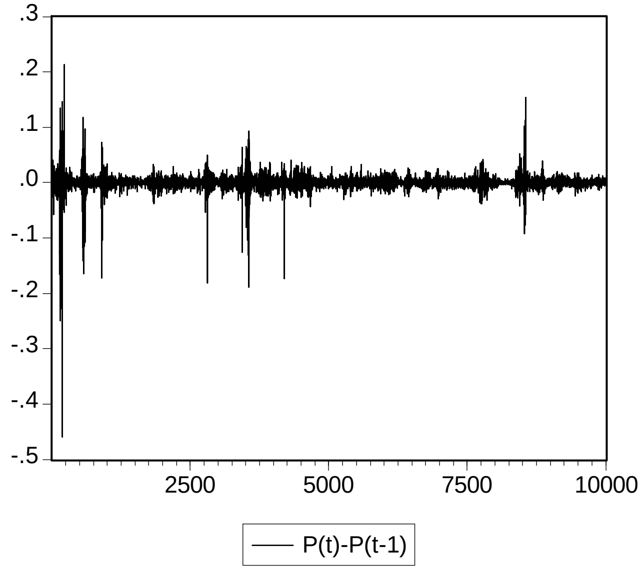

Figure 7.26: First price differences of a typical simulated series of variation B with $\xi = 5$ and $T = 1.0$.

and the market reaches an equilibrium with either all selling or buying. Due to this potential failure, a minor modification is designed to get back to the desired intermittent outcome. Because it is vital to have a sufficient amount of fundamentalists in the market, a further variation now allows for a minimum percentage of those traders. Now the price fluctuations always display the familiar pattern. Figure 7.26 shows a typical time series for ΔP in this variation. The result is obtained with a minimum of 5% of all traders being fundamentalists and a counter-reaction term $\xi = 3$. This $\xi-$value is applied to all following simulations.[52]

Statistical inferences agree with the findings of variation A. The next table gives the values for the index of stability as estimated by the parametric estimators. Results are obtained with a) a minimum of 5% of all traders that always follow the fundamental strategy and b) a minimum of 15% of all traders that always follow the fundamental strategy. The inclusion of variation b) is done to check the simulation for robustness. As can be seen, the differences in the outcome are not substantial.

[52] The critical temperature for this variation has slightly increased to 1.0.

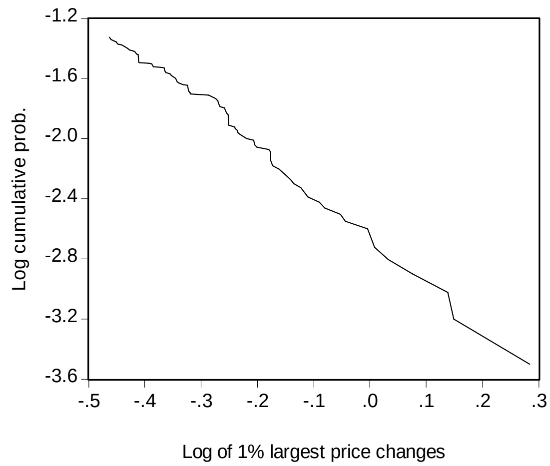

Figure 7.27: Cumulative probabilty of the 1% highest price changes on a log-log scale. The function is obtained from the highest positive price changes.

Table 7.7: Parametric estimations of the tail parameter

Series simulated with	Koutrouvelis	McCulloch	Nolan's MLE
Min. 5% fund. traders	1.889	1.846	1.849
Min. 15% fund. traders	1.804	1.837	1.828

Again, all estimations show a value less than 2, but the cumulative method and the Hill estimator both indicate a distribution with finite second moments. The plot of the empirical cumulative distributions of price fluctuations is consistent with a power-law, i.e. $P(\Delta P) \sim \Delta P^{-\alpha}$ with exponent $\alpha \approx 3$ for both variants. Figure 7.27 demonstrates that this variation reproduces the typical feature of a power-law behaviour for the cumulative probabilities of the (1%) largest values for ΔP.[53] The figure is similar to figure 4.5.

Table 7.8: Tail estimation by the cumulative method

Series simulated with	left tail	right tail
Min. 5% fund.traders	3.24	3.29
Min. 15% fund.traders	3.19	2.99

Table 7.9: Tail estimation by the Hill estimator

Series simulated with	p=0.01	p=0.025	p=0.05	p=0.1
Min. 5% fund.traders	4.21	3.817	3.001	2.3 8
Min. 15% fund.traders	4.06	3.903	2.88	2.44

[53] To be precise, the power-law has to written as $P(\Delta P > X) \sim \Delta P^{-\alpha}$, where X is some threshold value for the price changes, because estimation is only performed with the largest values and not all values of ΔP.

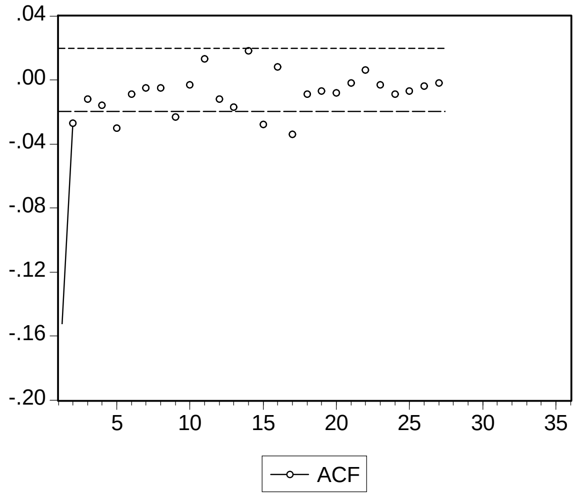

Figure 7.28: Autocorrelation of the price differences for variation B.

This accordance with variation A extends to the autocorrelations as well as the long memory properties. The lagged autocorrelation function for the first 36 price differences has again significant (negative) values only for the first 2 lags.

The long memory analysis again confirms the empirical literature: raw values for returns do not show significant deviations from the null hypothesis of an ordinary BM, but the absolute values as a measure of volatility all indicate the presence of a long range dependency.

Table 7.10: Long memory analysis for raw and absolute price changes

Raw Returns	R/S	GPH
Min. 5% fund. traders	0.688	0.0711 (1.0336)
Min. 15% fund. traders	0.557	0.021 (0.755)
Absolute Returns		
Min. 5% fund. traders	0.709*	0.1993 (3.719)
Min. 15% fund .traders	0.731*	0.1847 (6.5528)

The outcome of this more endogenously designed variation is qualitatively not different from variation A. Allowing for more flexibility in the weights of the opinion function as well as strategy changes and thus a transmission between the different trader groups does not alter the results substantially. This is a conclusion worth mentioning as variation A already showed that different variations of fundamentalists counter-reaction does not alter the outcome substantially. Hence, the simulation is robust to minor modifications that do not change the whole structure. In summation, this new Ising simulation is able to reproduce the main features of real stock markets, i.e.

- fat tailed distributions for price changes;
- time series with a multiscaling characteristic;
- short-time autocorrelations that go quickly to zero;
- no long run dependencies in the raw returns, but
- long memory in the volatility process.

7.4.4 First Conclusions

How are the results of stochastic simulations to evaluate? Naturally economists would like to put in more substance. For example, a well specified utility function from which actions are determined, a more detailed modelling of the fundamental element, or the incorporation of microstructure components are candidates for such economic elements. Compared with these requirements the approach of using Ising or percolation models is admittedly very stylised and has many limitations. First, there is no explicit decision process visible that can explain the actions of the traders in more detail, i.e. their exact motivation is still unknown. Second, there are no liquidity constrains for any trader in the market (except in the Iori model). Trend followers and fundamentalists can keep on loading their portfolios regardless of the resources they have already spent. On the other side, selling is unlimited as there is no portfolio that can become empty.

There is no doubt that each of the mentioned points indicates a missing in the analogy between a physical system and financial markets. Atoms are no intelligent people who adjust their behaviour to changing market situations much more flexible than by flipping between only three states. Removing all critical points in favour of a description more based on mainstream economics is possible and would definitely give a more realistic picture of a financial market. However, this overlooks on the one hand the ambition of these models and on the other hand the small informative values of those additional variations. First of all, physicists aim to produce parsimonious agent-based models that are in accordance with the reality. It seems to be that, for them, the more simple the model, the better it is. And it is in fact astonishing how much of the economic structure can be neglected while still reproducing the main stylised facts. On the other hand, too much microeconomic elements can destroy the particular network structure of the Ising-models which is esteemed to mirror one of most important features of financial markets: its self-reinforcing elements. And the fact that all of the main empirical findings are explained with such a simple structure leads to the assumption that a very important element in understanding the workings of stock markets has been discovered.

Chapter 8

Deterministic Simulation Models

Deterministic simulations are characterised by a more thorough and economically based description of artificial markets than the stochastic modellings. There, the decision to buy, hold or sell hinges on a probability function that was derived from principals of statistical physics. Here, the researcher explicitly describes how the decision is formed. This process can be tied to sound economic reasonings like utility theory, approaches of Behavioural Finance or other hypotheses of decision theory. However, differences should not be stretched too far. Both approaches have an inherent tendency to abstract from several attributes of financial markets that many economists may find indispensable. Moreover, deterministic simulations do also include some form of stochastic influence: news are specified in the same way as in the stochastic simulations.

Whereas in the Ising-related simulations the structure of the models is more or less fixed by the underlying physical system, deterministic models leave much more room for variations in the structure. The typical characteristics of deterministic financial market models are quickly outlined and although detailed descriptions are provided below, the next lines briefly summarise some main features. First, decisions of traders are often derived by applying utility theory, though it is not confined to this natural approach. A typical way to do this is to consider agents as investors who maximise expected utility coming from a portfolio of two assets: a risk-free and a risky one. By optimising the portfolio, depending on the evolution of the asset market, agents permanently act in order to reach the highest level of utility. The crucial point is that this optimisation must be done in the presence of an uncertain future. Therefore, expectations have to be formed. This situation is equal to all trader types.

The difference between traders comes into the model in form of heterogenous expectations. Fundamentalists evaluate the future price according to incoming news which are transformed into a new value for the asset by using a fundamental model. Chartists instead will try to recognise hidden trends in order to forecast asset prices. Besides these pure forms of trader types mixed strategies come into play by letting some traders form their expectation by weighting fundamental and technical analysis as before. Overall there are several ways in which trader influence each other. This can be modelled by including the behaviour of other traders as a third source of information. Other types of investors often recognised are portfolio rebalancers who try to keep a fixed ratio of riskless and risky assets, or liquidity traders who sell or buy dependent on financial constraints. Once the different investors have submitted their orders, an explicit price determination process sets in. Here, most models borrow from the known procedures of microeconomics like the Walrasian auctioneer. Typical exogenous factors comprise the modelling of the bond market with its return, the way in which (some) agents try to figure out the intrinsic value of the asset, the determination of the exact form of the utility function, or the number of strategies which can be chosen; the possibilities are numerous in any case. Having set-up the frame, the simulation evolves without any further interference from the researcher, just like in the stochastic models. The next sections review some of the most important models as well as early attempts to simulate financial markets. It should be mentioned that although these latter ones are the first microsimulations, the stochastic simulations have by now outpaced the deterministic approaches in number.

8.1 The Levy, Levy and Solomon Model

The models of Levy, Levy and Solomon (LLS henceforth, see LLS (1994, 1995)) are amongst the first microsimulations of financial markets undertaken with the help of physicists and have by now established themselves as a kind of basis around which a large number of present deterministic microsimulations are designed. This is due to its general platform that offers a wide range of variety. In a pedagogic version of their early works, LLS (2000) present a model in which different types of traders maximise their expected utility by trading two different types of assets: bonds that are assumed to yield a riskless return of r, and a single stock with a return R defined by

$$R_{t+1} = \frac{P_{t+1} + D_t}{P_t}. \tag{8.1}$$

Here D_t is the dividend paid in period t and P_t is as usual the price of the asset. It must be noted that (8.1) is an ex post equation. All investors have the same (power) utility function

$$U(W) = \frac{W^{1-\alpha}}{1-\alpha}, \tag{8.2}$$

where W is the wealth and α is a risk aversion parameter.[1] Wealth grows according to the following equation

$$W_{t+1} = W_t \left[(1-a)(1+r) + aR_{t+1} \right]. \tag{8.3}$$

The weights $(1-a)$ and a determine the partition of wealth into bonds and stocks. If e.g. $a = 0$ is fixed for all t, then W rises each period by the same amount, i.e. the riskless return. In the case of $a > 0$, W_{t+1} is no longer a sure outcome since P_{t+1} in equation (8.1) is not known and thus has to be estimated in t. This is the point where different expectations are incorporated. Traders are assumed to fall into two different groups: fundamentalists who use naturally a fundamental model in order to evaluate the assets and agents who employ some form of technical analysis. The next two sections will show how the two groups of investors exactly decide upon the optimal partition of W_t in order to get the highest (expected) utility.

8.1.1 Fundamentally Based Investors

LLS' choice of letting the utility function take the form of (8.2), thereby implying a constant relative risk aversion is made for two reason. The first is an empirical one: economists found the relative risk aversion to be nearly constant (see e.g. Friend and Blume (1975) or Gordon, Paradis and Rorke (1972)). The second is the simplifying implication that optimising (8.2) is independent of the time horizon (Samuelson, (1989 and 1994)). Thus, maximising expected utility only affords the calculation of the one period ahead wealth

$$\begin{aligned} \max U\left(E\left[W_{t+1}\right]\right) &= EU(W_{t+1}) \\ &= EU\left(W_t\left[(1-a)\left(1+r_{t+1}\right)+aE\left(R_{t+1}\right)\right]\right). \end{aligned} \tag{8.4}$$

The formation of expectations about future prices for the fundamentalists, called rational informed identical (RII) investors, is done with the help of a sensible model that explains how asset prices are build. Here, it is Gordon's dividend stream model that they use to calculate the price of the stock for the period $t+1$. They do this because they see the value of a stock as being determined by the discounted stream of all future dividends. According to the Gordon model, the one period ahead fundamental price of an asset is given by

$$P_{t+1}^F = \frac{E_{t+1}\left[D_{t+2}\right]}{k-g}, \tag{8.5}$$

where the superscript F symbolises the fundamental nature of this expression. The term in the numerator is the expectation of dividends two periods ahead, where the expectations are built in $t+1$. g is the expected growth of dividends and k is a discount factor. Now $E_{t+1}\left[D_{t+2}\right]$ can be built by a combination of g and the realised dividend observed in $t+1$, i.e.

[1] The functional form of (8.2) implies a constant relative risk aversion.

$$E_{t+1}\left[D_{t+2}\right] = D_{t+1}\left(1+g\right). \tag{8.6}$$

Dividends grow according to

$$D_{t+1} = D_t\left(1+z\right), \tag{8.7}$$

where the parameter z reflects the growth rate of dividends. However, z is no deterministic parameter but varies between z_1 and z_2 according to $\int_{z_1}^{z_2} f(z)dz$ and so traders have to form expectations about this parameter, hence

$$E_t\left[D_{t+1}\right] = D_t\left(1+E[z]\right), \tag{8.8}$$

Because of the fundamental nature of these traders, the model assumes $E[z] = \int_{z_1}^{z_2} f(z)dz = g$. Putting (8.6) and (8.8) together results in the following price equation for fundamentalists

$$P_{t+1}^F = \frac{D_t(1+E[z])(1+g)}{k-g} \tag{8.9}$$

Since $E_t\left[R_{t+1}\right]$ enters utility via equation (8.9), one arrives at the following expression for the expected utility

$$\begin{aligned} E_t\left[U\left(W_{t+1}^i\right)\right] &= \frac{\left(W_h^i\right)^{1-\alpha}}{1-\alpha}\int_{z_1}^{z_2}\left[\left(1-a\right)\left(1+r_F\right)+\right. \\ &\quad \left. +a\left(\frac{\frac{D_t(1+z)(1+g)}{k-g}+D_t\left(1+z\right)}{P_h}\right)\right]^{1-\alpha} f\left(z\right)dz. \end{aligned} \tag{8.10}$$

Equation (8.10) incorporates the variables P_h and W_h which stand for the hypothetical price and hypothetical wealth respectively. The reason for the inclusion of preliminary values is straightforward: at time t, investors do not know the equilibrium price of the next period. Thus, he has to calculate various outcomes for various hypothetical equilibrium prices. Only at the end of each round the equilibrium price is determined and $P_h = P_{t+1}$ is then used to form the new weights. LLS simplify equation (8.10) even more by letting z being uniformly distributed in the range $[z_1, z_2]$. They are now able to calculate the expected utility by

$$\begin{aligned} E_t\left[U\left(W_{t+1}^i\right)\right] &= \tfrac{\left(W_h^i\right)^{1-\alpha}}{(1-\alpha)(2-\alpha)}\tfrac{1}{(z_2-z_1)}\left(\tfrac{k-g}{k+1}\right)\tfrac{P_h}{aD_t}\times \\ \times&\left\{\left[\left(1-a\right)\left(1+r_f\right)+\tfrac{a}{P_h}\left(\tfrac{k+1}{k-g}\right)D_t\left(1+z_2\right)\right]^{(2-\alpha)}-\right. \\ -&\left.\left[\left(1-a\right)\left(1+r_f\right)+\tfrac{a}{P_h}\left(\tfrac{k+1}{k-g}\right)D_t\left(1+z_1\right)\right]^{(2-\alpha)}\right\}.[2] \end{aligned} \tag{8.11}$$

[2] For the derivation see LLS (2000).

There are two points worth mentioning in (8.11). The first concerns the possibility of still existing uncertainties. g and k are known parameters and z follows the known function $f(z)$, so there seems to be no uncertainty left. But knowing a distribution function does not mean knowing the exact value. The fact that z is a random variable induces a non reducible element of uncertainty. The second point touches the homogeneity of the fundamentalists. One might wonder if all RII investors really act identically given the same starting value $W^i_{t=0}$. The answer is yes. Given a starting ratio of risky and riskless assets that is the same for all agents and a homogenous risk parameter α, then demand or supply should also be equal among the RII investors.

8.1.2 Non Fundamental Orientated Investors

In this class traders are convinced that the actual price of stocks already reflects the true value. LLS call these trader the Efficient Market Believers (EMBs). Contrary to the RII investors who recognise a false price just by comparing their calculated $E_t\left(P^F_{t+1}\right)$ with the ex post realised price in $t+1$, EMBs never waste time and effort to beat the market. What they do is to use the m latest observations in order to estimate the ex ante distribution of returns:

$$P\left(R_{t+1} = R_{t-j}\right) = \frac{1}{m}. \tag{8.12}$$

This modelling of an EMB as proposed by LLS, however, should be strongly objected. The weak form of the EMH asserts that no excess profits can be made by trading on the information of past price records. One might wonder why people who believe in this hypothesis nevertheless employ a technical analysis. What are they trying to gain? On the other hand, calling these agents non-fundamental traders is justified precisely by this aspect.

After considering (8.12), expected utility of an EMB develops as

$$E_t\left[U\left(W_{t+1}\right)\right] = \frac{W_t^{1-\alpha}}{(1-\alpha)}\sum_{j=1}^{m}\left[(1-a)(1+r) + aR_{t-j}\right]^{1-\alpha}. \tag{8.13}$$

Solving (8.13) should lead to the optimal weighting factor a^*. LLS now introduce a bias in the sense that a^* is missed by a random variable $\varepsilon \overset{iid}{\sim} \left(0, \sigma^2\right)$. Accordingly

$$a = a^* + \varepsilon\ . \tag{8.14}$$

This noise effect is applied both to RII investors as well as to EMBs. Prices are determined so as to equate supply and demand. The supply side is given by a constant N shares offered each period. Demand is calculated through the maximisation of $E_t\left[U\left(W_{t+1}\right)\right]$. After having solved (8.11) and (8.13) for a^*, demand is simply given by

$$N_i\left(P_h\right) = \frac{a\left(P_h\right)W_h\left(P_h\right)}{P_h}\ . \tag{8.15}$$

Because of the difference in strategy, $N_i(P_h)$ constitutes the demand curve for the particular agent i. Aggregated demand is hence given by

$$\sum N_i\left(P_t^E\right) = \sum_i \frac{a_i\left(P_t^E\right) W_t\left(P_t^E\right)}{P_t^E} = N. \tag{8.16}$$

How is the equilibrium price reached? The mechanism starts in a first round by noting the gap between $\sum N_i(P_h)$ and N. If demand exceeds supply, P_h rises. The new price is used to calculate new optimal weights $(1-a^*)$ and a^*. Because $N(P_h)$ is decreasing in P, the gap closes. If excess demand still exists P_t rises again until eventually the market for stocks clears.

8.1.3 The Simulation

After starting values for k, g, P_0, D and W_0 are set, investors submit their orders. Then the process of finding the equilibrium price will need some time to eventually find P_1. This price is used to calculate the new numbers of shares each will hold until the next round. Then, the process starts again with the new orders. LLS now perform four different variants of simulations, i.e.

(i) with only RII are actively trading,

(ii) with RII and a small fraction of EMBs,

(iii) with RII and two different types of EMBs and finally,

(iv) with RII and a spectrum of heterogenous EMBs.

(i) This variation is completely driven by the uniformly distributed values of the dividend growth rate z. Because investors believe in a mean reverting process to P^f, fluctuations in prices are only caused by z. Since the realisations of z are not correlated, prices are also expected to show an uncorrelated random behaviour. In fact, the outcome is a series for the price that steadily goes upwards with no major fluctuations. The autocorrelation is near zero for all lags. Another result is the missing of any excess volatility in the sense of having a higher price fluctuation than the discounted dividend stream.[3] A last feature of this variant is the lack of volume. But this comes as no surprise, since all investors trade on the same information set. All investors come nearly to the same conclusion (only almost because of ε) and thus evaluate the asset equally. Investors want either to buy or sell and so the variation features a small trading volume.

(ii) The second variation introduces a small group of homogenous EMBs, with everything else left unaltered. Although this new subpopulation is only 5% large, price fluctuations now become more pronounced. However, it also shows a cyclical movement that is absent in real prices. LLS explain this periodicity exclusively with the behaviour of the EMBs. In the first phase (runs 1-150) price

[3] This is varified by LLS by applying Shiller's (1981) technique of volatility tests.

increments change randomly without major deviation from the upward trend in dividends and as a consequence EMBs also buy and sell assets randomly. Figure 8.1 shows the result of this variation. At a point like **a,** high dividends are accidently generated with a following high return. Now, EMBs set in with their chartist-like calculation and push the price further upward. In the course of the boom, rising prices generate rising returns and so all EMBs eventually build up a herd. LLS call this a positive feedback loop. At one point, 100% of the EMBs set investments are devoted to assets and this is the time where "the positive feedback loops 'runs out of gas' ".[4] From this point on, no further cash runs into assets but EMBs still hold no risk-free assets, so prices remain at a high level. 10 runs after point **b** the situation changes. With prices at high rates, dividend yields and returns are low. Consequently, the set of past returns that are considered by the EMBs becomes more and more filled with low returns. If this past record consists only of these low returns (point **c**) EMBs start investing in the risk-free assets and by doing this causing a crash. A now negative feedback loop sets in. The simulated time series displays a considerable amount of autocorrelation so all in all it is not a good approximation of a real time series. However, one of the stylised facts, namely the positive correlation between volatility and volume, is hit. During the upward phase (points **a**-**b**) believes about the future become more and more diverse between RII investors and EMBs. While the latter see a bright future in the price development, RII investors are more pessimistic since the price further and further deviates from its fundamental value. Consequently they are selling the assets that the EMBs buy and so a high trading volume is the outcome.

(iii) The next modification is just a minor one in that it allows for two EMB groups with different memory span over which they average last prices. Both groups have a ratio of 2%, leaving again a considerable high 96% for the RII investors. Because 10 is the average of 5 and 15, LLS expected a similar periodic pattern as in the last variant but to their surprise a rather irregular time series appeared. The authors explain the result by noting that the $m = 15$ strategy yields more wealth than $m = 5$ in the beginning, but then the more myopic traders dominate the market. Price development is thus determined to a large extend by the interaction between these two trading groups. Figure 8.2 shows the simulated price series.

(iv) In a last stage a wide spectrum of heterogenous EMBs is admitted, thereby causing even more complex patterns of interaction. Figure 8.3 gives the simulated time series. Here, 90% of all traders are RII investors. The rest are EMBs, each with a different time horizon m^i. The fluctuations are moderate until sudden and unpredictable crashes set in. Furthermore, cycles are not as periodic as in the other three cases.

The authors claim that autocorrelations resemble those found in real prices: a positive short term autocorrelation (lag 1-4) is followed by a negative autocorrelation in the long-run (mean-reversion). However, this is not exactly true

[4] LLS (2000, p. 183).

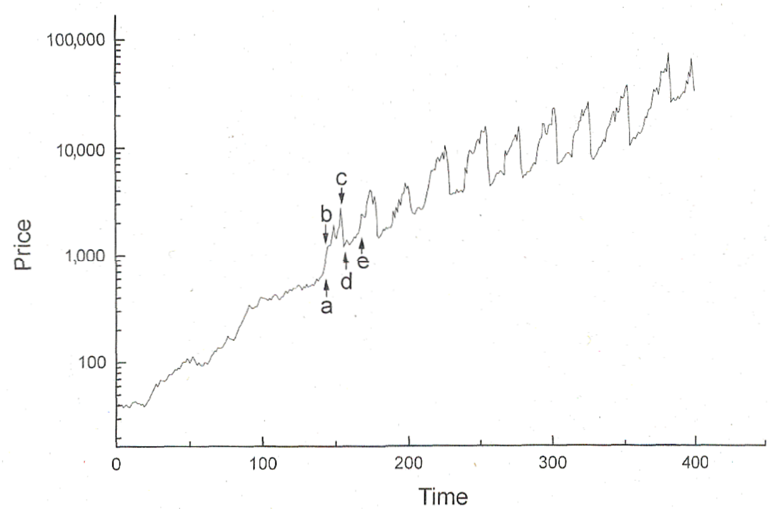

Figure 8.1: Price development in the LLS model with 5% EMBs and 95% RII investors. See LLS (2000, p. 162).

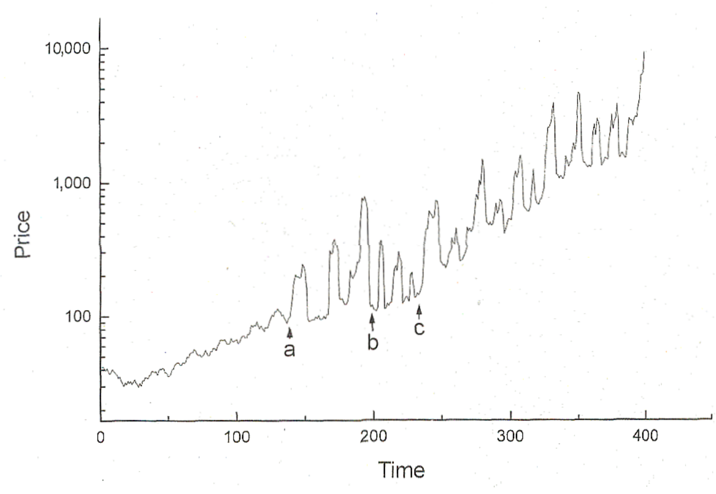

Figure 8.2: Price development in the LLS model with 2% EMBs with m=5 and 2% EMBs with m=15. See LLS (2000, p. 167).

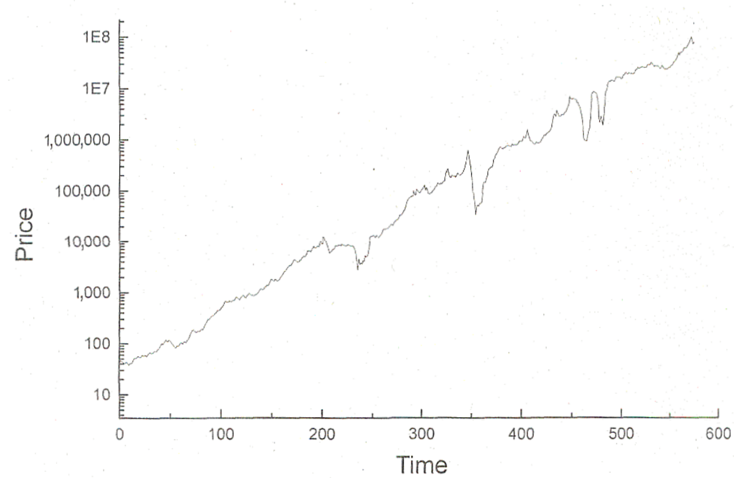

Figure 8.3: Price development in the LLS model with a wide spectrum of heterogenous EMB investors. See LLS (2000, p. 170).

recalling the results of the literature on that point. There are significant short-term correlations found, but these vanish after just a few lags, so how is the LSS simulation to evaluate? It has undoubtedly reproduced some of the main features of stock data, in particular

- the fast decay of the autocorrelation for price changes;
- excess volatility;
- a trading volume that becomes large in turbulent times;
- positive correlation between volume and absolute returns and
- price dynamics that have most of the time a smooth upward trending behaviour, interrupted by larger crashes.

However, compared with the results of the stochastic simulation it has some severe shortcomings. First, the quantitative measurement of the large values of ΔP is completely missing. Also, multiscaling is no topic and so is long memory. Nevertheless, it has a structure much more appreciable to economists because of its utility based decision process. As can be seen in section 8.3, many of the elements will appear in the own model.

8.2 Other Deterministic Simulations

8.2.1 The Stigler Model (1964)

The earliest work that can claim to incorporate at least some of the essential features of microsimulations for financial markets is the *Random Tender Stream* model of Stigler (1964). It is despite of its simplicity a notable pioneering work by the fact that only the last ten years have seen a resurgence of interest in applying numerical methods to describe financial markets. Although it does not contain explicitly modelled traders and the set-up it also hugely oversimplified, resulting price fluctuations show an unexpected erratic behaviour. In his model Stigler tries to contribute a model that exemplifies the possibility of an erratic price behaviour even in the vicinity of an efficient set-up with only mild random exogenous influences. To be more concrete, Stigler claims that market efficiency is realised whenever the following three requirements are met:

(i) a transaction takes place if either a bid exceeds the lowest ask price or an offer is below the highest bid price;

(ii) in all circumstances, higher bids are primarily fulfilled, i.e. before lower ones;

(iii) prices changes are limited by the costs of speculators to provide a market.

Stigler conceives an artificial market with 710,000 shares, where all three points above are fulfilled. His fixed equilibrium price is chosen to lie between \$ $29\frac{3}{4}$ and \$ 30. A random factor is then introduced that decides upon the sequence of bids and asks. This is done in the following way. First, a series of two-digit random numbers are generated. In each run one of these numbers is drawn from the list. The first digit of that particular number decides upon being a bid (if the first digit is even) or an ask (odd first digit) order. The second digit is used for the determination of the values. These are distributed equally around $29\frac{1}{4} - 30$, i.e. a 0 means \$ $28\frac{3}{4}$, a 1 \$ 29, a 2 \$ $29\frac{1}{4}$ and so on until 9 finally gives \$ 31. So if the selected number is 45, it represents a bid of \$ 30.

Stigler now observes the time series of the prices and the time elapsed until tenders have met each other. Surprisingly, transaction prices often deviate from the equilibrium range \$ $29\frac{3}{4}$ - \$ 30. Another interesting outcome is the long periods in which no trade takes place. This is astonishing considering the rather simple algorithm that ought to ensure a frequent turn-over. Of course, the Stigler simulation is not a full-blown model of a stock market. Actively trading agents with different strategies are clearly missing and so is any fundamental aspect. Even more important, there is no data available in order to compare it with empirical time series. Despite these deficiencies, it is historically the first approach to study the evolution of asset prices by a simulated mechanism.

8.2.2 The Kim-Markowitz Model (1989)

The Kim-Markowitz simulation is chronologically the next contribution of economic microsimulations. It does not derive the actions of the trader from an optimisation calculus but unlike the Stigler model where the tenders appear randomly it provides at least a rudimentary guideline for the traders how to act. Nonetheless, the set-up of the model is still quite artificial. It features two groups of investors who enter the market: *Rebalancers* who try to maintain a given proportion between the two assets (a stock and cash) and *Constant Proportion Portfolio Insurancers* (CPPI) who act according to the following rule: each time a plan (which has a predetermined duration of 65 runs[5]) starts, a floor is defined by

$$\text{Floor} = \text{Assets}\,(t_0) - \text{Cushion}\,(t_0)\,. \tag{8.17}$$

The cushion is a fraction (α) of assets hold at time t_i. This value is adjusted according to

$$\text{Cushion}\,(t_j) = \ \text{Assets}\,(t_j) - \ \text{Floor}. \tag{8.18}$$

The aim of the CPPI is to reach a stock position which is the multiple of the cushion, i.e.:

$$\text{Target Value of Stock } \ (t_j) = \ k \times \text{Cushion}\,(t_j)\,.^6 \tag{8.19}$$

The simulation itself works in the following way. In the beginning the portion of rebalancers and CPPI is determined randomly. Both trader types are equipped with the same starting value of the portfolio. The stock price is set equal to \$ 100 and each trader has an initial wealth of \$ 100,000. The proportion of the rebalancers in t_0 is not at their desired level of 5% but prototype I has 70% of his wealth stuck in stocks while for prototype II this portion is 3%. Kim and Markowitz introduce a permanent perturbation to investors portfolios by adding and withdrawing cash from the account. The time span between the disturbances is drawn from an exponential distribution with a mean of 10 runs. The amount of the contribution is uniformly distributed with a minimum of -8,000 and a maximum of +9,000. This periodical perturbation together with the non desired starting portion of the rebalancers triggers sell and buy orders of an amount needed to arrive at the required cash-stock-ratio. The restructuring is, however, not undertaken immediately after a shock hits a trader. In order to incorporate heterogeneity of the CPPIs, Kim and Markowitz let the individual

[5] One run equals one day.

[6] Kim and Markowitz (1989, p. 47), anticipating that the drescription may sound complicated, add an example in order to ease the understanding: "Suppose an investor ... has $a = 0.1$, $k = 5$, and Assets$_0$=\$ 100,000. Floor then is \$ 90,000, and stock held is \$ 50,000. Suppose that, before the stock price changes, \$ 10,000 is withdrawn. If the floor is not adjusted, the cushion becomes zero, and the investor is directed to sell all its shares. With the adjustment, the floor becomes \$ 81,000, the cushion \$ 9,000, and the investor is directed to sell \$ 5,000 of its shares."

plans start from different periods. This means that the insurance plan of CPPI No. 1 starts for example at date t_0, CPPI No. 2 at $t_1$etc.

The trading process is determined by the difference of incoming orders. Both rebalancers and CPPIs offer their bid and ask prices and the quantity they whish to sell or buy. While the quantity is determined by the individual strategy, the prices are calculated in the same way for each trader. Before bid and ask prices are released, each trader estimates the current stock price as the average of the best (highest) bid and the best (lowest) asked price. If no bids and offers are obtainable, $E_t[P_t]$ is determined as the last sale price. If only bids are present and no offers, the price is estimated as 1.01 times the best bid. In the case of offers only, $E_{t-1}[P_t]$ is equal to 0.99 times the best offer. In the case of a buy order, the bid price is then 1.01 times the estimated stock price, while for a sell order it is 0.99 $E[P_t]$. The starting value in period 1 is set to \$100. The orders are executed whenever two open positions meet each other. Of course, total agreements in price and quantity are scarce, so often the offer is executed only in part. If there is no match to the order, it is placed on a list together with all other open buy orders and remains there until a new offer eventually matches. It must be noted that the price building process of Kim and Markowitz does not allow for sudden large movements, because of the moderate steps up- and downwards. Hence, massive deviations from the average price level can only occur in the case of a huge difference between buy and sell orders that could not be balanced immediately.

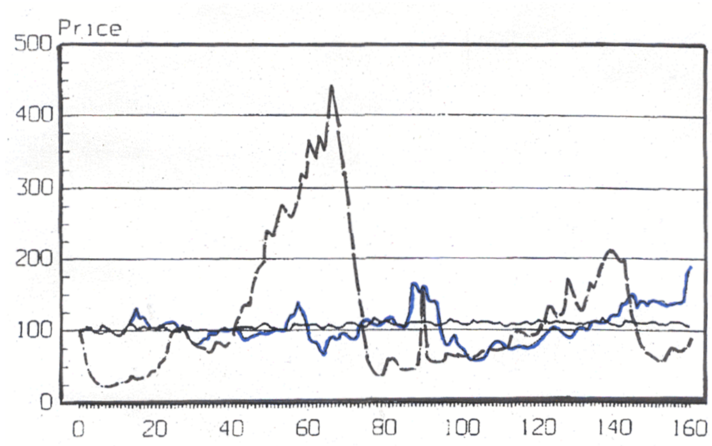

Figure 8.4: Price fluctuations in the Kim-Markowitz model. The solid black line is the simulation with no CPPI's, the solid blue line with 50 CPPI's and the dashed black line with 100 CPPI's. The more CPPI's, the greater the fluctuations. See Kim and Markowitz (1989, p. 50).

Kim and Markowitz now undertake six simulations with a changing fraction of CPPIs (with 0, 5, 25, 50, 75 and 100 CPPIs out of a total of 150 investors). The results seem to identify the presence of CPPIs and rebalancers at a ratio of $\frac{1}{3}$ to $\frac{2}{3}$, crashes with a magnitude equal to the 32% of the black Monday 1987 "... happen almost every quarter" (= 120 runs). Figure 26 shows three typical time series for CPPI portion of 0, 50, 100 and 160 trading days respectively. With 100 CPPIs the rally beginning at day 40 and ending in a crash shortly before the 80-th day is the most striking deviation from the mean $P = \$100$. There are of course plenty of caveats in the model that prevents a one-to-one application in order to explain the October crash of 1987. E.g. the incorporation of only two types of traders may seem to be rather small compared to reality. Furthermore, simulation results show a volume increase in phases of upward developments while crashes have noticed a downshift in prices that is highly correlated with transaction volume. This stands in contrast to the literature on volatility/volume correlation (see Karpoff (1987)). Nevertheless, even this simple model can account for some of the main empirical features of financial time series.

8.2.3 The Model of Arthur, Holland, LeBaron, Palmer and Tayler (1997)

The most striking feature of the Arthur et al. (1997) simulation is its strong emphasis on the evolution of different forecasting rules that are designed to predict future stock prices. These rules are decisive for the buy and sell actions of the traders, and because they change regularly depending on the market situation, they are to a large extend responsible for the price fluctuations in this simulation. The model is quite usual in having a formulation that allows to put agents broadly in two classes, which may be called fundamentalists and noise traders. Unlike most models, both types are not confined to a single prototype strategy but have a wide range of possible rules. Nevertheless, technical trading strategies are still a major factor since they contribute the positive feedback phenomena during price rallies or crashes. On the other side, fundamentally orientated agents do not pursue their strategy irrespective of their environment. They also take the actions of noise traders into account, or more precisely their success.[7]

The model itself consists of N agents who decide upon the best way to compose their portfolios. Utility is expressed by the negative exponential form

$$U(W) = -\exp(-\alpha W), \tag{8.20}$$

[7] The idea, again borrowed from DeLong et al. (1990) is that those seemingly irrational traders can bring the market further away from the true value than fundamentalist might expect, thus generating uncertainty even for agents with full knowledge. Thus, technical trading rules appear to have an edge in turbulent market conditions- even for the fundamentalists. And therefore, in an uncertain environment adaptive learning rules are the most appropriate mechanism in order to generate best results, i.e. highest profiles or utilities.

where α is the risk parameter and W denotes again wealth. Portfolios consist of risk free bonds with interest rate r and equities that pay dividends. These dividends D_t follow an exogenous given $AR(1)$ process of the form

$$D_t = D^* + P(D_{t-1} - D^*) + \varepsilon_t, \tag{8.21}$$

$$\text{with } \varepsilon \overset{iid}{\sim} (0, \sigma^2).$$

None of the agents has a knowledge about this functional form and has to estimate it. Arthur et al. (1997) assume that the predictions of the agents are normally distributed around a mean, $E_{i,t}[P_{t+1} + D_{t+1}]$, with variance $\sigma^2_{t,i,P+D}$. Under these conditions, demand for each agent i is given by

$$x_{i,t} = \frac{E_{i,t}\left[P_{t+1} + D_{t+1}\right] - P_t(1+r)}{\alpha \sigma^2_{t,i,P+D}},^{8} \tag{8.22}$$

where α is the degree of relative risk aversion. Total demand is then simply $\sum_{i=1}^{N} x_{i,t}$. The supply side is modelled by an infinite number of bonds and a stock that issues N units at all times. Equilibrium in the stock market therefore implies

$$\sum_{i=1}^{N} x_{i,t} = N, \tag{8.23}$$

which also determines the clearing price P_t.

The description so far does perfectly fit into a neoclassical set-up with only two assets traded. But here, the authors introduce an expectation process that gives room for a wide range of possible strategies. They allow each agent to have a set of linear forecasting models. These models constitute hypothesis about the temporally dominating forces acting on the market. For example, in times where technical trading prevails, forecasting with relatively simple rules like moving averages may serve as a better device than sophisticated fundamental research. However, market conditions do not last forever and different regimes afford the alteration of the strategy. These dynamics, the evolution of trading strategies in the face of changing markets, stand at the core of the Arthur et al. simulation.

Fixing the ideas about the forecasts, the mechanism by which the expectations are formed is now explained in more detail. In a first step, the situation of the market is characterised by a 13-bit array of 0's, 1's and #'s. Each bit corresponds to a prespecified situation. For example, the third bit might indicate: "the price has risen the last 3 periods", when it is 1, and otherwise 0. The 1 stands for confirmation, a 0 for denial; # is the symbol for "don't care". Hence, an array like (###0##11##0#1) represents a special, unique market state.

[8] See Arthur et al. (1997).

Arthur et al. call this a condition array. Forecasting is implemented by a linear combination of price and dividend, dependent on the structure of the condition array. The simulation features

$$E_t[P_{t+1} + D_{t+1}] = a(P_t + D_t) + b \tag{8.24}$$

as the forecasting expression where only the parameters a an b can change. The above given array may for example prescribe to set a to 0.90 and b to 0 in a specific economic environment, which is $E[P_{t+1} + D_{t+1}] = 90\%$ of last periods price plus dividend.[9] As other different conditions naturally arise during a whole simulation, they will also trigger off different forecasts. At each run, agents observe which of the arrays gives the best description of the market and then use the associated forecast rule. In the case of more than one best match, a linear combination of the rules is used. As an additional feature, the simulation allows for the generation of new forecasting rules that replace former unsuccessful ones.

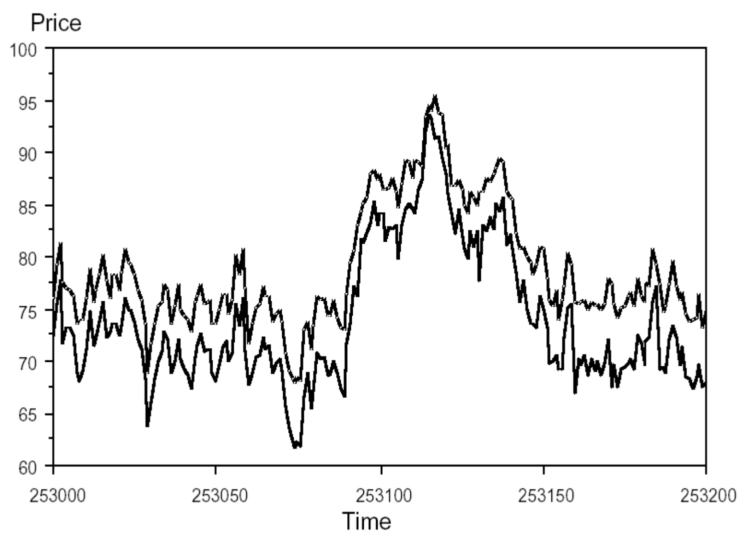

Figure 8.5: Two price charts in the Arthur et al. (1997) simulation. The series above represents the homogenous case while the lower price chart is obtained from the complex regime. See Arthur et al. (1997, p. 14).

The computer experiments when run with just a few agents and dividend changes that exhibit only minor variations show a price process close to a small number of fundamentally orientated rules between which they can choose, huge differences in the opinion about future prices are not observable, thus generating

[9] This may be sensible in a situation where technical tradings prevails and the mood is pessimistic.

also a low trading volume. This is not surprising. The picture changes when a much wider range of generic forecasting rules and a large number of agents is considered. Although the simulated price series still tracks the rational-expectations price quite close, the variance is much higher than in the former regime.

Moreover, bubbles and crashes appear to show up from time to time. These occasional larger deviations from P_t^F are accompanied by larger trading volumes and a more diversified group of traders. I.e. traders now adopt different rules and thus become more heterogenous in their believes about the future. This is the source of the fluctuations. As a consequence of the changing market situations, rules that are successful in a certain phase become eventually unreliable to the agents and therefore are exchanged much more frequently than in the more quite experiment. It is worth noting that the dynamics of the system can be considered as a good example of the way interactions influence or determine the behaviour of the macrovariable, the price: the more frequent and larger fluctuations are, the more complex are forecasting rules, thereby adding further heterogeneity. On the other hand, diversity is a prerequisite of substantial price changes. However, the market does not stay in a regime once it has reached it. Turbulent phases are followed by smoother times where the simulated prices are closer to the fundamental one. It is this feature that makes the Arthur et al. model a member of the class of self-organising complex systems. There is no external force that determines the dynamics of the rules.[10] All changes are endogenous and the emergence of herding is not induced by any of them.

The outcome of the Arthur et al. simulation corroborates the importance of elements like technical trading and the fact that only heterogeneity causes larger price fluctuations. However, their extent does not seem to fit well into the picture painted by the stylised facts of part two. Bubbles and crashes are too moderate compared with the empirical price records. Furthermore, like most simulations, the simulated time series are neither tested for possible multiscaling characteristics nor for long range dependencies. The experiment is thus best viewed as another example of how the introduction of many heterogenous trader influence the price dynamics.

8.2.4 The Model of Lux and Marchesi (1999)

At the centre of this simulation stands the goal to mimic three important empirical facts of financial markets, namely the fat-tailedness of returns, the clustering of volatility and the overall unpredictability of the price record itself. The three phenomena are obtained by introducing the usual types of market participants: fundamentalists and noise traders who are further subdivided into optimists and pessimists. Heterogeneity in this respect is, however, not the most important feature of the simulation. It is the way in which the transition between both groups are modelled. The fundamentally orientated investors are assumed to know the process $P_t^F - P_{t-1}^F = \varepsilon_t$, thus buying (selling) whenever the actual

[10] The arbitrary $AR(1)$ process for the dividends cannot account for the fluctuations.

price P_t is below (above) P_t^F.[11] P^F is equal to the discounted future returns (similar to the Gordon model). Lux and Marchesi alter the fundamental value of the asset regularly by introducing an exogenous disturbance term that is normally distributed.

Chartists on the other side orientate themselves on the behaviour of other traders and additionally try to capitalise hidden trends. Lux and Marchesi further subdivide this group into optimistic and pessimistic traders. The members of these subgroups buy a fixed amount of shares per run if they are optimistic and sell the same amount in the case of being pessimistic. A crucial point of this simulation is the interaction which comes in by the possibility of changing the attitude, either from optimistic to pessimistic, or the complete strategy (i.e. from a fundamental to a technical orientated or vice versa). Lux and Marchesi calculate the probabilities of transitions by

$$ve^{U_t}\Delta t, \text{ with } v > 0. \tag{8.25}$$

The term U_t captures the driving forces behind the strategy change. Chartists are concerned about the actual price trends and the overall market opinion which is measured by the ratio of optimists and pessimists. As an example one can take an optimistic chartist. If he faces a situation of a relatively long bearish phase and if he would furthermore be surrounded by a vast majority of pessimists, then the probability of a transition within the group of noise traders is high. On the other hand a balanced optimists/pessimists ratio with no obvious short term price trend leaves him with a high probability to stay in the optimistic group. Fundamentally orientated traders who believe in a kind of mean reverting process to the true value only change their strategy when excess profits generated by chartists become significantly larger than their own. For the same reason, noise traders switch regularly to a fundamental strategy. v is the parameter that determines how fast the strategy is re-evaluated and Δt is the smallest time increment in the simulation, i.e. one run.[12]

Endowed with a strategy behaviour for each trader which is determined randomly at the beginning, the transition probabilities and a starting ratio of fundamentalists and noise traders, the dynamics of the market generate endogenously determined price changing. These are calculated by the difference of demand and supply. The simulated time series of price changes shows some interesting features. First, fundamental and actual prices do not deviate from each other substantially (see figure 8.6). Despite the relative closeness of P and P^*, returns do not follow the pattern of a normally distributed variable. Instead, some intervals are characterised by extreme fluctuations that cannot come from the exogenous random factor. As a consequence, the *pdf* of the returns display pronounced tails and the authors estimate the tail-parameter to be around 2.64, reproducing the empirical results of part two in that it falls

[11] Note the coincidence with the model of Kaizoji, Bornholdt, and Fujiwara (2002).

[12] In a paper of Chen, Lux and Marchesi (1999) who use the same set-up for strategy transitions, the exact functional forms for the strategy transformations are provided.

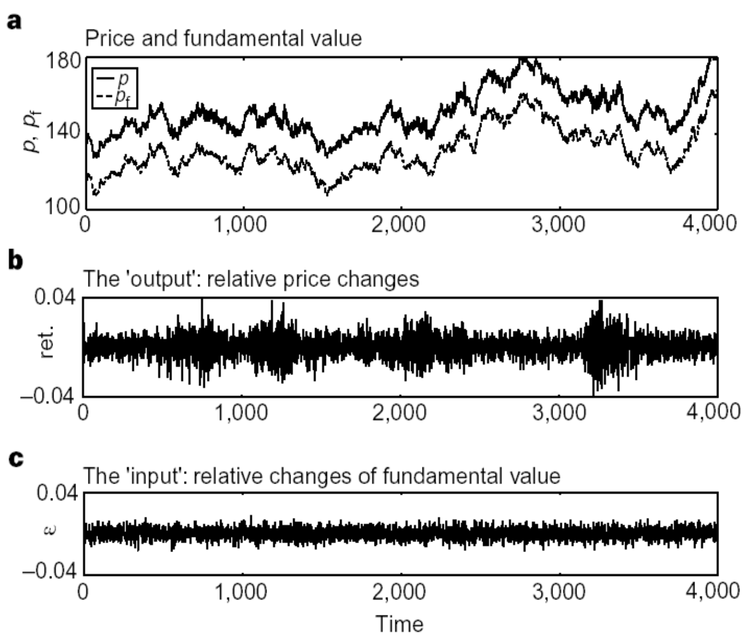

Figure 8.6: Panel a: Typical series for the price development in the Lux-Marchesi simulation. Beneath plotted is the fundamental values (p_f). The picture shows only a part of the longer series. Panel b: Log changes of the price. Panel c: Fluctuations of the fundamental value. They follow a normal distribution. Lux and Marchesi (1999, p. 498).

outside the LSD but still generates fat-tails! Using Lo's modified R/S-statistic, Lux and Marchesi also estimate the Hurst exponent for evaluating (comparing) the scaling characteristics of the simulated time series. Testing the raw returns first, H is estimated to lie around 0.5, and thus price change predictability is neglectable in the long run. This is roughly in line with the results of real data which shows only few signs of long memory. Taking absolute returns a long memory process with $H = 0.85$ turns up. Although this value is above the usually encountered estimations, it qualitatively shows a resemblance to the behaviour of absolute returns in empirical studies, supporting the hypothesis of a long-ranged dependence between the returns that comes from a heterogenous market.

8.3 A new Deterministic Simulation with Different Trader Types

A suitable platform for stochastic models where the functional form is given seems to be the Ising models. Deterministic models do not have this kind of platform but the minimal set of ingredients, a large number of interwoven agents who trade a single asset and try to optimise their utility by maximising their expected wealth, is equal among all previously performed simulations. With their different approaches these simulations have shown that price fluctuations of a more or less realistic form can result from various sources. Thus, the existing literature already offers a quite diversified picture that gives some strong indications about the important features. It would therefore be superfluous to invent a completely new structure. The following model aims to incorporate a large set of elements from the already existing literature while still preserving a flexible framework that is open to new variations.

8.3.1 The Model

As in the Ising models, N traders are located on a two dimensional lattice. The reason for such a structure has a simple pragmatic point that has nothing to do with stochastic simulations: it is the easiest way to incorporate local trends and influences on an individual level. And, this element can be eliminated without changes in the general structure of the model. Hence isolating its influence should be easy. While the Ising model treats financial agents as unobservable units whose decisions can only be described by probabilistic considerations, here they are modelled as utility optimising humans where the functional form is known. Agents are endowed with a prespecified amount of capital, which is here given as a diversified portfolio. This portfolio consists of two assets: a risky one that is traded on the artificial market and a riskless bond. Contrary to most simulations, investors are partioned into three distinct types: the obligatory fundamentalists and technical traders and as a third group the portfolio rebalancers taken from the Kim-Markowitz model.

8.3.1.1 Fundamentalists

Fundamentalists are assumed to maximise expected utility according to

$$\max E_t[U(W_{t+1})], \text{ with} \tag{8.26}$$

$W_{t+1} = W_t[(1-a)_t(1+r_{t+1}) + a_t R_{t+1}]$ representing wealth of period $t+1$.[13] R_t is the return of the bond while r_t, defined by

$$\frac{P_t - P_{t-1} + D_t}{P_{t-1}},$$

is the return of the risky asset. P_t and D_t are the price and dividend of the risky asset respectively, where the dividends follow a the simple autoregressive law

$$D_t = (1+g)D_{t-1} + \varepsilon_t, \text{ with } \varepsilon \overset{iid}{\sim} \left(0, \sigma^2\right).$$

This is identical to the LLS description of traders. Contrary to LLS where the weights a and $(1-a)$ are calculated numerically, one can show that this formulation implies an optimum investment proportion at time t, $(1-a)_t$, of

$$(1-a)_t = \frac{E_t[r_{t+1}] - R_t}{vol_t(r_{t+1})},^{14} \tag{8.27}$$

where vol_t is the volatility of the return.[15] The calculation of $E_t[r_{t+1}]$ involves the building of expectations for future prices and dividends. This will be the crucial point where heterogenous beliefs come in.

Here, just as in the LLS simulation, the Gordon growth model is employed, so the description is fairly brief. The Gordon growth model admittingly poses some severe restrictions on the variables and is thus surely not the most realistic choice. But this does not matter for the simulation. What matters only is the conformity of the theoretic premise with traders' actions. And this is given in the model because all fundamentalists calculate the expected value of stock for the next period according to the same calculus, i.e.

$$E_t[P_{t+1}] = \frac{E[D_{t+2}]}{(\rho - g)}, \tag{8.28}$$

where $E[D_{t+2}]$ is the expected dividend two periods ahead, the discount factor ρ is the required rate of return for equity investors[16] and g is the annual growth

[13] a and $(1-a)$ are the weights, just as in the LLS model. The main difference lies in the fact that the growth rate of D in the LLS simulation was determined by a function $\int_{z_1}^{z_2} f(z)dz$ while here g is deterministic but is regularly disturbed by a white noise term.

[14] For a derivation see Chiarella and He (2001). Compare also (8.22) in the Arthur et al. model.

[15] The simulation uses the historical volatilities. This is equal to all traders, so fundamentalists to not have an edge over noise-traders in this respect.

[16] ρ can be determined by various models of finance theory, e.g. the APT or the CAPM.

rate in dividends forever. Both factors, ρ and g, are assumed to be constant over time.[17] Dividends are modelled as a variable with a steady growth. In order to introduce a news element that regularly comes into the market from an exogenous source, g is disturbed every 10-th run by a white noise term identical to ε in the stochastic simulation.[18] Thus dividends grow with rate of g on average but swing around this trend on a short time scale. Expectations for dividends are simply given by $E_t[D_{t+1}] = (1+g)D_t$, and thus $E[D_{t+2}] = E_t[D_{t+1}](1+g)$. Hence fundamentalists have expectations that are correct on average, i.e. $E[D_{t+1}] = D_{t+1}$ because $E[\varepsilon] = 0$. Expected returns of the risky asset is thus given by

$$E[r_{t+1}] = \frac{E[P_{t+1}] - P_t + (1+g)D_t}{P_t}. \tag{8.29}$$

Once $(1-a)$ is determined according to (8.27), each trader is able to calculate the part of wealth that is assumed to deliver the highest expected wealth of the next period. Comparing this result with the temporary prevailing portion of risky and riskless assets determines the exact amount of demand or supply for the traded asset. This, however, has to take the price of that asset for the next period into account. Here, the $E_t[P_{t+1}]$ is taken as the point of orientation for the investors. Then, if $(1-a)_{t+1}W_t$ is the new value desired in the new portfolio, the number of assets offered or demanded by an individual trader, is given by

$$\frac{(1-a)_{t+1}W_t}{E_t[P_{t+1}]} = N_i^{D,S}. \tag{8.30}$$

For example, if $(1-a)_{t+1}W_t$ is less than the temporary held portion, then the investor is willing to sell N_i^S assets at least for price $E_t[P_{t+1}]$, which will be his offer price. If possible, he will sell them at a higher price but not for less. One important feature of the fundamentalists modellation to note is that it does not have a minority component as it was for the stochastic simulation above. The reason is that such a possible behaviour is already implicated by (8.27) for if the market is in a bubble-phase $E_t[r_{t+1}] - R_t$ is very small or even negative because of $E_t[P_{t+1}] - P_t \leq 0$. Then the weight for the risky asset is reduced to a very low value or zero.[19] In this case, fundamentalists will sell all their shares, thus bringing the price abruptly down.

8.3.1.2 Noise-traders

Noise traders are principally modelled in the same manner as fundamentalists. Their goal is to maximise expected utility which is a function of expected wealth just like the fundamentalists. The only difference is the calculation of $E_t[P_{t+1}]$. Contrary to fundamentalists they follow a technical rule which is in this case

[17] This alone shows the low representativeness of the model.

[18] The frequency with which the market is hit by new information can be varied.

[19] If $E_t[P_{t+1}] - P_t$ becomes negative, $(1-a)$ must also become negative. However, the model is not designed to have negative amounts of assets and so $(1-a)$ is set to zero in this case.

nothing else than a moving average of past growth rates. Let γ_t be the growth rate of prices from period $t-1$ to t, i.e.

$$\gamma_t = \frac{P_{t-1} + P_t}{P_{t-1}}.$$

The the traders calculate an average growth rate of e.g. the last four runs by

$$\bar{\gamma} = \frac{\gamma_t + \gamma_{t-1} + \gamma_{t-2} + \gamma_{t-3} + \gamma_{t-4}}{5}.$$

Now, they expect prices to develop like in the past, so expectations of future prices are calculated through

$$E[P_{t+1}] = \bar{\gamma} * P_t. \tag{8.31}$$

This element let noise trader extrapolate past trends, thereby strengthening the overall market situation. Although there are different noise traders with respect to the history which is taken for $E[P_{t+1}]$, in the simulations this will be $n = 5, 10, 25$, all of them share the same principal idea. This can be interpreted as following a global market trend.

Regarding $E_t[D_{t+1}]$ two principal ways of modelling chartists' dividend expectations exist. The first is to let the noise traders form their dividend forecasts by the same rule as fundamentalists. This would, however, imply an intimate knowledge of the fundamental process. Chartists can be thought of as individuals with at least a rudimentary understanding of a fundamental process, but they are not supposed to have a complete knowledge. Therefore, dividend expectations are built in the same way as price expectations, i.e. as a moving average over the last n runs. The simulation has $n = 2$ for simplicity. In contrast to the possibility of fundamentalists to act against a market trend, chartists are limited in their behaviour: they are always the driving force behind longer lasting price dynamics because of their feedback trading. The expected return for the chartists is now given by

$$E[r_{t+1}] = \frac{E[P_{t+1}] - P_t + \frac{D_t + D_{t-1}}{2}}{\rho - g}. \tag{8.32}$$

8.3.1.3 Rebalancers

Drawing on the work of Kim and Markowitz (1989), a third group of investors is introduced: the portfolio rebalancers. This type of trader does not employ utility theory in order to determine the weights of their wealth distribution. Instead, they always try to maintain a prescribed ratio of risky asset and bonds. Therefore, demand and supply is orientated by the amount by which this ratio is missed. Rebalancers do not have expectation functions for future prices attached to them. They buy and sell at every price. Hence they simply re-adjust the number of assets so that the desired portion of wealth held in assets is restored. As in the Kim-Markowitz simulation, rebalancers do not evaluate their portfolio every run but only each n-th run. In order to introduce heterogeneity, this group is subdivided into three equally large subgroups, each with $n = 10, 50, 100$. Another element of diversity is the fact that the rebalancers start with a different ratio where none exactly matches the desired proportion. This induces them to trade different amounts of assets. A last point is the ratio itself. Again the set of rebalancers is divided into three parts. One third whishes to hold 50% of wealth in bonds, another third with 25% and last with 75% of bonds as their target proportion. It is obvious that this type of trader is quite different from the first two. Rebalancers can be thought of as representatives of institutional traders who are obliged to fulfill certain restrictions in the construction of their portfolios. These restrictions are in keeping with reality as e.g. insurances and mutual funds are bound to invest only a prescribed percentage of capital in risky assets.

8.3.1.4 Local Influence

The next point concerns the local influence. This can be thought of as a network where individual traders exchange information about the latest developments. The question is how to model this element in a reasonable way. Just for convenience it is again incorporated via nearest neighbour influence, though not in the same manner as in the Ising models. Here, opinions from the surrounding environment are collected and averaged.[20] This averaged neighbour-opinion influences each individual trader according to

$$E[P_{t+1}] = bE[P^*_{t+1}] + (1-b)E[P^N_{t+1}]. \tag{8.33}$$

The weights b and $(1-b)$ mirror the strength of the two components. $E[P^*_{t+1}]$ denotes the own expectation, both for fundamental and noisy expectations and $E[P^N_{t+1}]$ is the averaged neighbour expectation. The higher b, the less local trends can change the opinion of the trader. In the case of $b = 1$ the model features no direct influence between the adjacent agents.[21]

8.3.1.5 Strategy Changes

A further important feature of the model is the way in which agents decide to change their strategy. First, the value of the portfolio for all agents is given by

$$P_t m_t + P^B_t m^B_t = A_t. \tag{8.34}$$

A_t is the value of the portfolio at time index t, P_t and P^B_t are the prices for assets and bonds respectively, while m_t and m^B_t denote their quantity. In order to keep things simple, P^B_t is held constant over the whole simulation period; m^B_t thus changes according to changes in $P_t m_t$. With constant bond prices, the evolution of A_t is mainly determined by the dynamics in the value of the risky

[20] If a trader is sourrounded by a rebalancer who does not have an expectation about future prices, only the remaining three neighbours are taken for the calculation of the local influence.

[21] Note that even a value of $b = 0$ does not lead to a duplication of the Ising model.

asset. Hence, it is now possible to take the development of A_t as a proxy of the success of a strategy. A_t is calculated in each run. The most successful strategy is that which gives the highest averaged value over a specified time period of n runs. Agents are assumed to observe the evolution of A_t for both types of traders (fundamental and technical traders) and decide periodically to join the group that has recently produced the best results. Rebalancers are not allowed to change their strategy because of the restrictions imposed on the composition of their portfolios. In order to preserve the variety of trading types, a minimum percentage for each group is set. Beyond it, free transition is allowed and hence situations where most traders follow the same strategy may well occur. In this respect, the simulation also keeps track of the evolution of the different winning trading strategies.

8.3.1.6 Price Determination

The last building stone of the simulation model is the price determination. Unlike the stochastic simulations, P_t is determined by a classical auctioneer model, where the auctioneer in charge sets the price as to maximise the traded volume. The exact procedure is quite simple. All traders who want to buy or sell are put on a list that includes their offers with quantity and ask and bid price respectively. The following table illustrates the mechanism with a simple example.

Table 8.1: Buy and sell orders

Buy orders		Sell orders	
Volume	Bid price	Volume	Ask price
100	market order	80	market order
60	182	120	180
80	181	40	181

This gives the following trading volumes:

Table 8.2: Trading volumes

Price	Buying volume	Selling volume	Volume of trades
180	240	200	200
181	240	240	240
182	160	240	160

Comparing the possible prices, one sees that $P = 181$ covers all offers: the market orders indicate buying at the lowest and selling at the highest possible price, i.e. they do not have limit orders like the four other traders. In case of $P = 180$, all want to buy but only 200 (80+120) are willing to sell, thus trading volume is 200. At $P = 182$, all of the selling orders could be fulfilled, but now only 160 (100+60) could be sold. Therefore, the new price would be fixed at 181. In the case of no trading because of only buying or selling orders without counterpart, the price is set at the old level plus 1% in the case of excess demand and minus 1% in the case of excess supply.

8.3.2 The Simulation

Before the simulation actually starts, a price history is created for the expectation building process of the technical traders. It contains 25 data points in order to serve even the largest backward looking agents. The record is calculated in order to end in a starting price of $P_0 = 100$. The market for the risky asset has a volume of $100,000$ shares and with P_0 at 100, the value of these shares amounts to $10,000,000$. These shares are distributed uniformly to the N traders. The starting capital A_0 is distributed according to a standard normal distribution with a mean given by $20,000$.[22] Given A_0, P_0 and the number of assets for each investor, $P_0^B m_0^B$ is determined by $A_0 - P_0\ m_0$. This is the procedure for fundamentalists and noise-traders. Because of the special importance of the ratio for rebalancers, $P_0^B m_0^B$ is there chosen to comply with the different starting non-desired ratios as described above. Other parameters values set at the beginning are:

- the minimum percentage of fundamentalists and noise traders respectively;
- the percentage of rebalancers (10%);
- the price (100) and return of the bonds (5%);
- the required rate of return for equity investors (6%)
- the frequency with which the portfolio is re-evaluated (50 runs).

With this starting configuration, the simulation starts by calculating the demand and supply orders according to (8.27) for fundamentalists and chartists. Then a new price is determined which in turn alters the fraction of wealth desired by all traders. The following results are obtained with two different combinations of the minimum percentage of fundamentalists and noise traders. The first is minimum of 5% for both groups while the second has 20% fundamentalists and again at least 5% noise traders. This is done to check the amount of differences in the outcome, just as in the own stochastic simulation.

As a first step in the statistical investigation, it may be fruitful to show a typical course of the price changes with associated *qq*-plot that indicates the presence of a non-normal distribution.[23] The series in figure 8.7 does not seem to have any cyclical movements, nor does it show large pronounced clusters of high fluctuations. The first is needed to let the series being unpredictable, but the second seems to contradict the empirical data. However, statistical analysis will prove the contrary. And the qq-plot already indicates non-normality. It should be stressed at this early point that the above result is fairly resistant to minor

[22] This is needed to keep roughly 50% of the wealth in bonds at the start (roughly because of the starting ratios for the different rebalancer types). The choice of 50% is purely arbitrary.

[23] The simulation needs a considerable time in order to arrive at a course that resembles real data. At the start, price fluctuations cause large disruptions in the disired proportion of wealth, which settle only after some initial runs that may be even up to a thousand periods.

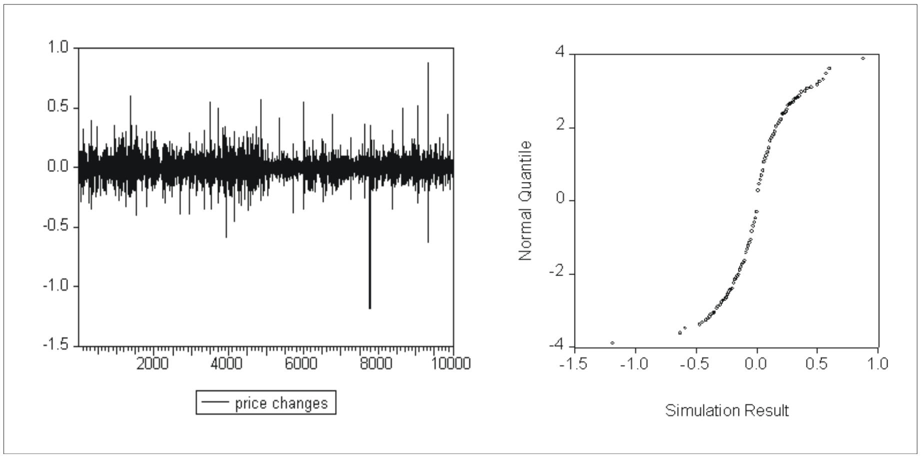

Figure 8.7: Price changes and associated qq-plot of the simulated time series. The result above is obtained with a minimum of 20% fundamentalists.

variations in the model. For example, changing ρ or R not too dramatically (i.e. ± 2 basis points) leaves the general appearance of the price development almost unaltered. Using different starting values for A_0 is without any consequence and so is the starting ratio of fundamentalists to noise traders. However, the weights in (8.27) are very sensitive to small changes in the values because they then often alternate between 0 and 1 which is implausible, or at least unrealistic and the resulting time series becomes cyclical. Therefore, the set-up needs a bit of fine tuning.

In the following, statistical inferences are made for the two variants (the one with a minimum of 5% and with a 20% minimum of fundamentalists). The first test concerns the distribution of the price changes. As for the stochastic simulation, the parametric estimators all yield values for the tail exponent well below 2. But also like former simulations, the cumulative estimation technique and the Hill estimator yield values way above 2. Figure 8.8 gives an example of how the (log) cumulative probabiltiy function only of the highest price increments looks like.

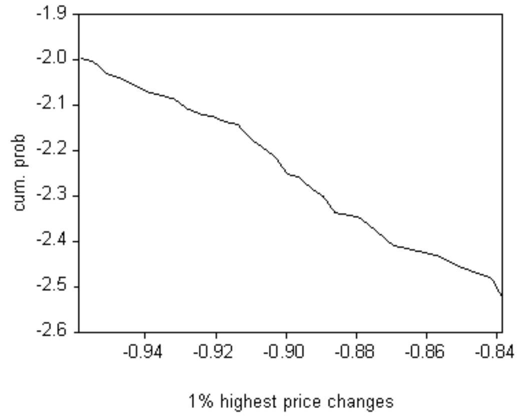

Figure 8.8: Cumulative probability of the 1% highest (positive) price changes. The relationship is presented on logarithmic scale. Estimation yields a (negative slope) of 3.19.

Table 8.3: Parametric estimation of the tails

Min. % of fundamentalists	Koutrouvelis	McCulloch	Nolan
5%	1.433	1.592	1.6
20%	1.489	1.61	1.67

Table 8.4: Tail estimation by the cumulative method

Min. % of fundamentalists	right	left
5%	3.313	3.49
20%	3.19	3.45

Table 8.5: Tail estimation by the Hill estimator

Min. % of fundamentalists.	p=0.1	p=0.05	p=0.025	p=0.01
5%	2.252	2.36	2.638	3.24
20%	1.987	2.177	2.227	2.983

The estimates show that the tails (positive and negative) are far more pronounced than the normal. This result indicates the existence of large deviations from the average values of ΔP. Hence crashes must exist in this simulation, just like in the stochastic model. But different to there, no explicite counter-reaction term ensures the bursting of the bubbles. So different reasons have to be found. An economic story for what is going on during a crash can be presented in the following way: Usually, chartists feed the market in the upward movement of the price. At one specific value of P_t, this current price exceeds the fundamental price expectation to such an extend that fundamental traders drastically reduce their weights in favour of bonds. As a consequence, the current price comes down. The sudden drop initiates chartists - at least the most myopic ones - to

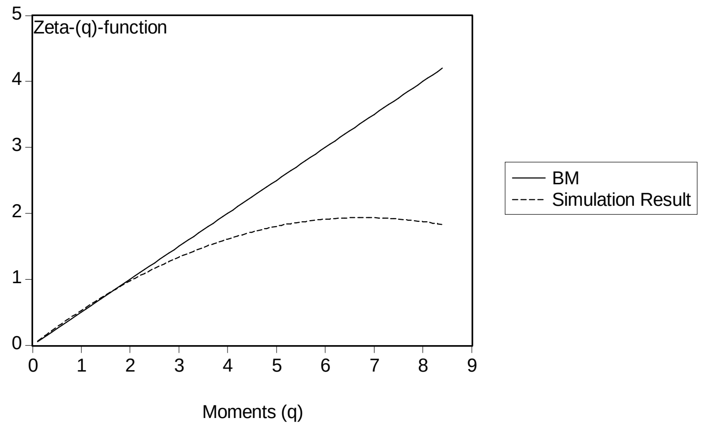

Figure 8.9: Zeta-(q)-function for the simulated series with a minimum percentage of 5% fundamentalists.

change their behaviour: they are now on the selling side thus bringing the price further down. However, as soon as markets prices are considerably below their true values, the fundamentalists again engage in buying the underpriced assets. This may explain the observed volatility clustering. Only when the turbulent phases come down and a more quite period prevails, noise-trader change their strategy, because trends are missing. This leads to a higher ratio of fundamentalists until a new bubble slowly starts. Quite interestingly, the two variants lead to similar outcomes concerning the tails, although it was maybe expected that a higher percentage of fundamentalists would have a smoothing effect on the price increments. But the difference his not large for all estimatiors.

In order to illustrate the scaling properties of the artificial price records, the zeta-(q)-function is estimated. The function yields the by now familiar non-linear appearance and is similar to some function of the empirical analysis. Hence, this outcome indicates that the q-dependence of $\zeta(q)$ for the artificial time series is strong and the corresponding graph can be viewed as being close to a multifractal. Furthermore, its curvarture is relatively strongly bent which indicates rather wild fluctuations. Recall that in part two it was said that an economic explanation for this stylised fact points towards the existence of heterogenous traders in the market. This feature is reproduced by this simulation, so it is tempting to assume that the present structure has confirmed the heterogenous-trader-assumption.

Now, the autocorrelation structure of price changes are investigated. In figure 8.10 values for the first-order autocorrelation are plotted against the first 36 lags.

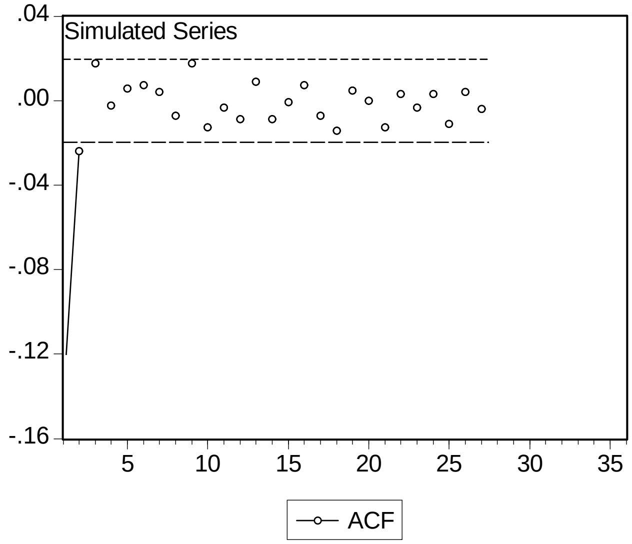

Figure 8.10: Autocorrelation function for price changes in the deterministic simulation.

The figure reveals the familiar picture of a fast decaying autocorrelation which yields at the beginning (first lag) a significant negative value. Yet already the second lag has no significant non-zero correlations. These results are compatible with the empirical findings of chapter 6.1. There, high-frequency data shows a similar pattern for the correlation function, sometimes with a positive value for the first lag. In any case, this stylised fact can only be attributed to time series with frequencies of less than a day. Larger time scales do not yield non-zero autocorrelations. The consequence for the simulation is to interpret it as a model that mimics trading on very short time scales. This may well be a point of citric for the description is designed to model agents that optimise their portfolios, which is probably not done every 10 seconds. Individuals usually build expectations that are not revised on such short time scales. Given the costs of searching for fundamental information and restructuring the portfolio, compositions are likely to hold over longer periods than a few seconds. However, theory demands a re-evaluation of positions whenever new information comes into the market, at least for the fundamentalists. If it is the case that such information comes in at a high frequency, then these traders will act on the same time scale. The problem is that news in the simulation do only occur in every 100-th run and thus cannot be responsible for the heavy trading.

A solution may be found by recalling the way in which optimal weights in (8.27) are calculated: weights will change whenever one of the parameters changes, and this is regularly the price P_t. Price changes induce a new expected return regardless of whether $E_t[P_{t+1}]$ has changed or not. Therefore $(1-a)_t$ in (8.27) is frequently changing hence inducing fundamentalists trading with at least a small volume. Chartists on the other hand do not have expected returns revised only because of changing prices but also because new prices enter the moving average series of past prices and lead to new expectations. Rebalancers as the third group become active whenever the desired distribution of their portfolios is disturbed which is the case for ΔP being large enough.[24] After all, this explains the trading frequency in the model but it may not be satisfactory for those who still object that portfolio restructuring in reality occurs less frequently.[25]

As a last point, results for the long-memory tests for the raw returns as well as for the absolute returns as a measure of volatility have to be reported. Statistically insignificant estimations of $\hat{H} = 0.599$ for the R/S statistic and $\hat{d} = 0.0893$ for the GPH method emerge for the raw returns. Absolute returns seem to display a long range autocorrelation. Here the significant estimation outcome is $\hat{H} = 0.782$ and $\hat{d} = 0.2513$ for the R/S statistic and GPH method respectively.[26]

It was mentioned that small differences in the parameter values have no big influences on the outcome. Therefore figure 8.11 shows the evolution of price changes for another variation of the simulation, now with parameter values $\rho = 5\%$, $R = 3\%$, 15% rebalancer and a minimum of 10% fundamentalists. As the figure shows, price changes are similar to the above result. This similarity extends to the statistical tests as all features are nearly the same, i.e. a non-linear zeta-(q)-function, fat tails, negative autocorrelations for the first lags and a long run dependency of absolute returns. The concordance with the first variations confirms the robustness of the model to small changes.

But what about some severe parameter changes that alter the behaviour of P significantly? First, allowing for fundamentalists and rebalancer only or chartists and rebalancer only has deep impacts on the outcome. Figure 8.12 shows the resulting price changes. Obviously, rebalancers are not able to bring in enough diversity to produce realistic time series. At the start the price changes are relatively high, but after a few runs, prices go upwards without end. This is due to the demand of fundamentalists. With no volume, prices have to be set by the auctioneer. If demand exceeds supply, prices go up, here 1% each

[24] Because minor disturbances do not induce activity.

[25] In order to compare the results with daily data, price changes for the surrogated stock market is computed for longer lags, i.e. $P_{t+\Delta t} - P_t$ is calculated for values of Δt of 5000 runs. The correlation functions now all get more closer to the daily records for there is no significant autocorrelation detectable anymore. The choice of $\Delta t = 5000$ runs may seems odd at first, but if it is true that the model replecates data with a such a high frequency as, say 10 seconds, then 5000 runs is merely a trading day.

[26] These results are obtained from the simulation with 5% fundamentalitsts. The variation with 20% yields insignificant estimates: $\hat{H} = 0.533$ and $\hat{d} = 0.153$ for the raw returns and significant values $\hat{H} = 0.714$ and $\hat{d} = 0.186$ for absolute returns.

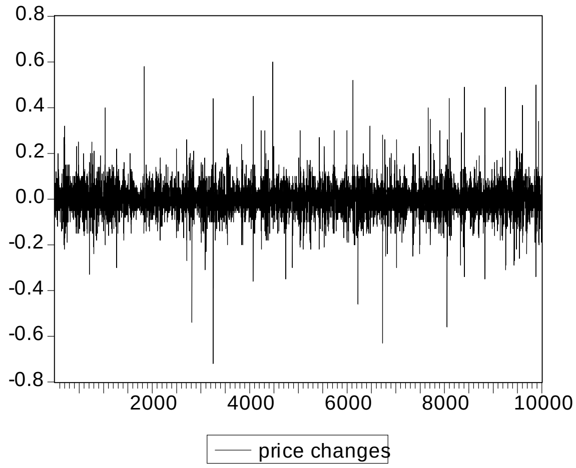

Figure 8.11: Price fluctuations for a parameter variation in the deterministic model.

round. This outcome prevails also for the rebalancer/noise-trader combination, so neither the first nor the second combination of trader-groups possesses sufficiently heterogenous expectations. Trading volume is extremely low after a few runs. Of course, fundamentalists do all have the same (fundamental) expectation and thus do not trade with each other. But rebalancers, once they have reached their prescribed ratio of shares and bonds are also not willing to trade with a substantial volume, and so the market eventually becomes almost illiquid. The same outcome emerges with chartists and rebalancers only. Although chartists have a form of diversity because of the different history length they use to calculate price expectations, it is not sufficiently diverse.

Interestingly, local influences do not play such an important role, too. Contrary to the Ising model, reducing the neighbourhood influence to zero does not destroy the complexity of the system. With $b = 0$ in (8.33), traders are isolated and do not base their decisions on other factors than their own opinions. However, the simulated time series remains complex and do not show huge differences to the series displayed in figure 8.11. It seems that deterministic formulations are in no need of local reinforcements that work as a kind of network. The interaction in the model as designed by the strategy changes is sufficient to build trends on the one side and a large enough counter-reaction to destroy the trends on the other side. In this respect, the simulation is in agreement with the performed deterministic simulations in the literature in the sense that none of these need a network component for their results. One may surmise that it is especially the Ising model that calls for a closer relationship between the agents, otherwise its mechanics does not work in the desired way. The deterministic simulations can thus claim to be more general than the simulations of chapter

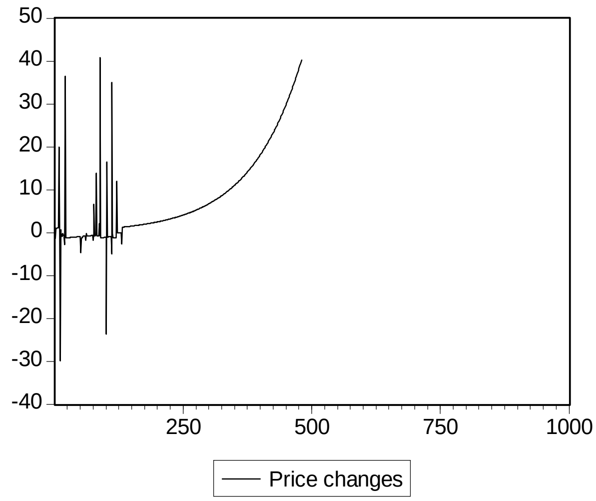

Figure 8.12: Price fluctuations in the variation with fundamentalists and rebalancer only.

7.

Having $b = 1$ leads to a smoothing of the price fluctuations but does not alter the outcome substantially. The reason for the quench of bubbles and crashes in this case may come form the fact that with $b = 1$ the traders all have a kind of "average opinion" which is not able to generate sudden huge price differences.

At the end the most interesting point should be stressed: there is no component that in itself can be made responsible for multifractality, long memory or the fat tails. The source of these peculiarities in the outcome cannot be traced back to a specific modellation of the individual elements, the fundamental process or any other component. A proximate conclusion would be that only the sum of all is able to produce such time series, and this is very much in the sense of complex systems.

Chapter 9

Conclusion

In conventional economics, current market prices are assumed to reflect all available fundamental information because traders have based their investment decisions exclusively on that information. This is the EMH that has emerged after a long debate to what extent market (stock) prices reflect fundamental influences. Keynes (1936), for example, still believed that prices are much more the outcome of collective psychological actions from irrational noise-traders, but the sixties and seventies of the last century witnessed a large body of studies in favour of the efficient view.

However, the upraising of market anomalies has questioned this conjecture. The controversial papers of Shiller (1981) and LeRoy and Porter (1981) claim that stock prices are too volatile to be in accordance with a present value model where future dividends are discounted with a constant rate. Other anomalies, such as the Monday and January effect and the still unresolved equity premium puzzle fuel the scepticism about efficient markets. Moreover, new considerations that model investors as agents with bounded rationality try to offer a more realistic approach of their behaviour and motivations.

Anomalies, though as a whole constituting important empirical facts, are not the most distinguished statistical features of financial markets. These are foremost the presence of some extreme events, many smaller disturbances and long periods of only minor price fluctuations that all can be connected by a simple power-law. Especially crashes that occur without major fundamental pre-announcements and then disappear without external forces, are intriguing subjects. However, financial markets are not the only systems that exhibit power-laws in their fluctuations. They are just among many other natural and social systems with many mutually interacting parts who constitute a complex organisation which occasionally drives itself out of equilibrium. Chapter 3.2 gives some compelling examples where natural phenomena show the same power-law for their fluctuations.

A central property of those complex systems was the ability to form collective behaviours among their constituent members. This kind of herding leads to the extreme events by the repetitive interactions of the parts. However, these interactions also ensure that the system comes back to an ordered state, without external interference. The network unfolds a degree of organisation which keeps the system in a fragile balance. From time to time turbulent phases emerge, however, they are followed by much longer periods of smooth fluctuations. The resulting time series mirrors this behaviour and produces stylised facts that can be extracted with statistical tools. These facts are common to all complex systems and must be seen as the decisive characteristics that distinguish complex systems from other systems. The empirical part of this work confirms the appearance of those empirical facts in the data sets of several stock markets. They are the fat-tailedness of price changes, implying a power-law; the multifractal nature of the time series and the absence of correlations in the raw returns except for high-frequent data, but strongly correlated volatilities where the time dependency extents over many periods.

Economic explanations for the underlying process that lead to such facts remain elusive. But the few tentative proposals all point to the existence of heterogeneity among traders. The chapter on simulations has taken up the idea of a complex financial system with many mutually interacting agents. It presents quite successful simulations - successful in the sense of producing time series that are statistically hard to distinguish from real data. In this respect they perform much better than the EMH.

However, there is still much room for improvements. For example, simulations of foreign exchange rate markets can be used to give a first hint about possible interventions to reduce undesired fluctuations. As a tool to defer speculative induced volatility, the often proposed Tobin Tax could be incorporated. A simulation may then be performed with varying tax values in order to infer at which level excess volatility is reduced. Excess volatility is thereby easily detected because the fundamental course of the exchange rate can be modelled by the simulator. In this manner, the artificial foreign exchange market can potentially point to how high obstacles to trade have to be. Other directions of future research may include models that use BFT much more than the simple simulations of this work. A step towards simulations, consistent with e.g. prospect theory is already done by LLS (2000), but this is definitely not end of that branch of microsimulations.

A rather obvious improvement of the Ising models would make the counter-reaction parameter endogenous. Chapter 7.4 has this important parameter still exogenous, but it is doubtful that agents never change their behaviour on that point. Another area of research is the development of models that also cover the other stylised facts, briefly mentioned at the beginning of part two. There is indeed a very active part of econophysists who focus on the explanation of cross-correlation between individual shares. Future simulations may also try to cope with problems that arise when analysing the microstructure of markets. Ising models as well as percolation models and deterministic simulations are principally able to account for market makers that have to take every order. Letting those market makers trade with each other - under the observation of uninformed investors - may lead to insights about which parameters or structures are responsible for a fast or slow prolefaration of information. Informational cascades are then not only based on purely theoretical considerations but can be enhanced by (computational) experiments, where different set-ups lead to different outcomes.

It is sometimes hoped that such simulations would allow the development of new and better than present forecast models. But the unpredictability postulate is strongly against this. Moreover, even if more reliable forecast could be made, the advantage of predicting the future would be self-destructing because observation by fellow traders induces them to copy the behaviour, thereby creating an efficient market. Thus profitable day-to-day forecasts are very unlikely. A way in which the idea of complex financial markets can improve upon existent theories as basis for predictions is the possibility to foresee extreme events. This is put forward by Johansen and Sornette (1999a,b) and Johansen, Sornette and Ledoit (1999). They claim to have found some form of log-periodic oscillations that can be interpreted as precursors of coming crashes. Even more, they are convinced to be able to predict the crash time, i.e. the time a complex system has reached its critical point where the behaviour of the market changes abruptly. In fact, they suggest to have unveiled a new mechanism, amplifying the size of fluctuations before a large catastrophe takes place and is thus less unpredictable than postulated by theory. The basic idea of the concept goes something like this: the motivation to buy assets is driven by expectations about future earnings. Although these expectations may first be based on sound economic factors, the investors slowly build up a self-fulfilling prophecy. This behaviour is explained by the assumption that traders are hierachielly intertwined and so influence each other in several ways, especially by enforcing the wish to act as other traders. Thereby, the slow growing bubble is fueled permanently by new traders, convinced to make a fortune in face of a bright economic future. This mechanism causes long-range correlations in the price process as the bubble needs a considerable time to fully blow up. However, at one stage, cooperative actions have reached a critical point where the market collapses within a short time interval. It is foremost the correlation, originating from the slowly build-up bubble that feeds the hope for a prediction based on past events.

Predictions must be evaluated on their success. Sornette (2003) reports of some successful forecasts but the number of applications is up to now not sufficient to prove the appropriateness of the method. Moreover, critics object the use of the equation that is the basis of the forecasts, because of a lack of a theoretical underpinning. Indeed, the concept does not offer a direct way to relate complex theory to the equation. But although a convincing model that substantiates the idea of precursors that are typical for complex systems is still warranted, this is a way to profit from the hypothesis of complex systems. And it would not be a small achievement considered the huge losses to people and the danger that may propagate to the real economy by giving false signals and so creating suboptimal distribution of financial resources.

Chapter 10

Bibliographie

Akgiray, V. and Lamoureux, C. (1989): "Estimation of Stable Law Parameters: A Comparative Study", Journal of Business and Economic Statistics, 7, 85-93.

Alvarez, L. W., Alvarez, W., Asaro, F. and Michel, H. V. (1980): "Extra Terrestrial Causes for the Cretaceous/Tertiary Extinctions," Science 208, 1095-1108.

Alvarez, W. and Asaro, F. (1990): "What caused the Mass Extinction: An Extraterrestrial Impact," Scientific American 263, 78-84.

Anderson, D. L. (2001): "How many Plates," Geology (Boulder) 30(5), 411-414.

Andersen, T. and Bollerslev, T. (1996): "DM-Dollar Volatility: Intraday Activity Patterns, Macroeconomic Announcements, and Longer Run Dependencies," NBER Working Papers 5783, National Bureau of Economic Research, Inc.

Andersen, T. and Bollerslev, T. (1997): "Intraday Periodicity and Volatility Persistence in Financial Markets," Journal of Empirical Finance 4, 115-158.

Andreev, A. V., Simons, B. D., Agam, O. and Altshuler, B. L. (1996): "Semiclassical Field Theory Approach to Quantum Chaos," Nuclear Physics B 482, 536-566.

Andreev, A. V., Simons, B. D., Agam, O. and Altshuler, B. L. (1996): "Semiclassical Field Theory Approach to Quantum Chaos," Nuclear Physics B 482, 536-566.

Aoki, M. (1996): *New Approaches to Macroeconomic Modeling: Evolutionary Stochastic Dynamics*, Cambridge University Press, New York.

Aoki, M. (2002): *Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Integrating Agents,* Cambridge University Press, New York.

Arad, R. W. (1980): "Parameter estimation for symmetric stable distribution," International Economic Review, 21, 209-220.

Arifovic, J. (1996): „The Behavior of the exchange rate in the genetic algorithm and experimental economics;“ Journal of Political Economy, 104, 510-541.

Arneodo, A., Muzy, J.-F. and Sornette, D. (1998): "Direct causal cascade in the stock market," Eur. Phys. Journal B 2, 277-282.

Arthur, W.B., Holland, B. Lebaron, B., Palmer, R. G., and Taylor, P. (1997): "Asset Pricing under Endogenous Expectations in an Artificial Stock Market," in: Arthur, W. B., Durlauf, S. and Lane, D., editors, *The Economy as an Evolving Complex System II*, Addison-Wesley, Redwood City, CA.

Bachelier, L. (1900): "Theorie de la Speculation," [Ph. D. thesis in mathematics], Annales Scientifiques de l'Ecole Normale Superieure III, 21-86.

Baillie, R. T. (1996): "Long memory processes and fractional integration econometrics," Journal of Econometrics, 73, 5-59.

Baillie, R., Bollerslev, T. and Mikkelsen, H. (1996): "Fractionally integrated generalized autoregressive conditional heteroskedasticity," Journal of Econometrics 74, 3-30.

Bak, P. , Tang, C. and Wiesenfeld, K. (1987): "Self-Organized Criticality. An Explanation of 1/f Noise," Physical Review Letters 59, 381-384.

Bak, P. (1996): *How Nature Works: The Science of Self-Organized Criticality*, Springer, New York.

Bak, P. and Paczuski, M. (1993): "Why Nature Is Complex," Physics World 6, 39-43.

Bak, P. and Tang, C. (1989): "Earthquakes as a Self-Organized Critical Phenomenon," Journal of Geophysical Research B94, 15635-15637.

Bak, P., Paczuski, M. and Shubik, M. (1997): "Price Variations in a Stock Market with Many Agents," Physica A 246, 430-453.

Balcilar, M. (2003): "Multifractality of the Istanbul and Moscow Stock Market Return," Emerging Markets and Finance and Trade, vol. 39, no.2, 5-46.

Bamberg, G. and Dorfleitner, G. (2001): "Is Traditional Capital Market Theory Consistent with Fat-Tailed Log-Returns," Zeitschrift für Betriebswirtschaft 72, 865-878.

Banarjee, A.V. (1992): "A simple model of herd behavior," The Quarterly Journal of Economics 107(3), 797-817.

Barberis, N., Shleifer, A. and Vishny, R. (1998): "A Model of Investor Sentiment," Journal of Financial Economics 49, 307-343.

Barber, B.Y., Odean, T. (2001): "Boys will be Boys: Gender, Overconfidence, and common Stock Investment," The Quarterly Journal of Economics 116, 261-292.

Barndorff-Nielsen, O. (1977): "Exponentially decreasing distributions for the logarithm of particle size," Proc. R. Soc. London A 353, 401-419.

Barndorff-Nielsen, O. E. and Prause, K. (2001): "Apparent scalling," Finance and Stochastics 5, 103-113.

Bartlett, M. S. (1946): "On the Theoretical Specification of Sampling Properties of Autocorrelated Time Series, Journal of the Royal Statistical Society, ser. B8, vol. 27.

Barkoulas, J. and Baum, C. F. (1996): "Long term dependence in stock returns," Economics Letters, 53, 253-259.

Barkoulas, J., Baum, C. F. and Travlos, N. (2000): "Long Memory in the Greek Stock Market," Journal of Applied Financial Economics, Vol. 10, 177-184.

Bar-Yam, Y. (1997): Dynamics of Complex Systems, Perseus Press, Cambridge.

Bascompte, J. and Sole, R. V. (1996): "Habitat Fragmentation and Extinction Thresholds in Spatially Explicit Models," Journal of Animal Ecology 65, 465-73.

Basu, S. (1977): "The Investment Performance of Common Stocks in Relation to their Price to Earnings Ratio: A Test of the Efficient Market Hypothesis," Journal of Finance 32, 663-682.

Basu, S. (1983): "The Relationship between Earnings Yield, Market Value, and Return for NYSE Common Stocks," Journal of Financial Economics 12, 126-441.

Baumol, W. J., and Quandt, R. E. (1964): "Rules of Thumb and Optimally Imperfect Decisions," American Economic Review 54, 23-46.

Beshadskii, A. (1999): "Multifractal critical phenomena in traffic and economic processes," Eur. Physical Journal B, 13, 595-599.

Beja, A. and Goldman, M. B. (1980): "On the dynamic behavior of prices in disequilibrium," Journal of Finance 35, 235-248.

Bell, D. E. (1982): "Regret in Decision-Making under Uncertainty," Operations Research 30, 961-981.

Bell, D. (1983): "Risk Premiums for Decision Regret," Management Science 29, 1156-1166.

Beran, J. (1994): *Statistics for Long-Memory Processes*, Chapman and Hall, New York.

Bergström, H. (1952): "On Some Expansion of Stable Distributions," Ark. Mathematicae II 18, 375-378.

Bernartzi, S. and Thaler, R. (1995): "Myopic Loss Aversion and the Equity Premium Puzzle," Quarterly Journal of Economics 110, 291-311.

Berry, A. C. (1941): "The Accuracy of the Gaussian approximation to the sum of independent variables," Trans. Amer. Math. Soc. 49, 122-136.

Bikchandani, S. and Sharma, S. (2000): "Herd behavior in financial markets," IMF Staff Papers 47, no. 3, 279-310.

Bikchandani, S., Hirshleifer, D. and Welch, I. (1992): "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," Journal of Political Economy 100 (5), 992-1026.

Billingsley, P. (1976): *Convergence of Probability Measures*, 2nd edition, John Wiley and Sons, New York.

Bingham, N. H., Goldie, C. and Teugels, J. L. (1989): *Regular Variation*, 2nd edition, Cambridge University Press, New York.

Black, F. (1988): "An Eqilibrium Model of the Crash," in: Stanley Fischer (ed.), NBER Macroeconomics Annual, MIT Press, 269-275.

Blanchard, O. (1979): ”Speculative bubbles, crashes and rational expectations,” Economics Letters, 3, 387-389.

Blanchard, O. and Watson, M. (1982): ”Bubbles, rational expectations and financial markets, in: P. Wachtel (ed), *Crisis in the Economic and Financial Structure*, Lexington Book, 295-316.

Blattberg, R. C. and Gonedes, N. J. (1974): “A Comparison of the Stable and Student Distributions as Statistical Model for Stock Prices,” Journal of Business 47, 244-280.

Bollabas, B. (1985): *Random Graphs.* New York: Academic Press.

Bollerslev, T. and Mikkelsen, H. O. (1996): “Modeling and Pricing Long Memory in Stock Market volatility,” Journal of Econometrics 73, 151-181.

Bollerslev, T. (1986): “Generalized Autoregressive Conditional Heteroskedasticity,” Journal of Econometrics 31, 307-327.

Borak, S., Härdle, W. and Weron, R. (2005): *Stable Distributions.* SFB 649 Discussion Paper 2005-008, Humboldt Universität.

Bornholdt, S. (2001): “Expectation bubbles in a spin model of markets: Intermittency from frustration across scales,” International Journal of Modern Physics C 12, 667-674.

Bouchaud, J. P. and Potters, M. (2001): “More stylized facts of financial markets: Leverage effect and downside correlations,” Science & Finance (CFM) Working Paper Archive 29960, Science & Finance, Capital Fund Management.

Bouchaud, J. P. and Potters, M. (2000): *Theory of Financial Risks: From Statistical Physics to Risk Management*, Cambridge University Press, New York.

Brant, R. (1984): “Approximate likelihood and probability calculations based on transforms,” Annals of Statistic 12, 989-1005.

Brav, A. and Heaton, J. (2001): *Competing Theories of Financial Anomalie*, Eleventh Annual Utah Winter Conference.

Breeden, D. (1979): “An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Oppurtunities,” Journal of Financial Economics 7, 265-296.

Breymann, W. , Ghashghaie, S. and P. Talkner (2000): “A Stochastic Cascade Model for FX Dynamics,” International Journal of Theoretical and Applied Finance, 3, 357–360.

Brock, W. (1993): “Pathways to Randomness in the Economy: Emergent Nonlinearity and Chaos in Economies and Finace,” Estudios Economicos, 8(1), 3-55.

Brock, W. and Durlauf, S. (1995): *Discrete Choice With Social Interaction I: Theory*, Mimeo, University of Wisconsin.

Brock, W. (1999): ”Scaling in economics: a reader’s guide. Industrial and Corporate Change, 8, 409-446.

Borsen, B. W. and Yang, S. R. (1990): ”Maximum Likelihood Estimates of Symmetrical Stable Distribution Parameters,” Communications in Statistics-Simulation and Computation, 19(4), 1459-1464.

Brown, S., Goetzmann, W. and Ross, S. (1995): "Survival," Journal of Finance 50, 853-873.

Bryson, M. C. (1974): "Heavy-Tailed Distributions: Properties and Tests," Technometrics 16, 61-68.

Bunde, A. and Havlin, S. (1991): *Fratals and Disordered Systems*. Springer-Verlag.

Campbell, J. Y. and Cochrane, J. (1999): "By force of Habit: A consumption–based Explanation of aggregate Stock Market Behaviour," Journal of Political Economy 107, 205-251.

Campbell, J. Y., Lo, A. W. and MacKinlay, A. (1997): *The Econometrics of Financial Markets*, Princeton University Press, Princeton.

Chan, L., Jegadeesh, N. and Lakonishok, J. (1996): "Momentum Strategies," Journal of Finance 51, 1681-1713.

Chang, I. and Stauffer, D. (1999): "Fundamental judgement in Cont-Bouchaud Herding model of market fluctuations," Physica A 264, 294-298.

Charbonneau, P., McIntosh, S. W., Liu, H.-L. and Bogdan, T. J. (2001): "Avalanche Models for Solar Flares," Solar Physics, 203, 321-353.

Chen, S. H., Lux, T. and Marchesi, M (1998): "Testing for non-linear structure in an artificial financial market," http://www.ge.infm.it/econophysics/papers/lux/nonlin.zip.

Cheung, Y.-W. and Lai Kon, S. (1995): "A search for long memory in international stock market returns, Journal of International Money and Finance, 14597-615.

Cheung, Y. W. and Menzie, D. C. (1999): "Are Macroeconomic Forecasts Informative? Cointegration Evidence from the ASA-NBER Surveys," NBER working paper No. 6926.

Chiarella, C. and He, X.-H. (2001): "Dynamics of Beliefs and Learning Under aL Processes - The Homogeneous Case," Research Paper Series 53, Quantitative Finance Research Centre, University of Technology, Sydney.

Chistyakov, V.P. (1964): "A theorem on sums of independent, positive random variables and its applications to branching random processes," Theory Probab. Appl. 9, 640-648.

Chowdury, D. and Stauffer, D. (1999): "A Generalized Spin Model of Financial Markets," Eur. Phys. J. B8, 477-482.

Clark, P. K. (1973): "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," Econometrica 41, 135-256.

Constantinides, G., Donaldson, J. B. and Mehra, R. (2001): "Junior Can't Borrow: A New Perspective on the Equity Premium Puzzle," manuscript, University of Chicago.

Constantinides, G. (1990): "Habit Formation: A Resolution of the Equity Premium Puzzle," Journal of Political Economy 98, 519-543.

Cont, R. (2001): "Deformation of implied volatility surfaces: An empirical analysis," in: Takayasu, H. (ed) Empirical Finance, Tokyo: Springer.

Cont, R. and Bouchaud, J.-P. (2000): "Herd Behavior and aggregate Fluctuations in Financial Markets," Marcoeconomic Dynamics, 4, 170-196.

Cootner, P. (1964): The Random Character of Stock Market Prices, MIT Press., Cambridge, Mass.

Coronel-Brizio, H. F. and Hernandez-Montoya, A. R. (2004): ”On fitting the Pareto-Levy distribution to stock market index data: selecting a suitable cutoff value,” arXiv:cond-mat/0411161 v1.

Courtault, J. M., Kabanov, Y., Bru, B., Crepel, P., Lebon, I. and Le Marchand, A. (2000): “Louis Bachelier on the centenary of Theorie de la Speculation,” Mathematical Finance, 10(3), 341-353.

Cross, F. (1973): “The Behavior of Stock Prices on Fridays and Monday,” Financial Analysts Journal 29, 67-69.

Cutler, D., Poterba, J. M. and Summers, L.H. (1989): “What moves Stock Prices,” Journal of Portfolio Management 16, 4-12.

Dacorogna, M. M., Gençay, R., Müller, U. A., Olsen, R. B., Pictet, O. V. (2001): *An Introduction to High-Frequency Finance.* Academic Press.

Dacorogna, M. M., Müller, U. A., Jost, C., Olsen, R. B., Pictet, O. V. and Ward, J. R. (1995): ”Heterogenous real-time trading strategies in the foreign exchange market,” European Journal of Finance, 1, 243-253.

Daniel, K., Hirshleifer, D. and Subrahmanyam, A. (1998): “Investor Psychology and Security Market under- and overreactions,” Journal of Finance 53, 1839-1885.

Danielson, J. and de Vries, C. G. (1997): “Tail index and quantile estimation with very high frequency data,” Journal of Empirical Finance 4, 241-257.

Dark, J. (2004): “Long Memory in the Volatility of the Australian All Ordinaries Index and the Share Price Index Futures,” Monash Econometrics and Business Statistics Working Papers 5/04, Monash University, Department of Econometrics and Business Statistics.

De Bondt, W. F. M. and Thaler, R. (1985): “Does the Stock Market overreact?,” Journal of Finance 40, 793-805.

De Finetti, B. (1937): “Foresight: Its logical laws, its subjective sources,” in: Studies in Subjective Probability (H. E. Kyburg and H. E. Smokler, eds.), Wiley, New York..

De Haan, L. and Resnick, S. I. (1979): “Derivatives of Regularly Varying Functions in Rd and Domains of Attraction of Stable Distributions,” Stoch. Proc. Appl. 8, 349-355.

De Haan, L. and Peng, L. (1998): “Comparison of tail index estimators,” Statistica Neerlandica, 52(1), 60-70.

DeLong, J. B. and Brecht, M. (1992): “Excess Volatility and the German Stock Market, 1876-1990,” NBER Working Papers 4054, National Bureau of Economic Research, Inc.

DeLong, J. B., Shleifer, A., Summers, L. and Waldmann, R. (1990): “Noise Trader Risk in Financial Markets,” Journal of Political Economy 98, 703-738.

Deheuvels, P., Häusler, E. and Mason, D. M. (1988): “Almost sure Convergence of the Hill Estimator,” Math. Proc. Cambridge Philos. Soc. 104, 371-381.

Dekkers, A. L., Einmahl, J. H. and de Haan, L. (1990): “A moment estimator for the index of an extreme value distribution,” Annals of Statistics

17, 1795-1832.

DePeretti, C. and Marimoutou, V. (2001): “Are the long memory tests really effective?,” Groupement de Recherche en Economie Quantitative d‘Aix-Marseille, Document de Travail, Nr. 02 A14.

Diamond, P. (1967): ”The role of a stock market in a general equilibrium model with technological uncertainty,” American Econimic Review, 57, 759-766.

Diamond, P. and Verrecchia (1981): ”Information aggregation in a noisy rational expectations model,” Journal of Financial Economics, 9, 221-235.

Diestel, R. (1996): *Graphentheorie.* Springer Verlag.

Ding, Z. and Granger, C. W. J. (1993): “A long memory property of Stock Market returns and a new model,” Jornal of Empirical Finance 1, 83-106.

Doob, J. L. (1953): *Stochastic Processes*, Wiley, New York.

Drees, H. and Kaufman, E. (1998): “Selecting the optimal fraction in univariate extreme value estimation,” Stochastic Processes and their Application, 75, 149-172.

Du Mouchel, W. H. (1971): *Stable Distributions in Statistical Inference,* Ph.D. dissertation, Department of Statistics, Yale University.

Du Mouchel, W. H. (1973): ”On the asymptotic normality of the maximum-likelihood estimate when sampling from a stabe distribution, Ann. Stat., 1, 948-957.

Du Mouchel, W. H. (1983): “Estimating the stable index a in order to measure tail thickness: A critique,” Ann. Statist. 11, 1019-1031.

Durlauf, S. (1991): “Multiple Equilibria and Persistence in Aggregate Fluctuations,” American Economic Review 81, 70-74.

Durlauf, S. (1994): “Path Dependence in Aggregate Output,” Industrial and Corporate Change 1, 149-172.

Eberlein, E. and Taqqu, M. (Eds) (1986): *Dependence in Probability and Statistics.* Birkhäuser, Bosten.

Eberlein, E. and Keller, U. (1999): “Hyperbolic Distributions in Finance,” Bernoulli 1, 281-299.

Edwards, W. (1968): *Conservatism in Human Information Processing,* in: Kleinmutz, B. (ed.), Formal Representation of Human Judgement, John Wiley and Sons, New York.

Eguiluz, M. and Zimmermann, M.G. (2000): „Transmission of Information and Herd Behaviour: An Application to Financial Markets,“ Phys. Rev. Lett. 85 (2000) 5659-5662.

Einstein, A. (1905): ”Die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen,” Annalen der Physik, 4. Folge, XVII, 549-560.

Elder, A. (1993): *Trading for living,* John Wiley and Sons, New York.

Elekdag, S. (2001): “Long Memory in the Volatility of Emerging Markets,” The ISE Finance Award Series Volume 3.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997): *Modelling Extremal Events*, Springer-Verlag, Heidelberg.

Engle, R. (1982): “Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflation,” Econometrica 50, 987-1008.

Engle, R. and Russell, J. (2002): "Analysis of High Freqeuncy Data," in Ait-Sahalia, Y. and Hansen, L. (eds.), Handbook of Financial Econometrics.

Epstein, L. and Zin, S. (1989): "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," Econometrica 57, 937-968.

Esseen, C. G. (1945): "Fourier Analysis of Distrbutions Functions. A Mathematical Study of the Laplace- Gaussian Law," Acta Math. 77, 1-125.

Falconer, K. (1990): *Fractal Geometry. Mathematical Foundations and Applications.* Wiley, New York.

Fama, E. (1965): "The Behavior of Stock Market Prices," Journal of Business 38, 34-105.

Fama, E. (1970): "Efficient Capital Markets: A Review of Theory and Empirical Work," Journal of Finance 25, 383-417.

Fama, E. (1991): "Efficient Capital Markets: II," Journal of Finance 46, 1575-1617.

Fama, E. (1998): "Market Efficiency, Long-term Returns and Behavioral Finance," Journal of Financial Economics 49, 283-306.

Fama, E. F. and Roll, R. (1968): "Some Properties of Symmetric Stable Distributions," Journal of American Statistics Association 63, 817-836.

Fama. E. F. and Roll, R. (1971): "Parameter Estimates for Symmetric Stable Distributions," Journal of American Statistical Association, 331-338.

Fama, E. and French, K. (1993): "Common Risk Factors in the Returns on Bonds and Stocks," Journal of Financial Economics 33, 3-56.

Fama, E. and French, K. (1998): "Value versus Growth: The international evidence," Journal of Finance 53, 1975-1999.

Feder, J. (1988): *Fractals.* Plenum Press, New York.

Feller, W. (1971): *An Introduction to Probability Theory and Its Application*, Vol. 2, Second Edition, Wiley and Sons, New York.

Ferris, S. P., Haugen, R. and Makhija, A. K. (1988): "Predicting Contemporary Volume with Historic Volume at Differential Price Levels: Evidence Supporting the Disposition Effect," Journal of Finance 43, 677-697.

Festinger, L.A. (1957): *A Theory of Cognitive Dissonance*, Stanford University Press, Stanford.

Feuerverger, A. and McDunnough, P. (1981): "On Efficient Inference in Symmetric Stable Laws and Processes", in: Csörgö, M., Dawson, D. A., Rao, J. N. K. and Saleh, A. K. (eds.): Statistics and Related Topics, North-Holland, Amsterdam.

Fillol, J. (2003): "Multifractality: Theory and Evidence. An Application to the French Stock Market, Economics Bulletin, 3, 1-12.

Fine, T. (1973): *Theories of Probability: An Examination of Foundations,* Academic Press, New York.

Fishburn, P.C. (1988): *Non-Linear Preference and Utility Theory,* John Hopkins University Press, Baltimore.

Föllmer, H. (1973): "On Entropy and Information Gain in Random Fields," Zur Wahrscheinlichkeitstheorie verw. Geb. 26, 207-217.

Frankel, J. A. and Froot, K. A. (1990): "Chartists, Fundamentalists, and Trading in the Foreign Exchange Market," American Economic Review, American Economic Association, vol. 80(2),181-185.

French, K. (1980): "Stock Returns and the Weekend Effect," Journal of Financial Economics 8, 55-70.

Friedman, M. (1953): "The Case for Flexible Exchange Rates," in: Essays in Positive Economics, University of Chicago Press, Chicago.

Friend, I. and Blume, M. E. (1975): "The Demand for Risky Assets," American Economic Review 65, 900-22.

Frisch, U. (1995): The Legacy of A. N. Kolmogorov, Cambridge University Press, Cambridge.

Galam, S. (1990): "Social Paradoxes of Majority Rule Voting and Renormalisation Group," Journal of Statistical Physics 61(3-4), 943-951.

Galam, S. (1997): "Rational Group Decision Making," Physica A 238, 66-80.

Galambos, S. and Kotz, S. (1978): *Characterizations of Probability Distributions,* Lecture Notes in Mathematics No. 675, Springer-Verlag, Heidelberg.

Galbraith. J. K. (1994): *A Short History of Financial Euphoria*, published by Viking Press.

Galbraith. J. K. (1997): *The Great Crash 1929,* Mariner Books, New York.

Garber, P.M., (1990): "Famous first bubbles," Journal of Economic Perspectives, vol.4 (2).

Garber, P. (2000): *Famous First Bubbles: The Fundamnentals of Early Manias,* MIT Press, Cambridge, Massachusetts.

Geweke, J. and Porter-Hudak, S. (1983): "The estimation and application of long memory time series models", Journal of Time Series Analysis 4, 221-38.

Ghashghaie, S., Breymann, W., Peinke, J., Talkner, P. and Dodge, Y. (1996): "Turbulent cascades in foreign exchange market, Natrure 381, 767.

Gibbons, M. and Hess, P. (1981): "Day of the week effects and asset returns," Journal of Business 54, 579-596.

Gibson, G. R. (1889): *The Stock Markets of London,* Paris and New York, G. P. Putnam's Sons, New York.

Gigerenzer, G. and Selten, R. (2002): *Bounded Rationality: The Adaptive Toolbox,* Cambridge, MA.

Gilli, M. and Kellezi, E. (2003): "An Application of Extreme Value Theory for Measuring Risk," unpublished working paper.

Gnedenko, B. V. and Kolmogorov, A. N. (1954): *Limit Theorems for Sums of Independent Random Variables,* Addison-Wesley, Cambridge, Massachusettes.#

Goetzmann, W. and Jorion, P. (1993): "Testing the Predictive Power of Dividend Yields," Journal of Finance 48, 663-679.

Goldberg, J. und von Nitsch, R. (2000): *Behavioral Finance.* Finanzbuch Verlag.

Goldie, C. and Klüppelberg, C. (1998): "Subexponential Distributions," in: A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy Tailed Distributions, 435-459, R. Adler, R. Feldman, M. Taqqu (Eds), Birkhäuser.

Gopikrishnan, P., Meyer, M., Amaral, L. A. and Stanley, H. E. (1998): "Inverse Cubic Law for the Distribution of Stock Price Variations," Eur. Phys. J. B 3, 139-140.

Gopikrishnan, P., Plerou, V., Amaral, L. A., Meyer, M., and Stanley, H. E. (1999): "Scaling of the distribution of fluctuations of financial market indices," Physical Review E, Vol. 60, No. 5, 5305-5316.

Gopikrishnan, P., Plerou, V., Liu, Y., Amaral, L. A., Gabaiyx, X. and Stanley, H. E. (2000): "Scaling and correlation in financial time series," Physica A, 287, 362-373.

Gordon, M., Paradis, G. E. and Rorke, C. H. (1972): "Experimental Evidence on Alternative Portfolio Decision Rules," American Economic Review 62, 107-118.

Gould, S. J. (1989): *Wonderful Life.* New York, Norton.

Graham, B. and Dodd, D. (1996): "Security Analysis," The Classic 1934 Edition New York: McGraw-Hill Trade.

Granger, C. W. J. (1966): "The typical spectral shape of an economic variable," Econometrica 34, 150-161.

Granger, C. W. J. (1981): "Some properties of time-series data and their use in econometric model specification," Journal of Econometrics 60, 203-233.

Granger, C. W. J. (1983): "Co-integrated variables and error correcting models," Discussion Paper 83-13, UCSD

Granger, C. W. J. and Joyuex, R. (1980): "An intoduction to long memory time series models and fractional differencing," Journal of Time Series Analysis 1, 15-39.

Grassberger, P. (1983):" On the fractal dimension of the Hénon attractor," Physics Letters 97A, 224-226.

Grassberger, P. and Procaccia, I. (1983): "Measuring the strageness of strage attractors," Physica 9D, 189-208.

Grasso, J. and Sornette, D. (1998): "Testing self-organized critically by induced seismicity," Journal of Geophys. Res., 103, B12, 29965-29987.

Gray, H., Zhang, N. and Woddward, W. A. (1989): "On generalized fractional processes," Journal of Time Series Analysis 10, 233-257.

Greene, M. and Fielitz, B. (1977): "Long term dependence in common stock returns," Journal of Financial Economics 4, 339-349.

Grossmann, S. and Stiglitz, J. (1980): "On the Impossibility of Informationally Efficient Markets," Economica 55, 437-460.

Gruen, D. W. R. and Gizychi, M. C. (1993): "Explaining Forward Discount Bias: Is it Anchoring?", Reserve Bank of Australia Research Discussion Paper No. 9307

Gumbel, E. J. (1958): *Statistics of Extremes.* Columbia University Press, New York.

Hagyard, M. J., Smith, Jr. J. B., Teuber, D., West, E. A. (1984), Solar Physics, 91, 115.

Haken, H. (1984): *Synergetik. Eine Einführung.* Springer-Verlag.

Hamilton, Rosanna L. (1995): *Earth's and Interior and Plate Tectonics.*

Hasbrouck, J. (1999): "Security Bid/Ask Dynamics with Discreteness and Clustering: Simple Strategies for Modeling and Estimation," Journal of Financial Markets, 2, 1-28.

Hausdorff; F. (1918): "Dimension und äußeres Maß," Mathematische Annalen, 79, 157-179.

Hauser, M. A., Kunst, R. M.and Reschenhofer, E. (1994): "Modelling exchange rates: long-run dependence versus conditional heteroscedasticity," Applied Financial Economics 4, 233-239.

Hawawini, G. and Keim, D. (1998): "The cross section of common stock returns: A Synthesis of the Evidence and Explanations," in: Security Market Imperfections in Worldwide Equity Markets, edited by Ziemba, W. T. and Keim, D., Cambridge University Press.

Heaton, J. and Lucas, D. (1996): "The Importance of Investor Heterogeneity and Financial Market Imperfections for the Behavior of Asset Prices," Carnegie Rochester Conference Series on Public Policy 42, 1-32.

Heaton, J. and Lucas, D. (1996): "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," Journal of Political Economy 104, 668-712.

Henry, O. T. (2002): "Long memory in stock returns: some international evidence," Applied Financial Economics, 12, 725-729.

Hentschel, H. G. E. and Procassia, I. (1983): "The Infinite Number of Generalized Dimensions of Fractals and Strange Attractors," Physica D 8, 391.

Hill, B. M. (1975): "A Simple General Approach to Inference About the Tail of a Distribution," Annals of Statistics 5, 1163–1174.

Hong, H. and Stein, J. (1999): "A Unified Theory of Underreaction, Momentum Trading and Overreaction in Asset Markets," Journal of Finance 54, 2143-2184.

Höpfner, R. and Rüschendorf, L. (1999): "Comparison of Estimators in Stable Models," Mathematical and Computer Modelling 29, 145-160.

Hosking, J. R. M. (1981): "Fractional differencing," Biometrica 68, 165-76.

Huang, Z. (2002): "The first 20 minutes in the Hong Kong stock market", arXiv:cond-mat/0006145 v1.

Hurst, H. E. (1951): "Long term storage capacity of reservoirs," Transactions of the American Society of Civil Engineers 116, 770-99.

Ibragimov, I. A. and Linnik, Y. V. (1971): *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen.

Ikenberry, D., Lakonishok, J. and Vermaelen, T. (1995): "Market Underreaction to open market share repurchases," Journal of Financial Economics 39, 181-208.

Ikenberry, D., Rankine, G. and Stice, E. (1996): "What do Stock Splits

really signal?," Journal of Financial and Quantitative Analysis 31, 357-375.
Iori, G. (2002): "A microsimulation of traders activity in the stock market: The role of heterogeneity, agents' interactions and trade frictions," Journal of Economic Behaviour and Organization 49, no. 1, 269-285.
Ito, K. and Matsuzaki, M. (1990): "Earthquakes as Self-Organized Critical Phenomena," Journal of Geophysical Research B95, 6853-6860.
Ivanova, K. and Ausloos, M. (1999): "Low q-moment multifractal analysis of Gold price, Dow Jones Industrial Average and BGL-USD exchange rate," European Physical Journal B 8, 665-669.

Jensen, M., (1978): "Some Anomalous Evidence Regarding Market Efficiency," Journal of Financial Economics, 6, 95-102.

Jegadeesh, N. and Titman, S. (1993): "Returns to buying Winners and selling Loosers: Implications for Stock Market Efficiency," Journal of Finance 48, 65-91.

Johnson, N., Kotz, S. and Balakrishnan, N. (1994): *Continuous Univariate Distributions,* Volume 1, 2nd edition, John Wiley & Sons, New York.

Johansen, A. and Sornette, D (1999a): "Modeling the stock market prior to large crashes," Eur. Phys. J. B 9:1, 167-174.

Johansen, A. and Sornette, D (1999b): "Financial "anti-bubbles": Log-periodicity in Gold and Nikkei collapses," Int. J. Mod. Phys. C 10(4), 563-575.

Johansen, A. and Sornette, D and Ledoit, O. (1999): "Predicting Financial Crashes using discrete scale invariance," Journal of Risk, vol. 1, number 4, 5-32.

Johnston, A. C. and Nava, S. (1985): "Recurrence Rates and Probability Estimates for the New Madrid Seismic Zone," Journal of Geophysical Research 90, 6737-6753.

Jost, K. L. (1995): "Creating a learning environment for teachers," Proceedings of the 1995 Annual National Convention of the Association for Educational Communications and Technology (AECT), (17th, Anaheim, CA). (ERIC Document Reproduction Service No. ED 383 313).

Kahnemann, D. and Tversky, A. (1973): "On the Psychology of Prediction," Psychological Review 80, 237-251.

Kahnemann, D. and Tversky, A. (1979): "Prospect Theory: An Analysis of Decision under Risk," Econometrica 47, 263-291.

Kaizoji, T., Bornholdt, S. and Fujiwara, Y. (2002): "Dynamics of price and trading volume in a spin model of stock markets with heterogeneous agents," Physica A 316, 441-452.

Kaizoji, T. (2000): "Speculative bubbles and crashes in stock markets: An interacting model of speculative activity," Physica A 287, 493-506.

Karagiannis, T., Faloutsos, M. and Riedi, R. (2002): "Long-Range Dependence: Now you see it, now you don't," Global Internet Symposium (in IEEE GLOBECOM), Taipei, Taiwan, November 17-21, 2002.

Karamata (1933): "Sur un mode de croissance régulière. Théorems fondamenteux." Bulletin Soc. Mat. France, 61, 55-62.

Kaufman, S. A. (1993): *The Origins of Order.* New York, Oxford: Oxford University Press.

Keane, S. M. (1983): *Stock Market Efficiency – Theory, Evidence, Implications,* Philip Allan Publishers Ltd., Oxford.

Keary, P., and Vine, F. (1996): *Global Tectonics*, Blackwell Scientific Publications, Oxford.

Keitt, T. H. and Marquet, P. (1996): "The introduced Hawaiian Avifauna reconsidered: Evidence for self-organized Criticality?," Journal of Theoretical Biology 182, 161-167.

Keynes, J. M. (1936): *The General Theory of Employment, Interest, and Money*, New York: Macmillan.

Khintchine, A. and Levy, P. (1936): "Sur les loi Stables," Acad. Sci. Pais 202, 374-376.

Kim, G. W., and Markowitz, H. (1989): "Investment Rules, Margin and Market Volatility," Journal of Portfolio Management 16, 47-52.

Kinney, W. R. and Rozeff, M. S. (1976): "Capital Market Seasonality: The Case of Stock Market Returns," Journal of Financial Economics 3, 376-402.

Klüppelberg, C. (1988): "Subexponential distributions and integrated tails," J. Appl. Probab. 35, 325–347.

Klüppelberg, C. and Mikosch, T. (1997): "Large Deviations of Heavy-Tailed Random Sums with Applications to Insurance and Finance," Journal of Applied Probabilities 34, 502-519.

Kocherlakota, N. (1996): "The Equity Premium: It is Still a Puzzle," Journal of Economic Literature 34, 42-71.

Koening, J. (1999): "Behavioral Finance: Examining thought processes for better investing," Trust & Investments 69, 17-23.

Kogon, S.M. and Williams, D. B. (1998): "On Characteristic Function Based Stable Distribution Parameter Estimation Techniques," in: A Practical Guide to Heavy Tails: Statistical Techniques for Analyzing Heavy Tailed Distributions, (R. Adler, R. Feldman, and M.S. Taqqu, eds.), Boston, MA: Birkhauser.

Koponen, I. (1995): "Analytic Approach to the Problem of Convergence of Truncated Levy Flights towards the Gaussian Stochastic Process," Phys. Rev. E 52, 1197-1199.

Koutrouvelis, I. A. (1980): "Regression-Type Estimation of the Parameters of Stable Laws," Journal of American Statistics Association 75, 918-928.

Koutrouvelis, I. A. (1981): "Large-sample quantile estimation in Pareto laws," Communications in Statistics, Part A - Thery and Methods 10, 189-201.

Krawiecki, A., Holyst, J. A. and Helbing, D. (2002): "Volatility Clustering and Scaling for Financial Time Series due to Attractor Bubbling," Physical Rev. Letters, vol. 89, No. 15, 158701-1-158701-4.

Krugman, P. (1996.): *The Self-Organizing Economy,* Blackwell Publishers, Malden, MA.

Lakonishok, J., Shleifer, A. and Vishny, R. (1994): "Contrarian Investment, Extrapolation and Risk," Journal of Finance 49, 1541-1578.

Leadbetter, M., Lindgren, G. and Rootzen, H. (1983): *Extremes and related properties of random sequences and processes,* Springer Series in Statistics, Springer-Verlag, New York.

Leland, H. and Rubinstein, M. (1988): "Comments on the Market Crash: Six months later," Journal of Economic Perspectives 2, 45-50.

LeRoy, S. F. and Porter, R. D. (1981): "The present-value relation: Tests based on implied variance bounds," Econometrica 49, 555-574.

Levy, M. and Solomon, S. (1996): "Power Laws Are Logarithmic Bolzmann Laws," Intl. J. Mod. Phys C 7, 595-601.

Levy, M., Levy, H. and Solomon, S. (1994): "A Microscopic Model of the Stock Market: Cycles, Booms and Crashes," Economics Letters 45,103-111.

Levy, M., Levy, H. and Solomon, S. (1995): "Microscopic Simulation of the Stock Market," Journal de Physique I France 5 (1995) 1087-1103.

Levy, M., Levy, H. and Solomon, S. (2000): *Microscopic Simulation of Financial Markets,* Academic Press, New York.

Levy, P. (1925): *Calcul des Probabilities,* Gauthier-Villars, Paris.

Lim, K. (1990): "Arbitrage and Price Behaviour of Nikkei Stock Index Futures," Journal of Futures Markets 12, 151-161.

Lintner, G. (1998): "Behavioral finance: Why investors make bad decisions," The Planner 13(1), 7-8.

Liu, W., Strong, N. and Xu, G.(1999): "The profitability of momentum investing," Journal of Business Finance and Accounting, 1043-1091.

Lo, A. (1991): "Long-term memory in stock market prices." Econometrica, 59, 1279-1313.

Lo, A. W. and MacKinlay, A. C. (1999): A non-random walk down the Wall Street. Princeton University Press.

Lobato, I. N. and Savin, N. E. (1997): "Real and Spurious Long-Memory Properties of Stock-Market Data," Journal of Business and Economic Statistics, 16, 261-283.

Loomes, G. and Sugden, R. (1982): "Regret theory: An alternative theory of rational choice under uncertainty," Economic Journal 92, 805-24.

Loomes, G. and Sugden, R. (1987): "Testing for regret and disappointment in choice under uncertainty," Economic Journal 97, 118-129.

Loretan, M. and Phillips, C. B. (1994): "Testing the Covariance Stationarity of Heavy-Tailed Time Series: An Overview of the Theory with Applications to Several Financial Instruments," Journal of Empirical Finance 1, 211-248.

Loughran, T. and Ritter, J. (1995): "The new Issues Puzzle," Journal of Finance 50, 23-51.

Loughran, T. and Ritter, J. (2001): "Why has IPO underpricing changed over time?," unpublished University of Florida Working Paper.

Lu, E. T. and Hamilton, R. J. (1991): "Avalanches and the Distribution of Solar Flares," Astrophysical Journal 189, 380-389.

Lucas, R. (1978): "Asset Prices in an Exchange Economy," Econometrica 46, 1429-1446.

Lux, T. (1996): "The Stable Paretian Hypothesis and the Frequency of Large Returns: An Examination of Major German Stocks," Applied Financial Economics, vol. 6 (1996), 463- 475

Lux, T. and Marchesi, M. (1999): "Volatility Clustering in Financial Markets: A Micro-Simulation of Interacting Agents," Nature 397, 498-500.

Mackay, C. (1852): *Memoirs of Extraordinary Popular Delusions and the Madness of Crowds,* 2nd edition, Office National Illustrated Library, London.

MacKinlay, A. and Ramaswamy, K. (1988): "Index-Future Arbitrage and the Behavior of Stock Index Future Prices," The Review of Financial Studies 1, No. 2, 137-158.

Main, I.G. (1995): "Earthquakes as critical phenomena: Implications for probabilistic seismic hazard analysis," Bull. Seismol. Soc. Am. 85, 1299-1308.

Malkiel, B. G. (2003): "The Efficient Market Hypothesis and Its Critics," Journal of Economic Perspectives 17 (1), 59-83.

Malone, T. W. and Rockart, J. F. (1991): "Computers, networks and the corporation," Scientific American 256, 128-136.

Mandelbrot, B.B. (1963a): "New Methods in Statistical Economics," Journal of Political Economy 71, 421-440.

Mandelbrot, B.B. (1963b): "The variation of certain speculative prices," The Journal of Business 36, 394–419.

Mandelbrot, B.B. (1966): "Forecasts of Future Prices, Unbiased Markets and Martingale Models," Journal of Business 39, 242-255.

Mandelbrot, B.B. (1966): "How ong is the coast of Great Britain? Statistical self similarity and fractal dimension," Scienes, 156, 636-638.

Mandelbrot, B. (1971): "How can Price be Arbitraged Efficiently? A Limit to the Validity of the Random Walk and Martingale Models," Review of Economics and Statistics 53, 225-236.

Mandelbrot, B B. (1972): " Statistical methodology for non-periodic cycles: from the covariance to R/S analysis." Annals of Economic and Social Measurement: 1, 257-288.

Mandelbrot, B B. (1977): *Fractals. Form, Chance and Dimension.* San Fransisco CA: W. H. Freeman and Company.

Mandelbrot, B B. (1982): *The Fractal Geometry of Nature.* New York NY: W. H. Freeman and Company.

Mandelbrot, B. B. and Wallis, J. R. (1968): "Noah, Joseph and operational hydrology." Water Resources Research: 4, 909-918.

Mandelbrot, B. B. and Ness, J. V. (1968): "Fractional Brownian Motions, Fractional Noises and Applications," SIAM Review 10, 422-437.

Mandelbrot, B. B., Fisher, A. and Calvet, L. (1997): "A Multifractal Model of Asset Returns. Cowles Foundation Discussion Paper No. 1164. Yale University.

Mankiw, N. G., Romer, D. and Shapiro, M. (1985): "An Unbiased Reexamination of Stock Market Volatility," Journal of Finance 40, 677-687.

Mantegna, R. N. and Stanley, H. E. (1994): "Stochastic Process with Ultraslow Convergence to a Gaussian: The Trucated Levy Flight," Phys. Rev. Lett. 73, 2946-2949.

Mantegna, R. N. and Stanley, H. E. (1995): “Scaling Behavior in the Dynamics of an Economic Index,” Nature 376, 46-49.

Mantegna, R. N. and Stanley, H. E. (1997): “Physics Investigation of Financial Markets,” in: Proceedings of the International School of Physics, edited by Mallamace, F. and Stanley, H. E., IOS Press, Amsterdam.

Mantegna, R. N. and Stanley, H. E. (2000): An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge.

Mason, D. M. (1982): “Laws of Large Numbers for Sums of Extreme Values,” Ann. Probab. 10, 756-764.

Matia, K., Amaral, L., Goodwin, S. and Stanley, E. (2002): “Different scaling behaviors of commodity spot and future prices,” Physical Review E, 66, 045101-1 045101-4.

McCulloch, J. H. (1986): “Simple Consistent Estimators of Stable Distribution Parameters,” Commun Statist. Simulation Comput. 15, 1109-1136.

McCulloch, J. H. (1997): “Measuring Tail Thickness to Estimate the Stable Index a: A Critique,” Journal of Business and Economic Statistics 15, 74-81.

McCulloch, J. H. (1998): ”Linear regression with stable disturbances,” in Adler, R. J., Feldman, R. E., Taqqu, M. S. (Eds.), *A Practical Guide to Heavy Tails:* Statistical Techniques for Analysing Heavy Tailed Distributions. Birkhäuser, Boston.

McGrattan, E. R. and Prescott, E. C. (2001): ”Is the Stock Market Overvalued?,” NBER Working Papers 8077, National Bureau of Economic Research, Inc.

McLeod, A. I. and Hippel, K. W. (1978): “Preservation of the Rescaled Range 1: A Reassesment of the Hurst Phenomenon,” Water Resources Research 14, 491-508.

Merton, R. (1987): “A sinple Model of Capital Market Equilibrium with incomplete Information,” Journal of Finance 42, 483-510.

Mehra, R. and Prescott, E. C. (1985): “The Equity Premium: A Puzzle,” Journal of Monetary Economics 15, 145-161.

Mehra, R. (2001): “The Equity Premium,” in: The Global Investor Book of Investing Rules, ed. by Jenks, P. and Eckett, S., Harriman House Ltd.

Menkhoff, L. (1998): “The Noise Trading Approach - Questionnaire Evidence from Foreign Exchange,” Journal of International Money and Finance 17, 547-564.

Mills, T. C. (1993): “Is there a Long-Term Memory in UK Stock Returns,” Applied Financial Economics 3(4), 303-306.

Mirowski; P. (1995): “Mandelbrot’s Economics After A Quarter Century,” Fractals, Vol. 3, No. 3, 581-600.

Müller, U. A., Dacorogna, M. M., Olsen, R. P., Pictet, O. V., Schwarz, M. and Morgenegg, C. (1990): ”Statistical Study of Foreign Exchange Rates, Empirical Evidence of a Price Change Scaling Law, and Intraday Analysis,” Journal of Banking and Finance, Volume 14, 1189-1208.

Müller, U. A., Dacorogna, M. M., Dave, R., Olsen, R., Pictet, O. V. and von Weizsäcker, J. (1997): "Volatilities of different time resolutions – Analyzing the dynamics of market components," Journal of Empirical Finance 4, 213-239.

Müller, U. A., Dacorogna, M. M., Pictet, O. V. (1998): "Heavy tails in high-frequency financial data", in: Adler, R. J., Feldman, R. E., Taqqu, M. S. (Eds.), *A Practical Guide to Heavy Tails:* Statistical Techniques for Analysing Heavy Tailed Distributions. Birkhäuser, Boston.

Muzy, F.-J., Delour, J., Barcy, E. (2000): "Modelling fluctuations of financial time series: from cascade process to stochastic volatility model," Eur. Phys. Journal B 17, 537-548.

Newey, W. K. and West, K. D. (1987): "A Simple Positive Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix," Econometrica, 55, 703-708.

Newman, M. E. (1997): "A model of mass extinction," Journal of Theoretical Biology 189, 235-52.

Nelson, C. and Kim, M. (1993): "Predictable Stock Returns: The Role of small Sample Bias," Journal of Finance 48, 641-661.

Nolan, J. (1997): "Maximum likelihood estimation and diagnostics for stable distributions," unpublished working paper.

Nolan, J. : program stable.exe available at: http://academic2.american.edu/~jpnolan

Northcraft, G. B. and Neale, M. A. (1987): "Expert, amateurs, and real estate: An anchoring-and-adjustment perpective on property pricing decisions," Organizational Behavior and Human Decision Processes 39, 228-241.

Odean, T. (1999): "Do investors trade too much?," American Economic Review 89, 1279-1298.

Orcutt, G. H., Caldwell, S. and Wertheimer, R. (1976): *Policy Exploration Through Microanalytik Simulation,* Urban Institute Press, Washington, DC.

Pagan, A. (1996): "The Econometrics of Financial Markets," Journal of Empirical Finance 3, 15-102.

Palagyi, Z. and Mantegna, R. N. (1999): "Empirical Investigation of Stock Price Dynamics in an Emerging Market," Physica A 269, 132-139.

Palomino, F. (1996): "Noise trading in small markets," Journal of Finance 51 (4), 1537–1550.

Paulson, A. S. and Delahanty, T. A. (1985): "Modified weighted squared error estimation procedures with special emphasis on the stable laws," Comm. in Statistics Simulation and Computation, 14(4), 927-972.

Pareto, V. (1897): *Cours d'Economie Politique*, Lausanne and Paris.

Parker, E. N. (1983): "Magnetic Neutral Sheets in Evolving Fields - Part two - Formation of the Solar Corona," Astrophysical Journal, 264, 642-647.

Parker, E. N. (1988): "Nanoflares and the solar X-ray corona," Astrophysical Journal, Part 1, vol. 220, 474-479.

Pasquini, M. and Serva, M. (2000): "Clustering of volatility as multiscal phenomenon," Eur. Physical Journal B, 16, 195-201.

Paulsen, A. S., Holcomb, E. W. and Leitch, R. (1975): "The estimation of the parameters of the stable laws," Biometrika 62, 163-170.

P**eitgen, H.-O., Jürgens, H. and Saupe, D. (1998):** *Bausteine des Chaos. Fraktale.* Rowohlt Taschenbuch Verlag GmbH.

Pickands, J. (1975): "Statistical Inference using extreme order Statistics," Annual Statistics 3, 119-131.

Popper, K. R. (1959): *The Logic of Scientific Discovery*, New York: Harper and Row.

Popper, K. R. (1979): *Die beiden Grundprobleme der Erkenntnistheorie*, Tübingen: J.C.B. Mohr(Paul Siebeck).

Poterba, J. M. and Summers, L. H. (1988): "Mean Reversion in Stock Prices? A Reappraisal of the Empirical Evidence," Review of Economic Studies 58, 515-528.

Porter-Hudak, S. (1990): "An application to the seasonal fractionaly differenced model to the monetary aggregates," Journal of the American Statistical Association 85, 338-344.

Press, S. J. (1972): "Estimation in univariate and mulitvariate stable distributions," JASA 67, 842-846.

Press, A. and Siever, C. (1986): *Earth,* W. H. Freeman, New York.

Priestley, M. B. (1981): *Non-Linear and Non-Stationary Time Series Analysis,* Academic Press, London.

Presidential Task Force on Market Mechanisms, Report of The Presidential Task Force on Market Mechanisms (Brady Comision Report). Washington D.C.: U.S. Government Printing Office 1988.

Rachev, T. S. and Mittnick, S. (2000): *Stable Paretian Models in Finance.* Wiley Series in Financial Economics and Quantitative Analysis.

Ramsey, F. P. (1931): *The foundations of mathematics and other logical essays,* Routledge & Kegan Paul, London.

Rasmusen, E. (2001): *Games and Information,* Blackwell Publishers, Malden, Ma.

Raup, D. M. and Sepkoski, J. J. (1984): "Periodicity of extinctions in the geologic past, " Proc. Natl. Acad. Sci. USA, 81, 801-805.

Raup, D. M. (1991): *Extinction: Bad Genes or Bad Luck,* Norton, New York.

Resnick, S. I. (1987): *Extreme Values, Regular Variation and Point Processes,* Springer, New York.

Resnick, S. I. (1997): "Discussion of the Danish Data on large Fire Insurance Losses," Astin Bulletin 27, 139-151.

Resnick, S. I. and Starica, C. (1995): "Consistency of Hill's estimator for dependent data," Journal of Applied Probabilities 32, 239-267.

Resnick, S. I. and Starica, C. (1997): "Smoothing the Hill Estimator," Advances in Applied Probability 29, 271-293.

Ridley, M. (1993): *Evolution,* Blackwell Scientific Publications, Cambridge, Ma.

Roberts, H. (1967): *Statistical versus Clinical Prediction of the Stock*

Market, unpublished manuscript, Center for Research in Security Prices, University of Chicago.

Robinson, P. M. (1992): "Semiparametric analysis of long-memory time series," Annals of Statistics 22, 515-39.

Roll, R. (1984): "A simple implicit measure of the effective bid-ask spread in an efficient market." Journal of Finance, 39, 1127-1159.

Roótzen, H., Leadbetter, M. R. and de Haan, L. (1992): Tail and quantile estimators for strongly mixing stationary processes. Report, Department of Statistics, University of North Carolina.

Rosenblatt, M. (1956): "Remarks on some nonparametric estimates of a density function," Annals Mathematical Statistics 27, 832-837.

Rubinstein, M. (1988): "Portfolio Insurance and Market Crash," Financial Analysts Journal 44, 38-47.

Rubinstein, M. (2000): "Comments on the 1987 Stock market crash: Eleven years later," Risks in Accumulation Products, Society of Actuaries.

Samorodnitsky, G. and Taqqu, M.S. (1994): *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance,* Chapman and Hall, New York.

Samuelson, P. (1963): "Risk and Uncertainty: A Fallacy of Large Numbers," Scientia 98, 108-113.

Samuelson, P. (1965): "Proof that properly anticipated prices fluctuate randomly," Industrial Management Review 6, 465-89.

Samuelson, P. (1989): "The $\sqrt{N}$ Law and Repeated Risktaking," Probability, Statistics, and Mathematics, Papers in Honor of Samuel Carlin. Academic Press, Inc.

Samuelson, P. (1994): "The Long-Term Case for Equities: And How It Can Be Oversold," Journal of Portfolio Management 21, 15–24.

Savage, L. J. (1954): *The Foundations of Statistics*, Wiley, New York.

Scharfstein, D. and Stein, J. (1990): "Herd Behaviour and Investment," American Economic Review 80, 465-489.

Schmitt, F., Schertzer, D. and Lovejoy, S. (1999): "Multifractal Analysis of Foreign Exchange Data," Applied Stochatic Models and Data Analysis, 15, 29-53.

Schroeder, M. (1991): *Fractals, Chaos, Power Laws,* Freeman, New York.

Sepkoski, J. J. Jr. (1993): "Ten Years in the Library: New Data confirm Paleontological Patterns," Paleobiology 19, 43-51.

Seuront, L., Schmitt, F., Lagadeuc, Y, Schertzer, D. and Lovejoy, S. (1999): "Universal multifractal analysis as a tool to charaterize multiscale intermittent patterns: example of phytoplankton distribution in turbulant coastal waters," Journal of Plankton Research, Vol. 21, No. 5, 877-922.

Shefron, H. and Stratman, M. (1985): "The Disposition to ride Winners Too Long and Sell Lowers Too Soon: Theory and Evidence," Journal of Finance 41, 774-790.

Shiller, R. J. (1981): "Do stock prices move too much to be justified by subsequent movements in dividends?," American Economic Review 71, 421-436.

Shiller, R. J. (1987): "Ultimate Sources of Aggregate Variability," NBER Working Papers 2129, National Bureau of Economic Research, Inc.

Shiller, R. (1989): *Market Volatility*, MIT Press, Cambridge.

Shiller, R.J. (1998): *Macro Markets: Creating Institutions for Managing Society's largest Economic Risks,* Oxford University Press, Oxford.

Shiller, R. J. (2000): *Irrational Exuberance, Princeton University Press*, Princeton, New Jersey.

Shiller,R.J.(2003): *The New Financial Order. Risk in the 21st Century*, Princeton University Press, Princeton, New Jersey.

Shiller, R. J. and Pound, J. (1987): Survey Evidence on Diffusion of Interest among Institutional Investors, Cowles Foundation Discussion Papers 794, Cowles Foundation, Yale University.

Shiryaev, A.N. (1999): *Essentials of Stochastic Finance*, World Scientic, Singapore.

Shleifer, Andrei (1999): *Inefficient Markets. An Introduction to Behavioral Finance.* Oxford University Press.

Siegel, J. (1998): *Stocks for the Long Run*, McGraw-Hill, New York.

Siegel, J. J. and Thaler, R. H. (1997): "Anomalies: The Equity Premium Puzzle," Journal of Economic Perspectives 11(1), 191–200.

Simon, Herbert (1957): "A Behavioral Model of Rational Choice," Quarterly Journal of Economics 69, 99–118.

Sinai, Y. G. (1976): *Introduction to Ergodic Theory*, translated by V. Scheffer, Princeton University Press, Princeton.

Sornette, A. and Sornette, D. (1989): "Self-Organized Criticality and Earthquakes," Europhysics Letters 9, 197-212.

Sornette, D. (2003): *Why Stock Markets Crash: Critical Events in Complex Financial systems.* Princeton University Press.

Slovic, P. (1972): "Psychological study of human judgment: Implications for investment decision making," The Journal of Finance 27(4), 779-799.

Sznajd-Weron, K. and Weron, R. (2002): "A simple model of price formation," Int. J. Mod. Phys. C 13, 115.

Stauffer, D. and Aharony, A. (1994): *An Introduction to Percolation Theory.* Taylor & Francis.

Stauffer, D. and Penna, T. J. P. (1998): "Crossover in the Cont-Bouchaud percolation model for market fluctuations," Physica A 256, 284-290.

Stauffer, D. and Sornette, D. (1999): "Self-Organized Percolation Model for Stock Market Fluctuations," Physica A 271, 496-506.

Stauffer, D. and Jan, N. (2000): "Sharp peaks in the percolation model for stock markets," Physica A, 277, 215.

Stigler, George (1964): "A Theory of Oligopoly," Journal of Political Economy 72, 44-61.

Swalm, R. O. (1966): "Utility Theory: Insights into Risk Taking," Harvard Business Review 47, 123-136.

Taqqu, M. S. (1975): "Weak convergence to fractional Brownian to the Rosenblatt process," Zeitschrift für Wahrscheinlichkeit und verwandte Gebiete, 31, 287-302.

Taqqu, M.S.(1977): "Law of the iterated logarithm for sums on non-linear functions of Gaussian random variables," Zeitschrift für Wahrscheinlichkeit und verwandte Gebiete, 40, 203-238.

Taylor, S. (1986): *Modelling Financial Time Series.* John Wiley, New York.

Tisdell, C. (1996): *Bounded rationality and economic evolution*, Edward Elgar, Cheltenham.

Tolvi, J. (2003): "Long memory in a small stock market." Economics Bulletin, University of Turku.

Uchaikin, V. V. and Zolotarev, V. M. (1999): *Chance and Stability: Stable distribution and their applications*, Academic Publishers, Brill.

Van Damme, A. (1976): *Aanteekeningen Betreffende de Geschiedenis der Bloembollen*, Haarlem 1899-1903, Boerhaave Press, Leiden.

Vanderwall, N. and Ausloos, M. (1998): "Sparseness and roughless of foreign exchange rates," International Journal of Modern Physics C9, 711-719.

Vassillicos, J. C., Demos, A. and Tata, F. (1993): "No Evidence of Chaos but some Evidence of Multifractals in the Foreign Exchange and the Stock Market," in: Crilly, A. J., Earnshaw, R. A. and Jones, H. (eds), Applications of Fractals and Chaos. Berlin: Springer.

Von Neumann, J. and Morgenstern, O. (1944): *Theory of Games and Economic Behaviour*, Princeton University Press, Princeton, New Jersey.

Voss, R. (1988): "Fractals in Nature", in: *The Science of Fractal Images,* Peitgen, H.-O. and Saupe, D. (Ed.), Springer-Verlag, New York.

Wang, B. H. and Hui, P. M. (2001): "The distribution and scaling of fluctuations for Hang Seng index in Hong Kong stock market," European Physical Journal B, 20, 573-579.

Wang, S. (2003): "High Frequency Returns - Mean reversion and volatility persistence." Working paper. University of Manchester.

Wegener, A. (1912): "Die Entstehung der Kontinente," Geologische Rundschau 3, 276–292.

Weil, P. (1989): "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," Journal of Monetary Economics 24, 401-421.

Weron, R. (1995): "Performance of the estimators of stable law parameters." Research Report HSC/95/1, Wroclaw University of Technology.

Weron, R. (2001): "Lévy-stable distributions revisited: Tail index > 2 does not exclude the Lévy-stable regime," International Journal of Modern Physics C, Vol. 12, No. 2, 209-223.

White, R. and Engelen, G. (1993): "Cellular Automata and Fractal Urban Form: A Cellular Modeling Approach to the Evolution of Urban Land Use Patterns," Environment and Planning A 25, 1175-1199.

Williams, J. B. (1938): *The Theory of Investment Value*, Harvard University Press, Cambridge.

www.economy.com/freelunch

Yodav, P. K. and Pope, P. (1990): "Mean Reversion in Stock Index Futures Mispricing: Evidence from the US and the UK," New York University Salomon Center Working Paper. Presented at the French Fiance Association Conference, the American Finance Association Conference and the World Econometric Congress (Japan).

Young, L.-S. (1982): "Dimension, Entropy, and Lyapunov Exponents," Ergodic Theory and Dynamic Systems, 2, 109.

Young, W. (2000): "Parallel Trading," Actice Trader, 22-25.

Zhang, Y.-C. (1998): "Modeling Market Mechanism with Evolutionary Games," Europhys. News 29, 51.

Zipf, G. K. (1949): *Human behavior and the principle of least efford,* Addison-Wesley Press, Cambridge, Mass.

Chapter 11

Appendix: The Ising Model

11.1 The Ising Model

In 1925 Ernst Ising constructed a simple yet powerful model for mimicking empirically observed facts about ferromagnetic materials. With all its variations, it has nowadays become the most thoroughly researched model in the field of statistical physics. This appendix chapter aims to stretch out the main ideas and concepts of the model. It will not go deeply in the more sophisticated topics of condensed matter physics, nor does it give any detailed explanation for well known physical facts.

11.1.1 The Background

In atoms, electrons, moving around their nucleons, build very small magnetic fields, just as the electrical current flowing in a loop does. In this respect, atoms behave like ordinary ferromagnets and can be thought of in terms of small magnet vectors pointing e.g. either to the north or south. Since every solid-state consists of many atoms, the essential premise behind the Ising model is that the magnetism of a bulk material is made up of the combined magnetic (dipole) moments of the atoms within the material. Now, usually these atomistic magnets are pointing in random directions, and so by cancelling each other out leave the material with no magnetic field at all. This is not the case for ferromagnets. There, large clusters of atoms show in the same direction giving a total magnetic field which is not zero.

In general, the question of whether and how strong a magnetic field may show up hinges on the balance of two following opposing principles:

11.1.2 Energy Minimisation

The interactions between the atomic-scale magnets (also called "spins") is such that the lowest energy configuration with two spins has the two spin vectors pointing in the same direction. From the perspective of energy alone, the lowest energy state of a large chunk of matter would have all the little spin vectors aligned, giving huge total magnetic fields!

11.1.3 Entropy Maximisation

The configuration in which all atomic magnets line up is one very special case out of an incomprehensibly large number of possible configurations. Unless there is a huge "energy cost" for an individual spin which is not lined up with its neighbours, the sheer number of possible unaligned configurations completely swamps the one unique ground state with all spins' orientation equal. Hence a macroscopic-sised system shows then no net magnetization. The randomness of the real configuration (and randomness is essentially what entropy measures) tends to wash out the large scale magnetism predicted by energy considerations alone.

Of course, it is impossible to say a priori which principle will carry through energy minimisation or entropy maximisation, but probability theory helps to characterise the relative importance of the numerous microscopic configurations: Suppose one has N spins and all of them can only point either up $(+1)$ or down (-1). Let S be the set of all possible configurations

$$S = (S_1, S_2, ..., S_n), \tag{A.1}$$

then, the probability that a particular state (S_i) actually occurs is given by the Boltzmann probability distribution function:

$$\pi(S_i) = \frac{e^{-E(S_i)/k_B T}}{\sum\limits_{S_i} e^{-E(S_i)/k_B T}} \tag{A.2}$$

where $E(S_i)$ is the energy of the system when it is in state S_k, T is the (absolute) temperature, k_B is the Boltzmann constant and $\sum\limits_{S_i}$ is the sum over all possible states.[1] In order to consider the importance of the temperature, imagine a solid-state with a very high T. Then the atoms move around randomly, showing us specific pattern of up's and down's. In this case entropy dominates and we have on average a zero magnetic field. To gain some insight consider two different states with $E(S_i) < E(S_k)$. The relative probability that the system is in the two states is given by

$$\frac{P[S_i]}{P[S_k]} = e^{-\Delta E_i / B} \tag{A.3}$$

$$\Delta E_i = E(S_i) - E(S_k) < 0. \tag{A.4}$$

At high temperatures (i.e. for much larger than the energy difference $|\Delta E_i|$), the system becomes equally likely to be in either of the states S_i or S_k – that

[1] this ensures that the sum of the probabilities $\sum\limits_i P(S_i)$ is one.

is, randomness and entropy "win". on the other hand, if the energy difference is much larger than B, the system is far more likely to be in the lower energy state.

11.2 The Simulation

The Ising model assumes that every spin is located on one site of a square lattice and can only have two states, i.e. $S_i = \pm 1$.The interaction of the spins is captured by the tendency of each site to force its nearest lattice neighbour into the same state it has itself (with exchange energy J). Usually an external force (or magnetic field) S_k is added to the model. S_k is acting on all sites and influence them to reach either $+1$ or -1. The energy of any particular state within the system is expressed through

$$E = -J \sum_{(ij)} S_i S_j - B \sum_i S_i. \tag{A.5}$$

The total magnetic field is calculated via

$$M = \sum_i S_i. \tag{A.6}$$

Obviously, if the spins are distributed totally randomly, $\langle M \rangle$ should be around zero.[2]

The system, left to its own evolves through a number of independent random flips (changes from one state to another). An individual spin flip causes an energy change of

$$\Delta E_i = E_{New} - E_{Old} = E(S_i) - E(S_k), \tag{A.7}$$

where k is the present state and; is the present state and; is the possible new one. And, according to the standard Boltzmann distribution, the likelihood that this transition occurs is of the form:

The following algorithm, named after its inventor the Metropolis-algorithm, attempts to mimic the nature of the evolution of a ferromagnetic solid matter by five consecutive steps that have to be repeated for all N spins:

1. Select a spin to be investigated.
2. Calculate the energy change $\Delta E = E_{new} - E_{old}$ associated with a possible spin flip of S_i into $-S_i$.
3. Compare a uniformly distributed random number z, with $0 < z < 1$, with the thermal probability $p = \exp(-\Delta E_i / k_B T)$ to flip that spin.

[2] $\langle \cdot \rangle$ again indicates averaged values.

4. Flip spin, $S_i = -S_i$, if and only if $z < p$.
5. Use the present configuration, whether S_i was flipped or not, in order to calculate any desired averages, e.g. $\langle M \rangle$.

Chapter 12

List of Symbols

Chapter 2:

Ω = Information set
P = Price of an asset
μ = Drift parameter
S_n = Sum of n random variables
F(x) = Distribution function of the random variable x
$\bar{F}$(x) = Tail of the distribution funtion
P(x) = Probability of x, with x = random variable
M_n = Maximum value of S_n
Var (x) = Variance of x
P(x) = Price of an asset x at time t

Chapter 3:

M = Magnitude of an earthquake
f(E) = Frequency spectrum of energy

Chapter 4:

P(x) = Probability of x
x, y = Random variables
F(x) = Distribution function of the random variable x
$\bar{F}$(x) = Tail of the distribution funtion
Λ_α= Gumbel distribution
$\Phi\alpha$ = Fréchet distribution
$\Psi\alpha$ = Weibull distribution
$\alpha, \beta, \mu, \sigma$ = Parameters that define a Lévy-stable-distribution

Chapter 5:

Chapter 5.1

L(δ) = Lengt of the fractal
δ = Scaling factor
r = Size of boxes
μ = Number of points in the box
D_E = Euclidean dimension
D_T = Topological dimension
D_s = self-similar dimension
D_b = box dimension
D_p = pointwise dimension
Chapter 5.2
H = Hölder exponent
B_H = Fractal Brownian Motion with Hölder exponent H
ζ(q) = Legendre transformation with index q

Chapter 6:
ρ(k) = Autocorrelation of lag-length k
γ(k) = Covariance of x with lag length k
Q_k = Rescaling statistic of Hust
$\tilde{Q}_k$ = Modified rescaling statistic of Lo
f(ω) = Spectral density, ω = frequency
I(ω) = Periodogramm with frequency ω

Chapter 7:
Chapter 7.1
W(N) = Total number of combinations
A = Set of choices or possible actions
Chapter 7.2
A = Influence of exogenous forces
B = Influence of the market trend on traders opion
rn = Random number
Chapter 7.3
C = Influence of the market trend on traders opion; equivilant to B in chapter 7.2.2
M = Magnetisation
Vo = Trading Volume
Chapter 7.4
a, b = Weighting factors
S_i = State of agent i
S = Set of microstates S_i
P(N_1, N_2, ...) = Multinominal distribution
y = Aggregation of microstates
F = Entropy of a system
$\pi(c_i)$ = probability of being in state c_i
Z(β) = Partion function

H_i = Hamiltion function for agent i
E_i = Energy function of agent i

Chapter 8:
Chapter 8.2.1
U(W) = Utility function with argument w = wealth
a = Weighting factor in the utility function
r = Riskless return
R = Returm of a stock
k = Discount factor
z = Rate of groth of dividends
N = Demand for risky assets
Chapter 8.2.3
x = Demand for assets
a, b = Parameter values
Chapter 8.2.4
ν = Velocity of reevaluation of the success of an investment strategy
U = Parameter value that determines the strategy changes
Chapter 8.3
ρ = Discount factor
g = Annual growth rate of dividends
R = Return of bonds
r = Return of risky assets
N = Demand for risky assets
γ = Growth rate of asset prices
m = Quantaty of assets and bonds hold by an individual trader