Non–stationary Sibling Wavelet Frames on Bounded Intervals

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Dedic această lucrare acelora,
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Introduction

Frame Theory is a modern branch of Harmonic Analysis. It has its roots in Communication Theory and Quantum Mechanics. Frames are overcomplete and stable families of functions which provide non–unique and non–orthogonal series representations for each element of the space.
The first milestone was set 1946 by Gabor with the paper "Theory of communications" [33]. He formulated a fundamental approach to signal decomposition in terms of elementary signals generated by translations and modulations of a Gaussian. The frames for Hilbert spaces were formally defined for the first time 1952 by Duffin&Schaeffer in their fundamental paper "A class of nonharmonic Fourier series" [30]. They also coined the term "frame" in the mentioned article. The breakthrough of frames came 1986 with Daubechies, Grossmann and Meyer's paper "Painless nonorthogonal expansions" [24]. Since then a lot of scientists have been investigating frames from different points of view.
In this thesis we study non–stationary sibling frames, in general, and the possibility to construct such function families in spline spaces, in particular. Our work follows a theoretical, constructive track. Nonetheless, as demonstrated by several papers by Daubechies and other authors, frames are very useful in various areas of Applied Mathematics, including Signal and Image Processing, Data Compression and Signal Detection. The overcompleteness of the system incorporates redundant information in the frame coefficients. In certain applications one can take advantage of these correlations.

The content of this thesis can be split naturally into three parts: Chapters 1-3 introduce basic definitions, necessary notations and classical results from the General Frame Theory, from B–Spline Theory and on non–stationary tight wavelet spline frames. Chapters 4–5 describe the theory we developed for sibling frames on an abstract level. The last chapter presents an explicit construction of a certain class of non–stationary sibling spline frames with vanishing moments in $L_2[a, b]$ which exemplifies and thus proves the applicability of our theoretical results from Chapters 4–5.

Let us describe the chapters in more detail.

Chapter 1 specifies the early roots of Frame Theory and introduces terminology and definitions from this field which will be used throughout this work. Further it presents basic results on and examples of frames in Hilbert spaces (see Examples 1.3, 1.5, 1.7, 1.10). By including this chapter the author intended to enable readers which are not especially acquainted with Frame Theory to understand the main ideas behind frame systems and their duals. These are needed for the comprehension of the rest of this thesis. As a principle of writing we did the best possible to make this thesis self–contained. Classical handbooks, recent monographs, fundamental research papers and survey articles from Wavelet/Frame Theory are cited for further – more detailed – reading.

Chapter 2 collects a variety of results on B–splines of order m on a bounded interval $[a, b]$. It is, of course, beyond the scope of this thesis to give a comprehensive survey on the subject. Instead we confine ourselves to compiling those results which are directly related to the present work. For a more detailed description of B–splines the reader is referred to the classical monographies [26, 59, 28].

Section 2.1 describes the underlying knot sequence $\mathbf{t}$ with stacked boundary knots and certain spline spaces – the so–called Schoenberg spaces $\mathcal{S}_m(\mathbf{t}, [a, b])$. Section 2.2 introduces the normalized B–splines $N_{\mathbf{t};m,k}$ along with some of their essential properties such as recursion, partition of unity and stability. Considerations on the Gramian associated to the L_2–normalized B–spline functions $N^B_{\mathbf{t};m,k}$, on the reproducing kernel of the function space $\mathcal{S}_m(\mathbf{t}, [a, b])$ and on the dual B–spline basis are presented in Section 2.3. The refinement matrix P is obtained for the B–spline case from the Oslo algorithm. For $\mathbf{t} \subset \tilde{\mathbf{t}}$ it represents the connection between the Schoenberg spaces $\mathcal{S}_m(\mathbf{t}, [a, b]) \subset \mathcal{S}_m(\tilde{\mathbf{t}}, [a, b])$, and thus between two consecutive approximation spaces of the spline multiresolution analysis of $L_2[a, b]$ considered in Chapters 3 and 6. This is summarized in Section 2.4. Derivatives of B–splines play a key rôle in our constructions of spline sibling frames. They ensure the existence of the desired vanishing moments for the framelets. A matrix formulation for the B–spline derivatives is given in Section 2.5. It is further used in our MATLAB implementations, as well as for the explicit formulation of the frame and dual frame elements in Chapter 6.

In order to exemplify in a unified presentation all notions discussed in this chapter, we consider the admissible knot sequence of order 4 of Quak (see [51, p.144]) and push it consequently through Examples 2.5, 2.8, 2.9, 2.11, 2.14.

The non–uniform B–splines on bounded intervals are thus building blocks for our framelets from Chapter 6. This is due to their valuable properties such as local character, numerical stability and efficient evaluation. We often revert throughout this thesis to their properties depicted centrally in Chapter 2.

In the first part of Chapter 3 we describe the non–stationary multiresolution analysis setting under which we will work in Chapters 3-6. After discussing the general MRA we also present in detail the spline MRA of $L_2[a, b]$, which provides the approximation spaces for our construction in Chapter 6. Further we summarize the considerations on normalized tight spline frames of Chui, He and Stöckler and some of their results from [18]. Our work detailed in Chapters 4-6 is meant to extend and supplement their theory for bounded intervals.

Chui, He and Stöckler develop in [18] an explicit matrix formulation for the unique approximate dual with minimal support for the B–spline basis. It plays an important rôle in our construction of sibling spline frames, too. Therefore, we present the algorithm in Section 3.3. The approximate dual matrix has an associated approximate kernel. Chui, He and Stöckler's result on the boundedness of this kernel is discussed in Section 3.4. Their factorization steps for obtaining the (dual) frame coefficient matrices Q_j, $\tilde{Q}_j$, $j \geq 0$, are given in Section 3.5.

The construction of Chui, He and Stöckler provides automatically the canonical dual $\tilde{\mathcal{E}}$ for their function system $\mathcal{E}$, because in the normalized tight case $Q_j \equiv \tilde{Q}_j$ for all $j \geq 0$ and $\mathcal{E} \equiv \tilde{\mathcal{E}}$. Thus, after factorizing they do not have to verify any boundedness condition for the (dual) frame elements. In the sibling frame case this is more complicated. After factorizing asymmetrically ($Q_j \not\equiv \tilde{Q}_j$) we have to verify the existence of finite Bessel bounds for both: the frame and its dual. To our knowledge, this thesis represents the first work investigating this kind of questions in a constructive manner. We first propose explicitely formulated functions as candidates for sibling spline frames and secondly we give concrete Bessel bounds for these function families. In order to be able to do this in Chapter 6 for certain sibling spline frames we develop some general tools in Chapter 5.

Our personal contribution to Chapters 1–3 refers to the careful selection of classical material – which is needed for the understanding of the rest of this thesis – and to the brief, structured and unified presentation with relevant examples and figures generated by our MATLAB implementations. By contrast, Chapters 4–6 contain our own results, as detailed below. Let us include at this point an overview of the sub–goals of our research work during the last years. We pursued the following issues:

a) to formulate a general construction principle for non–stationary sibling wavelet frames on bounded intervals;

b) to find easily verifiable sufficient conditions for structured function systems to build Bessel families in $L_2[a, b]$;

c) to clarify the relation between our non–stationary wavelet setting and the different localization concepts appearing in modern Harmonic Analysis literature;

d) to specify general construction schemes for non–stationary sibling spline frames on bounded intervals and to prove the correctness of these schemes;

e) to give concrete examples of non–stationary sibling spline frames on bounded intervals and to visualize them through plots generated by MATLAB implementations.

As already mentioned, the results of our research on the above issues are presented in Chapters 4–6 of this thesis.

Chapter 4 deals with our extension of the general construction principle of non–stationary wavelet frames from the tight case – presented in Chapter 3 – to the non–tight (= sibling) case on which our present work focuses. We apply this principle in Chapter 6 in order to give a general construction scheme for certain non–stationary sibling spline frames of order m with L vanishing moments ($m \in I\!N$, $m \geq 2$, $1 \leq L \leq m$), as well as some concrete illustrative examples.

In Section 4.1 we define and study some appropriate tools for our further investigations of sibling frames: bilinear forms T_{S_j} and kernels K_{S_j} associated to real matrices S_j. In Theorem 4.1 we present some inheritance properties of these entities.

Section 4.2 includes the definition of sibling frames. Two function families of a certain structure constitute sibling frames, if they are Bessel families – i.e., they verify conditions (4.10) and (4.11) – and are dual – i.e., they verify (4.12). In Proposition 4.4 we describe sufficient conditions for the boundedness of the bilinear forms T_{S_j} and in Proposition 4.5 some for the monotonicity of the associated quadratic forms.

In Section 4.3 we discuss in detail the duality relation (4.12) between two Bessel families in several situations. Theorem 4.6 presents for the general case two conditions which are necessary and sufficient for the existence of the duality relation (4.12). With this result we extend Theorem 1 from [18]. In Theorem 4.8 we prove the following: in case the kernels K_{S_j} form a uniformly bounded approximate identity, the bilinear forms T_{S_j} verify the first of the necessary and sufficient conditions from Theorem 4.6. In the non–stationary spline setting described in Chapters 2&3, if the matrices S_j are chosen to be the approximate dual matrices of Chui, He and Stöckler $S^B_{\mathbf{t}_j;m,L}$, then it follows that the associated kernels fulfill all assumptions of Theorem 4.8. Therefore, in this special case, the first condition from Theorem 4.6 is verified. On the basis of the second condition from the theorem mentioned we are able to formulate the general construction principle for sibling spline frames:

In order to obtain sibling spline frames of $L_2[a,b]$ we have to factorize the matrices

$$S^B_{\mathbf{t}_{j+1};m,L} - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot S^B_{\mathbf{t}_j;m,L} \cdot \left(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\right)^T$$

appropriately into $Q_j \cdot \tilde{Q}_j^T$, i.e., we have to determine coefficient matrices Q_j and $\tilde{Q}_j$ such that the Bessel conditions (4.10) and (4.11) are satisfied.

Thus, Chapter 4 presents in Subsection 4.3.3 the motivation for our investigations in Chapter 5. We need some sufficient conditions on the function families defined by the coefficient matrices Q_j and $\tilde{Q}_j$ which are as simple as possible, in order to be able to verify easily if some concrete spline families are Bessel families (and thus sibling frames) or not.

In Chapter 5 we develop general strategies for proving the boundedness of certain linear operators. These will enable us to check in Chapter 6 the Bessel condition for concrete spline systems which are our candidates for sibling spline frames.
This chapter contains the core material about Bessel families with a certain structure and corresponding Bessel bounds. A central part deals with sufficient conditions on function families for the existence of upper bounds. Some results concerning multivariate Bessel families are also included. Chapter 5 is organized as follows.

Section 5.1 introduces the notions of Bessel family, Bessel sequence and Bessel bound. Theorem 5.4 (from Young [65]) gives a characterization of the Bessel property of a function family in terms of the Gramian associated to this family. More precisely, this result relates the Bessel property to the boundedness of the matrix operator defined by the Gramian. Thus, it enables us to rephrase results on the boundedness of certain linear operators in terms of the Bessel property for some function families. We will make use of this possibility several times in Chapter 5.

Section 5.2 presents the discrete form of Schur's Lemma (see Ladyženskiĭ [42]). This classical tool formulates easily verifiable conditions on infinite matrices which guarantee boundedness for the associated linear operators on l_2. Furthermore, it gives concrete upper bounds for the operator norms. These are directly related to the Bessel bounds we are interested in. Because of the importance of this lemma for our subsequent results we included a short proof.

Section 5.3 introduces and summarizes essential properties of Meyer's stationary vaguelettes families from [48]. Meyer introduces the concept 'vaguelettes' in order to describe a family of continuous functions which are indexed by the same scheme as the wavelets and are 'wavelet–like'. Thus he described a wide collection of systems which share essential qualitative features like localization, oscillation and regularity. The non–stationary function families we will introduce and study further in this chapter exhibit in principle the same features, but are adapted to the non–stationary setting. In order to illustrate these features we constructed a general vaguelettes family for the d–dimensional case, as well as concrete examples for the one– and two– dimensional cases with both: bounded and unbounded supports (see Example 5.9 and Figures 5.4, 5.5, 5.7.a, 5.8).
The essential support of a function with good decay properties is used in the literature often without a rigorous definition. We inserted in this section (in connection to the localization property of Meyer's vaguelettes) a rather long remark on essential supports (see Remark 5.8 and Figures 5.2, 5.3, 5.4, 5.7.b) which hopefully clarifies our understanding of this concept. We want to emphasize that the content of this remark (especially the notions 'geometrical

essential support' and 'abstract essential support') corresponds to our own intuitive understanding and will be used in this form throughout this thesis.
The starting point for our extensions from the stationary to the non–stationary situation in Sections 5.8 and 5.11 was Theorem 5.10 of Meyer (see [48]). It states that every d–dimensional dyadic stationary vaguelettes family is a Bessel family in $L_2(\mathbb{R}^d)$.
From Definition 5.7 one also can deduce the localization point of a vaguelette function in the scale–time half–space $\mathbb{R}_+ \times \mathbb{R}^d$. In order to be able to accomplish the above mentioned extension one has to understand deeply the distribution of the localization points of a function family and its rôle in connection to the Bessel property. These aspects are discussed in more detail in the following sections of Chapter 5.

Section 5.4 contains – amongst others – our answer to sub–goal c). In modern Harmonic Analysis literature one can find *two essentially different localization concepts* for families of structured functions which do not seem to be compatible with each other. The first one has been developed in parallel by Frazier&Jawerth [32] and Meyer [48] for the canonic tiling of the scale–time space in the wavelet case. The second can be found in papers by Gröchenig (see e.g. [37]); it is formulated for the regular tiling of the time–frequency space in the Gabor case.
Because of the fundamental difference between the two structures (see Figure 5.10 and Figure 5.9) different distance functions between points have to be used: a hyperbolic metric in the wavelet case and the Euclidian distance in the Gabor case. Until now no approach was found in order to unify or bridge these two theories. For our purposes we follow the first one and we present in Section 5.4 some central concepts and main results from [32] which are directly connected to our subsequent considerations.
Definition 5.12 presents the notions 'almost diagonal matrix' and 'almost diagonal linear operator' for the stationary case $l_2(\mathcal{Q})$. With the aid of the generalized Poincaré metric (see Definition 5.13) Proposition 5.16 gives a characterization of almost diagonal matrices in terms of exponential localization. Theorem 5.17 formulates a boundedness criterion (see [32]): an almost diagonal operator on $l_2(\mathcal{Q})$ is bounded on $l_2(\mathcal{Q})$. Using Theorem 5.4 one can rephrase Theorem 5.17 as follows: every function family $\{f_Q\}_{Q\in\mathcal{Q}}$ from $L_2(\mathbb{R}^d)$ with almost diagonal Gram matrix on $l_2(\mathcal{Q})$ is a Bessel family in $L_2(\mathbb{R}^d)$. This compact presentation of some of Frazier&Jawerth results from their very extensive paper [32] reveals the existence of a common strategy of Meyer and Frazier&Jawerth in proving the Bessel property for some function systems.
We stress in Remark 5.19 the fact that the general strategies behind the approaches of Meyer and Frazier&Jawerth are basically the same. We did not find this parallelism mentioned anywhere in the literature. We detected it during our intensive reading of [48] and [32] in connection with some papers by Gröchenig. Note that neither Theorem 5.17/5.18 (see [32, Theorem 3.3], nor Theorem 5.10 (see [48, Theorem 2 on p. 270]) are formulated in the original papers in terms of Bessel families. Thus the connections to our purpose were not quite direct.
For our sibling frame candidates in Chapter 6 we will follow a scheme similar to that of Meyer and Frazier&Jawerth in order to prove the Bessel property for them. Therefore, we formulated in the remainder of Chapter 5 our extension of this scheme from the stationary to the non–stationary situation.

Section 5.5 introduces through Definition 5.20 the concept of almost diagonality of a bi–infinite matrix w.r.t. a given collection of closed and bounded intervals $\{I_\lambda\}_{\lambda\in\Lambda}$ of the real line (i.e., this represents the non–stationary univariate case). For the case $d = 1$ our concept fully generalizes the one for the stationary situation presented in Definition 5.12. In Proposition 5.22 we give a characterization of almost diagonal matrices on $l_2(\Lambda)$ in terms of exponential localization.

For both the non–compact and compact cases sufficient conditions on a non–stationary function family (i.e., vanishing moment, boundedness and decay, Hölder continuity) are given for the associated Gramian to be almost diagonal in the sense of Definition 5.20. Explicit values of the constants C and ε figuring in Definition 5.20 are given. These are depending exclusively on the parameters in the assumptions made. As immediate consequences of the two main results (Theorem 5.24 and Theorem 5.26) we are able to formulate in Propositions 5.25 and 5.27 the corresponding exponential localization statements for the function families. As an example for the compact case, at the end of Section 5.5 we present and discuss in detail a family of suitably normalized differentiated B–splines w.r.t. certain nested knot sequences all contained in an interval $[a, b]$ (see Example 5.28 and Figures 5.11–5.16). We prove further that the sufficient conditions from Theorem 5.26 are satisfied for this function family. We thus obtain the almost diagonality of the Gramian and the corresponding exponential localization.
We summarize the results of Section 5.5 as follows:
We prove for two types of function families the almost diagonality of the associated Gramian. We have thus carried over the first step of Meyer's and Frazier&Jawerth's scheme to the non–stationary one–dimensional case (see Remark 5.19).
For the second step, namely the boundedness of the operator associated to the Gram matrix in the non–stationary case extra tools had to be designed in order to be able to proceed further. This is done in Sections 5.6 and 5.7, after shortly recalling the analogous concepts for the Gabor case.

The most important ingredient for our extension from the stationary to the non–stationary setting in the wavelet case is a separation concept for the irregularly distributed localization points of the function family under discussion. Such concepts exist for the Gabor setting in earlier work by Young and Gröchenig; these are briefly reviewed in three definitions at the beginning of Section 5.6. We stress the fact that these concepts match only the Gabor case and cannot be carried over to the wavelet situation. Our appropriate separation concept for the latter case is presented in Definition 5.32, i.e., a countable family of compact intervals $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ with $I_\lambda = [c_\lambda, b_\lambda]$ is called relatively separated, if there exists a finite overlapping constant for these intervals, i.e.,

$$\exists D_2 > 0 \quad \forall J \subset I\!R \quad \text{bounded interval:} \quad \#\Lambda_J \leq D_2,$$
$$\text{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \ : \ |I_\lambda| \in \left[\frac{|J|}{2}, |J| \right], \ c_\lambda \in J \right\}.$$

This concept is – to our knowledge – new (it was first introduced in our preprint [4]) and it enables us to prove in the next section a general boundedness result for almost diagonal matrices. In Example 5.33 and Figures 5.17–5.22 we present some overlapping situations for the one–dimensional dyadic stationary vaguelettes family with compact supports from Example 5.9. We further prove in this example that the corresponding overlapping constant takes the value $D_2 = 10$.

In Section 5.7, after proving a technical lemma concerning some Riemann–type sums, we present in Theorem 5.35 a general boundedness result for certain linear operators in the non–stationary univariate setting, i.e., we prove that every matrix which is almost diagonal w.r.t. a relatively separated family of intervals defines a bounded operator on l_2. This is the central result of our non–stationary theory in Chapter 5 and its proof – which follows the "hard analysis" track of Frazier&Jawerth and Meyer (according to Gröchenig's phrasing) – makes use of Schur's Lemma. By looking at the proof thoroughly, it can be seen that the existence of an overlapping constant is indispensable for showing that certain infinite series converge. Our result generalizes Theorem 5.17 of Frazier&Jawerth (see [32, Theorem 3.3]). In Corollary 5.36 we give a general bound for the operator norm.

In Section 5.8 we focus on function families with compact supports. Our motivation is the subsequent construction of sibling spline frames on a compact interval. This section makes essential use of our results in Section 5.5.
In Definition 5.37 we propose a generalization of Meyer's stationary vaguelettes for the one–dimensional non–stationary case and for functions with compact support, namely the *univariate non–stationary vaguelettes with compact support*. In this generalized case it is necessary to introduce a so–called 'finite overlapping constant' for the function family in order to prove the desired boundedness result (see Theorem 5.39 which generalizes [32, Theorem 3.3]). It implies the separation of the supports of the function family in the sense of Definition 5.32 and enables us to determine Bessel bounds (see Theorem 5.40). It should be noted that in the stationary case the existence of such overlapping constant is automatically given (see Proposition 5.42).

In Section 5.9 we propose a generalization of Meyer's stationary vaguelettes for the one–dimensional non–stationary case and for functions with infinite support, namely the *univariate non–stationary vaguelettes with infinite support* (see Definition 5.46). In Theorem 5.47 we extend again Theorem 5.17 (see [32, Theorem 3.3]) by giving the corresponding boundedness criterion and Theorem 5.48 contains the desired Bessel bound.

In Sections 5.10 and 5.11 we present results for the d–variate case. The basic tool is a tensor product approach, and thus many of our previous univariate results carry over in a very natural fashion. We restricted ourselves in this latter part to the compactly supported case. We choose as measure for our cuboids the length of their diagonal.
In Definition 5.49 we propose a generalization of the almost diagonality for the non–stationary multivariate case. Proposition 5.51 contains a characterization of almost diagonal matrices in terms of exponential localization. Theorem 5.52 presents sufficient conditions for the almost diagonality of a Gram matrix. The localization property of multivariate function families with compact support is described in Proposition 5.53. Next we extend the definition of a compactly supported non–stationary vaguelettes family to the d–dimensional case by proposing Definition 5.54. Our multivariate boundedness result can be found in Theorem 5.55. It generalizes Theorem 5.17 of Frazier&Jawerth (see [32, Theorem 3.3] for the multivariate case). The Bessel bounds for the multivariate case are explicitely given in Theorem 5.56. We depicted in Proposition 5.57 the overlapping constants for the d–dimensional dyadic stationary case with disjoint supports on each level. For illustrative purposes some possible situations of overlapping are presented for $d = 2$ in Figures 5.23–5.25.

In Chapter 6 we give concrete examples of sibling spline frames in $L_2[a, b]$ demonstrating hereby that our theory developed in Chapter 5 can indeed be used to check the Bessel property for spline functions families.
Section 6.1 presents our general construction scheme for sibling spline frames of order m. It provides explicit matrix formulations for both the frame and the dual frame elements. These are based on the matrix formulations for the derivatives of B–splines we presented in Chapter 2. Thus all framelets exhibit L vanishing moments ($1 \leq L \leq m$).
In Section 6.2 we study in detail the structure of the matrix Z (involved in the construction of the dual frame) in order to obtain a useful estimate for its elements. Subsection 6.2.1 presents the general situation and Subsection 6.2.2 details the case of bounded refinement rate between adjacent multiresolution levels. In this specific situation we are able to formulate an estimate for the elements of Z which enables us further to prove the Bessel property for the dual frame (see Remark 6.14 and Proposition 6.15 in particular). In Subsection 6.2.3 we develop an example which visualizes the structures of the matrices involved (i.e., the matrices V, P and Z). The local character of the blocks PVP^T – which yield by summation the matrix Z – is illustrated in Figure 6.1.

A general construction scheme for quasi–uniform sibling spline frames is discussed in depth in Section 6.3. In Conditions 6.17 we collected all properties we need for the knot sequences $\mathbf{t}_j$ in order to prove the Bessel property for the function systems defined in (6.19)–(6.20). Each condition plays its special part in the subsequent development. These connections can be seen from our proofs. In Proposition 6.21 we give estimates for the elements of the matrix Z in the setting characterized by Conditions 6.17. The verification of the sufficient conditions we formulated in Theorem 5.40 is carried out in full detail in Propositions 6.22, 6.24, 6.25, 6.26, 6.27 and 6.28. Explicit and useful estimates for the derivatives of B–splines are given in Proposition 6.23. Summarizing the assertions proved in Section 6.3 we formulate Theorem 6.29 which states the Bessel property of the families proposed in (6.19)–(6.20). Therefore, the systems given in (6.19)–(6.20) are sibling frames.
Concrete examples for sibling spline frames of order 4 with one, two, three and four vanishing moments (for the same refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ of the knot sequences) are given in Example 6.30. The corresponding spline families are visualized in Figures 6.2–6.17. In order to obtain these illustrations we implemented all algorithms in MATLAB. Sibling frames of order 5 with one and two vanishing moments (for two consecutive refinements $\mathbf{t}_0 \subset \mathbf{t}_1 \subset \mathbf{t}_2$) can be found in Example 6.31. Figures 6.19–6.26 depict the corresponding frame and dual frame elements.
Finally, in Section 6.5 an outlook for further research completes our work. It points out how a construction scheme with local character for sibling spline frames can be built upon our theory from Chapters 4-5 and our investigations in Chapter 6. The local character of the scheme is illustrated in Example 6.32.

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Chapter 1

Some Basics of Frame Theory

Hilbert frames are overcomplete and stable families of functions from a Hilbert space which provide non–unique and non–orthogonal series representations for each element of the space. The overcompleteness of the system incorporates redundant information (built–in correlations) in the frame coefficients, such that the loss of some of them – for example during transmission – does not necessarily imply loss of information.

Frame systems play an important rôle in both Pure and Applied Mathematics. Frames have been used with success in Signal Processing, Image Processing, Data Compression, Sampling Theory, Optics, Filterbanks, Signal Detection, etc. In the development of theoretical results they proved to be a useful tool - for example - in the study of Besov spaces and in Banach Space Theory. Frame Theory is a modern branch of Harmonic Analysis.

The main issue of this introductory chapter is to present basic definitions and terminology from Frame Theory which will be used throughout this work. We will first line out some of the early roots of this theory.

1.1 Early development of frame theory

As very often is the case, Frame Theory does not have its roots in Mathematics, but in applied areas like Communication Theory and Quantum Mechanics. In the following we give a short account of authors and papers who initiated the study of (predecessors of) frame systems.

1. In 1946 Dennis Gabor (1900–1979) formulated in Communication Theory (see [33]) a fundamental approach to signal decomposition in terms of elementary signals of the form
$$g_{m,n}(x) = e^{2\pi imx} g(x-n), \qquad m,n \in \mathbb{Z},$$
generated by translations and modulations of a Gaussian $g(x) = e^{-\alpha x^2}$.

2. The idea to represent a function f in terms of the time–frequency shifts of a single atom g did not only originate in Communication Theory but also in Quantum Mechanics. In order to expand general functions (quantum mechanical states) with respect to states with minimal uncertainty John von Neumann (1903–1957) introduced 1932 a set of coherent states on a lattice (see [50]) which are essentially the same used by Gabor.

3. Frames for Hilbert spaces were formally defined in 1952 by Richard James Duffin (1909-1996) and Albert Charles Schaeffer in the context of nonharmonic Fourier series (see [30]); they also coined the term "frame" (see [39, Introduction (by J.J. Benedetto)]). For functions $f \in L_2[0,1]$, they considered expansions in terms of translations of a

Gaussian g and modulations $\exp(i\lambda_n x)$ where $\lambda_n \neq 2\pi n$ and generalised in this way Gabor's approach.

4. 1980 Robert Michael Young reconsidered frames in his book "An Introduction to Nonharmonic Fourier Series" [65, Ch. 4, Sect. 7: The Theory of Frames], but at that time frames did not generate much interest outside nonharmonic Fourier series.

5. The breakthrough of frames came 1986 with Ingrid Daubechies, Alexander Grossmann & Yves Meyer's paper "Painless nonorthogonal expansions" [24].

Since then a lot of scientists from different fields have become interested in and done research on frames.

A fundamental paper by Michael Frazier and Björn Jawerth (see [32]) which simplifies, extends and unifies a variety of results from Harmonic Analysis and moreover discusses several decomposition methods and representation formulae is of great importance for our subsequent work.

Classical handbooks in Wavelet/Frame Theory are those of Young [65], Meyer [47]–[49], Daubechies [23], Chui [15] and Mallat [44]. Recent monographs on these subjects were written – amongst others – by Gröchenig [36] and Christensen [14]. Fundamental survey articles on the frame topic were compiled by Heil&Walnut [38], Casazza [13] and Chui&Stöckler [16]. In [39] Heil and Walnut brought together the seminal papers that presented the ideas from which Wavelet/Frame Theory evolved, as well as those major papers that developed the theory into its current form. The introduction of this volume by John J. Benedetto contains much historical information.

1.2 Abstract Hilbert space frames

We begin by giving a short self–contained exposition of Hilbert frames suitable for our work in subsequent sections. More detailed fundamentals of Hilbert frame theory can be found in [30, 65, 23, 15].

1.2.1 Definition, remarks, examples

Let Λ be a countable index set and $\mathcal{E} = \{e_\lambda\}_{\lambda\in\Lambda}$ be a family of elements in a separable Hilbert space $\mathcal{H}$ endowed with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{\mathcal{H}}$.
For $f \in \mathcal{H}$ we call $(\langle f, e_\lambda\rangle)_{\lambda\in\Lambda}$ the moment sequence of f w.r.t. $\mathcal{E}$ and denote it by $\langle f, \mathcal{E}\rangle$. The Gram matrix of $\mathcal{E}$ will be denoted by $Gram(\mathcal{E})$.

Definition 1.1 *(Hilbert frame, Bessel family, see [30, Section 3: Abstract frames])*

a) Let Λ, $\mathcal{E}$ and $\mathcal{H}$ be given as above. $\mathcal{E}$ is called a (Hilbert) frame of $\mathcal{H}$ if there exist constants A and B $(0 < A \leq B < \infty)$ such that

$$A \cdot \|f\|^2_{\mathcal{H}} \leq \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2 \leq B \cdot \|f\|^2_{\mathcal{H}} \qquad \text{for all } f \in \mathcal{H}. \tag{1.1}$$

Any constants A and B satisfying (1.1) are called lower and upper bound of the frame $\mathcal{E}$, respectively. The sharpest possible constants A and B are called optimal frame bounds and are denoted by A_{opt} and B_{opt}. If one can choose equal frame bounds, then the frame is called tight, i.e.,

$$\sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2 = A \cdot \|f\|^2_{\mathcal{H}} \qquad \text{for all } f \in \mathcal{H}. \tag{1.2}$$

The inequalities (1.1) are called 'the frame condition' and the components $\langle f, e_\lambda \rangle$ are the frame coefficients. The frame elements e_λ are often called framelets.

b) *A function system $\mathcal{E}$ which satisfies only the second inequality in (1.1) is called Bessel family with Bessel bound $\sqrt{B}$.*

Note that the second inequality in (1.1) can be written equivalently in the adjoint form

$$\left\| \sum_{\lambda \in \Lambda} c_\lambda e_\lambda \right\|_{\mathcal{H}}^2 \leq B \cdot \|\mathbf{c}\|_{l_2}^2 \qquad \text{for all } \mathbf{c} = (c_\lambda)_{\lambda \in \Lambda} \in l_2(\Lambda).$$

Hilbert frames can be considered to be a natural generalization of Riesz bases, as a comparison with the following definition suggests.

Definition 1.2 *(Riesz basis)*
Let Λ, $\mathcal{E}$ and $\mathcal{H}$ be given as above. $\mathcal{E}$ is called a Riesz basis of $\mathcal{H}$ if

a) *$\mathcal{E}$ is complete, i.e., $\overline{span}\, \mathcal{E} = \mathcal{H}$, and*

b) *there exist constants A and B $(0 < A \leq B < \infty)$ such that*

$$A \cdot \|\mathbf{c}\|_{l_2}^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda e_\lambda \right\|_{\mathcal{H}}^2 \leq B \cdot \|\mathbf{c}\|_{l_2}^2 \tag{1.3}$$

holds for all $\mathbf{c} = (c_\lambda)_{\lambda \in \Lambda} \in l_2(\Lambda)$. The constants A and B are called lower and upper Riesz bound, respectively.

A tight frame can always be normalized such that $A = B = 1$. Normalized tight frames generalize orthonormal (wavelet) bases, i.e., there holds

$$\sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle e_\lambda = f, \qquad f \in \mathcal{H},$$

but in general the frame elements are neither mutually orthogonal nor linearly independent. The frame condition expresses the (numerical) stability of the family $\mathcal{E}$ in the sense of norm equivalence:

$$\|\langle f, \mathcal{E} \rangle\|_{l_2(\Lambda)} \asymp \|f\|_{\mathcal{H}} \qquad \text{for all } f \in \mathcal{H}. \tag{1.4}$$

Example 1.3 *(The classical example of tight frames, see [23])*
We consider in the 2–dimensional Hilbert space $\mathcal{H} = \mathbf{C}^2$ the vectors

$$e_1 = (0, 1), \quad e_2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2} \right), \quad e_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right)$$

(see Figure 1.1). For arbitrary $v \in \mathcal{H}$ one gets

$$\sum_{j=1}^{3} | < v, e_j > |^2 = \frac{3}{2} \|v\|^2,$$

which implies the tight frame property of the system $\{e_1, e_2, e_3\}$ with bound

$$A = B = \frac{3}{2} = 1.5\,.$$

This redundancy ratio $\frac{3}{2}$ reflects that we work with a system of cardinality 3 in a 2–dimensional space. □

For applications such as compression it is crucial that the framelets possess some cancellation property to be introduced next.

Definition 1.4 *(Vanishing moments)*
A function $f \in L_2[a,b]$ has L vanishing moments ($L \in \mathbb{N}$) if

$$\int_a^b x^\nu \cdot f(x)\,dx = 0 \qquad \text{for } 0 \le \nu \le L-1.$$

The following frame example is historically important: Alex Grossmann (Croatian physicist working in France) and Jean P. Morlet (French geophysicist) used this frame in the numerical analysis of seismic signals (see [34]).

Example 1.5 *(Non–tight wavelet frame for $L_2(\mathbb{R})$ generated by the 'Mexican hat' function, see [23, p. 75])*
We consider the function Ψ to be the second derivative of the Gaussian $e^{-t^2/2}$, normalized such that

$$\|\Psi\|_{L_2} = 1 \quad \text{and} \quad \Psi(0) > 0.$$

Therefore, we obtain the so–called generatrix

$$\Psi(t) = \frac{2}{\sqrt[4]{9\pi}}(1-t^2)e^{-t^2/2}$$

with good localization in both time (around $t_0 = 0$) and in frequency (around $\omega_0 = \pm\sqrt{2}$; see Figure 1.2). After dilation, translation and normalization we get the framelets

$$\psi_{m,n}(t) := 2^{-m/2}\Psi(2^{-m}t - n), \quad m,n \in \mathbb{Z}, \tag{1.5}$$

having two vanishing moments (see Figure 1.3 and Remark 1.6). Daubechies' bounds for this frame are

$$A = 3.223 \text{ and } B = 3.596$$

(see [23, Table 3.1 on p. 77]), which imply

$$\frac{B}{A} - 1 = 0.116 << 1$$

and thus the property to be 'almost tight' and good reconstruction of functions $f \in L_2(\mathbb{R})$ from their frame coefficients $\{\langle f, \psi_{m,n}\rangle\}_{(m,n)\in\mathbb{Z}}$ by the iterative algorithm (see [23, p. 61ff] and [30, Theorem III]). The redundancy ratio is given in this case by

$$\frac{A+B}{2} = 3.41. \qquad \square$$

Remark 1.6 *The following are basic properties of the framelets $\psi_{m,n}$ defined by*

$$\psi_{m,n}(t) := 2^{-m/2}\Psi(2^{-m}t - n), \quad m,n \in \mathbb{Z},$$

from some generatrix $\Psi \in L_2(\mathbb{R})$:

a) $\|\psi_{m,n}\|_2 = \|\Psi\|_2$;

b) $\|\psi_{m,n}\|_\infty = 2^{-m/2} \cdot \|\Psi\|_\infty$;

c) if $\text{supp}\,(\Psi) = [a,b]$, *then* $\text{supp}\,\psi_{m,n} = [2^m(a+n), 2^m(b+n)]$;

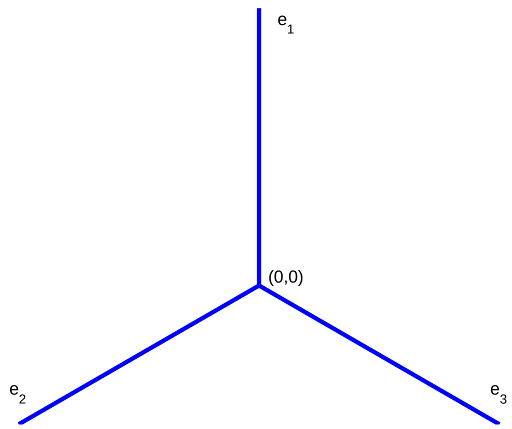

Figure 1.1: Example of tight Hilbert frame in $\mathbf{C}^2$ with three elements: $e_1 = (0,1)$, $e_2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $e_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

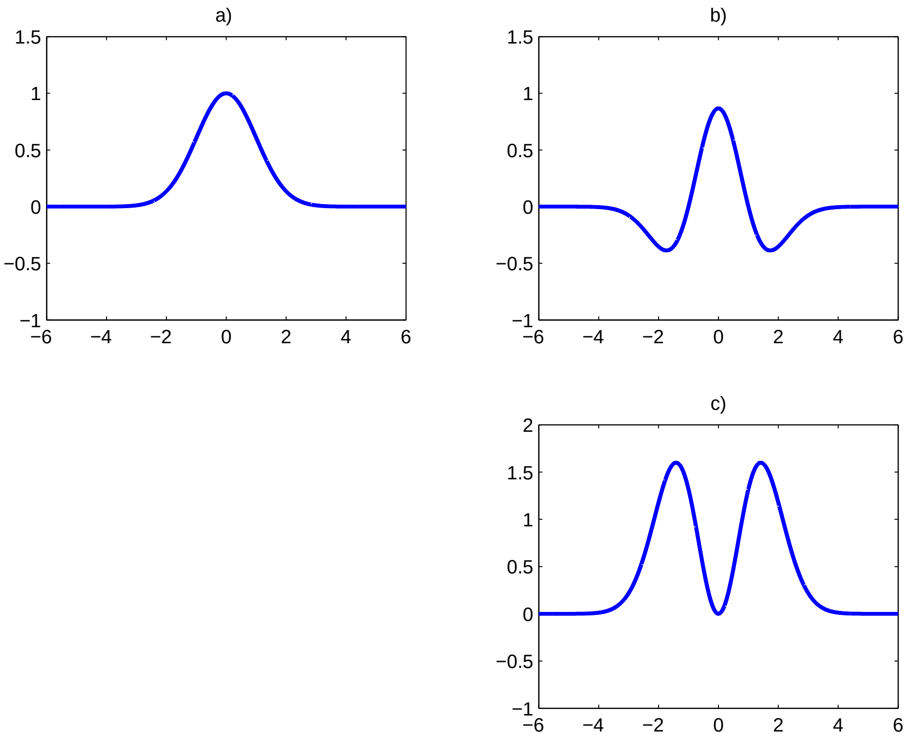

Figure 1.2: a) Gaussian $e^{-t^2/2}$. b) 'Mexican hat' function $\Psi(t) = \frac{2}{\sqrt[4]{9\pi}}(1-t^2)e^{-t^2/2}$, localized in time domain around $t_0 = 0$. c) Fourier transform of Ψ: $\hat{\Psi}(\omega) = \sqrt[4]{\frac{64\pi}{9}} \cdot \omega^2 e^{-\omega^2/2}$, which implies the localization of Ψ in frequency domain around $\omega_0 = \pm\sqrt{2}$.

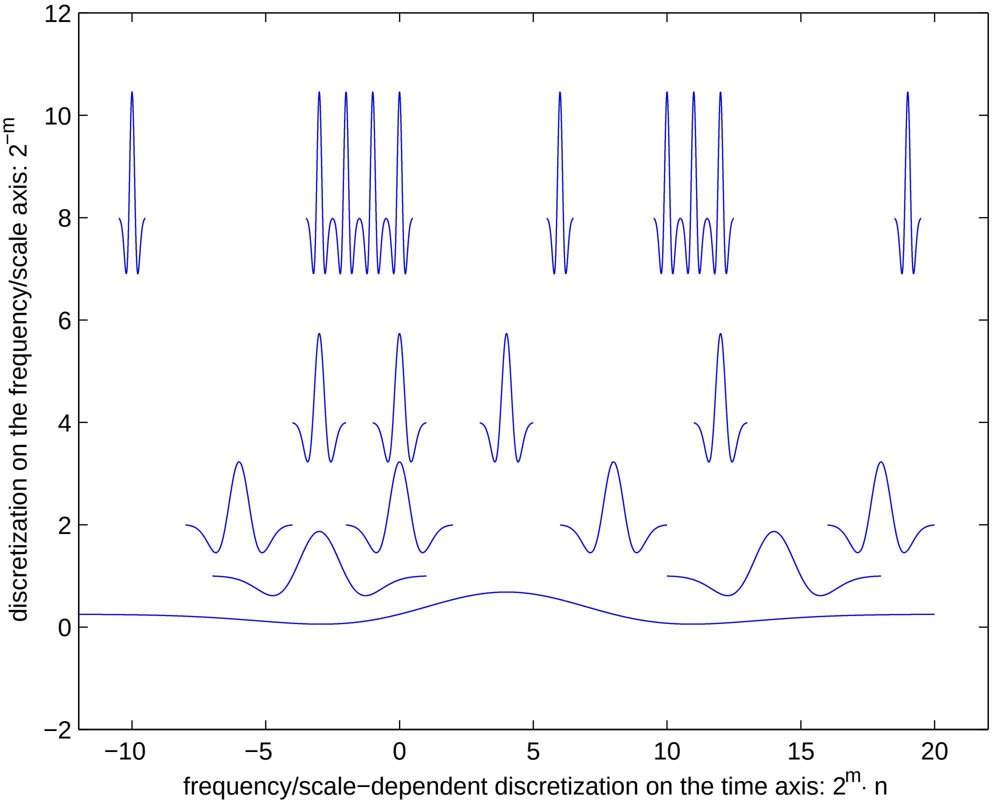

Figure 1.3: 'Mexican hat' framelets $\psi_{m,n} := 2^{-m/2}\Psi(2^{-m}\cdot - n)$ for index pairs: $(m,n) \in \{(2,1), (0,-3), (0,14), (-1,-12), (-1,0), (-1,16), (-1,36), (-2,-12), (-2,0), (-2,16), (-2,48), (-3,-80), (-3,-24), (-3,-16), (-3,-8), (-3,0), (-3,48), (-3,80), (-3,88), (-3,96), (-3,152)\}$ (order: bottom–up and left–right).

d) length(supp $\psi_{m,n}$) $= 2^m \cdot$ length(supp Ψ);

e) if Ψ is symmetric w.r.t. $c = \frac{a+b}{2}$, then $\psi_{m,n}$ is symmetric w.r.t. $c + 2^m n$;

f) if Ψ is localized in time domain around t_0, then $\psi_{m,n}$ is localized in time domain around $t_0 + 2^m n$;

g) if Ψ is localized in frequency domain around ω_0, then $\psi_{m,n}$ is localized in frequency domain around $2^{-m}\omega_0$.

In the special case $\Psi =$'Mexican hat' function the above properties can be verified by means of Figure 1.3.

The main directions of frame theory are:

a) the abstract frame theory (in Hilbert and Banach spaces) where general families $\{f_i\}_{i\in\mathbb{N}}$ are considered;

b) the Gabor (or Weyl–Heisenberg) frame theory where function families of type

$$g_{m,n}(x) := e^{imx} g(x-n), \qquad (m,n) \in \mathbb{Z}^2,$$

are studied;

c) the wavelet frame theory where the systems have the structure

$$\psi_{m,n} := 2^{-m/2}\Psi(2^{-m}x - n), \qquad (m,n) \in \mathbb{Z}^2,$$

or some generalization of it.

The constructions presented in Chapters 3, 4 and 6 belong to part c). Chui, He and Stöckler (see [18] and Chapter 3) deal with the tight frame case, i.e., they verify for their function systems the identity (1.2). We construct in Chapters 4&6 non–tight frames and therefore we have to prove for our families the boundedness described by (1.1).

1.2.2 Operators and duals associated to a frame

The aim of this subsection is to introduce natural operators associated to a frame and summarize their basic properties. The frame definition/condition appears in many different forms in the literature. In order to stress their equivalence we incorporated several of them in the following.

Let Λ and $\mathcal{H}$ be defined as in Section 1.2.1. For the frame $\mathcal{E} = \{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ with bounds A and B we define the following operators:

• the analysis (decomposition, coefficient) operator

$$F := F_{\mathcal{E}} : \mathcal{H} \to l_2(\Lambda), \qquad F(f) := \langle f, \mathcal{E}\rangle \quad \text{for all } f \in \mathcal{H},$$

which is linear and bounded with $\|F\| \le B^{1/2}$ according to the second inequality in (1.1). The frame condition reads now

$$A \cdot \|f\|^2_{\mathcal{H}} \le \|Ff\|^2_{l_2(\Lambda)} \le B \cdot \|f\|^2_{\mathcal{H}}, \qquad f \in \mathcal{H},$$

and condition (1.4) can be expressed as

$$\|Ff\|_{l_2(\Lambda)} \asymp \|f\|_{\mathcal{H}} \qquad \text{for all } f \in \mathcal{H};$$

• the synthesis (reconstruction) operator

$$F^* := F^*_{\mathcal{E}} : l_2(\Lambda) \to \mathcal{H}, \qquad F^*\mathbf{c} = \sum_{\lambda\in\Lambda} c_\lambda e_\lambda \quad \text{for all } \mathbf{c} = \{c_\lambda\}_{\lambda\in\Lambda} \in l_2(\Lambda),$$

which is the adjoint of the analysis operator and also linear and bounded with $\|F^*\| = \|F\| \leq B^{1/2}$. An equivalent form for (1.3) is

$$A \cdot \|\mathbf{c}\|^2_{l_2(\Lambda)} \leq \|F^* c\|^2_{\mathcal{H}} \leq B \cdot \|\mathbf{c}\|^2_{l_2(\Lambda)}$$

with $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in l_2(\Lambda)$, and one for (1.4) is

$$\|F^*\mathbf{c}\|_{\mathcal{H}} \asymp \|\mathbf{c}\|_{l_2(\Lambda)}, \qquad \mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in l_2(\Lambda);$$

• the frame operator

$$S := S_{\mathcal{E}} := F^*_{\mathcal{E}} F_{\mathcal{E}} : \mathcal{H} \to \mathcal{H}, \qquad S(f) = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle e_\lambda \quad \text{for all } f \in \mathcal{H},$$

which is linear, bounded with $\|S\| = \|F\|^2 \leq B$, positive definite, invertible, self–adjoint and is an isomorphism on $\mathcal{H}$. The frame condition can be written now as

$$A \cdot Id_{\mathcal{H}} \leq S \leq B \cdot Id_{\mathcal{H}}, \tag{1.6}$$

where $Id_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. The optimal upper frame bound for $\mathcal{E}$ is $B_{opt} = \|S\|$. In particular, $\mathcal{E}$ is a tight frame if and only if $S = A \cdot Id_{\mathcal{H}}$;

• the inverse frame operator

$$S^{-1} := S^{-1}_{\mathcal{E}} : \mathcal{H} \to \mathcal{H},$$

which is linear, bounded with $\|S^{-1}\| \leq A^{-1}$, positive definite, invertible, self–adjoint and is an isomorphism on $\mathcal{H}$, too. We can write

$$B^{-1} \cdot Id_{\mathcal{H}} \leq S^{-1} \leq A^{-1} \cdot Id_{\mathcal{H}}.$$

The optimal lower frame bound for $\mathcal{E}$ is $A_{opt} = \|S^{-1}\|^{-1}$. The family $\tilde{\mathcal{E}} := \{S^{-1}e_\lambda\}_{\lambda\in\Lambda}$ is a frame with bounds B^{-1} and A^{-1}, the so–called *canonical dual frame* of $\mathcal{E}$. Every $f \in \mathcal{H}$ can be non–orthogonally expanded as

$$\sum_{\lambda\in\Lambda} \langle f, S^{-1}e_\lambda\rangle e_\lambda = f = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle S^{-1} e_\lambda, \tag{1.7}$$

where both series converge unconditionally[1] in $\mathcal{H}$. The identities (1.7) express possibilities of recovering f when 'discrete information' in form of $\langle f, S^{-1}e_\lambda\rangle$ or $\langle f, e_\lambda\rangle$ is given. They are called reconstruction formulae. In the tight frame case we have $S^{-1} = \frac{1}{A} \cdot Id_{\mathcal{H}}$ and thus the canonical dual $\tilde{\mathcal{E}} = \{\frac{1}{A} e_\lambda\}_{\lambda\in\Lambda}$. The canonical dual of a normalized tight frame is the frame itself, i.e., $\mathcal{E} \equiv \tilde{\mathcal{E}}$.

Example 1.7 *The canonical dual of the frame* $\{e_1, e_2, e_3\}$ *from Example 1.3 is*

$$\left\{\frac{2}{3}e_1, \frac{2}{3}e_2, \frac{2}{3}e_3\right\}.$$

The canonical dual of the frame in Example 1.5 cannot be given in such a direct way; first one has to determine the inverse of the associated frame operator. □

[1] A series $\sum_{n\in\mathbb{N}} a_n$ converges unconditionally if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n\in\mathbb{N}} a_{\sigma(n)}$ converges. Every absolutely convergent series is unconditionally convergent, but the converse implication does not hold in general.

The canonical dual of a frame – as defined above – is not a uniquely defined function system which satisfies the relations (1.7). For a given frame there exist several associated function families which provide series representations of the type (1.7) – as presented in the following.

Definition 1.8 *(Dual frame, see [30, definition of conjugate frames])*
Let Λ and $\mathcal{H}$ be defined as above. Further let $\mathcal{E} = \{e_\lambda\}_{\lambda\in\Lambda}$ be a frame in $\mathcal{H}$. A function family $\tilde{\mathcal{E}} := \{\tilde{e}_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}$ is called dual frame of $\mathcal{E}$ if

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde{e}_\lambda\rangle e_\lambda \qquad \text{for all } f \in \mathcal{H}. \tag{1.8}$$

Recall that the canonical dual frame is defined by the inverse frame operator (see (1.7)).

Proposition 1.9 *(Characterizations of dual frames, see [14, Lemma 5.6.2])*
Let Λ and $\mathcal{H}$ be defined as above. Further let $\mathcal{E} = \{e_\lambda\}_{\lambda\in\Lambda}$ and $\tilde{\mathcal{E}} := \{\tilde{e}_\lambda\}_{\lambda\in\Lambda}$ be two Bessel families in $\mathcal{H}$. Then the following assertions are equivalent:

a) $f = \sum_{\lambda\in\Lambda} \langle f, \tilde{e}_\lambda\rangle e_\lambda$ *for all* $f \in \mathcal{H}$;

b) $f = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle \tilde{e}_\lambda$ *for all* $f \in \mathcal{H}$;

c) $\langle f, g\rangle = \sum_{\lambda\in\Lambda} \langle f, e_\lambda\rangle \cdot \langle \tilde{e}_\lambda, g\rangle$ *for all* $f, g \in \mathcal{H}$.

In case the equivalent conditions are satisfied, $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are dual frames for $\mathcal{H}$.

Note that in order to be able to process both steps – decomposition and reconstruction of some signal $f \in \mathcal{H}$ – one has to know both: the frame and some dual of it (but not necessarily the canonical dual).

The canonical dual is special in the sense that it provides a moment sequence with minimal l_2 norm (see [30, Lemma VIII]). But this is not always the most important issue: there are cases where other criteria are more relevant; and thus other duals should be constructed and considered for the processing. Our subsequent construction of spline frames with associated duals will be guided by the goal to obtain framelets with local support and vanishing moments. These are the criteria of interest to us.

Example 1.10 *Recalling Examples 1.3 and 1.7 we get the representation*

$$v = \sum_{j=1}^{3} \langle v, \frac{2}{3}\, e_i\rangle\, e_i$$

for all vectors v in $\mathcal{H}$. One can also prove (see [23, Ch. 3]) that by means of the parameter $\alpha \in \mathbf{C}$ the formula

$$v = \sum_{j=1}^{3} \left[\langle v, \frac{2}{3}\, e_i\rangle + \alpha \right] \cdot e_i$$

gives all possible superpositions of the frame elements $\{e_1, e_2, e_3\}$ which yield a given vector $v \in \mathcal{H}$. Simple computations lead to the identity

$$\left\| \left(\langle v, \frac{2}{3} e_i\rangle + \alpha \right)_{i=1,2,3} \right\|_{l_2}^2 = \left\| \left(\langle v, \frac{2}{3} e_i\rangle \right)_{i=1,2,3} \right\|_{l_2}^2 + 3|\alpha|^2,$$

which demonstrates the above mentioned minimality property. □

In Chapter 3 we summarize the construction of normalized tight spline frames of Chui, He and Stöckler from [18]. As mentioned before, from the frame construction they also automatically obtain the canonical dual for their function system (i.e., $\mathcal{E} \equiv \tilde{\mathcal{E}}$). In Chapters 4&6 we present our construction of non–tight spline frames. In this different situation we have to give explicit formulae for both: the frame and some dual frame elements.

Chapter 2

B–Splines on a Bounded Interval

It is virtually impossible to reconstruct when B–splines were introduced for the first time. Schoenberg states in [57] that they were already known to Laplace in connection with their rôle as density functions in Probability Theory. Also Favard [31] used them, but without calling them splines. However, in practically the whole literature Isaac Jacob Schoenberg is recognized as "the father of splines" since the systematic study of splines began with his work in the 1940s. The key paper is [22] where on page 71 it is mentioned that Schoenberg already in 1945 wrote an article on the topic which was published as an abstract in 1947 (see [21]) only. The latter abstract was the basis for [22]. At that time the B–splines were still called basis spline curves; the abbreviation B–spline is due to Schoenberg himself (see [58]).

The history of mathematics is full of surprises. It was only in 2003 that de Boor and Pinkus published a note on "The B–spline recurrence relations of Chakalov and of Popoviciu" [27], thus showing that significant parts of the theory already existed in the 1930s in both the Bulgarian and Romanian literatures.

Splines in general, and B–splines in particular, have become a widely used tool in numerical computation and Computer Aided Geometric Design, for example. They are also indispensable for the investigation of theoretical questions occuring for instance in Quantitative Approximation Theory and many other fields of mathematics.

It is, of course, beyond the scope of this thesis to give a comprehensive survey on the subject. Instead we confine ourselves to compiling those results which are directly related to the present work. For a more detailed description the reader is referred to the monographies by de Boor [26], Schumaker [59] and DeVore&Lorentz [28].

2.1 Knot sequences and Schoenberg spaces

In order to define piecewise polynomials of a given order $m \in \mathbb{N}$ over a compact interval $[a, b]$, we have to specify the break points where two adjacent polynomial pieces meet.

Definition 2.1 *(Admissible knot sequence, see [51] and [28, Ch. 5])*
For given $m \in \mathbb{N}$ and $[a, b] \subset \mathbb{R}$ we will call the vector

$$\mathbf{t} := \{t_{-m+1} = \cdots = t_0 = a < t_1 \leq t_2 \leq \cdots \leq t_N < t_{N+1} = \cdots = t_{N+m} = b\} \tag{2.1}$$

an admissible knot sequence of order m in the interval $[a, b]$ if the multiplicity of any knot does not exceed m, i.e., $t_k < t_{k+m}$ for all possible k. The constant N describes the number of all (not necessarily distinct) inner knots of the sequence $\mathbf{t}$.
The information contained in $\mathbf{t}$ can be described alternatively through a vector of distinct inner knots

$$\theta := \{t_1 = \theta_1 < \theta_2 < \cdots < \theta_l = t_N\} \tag{2.2}$$

in combination with a vector of corresponding multiplicities

$$\mu := (\mu_1, \mu_2, \ldots, \mu_l). \tag{2.3}$$

The information on the stacked boundary knots is always the same and can thus be added automatically when passing from the (θ, μ) setting to the extended knot sequence $\mathbf{t}$. The constant l describes the number of distinct inner knots in $\mathbf{t}$.

For the subsequent construction of spline frames in $L_2[a, b]$ the following spaces will be needed.

Definition 2.2 *(Schoenberg space, see [28, Ch. 5])*
For given $m \in \mathbb{N}$, $[a, b] \subset \mathbb{R}$ and (θ, μ) as described in (2.2,2.3) we define the Schoenberg space

$$\mathcal{S}_m(\theta, \mu, [a, b])$$

to be the space of all functions $S : [a, b] \to \mathbb{R}$ which are piecewise polynomials of degree less than or equal to $m - 1$ on each interval (θ_i, θ_{i+1}) and also on $[a, \theta_1)$ and $(\theta_l, b]$; at least one of the polynomials should be of exact degree $m - 1$. The function S and its derivatives are defined at the breakpoints θ_i by continuity from the left or from the right.
The elements of the Schoenberg space $\mathcal{S}_m(\theta, \mu, [a, b])$ are called splines of order m on the interval $[a, b]$ with defects

$$(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_l - 1)$$

at the breakpoints $\theta = (\theta_1, \theta_2, \ldots, \theta_l)$, because the function S possesses in a neighborhood of θ_i the smoothness $C^{(m-1-\mu_i)}$ and the highest possible degree of smoothness $(m-2)$ is attained in simple knots. The space C^{-1} contains functions with discontinuity points.
Alternatively, we will denote the Schoenberg space $\mathcal{S}_m(\theta, \mu, [a, b])$ also by $\mathcal{S}_m(\mathbf{t}, [a, b])$ if (θ, μ) describes the sequence $\mathbf{t}$ as presented in Definition 2.1.

$\mathcal{S}_m(\mathbf{t}, [a, b])$ is a finite–dimensional Hilbert space w.r.t. the scalar product

$$\langle s_1, s_2 \rangle := \int_a^b s_1(x) s_2(x)\, dx.$$

Independently of (θ, μ) there always holds

$$\Pi_{m-1}[a, b] \subseteq \mathcal{S}_m(\theta, \mu, [a, b]) \subset L_2[a, b]. \tag{2.4}$$

These inclusions are important for the general construction scheme of spline frames with vanishing moments in $L_2[a, b]$, see Theorem 3.10, for example.

In order to specify a basis for the Schoenberg space we introduce the truncated power functions

$$(x - a)_+^k := \begin{cases} (x - a)^k, & x \geq a, \\ 0, & x < a. \end{cases}$$

Theorem 2.3 *(Basis for the Schoenberg space, see [28, Ch. 5, Theorem 1.1])*
The spline space $\mathcal{S}_m(\theta, \mu, [a, b])$ – as defined in Definition 2.2 – has the basis

$$S_{-j}(x) := \frac{(x - a)^j}{j!}, \qquad j = 0, \ldots, m - 1, \tag{2.5}$$

$$S_{i,j}(x) := \frac{(x - \theta_i)_+^j}{j!}, \qquad j = m - \mu_i, \ldots, m - 1, \quad i = 1, \ldots, l. \tag{2.6}$$

With the notation from Definition 2.1 we have

$$\dim\, \mathcal{S}_m(\theta, \mu, [a, b]) = m + \sum_{i=1}^{l} \mu_i = m + N. \tag{2.7}$$

Because this canonical basis for the Schoenberg space has some serious disadvantages (it is not local, it is numerically unstable) another – less obvious – basis with much better features was introduced by Curry and Schoenberg in their 1966 paper [22]. Its elements are called B–splines (basic splines) and these functions have the smallest possible support in the corresponding Schoenberg space. B–splines – along with some of their most important properties – are treated in the remaining sections of this chapter.

2.2 Non–uniform B–splines on bounded intervals

Schoenberg considered in his 1946 paper [57] only B–splines over equidistant knots. However, he noticed at that time already that it is possible to define B–splines by means of divided differences. It was Curry who proposed the generalization for non–uniform knot sequences which we will introduce next and use throughout this thesis. For several other possible definitions and a lot of historical information see the survey paper by Quak [52].

Definition 2.4 *(Normalized B–splines)*
Let be given: the order $m \in \mathbb{N}$ of the splines, the interval $[a,b] \subset \mathbb{R}$ and the admissible knot sequence $\mathbf{t}$ of order m over $[a,b]$ with N inner knots, $N \in \mathbb{N}_0$. For

$$k \in \mathbb{M}_{\mathbf{t};m} := \{-m+1, \ldots, N\}$$

we define the normalized B–splines $N_{\mathbf{t};m,k}$ by means of divided differences as follows:

$$N_{\mathbf{t};m,k}(x) := (t_{m+k} - t_k) \cdot [t_k, t_{k+1}, \ldots, t_{k+m}](x-t)_+^{m-1}, \qquad x \in [a,b]. \tag{2.8}$$

The B–splines $N_{\mathbf{t};m,k}$ form a basis of the Schoenberg space $\mathcal{S}_m(\theta, \mu, [a,b])$ – see [22, Section I.4]) for the original proof –, where (θ, μ) contain the information from the knot sequence $\mathbf{t}$. The aforementioned drawback of non–locality is eliminated because the B–splines have local support, i.e.,

$$\textit{supp}\ N_{\mathbf{t};m,k} = [t_k, t_{m+k}].$$

The pointwise recursion

$$N_{\mathbf{t};m,k}(x) = \frac{x - t_k}{t_{k+m-1} - t_k} N_{\mathbf{t};m-1,k}(x) + \frac{t_{k+m} - x}{t_{k+m} - t_{k+1}} N_{\mathbf{t};m-1,k+1}(x) \tag{2.9}$$

with initialization

$$N_{\mathbf{t};1,k}(x) := \chi_{[t_k, t_{k+1})}(x) \tag{2.10}$$

is a stable procedure, due to the positivity of both the weights and the B–splines on the right–hand side in (2.9). In the case of multiple knots, if one of the denominators in (2.9) vanishes, then the whole term should be set by convention equal to zero. As (2.10) reveals we consider here right–continuous B–splines. Therefore – also for the computational part – one has to "complete" the definition and the recursion (2.9) by setting the rightmost B–spline equal to 1 at b. We used the procedure (2.9)–(2.10) to evaluate B–splines in our implementations for sibling spline frames (see Chapter 6).

In the special case $N = 0$ (no inner knots, only stacked boundary knots in $\mathbf{t}$) the B–splines of order m over $[a,b]$ coincide with the Bernstein basic functions of degree $m-1$ on the same interval, namely

$$\begin{aligned} p_{m-1,k}(x) &:= \frac{1}{(b-a)^{m-1}} \cdot \binom{m-1}{k} (x-a)^k (b-x)^{m-1-k} \\ &= N_{\mathbf{t};m,-m+k+1}(x), \qquad k = 0,1,\ldots,m-1, \quad x \in [a,b]. \end{aligned}$$

The B–splines $N_{\mathbf{t};m,k}$ are normalized such that they form a partition of unity, i.e., for each $x \in [a,b]$ there holds

$$\sum_{k=-m+1}^{N} N_{\mathbf{t};m,k}(x) = 1.$$

The above basis constitutes the fundamental building blocks for so–called variation–diminishing Schoenberg splines on $[a,b]$ and many of the "B–spline curves" as used in Computer Aided Geometric Design. For several results related to quantitative aspects and a number of pertinent references see our notes [8, 9, 2, 10].

In order to illustrate the notions and results of this chapter the following example will be systematically extended and discussed.

Example 2.5 *(Non–uniform cubic B–splines, see [51, p. 144])*
We consider the following admissible knot sequence of order $m = 4$ over $[0,1]$:

$$\begin{aligned} \mathbf{t} &= \left\{0,\, 0,\, 0,\, 0,\, \frac{1}{6},\, \frac{1}{4},\, \frac{1}{2},\, \frac{1}{2},\, \frac{2}{3},\, 1,\, 1,\, 1,\, 1\right\} \\ &= \{t_{-3}, t_{-2}, t_{-1}, t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\}. \end{aligned} \tag{2.11}$$

It has $N = 5$ inner knots. The equivalent description according to Definition 2.1 is

$$(\theta, \mu) = \left(\left\{\frac{1}{6},\, \frac{1}{4},\, \frac{1}{2},\, \frac{2}{3}\right\},\, (1,\, 1,\, 2,\, 1)\right)$$

with $l = 4$ distinct inner knots. The B–spline basis $[N_{\mathbf{t};4,k}]_{k \in I\!M_{\mathbf{t};4}}$ of the corresponding Schoenberg space $\mathcal{S}_4(\mathbf{t}, [0,1])$ contains 9 elements, as visualized in Figure 2.1. □

In the sequel we will be interested in L_2–normalized B–splines (denoted by $N^B_{\mathbf{t};m,k}$).

Definition 2.6 *(Weighted knot differences, L^2–normalized B–splines)*
Under the general assumptions of Definition 2.4 weighted knot differences for the vector $\mathbf{t}$ are given by

$$d_{\mathbf{t};m,k} := \frac{t_{m+k} - t_k}{m} = \int_{t_k}^{t_{m+k}} N_{\mathbf{t};m,k}(x)\, dx, \tag{2.12}$$

for all possible values of k. The L^2–normalized B–splines are defined by

$$N^B_{\mathbf{t};m,k} := (d_{\mathbf{t};m,k})^{-1/2} \cdot N_{\mathbf{t};m,k}, \qquad k \in I\!M_{\mathbf{t};m}. \tag{2.13}$$

We will use the row[1] vector notation $\mathbf{\Phi}^B_{\mathbf{t};m} := [N^B_{\mathbf{t};m,-m+1}, \ldots, N^B_{\mathbf{t};m,N}]$ for brevity.

The L_2–normalized B–splines $N^B_{\mathbf{t};m,k}$, $k \in I\!M_{\mathbf{t};m}$, define a Riesz basis for the corresponding Schoenberg space $\mathcal{S}_m(\mathbf{t}, [a,b])$ as expressed in the next theorem.

Theorem 2.7 *(Stability of B–splines, see [26, p. 156] and [28, Ch. 5, Theorem 4.2])*
For each $m \in I\!N$ there exists a constant $D_m > 0$ (independent of the knot vector $\mathbf{t}$) such that for all $\{c_k\}_{k \in I\!M_{\mathbf{t};m}} \in l_2(I\!M_{\mathbf{t};m})$ there holds

$$D_m \|\{c_k\}_{k \in I\!M_{\mathbf{t};m}}\|^2_{l_2(I\!M_{\mathbf{t};m})} \leq \left\| \sum_{k \in I\!M_{\mathbf{t};m}} c_k \cdot N^B_{\mathbf{t};m,k} \right\|^2_{L_2[a,b]} \leq \|\{c_k\}_{k \in I\!M_{\mathbf{t};m}}\|^2_{l_2(I\!M_{\mathbf{t};m})}. \tag{2.14}$$

[1]Throughout this thesis we use row vector notation. The superscript B will always denote L_2–normalization.

Example 2.8 *(Non–uniform cubic B–splines)*
We continue the discussion in Example 2.5 and obtain the weight vector

$$\begin{aligned}[d_{\mathbf{t};4,k}]_{k\in I\!M_{\mathbf{t};4}} &= \frac{1}{4}\,[\,t_1-t_{-3},\, t_2-t_{-2},\, t_3-t_{-1},\, \dots, t_8-t_4,\, t_9-t_5\,] \\ &= \left[\frac{1}{24},\frac{1}{16},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{3}{16},\frac{1}{8},\frac{1}{8},\frac{1}{12}\right].\end{aligned}$$

The L^2–normalized B–spline basis $[N^B_{\mathbf{t};4,k}]_{k\in I\!M_{\mathbf{t};4}}$ of the Schoenberg space $\mathcal{S}_4(\mathbf{t},[0,1])$ is depicted in Figure 2.2. □

2.3 The dual B–spline basis

In order to represent each element of $\mathcal{S}_m(\mathbf{t},[a,b])$ in terms of the basis $[N^B_{\mathbf{t};m,k}]_{k\in I\!M_{\mathbf{t};m}}$ one needs a dual basis $[D^B_{\mathbf{t};m,k}]_{k\in I\!M_{\mathbf{t};m}}$ in the sense that $\langle N^B_{\mathbf{t};m,k}, D^B_{\mathbf{t};m,l}\rangle = \delta_{k,l}$ for all $k,l \in I\!M_{\mathbf{t};m}$. Having such function system available one can write

$$s = \sum_{k\in I\!M_{\mathbf{t};m}} \langle s, D^B_{\mathbf{t};m,k}\rangle N^B_{\mathbf{t};m,k}, \qquad \text{for all } s \in \mathcal{S}_m(\mathbf{t},[a,b]).$$

The Gram matrix $\Gamma^B_{\mathbf{t};m}$ of $\mathbf{\Phi}^B_{\mathbf{t};m} = \left[N^B_{\mathbf{t};m,k}\right]_{k\in I\!M_{\mathbf{t};m}}$, given by

$$\Gamma^B_{\mathbf{t};m} := \int_a^b \mathbf{\Phi}^B_{\mathbf{t};m}(x)^T \cdot \mathbf{\Phi}^B_{\mathbf{t};m}(x)\,dx = \left[(d_{\mathbf{t};m,k}\cdot d_{\mathbf{t};m,l})^{-1/2}\langle N_{\mathbf{t};m,k}, N_{\mathbf{t};m,l}\rangle\right]_{k,l\in I\!M_{\mathbf{t};m}} \tag{2.15}$$

is a symmetric positive definite matrix (see [41]) which defines the dual Riesz basis

$$\tilde{\mathbf{\Phi}}^{\Gamma,B}_{\mathbf{t};m} := \mathbf{\Phi}^B_{\mathbf{t};m}\cdot\left(\Gamma^B_{\mathbf{t};m}\right)^{-1} =: \left[D^B_{\mathbf{t};m,k}\right]_{k\in I\!M_{\mathbf{t};m}} \tag{2.16}$$

for the Schoenberg space $\mathcal{S}_m(\mathbf{t},[a,b])$.

Example 2.9 *(Non–uniform cubic B–splines)*
For $\mathbf{t}$ *in (2.11) we obtain the banded Gram matrix*

$$\Gamma^B_{\mathbf{t};4} = \begin{pmatrix} 0.5714 & 0.2670 & 0.0550 & 0.0037 & 0 & 0 & 0 & 0 & 0 \\ 0.2670 & 0.4063 & 0.2292 & 0.0363 & 0.0000 & 0 & 0 & 0 & 0 \\ 0.0550 & 0.2292 & 0.4690 & 0.2855 & 0.0484 & 0.0026 & 0 & 0 & 0 \\ 0.0037 & 0.0363 & 0.2855 & 0.4329 & 0.2232 & 0.0249 & 0 & 0 & 0 \\ 0 & 0.0000 & 0.0484 & 0.2232 & 0.4621 & 0.2115 & 0.0068 & 0.0004 & 0 \\ 0 & 0 & 0.0026 & 0.0249 & 0.2115 & 0.5359 & 0.2390 & 0.0835 & 0.0085 \\ 0 & 0 & 0 & 0 & 0.0068 & 0.2390 & 0.3683 & 0.2730 & 0.0726 \\ 0 & 0 & 0 & 0 & 0.0004 & 0.0835 & 0.2730 & 0.4063 & 0.2670 \\ 0 & 0 & 0 & 0 & 0 & 0.0085 & 0.0726 & 0.2670 & 0.5714 \end{pmatrix}.$$

The dual Riesz basis is obtained as follows.

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{\Gamma,B}_{\mathbf{t};4} &= \mathbf{\Phi}^B_{\mathbf{t};4}\cdot\left(\Gamma^B_{\mathbf{t};4}\right)^{-1} \\ &= \mathbf{\Phi}^B_{\mathbf{t};4}\cdot\begin{pmatrix} 2.8452 & -2.6476 & 1.5417 & -1.0244 & 0.4212 & -0.1974 & 0.1983 & -0.1133 & 0.0307 \\ -2.6476 & 6.6081 & -4.7908 & 3.3027 & -1.3829 & 0.6506 & -0.6538 & 0.3735 & -0.1011 \\ 1.5417 & -4.7908 & 7.6342 & -5.9204 & 2.6113 & -1.2414 & 1.2485 & -0.7135 & 0.1932 \\ -1.0244 & 3.3027 & -5.9204 & 7.9496 & -4.1114 & 2.0101 & -2.0265 & 1.1593 & -0.3140 \\ 0.4212 & -1.3829 & 2.6113 & -4.1114 & 5.1816 & -2.9446 & 3.0031 & -1.7263 & 0.4687 \\ -0.1974 & 0.6506 & -1.2414 & 2.0101 & -2.9446 & 4.7527 & -5.0698 & 2.9672 & -0.8128 \\ 0.1983 & -0.6538 & 1.2485 & -2.0265 & 3.0031 & -5.0698 & 11.8233 & -8.6119 & 2.5971 \\ -0.1133 & 0.3735 & -0.7135 & 1.1593 & -1.7263 & 2.9672 & -8.6119 & 10.0279 & -3.6354 \\ 0.0307 & -0.1011 & 0.1932 & -0.3140 & 0.4687 & -0.8128 & 2.5971 & -3.6354 & 3.1307 \end{pmatrix}. \;\square\end{aligned}$$

The reproducing kernel of $\mathcal{S}_m(\mathbf{t},[a,b])$ is thus given by

$$K^B_{\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}}(x,y) := \sum_{k=-m+1}^{N} N^B_{\mathbf{t};m,k}(x)\cdot D^B_{\mathbf{t};m,k}(y) = \mathbf{\Phi}^B_{\mathbf{t};m}(x)\cdot\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}\cdot\left(\mathbf{\Phi}^B_{\mathbf{t};m}(y)\right)^T. \tag{2.17}$$

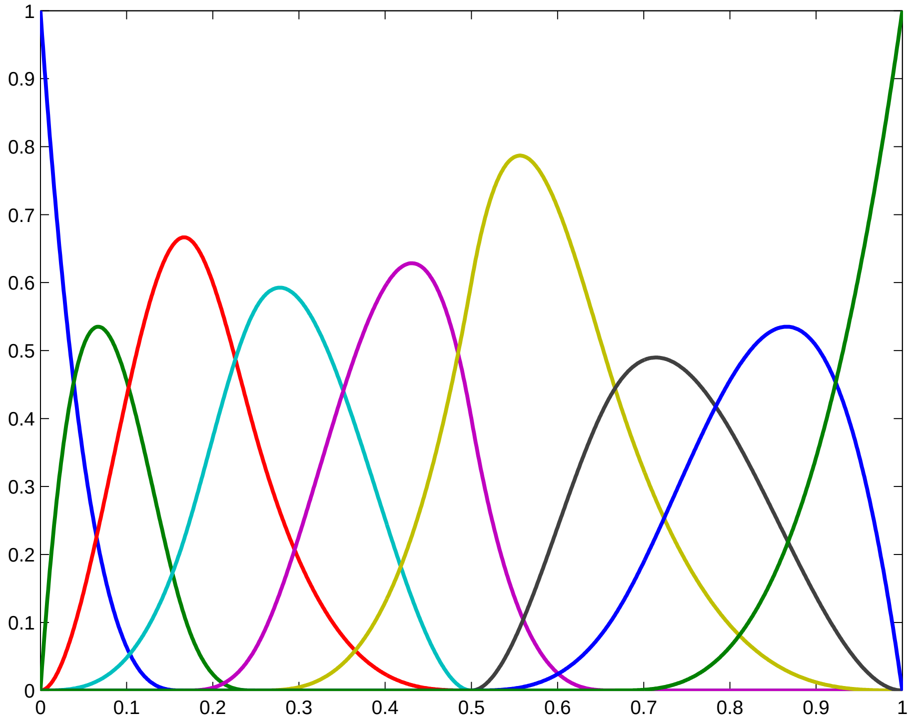

Figure 2.1: Non–uniform cubic B–spline basis $[N_{\mathbf{t};4,k}]_{k\in I\!M_{\mathbf{t};4}}$ on $[0,1]$ w.r.t. the knot sequence $\mathbf{t}$ in (2.11).

Figure 2.2: Non–uniform cubic B–spline basis $[N^B_{\mathbf{t};4,k}]_{k\in I\!M_{\mathbf{t};4}}$ on $[0,1]$ w.r.t. the knot sequence $\mathbf{t}$ in (2.11).

I.e., we have

$$\int_a^b s(y) \cdot K^B_{\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}}(x,y)\,dy = s(x), \qquad \text{for all } s \in \mathcal{S}_m(\mathbf{t},[a,b]). \tag{2.18}$$

Due to the locality of the B–splines, the Gramian is a banded matrix, but its inverse $\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}$ in general is not. This is a drawback, in addition to the problem of giving convenient representations of the entries of $\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}$. The dual basis in (2.16) does not have local character and thus the kernel (2.17) is hard to handle.

For the construction of tight spline frames Chui, He and Stöckler eliminated this drawback by introducing (see [18] and Section 3.3) a so–called approximate dual $\tilde{\mathbf{\Phi}}^{S,B}_{\mathbf{t};m}$ for the Riesz basis $\mathbf{\Phi}^B_{\mathbf{t};m} = \left[N^B_{\mathbf{t};m,k}\right]_{k\in I\!\!M_{\mathbf{t};m}}$ with the aid of a **banded** symmetric positive semi–definite matrix $S^B_{\mathbf{t};m}$ constructed directly from the knot sequence $\mathbf{t}$ (for details see Section 3.3), i.e.,

$$\tilde{\mathbf{\Phi}}^{S,B}_{\mathbf{t};m} := \mathbf{\Phi}^B_{\mathbf{t};m} \cdot S^B_{\mathbf{t};m}. \tag{2.19}$$

With the corresponding **approximate** kernel

$$K^B_{S^B_{\mathbf{t};m}}(x,y) := \mathbf{\Phi}^B_{\mathbf{t};m}(x) \cdot S^B_{\mathbf{t};m} \cdot \left(\mathbf{\Phi}^B_{\mathbf{t};m}(y)\right)^T, \tag{2.20}$$

they cannot provide any more the full reproduction property (2.18). Nonetheless, for the construction of tight spline frames they do not need this strong property. One still has the reproduction of some polynomial space Π_{L-1}, i.e.,

$$\int_a^b p(y) \cdot K^B_{S^B_{\mathbf{t};m}}(x,y)\,dy = p(x) \qquad \text{for all } p \in \Pi_{L-1}[a,b]. \tag{2.21}$$

Moreover, this approach has significant theoretical and numerical advantages (for details see Chapter 3).

Shadrin proved the following result for the reproducing kernel of $\mathcal{S}_m(\mathbf{t},[a,b])$ defined in (2.17). Theorem 2.10 is implied by Shadrin's solution of a more general problem of de Boor posed in 1972. Chui, He and Stöckler obtain the same property for their approximate kernel (2.20) in [18], see also Section 3.4.

Theorem 2.10 *(Uniform boundedness of the kernel, see [56, Theorem I])*
There exists a constant C_m independent of the knot vector $\mathbf{t}$ and the interval $I = [a,b]$, such that

$$\sup_{x\in I} \int_I \left| K^B_{\left(\Gamma^B_{\mathbf{t};m}\right)^{-1}}(x,y)\right| dy \le C_m. \tag{2.22}$$

2.4 Refining the B–spline basis via knot insertion

Let $\mathbf{t}$ be an admissible knot sequence of order m in the interval $[a,b]$ (in the sense of Definition 2.1). Consider a refined sequence $\tilde{\mathbf{t}}$ of $\mathbf{t}$ such that $\tilde{\mathbf{t}}$ is again admissible of order m. The relation $\mathbf{t} \subset \tilde{\mathbf{t}} \subset [a,b]$ implies

$$\mathcal{S}_m(\mathbf{t},[a,b]) \subset \mathcal{S}_m(\tilde{\mathbf{t}},[a,b]) \subset L_2[a,b].$$

The refinement relations of the corresponding B–splines are given by

$$\begin{aligned} \mathbf{\Phi}_{\mathbf{t};m} &= \mathbf{\Phi}_{\tilde{\mathbf{t}};m} \cdot P_{\mathbf{t},\tilde{\mathbf{t}};m}, \\ \mathbf{\Phi}^B_{\mathbf{t};m} &= \mathbf{\Phi}^B_{\tilde{\mathbf{t}};m} \cdot P^B_{\mathbf{t},\tilde{\mathbf{t}};m}, \end{aligned} \tag{2.23}$$

with refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ obtained from the Oslo algorithm (see [19] for the original version and [51] for an elegant matrix formulation) and

$$P^B_{\mathbf{t},\tilde{\mathbf{t}};m} := diag\left(d^{1/2}_{\tilde{\mathbf{t}};m,0}\right) \cdot P_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot diag\left(d^{-1/2}_{\mathbf{t};m,0}\right) \tag{2.24}$$

with weights $d_{\tilde{\mathbf{t}};m,0}$ and $d_{\mathbf{t};m,0}$ defined by (2.12).

A very comprehensive survey of the Oslo algorithm and related techniques with many aspects to be considered can be found in [64].

The matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ generated by the Oslo algorithm has non–negative entries and the elements of each row sum up to 1. The row indices indicate the basis functions $\mathbf{\Phi}_{\tilde{\mathbf{t}};m}$ and the column indices refer to the basis functions $\mathbf{\Phi}_{\mathbf{t};m}$. This matrix is sparse in the following sense: an entry $p_{r,s}$ does not vanish only in the case

$$supp(N_{\tilde{\mathbf{t}};m,r}) \subset supp(N_{\mathbf{t};m,s}). \tag{2.25}$$

Example 2.11 *(Non–uniform cubic B–splines continued)*
For the refinement

$$\tilde{\mathbf{t}} = \left\{0,\, 0,\, 0,\, 0,\, \frac{1}{9},\, \frac{1}{6},\, \frac{1}{5},\, \frac{2}{9},\, \frac{1}{4},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{3}{5},\, \frac{2}{3},\, \frac{5}{6},\, \frac{5}{6},\, 1,\, 1,\, 1,\, 1\right\} \tag{2.26}$$

of the knot sequence $\mathbf{t}$ *in (2.11) we obtain from the Oslo algorithm the following matrix:*

$$P_{\mathbf{t},\tilde{\mathbf{t}};4} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5/9 & 4/9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/9 & 32/45 & 8/45 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/45 & 28/45 & 16/45 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 29/60 & 1/60 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5/6 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2/5 & 3/5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4/25 & 21/25 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4/5 & 1/5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4/15 & 3/5 & 2/15 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/9 & 4/9 & 4/9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 7/12 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The B–spline basis $[N_{\tilde{\mathbf{t}};4,k}]_{k\in I\!M_{\tilde{\mathbf{t}};4}}$ *contains* 16 *elements, as visualized in Figure 2.3. The B–spline basis* $[N_{\mathbf{t};4,k}]_{k\in I\!M_{\mathbf{t};4}}$ *is plotted in Figure 2.1.*
By comparing these graphical representations the relation between the sparsity of $P_{\mathbf{t},\tilde{\mathbf{t}};4}$ *and the inclusion (2.25) becomes clear. Furthermore, with*

$$\begin{aligned}
d_{\tilde{\mathbf{t}};4,0} &= \left[\frac{1}{36}, \frac{1}{24}, \frac{1}{20}, \frac{1}{18}, \frac{5}{144}, \frac{1}{12}, \frac{3}{40}, \frac{5}{72}, \frac{7}{80}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}, \frac{1}{10}, \frac{1}{12}, \frac{1}{24}, \frac{1}{24}\right], \\
d_{\mathbf{t};4,0} &= \left[\frac{1}{24}, \frac{1}{16}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{16}, \frac{1}{8}, \frac{1}{8}, \frac{1}{12}\right],
\end{aligned}$$

we obtain

$$P^B_{\mathbf{t},\tilde{\mathbf{t}};4} = \begin{pmatrix}
0.8165 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3333 & 0.5443 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.4969 & 0.2811 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.1048 & 0.4741 & 0.1185 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0166 & 0.3279 & 0.1874 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.4082 & 0.3946 & 0.0136 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.6455 & 0.1291 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.7454 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.3347 & 0.4099 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0924 & 0.3960 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5333 & 0.1633 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1778 & 0.4899 & 0.1089 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0811 & 0.3975 & 0.3975 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.1361 & 0.4763 & 0.2500 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2887 & 0.3536 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7071
\end{pmatrix}. \qquad \square$$

The orthogonal projection of $L_2[a,b]$ onto $\mathcal{S}_m(\mathbf{t},[a,b])$ and its orthogonal complement relative to $\mathcal{S}_m(\tilde{\mathbf{t}},[a,b])$ are given by

$$f \mapsto \int_a^b f(y) \cdot K^B_{(\Gamma^B_{\mathbf{t};m})^{-1}}(\cdot, y)\, dy \qquad \text{and} \tag{2.27}$$

$$f \mapsto \int_a^b f(y) \cdot \left[K^B_{\left(\Gamma^B_{\tilde{\mathbf{t}};m}\right)^{-1}}(\cdot, y) - K^B_{(\Gamma^B_{\mathbf{t};m})^{-1}}(\cdot, y) \right] dy, \tag{2.28}$$

respectively. We also have

$$\begin{aligned}
& K^B_{\left(\Gamma^B_{\tilde{\mathbf{t}};m}\right)^{-1}}(x, y) - K^B_{(\Gamma^B_{\mathbf{t};m})^{-1}}(y, y) \\
& \overset{(2.17)}{=} \mathbf{\Phi}^B_{\tilde{\mathbf{t}};m}(x) \cdot \left(\Gamma^B_{\tilde{\mathbf{t}};m}\right)^{-1} \cdot \left(\mathbf{\Phi}^B_{\tilde{\mathbf{t}};m}(y)\right)^T - \mathbf{\Phi}^B_{\mathbf{t};m}(x) \cdot \left(\Gamma^B_{\mathbf{t};m}\right)^{-1} \cdot \left(\mathbf{\Phi}^B_{\mathbf{t};m}(y)\right)^T \\
& \overset{(2.23)}{=} \mathbf{\Phi}^B_{\tilde{\mathbf{t}};m}(x) \cdot \left[\left(\Gamma^B_{\tilde{\mathbf{t}};m}\right)^{-1} - P^B_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot \left(\Gamma^B_{\mathbf{t};m}\right)^{-1} \cdot \left(P^B_{\mathbf{t},\tilde{\mathbf{t}};m}\right)^T \right] \cdot \left(\mathbf{\Phi}^B_{\tilde{\mathbf{t}};m}(y)\right)^T .
\end{aligned}$$

Thus the orthogonal complement of $\mathcal{S}_m(\mathbf{t},[a,b])$ in $\mathcal{S}_m(\tilde{\mathbf{t}},[a,b])$ is characterized through the symmetric positive semi–definite matrix

$$\left(\Gamma^B_{\tilde{\mathbf{t}};m}\right)^{-1} - P^B_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot \left(\Gamma^B_{\mathbf{t};m}\right)^{-1} \cdot \left(P^B_{\mathbf{t},\tilde{\mathbf{t}};m}\right)^T .$$

As mentioned in the previous section, in [18] Chui, He and Stöckler introduced approximate duals $\tilde{\mathbf{\Phi}}^{S,B}_{\mathbf{t};m}$ and $\tilde{\mathbf{\Phi}}^{S,B}_{\tilde{\mathbf{t}};m}$ by means of certain symmetric positive semi–definite matrices $S^B_{\mathbf{t};m}$ and $S^B_{\tilde{\mathbf{t}};m}$, respectively. These satisfy an analogous property as the aforementioned one, i.e., $S^B_{\tilde{\mathbf{t}};m} - P^B_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot S^B_{\mathbf{t};m} \cdot \left(P^B_{\mathbf{t},\tilde{\mathbf{t}};m}\right)^T$ is symmetric and positive semi–definite (see Section 3.5). For a related discussion of positive semi–definite matrices of this special type see Section 2.1 in our note [3] and Proposition 3 there in particular.
The operator

$$f \mapsto \int_a^b f(y) \cdot K^B_{S^B_{\mathbf{t};m}}(\cdot, y)\, dy \tag{2.29}$$

associated to $S^B_{\mathbf{t};m}$ does not describe the ortho–projection from $L_2[a,b]$ onto $\mathcal{S}_m(\mathbf{t},[a,b])$ any more, but for some polynomial space $\Pi_{L-1} \subset \mathcal{S}_m(\mathbf{t},[a,b])$ it preserves the projection property in the sense that

$$\langle p, \tilde{\mathbf{\Phi}}^{S,B}_{\mathbf{t};m} \rangle = \langle p, \tilde{\mathbf{\Phi}}^{\Gamma,B}_{\mathbf{t};m} \rangle \qquad \text{for all } p \in \Pi_{L-1}[a,b],$$

and it thus implies additional advantages as already mentioned (see also Chapter 3).

Remark 2.12 *(see [51, Section 3])*
It is known that for nested admissible sequences $\mathbf{t} \subset \tilde{\mathbf{t}} \subset \tilde{\tilde{\mathbf{t}}} \subset [a,b]$ the following factorization for the matrices given by the Oslo algorithm holds:

$$P_{\mathbf{t},\tilde{\tilde{\mathbf{t}}};m} = P_{\tilde{\mathbf{t}},\tilde{\tilde{\mathbf{t}}};m} \cdot P_{\mathbf{t},\tilde{\mathbf{t}};m}. \tag{2.30}$$

2.5 Derivatives of B–splines

Derivatives of B–splines are essential for the construction of spline frames with vanishing moments. We now introduce the notations needed and several results which we will refer to later in this thesis.

For the first derivative of a B–spline (which is again a spline) the following is known:

$$N'_{\mathbf{t};m,k}(x) = \frac{m-1}{t_{k+m-1} - t_k} N_{\mathbf{t};m-1,k}(x) - \frac{m-1}{t_{k+m} - t_{k+1}} N_{\mathbf{t};m-1,k+1}(x). \qquad (2.31)$$

For this formula to be true in all cases one has to adopt the same convention as for the recursion (2.9), i.e., in the case of multiple knots, if one of the denominator vanishes, then the whole term should be set equal to zero.
The knot sequence $\mathbf{t}$ has boundary knots with multiplicity m. This always implies the annulation of the denominators $(t_0 - t_{-m+1})$ and $(t_{N+m} - t_{N+1})$. Therefore we get for the boundary derivatives $N'_{\mathbf{t};m,-m+1}$ and $N'_{\mathbf{t};m,N}$ only a one term formula in (2.31).

In order to be able to present matrix formulations for higher order derivatives of B–splines – as introduced in [18] and needed further in this thesis – we need some more notation.

Let $\nu \in I\!N_0$, $0 \leq \nu \leq m$, be a new parameter indicating a certain increase of the order m of the B–splines considered. For $0 \leq \nu \leq m$ define the index sets

$$I\!M_{\mathbf{t};m,\nu} := \{-m+1, \ldots, N-\nu\}.$$

Note that $I\!M_{\mathbf{t};m,0}$ was introduced earlier in this chapter as $I\!M_{\mathbf{t};m}$.

The B–splines $N_{\mathbf{t};m,k}$, $k \in I\!M_{\mathbf{t};m,0}$, of order m over the knot sequence $\mathbf{t}$ are defined in (2.8). By $N_{\mathbf{t};m+\nu,k}$ we denote the B–splines of order $m+\nu$ ($\nu \geq 1$) over **the same** knot sequence $\mathbf{t}$, i.e.,

$$N_{\mathbf{t};m+\nu,k}(x) := (t_{k+m+\nu} - t_k) \cdot [t_k, \ldots, t_{k+m+\nu}](\cdot - x)_+^{m+\nu-1}, \qquad (2.32)$$

for $k \in I\!M_{\mathbf{t};m,\nu}$ and $x \in [a,b]$. Note that we do not consider here the whole B–spline basis of order $m+\nu$ ($\nu \geq 1$). We **do not** expand the multiplicity of the boundary knots from m to $m+\nu$ and this implies the lack of some boundary elements of the basis for the Schoenberg space of order $m+\nu$ ($\nu \geq 1$). The new B–spline vectors are then given by

$$\mathbf{\Phi}_{\mathbf{t};m+\nu} \;:=\; [N_{\mathbf{t};m+\nu,k}]_{k \in I\!M_{\mathbf{t};m,\nu}}, \quad 0 \leq \nu \leq m. \qquad (2.33)$$

Note that the vector $\mathbf{\Phi}_{\mathbf{t};m+\nu}$ has ν entries less than $\mathbf{\Phi}_{\mathbf{t};m}$. We consider this special setting because it turned out to be the appropriate one for the construction of spline frames with vanishing moments (see [18] and Section 3.5).

For the case $\nu = 1$, from (2.31) we obtain the following representation for the first derivatives:

$$N'_{\mathbf{t};m+1,k}(x) = \frac{m}{t_{k+m} - t_k} N_{\mathbf{t};m,k}(x) - \frac{m}{t_{k+m+1} - t_{k+1}} N_{\mathbf{t};m,k+1}(x), \qquad k \in I\!M_{\mathbf{t};m,1}. \qquad (2.34)$$

Note that in this setting we do not encounter any difficulties for the boundary elements as presented at the beginning of this section, i.e., in (2.34) we always have 2 terms on the right hand side. This property is inherited by the higher derivatives $\frac{d^\nu}{dx^\nu} N_{\mathbf{t};m+\nu,k}(x)$.

For $0 \leq \nu \leq m$ we extend the definition of the weights from (2.12) to

$$d_{\mathbf{t};m,\nu,k} := \frac{t_{k+m+\nu} - t_k}{m+\nu} = \int_{t_k}^{t_{k+m+\nu}} N_{\mathbf{t};m+\nu,k}(x)\, dx\,, \qquad (2.35)$$

and introduce the sequences of weights

$$d_{\mathbf{t};m,\nu} := [d_{\mathbf{t};m,\nu,k}]_{k \in I\!M_{\mathbf{t};m,\nu}}\,. \qquad (2.36)$$

The L^2–normalization of the B–splines in (2.32) is given by

$$N^B_{\mathbf{t};m+\nu,k} := d^{-1/2}_{\mathbf{t};m,\nu,k} \cdot N_{\mathbf{t};m+\nu,k}, \quad k \in I\!M_{\mathbf{t};m,\nu}, \quad 0 \le \nu \le m, \tag{2.37}$$

and the corresponding B–spline vectors are

$$\mathbf{\Phi}^B_{\mathbf{t};m+\nu} := [N^B_{\mathbf{t};m+\nu,k}]_{k \in I\!M_{\mathbf{t};m,\nu}} = \mathbf{\Phi}_{\mathbf{t};m+\nu} \cdot diag\left(d^{-1/2}_{\mathbf{t};m,\nu}\right), \qquad 0 \le \nu \le m. \tag{2.38}$$

For the representation of differences of order 1 (as they appear in (2.34) on the right hand side) we define the matrix

$$\Delta_n := \begin{pmatrix} 1 & & & & \mathbf{0} \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \\ \mathbf{0} & & & & -1 \end{pmatrix}_{n \times (n-1)} . \tag{2.39}$$

Thus we can rewrite (2.34) in the form

$$\frac{d}{dx} \mathbf{\Phi}_{\mathbf{t};m+1} = \mathbf{\Phi}_{\mathbf{t};m} \cdot diag\left(d^{-1}_{\mathbf{t};m,0}\right) \cdot \Delta_{N+m}. \tag{2.40}$$

For higher order differentiation of B–splines we introduce the bi–diagonal matrices

$$D_{\mathbf{t};m,\nu} := diag\left(d^{-1}_{\mathbf{t};m,\nu}\right) \cdot \Delta_{N+m-\nu}, \tag{2.41}$$

and

$$D^B_{\mathbf{t};m,\nu} := diag\left(d^{-1/2}_{\mathbf{t};m,\nu}\right) \cdot \Delta_{N+m-\nu} \cdot diag\left(d^{-1/2}_{\mathbf{t};m,\nu+1}\right) \tag{2.42}$$

$$= diag\left(d^{1/2}_{\mathbf{t};m,\nu}\right) \cdot D_{\mathbf{t};m,\nu} \cdot diag\left(d^{-1/2}_{\mathbf{t};m,\nu+1}\right), \tag{2.43}$$

where $\nu \in \{0, 1, \cdots, m-1\}$. By iteration one gets from (2.34) the following result.

Proposition 2.13 *(Matrix representation of higher order derivatives of B–splines, see [18, Section 4])*
For the B–splines defined in (2.32) the following formula for differentiation of order L ($1 \le L \le m$) is true:

$$\frac{d^L}{dx^L} \mathbf{\Phi}_{\mathbf{t};m+L}(x) = \mathbf{\Phi}_{\mathbf{t};m}(x) \cdot E_{\mathbf{t};m,L} \tag{2.44}$$

with differentiation matrix of order L (representing the difference operator of order L) defined by

$$E_{\mathbf{t};m,L} := D_{\mathbf{t};m,0} \cdot D_{\mathbf{t};m,1} \cdot \ldots \cdot D_{\mathbf{t};m,L-1} = \prod_{\nu=0}^{L-1} D_{\mathbf{t};m,\nu}. \tag{2.45}$$

Furthermore,

$$\frac{d^L}{dx^L} \mathbf{\Phi}^B_{\mathbf{t};m+L}(x) = \mathbf{\Phi}^B_{\mathbf{t};m}(x) \cdot E^B_{\mathbf{t};m,L} \tag{2.46}$$

with

$$E^B_{\mathbf{t};m,L} := D^B_{\mathbf{t};m,0} \cdot D^B_{\mathbf{t};m,1} \cdot \ldots \cdot D^B_{\mathbf{t};m,L-1} = \prod_{\nu=0}^{L-1} D^B_{\mathbf{t};m,\nu} \tag{2.47}$$

$$= diag\left(d^{1/2}_{\mathbf{t};m,0}\right) \cdot E_{\mathbf{t};m,L} \cdot diag\left(d^{-1/2}_{\mathbf{t};m,L}\right). \tag{2.48}$$

At this point we emphasize the fact that the differentiation matrices $E^B_{\mathbf{t};m,L}$ are constructed directly from the knot sequence $\mathbf{t}$ by the following steps:

$$\mathbf{t} \quad \rightarrow \quad d_{\mathbf{t};m,\nu} \quad (0 \le \nu \le L) \quad \rightarrow \quad D^B_{\mathbf{t};m,\nu} \quad (0 \le \nu \le L-1) \quad \rightarrow \quad E^B_{\mathbf{t};m,L}.$$

Example 2.14 *(Non–uniform cubic B–splines)*
For $\mathbf{t}$ *in (2.11) we obtain the differentiation matrix of order* $L = 3$ *as*

$$\begin{aligned}
E_{\mathbf{t};4,3} &= D_{\mathbf{t};4,0} \cdot D_{\mathbf{t};4,1} \cdot D_{\mathbf{t};4,2} \\
&= \begin{pmatrix}
24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{16}{3} & \frac{16}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -12
\end{pmatrix} \\
&\quad \cdot \begin{pmatrix}
20 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & -10 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{15}{2} & \frac{15}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -6 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{20}{3} & \frac{20}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & -10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & -10
\end{pmatrix} \cdot \begin{pmatrix}
12 & 0 & 0 & 0 & 0 & 0 \\
-12 & 12 & 0 & 0 & 0 & 0 \\
0 & -9 & 9 & 0 & 0 & 0 \\
0 & 0 & -6 & 6 & 0 & 0 \\
0 & 0 & 0 & -\frac{36}{5} & \frac{36}{5} & 0 \\
0 & 0 & 0 & 0 & -8 & 8 \\
0 & 0 & 0 & 0 & 0 & -12
\end{pmatrix} \\
&= \begin{pmatrix}
5760 & 0 & 0 & 0 & 0 & 0 \\
-7680 & 1920 & 0 & 0 & 0 & 0 \\
2880 & -2640 & 720 & 0 & 0 & 0 \\
-960 & 2220 & -1620 & 360 & 0 & 0 \\
0 & -540 & 1188 & -\frac{4968}{5} & \frac{1728}{5} & 0 \\
0 & 0 & -192 & \frac{3392}{5} & -\frac{34688}{45} & \frac{2560}{9} \\
0 & 0 & 0 & -384 & \frac{4352}{3} & -\frac{6080}{3} \\
0 & 0 & 0 & 0 & -640 & 2560 \\
0 & 0 & 0 & 0 & 0 & -1440
\end{pmatrix}.
\end{aligned}$$

The third derivatives of the B–splines $\mathbf{\Phi}_{\mathbf{t};7}$ *can now be computed by*

$$\frac{d^3}{dx^3}\mathbf{\Phi}_{\mathbf{t};7}(x) = \mathbf{\Phi}_{\mathbf{t};4}(x) \cdot E_{\mathbf{t};4,3}.$$

For example we obtain from the building blocks $N_{\mathbf{t};4,-1}$*,* $N_{\mathbf{t};4,0}$*,* $N_{\mathbf{t};4,1}$ *and* $N_{\mathbf{t};4,2}$ *the spline* $s \in \mathcal{S}_4(\mathbf{t},[0,1])$ *defined by*

$$\begin{aligned}
s(x) &:= \frac{d^3}{dx^3}N_{\mathbf{t};7,-1}(x) \qquad (2.49)\\
&= 720 \cdot N_{\mathbf{t};4,-1}(x) - 1620 \cdot N_{\mathbf{t};4,0}(x) + 1188 \cdot N_{\mathbf{t};4,1}(x) - 192 \cdot N_{\mathbf{t};4,2}(x)
\end{aligned}$$

over the knots $\{t_{-1}, t_0, \ldots, t_6\} = \left\{0, 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 1\right\}$ *with supp* $(s) = [t_{-1}, t_6] = [0,1]$ *and three vanishing moments (see Figure 2.4).* □

The construction principle of tight spline frames with vanishing moments of Chui, He and Stöckler [18] is built upon the fact that a spline $s \in \mathcal{S}_m(\mathbf{t},[a,b])$ has L vanishing moments, if and only if it is the Lth derivative of a spline S of order $m+L$ w.r.t. the same knot vector $\mathbf{t}$ (as described at the beginning of this section). S can be chosen such that its derivatives $S^{(\nu)}$, $0 \le \nu \le L-1$, vanish at both endpoints of the interval $[a,b]$ (see [18, p. 156]). This principle is visualized for a concrete situation in Example 2.14 and will be adopted in the sequel also for our construction of sibling spline frames in Chapters 4&6.

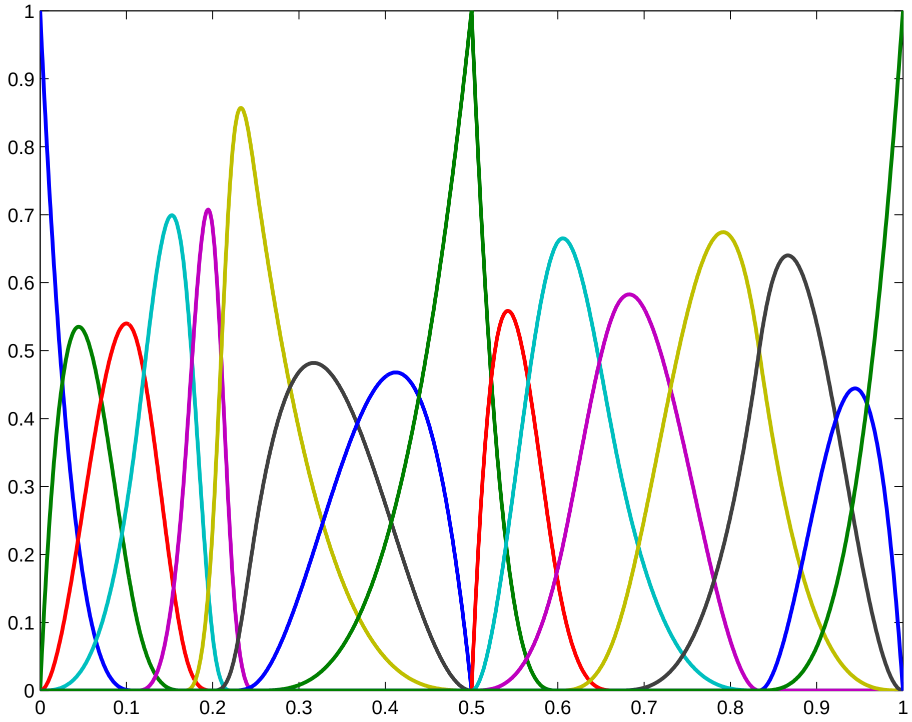

Figure 2.3: Non–uniform cubic B–spline basis $[N_{\tilde{\mathbf{t}};4,k}]_{k\in I\!M_{\tilde{\mathbf{t}};4}}$ on $[0,1]$ w.r.t. the knot sequence $\tilde{\mathbf{t}}$ in (2.26).

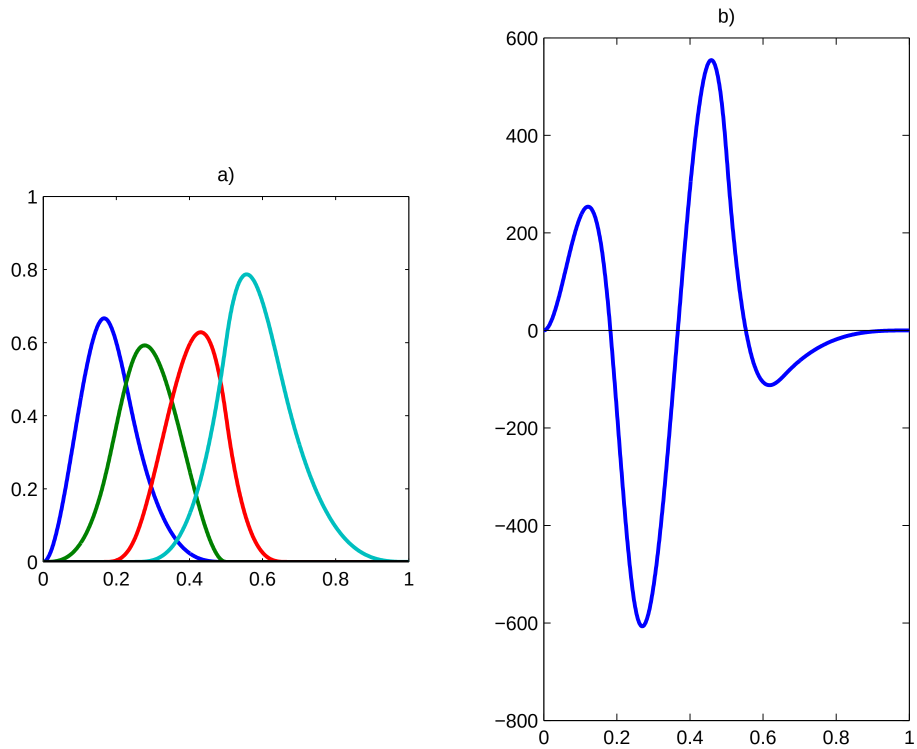

Figure 2.4: a) From left to right: $N_{\mathbf{t};4,-1}$, $N_{\mathbf{t};4,0}$, $N_{\mathbf{t};4,1}$ and $N_{\mathbf{t};4,2}$ with $\mathbf{t}$ from (2.11). These are the building blocks for the spline depicted on the right hand side. b) The spline $s \in \mathcal{S}_4(\mathbf{t},[0,1])$ defined in (2.49) with three vanishing moments.

Chapter 3

Non–stationary MRA Tight Frames on Bounded Intervals

The aim of this chapter is to describe the non–stationary multiresolution analysis (MRA) setting under which we will work in the sequel and to give a brief review of non–stationary MRA tight frames of $L_2[a,b]$ as introduced and studied by Chui, He & Stöckler in [18]. The latter paper presents a general construction principle as well as practical procedures for non–stationary tight wavelet frames with maximal number of vanishing moments and minimal support on a compact interval $[a,b]$ of the real line.

The most important ingredient in [18] is the so–called *approximate dual matrix* which determines an approximate dual (basis) of a given finite basis. This notion is introduced for the first time in [18, Definition 3.1] and it extends the concept of vanishing moment recovery (VMR) function introduced in [17, Section 3] (and the notion of fundamental function of multiresolution introduced in [54, Subsection 6.1] and adopted in [25] for recovering vanishing moments) to the matrix formulation.

3.1 The non–stationary MRA framework

In order to systematically construct orthonormal wavelet bases Mallat and Meyer introduced in 1986 the multiresolution analysis (or multiscale approximation) as a general tool in approximation theory and signal analysis. Thus they provided a natural framework for the understanding of wavelet bases and provided a well structured scheme which describes the various refinement steps clearly, such that this technique became accessible to engineers for practical implementation (see [44]).

For a detailed review of the classical (i.e., stationary) MRA see, e.g., [23]. Daubechies [23] resumes: "The history of the formulation of multiresolution analysis is a beautiful example of applications stimulating theoretical development. When he first learned about the Meyer basis, Mallat was working on image analysis, where the idea of studying images simultaneously at different scales and comparing the results had been popular for many years. This stimulated him to view orthonormal wavelet bases as a tool to describe mathematically the "increment of information" needed to go from a coarse approximation to a higher resolution approximation. This insight crystallized into multiresolution analysis." The classical MRA has been extended in different ways with the purpose of removing some of its constraints. For our further work we need the following generalization.

Definition 3.1 *(Non–stationary multiresolution analysis on a compact interval)*
Let $I := [a,b]$ be a bounded interval on the real axis $\mathbb{R}$ and let

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots$$

be a sequence of nested finite–dimensional subspaces of $L_2(I)$ which are dense in the space, i.e.,

$$clos_{L_2}\left(\cup_{j\geq 0}V_j\right) = L_2(I).$$

Further consider each V_j to be spanned by a finite system

$$\mathbf{\Phi}_j := [\phi_{j,k};\ 1 \leq k \leq M_j], \tag{3.1}$$

where $M_j \geq \dim V_j$. The refinement relation of $V_j \subset V_{j+1}$ is described by a real matrix

$$P_j = [p_{k,l}^{(j)}]$$

of dimension $M_{j+1} \times M_j$ and reads as follows:

$$\mathbf{\Phi}_j = \mathbf{\Phi}_{j+1} \cdot P_j. \tag{3.2}$$

The triplet

$$\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$$

is called non–stationary MRA of $L_2(I)$. j serves as an index for the different levels (scales, resolutions) of the MRA. The V_j's are called approximation spaces of the MRA.

In order to approximate a function $f \in L_2(I)$ via a MRA $(V_j)_{j\geq 0}$, the natural starting point is to search for an approximation f_{j_0} in a certain space V_{j_0}. If no element in this space approximates f well enough, then one obtains a better approximation by choosing a larger value j_1 – i.e., a higher resolution – and by searching for a new approximation f_{j_1} in the larger space V_{j_1}. This approximation can be expressed as a linear combination of the functions in $\mathbf{\Phi}_{j_1}$.

In order to express the "difference" (or the "amount of details") between the approximation spaces V_j one needs some building blocks which, in our case, will be the frame elements $\psi_{j,k}$. In other cases the rôle of the building blocks will be played by the elements of a Riesz, or a wavelet basis.

Note that no conditions of uniform refinement are required in Definition 3.1 (neither shift invariance, nor dilation invariance). Here non–stationarity refers to both irregular "shifts" on the same level (see (3.1)) and non–uniform refinement from one level to the next one (see (3.2)) which are allowed in the described MRA. This general framework is used in [18] and the special case of the spline MRA generated by a sequence of nested knot sequences which are dense in I is included (see, e.g., [43, 18]).

Definition 3.2 *(The non–stationary spline MRA on a bounded interval)*
Let $m \in I\!N$ be the order of the B–splines and let

$$\mathbf{t}_0 \subset \cdots \subset \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \cdots \subset I := [a, b] \tag{3.3}$$

be a <u>dense</u> sequence of finite knot vectors for the interval $[a, b] \subset I\!R$. Each vector $\mathbf{t}_j$ has N_j inner knots and is admissible in the sense of Definition 2.1. Let $\mathcal{S}_m(\mathbf{t}_j, [a, b])$ be the Schoenberg space generated by the knot sequence $\mathbf{t}_j$, as presented in Definition 2.2. It will play the rôle of the approximation space on the level j, because $\mathbf{t}_j \subset \mathbf{t}_{j+1} \subset [a, b]$ implies

$$\mathcal{S}_m(\mathbf{t}_j, [a, b]) \subset \mathcal{S}_m(\mathbf{t}_{j+1}, [a, b]) \subset L_2[a, b],$$

and

$$clos_{L_2}\left(\cup_{j\geq 0}\, \mathcal{S}_m(\mathbf{t}_j, [a, b])\right) = L_2[a, b]$$

(see [28, Ch. 7, Theorem 7.3]).

The L^2–normalized B–splines over the knot sequence $\mathbf{t}_j$ are denoted by $N^B_{\mathbf{t}_j;m,k}$ (see Definition 2.6). Since $\{N^B_{\mathbf{t}_j;m,k} : k \in I\!M_{\mathbf{t}_j;m}\}$ is a Riesz basis for the Schoenberg space $\mathcal{S}_m(\mathbf{t}_j, [a,b])$ we consider this vector to be $\mathbf{\Phi_j}$ in the sense of Definition 3.1.

The refinement $\mathcal{S}_m(\mathbf{t}_j, [a,b]) \subset \mathcal{S}_m(\mathbf{t}_{j+1}, [a,b])$ is characterized by a matrix $P_{\mathbf{t}_j,\mathbf{t}_{j+1};m}$ which may be computed by the Oslo algorithm (see [51]).
Thus the triplet

$$\left\{(\mathcal{S}_m(\mathbf{t}_j, [a,b]))_{j\geq 0}, (\{N^B_{\mathbf{t}_j;m,k} : k \in I\!M_{\mathbf{t}_j;m}\})_{j\geq 0}, (P_{\mathbf{t}_j,\mathbf{t}_{j+1};m})_{j\geq 0}\right\} \tag{3.4}$$

is a non–stationary MRA of $L_2(I)$ in the sense of Definition 3.1, the so–called non–stationary spline MRA.

For the frame elements to be useful in applications, we will require in the sequel the same localization property of the function vectors $\mathbf{\Phi}_j$ as in [18]. For brevity we write $I\!M_j := \{1, \ldots, M_j\}$.

Definition 3.3 *(Locally supported function family, see [18, Definition 2.1])*
A function family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} := \{[\phi_{j,k};\ 1 \leq k \leq M_j]\}_{j\geq 0}$$

is said to be locally supported, if the sequence of the maximal support lengths on each level

$$h_j := h(\mathbf{\Phi}_j) := \max_{k\in I\!M_j} length(supp\ \phi_{j,k})$$

converges to zero.

Remark 3.4 *In the spline case described in Definition 3.2 the family*

$$\mathbf{\Phi}^B_m := \left\{\mathbf{\Phi}^B_{\mathbf{t}_j;m}\right\}_{j\geq 0} = \left(\left\{N^B_{\mathbf{t}_j;m,k} : k \in I\!M_{\mathbf{t}_j;m}\right\}\right)_{j\geq 0}$$

is locally supported due to the density of the nested knot sequences in $[a,b]$. The sequence of maximal support lengths $\left\{h\left(\mathbf{\Phi}^B_{\mathbf{t}_j;m}\right)\right\}_{j\geq 0}$ is in this case monotonically decreasing; this is a positive aspect for applications.

The main tools in characterizing MRA tight frames will be the entities defined next.

Definition 3.5 *(Quadratic form, kernel, see [18, Definition 2.2])*
Let $I = [a,b]$ be a compact interval of the real axis. For a finite family $\mathbf{\Phi} = [\phi_k]_{k\in I\!M}$ from $L_2(I)$ with cardinality M ($\{1, \ldots, M\} =: I\!M$) and a real matrix $S = [s_{k,l}]_{k,l\in I\!M}$ we define

(i) the corresponding quadratic form

$$\begin{aligned} T_S(f) &:= [\langle f, \phi_k\rangle]_{k\in I\!M} \cdot S \cdot [\langle f, \phi_k\rangle]^T_{k\in I\!M} \\ &=: \langle f, \mathbf{\Phi}\rangle \cdot S \cdot \langle f, \mathbf{\Phi}\rangle^T, \qquad f \in L_2(I), \end{aligned} \tag{3.5}$$

(ii) the corresponding symmetric kernel

$$\begin{aligned} K_S(x,y) &:= \Phi(x) \cdot S \cdot \Phi^T(y) \\ &= \sum_{k,l\in I\!M} \phi_k(x) \cdot s_{k,l} \cdot \phi_l(y), \qquad x, y \in I. \end{aligned} \tag{3.6}$$

$T_S(f)$ and K_S are related by

$$T_S(f) = \int_I f(x) \int_I f(y) \cdot K_S(x,y)\, dydx, \qquad f \in L_2(I).$$

3.2 Non–stationary MRA tight frames on bounded intervals

MRA tight frames can be viewed as a natural generalization of orthonormal wavelet bases. By allowing redundancy, one gains the flexibility to achieve some desired additional properties for the elements of the system. A systematical study of MRA frames in $L_2(I\!R^d)$ was initiated by Ron & Shen (see [54, 55]). Two recent parallel independent developments [17, 25] showed that compactly supported orthonormal wavelet bases of $L_2(I\!R)$ can be replaced by compactly supported tight frames in $L_2(I\!R)$ to achieve analytical formulations and symmetry for the frame elements, while retaining the same order of vanishing moments needed for practical applications such as compression.

For stationary orthonormal wavelet bases in the bounded interval setting, it is always possible to adopt the corresponding orthonormal wavelets from the real–line setting as interior wavelets and to construct only additional boundary elements for the basis. For non–orthonormal MRA tight frames in general, it is not clear how to adopt the tight frame elements from the real–line setting for the bounded interval case (see [18, Section 10]). This is a clear distinction between the theory of tight frames and that of orthonormal wavelets for the bounded interval setting and at the same time the motivation for the ansatz in [18], where constructive schemes for all spline frame elements are presented.

Definition 3.6 *(Non–stationary MRA tight frame, see [18, Definition 2.3])*
With $I := [a,b]$ let

$$\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$$

be a non–stationary MRA of $L_2(I)$ with locally supported family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} := \{[\phi_{j,k};\ 1 \leq k \leq M_j]\}_{j\geq 0}.$$

Further let S_0 be a symmetric positive semi–definite (spsd) matrix with associated quadratic form T_{S_0} (the so–called ground level component).
Then the family $\mathbf{\Psi}$ determined by the sequence of coefficient matrices $\{Q_j\}_{j\geq 0}$ ($\dim Q_j = M_{j+1} \times N_j$) through the relations

$$\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\geq 0} := \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\geq 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\geq 0} \tag{3.7}$$

is called a non–stationary MRA tight frame of $L_2(I)$ w.r.t. T_{S_0}, if

$$T_{S_0}(f) + \sum_{j\geq 0}\sum_{k\in I\!N_j} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|_{L_2}^2 \qquad \text{for all } f \in L_2(I). \tag{3.8}$$

$I\!N_j$ denotes the set $\{1,\ldots,N_j\}$. The parameter N_j describes the number of frame elements $\psi_{j,k}$ on the level j. If (3.8) holds, the Q_j's are named frame coefficient matrices.

The following theorem is one of the main results in the general theory of MRA tight frames developed by Chui, He and Stöckler in [18].

Theorem 3.7 *(Characterization of non–stationary MRA tight frames, see [18, Theorem 2.4])*
With $I := [a,b]$ let

$$\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$$

be a non–stationary MRA of $L_2(I)$ with locally supported family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} := \{[\phi_{j,k};\ 1 \leq k \leq M_j]\}_{j\geq 0}.$$

Further let T_{S_0} be the quadratic form associated to a spsd matrix S_0 which satisfies for all $f \in L_2(I)$ the condition $T_{S_0}(f) \le \|f\|^2$.

Then $\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\ge 0} := \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\ge 0}$ defines a non–stationary MRA tight frame w.r.t. T_{S_0} if and only if there exist spsd matrices S_j of dimension $M_j \times M_j$, $j \ge 1$, such that the following conditions hold:

(i) the quadratic forms T_{S_j} satisfy

$$\lim_{j\to\infty} T_{S_j}(f) = \|f\|^2_{L_2}, \qquad f \in L_2(I);$$

(ii) for each $j \ge 0$, the matrices Q_j, S_j and S_{j+1} satisfy the identity

$$S_{j+1} - P_j S_j P_j^T = Q_j Q_j^T.$$

The existence of vanishing moments for framelets is connected to some property of the quadratic forms from Theorem 3.7, as presented in the next result.

Theorem 3.8 *(Characterization of non–stationary MRA tight frames with vanishing moments, see [18, Theorem 2.6])*
With $I := [a, b]$ let

$$\{(V_j)_{j\ge 0}, (\mathbf{\Phi}_j)_{j\ge 0}, (P_j)_{j\ge 0}\}$$

be a non–stationary MRA of $L_2(I)$ with locally supported family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\ge 0} = \{[\phi_{j,k};\ 1 \le k \le M_j]\}_{j\ge 0}$$

and with $\Pi_{L-1} \subset V_0$ for some $L \in I\!N$. Let T_{S_0} be the quadratic form associated to a spsd matrix S_0 which satisfies for all $f \in L_2(I)$ the condition $T_{S_0}(f) \le \|f\|^2$, and let $\mathbf{\Psi}$ be the function family given by

$$\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\ge 0} := \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\ge 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\ge 0}.$$

Then the functions $\psi_{j,k}$ have L vanishing moments and define a non–stationary MRA tight frame w.r.t. T_{S_0}, if and only if there exist spsd matrices S_j of dimensions $M_j \times M_j$ ($j \ge 1$), such that conditions (i)–(ii) of Theorem 3.7 hold and, moreover,

(iii) $T_{S_j}(p) = \|p\|^2_{L_2} \qquad$ *for all* $p \in \Pi_{L-1}[a, b], \quad j \ge 1.$

Definition 3.9 below introduces the essential ingredient for the characterization and construction of non–stationary tight MRA frames with vanishing moments.

Definition 3.9 *(Approximate dual in $L_2[a, b]$, see [18, Definition 3.1])*
Let Φ be a basis of a finite–dimensional subspace V of $L_2[a, b]$ and let $L \in I\!N$ be an integer such that $\Pi_{L-1}[a, b] \subset V$. For an spsd matrix S, the function vector

$$\Phi^S := \Phi \cdot S$$

is called an approximate dual (basis) of Φ of order L, if the following polynomial reproduction property holds:

$$\int_a^b p(y) \cdot K_S(x, y)\, dy = p(x), \qquad \text{for all } p \in \Pi_{L-1}[a, b],$$

with K_S from (3.6). We call S the approximate dual matrix.

With the aid of this concept Chui, He and Stöckler proved the following theorem revealing the general construction principle of tight spline frames from [18].

Theorem 3.10 *(Characterization of non–stationary MRA tight frames with vanishing moments by means of approximate duals, see [18, Theorem 2.6, Corollary 3.3])*

Let

- $\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$, *with* $\Pi_{L-1} \subset V_0$ *for some* $L \in I\!N$, *be a non–stationary MRA of* $L_2[a,b]$ *with locally supported* <u>*bases*</u>

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} = \{[\phi_{j,k};\ 1 \leq k \leq M_j]\}_{j\geq 0};$$

- S_0 *be a spsd matrix such that*

$$T_{S_0}(f) \leq \|f\|_{L_2}^2, \qquad f \in L_2[a,b],$$

$$\int_a^b p(y) \cdot K_{S_0}(x,y)\, dy = p(x), \qquad p \in \Pi_{L-1}[a,b];$$

- *the function family* $\mathbf{\Psi}$ *be given as*

$$\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\geq 0} := \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\geq 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\geq 0},$$

  with suitable matrices Q_j, $j \geq 0$.

Then the following holds:
The functions $\psi_{j,k}$ *have* L *vanishing moments and define a non–stationary MRA tight frame w.r.t.* T_{S_0}, *if and only if there exist spsd matrices* S_j *of dimensions* $M_j \times M_j$ $(j \geq 1)$, *such that conditions (i)–(ii) of Theorem 3.7 hold and* S_j *defines an approximate dual of* $\mathbf{\Phi}_j$ *of order* L *for all* $j \geq 1$.

The authors of [18] construct for the spline case concrete approximate dual matrices S_j which satisfy the conditions in Theorem 3.10 and obtain in this way non–stationary tight spline frames with vanishing moments. These approximate dual matrices are presented in some detail in the next section.

3.3 Construction of the minimally supported approximate dual of the B–spline basis

Chui, He and Stöckler develop in [18] an explicit formulation for the unique approximate dual with minimal support for the B–spline basis. We use exactly the same approximate dual in our forthcoming considerations for sibling frames. Therefore, we recall briefly the 4–step construction scheme for the approximate dual matrix, as given in [18, Section 5].

Let $m \in I\!N$ be the order of B–splines over the admissible knot sequence $\mathbf{t} \subset [a,b]$ with N inner knots. The whole basis is denoted by $\mathbf{\Phi}_{\mathbf{t};m}$. The order of the approximate dual is denoted by $L \in I\!N$, $(1 \leq L \leq m)$.

Algorithm 3.11 *(Minimally supported approximate dual of order L for the B–spline basis)*

- **Input***:* m, L, $\mathbf{t}$.
- **Output***:* $S_{\mathbf{t};m,L}$ *and* $\mathbf{\Phi}_{\mathbf{t},m} \cdot S_{\mathbf{t};m,L}$.

- **Procedure**:

  1. *Generalized Marsden coefficients (see [18, Subsection 5.1]) are defined as homogeneous polynomials of degree 2ν with variable number of arguments ($= r$) through*

$$
\begin{aligned}
&F_\nu : \mathbb{R}^r \to \mathbb{R} \\
&F_\nu(x_1, \ldots, x_r) := \sum_{\substack{1 \le i_1, \ldots, i_{2\nu} \le r \ \text{distinct} \\ i_1 > i_3 > \cdots > i_{2\nu-1} \\ i_{2j-1} > i_{2j} \ \text{for} \ 1 \le j \le \nu}} y(x_{i_1}, \ldots, x_{i_{2\nu}}) \qquad (3.9)
\end{aligned}
$$

  with

$$y(x_{i_1}, \ldots, x_{i_{2\nu}}) := (x_{i_1} - x_{i_2})^2 \cdot (x_{i_3} - x_{i_4})^2 \cdot \ldots \cdot (x_{i_{2\nu-1}} - x_{i_{2\nu}})^2. \qquad (3.10)$$

  2. *β–coefficients (see [18, Subsection 5.2]) are defined by*

$$\beta_{\mathbf{t};m,0,k} := 1, \quad k \in \mathbb{M}_{\mathbf{t};m,0}, \qquad (3.11)$$

  and for $\nu \in \{1, 2, \ldots, L-1\}$ by

$$\beta_{\mathbf{t};m,\nu,k} := \frac{m!(m-\nu-1)!}{(m+\nu)!(m+\nu-1)!} F_\nu(t_{k+1}, \ldots, t_{k+m+\nu-1}), \quad k \in \mathbb{M}_{\mathbf{t};m,\nu-1}. \qquad (3.12)$$

  3. *For $\nu \in \{0, \ldots, L-1\}$ define*

$$\text{the elements:} \quad u_{\mathbf{t};m,\nu,k} := \frac{m+\nu}{t_{k+m+\nu} - t_k} \beta_{\mathbf{t};m,\nu,k}, \quad k \in \mathbb{M}_{\mathbf{t};m,\nu}, \qquad (3.13)$$

$$\text{the sequences:} \quad u_{\mathbf{t};m,\nu} := [u_{\mathbf{t};m,\nu,k}]_{k \in \mathbb{M}_{\mathbf{t};m,\nu}}, \qquad (3.14)$$

$$\text{the matrices:} \quad U_{\mathbf{t};m,\nu} := \mathit{diag}\,(u_{\mathbf{t};m,\nu}), \qquad (3.15)$$

  (see [18, Subsection 5.2]).

  4. *The minimally supported approximate dual of $\mathbf{\Phi}_{\mathbf{t};m}$ of order L is given by $\mathbf{\Phi}_{\mathbf{t},m} \cdot S_{\mathbf{t};m,L}$ with*

$$S_{\mathbf{t};m,L} := U_{\mathbf{t};m,0} + \sum_{\nu=1}^{L-1} E_{\mathbf{t};m,\nu} \cdot U_{\mathbf{t};m,\nu} \cdot E_{\mathbf{t};m,\nu}^T \qquad (3.16)$$

  and with matrices $E_{\mathbf{t};m,\nu}$ defined in (2.45). For further details on the existence, the support minimality and the uniqueness of the above mentioned approximate dual see [18, Subsections 5.2, 5.4, 5.6], especially Theorems 5.6 and 5.11.

The matrix $S_{\mathbf{t};m,L}$ has dimension $(m+N) \times (m+N)$, is symmetric, non–singular and banded with bandwidth[1] L. The corresponding kernel (defined in (3.6)) has the form

$$K_{S_{\mathbf{t};m,L}}(x, y) = \sum_{\nu=0}^{L-1} \sum_{k \in \mathbb{M}_{\mathbf{t};m,\nu}} u_{\mathbf{t};m,\nu,k} \cdot \frac{d^{2\nu}}{dx^\nu dy^\nu} N_{\mathbf{t};m+\nu,k}(x) N_{\mathbf{t};m+\nu,k}(y). \qquad (3.17)$$

Note that the approximate dual matrix $S_{\mathbf{t};m,L}$ is constructed directly from the knot sequence $\mathbf{t}$ as follows:

$$\mathbf{t} \ \xrightarrow{F_\nu} \ \underset{(0 \le \nu \le L-1)}{[\beta_{\mathbf{t};m,\nu,k}]_{k \in \mathbb{M}_{\mathbf{t};m,\nu}}} \ \longrightarrow \ \underset{(0 \le \nu \le L-1)}{U_{\mathbf{t};m,\nu}} \ \xrightarrow[(0 \le \nu \le L-1)]{E_{\mathbf{t};m,\nu}} \ S_{\mathbf{t};m,L}\,.$$

[1] In this setting we use the following definition: A matrix $A = \{a_{ik}\}_{i,k}$ has bandwidth L, if $|i-k| \ge L$ always implies $a_{ik} = 0$.

For the L_2–normalization one obtains:

$$u^B_{\mathbf{t};m,\nu} := u_{\mathbf{t};m,\nu} \cdot diag\left(d_{\mathbf{t};m,\nu}\right), \tag{3.18}$$

$$U^B_{\mathbf{t};m,\nu} := diag\left(d^{1/2}_{\mathbf{t};m,\nu}\right) \cdot U_{\mathbf{t};m,\nu} \cdot diag\left(d^{1/2}_{\mathbf{t};m,\nu}\right) = diag\left(d_{\mathbf{t};m,\nu}\right) \cdot U_{\mathbf{t};m,\nu} \tag{3.19}$$

$$= diag\left(u^B_{\mathbf{t};m,\nu}\right), \tag{3.20}$$

$$S^B_{\mathbf{t};m,L} := diag\left(d^{1/2}_{\mathbf{t};m,0}\right) \cdot S_{\mathbf{t};m,L} \cdot diag\left(d^{1/2}_{\mathbf{t};m,0}\right) = diag\left(d_{\mathbf{t};m,0}\right) \cdot S_{\mathbf{t};m,L} \tag{3.21}$$

$$= U^B_{\mathbf{t};m,0} + \sum_{\nu=1}^{L-1} E^B_{\mathbf{t};m,\nu} \cdot U^B_{\mathbf{t};m,\nu} \cdot \left(E^B_{\mathbf{t};m,\nu}\right)^T, \tag{3.22}$$

$$K^B_{S^B_{\mathbf{t};m,L}} := \mathbf{\Phi}^B(x) \cdot S^B_{\mathbf{t};m,L} \cdot \left(\mathbf{\Phi}^B(y)\right)^T = K_{S_{\mathbf{t};m,L}}. \tag{3.23}$$

3.4 Approximate kernels for the B–spline case

Here we briefly recall Chui, He and Stöckler's result on the boundedness of their approximate kernel $K_{S_{\mathbf{t};m,L}}$ introduced in (3.17), cf. Section 2.3. This property is sufficient for the proof of condition (i) in Theorem 3.7 also appearing in Theorem 3.10.

Theorem 3.12 *(Normalization and uniform boundedness of the approximate kernel, see [18, Theorem 5.12])*
The kernel (3.17) associated to the minimally supported approximate dual of order L ($1 \le L \le m$) of the B–spline basis of order $m \in \mathbb{N}$ over the admissible knot vector $\mathbf{t} \subset [a,b] =: I$ satisfies the following properties:

(i) normalization to 1, i.e., we have

$$\int_I K_{S_{\mathbf{t};m,L}}(x,y)\,dy = 1 \qquad \text{for all } x \in I; \tag{3.24}$$

(ii) uniform boundedness, i.e., there holds

$$\int_I \left|K_{S_{\mathbf{t};m,L}}(x,y)\right|\,dy \le C \qquad \text{for all } x \in I \tag{3.25}$$

with constant

$$C = C(m,L) = \sum_{\nu=0}^{L-1} 2^\nu \binom{m+\nu-1}{\nu}$$

independent of the knot vector $\mathbf{t}$ and the interval I.

Example 3.13 *For low orders of splines we get the following concrete constants $C(m,L)$.*

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
$m=1$	1					
$m=2$	1	5				
$m=3$	1	7	31			
$m=4$	1	9	49	209		
$m=5$	1	11	71	351	1471	
$m=6$	1	13	97	545	2561	10625

The uniform boundedness of the kernel leads to the following result.

Theorem 3.14 *(see [18, Theorem 6.1])*
The quadratic forms associated to the minimally supported approximate duals of order L $(1 \leq L \leq m)$ of the B–spline bases of order $m \in I\!N$ over the admissible knot vectors $\mathbf{t}_j$ with

$$\mathbf{t}_0 \subset \mathbf{t}_1 \subset \ldots \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \ldots \subset [a,b],$$
$$\lim_{j\to\infty} h(\mathbf{t}_j) = 0,$$

satisfy the following property:

$$\lim_{j\to\infty} T_{S_{\mathbf{t}_j;m,L}}(f) = \|f\|_{L_2}^2, \qquad f \in L_2[a,b].$$

Thus Theorem 3.14 explains that condition (i) in Theorem 3.7 is always satisfied if one chooses the matrices S_j to be the approximate dual matrices $S_{\mathbf{t}_j;m,L}$.

3.5 Tight spline frames

Theorem 3.10 provides the general principle for the construction of tight spline frames in [18]. The matrices $S_{\mathbf{t}_j;m,L}$ determine approximate duals of order L and they satisfy condition (i) in Theorem 3.7. By constructing matrices Q_j via factorizing the left hand side in condition (ii) from Theorem 3.7, i.e.,

$$S_{\mathbf{t}_{j+1};m,L} - P_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot S_{\mathbf{t}_j;m,L} \cdot P^T_{\mathbf{t}_j,\mathbf{t}_{j+1};m} = Q_j Q_j^T,$$

one obtains a tight MRA frame

$$\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\geq 0} := \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\geq 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\geq 0}$$

with framelets $\psi_{j,k}$ possessing L vanishing moments. The next assertion shows that this factorization always exists.

Theorem 3.15 *(Positive semi–definiteness, see [18, Theorem 5.7])*
For $L = 1, 2, \ldots, m$ and $j \geq 0$ the matrices

$$S_{\mathbf{t}_{j+1};m,L} - P_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot S_{\mathbf{t}_j;m,L} \cdot P^T_{\mathbf{t}_j,\mathbf{t}_{j+1};m}$$

are positive semi–definite.

For the linear and the cubic case examples of tight spline frames with vanishing moments can be found in [18, Section 7].

In the sequel we drop the index j and describe step by step the above mentioned factorization for the refinement $\mathbf{t} \subset \tilde{\mathbf{t}} \subset [a,b]$. This is also relevant for the forthcoming construction scheme for sibling spline frames.

The first step of the factorization.
Denote the cardinality of the refinement by $M := \#(\tilde{\mathbf{t}} \setminus \mathbf{t})$ and the intermediate knot vectors by

$$\mathbf{t} = \mathbf{s_0} \subset \mathbf{s_1} \subset \cdots \subset \mathbf{s_{M-1}} \subset \mathbf{s_M} = \tilde{\mathbf{t}}, \tag{3.26}$$

where $\#(\mathbf{s_{k+1}} \setminus \mathbf{s_k}) = 1$ for all possible k. In this procedure the M new knots are inserted one by one, in order, from the left to the right. $N_{\mathbf{s_k}}$ denotes the number of inner knots in the sequence $\mathbf{s_k}$ and $s_i^{(k)}$ some element of the sequence $\mathbf{s_k}$.

The first step (see [18, Theorem 5.7]) consists in factorizing out on both sides the differentiation matrices $E_{\tilde{\mathbf{t}};m,L}$ presented in Chapter 2:

$$S_{\tilde{\mathbf{t}};m,L} - P_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot S_{\mathbf{t};m,L} \cdot \left(P_{\mathbf{t},\tilde{\mathbf{t}};m}\right)^T = E_{\tilde{\mathbf{t}};m,L} \cdot Z_{\mathbf{t},\tilde{\mathbf{t}};m,L} \cdot \left(E_{\tilde{\mathbf{t}};m,L}\right)^T . \tag{3.27}$$

The remaining symmetric positive semi–definite matrix $Z_{\mathbf{t},\tilde{\mathbf{t}};m,L}$ (see [18, proof of Theorem 5.7 on p. 168]) has the representation (see [18, Theorem 5.7])

$$Z_{\mathbf{t},\tilde{\mathbf{t}};m,L} = \sum_{k=1}^{M} P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}, \tag{3.28}$$

where $P_{\mathbf{s_k},\mathbf{s_M};m+L}$ are the refinement matrices given by the Oslo algorithm and the diagonal matrices

$$V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} = diag\left(v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}\right) \tag{3.29}$$

have non–negative entries denoted as follows:

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} = [v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l}]_{l\in I\!M_{\mathbf{s_k};m,L-1}}. \tag{3.30}$$

For the knot $\{\tau\} := \mathbf{s_k} \setminus \mathbf{s_{k-1}}$ and the corresponding index ρ defined via $\tau \in \left[s_\rho^{(k-1)}, s_{\rho+1}^{(k-1)}\right)$ we get the following representation for the entries (see [18, Lemma 5.9]):

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} = \begin{cases} \dfrac{\left(s^{(k-1)}_{l+m+L-1}-\tau\right)\left(\tau-s^{(k-1)}_l\right)}{(m+L-1)\left(s^{(k-1)}_{l+m+L-1}-s^{(k-1)}_l\right)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} & \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise} , \end{cases}$$

$$= \begin{cases} \dfrac{\left(s^{(k)}_{l+m+L}-s^{(k)}_{\rho+1}\right)\left(s^{(k)}_{\rho+1}-s^{(k)}_l\right)}{(m+L-1)\left(s^{(k)}_{l+m+L}-s^{(k)}_l\right)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} & \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise} , \end{cases}$$

$$= \begin{cases} \dfrac{s^{(k)}_{l+m+L}-s^{(k)}_{\rho+1}}{s^{(k)}_{l+m+L}-s^{(k)}_l} \cdot \dfrac{s^{(M)}_{\rho+1}-s^{(M)}_l}{m+L-1} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} & \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise} , \end{cases}$$

with β–coefficients defined by (3.11)–(3.12). □

The second step of the factorization.
The second step consists in symmetrically factorizing the above matrix $Z_{\mathbf{t},\tilde{\mathbf{t}};m,L}$, i.e.,

$$Z_{\mathbf{t},\tilde{\mathbf{t}};m,L} = A_{\mathbf{t},\tilde{\mathbf{t}};m,L} \cdot A^T_{\mathbf{t},\tilde{\mathbf{t}};m,L},$$

either by the Cholesky method or by some other technique (see [18, Section 7]). One then obtains the frame coefficient matrix $Q_{\mathbf{t};m,L}$ associated to the refinement $\mathbf{t} \subset \tilde{\mathbf{t}}$ by setting

$$Q_{\mathbf{t};m,L} := E_{\tilde{\mathbf{t}};m,L} \cdot A_{\mathbf{t},\tilde{\mathbf{t}};m,L}.$$

With $\mathbf{t} := \mathbf{t}_j$ and $\tilde{\mathbf{t}} := \mathbf{t}_{j+1}$ one thus obtains the frame coefficient matrices

$$Q_j := Q_{\mathbf{t}_j;m,L} := E_{\mathbf{t}_{j+1};m,L} \cdot A_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$$

and the tight spline framelets of order m with L vanishing moments

$$\begin{aligned}\mathbf{\Psi}_j &= \mathbf{\Phi}_{\mathbf{t}_{j+1};m} \cdot Q_j = \mathbf{\Phi}_{\mathbf{t}_{j+1};m} \cdot E_{\mathbf{t}_{j+1};m,L} \cdot A_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} \\ &= \frac{d^L}{dx^L}\mathbf{\Phi}_{\mathbf{t}_{j+1};m+L} \cdot A_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} \qquad j \geq 0. \qquad \square\end{aligned}$$

For the L_2–normalization we accordingly obtain the following formulae. Note the differences in comparison to the previous description.

$$S^B_{\tilde{\mathbf{t}};m,L} - P^B_{\mathbf{t},\tilde{\mathbf{t}};m} \cdot S^B_{\mathbf{t};m,L} \cdot \left(P^B_{\mathbf{t},\tilde{\mathbf{t}};m}\right)^T = E^B_{\tilde{\mathbf{t}};m,L} \cdot Z^B_{\mathbf{t},\tilde{\mathbf{t}};m,L} \cdot \left(E^B_{\tilde{\mathbf{t}};m,L}\right)^T \tag{3.31}$$

with

$$\begin{aligned} Z^B_{\mathbf{t},\tilde{\mathbf{t}};m,L} &= \sum_{k=1}^{M} P^B_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot \left(P^B_{\mathbf{s_k},\mathbf{s_M};m+L}\right)^T & (3.32)\\ &= diag\left(d^{1/2}_{\tilde{\mathbf{t}};m,L}\right) \cdot Z_{\mathbf{t},\tilde{\mathbf{t}};m,L} \cdot diag\left(d^{1/2}_{\tilde{\mathbf{t}};m,L}\right) & (3.33)\\ &= diag\left(d_{\tilde{\mathbf{t}};m,L}\right) \cdot Z_{\mathbf{t},\tilde{\mathbf{t}};m,L} & (3.34)\end{aligned}$$

and

$$P^B_{\mathbf{s_k},\mathbf{s_M};m+L} = diag\left(d^{1/2}_{\mathbf{s_M};m,L}\right) \cdot P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot diag\left(d^{-1/2}_{\mathbf{s_k};m,L}\right). \tag{3.35}$$

Furthermore,

$$\begin{aligned} V^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} &= diag\left(d^{1/2}_{\mathbf{s_k};m,L}\right) \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot diag\left(d^{1/2}_{\mathbf{s_k};m,L}\right) & (3.36)\\ &= diag\left(d_{\mathbf{s_k};m,L}\right) \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} & (3.37)\\ &= diag\left(v^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}\right) & (3.38)\end{aligned}$$

with

$$\begin{aligned} v^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} &= v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot diag\left(d_{\mathbf{s_k};m,L}\right) & (3.39)\\ &= [v^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l}]_{l\in M_{\mathbf{s_k};m,L-1}}. & (3.40)\end{aligned}$$

For the knot $\{\tau\} := \mathbf{s_k} \setminus \mathbf{s_{k-1}}$ and the corresponding index ρ defined as $\tau \in \left[s^{(k-1)}_{\rho}, s^{(k-1)}_{\rho+1}\right)$ we obtain the following representation which differs from the one for $v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l}$.

$$\begin{aligned} v^B_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} &= \begin{cases} \frac{\left(s^{(k-1)}_{l+m+L-1}-\tau\right)\left(\tau-s^{(k-1)}_{l}\right)}{(m+L-1)(m+L)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} \quad \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \leq l \leq \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{\left(s^{(k)}_{l+m+L}-s^{(k)}_{\rho+1}\right)\left(s^{(k)}_{\rho+1}-s^{(k)}_{l}\right)}{(m+L-1)(m+L)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} \quad \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \leq l \leq \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{\left(s^{(k)}_{l+m+L}-s^{(k)}_{\rho+1}\right)\left(s^{(M)}_{\rho+1}-s^{(M)}_{l}\right)}{(m+L-1)(m+L)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} \quad \text{for} \\ \qquad \max(\rho+2-m-L, 1-m) \leq l \leq \min(\rho, N_{\mathbf{s_k}}-\rho+1), \\ 0 \quad \text{otherwise}. \end{cases}\end{aligned}$$

The symmetric factorization

$$Z^B_{\mathbf{t},\tilde{\mathbf{t}};m,L} = A^B_{\mathbf{t},\tilde{\mathbf{t}};m,L} \cdot \left(A^B_{\mathbf{t},\tilde{\mathbf{t}};m,L}\right)^T$$

leads to

$$Q^B_{\mathbf{t};m,L} := E^B_{\tilde{\mathbf{t}};m,L} \cdot A^B_{\mathbf{t},\tilde{\mathbf{t}};m,L}.$$

With $\mathbf{t} := \mathbf{t}_j$ and $\tilde{\mathbf{t}} := \mathbf{t}_{j+1}$ one thus obtains the frame coefficient matrices

$$Q^B_j := Q^B_{\mathbf{t}_j;m,L} := E^B_{\mathbf{t}_{j+1};m,L} \cdot A^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$$

and the L_2–normalized tight spline framelets of order m with L vanishing moments

$$\begin{aligned}\mathbf{\Psi}^B_j &= \mathbf{\Phi}^B_{\mathbf{t}_{j+1};m} \cdot Q^B_j = \mathbf{\Phi}^B_{\mathbf{t}_{j+1};m} \cdot E^B_{\mathbf{t}_{j+1};m,L} \cdot A^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} \\ &= \frac{d^L}{dx^L}\mathbf{\Phi}^B_{\mathbf{t}_{j+1};m+L} \cdot A^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} \qquad\qquad j \geq 0.\end{aligned}$$

Chapter 4

The General Construction Principle for Non–stationary Sibling Frames

We are interested in locally supported frames of $L_2[a,b]$ and corresponding duals which are defined from a non–stationary MRA, in general, and especially from the non–stationary B–spline MRA of a given order $m \geq 2$ on a bounded interval.

This chapter presents the general construction principle for non–stationary sibling frames and thus it supplements the general theory of non–stationary tight frames of Chui, He and Stöckler from [18]. This principle will be applied in Chapter 6 in order to give construction schemes for non–stationary sibling spline frames, as well as some concrete examples.

After defining the notion of sibling frames and some appropriate tools for their study, we give some general characterization for the duality relation between two Bessel families and also sufficient conditions in some special cases.

4.1 Bilinear forms and kernels

In this section, for a finite function family and a real matrix, we introduce and study two entities: a bilinear form and a kernel. They will be our tools in characterizing sibling frames. In [18] the authors used for the characterization of tight frames the same kernel as we do and the quadratic form associated to the bilinear form from below (see Definition 3.5).

Let $I = [a,b]$ be a compact interval on the real axis. For a finite family $\mathbf{\Phi}_j = [\phi_{j,k}]_{k \in \mathbb{M}_j}$ from $L_2(I)$ with cardinality M_j ($\mathbb{M}_j := \{1, \ldots, M_j\}$) and a real matrix $S_j = [s_{k,l}^{(j)}]_{k,l \in \mathbb{M}_j}$ we consider:

a) the associated bilinear form

$$\begin{aligned} T_{S_j}(f,g) &:= [\langle f, \phi_{j,k} \rangle]_{k \in \mathbb{M}_j} \cdot S_j \cdot [\langle g, \phi_{j,k} \rangle]_{k \in \mathbb{M}_j}^T \\ &=: \langle f, \mathbf{\Phi}_j \rangle \cdot S_j \cdot \langle g, \mathbf{\Phi}_j \rangle^T, \quad f, g \in L_2(I), \end{aligned} \tag{4.1}$$

b) the associated kernel

$$\begin{aligned} K_{S_j}(x,y) &:= \mathbf{\Phi}_j(x) \cdot S_j \cdot \mathbf{\Phi}_j^T(y) \\ &= \sum_{k,l \in \mathbb{M}_j} \phi_{j,k}(x) \cdot s_{k,l}^{(j)} \cdot \phi_{j,l}(y), \quad x, y \in I. \end{aligned} \tag{4.2}$$

$T_{S_j}(f,g)$ and $K_{S_j}(x,y)$ inherit the symmetry and definiteness properties of the matrix S_j. Furthermore, they are related by

$$T_{S_j}(f,g) = \int_I f(x) \int_I g(y) \cdot K_{S_j}(x,y)\, dy dx, \quad f, g \in L_2(I).$$

Our next result describes inheritance properties of T_{S_j} and K_{S_j}.

Theorem 4.1 *(Inheritance properties of T_{S_j} and K_{S_j}, see [3])*
Let $\mathbf{\Phi}_j$, $j \geq 0$, be finite families from $L_2(I)$, $\mathbf{\Phi}_j$ with cardinality M_j, such that a refinement relation of the form $\mathbf{\Phi}_j = \mathbf{\Phi}_{j+1} \cdot P_j$ exists for all $j \geq 0$, where the P_j's are real matrices of dimensions $M_{j+1} \times M_j$.
Furthermore, let the families $\mathbf{\Psi} = \{\mathbf{\Psi}_j\}_{j\geq 0}$ and $\tilde{\mathbf{\Psi}} = \{\tilde{\mathbf{\Psi}}_j\}_{j\geq 0}$ have the structure

$$\begin{aligned} \mathbf{\Psi}_j &:= \mathbf{\Phi}_{j+1} \cdot Q_j =: [\psi_{j,k}]_{k\in I\!N_j}, \quad j \geq 0, \\ \tilde{\mathbf{\Psi}}_j &:= \mathbf{\Phi}_{j+1} \cdot \tilde{Q}_j =: [\tilde{\psi}_{j,k}]_{k\in I\!N_j}, \quad j \geq 0, \end{aligned}$$

where Q_j and $\tilde{Q}_j$ are real matrices of dimensions $M_{j+1} \times N_j$ ($I\!N_j := \{1, \ldots, N_j\}$).
If there exists a sequence of real matrices $(S_j)_{j\geq 0}$ which are related by the recurrence

$$S_{j+1} = P_j S_j P_j^T + Q_j \tilde{Q}_j^T, \quad j \geq 0, \tag{4.3}$$

then the following statements hold.

a) The associated bilinear forms T_{S_J} on $L_2(I)^2$ w.r.t. $\{\mathbf{\Phi}_J\}_{J\geq 0}$ inherit this structure. They satisfy the recurrence relation

$$T_{S_{J+1}}(f, g) = T_{S_J}(f, g) + \sum_{l\in I\!N_J} \langle f, \psi_{J,l}\rangle \langle g, \tilde{\psi}_{J,l}\rangle \tag{4.4}$$

and the representation formula

$$T_{S_{J+1}}(f, g) = T_{S_0}(f, g) + \sum_{j=0}^{J} \sum_{l\in I\!N_j} \langle f, \psi_{j,l}\rangle \langle g, \tilde{\psi}_{j,l}\rangle \tag{4.5}$$

for all $f, g \in L_2(I)$ and all $J \geq 0$.

b) The associated kernels K_{S_j} w.r.t. $\{\mathbf{\Phi}_j\}_{j\geq 0}$ inherit this structure, i.e., there holds the recurrence

$$K_{S_{J+1}}(x, y) = K_{S_J}(x, y) + \sum_{k\in I\!N_J} \psi_{J,k}(x) \cdot \tilde{\psi}_{J,k}(y),$$

and further we get the representation formula

$$K_{S_{J+1}}(x, y) = K_{S_0}(x, y) + \sum_{j=0}^{J} \sum_{k\in I\!N_j} \psi_{j,k}(x) \cdot \tilde{\psi}_{j,k}(y),$$

both being valid for all $x, y \in I$ and any $J \geq 0$.

Proof. Direct computations give

$$\begin{aligned} T_{S_{J+1}}(f, g) &= \langle f, \mathbf{\Phi}_{J+1}\rangle \cdot S_{J+1} \cdot \langle g, \mathbf{\Phi}_{J+1}\rangle^T \\ &= (\langle f, \mathbf{\Phi}_{J+1}\rangle \cdot P_J) \cdot S_J \cdot (\langle g, \mathbf{\Phi}_{J+1}\rangle \cdot P_J)^T \\ &\quad + (\langle f, \mathbf{\Phi}_{J+1}\rangle \cdot Q_J) \cdot \left(\langle g, \mathbf{\Phi}_{J+1}\rangle \cdot \tilde{Q}_J\right)^T \\ &= \langle f, \mathbf{\Phi}_{J+1} \cdot P_J\rangle \cdot S_J \cdot \langle g, \mathbf{\Phi}_{J+1} \cdot P_J\rangle^T \\ &\quad + \langle f, \mathbf{\Phi}_{J+1} \cdot Q_J\rangle \cdot \langle g, \mathbf{\Phi}_{J+1} \cdot \tilde{Q}_J\rangle^T \\ &= \langle f, \mathbf{\Phi}_J\rangle \cdot S_J \cdot \langle g, \mathbf{\Phi}_J\rangle^T + \langle f, \mathbf{\Psi}_J\rangle \cdot \langle g, \tilde{\mathbf{\Psi}}_J\rangle^T \\ &= T_{S_J}(f, g) + \sum_{l\in I\!N_J} \langle f, \psi_{J,l}\rangle \langle g, \tilde{\psi}_{J,l}\rangle, \quad \forall J \geq 0, \\ &= T_{S_0}(f, g) + \sum_{j=0}^{J} \sum_{l\in I\!N_j} \langle f, \psi_{j,l}\rangle \langle g, \tilde{\psi}_{j,l}\rangle, \quad \forall J \geq 0, \end{aligned}$$

and

$$\begin{aligned}
K_{S_{J+1}}(x,y) &= \mathbf{\Phi}_{J+1}(x)\cdot S_{J+1}\cdot \mathbf{\Phi}_{J+1}^T(y)\\
&= (\mathbf{\Phi}_{J+1}(x)\cdot P_J)\cdot S_J\cdot (\mathbf{\Phi}_{J+1}(y)\cdot P_J)^T\\
&\quad + (\mathbf{\Phi}_{J+1}(x)\cdot Q_J)\left(\mathbf{\Phi}_{J+1}(y)\cdot \tilde{Q}_J\right)^T\\
&= \mathbf{\Phi}_J(x)\cdot S_J\cdot \mathbf{\Phi}_J^T(y) + \mathbf{\Psi}_J(x)\cdot \tilde{\mathbf{\Psi}}_J^T(y)\\
&= K_{S_J}(x,y) + \sum_{k\in I\!N_J} \psi_{J,k}(x)\cdot \tilde{\psi}_{J,k}(y), \quad \forall J\geq 0,\\
&= K_{S_0}(x,y) + \sum_{j=0}^{J}\sum_{k\in I\!N_j} \psi_{j,k}(x)\cdot \tilde{\psi}_{j,k}(y), \quad \forall J\geq 0. \qquad \square
\end{aligned}$$

We emphasize here, that – in order to get the above recurrence relations for T_{S_j} and K_{S_j} – we didn't have to assume any special properties of the matrices in use (such as spsd); relation (4.3) between the matrices was the crucial point.

4.2 Non–stationary MRA sibling frames on bounded intervals

The notion of affine sibling frames of $L_2(I\!R)$ was introduced for the first time in [17, Definition 1] in order to achieve more flexibility and thus additional properties for the frame elements such as symmetry (or anti–symmetry), small support, high order of vanishing moments, approximate shift–invariance and inter–orthogonality. A parallel and independent development of some similar and overlapping results is presented in [25].

[18] presents a general construction scheme as well as practical procedures for (non–affine, non–stationary) tight wavelet frames with maximal number of vanishing moments and minimal support on a compact interval of the real line.

We will present here a more general (i.e., non–affine, non–stationary, non–tight) approach for sibling frames of $L_2(I)$, where I is a compact interval of $I\!R$. To our knowledge this approach has not been studied so far.

In analogy to Definition 3.6 we present next the notion of an MRA frame, in the form we use it in the remainder of this thesis.

Definition 4.2 *(Non–stationary MRA frame on a compact interval)*
With $I := [a, b]$ let

$$\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$$

be a non–stationary MRA of $L_2(I)$ with locally supported family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} := \{[\phi_{j,k};\ 1\leq k\leq M_j]\}_{j\geq 0}.$$

Further let S_0 be a spsd matrix with associated bilinear form T_{S_0}.

Then the family $\mathbf{\Psi}$ determined by the sequence of coefficient matrices $\{Q_j\}_{j\geq 0}$ ($\dim Q_j = M_{j+1}\times N_j$) through the relations

$$\mathbf{\Psi} := \{\mathbf{\Psi}_j\}_{j\geq 0} := \{\mathbf{\Phi}_{j+1}\cdot Q_j\}_{j\geq 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\geq 0} \tag{4.6}$$

is called a non–stationary MRA frame of $L_2(I)$ w.r.t. T_{S_0}, if there exist constants A and B ($0 < A \leq B < \infty$) such that

$$A\cdot \|f\|_{L_2}^2 \leq T_{S_0}(f,f) + \sum_{j\geq 0}\sum_{k\in I\!N_j} |\langle f,\psi_{j,k}\rangle|^2 \leq B\cdot \|f\|_{L_2}^2, \qquad f\in L_2(I). \tag{4.7}$$

$I\!N_j$ denotes the set $\{1,\ldots,N_j\}$.

We are now in the position to define sibling frames.

Definition 4.3 *(Non–stationary MRA sibling frames on a compact interval, see [3])* *With $I := [a, b]$ let*

$$\{(V_j)_{j\geq 0}, (\mathbf{\Phi}_j)_{j\geq 0}, (P_j)_{j\geq 0}\}$$

be a non–stationary MRA of $L_2(I)$ with locally supported family

$$\mathbf{\Phi} := \{\mathbf{\Phi}_j\}_{j\geq 0} := \{[\phi_{j,k};\ 1 \leq k \leq M_j]\}_{j\geq 0}.$$

Further let T_{S_0} be the bilinear form associated to a spsd matrix S_0.
Then the families $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ determined by the sequences of coefficient matrices $\{Q_j\}_{j\geq 0}$ and $\{\tilde{Q}_j\}_{j\geq 0}$ $(\dim Q_j = \dim \tilde{Q}_j = M_{j+1} \times N_j)$ through the relations

$$\mathbf{\Psi} = \{\mathbf{\Psi}_j\}_{j\geq 0} = \{\mathbf{\Phi}_{j+1} \cdot Q_j\}_{j\geq 0} = \{[\psi_{j,k}]_{k\in I\!N_j}\}_{j\geq 0}, \tag{4.8}$$

$$\tilde{\mathbf{\Psi}} = \{\tilde{\mathbf{\Psi}}_j\}_{j\geq 0} = \{\mathbf{\Phi}_{j+1} \cdot \tilde{Q}_j\}_{j\geq 0} = \{[\tilde{\psi}_{j,k}]_{k\in I\!N_j}\}_{j\geq 0}, \tag{4.9}$$

constitute non–stationary sibling frames of $L_2(I)$ w.r.t. T_{S_0}, if the following conditions are satisfied.

a) They are Bessel families, i.e., there exist constants B and $\tilde{B}$ with $0 < B, \tilde{B} < \infty$ such that for all $f \in L_2(I)$ hold

$$T_{S_0}(f, f) + \sum_{j\geq 0}\sum_{k\in I\!N_j} |\langle f, \psi_{j,k}\rangle|^2 \leq B \cdot \|f\|^2_{L_2}, \tag{4.10}$$

$$T_{S_0}(f, f) + \sum_{j\geq 0}\sum_{k\in I\!N_j} |\langle f, \tilde{\psi}_{j,k}\rangle|^2 \leq \tilde{B} \cdot \|f\|^2_{L_2}. \tag{4.11}$$

b) They are dual, i.e., for all $f, g \in L_2(I)$ we have the identity

$$T_{S_0}(f, g) + \sum_{j\geq 0}\sum_{k\in I\!N_j} \langle f, \psi_{j,k}\rangle\langle\tilde{\psi}_{j,k}, g\rangle = \langle f, g\rangle. \tag{4.12}$$

$I\!N_j$ denotes the set $\{1, \ldots, N_j\}$. The parameter N_j describes the number of frame (and dual frame) elements on the level j. The matrices Q_j (and $\tilde{Q}_j$) are called frame coefficient matrices (and dual frame coefficient matrices, respectively).

Note that in this case both families $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ are indeed non–stationary MRA frames of $L_2(I)$ in the sense of Definition 4.2. Using the duality relation (4.12) one can prove that the lower frame bound of $\mathbf{\Psi}$ is $\tilde{B}^{-1}$ and that of $\tilde{\mathbf{\Psi}}$ is B^{-1}. The finite numbers N_j of frame (and dual frame) elements on the corresponding levels j ($j \geq 0$) are parameters which have to be concretized in the construction of sibling frames (see our examples in Chapter 6, for example). They possess some degree of freedom (for details see the remarks after Definition 2.3 in [18]) and they govern the redundancy degree of the frame system.

The assumption on $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ to be Bessel families is not needed for tight frames. For this special case the boundedness is contained in the duality relation (i.e., all three conditions in the above definition collapse to one identity). Unlike this, in the (non–tight) sibling frame case one has to find suitable (necessary and) sufficient conditions for the boundedness. This will be done in Chapter 5.

For the remainder of this section let $\mathbf{\Phi} = \{\mathbf{\Phi}_j\}_{j\geq 0}$ be a locally supported family with $\#\mathbf{\Phi}_j = M_j$ and $\mathbf{\Phi}_j = \mathbf{\Phi}_{j+1} \cdot P_j$. The matrix S_0 defines the bilinear form T_{S_0}. Further let $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ be function families as defined by (4.8) and (4.9) and satisfying the Bessel conditions (4.10) and (4.11).

Using identity (4.5) and applying the Cauchy–Schwarz inequality we get the result below.

Proposition 4.4 *(Boundedness of the bilinear forms, see [3])*
If the matrix S_0 is symmetric positive definite, then the bilinear forms T_{S_J} $(J \geq 1)$ are bounded from above as follows:

$$|T_{S_J}(f,g)| \leq |T_{S_0}(f,g)| + \sqrt{B\tilde{B}} \cdot \|f\|_{L_2} \cdot \|g\|_{L_2}, \qquad f,g \in L_2(I). \tag{4.13}$$

If, in addition, $\mathbf{\Phi}_0$ is a Bessel family with Bessel bound B_0, then the T_{S_J}'s are uniformly bounded, i.e, there holds

$$|T_{S_J}(f,g)| \leq \left(B_0 \cdot \|S_0\|_2 + \sqrt{B\tilde{B}}\right) \cdot \|f\|_{L_2} \cdot \|g\|_{L_2}, \qquad f,g \in L_2(I), \tag{4.14}$$

where by $\|S_0\|_2$ we denote the spectral norm of S_0.

Proof. The following computations yield the desired estimates.

$$\begin{aligned}
\left|T_{S_{J+1}}(f,g)\right| &\overset{(4.5)}{\leq} |T_{S_0}(f,g)| + \sum_{j=0}^{J}\sum_{l\in\mathbb{N}_j} \left|\langle f,\psi_{j,l}\rangle\langle g,\tilde{\psi}_{j,l}\rangle\right| \\
&\overset{CS}{\leq} |T_{S_0}(f,g)| + \left(\sum_{j=0}^{J}\sum_{l\in\mathbb{N}_j} |\langle f,\psi_{j,l}\rangle|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=0}^{J}\sum_{l\in\mathbb{N}_j} \left|\langle g,\tilde{\psi}_{j,l}\rangle\right|^2\right)^{\frac{1}{2}} \\
&\leq |T_{S_0}(f,g)| + \left(T_{S_0}(f,f) + \sum_{j=0}^{J}\sum_{l\in\mathbb{N}_j} |\langle f,\psi_{j,l}\rangle|^2\right)^{\frac{1}{2}} \cdot \\
&\qquad \cdot \left(T_{S_0}(g,g) + \sum_{j=0}^{J}\sum_{l\in\mathbb{N}_j} \left|\langle g,\tilde{\psi}_{j,l}\rangle\right|^2\right)^{\frac{1}{2}} \\
&\overset{Bessel}{\leq} |T_{S_0}(f,g)| + \sqrt{B} \cdot \|f\|_{L_2} \cdot \sqrt{\tilde{B}} \cdot \|g\|_{L_2} \\
&= |T_{S_0}(f,g)| + \sqrt{B\tilde{B}} \cdot \|f\|_{L_2} \cdot \|g\|_{L_2} \\
&\overset{(4.1)}{=} \left|\langle f,\mathbf{\Phi}_0\rangle \cdot S_0 \cdot \langle g,\mathbf{\Phi}_0\rangle^T\right| + \sqrt{B\tilde{B}} \cdot \|f\|_{L_2} \cdot \|g\|_{L_2} \\
&\leq \|\langle f,\mathbf{\Phi}_0\rangle\|_{l_2} \cdot \|S_0\|_2 \cdot \|\langle g,\mathbf{\Phi}_0\rangle\|_{l_2} + \sqrt{B\tilde{B}} \cdot \|f\|_{L_2} \cdot \|g\|_{L_2} \\
&\overset{Bessel}{\leq} \|S_0\|_2 \cdot \sqrt{B_0} \cdot \|f\|_{L_2} \cdot \sqrt{B_0} \cdot \|g\|_{L_2} + \sqrt{B\tilde{B}} \cdot \|f\|_{L_2} \cdot \|g\|_{L_2} \\
&= \left(B_0 \cdot \|S_0\|_2 + \sqrt{B\tilde{B}}\right) \cdot \|f\|_{L_2} \cdot \|g\|_{L_2}, \qquad f,g \in L_2(I). \qquad \square
\end{aligned}$$

The monotonicity of the sequence of bilinear forms $(T_{S_J})_J$ is not ensured in the general case, but we can state the following.

Proposition 4.5 *(Monotonicity of the quadratic forms, see [3])*
If all the matrices $Q_J\tilde{Q}_J^T$ are symmetric positive (negative) semi–definite, then the sequence of quadratic forms $(T_{S_J})_J$ is monotonically increasing (decreasing, respectively).
Obviously, if the matrices are definite then we get strict monotonicity.

Proof. For all $f \in L_2(I)$ we obtain for the spsd matrix $Q_J\tilde{Q}_J^T$ the following:

$$\begin{aligned}
& \langle f,\mathbf{\Phi}_{J+1}\rangle \cdot Q_J\tilde{Q}_J^T \cdot \langle f,\mathbf{\Phi}_{J+1}\rangle^T \geq 0 \\
\Leftrightarrow\ & \langle f,\mathbf{\Phi}_{J+1} \cdot Q_J\rangle \cdot \langle f,\mathbf{\Phi}_{J+1}\tilde{Q}_J\rangle^T \geq 0 \\
\Leftrightarrow\ & \langle f,\mathbf{\Psi}_J\rangle \cdot \langle f,\tilde{\mathbf{\Psi}}_J\rangle^T \geq 0 \\
\Leftrightarrow\ & \sum_{l\in\mathbb{N}_J} \langle f,\psi_{J,l}\rangle \cdot \langle f,\tilde{\psi}_{J,l}\rangle \geq 0.
\end{aligned}$$

Combining this with (4.4) for $g = f$ we obtain the desired result for the sequence of quadratic forms $(T_{S_J})_J$. The other cases are analogous. □

4.3 Characterization of the duality relation

4.3.1 The general case

In this section we present and discuss necessary and sufficient conditions for the existence of the duality relation (4.12). Our next result generalizes Theorem 1 in [18] and represents one of the main building blocks of the subsequent construction principle of sibling frames.

Theorem 4.6 *(Characterization of the duality of two Bessel families, see [3])*
Let $\mathbf{\Phi} = \{\mathbf{\Phi}_j\}_{j\geq 0}$ be a locally supported family with $\#\mathbf{\Phi}_j = M_j$ and $\mathbf{\Phi}_j = \mathbf{\Phi}_{j+1} \cdot P_j$. The matrix S_0 defines the ground level component T_{S_0}. Furthermore, let $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ be function families as defined by (4.8) and (4.9) and satisfying the Bessel conditions (4.10) and (4.11).
Then $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ are dual (and thus sibling frames w.r.t. T_{S_0}), if and only if there exists a sequence of matrices $(S_j)_{j\geq 1}$, $\dim S_j = M_j$, such that

a) the bilinear forms T_{S_j} satisfy

$$\lim_{j\to\infty} T_{S_j}(f,g) = \langle f,g\rangle, \quad f,g \in L_2(I); \tag{4.15}$$

b) for every $j \geq 0$ we have

$$S_{j+1} - P_j S_j P_j^T = Q_j \tilde{Q}_j^T. \tag{4.16}$$

Proof. Let f and g be two arbitrarily fixed functions from $L_2(I)$.

Sufficiency. According to Theorem 4.1, property (4.16) implies identity (4.5) which combined with (4.15) gives the desired duality relation.

Necessity. S_0 is given and for $j \geq 1$ we define the matrices recursively by

$$S_{j+1} := P_j S_j P_j^T + Q_j \tilde{Q}_j^T.$$

Thus condition (4.16) is satisfied. Equation (4.5) follows by an application of Theorem 4.1. Thus the duality relation implies the convergence of the sequence $(T_{S_J}(f,g))_J$ and, therefore, the desired limit (4.15). □

Note that identity (4.16) describes the relation of all the matrices involved in the definition of sibling frames and thus it points out their interplay in the construction process.

4.3.2 The approximate identity case

Next we want to find some special cases where the limit (4.15) exists. For this purpose we consider the kernels introduced in (4.2) and we follow a classical and well studied approach from Approximation Theory which is rather useful in our Hilbert frame setting. We first recall the definition of an approximate identity.

Definition 4.7 *(Approximate identity, see [12, 28])*
Let $I = [a,b]$ be a compact interval of the real line and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of kernels $K_n : I^2 \to \mathbb{R}$.

i) $(K_n)_{n\in I\!N}$ *is called approximate identity if the functions* K_n *are continuous and satisfy the following properties.*

a) Normalization:

$$\int_I K_n(x,t)\,dt \to 1$$

uniformly in $x \in I$ *when* $n \to \infty$*;*

b) Uniform boundedness w.r.t. n*: for every* $x \in I$ *there exists* $M(x) > 0$ *such that for all* $n \in I\!N$

$$\int_I |K_n(x,t)|\,dt \le M(x);$$

c) Localization: for every $\delta \in (0, |I|]$ *we have*

$$\int_{|x-t|\ge\delta} |K_n(x,t)|\,dt \to 0$$

uniformly in x *for* $n \to \infty$*.*

ii) If the boundedness constant $M(x)$ *does not depend on the variable* x*, then we call the approximate identity* $(K_n)_{n\in I\!N}$ *uniformly bounded.*

A fundamental result from Approximation Theory (see, e.g., [12, Theorem 2.1 on p. 5]) states that for an approximate identity $(K_n)_{n\in I\!N}$, for every continuous function $f : I \to I\!R$ and every $x \in I$ we have the following convergence:

$$\int_I K_n(x,t)\cdot f(t)\,dt \to f(x) \quad \text{for } n \to \infty. \tag{4.17}$$

Furthermore, this convergence is uniform in x if the approximate identity is uniformly bounded.

This powerful result will be the main ingredient in the proof of Theorem 4.8 below. It presents some sufficient conditions on the kernels K_{S_j} introduced in (4.2) in order to obtain (4.15).

Theorem 4.8 *(Sufficient conditions on the kernels* K_{S_j}*, see [3])*
If the kernels $(K_{S_j})_{j\ge 0}$ *form a uniformly bounded approximate identity, then the bilinear forms* $(T_{S_j})_{j\ge 0}$ *form a bilinear approximation method of the scalar product operator on* $L_2(I^2)$*, i.e., identity (4.15) holds.*

Proof. Let f and g be real continuous functions on I. Without loss of generality we assume that $f \not\equiv 0$. By using the aforementioned result for uniformly bounded approximate identities, for each $\varepsilon > 0$, there exists $N_\varepsilon \in I\!N$ such that for all $j \ge N_\varepsilon$ there holds

$$\begin{aligned} |T_{S_j}(f,g) - <f,g>| &= \left|\int_I f(x)\left[\int_I g(y)\cdot K_{S_j}(x,y)dy - g(x)\right]dx\right| \\ &\le \int_I |f(x)|\cdot\left|\int_I g(y)\cdot K_{S_j}(x,y)dy - g(x)\right|dx \\ &< \int_I |f(x)|\cdot\frac{\varepsilon}{\|f\|_1}\,dx = \varepsilon. \end{aligned}$$

Applying further a density argument we get the desired limit for all functions in the space $L_2(I)$. □

Proposition 4.4 and Theorem 4.8 generalize Theorem 9 in [18]. Theorem 4.8 is of importance for our subsequent construction of sibling frames because its requirements can be easily verified in practical situations; normally a direct verification of (4.15) is much more difficult.

Remark 4.9 *Note that the kernels K_{S_j} from (4.2) are continuous if we choose families $\mathbf{\Phi}_j$ of continuous functions. Condition c) in Definition 4.7 is satisfied if all the matrices S_j have a fixed maximal bandwidth and the function family $\mathbf{\Phi} = \{\mathbf{\Phi}_j\}_{j\geq 0}$ is locally supported in the sense of Definition 3.3. Namely, in this case the integral appearing in condition c) of Definition 4.7 is equal to zero for indices j large enough.*

4.3.3 The spline case

In the spline setting described in Chapters 2&3 one usually chooses $\mathbf{\Phi}_j$ to be the (suitably normalized) B–spline basis of $V_j = \mathcal{S}_m(\mathbf{t}_j, [a,b])$, namely the Riesz basis $\mathbf{\Phi}^B_{\mathbf{t}_j;m} = [N^B_{\mathbf{t}_j;m,k}]_{k\in I\!M_{\mathbf{t}_j;m,0}}$. Furthermore, if the matrices S_j are chosen to be the approximate dual matrices of Chui, He and Stöckler $S^B_{\mathbf{t}_j;m,L}$ (constructed directly and only from the knot sequences $\mathbf{t}_j \subset [a,b]$, as presented in Section 3.3), then it follows immediately that $\left(K^B_{S^B_{\mathbf{t}_j;m,L}}\right)_{j\geq 0}$ defined in (3.23) are continuous and local (confer Remark 4.9).

The uniform boundedness of the kernels $K^B_{S^B_{\mathbf{t}_j;m,L}}(x,y)$ w.r.t. both j and x, as well as their normalization, is given by Theorem 3.12 in combination with (3.23).

Therefore, in the non–stationary spline case discussed in Chapters 2 and 3 the kernels defined by the approximate dual matrices of Chui, He and Stöckler are suitable because they fulfill all the assumptions of Theorem 4.8 and thus property (4.15) holds for the bilinear forms

$$T^B_{S^B_{\mathbf{t}_j;m,L}}(f,g) \;:=\; \langle f, \mathbf{\Phi}^B_{\mathbf{t}_j;m}\rangle \cdot S^B_{\mathbf{t}_j;m,L} \cdot \langle g, \mathbf{\Phi}^B_{\mathbf{t}_j;m}\rangle^T, \quad f,g \in L_2[a,b].$$

Considering the matrix sequence $\{S_j\}_{j\geq 0} = \left\{S^B_{\mathbf{t}_j;m,L}\right\}_{j\geq 0}$ we can formulate on the basis of Theorem 4.6 the following **general construction principle for sibling spline frames**.

In order to obtain sibling spline frames of $L_2[a,b]$ we have to factorize the matrices

$$S^B_{\mathbf{t}_{j+1};m,L} - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot S^B_{\mathbf{t}_j;m,L} \cdot \left(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\right)^T$$

appropriately into $Q_j \cdot \tilde{Q}_j^T$, i.e., we have to determine coefficient matrices Q_j and $\tilde{Q}_j$ such that the Bessel conditions (4.10) and (4.11) are satisfied.

Because the Bessel conditions (4.10) and (4.11) cannot in general be verified directly for some structured function families $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ as those defined in (4.8) and (4.9), we need special techniques for doing this check.

In Chapter 5 we develop general strategies for proving the boundedness of certain linear operators. This will permit in Chapter 6 to check the Bessel condition for concrete spline systems which are our candidates for sibling spline frames.

Chapter 5

Vaguelettes Systems and Localization Theory

We are interested in necessary and sufficient conditions for a function family to possess a so-called Bessel bound (see Definition 1.1). Such criteria are interesting among others in the case of sibling frames where such Bessel bounds have to exist both for the frame and for its dual (see [17, 3] and Chapter 4&6).

In his monograph [48], published in 1990, Meyer introduced some general stationary systems of functions with 'minimal' regularity and proved that these are Bessel families in $L_2(I\!R)$.

At about the same time, the fundamental paper by Frazier and Jawerth [32] appeared. In a very general stationary setting the authors investigate several types of countable families of functions with convenient properties and also prove their boundedness.

These approaches provide the stationary localization theory for the wavelet tiling of the time–frequency plane and complement the localization theory of Gröchenig for the Gabor setting. Gröchenig presents results also for the non–stationary case, by considering families of functions with localization points[1] which constitute a relatively separated set w.r.t. the Gabor setting. In the sequel we introduce a new separation concept for the wavelet situation (see Section 5.6).

Mixing, adapting and extending the ideas of Meyer and Frazier&Jawerth to the compactly supported non–stationary case we prove a general boundedness result (see Theorem 5.35) followed by other results on the existence of Bessel bounds for some function systems. This is the relevant setting for our subsequent concrete constructions of sibling frames on a bounded interval $[a, b]$ in the spline multiresolution analysis of $L_2[a, b]$ defined by nested knot sequences where maximal knot distance converges to zero (see Chapter 6). Such constructions are done in the present thesis for the first time.

5.1 The Bessel property

Bessel sequences were introduced and studied extensively by Bari in her 1951 paper [1]. Some initial considerations on this topic can be found in Boas [11]. A detailed review of this topic is presented in Young's book [65, Ch. 4].

Let Λ be a countable index set and $\mathcal{G} = \{g_\lambda\}_{\lambda\in\Lambda}$ be a family of functions in a separable Hilbert space $\mathcal{H}$ endowed with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{\mathcal{H}}$. For a function $g \in \mathcal{H}$

[1] Intuitively, one can visualize the localization points as those points close to which the function exhibits its essential features (such as absolute mass and vanishing moment). This is an informal and intuitive notion, though.

we call $(\langle g, g_\lambda \rangle)_{\lambda\in\Lambda}$ the moment sequence of g w.r.t. $\mathcal{G}$. The Gram matrix of $\mathcal{G}$ is denoted by $Gram(\mathcal{G})$.

Since we intend this chapter to be self–contained, we recall the definitions of a Bessel family and a Bessel bound in the form used from now on in this thesis.

Definition 5.1 *(Bessel family, Bessel bound, see [65, Ch. 4, Sect. 2])*
Let Λ, $\mathcal{G}$ and $\mathcal{H}$ be given as above. $\mathcal{G}$ is called a Bessel family if there exists a constant B $(0 < B < \infty)$ such that one of the following equivalent conditions is satisfied.

$$a)\quad \left\| (\langle g, g_\lambda \rangle)_{\lambda\in\Lambda} \right\|_{l_2} \leq B \cdot \|g\|_{\mathcal{H}} \qquad \text{for all } g \in \mathcal{H};$$

$$b)\quad \left\| \sum_{\lambda\in\Lambda} c_\lambda g_\lambda \right\|_{\mathcal{H}} \leq B \cdot \|\mathbf{c}\|_{l_2} \qquad \text{for all } \mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in l_2(\Lambda).$$

Such constant B is called Bessel bound[2] *of $\mathcal{G}$ and the sharpest possible constant B is called optimal Bessel bound of $\mathcal{G}$ (denoted by B_{opt}). For $\Lambda = \mathbb{N}$ we get a Bessel sequence.*

The equivalence between part a) and b) in Definition 5.1 can be verified as follows.

If the 'analysis operator' (or 'decomposition operator') of the family $\mathcal{G}$ is defined formally as the linear map

$$T_{\mathcal{G}} = T : \mathcal{H} \to l_2(\Lambda), \qquad Tg := (\langle g, g_\lambda \rangle)_{\lambda\in\Lambda}\,,$$

then the 'synthesis operator' (or 'reconstruction operator') of the family $\mathcal{G}$

$$T^*_{\mathcal{G}} = T^* : l_2(\Lambda) \to \mathcal{H}, \qquad T^*\mathbf{c} := \sum_{\lambda\in\Lambda} c_\lambda g_\lambda \quad \text{where } \mathbf{c} = (c_\lambda)_{\lambda\in\Lambda},$$

is the formal adjoint of $T_{\mathcal{G}}$. Condition (a) is equivalent to

$$T_{\mathcal{G}} \text{ is well–defined and bounded from } \mathcal{H} \text{ to } l_2(\Lambda) \text{ with } \|T_{\mathcal{G}}\| \leq B$$

and condition (b) to

$$T^*_{\mathcal{G}} \text{ is well–defined and bounded from } l_2(\Lambda) \text{ to } \mathcal{H} \text{ with } \|T^*_{\mathcal{G}}\| \leq B.$$

Note that the optimal Bessel bound of $\mathcal{G}$ from Definition 5.1 is equal to $\|T_{\mathcal{G}}\| = \|T^*_{\mathcal{G}}\|$.

Definition 5.2 *We will call*

$$S_{\mathcal{G}} := S := T^*T : \mathcal{H} \to \mathcal{H}, \qquad Sg := \sum_{\lambda\in\Lambda} \langle g, g_\lambda \rangle g_\lambda,$$

the operator associated to the family $\mathcal{G} = \{g_\lambda\}_{\lambda\in\Lambda}$.

Definition 5.3 *Let Λ be a countable index set, $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ and $M = (m_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ be an arbitrary $\mathbb{K}$–matrix. The matrix operator defined by M is*

$$\mathbb{K}^\Lambda \ni \mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \mapsto M\mathbf{c} = \left(\sum_{\lambda'\in\Lambda} m_{\lambda,\lambda'} \cdot c_{\lambda'} \right)_{\lambda\in\Lambda} \in \mathbb{K}^\Lambda.$$

Note that $TT^* : l_2(\Lambda) \to l_2(\Lambda)$ is the matrix operator defined by $Gram(\mathcal{G})$.

The Bessel property is related to the Gram matrix in the following way.

[2] Note that the present definition of a Bessel bound is compatible with the one in Definition 1.1.

Theorem 5.4 *(Characterization of the Bessel property in terms of the Gramian, see [65, Ch. 4, Sect. 2])*
Let Λ be a countable index set and $\mathcal{G} = \{g_\lambda\}_{\lambda\in\Lambda}$ be a family of functions in a separable Hilbert space $\mathcal{H}$.
$\mathcal{G}$ is a Bessel family with bound B if and only if $Gram(\mathcal{G})$ defines a bounded matrix operator M on $l_2(\Lambda)$ with

$$\|M\|_{l_2(\Lambda)\to l_2(\Lambda)} \leq B^2.$$

Proof. The necessity is obvious. The sufficiency follows directly from an application of the Cauchy–Schwarz inequality to the left–hand term in Definition 5.1.b). □

Example 5.5 *(Bessel sequence, see [65])*
The sequence $\{1, t, t^2, \cdots, t^n, t^{n+1}, \cdots\}$ forms a Bessel sequence in $L_2[0,1]$. The corresponding Gram matrix is the Hilbert matrix with entries

$$h_{i,j} = \int_0^1 t^{i+j}\,dt = \frac{1}{i+j+1}.$$

Hilbert's inequality (see, e.g., [65, p. 134])

$$\sum_{m,n=0}^{\infty} \frac{|c_m c_n|}{m+n+1} \leq \pi \sum_{n=0}^{\infty} |c_n|^2$$

implies the inequality in Definition 5.1.b) with bound $B = \sqrt{\pi}$. □

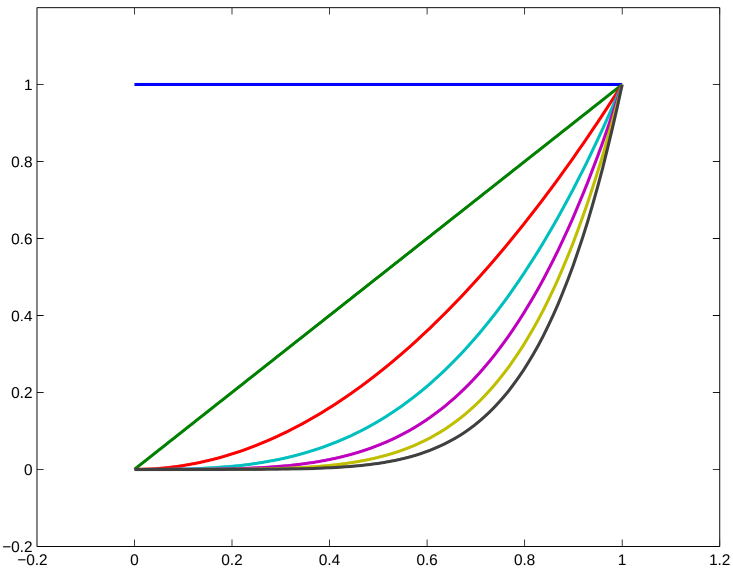

Figure 5.1: The Bessel sequence $\{1, t, t^2, \cdots\} \subset L_2[0,1]$ with bound $\sqrt{\pi}$.

In the sequel, we investigate under which conditions some generalized systems are Bessel families. According to Theorem 5.4 it is necessary and sufficient to check the boundedness of the matrix operator mentioned. To this end, in the proofs of our results one of the main ingredients is Schur's lemma.

5.2 Schur's lemma

We are interested in simple, easily verifiable conditions on infinite matrices which guarantee boundedness for the associated linear operators on l_2. Moreover, we are interested in finding upper bounds for the operator norm. A classical tool for this is Schur's lemma.

Issai Schur (1875 – 1941) formulated this result in 1911 in the fundamental paper *"Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen"* (see [60]). In 1971 Ladyženskiĭ published the paper "On a Lemma of Schur" (in Russian, see [42][3]) drawing again attention to the discrete form of this result. The article contains – amongst others – a strong form of Schur's Lemma for the space l_p (see Theorem 1, p. 140).

As McCarthy writes in his brief note on interpolation in Operator Theory [46], "the first theorem on interpolation of linear operators was proved in 1911 by I. Schur [60], who showed that if T maps l_1 to l_1 and l_∞ to l_∞, then T maps l_2 to l_2, and

$$\|T\|_{l_2} \le \|T\|_{l_1}^{1/2} \cdot \|T\|_{l_\infty}^{1/2}.$$

This was extended by M. Riesz in 1926 [53], when he proved that if T is bounded from $L_{p_0}(\mu)$ to $L_{p_0}(\mu)$ and from $L_{p_1}(\mu)$ to $L_{p_1}(\mu)$, then it is bounded from $L_p(\mu)$ to $L_p(\mu)$, for all $1 \le p_0 \le p \le p_1 \le \infty$. In the late 1930's, two quite different proofs of (generalizations of) Riesz's theorem were found: G. Thorin found a proof using complex analysis [62], and J. Marcinkiewicz a real variable proof [...] announced in [45] [...]; A. Zygmund gave a proof in [66]."

For our purposes we need the following form of Schur's lemma.

Lemma 5.6 *(Discrete form of Schur's lemma for the space l_2, see [42, Lemma on p. 139 and Corollary 2 on p. 143])*
Let Λ be a countable index set and $M = (m_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ be a real matrix. If there exist $a_1 > 0$, $a_2 > 0$ and $(\omega_\lambda)_{\lambda\in\Lambda}$, a sequence of positive numbers, such that

$$S_\lambda := \sum_{\lambda'\in\Lambda} |m_{\lambda,\lambda'}| \cdot \omega_{\lambda'} \le a_1 \cdot \omega_\lambda \qquad \text{for all } \lambda \in \Lambda, \tag{5.1}$$

and

$$S_{\lambda'} := \sum_{\lambda\in\Lambda} |m_{\lambda,\lambda'}| \cdot \omega_{\lambda} \le a_2 \cdot \omega_{\lambda'} \qquad \text{for all } \lambda' \in \Lambda, \tag{5.2}$$

then the matrix operator $M : l_2(\Lambda) \to l_2(\Lambda)$ is well–defined and bounded on $l_2(\Lambda)$ with

$$\|M\|_{l_2(\Lambda)\to l_2(\Lambda)} \le \sqrt{a_1 \cdot a_2}.$$

Because of the importance of this lemma for our results we include a short proof.

Proof. Let $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ be an arbitrary sequence from $l_2(\Lambda)$. The following estimates imply the assertion.

$$\begin{aligned} \|M\mathbf{x}\|^2 &= \sum_{\lambda\in\Lambda} \left| \sum_{\lambda'\in\Lambda} m_{\lambda,\lambda'} \cdot x_{\lambda'} \right|^2 \\ &\le \sum_{\lambda\in\Lambda} \left[\sum_{\lambda'\in\Lambda} |m_{\lambda,\lambda'}| \cdot |x_{\lambda'}| \right]^2 \end{aligned}$$

[3]The author found this reference in [61].

$$
\begin{aligned}
&\overset{CS}{\le} \sum_{\lambda\in\Lambda}\left[\sum_{\lambda'\in\Lambda}|m_{\lambda,\lambda'}|\cdot\omega_{\lambda'}\right]\cdot\left[\sum_{\lambda'\in\Lambda}\omega_{\lambda'}^{-1}\cdot|m_{\lambda,\lambda'}|\cdot|x_{\lambda'}|^2\right]\\
&\overset{(5.1)}{\le} \sum_{\lambda\in\Lambda}a_1\omega_\lambda\cdot\left[\sum_{\lambda'\in\Lambda}\omega_{\lambda'}^{-1}\cdot|m_{\lambda,\lambda'}|\cdot|x_{\lambda'}|^2\right]\\
&= a_1\sum_{\lambda'\in\Lambda}\omega_{\lambda'}^{-1}\cdot|x_{\lambda'}|^2\sum_{\lambda\in\Lambda}|m_{\lambda,\lambda'}|\cdot\omega_\lambda\\
&\overset{(5.2)}{\le} a_1\sum_{\lambda'\in\Lambda}\omega_{\lambda'}^{-1}\cdot|x_{\lambda'}|^2\cdot a_2\omega_{\lambda'}\\
&= a_1a_2\cdot\|\mathbf{x}\|_2^2. \qquad \square
\end{aligned}
$$

In the literature the discrete form of Schur's lemma is typically formulated as a sufficient condition for the boundedness. In [42, Theorem 1] Ladyženskiĭ proves, for matrices with non–negative entries, the necessity as well.

The next section summarizes essential properties of Meyer's stationary Bessel families from [48]. In order to illustrate the features of such families we include examples for the one– and two–dimensional cases.

5.3 Meyer's stationary vaguelettes

The concept 'vaguelettes' was introduced by Meyer in his monograph [48], published in 1990, in order to describe a family of continuous functions $f_{j,k} : I\!R^d \to \mathbf{C}$ which are indexed by the same scheme as the wavelets ($(j,k) \in Z\!\!\!Z \times Z\!\!\!Z^d$) and are 'wavelet–like'. Thus he described a wide collection of systems which share essential qualitative features like localization, oscillation and regularity.

Vaguelettes systems were used successfully by Donoho in the study of inverse problems (see [29]). His motivation was the fact that significant types of differential and integral operators transform wavelet bases into vaguelettes systems. Isac and Vuza introduce in [40] a definition of vaguelettes systems having higher degrees of regularity than those of Meyer and they prove some Bessel–type inequalities in Besov spaces.

Definition 5.7 *(Stationary vaguelettes family, see Meyer [48, p. 270, Definition 3])*
A family of continuous functions $\mathcal{F} = \{f_{j,k} : I\!R^d \to \mathbf{C},\, j \in Z\!\!\!Z,\, k \in Z\!\!\!Z^d\}$, is called d–dimensional dyadic stationary vaguelettes family if there exist constants $\alpha > \beta > 0$ and $C > 0$ such that for all $(j,k) \in Z\!\!\!Z \times Z\!\!\!Z^d$ and all $x, x' \in I\!R^d$

$$\int_{I\!R^d} f_{j,k}(x)\,dx = 0, \tag{5.3}$$

$$|f_{j,k}(x)| \le C\cdot 2^{dj/2}\cdot\Big(1+|2^jx-k|\Big)^{-(d+\alpha)} =: g_{j,k}(x), \tag{5.4}$$

$$|f_{j,k}(x)-f_{j,k}(x')| \le C\cdot 2^{(d/2+\beta)j}\cdot|x-x'|^\beta. \tag{5.5}$$

(5.3)–(5.5) are called the vaguelettes conditions.

Remark 5.8 *(Essential support)*

a) The concept 'essential support' of a function $f = f_{j,k}$ with good decay properties (like in (5.4), for example) is used in the literature often without rigorous definition, but with the following intuitive meaning: the essential support is a finite interval which *contains the most important features of the function. Due to the good decay properties of the function under discussion, outside this support the graph of f consists mainly of asymptotic tails which can be neglected in certain considerations. In order to stress the difference to another concept (which is presented in part b) of this remark) we call the above detailed notion 'geometrical essential support' of f. We are aware of the fact that the above assertions do not imply uniqueness for the geometrical essential support of a function. An example is illustrated in Figure 5.2. For functions with compact supports one can consider the geometrical essential support to be the whole support of the function.*

b) *The localization property (5.4) is usually interpreted in the following way in the literature: the function $f_{j,k}$ has as 'essential support' the standard dyadic cube*

$$Q_{j,k} = [2^{-j}k, 2^{-j}(k+1)) \subset I\!R^d$$

with side length $l(Q_{j,k}) = 2^{-j}$, volume $vol(Q_{j,k}) = |Q_{j,k}| = 2^{-dj}$ and left lower corner $c(Q_{j,k}) = 2^{-j}k$. A concrete example - which illustrates this intuitively clear concept of 'essential support' of the function $f_{j,k}$ with good decay properties of the type $|f_{j,k}(x)| \leq g_{j,k}(x)$) - is given in Figure 5.2. Note that geometrically the standard dyadic cube $Q_{j,k}$ represents only the right half of a possible geometrical essential support $[2^{-j}(k-1), 2^{-j}(k+1))$ of the function $f_{j,k}$. We call the dyadic cube $Q_{j,k}$ the 'abstract essential support' of the function $f_{j,k}$.

c) *The interesting question at this point is the following: "Why were the abstract essential supports introduced, instead of working with the geometrical essential supports which conform better to our intuitive associations?"*

   # *Note that there exists a one–to–one relation between the cubes $Q_{j,k}$ and the functions $f_{j,k}$ with $|f_{j,k}(x)| \leq g_{j,k}(x)$ (see (5.4)). Thus the abstract essential supports of the functions $f_{j,k}$ from one fixed level j determine a (disjoint) partition of the real line. This property is visualized in Figure 5.3 for the special case of functions $g_{j,k}$ from the level $j = 1$. It represents an essential ingredient in Meyer's proof for Theorem 5.10 (when an infinite sum is identified to be a Riemann sum in order to pass to an integral).*

   # *We emphasize at this point that the function $f_{j,k}$ is not reduced during computations to the information on the abstract essential support so that some loss or inexactness might occur. The information over the entire support is needed, is given and is used in computations. Only in order to be able to prove that some infinite sums are finite, one needs the above mentioned one–to–one relation between the functions $f_{j,k}$ and the elements $Q_{j,k}$ of a disjoint partition of the real line (follow to this end the proof of Theorem 2 in [48, p. 270–271], for example).*

   # *Furthermore, the abstract essential supports have an impact on the right hand sides of (5.4) and (5.5) through the entities $l(Q_{j,k})$, $|Q_{j,k}|$ and $c(Q_{j,k})$. With this terminology one can transform (5.4) and (5.5) into*

$$|f_{j,k}(x)| \leq C \cdot |Q_{j,k}|^{-1/2} \cdot [1 + l(Q_{j,k})^{-1} \cdot |x - c(Q_{j,k})|]^{-(d+\alpha)}, \tag{5.6}$$

$$|f_{j,k}(x) - f_{j,k}(x')| \leq C \cdot |Q_{j,k}|^{-1/2} \cdot \left(\frac{|x - x'|}{l(Q_{j,k})}\right)^{\beta}. \tag{5.7}$$

   # *It is obvious that in general the geometrical essential supports do not form a (disjoint) partition, namely at least two consecutive are overlapping. For the special*

case depicted in Figure 5.2 each two consecutive intervals $[2^{-j}(k-1), 2^{-j}(k+1))$ *are overlapping. One could also consider as geometrical essential supports for the functions* $f_{j,k}$ *the dilated cubes* $\left[2^{-j}\left(k-\frac{1}{2}\right), 2^{-j}\left(k+\frac{1}{2}\right)\right)$ *– which are disjoint – but this is rather inconvenient for computations.*

Some d–dimensional dyadic stationary vaguelettes systems generated from only one function are given in the following example.

Example 5.9 *(Multivariate dyadic stationary vaguelettes families)*
Let $F : \mathbb{R}^d \to \mathbf{C}$ *be a continuous function for which constants* $C > 0$ *and* $\alpha > \beta > 0$ *exist, such that for all* $x, x' \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} F(x)\,dx = 0,$$

$$|F(x)| \le C \cdot (1 + |x|)^{-(d+\alpha)},$$

$$|F(x) - F(x')| \le C \cdot |x - x'|^{\beta}.$$

We define for $j \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^d$ *the function* $f_{j,k} : \mathbb{R}^d \to \mathbf{C}$ *by dilation, translation and normalization from the generatrix* F *through*

$$f_{j,k}(x) := 2^{dj/2} \cdot F(2^j x - k), \qquad x \in \mathbb{R}^d.$$

The resulting affine collection $\{f_{j,k},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ *is a dyadic stationary vaguelettes family in the sense of Definition 5.7.*

a) *Concrete examples for the case of unbounded supports.*

In the univariate case one possible choice for the generating function F *is*

$$F_1 : \mathbb{R} \to \mathbb{R}, \qquad F_1(x) := x \cdot e^{-x^2}$$

(see Figure 5.4). Conditions (5.4) and (5.5) are satisfied for $\beta = 1$, $\alpha = 1.1$ *and* $C = 2$.

A concrete example for the bidimensional case is

$$F_2 : \mathbb{R}^2 \to \mathbb{R}, \qquad F_2(x, y) := xy \cdot e^{-(x^2+y^2)},$$

obtained from F_1 *by the tensor product approach (see Figure 5.5 and for more details on the mentioned approach our papers [5, 6, 7]). Conditions (5.4) and (5.5) are satisfied for* $\beta = 1$, $\alpha = 1.1$ *and* $C = 1$.

Figure 5.6 depicts the relation (5.4), as well as the abstract essential support for the case that the functions $f_{j,k}$ *are generated by* F_1.

b) *Concrete examples for the case of bounded supports.*

In the univariate case one possible choice for the generating function F *is*

$$F_3 : [-1.5, 1.5] \to \mathbb{R}, \qquad F_3(x) := x \cdot (x-1)(x+1)$$

(see Figure 5.7.a)). Conditions (5.4) and (5.5) are satisfied for $\beta = 1$, $\alpha = 1.5$ *and* $C = 9$. *Relation (5.4) and the abstract essential support are illustrated for the case* $f_{0,0} = F_3$ *in Figure 5.7.b).*

A concrete example for the bidimensional case is

$$F_4 : [-1.5, 1.5]^2 \to \mathbb{R}, \qquad F_4(x, y) := xy \cdot (x^2 - 1) \cdot (y^2 - 1),$$

obtained from F_3 *by the tensor product approach (see Figure 5.8). Conditions (5.4) and (5.5) are satisfied for* $\beta = 1$, $\alpha = 1.5$ *and* $C = 16$. □

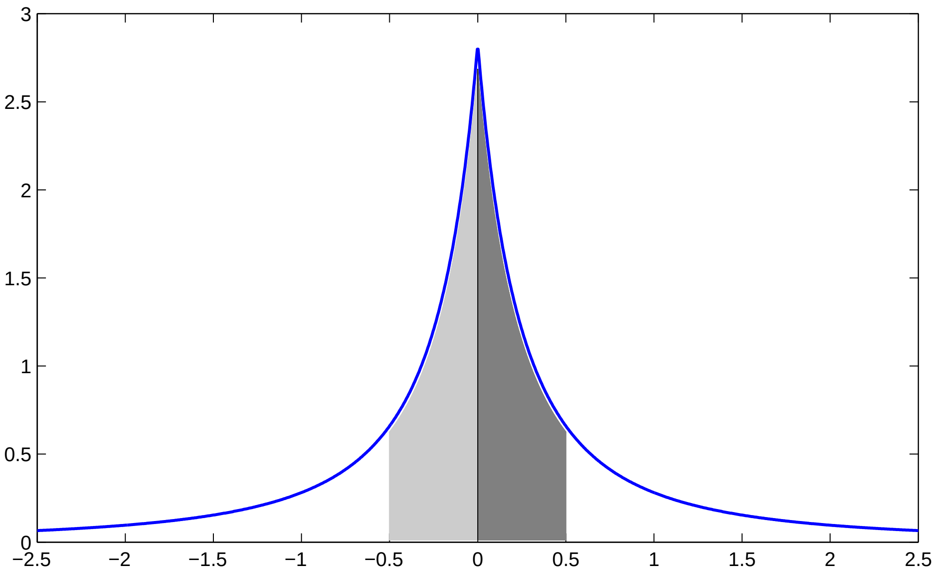

Figure 5.2: Graph of the function $g_{j,k}(x) = C \cdot 2^{dj/2} \cdot (1 + |2^j x - k|)^{-(d+\alpha)}$ – which describes the good decay property of the function $f_{j,k}$ (see (5.4)) – for the particular values $d = 1$, $j = 1$, $k = 0$, $C = 2$, $\alpha = 1.1$. The whole marked area corresponds to the integral of $g_{1,0}$ over $[-0.5, 0.5)$. This interval can be interpreted as geometrical essential support of $f_{1,0}$. The darker right part corresponds to the interval $[2^{-j}k, 2^{-j}(k+1)) = [0, 0.5)$, which represents the abstract essential support of the function $f_{1,0}$.

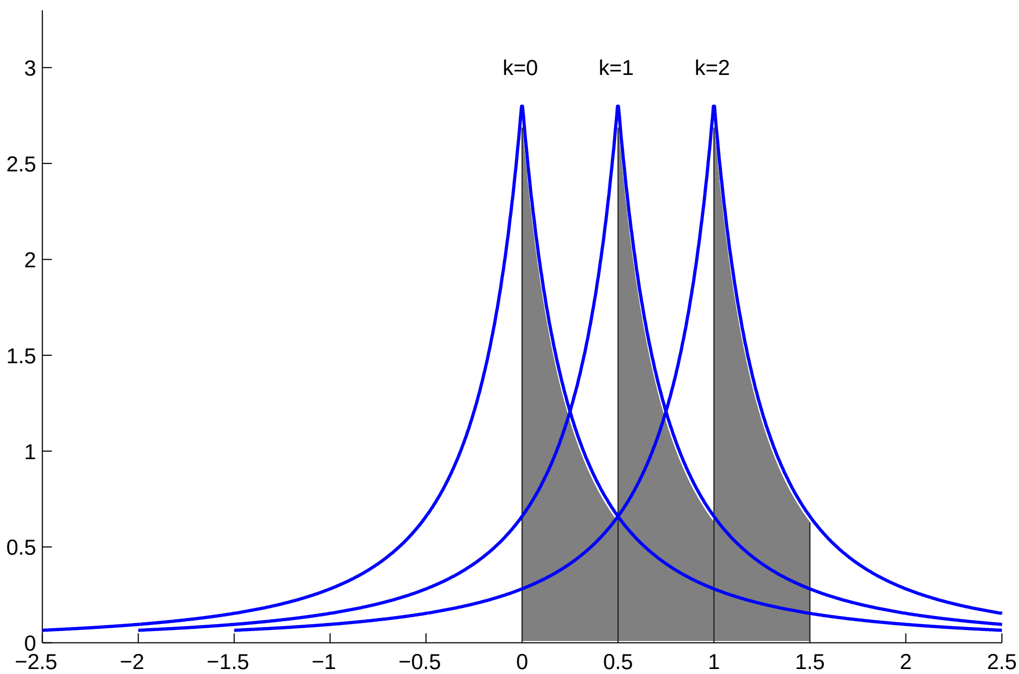

Figure 5.3: Graphs of the function $g_{j,k}(x) = C \cdot 2^{dj/2} \cdot (1 + |2^j x - k|)^{-(d+\alpha)}$ for the particular values $d = 1$, $j = 1$, $k \in \{0, 1, 2\}$, $C = 2$, $\alpha = 1.1$. Note that there exists a one–to–one relation between the cubes $Q_{1,k} = [k/2, (k+1)/2)$, $k \in \mathbb{Z}$, and the functions $f_{1,k}$ with $|f_{1,k}(x)| \leq g_{1,k}(x)$, $k \in \mathbb{Z}$. Thus the abstract essential supports of the functions $f_{j,k}$ from one fixed level j determine a (disjoint) partition of the real line.

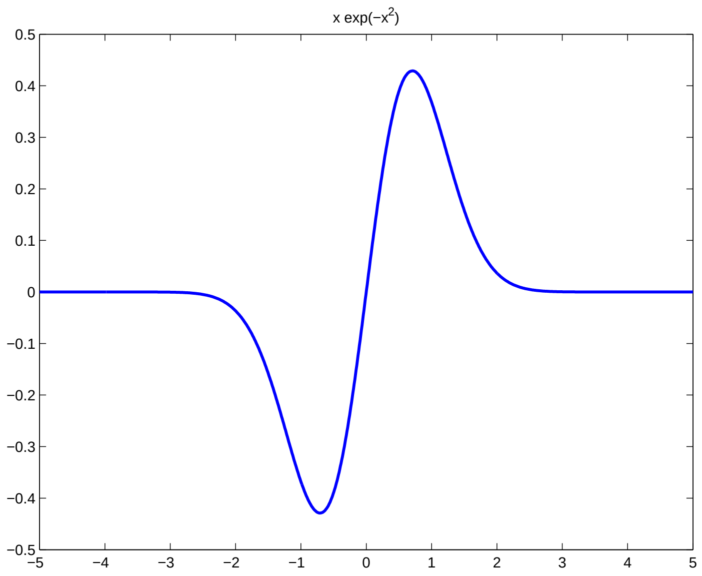

Figure 5.4: Generating function $F_1(x) = x \cdot e^{-x^2}$ without compact support for a one–dimensional dyadic stationary vaguelettes family.

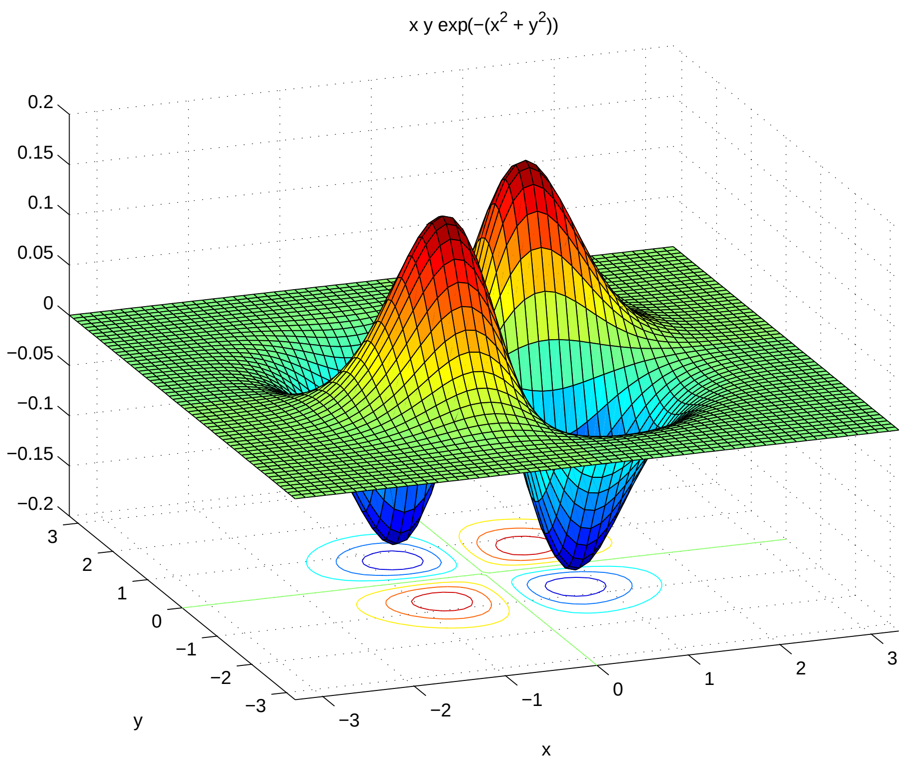

Figure 5.5: Generating function $F_2(x,y) = xy \cdot e^{-(x^2+y^2)}$ without compact support for a two–dimensional dyadic stationary vaguelettes family.

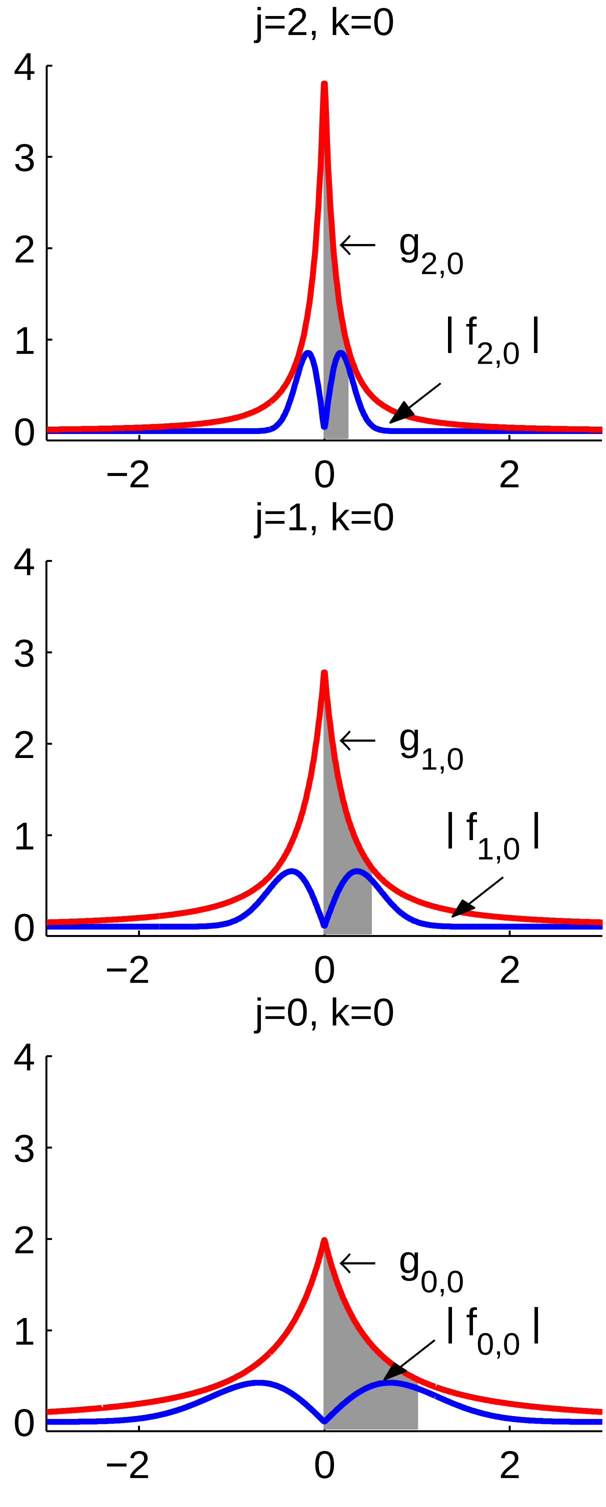

Figure 5.6: Relation (5.4) and the abstract essential support are depicted for the case that the functions $f_{j,k}$ are generated by dilation, translation and normalization from the generatrix $F_1(x) := x \cdot e^{-x^2}$.

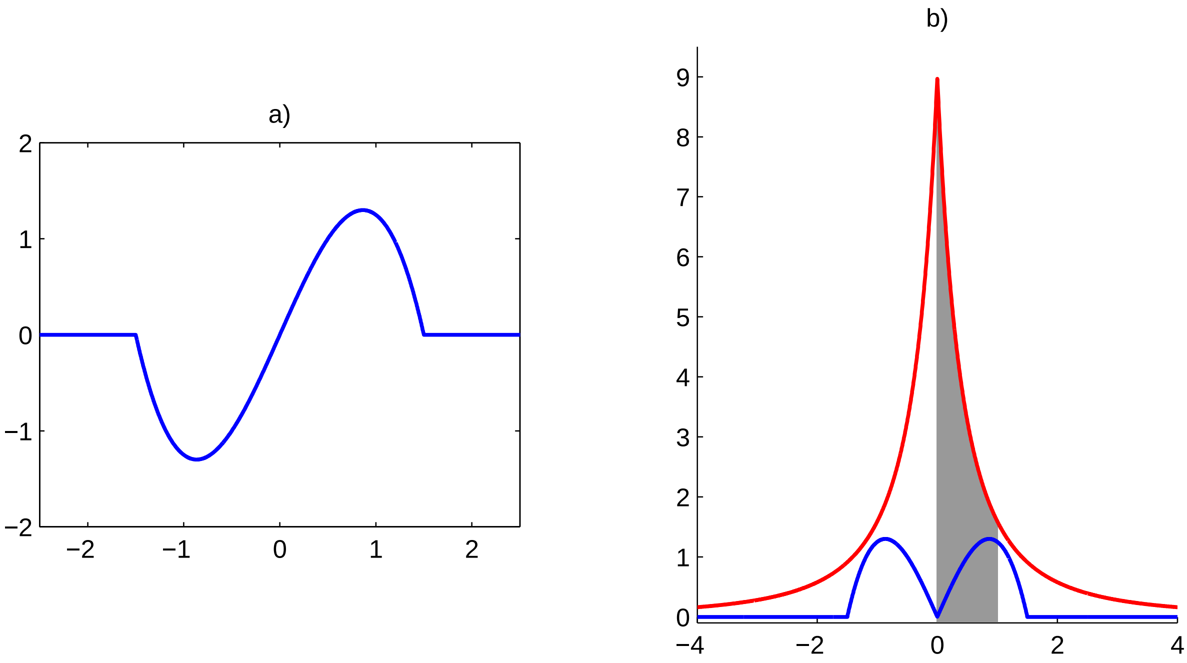

Figure 5.7: a) Generating function $F_3(x) = x \cdot (x-1)(x+1)$ with compact support $[-1.5, 1.5]$ for a one–dimensional dyadic stationary vaguelettes family. b) Relation (5.4) and the abstract essential support are illustrated for the case $f_{0,0} = F_3$.

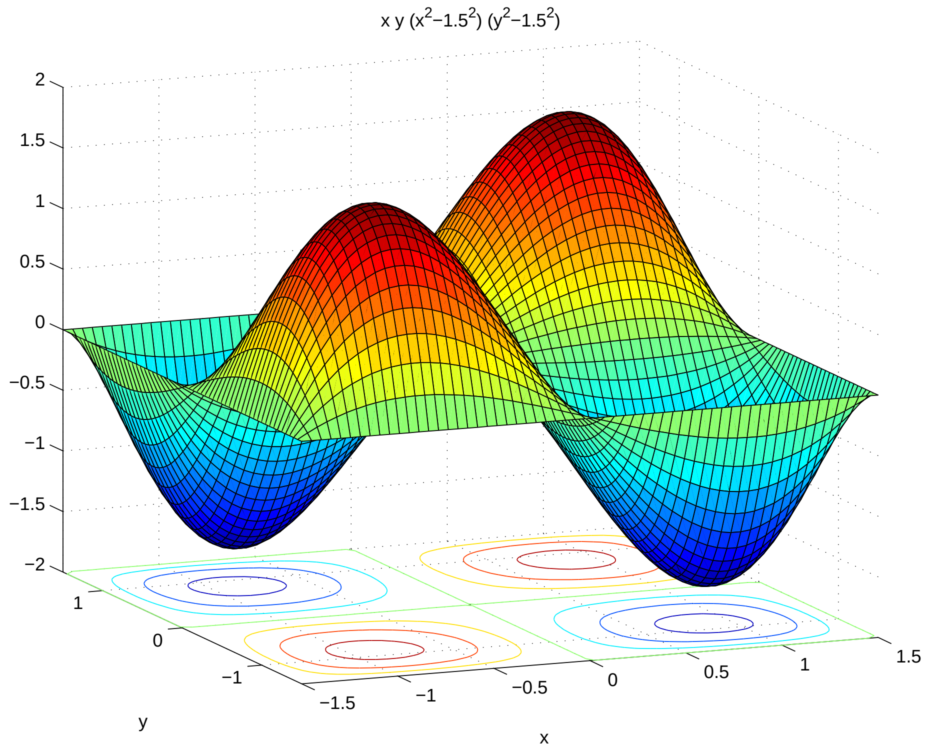

Figure 5.8: Generating function $F_4(x, y) = xy \cdot (x^2 - 1) \cdot (y^2 - 1)$ with compact support $[-1.5, 1.5]^2$ for a two–dimensional dyadic stationary vaguelettes family.

The starting point for our extensions from Sections 5.8 and 5.11 will be Definition 5.7 and the next result.

Theorem 5.10 *(Bessel families, see Meyer [48])*
Every d–dimensional dyadic stationary vaguelettes family is a Bessel family in $L_2(\mathbb{R}^d)$.

Proof. The main ideas can be found in [48, p. 270–271]. □

The previous theorem expresses the fact that vaguelettes are 'almost orthogonal'. From Definition 5.7 one also can deduce that the vaguelette $f_{j,k}$ is localized around the point $(2^{-j}, 2^{-j}k)$ in the scale–time half–space $\mathbb{R}_+ \times \mathbb{R}^d$. These aspects will be discussed in more detail in the following section, which also discusses some Bessel families of Frazier&Jawerth from [32].

5.4 Localization theory of Frazier and Jawerth

In modern Harmonic Analysis literature one can find *two essentially different localization concepts* for families of structured functions which do not seem to be compatible with each other. The first one has been developed in parallel by Frazier&Jawerth [32] and Meyer [48] for the canonic tiling of the scale–time space in the wavelet case. The second can be found in papers by Gröchenig (see e.g. [37]); it is formulated for the regular tiling of the time–frequency space in the Gabor case.

Because of the fundamental difference between the two structures (see Figure 5.10 and Figure 5.9) different distance functions between points have to be used: a hyperbolic metric in the wavelet case and the Euclidian distance in the Gabor case. Until now no approach has been found in order to unify or bridge these two theories. For our purposes in the sequel we will follow the first one and present next some central concepts and main results from [32] which are directly connected to our further considerations.

In the stationary wavelet case the localization points of the functions are distributed in a regular manner in the scale–time space $\mathbb{R} \times \mathbb{R}^d$, creating a regular grid. For the upper half–space $\mathbb{R}_+ \times \mathbb{R}^d$ one usually uses dyadic sampling 2^{-j}, $j \in \mathbb{Z}$, on the scale axis and equidistant scale–dependent sampling of the time axis: $2^{-j}k$, $k \in \mathbb{Z}^d$, for fixed scale j (see Figure 5.10 for the case $d = 1$). Therefore, the points of the grid from the upper half–space (and implicitely the associated sampling functions themselves) can be indexed either by the pairs $(j, k) \in \mathbb{Z} \times \mathbb{Z}^d$ (notation à la Meyer), or by the standard dyadic cubes $Q = Q_{j,k} \in \mathbb{Z}^d$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ (notation à la Frazier&Jawerth).

Definition 5.11 *For $j \in \mathbb{Z}$ and $(k_1, \ldots, k_d) = k \in \mathbb{Z}^d$ let $Q_{j,k}$ be the standard dyadic cube*

$$Q_{j,k} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \;:\; k_i \le 2^j x_i < k_i + 1,\;\; i = 1, \ldots, d\}.$$

We denote the whole collection of dyadic cubes in $\mathbb{R}^d$ by $\mathcal{Q}$, the "lower left corner" $2^{-j}k$ of $Q = Q_{j,k}$ by c_Q and the side length 2^{-j} of $Q = Q_{j,k}$ by $l(Q)$.

One of the main concepts in Frazier&Jawerth's theory is the almost diagonality of bi–infinite matrices. A matrix $A = (a_{Q,P})_{(Q,P)\in\mathcal{Q}^2}$ has the diagonalization property if its entries $|a_{Q,P}|$ decay at a certain rate away from the diagonal (i.e., when $Q = P$). This means that $|a_{Q,P}|$ must decay as $l(Q)/l(P)$ goes to 0 or ∞, and as P and Q get apart from each other.

Definition 5.12 *(Almost diagonal matrix on $l_2(\mathcal{Q})$, see [32, p. 53])*
Let $\mathcal{Q}$ be the set of all dyadic cubes from $\mathbb{R}^d$.

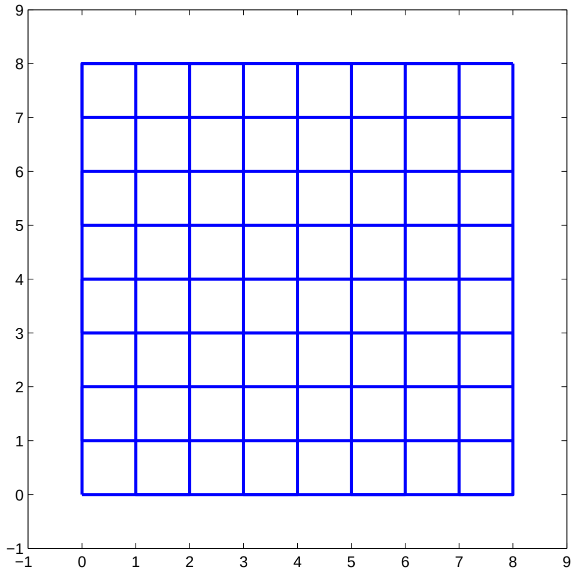

Figure 5.9: Canonic tiling of the time–frequency space $\mathbb{R}^2$ in the Gabor case.

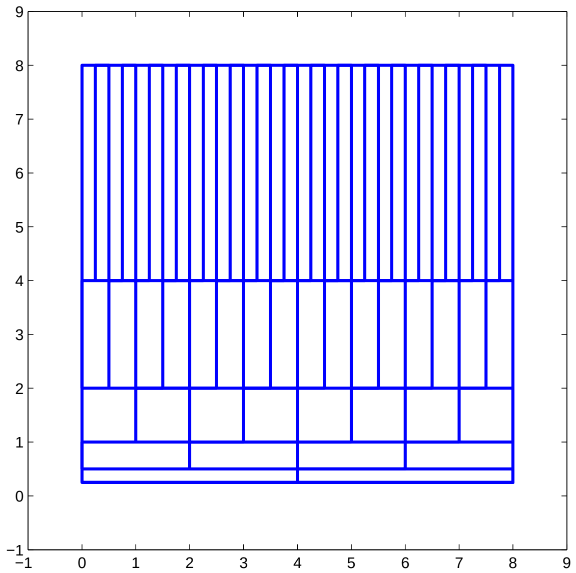

Figure 5.10: Standard tiling of the time–scale space $\mathbb{R}^2$ in the wavelet case.

a) The real or complex matrix $A = (a_{Q,P})_{(Q,P)\in\mathcal{Q}^2}$ is called almost diagonal on $l_2(\mathcal{Q})$ if there exist $C > 0$ and $\varepsilon > 0$ such that

$$|a_{Q,P}| \le C \cdot \left(1 + \frac{|c_Q - c_P|}{\max\{l(P), l(Q)\}}\right)^{-d-\varepsilon} \cdot \min\left\{\frac{l(Q)}{l(P)}, \frac{l(P)}{l(Q)}\right\}^{(d+\varepsilon)/2} \tag{5.8}$$

holds for all possible choices of $(Q,P) \in \mathcal{Q}^2$. By $|c_Q - c_P|$ we denote the Euclidian distance between the two points c_Q and c_P from $I\!R^d$.

b) A linear operator $A : l_2(\mathcal{Q}) \to l_2(\mathcal{Q})$ is called almost diagonal if its associated matrix possesses this property.

In order to express localization properties of almost diagonal matrices one needs a distance function on $\mathcal{Q}$. To this end the Poincaré metric for the upper half–plane has been generalized as follows.

Definition 5.13 *(Generalized Poincaré metric, see [32, p. 53])*
Let $G = \{(x,t) : x \in I\!R^d, t > 0\}$ be the group with multiplication

$$(x,t)\cdot(y,s) = (sx + y, ts).$$

The generalized Poincaré metric d_{Pc} in G is defined by

$$d_{Pc}((x,t),(y,s)) := \ln\sqrt{\frac{1+\rho((x,t),(y,s))}{1-\rho((x,t),(y,s))}} \tag{5.9}$$

with

$$\rho((x,t),(y,s)) := \sqrt{\frac{|x-y|^2 + (s-t)^2}{|x-y|^2 + (s+t)^2}}, \tag{5.10}$$

where $|x-y|$ denotes the Euclidean distance between x and y.

From this hyperbolic metric in G a distance function in $\mathcal{Q}$ is obtained by setting

$$d_{\mathcal{Q}}(P,Q) := d_{Pc}((c_P, l(P)), (c_Q, l(Q))) \qquad \text{for all } (P,Q) \in \mathcal{Q}^2. \tag{5.11}$$

The equivalence stated in the following lemma emphasizes the deep connection between Definition 5.12 and Definition 5.13. Therefore we include a proof of this result.

Lemma 5.14 *(see [32, p. 54])*
Under the general conditions of Definition 5.13 there holds

$$\sqrt{\frac{1+\rho((x,t),(y,s))}{1-\rho((x,t),(y,s))}} \approx \max\left\{\sqrt{\frac{t}{s}}, \sqrt{\frac{s}{t}}\right\} \cdot \left(1 + \frac{|x-y|}{\max\{t,s\}}\right),$$

where $\approx$ means that each term can be majorated by a finite constant (independent of the variables involved) times the other term.

Proof. $t > 0$ and $s > 0$ imply $\rho \in [0,1)$. For notational simplicity consider the case $s = \min\{t,s\}$ and $t = \max\{t,s\}$; the opposite case follows in an analogous way. We obtain

$$\sqrt{\frac{1+\rho((x,t),(y,s))}{1-\rho((x,t),(y,s))}} \;=\; \frac{1+\rho((x,t),(y,s))}{2} \cdot \sqrt{\frac{(s+t)^2}{st}} \cdot \sqrt{1 + \frac{|x-y|^2}{(s+t)^2}}$$

$$
\begin{aligned}
&\leq \; 2\cdot\sqrt{\frac{t}{s}}\cdot\sqrt{1+\frac{|x-y|^2}{(s+t)^2}}\\
&\leq \; 2\cdot\sqrt{\frac{t}{s}}\cdot\left(1+\frac{|x-y|}{s+t}\right)\\
&\leq \; 2\cdot\max\left\{\sqrt{\frac{t}{s}},\sqrt{\frac{s}{t}}\right\}\cdot\left(1+\frac{|x-y|}{\max\{s,t\}}\right).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\max&\left\{\sqrt{\frac{t}{s}},\sqrt{\frac{s}{t}}\right\}\cdot\left(1+\frac{|x-y|}{\max\{s,t\}}\right)\\
&= 1\cdot\sqrt{\frac{t}{s}}\cdot\left(1+\frac{|x-y|}{t}\right)\\
&\leq [1+\rho((x,t),(y,s))]\cdot\sqrt{\frac{t^2}{st}}\cdot\left(1+\frac{|x-y|}{\frac{s+t}{2}}\right)\\
&\leq [1+\rho((x,t),(y,s))]\cdot\sqrt{\frac{(s+t)^2}{st}}\cdot 2\sqrt{2}\sqrt{1+\frac{|x-y|^2}{(s+t)^2}}\\
&= 4\sqrt{2}\cdot\sqrt{\frac{1+\rho((x,t),(y,s))}{1-\rho((x,t),(y,s))}}. \qquad \square
\end{aligned}
$$

The last proof also shows the validity of the next result.

Lemma 5.15 *Under the settings of Definition 5.13 we have*

$$
\left[\max\left\{\sqrt{\frac{t}{s}},\sqrt{\frac{s}{t}}\right\}\cdot\left(1+\frac{|x-y|}{\max\{t,s\}}\right)\right]^{-1} \leq 2\cdot\left[\sqrt{\frac{1+\rho((x,t),(y,s))}{1-\rho((x,t),(y,s))}}\,\right]^{-1}.
$$

The following result constitutes an important characterization of almost diagonal matrices in terms of the distance function $d_{\mathcal{Q}}$ which is essential in the localization theory of Frazier&Jawerth.

Proposition 5.16 *(Characterization of almost diagonal matrices, see [32, p. 54])*
$A = (a_{Q,P})_{(Q,P)\in\mathcal{Q}^2}$ is an almost diagonal matrix on $l_2(\mathcal{Q})$ if and only if there exist $C>0$ and $\varepsilon>0$ such that

$$
|a_{Q,P}| \leq C\cdot e^{-(d+\varepsilon)\cdot d_{\mathcal{Q}}(P,Q)} \qquad \textit{for all } (P,Q)\in\mathcal{Q}^2.
$$

Proof. An application of Lemma 5.14 gives the desired equivalence. □

Proposition 5.16 characterizes the off–diagonal decay of an almost diagonal matrix; this type of matrices feature exponential localization w.r.t. the distance $d_{\mathcal{Q}}$. Moreover, the associated linear operators possess the following property.

Theorem 5.17 *(Boundedness criterion, see Frazier&Jawerth [32, Theorem 3.3])*
An almost diagonal operator on $l_2(\mathcal{Q})$ is bounded on $l_2(\mathcal{Q})$.

Proof. See [32, p. 54f]. □

Using Theorem 5.4 one can rephrase Theorem 5.17 as follows:

Theorem 5.18 *(Bessel families)*
Every function family $\{f_Q\}_{Q\in\mathcal{Q}}$ from $L_2(\mathbb{R}^d)$ with almost diagonal Gram matrix on $l_2(\mathcal{Q})$ is a Bessel family in $L_2(\mathbb{R}^d)$.

Remark 5.19 *(Common strategy of Meyer and Frazier&Jawerth)*
There is a common scheme in the work of Meyer and Frazier&Jawerth to prove that a function system constitutes a Bessel family.
In the first part of the proof of Theorem 5.10 Meyer shows a property similar to (5.8) for the entries of the Gram matrix associated to the vaguelettes family. Based on this result and applying Schur's lemma he proves in the second part the boundedness of the operator associated to the Gram matrix which - according to Theorem 5.4 - implies the Bessel property for the vaguelettes family.
We thus stress the fact that the general strategies behind the approaches of Meyer and Frazier&Jawerth are basically the same.

For our sibling frame candidates in Chapter 6 we will follow a scheme similar to the above in order to prove the Bessel property for them. Therefore, next we give our extension from the stationary to the non–stationary situation.

5.5 Almost diagonality in the non–stationary univariate case

In this section we introduce the concept of almost diagonality of a bi–infinite matrix $A = (a_{\lambda,\lambda'})_{\lambda,\lambda'}$ w.r.t. a given collection of closed and bounded intervals of the real line. For the case $d = 1$ this concept fully generalizes the one for the stationary situation presented in Definition 5.12.

For both the non–compact and compact cases sufficient conditions will be given for the Gramian of a non–stationary function system to be almost diagonal in the sense of Definition 5.20. Explicit values of the constants C and ε figuring in Definition 5.20 will be given. These will depend exclusively on the parameters in the assumptions made.

As immediate consequences of the two main results (Theorem 5.24 and Theorem 5.26) we will be able to formulate the corresponding exponential localization statements.

As an example for the compact case, at the end of the section we discuss a family of suitably normalized differentiated B–splines w.r.t. certain nested knot sequences all contained in an interval $[a, b]$.

Next we introduce and discuss the concept of almost diagonal matrices in the one–dimensional non–stationary setting.

Definition 5.20 *(Almost diagonal matrix on $l_2(\Lambda)$)*
Let Λ be a countable index set and $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ a collection of compact intervals on the real line. The length of I_λ will be denoted by $|I_\lambda|$ and its left endpoint by c_λ.

a) A matrix $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ is called almost diagonal on $l_2(\Lambda)$ w.r.t. $\{I_\lambda\}_{\lambda\in\Lambda}$ if there exist $C > 0$ and $\varepsilon > 0$ such that

$$|a_{\lambda,\lambda'}| \leq C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-1-\varepsilon} \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2} \tag{5.12}$$

holds for all possible choices of $(\lambda, \lambda') \in \Lambda^2$.

b) A linear operator $A : l_2(\Lambda) \to l_2(\Lambda)$ is called almost diagonal if its associated matrix possesses this property.

Remark 5.21 *Notice that the first pair of parentheses on the right–hand side in (5.12) describes the decay of $|a_{\lambda,\lambda'}|$ when I_λ and $I_{\lambda'}$ get apart from each other (such that the difference $|c_\lambda - c_{\lambda'}|$ becomes big). The second one explains the decay in the case that $\frac{|I_\lambda|}{|I_{\lambda'}|}$ goes to 0 or ∞. Therefore, if one of the mentioned decay properties happens to be intrinsic for a particular matrix, one drops the corresponding pairs of parentheses and obtains a simpler condition to be proved for the entries of the matrix.*

In the sequel we will encounter the case where for I_λ and $I_{\lambda'}$ sufficiently apart from each other the entries $a_{\lambda,\lambda'}$ of the matrix A will turn out to be zero. This is always the case when the matrix A is the Gram matrix of a function family with compact supports I_λ, $\lambda \in \Lambda$. In this case we will estimate $|a_{\lambda,\lambda'}|$ by

$$C \cdot 1 \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2}.$$

Although the first pair of parentheses can always be estimated from above by 1, i.e.,

$$\left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-1-\varepsilon} \le 1,$$

in certain other cases it is necessary to have a better bound. This is always the case when the matrix A is the Gram matrix of a function family with unbounded supports, with I_λ being the corresponding (geometrical or abstract) essential supports.

We emphasize that in our work consideration of the relation between $|I_\lambda|$ and $|I'_\lambda|$, i.e., of

$$\min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2},$$

is indispensable in both the compact and the non–compact support cases.

In the setup of Definition 5.20 we get a distance function in $\mathcal{I}$ by setting

$$d_{\mathcal{I}}(I_\lambda, I_{\lambda'}) := d_{Pc}\left((c_\lambda, |I_\lambda|), (c_{\lambda'}, |I_{\lambda'}|)\right) \qquad \text{for all } (\lambda, \lambda') \in \Lambda^2, \tag{5.13}$$

where d_{P_c} is the Poincaré metric defined in (5.9) and (5.10).

As a direct application of Lemma 5.14 we arrive at the following exponential localization of an almost diagonal matrix in the non–stationary case.

Proposition 5.22 *(Characterization of almost diagonal matrices on $l_2(\Lambda)$)*
Let Λ be a countable index set and let $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ be a system of compact intervals on the real line.

a) $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ is an almost diagonal matrix on $l_2(\Lambda)$ w.r.t. $\mathcal{I}$ in the sense of Definition 5.20 if and only if there exist $\varepsilon > 0$ and $C' > 0$ such that

$$|a_{\lambda,\lambda'}| \le C' \cdot e^{-(1+\varepsilon)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})} \tag{5.14}$$

for all $(\lambda, \lambda') \in \Lambda^2$ with the distance function $d_{\mathcal{I}}$ defined in (5.13).

b) Moreover, if $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ is an almost diagonal matrix on $l_2(\Lambda)$ w.r.t. $\mathcal{I}$ in the sense of Definition 5.20, then with ε and C from Definition 5.20.a) we obtain in (5.14) for C' the value $2^{1+\varepsilon}C$.

Proof. a) An application of Lemma 5.14 with $x := c_\lambda$, $y := c_{\lambda'}$, $t := |I_\lambda|$ and $s := |I_{\lambda'}|$ gives the desired equivalence, i.e.,

$$\begin{aligned}
& |a_{\lambda,\lambda'}| \le C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-1-\varepsilon} \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2} \\
\Leftrightarrow \quad & |a_{\lambda,\lambda'}| \le C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-(1+\varepsilon)} \cdot \max\left\{\sqrt{\frac{|I_\lambda|}{|I_{\lambda'}|}}, \sqrt{\frac{|I_{\lambda'}|}{|I_\lambda|}}\right\}^{-(1+\varepsilon)} \\
\overset{\text{Lemma 5.14}}{\Leftrightarrow} \quad & |a_{\lambda,\lambda'}| \le C' \cdot \left(\sqrt{\frac{1 + \rho((c_\lambda, |I_\lambda|), (c_{\lambda'}, |I_{\lambda'}|))}{1 - \rho((c_\lambda, |I_\lambda|), (c_{\lambda'}, |I_{\lambda'}|))}}\right)^{-(1+\varepsilon)} \\
\overset{(5.9),(5.13)}{\Leftrightarrow} \quad & |a_{\lambda,\lambda'}| \le C' \cdot e^{-(1+\varepsilon)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})}.
\end{aligned}$$

Part b) follows from part a) when Lemma 5.15 is applied. □

In the sequel we need a technical lemma.

Lemma 5.23 *For $a \in (0, 1]$, $\alpha > 0$ and*

$$f_a(x) := \frac{1}{(1 + a|x|)^{1+\alpha}}, \qquad \forall\, x \in \mathbb{R},$$

the following estimate holds for the convolution of f_1 and f_a :

$$(f_1 * f_a)(z) \le C(\alpha) \cdot f_a(z), \qquad \forall\, z \in \mathbb{R},$$

where

$$C(\alpha) = \frac{2}{\alpha}\left[1 + 2^{2+\alpha}\right].$$

Proof. Let $z \ge 0$. After the decomposition

$$\begin{aligned}
I &:= (f_1 * f_a)(z) \\
&= \int_{|y-z|\ge z} f_1(y) f_a(z-y)\, dy + \int_{|y-z|\le z} f_1(y) f_a(z-y)\, dy =: I_1 + I_2
\end{aligned}$$

one obtains

$$I_1 \le \frac{2}{\alpha} \cdot f_a(z) \qquad \left(\text{since } \int_{\mathbb{R}} f_1(y)\, dy = \frac{2}{\alpha}\right),$$

and

$$\begin{aligned}
I_2 &= \int_0^z f_1(y) f_a(z-y)\, dy + \int_z^{2z} f_1(y) f_a(z-y)\, dy \\
&\le 2\int_0^z f_1(y) f_a(z-y)\, dy \\
&= 2 \cdot f_a(z) \int_0^z \left[\frac{1 + az}{(1+y)(1 + az - ay)}\right]^{1+\alpha} dy \\
&\overset{\substack{\text{expansion to}\\ \text{part. fract.}}}{=} 2 \cdot f_a(z) \cdot \left(\frac{1+az}{1 + a + az}\right)^{1+\alpha} \int_0^z \left[\frac{1}{1+y} + \frac{a}{1 + az - ay}\right]^{1+\alpha} dy \\
&\le 2 \cdot f_a(z) \cdot \int_0^z \left[\frac{1}{1+y} + \frac{1}{1 + z - y}\right]^{1+\alpha} dy \\
&\le 2^{3+\alpha} \cdot f_a(z) \cdot \int_0^{\frac{z}{2}} \frac{dy}{(1+y)^{1+\alpha}} \\
&\le \frac{2^{3+\alpha}}{\alpha} \cdot f_a(z).
\end{aligned}$$

The result for $z < 0$ follows by symmetry. $\square$

We first consider the situation of function families with non–compact supports.

Theorem 5.24 *(Sufficient conditions for the almost diagonality of a Gram matrix)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(I\!R)$ with (geometrical or abstract) essential supports $\{I_\lambda = [c_\lambda, b_\lambda]\}_{\lambda\in\Lambda}$, satisfying the following conditions:

a) $\displaystyle\int_{I\!R} \psi_\lambda(x)\, dx = 0$ *for all* $\lambda \in \Lambda$ (*vanishing moment*);

b) $\exists\, \alpha > 0 \quad \exists\, C_1 > 0 \quad \forall\lambda \in \Lambda \quad \forall x \in I\!R:$

$$|\psi_\lambda(x)| \le C_1 \cdot |I_\lambda|^{-1/2} \left[1 + |I_\lambda|^{-1} \cdot |x - c_\lambda|\right]^{-(1+\alpha)}$$

(*boundedness and decay*);

c) $\exists \beta \in (0, \alpha) \quad \exists\, C_2 > 0 \quad \forall\lambda \in \Lambda \quad \forall x, x' \in I\!R \quad (x \neq x'):$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \le C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |x - x'|^\beta$$

(*Hölder condition*).

Then $Gram(\Psi)$ is an almost diagonal matrix on $l_2(\Lambda)$ w.r.t. $\{I_\lambda\}_{\lambda\in\Lambda}$ with exponent

$$\varepsilon = \frac{2\alpha\beta}{1 + \alpha + 2\beta} \tag{5.15}$$

and constant

$$C = 2C_1 \cdot \left[\left(1 + 2^{2+\alpha}\right) \frac{C_1}{\alpha}\right]^{\frac{1+2\beta}{1+\alpha+2\beta}} \left(\frac{C_2}{\alpha - \beta}\right)^{\frac{\alpha}{1+\alpha+2\beta}}, \tag{5.16}$$

i.e., with the mentioned constants there holds

$$|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \le C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-1-\varepsilon} \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2}$$

for all $(\lambda, \lambda') \in \Lambda^2$.

Proof. With the notations from Lemma 5.23 and the substitution $y = |I_{\lambda'}|^{-1}(x - c_{\lambda'})$ a first estimate for the elements $\langle\psi_\lambda, \psi_{\lambda'}\rangle$ of the Gram matrix in the case $|I_{\lambda'}| \le |I_\lambda|$ is given by

$$\begin{aligned}
|\langle\psi_\lambda, \psi_{\lambda'}\rangle| &\le \int_{I\!R} |\psi_\lambda(x)| \cdot |\psi_{\lambda'}(x)|\, dx \\
&\overset{b)}{\le} C_1^2 \left(|I_\lambda| \cdot |I_{\lambda'}|\right)^{-\frac{1}{2}} \cdot \\
&\quad \cdot \int_{I\!R} \left[1 + |I_\lambda|^{-1} \cdot |x - c_\lambda|\right]^{-(1+\alpha)} \cdot \left[1 + |I_{\lambda'}|^{-1} \cdot |x - c_{\lambda'}|\right]^{-(1+\alpha)} dx \\
&\overset{Subst.}{=} C_1^2 \left(|I_\lambda| \cdot |I_{\lambda'}|\right)^{-\frac{1}{2}} \int_{I\!R} \left[1 + \frac{|I_{\lambda'}|}{|I_\lambda|} \cdot \left|y + \frac{c_{\lambda'} - c_\lambda}{|I_{\lambda'}|}\right|\right]^{-(1+\alpha)} \cdot \\
&\qquad \cdot [1 + |y|]^{-(1+\alpha)} \cdot |I_{\lambda'}|\, dy
\end{aligned}$$

$$
\begin{aligned}
&= && C_1^2 \cdot \left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{-\frac{1}{2}} \cdot (f_a * f_1)(z) \\
& && \qquad \text{with } a = \frac{|I_{\lambda'}|}{|I_\lambda|} \in (0,1]; \quad z = -\frac{c_{\lambda'} - c_\lambda}{|I_{\lambda'}|}; \quad f_a(y) := \frac{1}{(1+a|y|)^{1+\alpha}} \\
&\overset{L.\,5.23}{\leq} && C_1^2 \cdot \left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{-\frac{1}{2}} \cdot \frac{2}{\alpha}\left[1 + 2^{2+\alpha}\right] \cdot f_a(z) \\
&= && \frac{2C_1^2}{\alpha}\left[1 + 2^{2+\alpha}\right] \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{\frac{1}{2}} \cdot \left(1 + \frac{|c_{\lambda'} - c_\lambda|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-(1+\alpha)} \\
&=: && M_1.
\end{aligned}
$$

For symmetry reasons we get the same result in the opposite case. Properties a),b),c) imply the second estimate

$$
\begin{aligned}
|\langle \psi_\lambda, \psi_{\lambda'} \rangle| &\overset{a)}{=} \left| \int_{\mathbb{R}} [\psi_\lambda(x) - \psi_\lambda(c_{\lambda'})] \cdot \overline{\psi_{\lambda'}(x)}\, dx \right| \\
&\overset{b),c)}{\leq} C_1 C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |I_{\lambda'}|^{-\frac{1}{2}+\beta} \cdot \\
&\qquad \cdot \int_{\mathbb{R}} \left(|I_{\lambda'}|^{-1} \cdot |x - c_{\lambda'}|\right)^\beta \cdot \left[1 + |I_{\lambda'}|^{-1} \cdot |x - c_{\lambda'}|\right]^{-(1+\alpha)} dx \\
&\leq C_1 C_2 \cdot \left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{\frac{1+2\beta}{2}} \int_{\mathbb{R}} \left[1 + |I_{\lambda'}|^{-1} \cdot |x - c_{\lambda'}|\right]^{-(1+\alpha-\beta)} \cdot |I_{\lambda'}|^{-1} dx \\
&= \frac{2C_1C_2}{\alpha - \beta} \cdot \left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{\frac{1+2\beta}{2}}.
\end{aligned}
$$

Because of the symmetry of this expression we finally get

$$
|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \leq \frac{2C_1C_2}{\alpha - \beta} \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{\frac{1+2\beta}{2}} =: M_2.
$$

Putting $M_3 := M_2^\theta \cdot M_1^{1-\theta}$ with M_1 and M_2 from above and $\theta := \frac{\alpha}{1+\alpha+2\beta} \in (0,1)$ implies the desired property for the Gram matrix with ε given in (5.15) and constant C from (5.16). □

Combining Proposition 5.22 with Theorem 5.24 we get the following result.

Proposition 5.25 *(Localization property for a function family with unbounded supports)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(\mathbb{R})$ with (geometrical or abstract) essential supports $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$, satisfying conditions a)–c) in Theorem 5.24. With ε and C from Theorem 5.24 there holds

$$
|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \leq 2^{1+\varepsilon} C \cdot e^{-(1+\varepsilon)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})} \qquad \textit{for all } (\lambda, \lambda') \in \Lambda^2.
$$

In the sequel we will see how much the proof is simplified in the case of compact supports as compared to the one of unbounded supports.

Theorem 5.26 *(Sufficient conditions for the almost diagonality of a Gram matrix)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(X)$, $X \in \{\mathbb{R}, (-\infty, b], [a, \infty), [a, b]\}$, with compact supports

$$
\mathit{supp}\, \psi_\lambda \subseteq [c_\lambda, b_\lambda] =: I_\lambda, \qquad \lambda \in \Lambda,
$$

satisfying the following conditions:

a) $\displaystyle\int_{I_\lambda} \psi_\lambda(x)\, dx = 0$ *for all* $\lambda \in \Lambda$ (*vanishing moment*);

b) $\exists C_1 > 0 \quad \forall \lambda \in \Lambda : \qquad \|\psi_\lambda\|_\infty \le C_1 \cdot |I_\lambda|^{-1/2}$ (*boundedness*);

c) $\exists \beta > 0 \quad \exists C_2 > 0 \quad \forall \lambda \in \Lambda \quad \forall x, x' \in I_\lambda \quad (x \neq x') :$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \le C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |x - x'|^\beta \qquad (\textit{Hölder continuity}).$$

Then $Gram(\Psi)$ *is an almost diagonal matrix on* $l_2(\Lambda)$ *w.r.t.* $\{I_\lambda\}_{\lambda\in\Lambda}$ *with exponent* $\varepsilon = 2\beta$ *and constant* $C = \frac{C_1 C_2}{\beta+1}$*, i.e., for all* $(\lambda, \lambda') \in \Lambda^2$ *we have*

$$|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \begin{cases} \le \dfrac{C_1 C_2}{\beta+1} \cdot \min\left\{ \dfrac{|I_{\lambda'}|}{|I_\lambda|}, \dfrac{|I_\lambda|}{|I_{\lambda'}|} \right\}^{(1+2\beta)/2} & \textit{if} \quad \mathring{I}_\lambda \cap \mathring{I}_{\lambda'} = \emptyset, \\ = 0 & \textit{otherwise.} \end{cases}$$

$\mathring{I}_\lambda$ *denotes the interior of the interval* I_λ.

Proof. For all indices $\lambda, \lambda' \in \Lambda$ with $\mathring{I}_\lambda \cap \mathring{I}_{\lambda'} = \emptyset$ we have $a_{\lambda,\lambda'} := \langle \psi_\lambda, \psi_{\lambda'} \rangle = 0$ and we can therefore replace in our subsequent considerations the first parentheses in (5.12) by 1 (see also Remark 5.21). For indices $\lambda, \lambda' \in \Lambda$ with $\mathring{I}_\lambda \cap \mathring{I}_{\lambda'} \neq \emptyset$, we get

$$\begin{aligned} |\langle \psi_\lambda, \psi_{\lambda'} \rangle| &= \left| \int_{I_\lambda \cap I_{\lambda'}} \psi_\lambda(x) \cdot \overline{\psi_{\lambda'}(x)}\, dx \right| \\ &\overset{a)}{=} \left| \int_{I_\lambda \cap I_{\lambda'}} [\psi_\lambda(x) - \psi_\lambda(c_{\lambda'})] \cdot \overline{\psi_{\lambda'}(x)}\, dx \right| \\ &\le \int_{I_\lambda \cap I_{\lambda'}} |\psi_\lambda(x) - \psi_\lambda(c_{\lambda'})| \cdot |\psi_{\lambda'}(x)|\, dx \\ &\overset{b),c)}{\le} C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot C_1 \cdot |I_{\lambda'}|^{-1/2} \int_{I_\lambda \cap I_{\lambda'}} |x - c_{\lambda'}|^\beta\, dx. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |\langle \psi_\lambda, \psi_{\lambda'} \rangle| &\le C_1 C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |I_{\lambda'}|^{-1/2} \int_{I_{\lambda'}} (x - c_{\lambda'})^\beta\, dx \\ &= C_1 C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |I_{\lambda'}|^{-1/2} \cdot \frac{|I_{\lambda'}|^{\beta+1}}{\beta+1} \\ &= \frac{C_1 C_2}{\beta+1} \cdot \left(\frac{|I_{\lambda'}|}{|I_\lambda|} \right)^{(1+2\beta)/2}. \end{aligned}$$

By symmetry we also get

$$|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \le \frac{C_1 C_2}{\beta+1} \cdot \left(\frac{|I_\lambda|}{|I_{\lambda'}|} \right)^{(1+2\beta)/2},$$

which yields the desired estimate:

$$|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \le \frac{C_1 C_2}{\beta+1} \cdot \min\left\{ \frac{|I_{\lambda'}|}{|I_\lambda|}, \frac{|I_\lambda|}{|I_{\lambda'}|} \right\}^{(1+2\beta)/2}. \qquad \square$$

Combining Proposition 5.22 with Theorem 5.26 we obtain the following localization property.

Corollary 5.27 *(Localization property for a function family with compact supports)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(X)$, $X \in \{I\!R, (-\infty, b], [a, \infty), [a, b]\}$, with compact supports $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$.
If Ψ satisfies conditions a)–c) in Theorem 5.26, then we have

$$|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \le 2^{1+2\beta} \cdot \frac{C_1C_2}{\beta+1} \cdot e^{-(1+2\beta)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})} \qquad \text{for all } (\lambda, \lambda') \in \Lambda^2,$$

with distance function $d_{\mathcal{I}}$ defined in (5.13).

In the following example we present a function family which satisfies the conditions in Theorem 5.26 for the case $L_2[a, b]$.

Example 5.28 *(Exponentially localized spline family)*
We consider the non–stationary spline MRA on the bounded interval $[a, b]$ generated by the dense sequence of finite knot vectors $\mathbf{t}_0 \subset \cdots \subset \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \cdots \subset [a, b]$ as defined in Chapter 3. Each $\mathbf{t}_j$ has N_j interior knots of multiplicity **at most $(\mathbf{m-1})$** *and stacked boundary knots of maximal multiplicity m ($m \ge 2$).*

The L_2–normalized B–splines of order m over the knot sequence $\mathbf{t}_j$ are denoted as usual by $[N^B_{\mathbf{t}_j;m,k}]_{k\in I\!N_j}$ with $I\!N_j := \{-m+1, \ldots, N_j\}$, and weighted knot differences are defined by

$$d_{\mathbf{t}_j;m,\nu,k} := \frac{t^{(j)}_{m+k+\nu} - t^{(j)}_k}{m+\nu}, \qquad 0 \le \nu \le m.$$

The family

$$\mathbf{\Phi}^B := \cup_{j\ge0}\mathbf{\Phi}^B_{\mathbf{t}_j;m} := \cup_{j\ge0}[N^B_{\mathbf{t}_j;m,k}]_{k\in I\!N_j}$$

contains the building blocks we need for our construction of an exponentially localized family of compactly supported functions.

Candidates for the family $\mathbf{\Psi}$ in Theorem 5.26 are given by

$$\psi_{j,k}(x) := norm_{j,k} \cdot \left(N^B_{\mathbf{t}_j;m+1,k}\right)'(x), \quad k \in \{-m+1, \ldots, N_j - 1\}, \quad j \ge 0, \tag{5.17}$$

with normalization

$$norm_{j,k} := \min\{d_{\mathbf{t}_j;m,0,k}; d_{\mathbf{t}_j;m,0,k+1}\} \cdot \frac{\min\{d_{\mathbf{t}_j;m-1,0,k}; d_{\mathbf{t}_j;m-1,0,k+1}; d_{\mathbf{t}_j;m-1,0,k+2}\}}{d_{\mathbf{t}_j;m,1,k}}, \tag{5.18}$$

where the second minimum is considered only over the non–zero elements[4]*. In this case $\mathbf{\Psi}$ is indexed by $\lambda := (j, k)$ from the countable index set*

$$\Lambda := \cup_{j\in I\!N_0}\left(\{j\} \times \{-m+1, \ldots, N_j - 1\}\right)$$

and has thus the structure $\mathbf{\Psi} = \{\mathbf{\Psi}_{\mathbf{t}_j}\}_{j\ge0} = \{[\psi_{j,k}]_k\}_{j\ge0}$. Obviously, each $\psi_{j,k}$ has one vanishing moment, is continuous and compactly supported with support $I_{j,k} = [t^{(j)}_k, t^{(j)}_{k+m+1}]$. The spline functions from (5.17) can be computed by

$$\mathbf{\Psi}_{\mathbf{t}_j} = \mathbf{\Phi}^B_{\mathbf{t}_j;m} \cdot E^B_{\mathbf{t}_j;m,1} \cdot diag\left(norm(\mathbf{t}_j)\right), \qquad j \ge 0,$$

[4]At the boundary of each knot sequence $\mathbf{t}_j$ we considered stacked knots of multiplicity m and thus we always obtain $d_{\mathbf{t}_j;m-1,0,-m+1} = 0$ and $d_{\mathbf{t}_j;m-1,0,N+1} = 0$. We do not consider these numbers when determining the second *min* in (5.18). This is justified by the fact that the corresponding terms in the formula for the first derivative of B–splines are set by convention equal to zero (see Section 2.5).

where $E^B_{\mathbf{t}_j;m,1}$ *are the differentiation matrices defined in Section 2.5 and*

$$norm(\mathbf{t}_j) := [norm_{j,k}]_{k\in\{-m+1,\dots,N_j-1\}}.$$

For $m = 4$ *and* $\mathbf{t}$ *defined in (2.11) we obtain the cubic spline family* $\mathbf{\Psi_t}$ *with* 8 *members depicted in Figure 5.12. For* $m = 4$ *and the refinement* $\tilde{\mathbf{t}}$ *of* $\mathbf{t}$ *defined in (2.26) we obtain the cubic spline family* $\mathbf{\Psi}_{\tilde{\mathbf{t}}}$ *with* 15 *elements visualized in Figure 5.15. For these two concrete situations the influence of the normalization factors*

$$\begin{aligned} norm(\mathbf{t}) &= [\,0.046\ \ 0.034\ \ 0.104\ \ 0.104\ \ 0.083\ \ 0.115\ \ 0.138\ \ 0.092\,], \\ norm(\tilde{\mathbf{t}}) &= [\,0.031\ \ 0.039\ \ 0.042\ \ 0.019\ \ 0.012\ \ 0.031\ \ 0.097\ \ 0.031 \\ &\qquad 0.017\ \ 0.021\ \ 0.069\ \ 0.065\ \ 0.058\ \ 0.035\ \ 0.069\,], \end{aligned}$$

on the first order derivatives on the right hand side of (5.17) can be noticed in Figures 5.13 and 5.16, respectively.

The estimate

$$\begin{aligned} |\psi_{j,k}(x)| &= norm_{j,k}\cdot\left|\left(N^B_{\mathbf{t}_j;m+1,k}\right)'(x)\right| \\ &\overset{(5.18),(2.37),(2.34)}{\leq} \frac{m+1}{m-1}\cdot\min\{d_{\mathbf{t}_j;m,0,k};d_{\mathbf{t}_j;m,0,k+1}\}\cdot d^{-1/2}_{\mathbf{t}_j;m,1,k} \\ &\qquad\cdot\left|d^{-1}_{\mathbf{t}_j;m,0,k}\cdot N_{\mathbf{t}_j;m,k}(x) - d^{-1}_{\mathbf{t}_j;m,0,k+1}\cdot N_{\mathbf{t}_j;m,k+1}(x)\right| \\ &\leq \frac{(m+1)^{3/2}}{m-1}\cdot|I_{j,k}|^{-1/2}\cdot\min\{d_{\mathbf{t}_j;m,0,k};d_{\mathbf{t}_j;m,0,k+1}\} \\ &\qquad\cdot\max\left\{d^{-1}_{\mathbf{t}_j;m,0,k}\cdot N_{\mathbf{t}_j;m,k}(x)\ ;\ d^{-1}_{\mathbf{t}_j;m,0,k+1}\cdot N_{\mathbf{t}_j;m,k+1}(x)\right\} \\ &\leq \frac{(m+1)^{3/2}}{m-1}\cdot|I_{j,k}|^{-1/2} \end{aligned}$$

provides the boundedness of $\psi_{j,k}$ *with* $C_1 = \frac{(m+1)^{3/2}}{m-1} = \mathcal{O}\left(m^{1/2}\right)$*. The Hölder continuity of* $\psi_{j,k}$ *is given by*

$$\begin{aligned} |\psi_{j,k}(x)-\psi_{j,k}(x')| &= norm_{j,k}\cdot\left|\left(N^B_{\mathbf{t}_j;m+1,k}\right)'(x) - \left(N^B_{\mathbf{t}_j;m+1,k}\right)'(x')\right| \\ &\leq norm_{j,k}\cdot\left\|\left(N^B_{\mathbf{t}_j;m+1,k}\right)''\right\|_{L_\infty;(x,x')}\cdot|x-x'| \end{aligned} \tag{5.19}$$

(where $\|\cdot\|_{L_\infty;(x,x')}$ *stands for the essential supremum on* (x,x')*) with*

$$norm_{j,k}\cdot\left\|\left(N^B_{\mathbf{t}_j;m+1,k}\right)''\right\|_{L_\infty;(x,x')} \overset{(5.18),(2.37),(2.34)}{=}$$

$$\min\{d_{\mathbf{t}_j;m,0,k};d_{\mathbf{t}_j;m,0,k+1}\}\cdot\min\{d_{\mathbf{t}_j;m-1,0,k};d_{\mathbf{t}_j;m-1,0,k+1};d_{\mathbf{t}_j;m-1,0,k+2}\}\cdot d^{-3/2}_{\mathbf{t}_j;m,1,k}$$

$$\cdot\Big\|d^{-1}_{\mathbf{t}_j;m,0,k}\cdot\left[d^{-1}_{\mathbf{t}_j;m-1,0,k}\cdot N_{\mathbf{t}_j;m-1,k} - d^{-1}_{\mathbf{t}_j;m-1,0,k+1}\cdot N_{\mathbf{t}_j;m-1,k+1}\right]$$

$$-d^{-1}_{\mathbf{t}_j;m,0,k+1}\cdot\left[d^{-1}_{\mathbf{t}_j;m-1,0,k+1}\cdot N_{\mathbf{t}_j;m-1,k+1} - d^{-1}_{\mathbf{t}_j;m-1,0,k+2}\cdot N_{\mathbf{t}_j;m-1,k+2}\right]\Big\|_{\infty;(x,x')}$$

$$\leq \min\{d_{\mathbf{t}_j;m,0,k};d_{\mathbf{t}_j;m,0,k+1}\}\cdot\min\{d_{\mathbf{t}_j;m-1,0,k};d_{\mathbf{t}_j;m-1,0,k+1};d_{\mathbf{t}_j;m-1,0,k+2}\}\cdot d^{-3/2}_{\mathbf{t}_j;m,1,k}$$

$$\cdot 2\cdot\max\left\{\left\|d^{-1}_{\mathbf{t}_j;m,0,k}\cdot\left[d^{-1}_{\mathbf{t}_j;m-1,0,k}\cdot N_{\mathbf{t}_j;m-1,k} - d^{-1}_{\mathbf{t}_j;m-1,0,k+1}\cdot N_{\mathbf{t}_j;m-1,k+1}\right]\right\|_{\infty;(x,x')},\right.$$

$$\left\| d^{-1}_{\mathbf{t}_j;m,0,k+1} \cdot \left[d^{-1}_{\mathbf{t}_j;m-1,0,k+1} \cdot N_{\mathbf{t}_j;m-1,k+1} - d^{-1}_{\mathbf{t}_j;m-1,0,k+2} \cdot N_{\mathbf{t}_j;m-1,k+2} \right] \right\|_{\infty;(x,x')} \Bigg\}$$

$$\overset{N_{\mathbf{t};m,\nu,k}(x)\geq 0}{\leq} \quad 2 \cdot d^{-3/2}_{\mathbf{t}_j;m,1,k} \cdot$$

$$\cdot \min\{d_{\mathbf{t}_j;m,0,k}; d_{\mathbf{t}_j;m,0,k+1}\} \cdot \min\{d_{\mathbf{t}_j;m-1,0,k}; d_{\mathbf{t}_j;m-1,0,k+1}; d_{\mathbf{t}_j;m-1,0,k+2}\} \cdot$$

$$\cdot \max\Bigg\{ \max\Big\{ \left\| d^{-1}_{\mathbf{t}_j;m,0,k} \cdot d^{-1}_{\mathbf{t}_j;m-1,0,k} \cdot N_{\mathbf{t}_j;m-1,k} \right\|_{\infty;(x,x')}, \left\| d^{-1}_{\mathbf{t}_j;m,0,k} \cdot d^{-1}_{\mathbf{t}_j;m-1,0,k+1} \cdot N_{\mathbf{t}_j;m-1,k+1} \right\|_{\infty;(x,x')} \Big\},$$

$$\max\Big\{ \left\| d^{-1}_{\mathbf{t}_j;m,0,k+1} \cdot d^{-1}_{\mathbf{t}_j;m-1,0,k+1} \cdot N_{\mathbf{t}_j;m-1,k+1} \right\|_{\infty;(x,x')}, \left\| d^{-1}_{\mathbf{t}_j;m,0,k+1} \cdot d^{-1}_{\mathbf{t}_j;m-1,0,k+2} \cdot N_{\mathbf{t}_j;m-1,k+2} \right\|_{\infty;(x,x')} \Big\} \Bigg\}$$

$$\overset{N_{\mathbf{t};m,\nu,k}(x)\leq 1}{\leq} \quad 2 \cdot (m+1)^{3/2} \cdot |I_{j,k}|^{-3/2} \cdot \max\{\max\{1,1\}, \max\{1,1\}\}$$

$$= 2 \cdot (m+1)^{3/2} \cdot |I_{j,k}|^{-3/2}.$$

Thus $\beta = 1$ *and* $C_2 = 2 \cdot (m+1)^{3/2} = \mathcal{O}\left(m^{3/2}\right)$.

Note that the estimate (5.19) holds even in cases when a knot has multiplicity $m-1$. *This maximal multiplicity implies the continuity and the piecewise differentiability of the first derivative of the B–spline of order* $m+1$ *(A concrete situation is depicted in Figure 5.14:* $m = 4$ *and* $\tilde{\mathbf{t}}$ *from (2.26) with* $t_6 = t_7 = t_8 = \frac{1}{2}$*).*

The above computations in combination with Theorem 5.26 imply

$$\varepsilon = 2\beta = 2,$$
$$C = \frac{C_1 C_2}{\beta + 1} = \frac{1}{2} \cdot \frac{(m+1)^{3/2}}{m-1} \cdot 2(m+1)^{3/2} = \frac{(m+1)^3}{m-1} = \mathcal{O}\left(m^2\right).$$

Thus the entries of the Gramian associated to $\mathbf{\Psi}$ *exhibit almost diagonality in the following fashion:*

$$|\langle \psi_{j,k}, \psi_{j',k'} \rangle| \leq \frac{(m+1)^3}{m-1} \cdot \min\left\{ \frac{|I_{j,k}|}{|I_{j',k'}|}, \frac{|I_{j',k'}|}{|I_{j,k}|} \right\}^{3/2}. \tag{5.20}$$

Furthermore, taking into account Proposition 5.27, the exponential localization is expressed in this spline case by

$$\begin{aligned} |\langle \psi_{j,k}, \psi_{j',k'} \rangle| &\leq 8 \cdot \frac{(m+1)^3}{m-1} \cdot e^{-3 \cdot d_{\mathcal{I}}(I_{j,k}, I_{j',k'})}, \qquad \textit{with} \\ \mathcal{I} &= \{I_{j,k}\}_{(j,k) \in \bigcup_{j \in \mathbb{N}_0} (\{j\} \times \{-m+1, \ldots, N_j - 1\})} \\ &= \left\{[t^{(j)}_k, t^{(j)}_{k+m+1}]\right\}_{(j,k) \in \bigcup_{j \in \mathbb{N}_0} (\{j\} \times \{-m+1, \ldots, N_j - 1\})}. \end{aligned} \tag{5.21}$$

For splines of low orders we get the following concrete constants on the right hand sides of Theorem 5.26.b), Theorem 5.26.c), (5.20) and (5.21), respectively.

		$C_1 = \frac{(m+1)^{3/2}}{m-1}$	$C_2 = 2(m+1)^{3/2}$	$C = \frac{(m+1)^3}{m-1}$	$8 \cdot \frac{(m+1)^3}{m-1}$
linear splines	$m = 2$	$\sim 5,2$	$\sim 10,39$	27	216
quadratic splines	$m = 3$	4	16	32	256
cubic splines	$m = 4$	$\sim 3,73$	$\sim 22,36$	$41,\bar{6}$	$333,\bar{3}$
quartic splines	$m = 5$	$\sim 3,67$	$\sim 29,39$	54	432
quintic splines	$m = 6$	$\sim 3,70$	$\sim 37,04$	68,6	548,8

□

We summarize the results of this section as follows:
We have proved for two types of function families the almost diagonality of the associated Gramian. We have thus carried over the first step of Meyer's and Frazier&Jawerth's scheme to the non–stationary one–dimensional case (see Remark 5.19).
For the second step, namely the boundedness of the operator associated to the Gram matrix in the non–stationary case extra tools have to be designed in order to be able to proceed further. This will be done in Sections 5.6 and 5.7, after shortly recalling the analogous concepts for the Gabor case.

5.6 Separation concept

The most important ingredient for our extension from the stationary to the non–stationary setting in the wavelet case is a separation concept for the irregularly distributed localization points of the function family under discussion. Such concepts exist for the Gabor setting in earlier work by Young and Gröchenig; these will be briefly reviewed in the following three definitions.

We stress the fact that these concepts match only the Gabor case and cannot be carried over to the wavelet situation (see also the introductory part of Section 5.4). Our appropriate separation concept for the latter case will be presented at the end of this section.

The separation principle for a set of points on the real line or from the complex plane is used by Young [65] in the study of entire functions of exponential type.

Definition 5.29 *(see [65, Ch. 2])*
A sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ of real or complex numbers is said to be separated if for some positive number δ

$$|\lambda_n - \lambda_m| \geq \delta \quad \text{whenever } n \neq m.$$

Because of the separation property the set of points is nowhere dense.

In his non–stationary localization theory for Gabor frames Gröchenig used exactly this concept as an essential ingredient. He employs points λ_n as indices of the function family to be discussed and formulates the separation condition for index sets relative to the Gabor tiling of the time–frequency plane as follows.

Definition 5.30 *(see, e.g., [20])*
Let $\Lambda \subset \mathbb{R}^d$ be a countable index set.

a) Λ is called separated if $\delta > 0$ exists such that

$$\inf_{\substack{\lambda,\lambda'\in\Lambda \\ \lambda\neq\lambda'}} \|\lambda - \lambda'\|_{\mathbb{R}^d} \geq \delta.$$

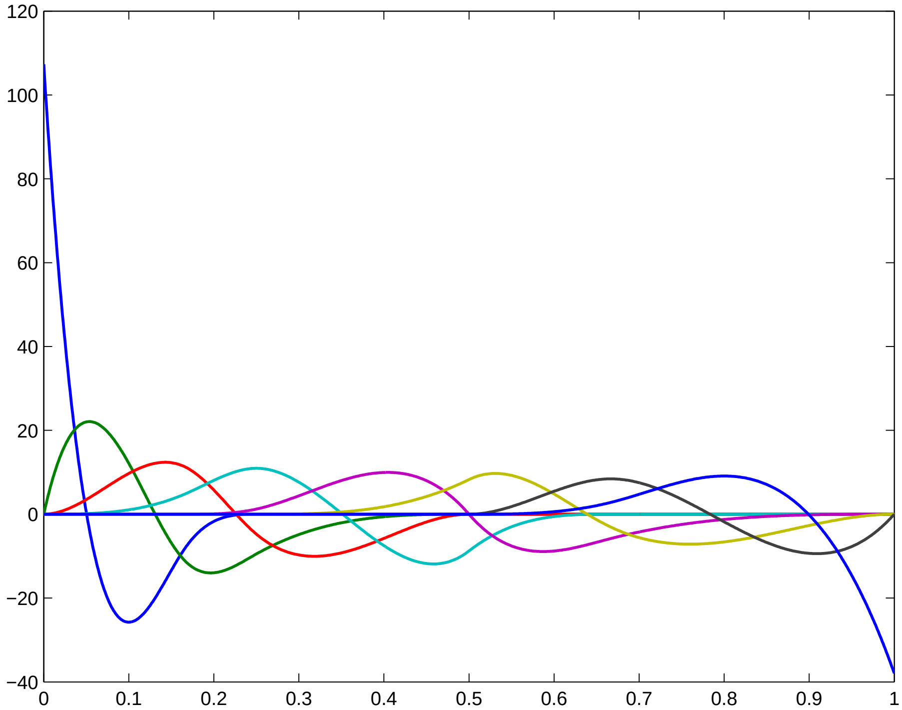

Figure 5.11: The first order derivatives on the right hand side of (5.17) for $m = 4$ and $\mathbf{t}$ from (2.11).

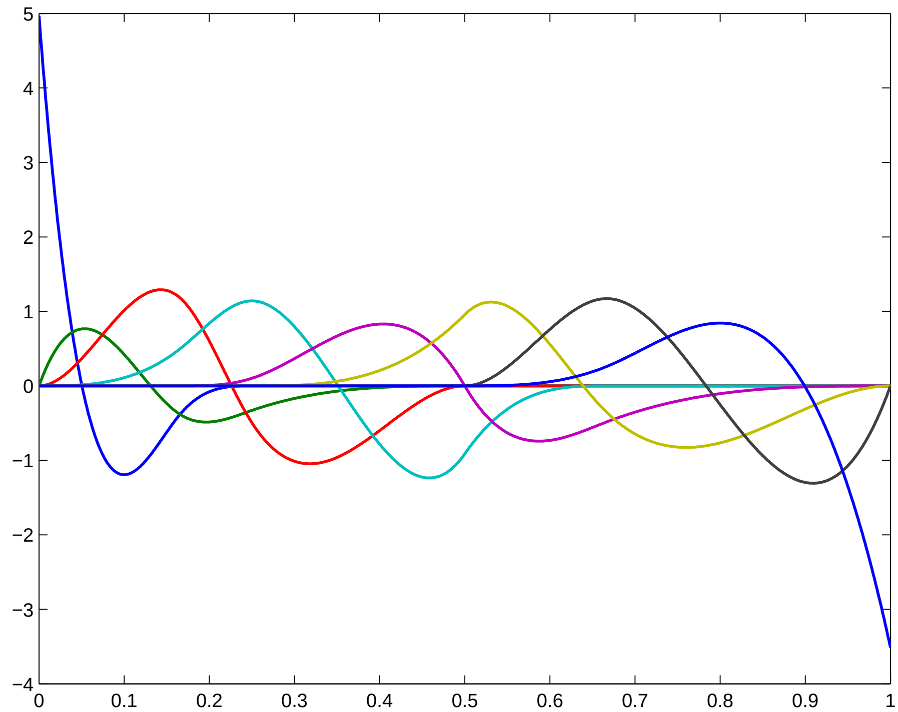

Figure 5.12: The exponentially localized spline family with one vanishing moment defined by (5.17) for $m = 4$ and $\mathbf{t}$ from (2.11). Note the influence of the normalization factors on the derivatives depicted in Figure 5.11.

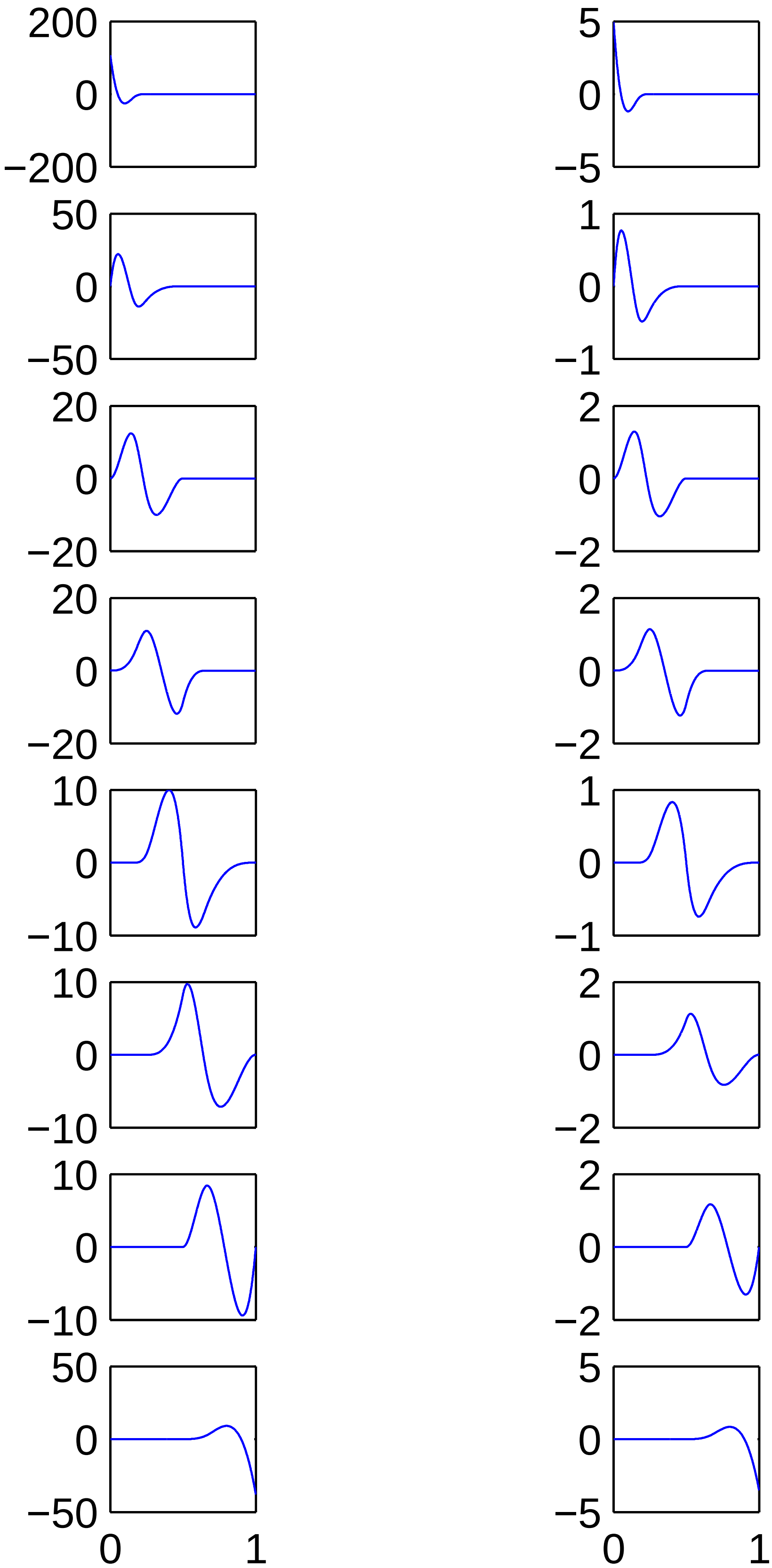

Figure 5.13: The first column shows the derivatives from Figure 5.11; the second column presents for direct comparison the splines from Figure 5.12.

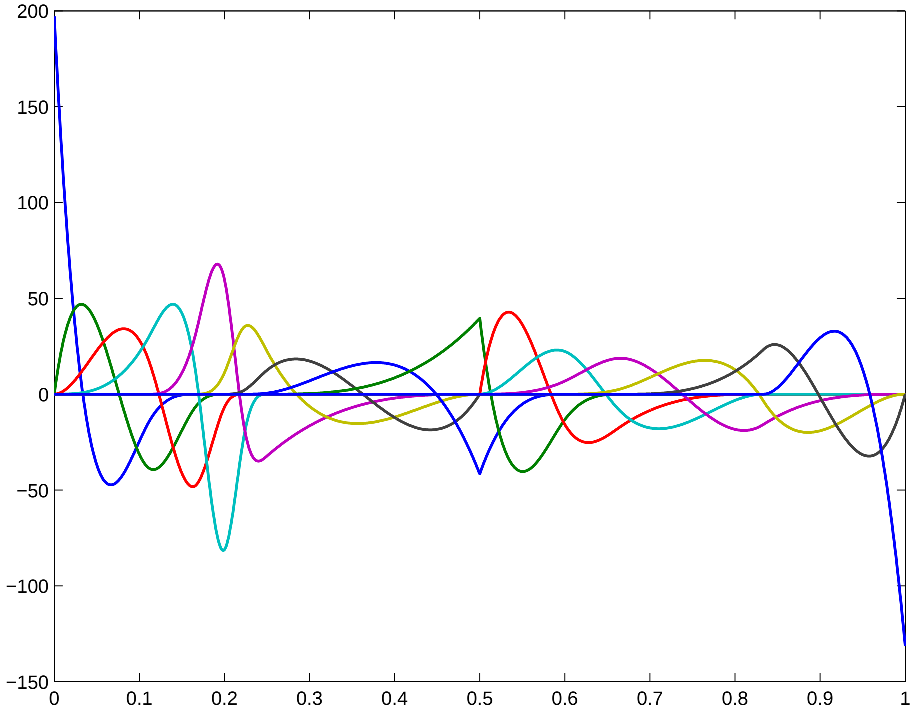

Figure 5.14: The first order derivatives on the right hand side of (5.17) for $m = 4$ and $\tilde{\mathbf{t}}$ from (2.26).

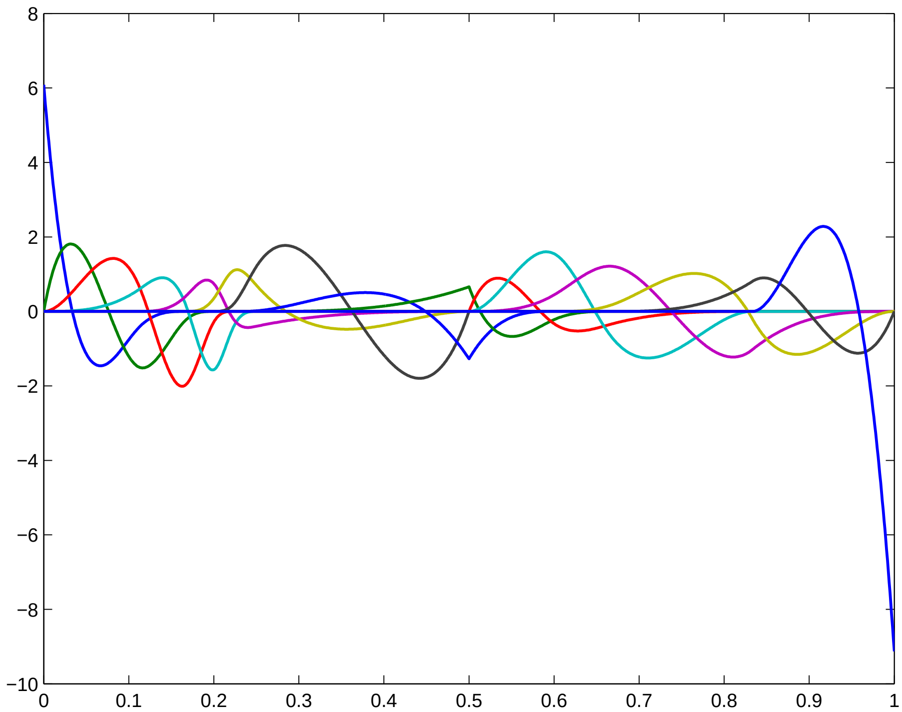

Figure 5.15: The exponentially localized spline family with one vanishing moment defined by (5.17) for $m = 4$ and $\tilde{\mathbf{t}}$ from (2.26). Note the influence of the normalization factors on the derivatives depicted in Figure 5.14.

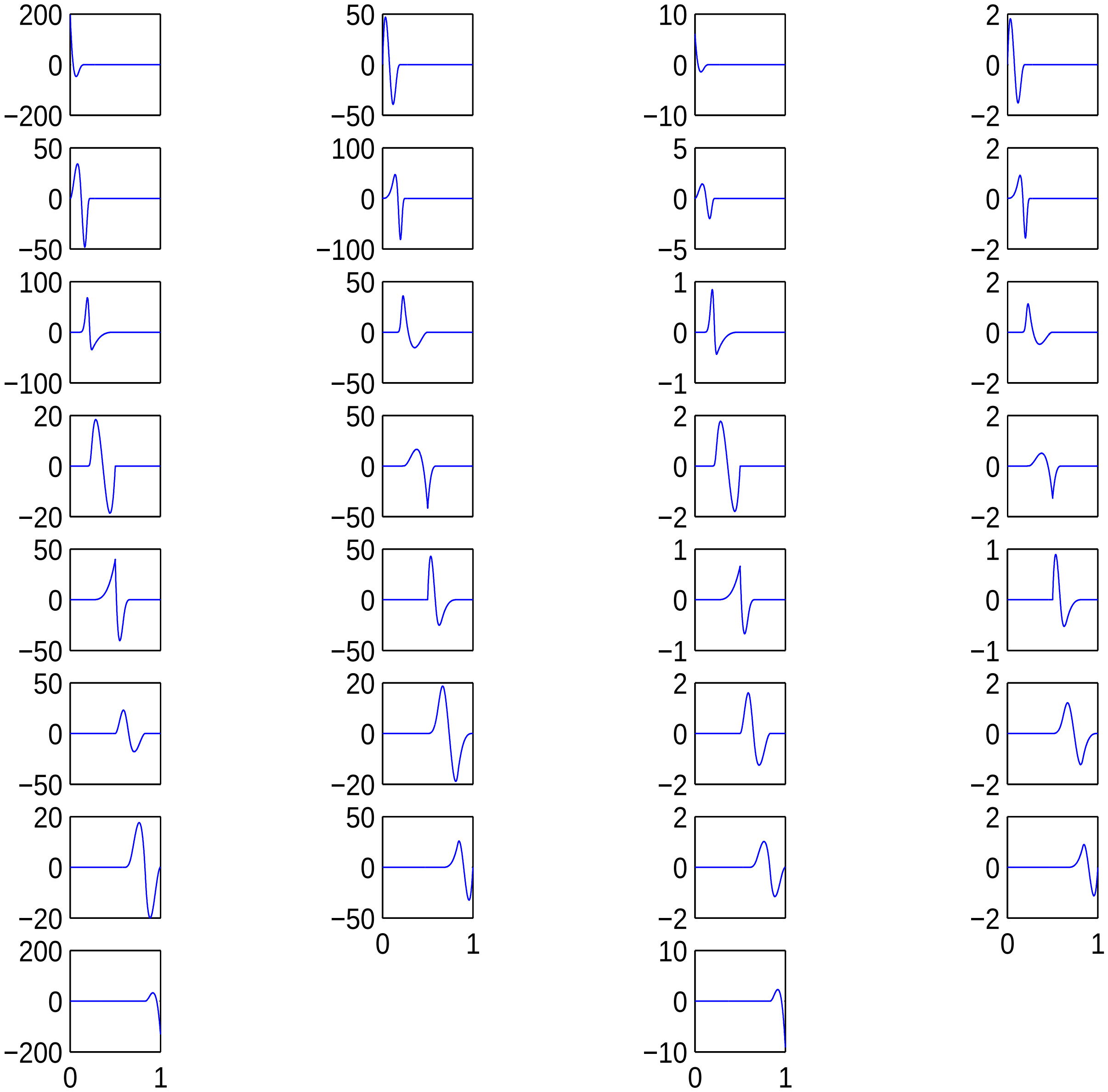

Figure 5.16: The first two columns show the derivatives from Figure 5.14; the last two columns present for direct comparison the splines from Figure 5.15.

b) Λ *is called relatively separated if it is a finite union of separated sets.*

c) *When* Λ *is used to index a family of functions on* $I\!R^d$*, the index* λ *in* f_λ *indicates that the essential support of* f_λ *is centered at* λ*.*

Sometimes Gröchenig uses an equivalent, but more geometric formulation for Definition 5.30.b).

Definition 5.31 *(see [20])*
The index set $\Lambda \subset I\!R^d$ *is called relatively separated if*

$$\sup_{k \in \mathbb{Z}^d} card \left\{ \lambda \in \Lambda \ : \ \lambda \in k + [0,1]^d \right\} =: \nu < \infty.$$

For our purpose we need a new separation concept relative to the wavelet tiling of the scale–time space. It will be introduced next.

Definition 5.32 *(Separation for the wavelet case)*
A countable family of compact intervals $\mathcal{I} = \{I_\lambda\}_{\lambda \in \Lambda}$ *with* $I_\lambda = [c_\lambda, b_\lambda]$ *is called relatively separated, if there exists a finite overlapping constant for these intervals, i.e,*

$$\exists D_2 > 0 \quad \forall J \subset I\!R \quad \textit{bounded interval:} \quad \#\Lambda_J \le D_2,$$

$$\textit{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \ : \ |I_\lambda| \in \left[\frac{|J|}{2}, |J| \right], \ c_\lambda \in J \right\}. \tag{5.22}$$

This concept enables us to prove in the next section a boundedness result for almost diagonal matrices. By looking at the proof of this result thoroughly, it can be seen that the existence of an overlapping constant is indispensable for proving that certain infinite sums are finite and that the condition $c_\lambda \in J$ can be replaced by $I_\lambda \cap J \neq \emptyset$.

Example 5.33 *(Overlapping constants)*
We presented in Example 5.9 a dyadic stationary vaguelettes family with compact support generated by dilation, translation and normalization from the generatrix $F_3 : [-1.5, 1.5] \to I\!R$, $F_3(x) := x \cdot (x-1)(x+1)$ *through* $f_{j,k}(x) := 2^{dj/2} \cdot F(2^j x - k)$, $x \in I\!R$*.* F_3 *has the support* $I = [-1.5, 1.5]$ *and the affine family* $\{f_{j,k}\}_{(j,k) \in \mathbb{Z}^2}$ *possesses the corresponding supports*

$$\mathcal{I} = \left\{ I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k)) \right\}_{(j,k) \in \mathbb{Z}^2}.$$

They are not a (disjoint) partition of $I\!R$*, but overlapping. For some special cases of* J*, the overlappings (5.22) are detailed below. The corresponding figures depict the functions* $f_{j,k}$ *in order to visualize their supports* $I_{j,k}$*.*

$J = [1.5, 2.5)$	$\|J\| = 1$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [0.5, 1]$	$l = 0.75$	$\#\Lambda_J = 3$	*Fig. 5.17*
$J = [1.5, 5.5)$	$\|J\| = 4$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [2, 4]$	$l = 3$	$\#\Lambda_J = 4$	*Fig. 5.19*
$J = [1.5, 5.5]$	$\|J\| = 4$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [2, 4]$	$l = 3$	$\#\Lambda_J = 5$	*Fig. 5.19*
$J = [1.5, 7)$	$\|J\| = 5.5$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [2.75, 5.5]$	$l = 3$	$\#\Lambda_J = 6$	*Fig. 5.21*
$J = [1.5, 7.5)$	$\|J\| = 6$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [3, 6]$	$l \in \{3, 6\}$	$\#\Lambda_J = 9$	*Fig. 5.18*
$J = [1.5, 4.5]$	$\|J\| = 3$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [1.5, 3]$	$l \in \{1.5, 3\}$	$\#\Lambda_J = 10$	*Fig. 5.20*
$J = [1.25, 4.25]$	$\|J\| = 3$	$\left[\frac{\|J\|}{2}, \|J\|\right] = [1.5, 3]$	$l \in \{1.5, 3\}$	$\#\Lambda_J = 10$	*Fig. 5.22*

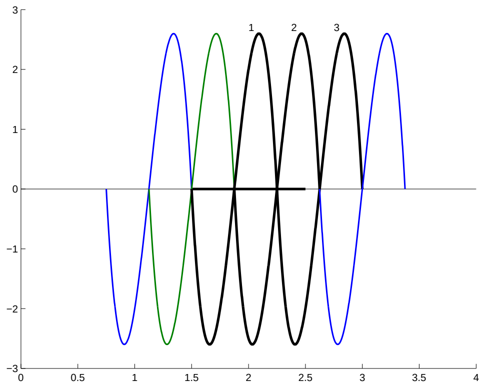

Figure 5.17: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.5, 2.5)$, $|J| = 1$, $\left[\frac{|J|}{2}, |J|\right] = [0.5, 1]$, $l = 0.75$ ($j = 2$): $\#\Lambda_J = 3$.

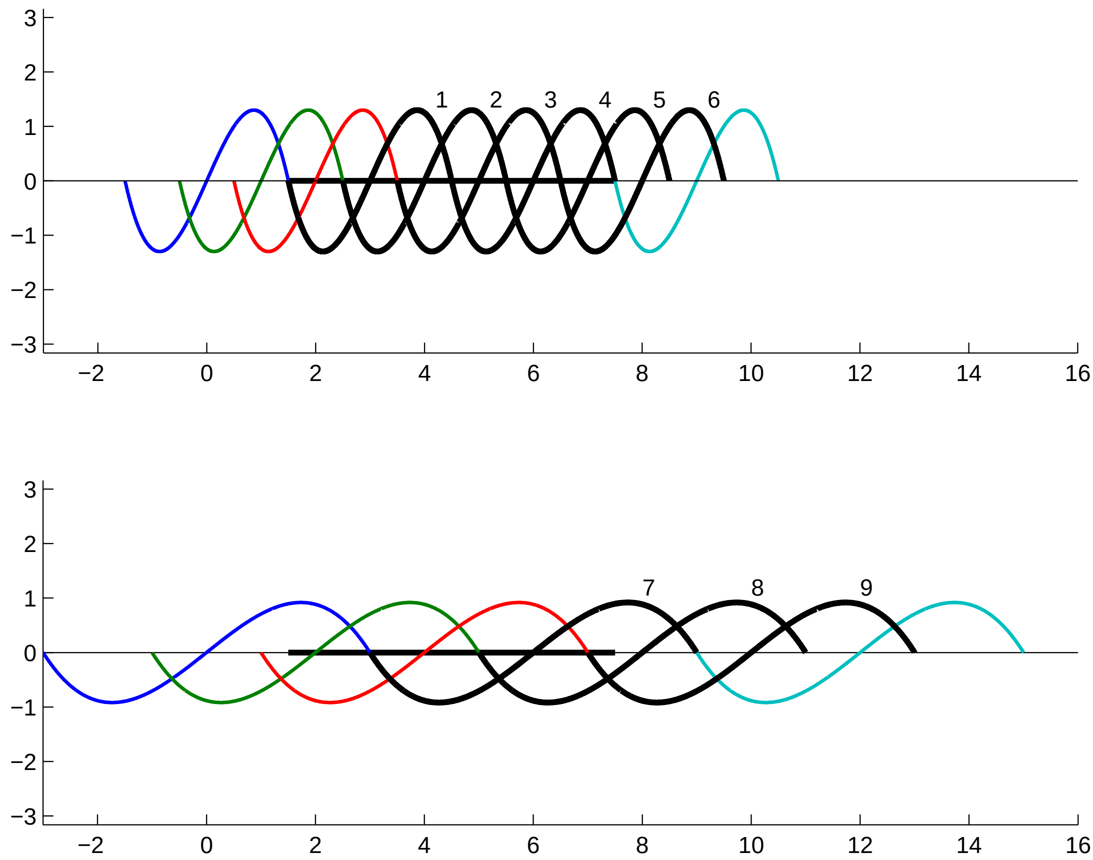

Figure 5.18: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.5, 7.5)$, $|J| = 6$, $\left[\frac{|J|}{2}, |J|\right] = [3, 6]$, $l \in \{3, 6\}$ ($j \in \{0, -1\}$): $\#\Lambda_J = 9$.

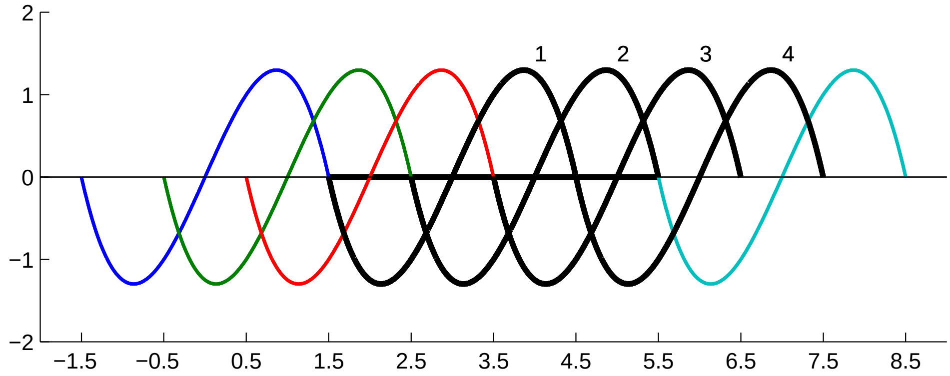

Figure 5.19: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.5, 5.5)$, $|J| = 4$, $\left[\frac{|J|}{2}, |J|\right] = [2, 4]$, $l = 3$ $(j = 0)$: $\#\Lambda_J = 4$. For $J = [1.5, 5.5]$ we obtain $\#\Lambda_J = 5$.

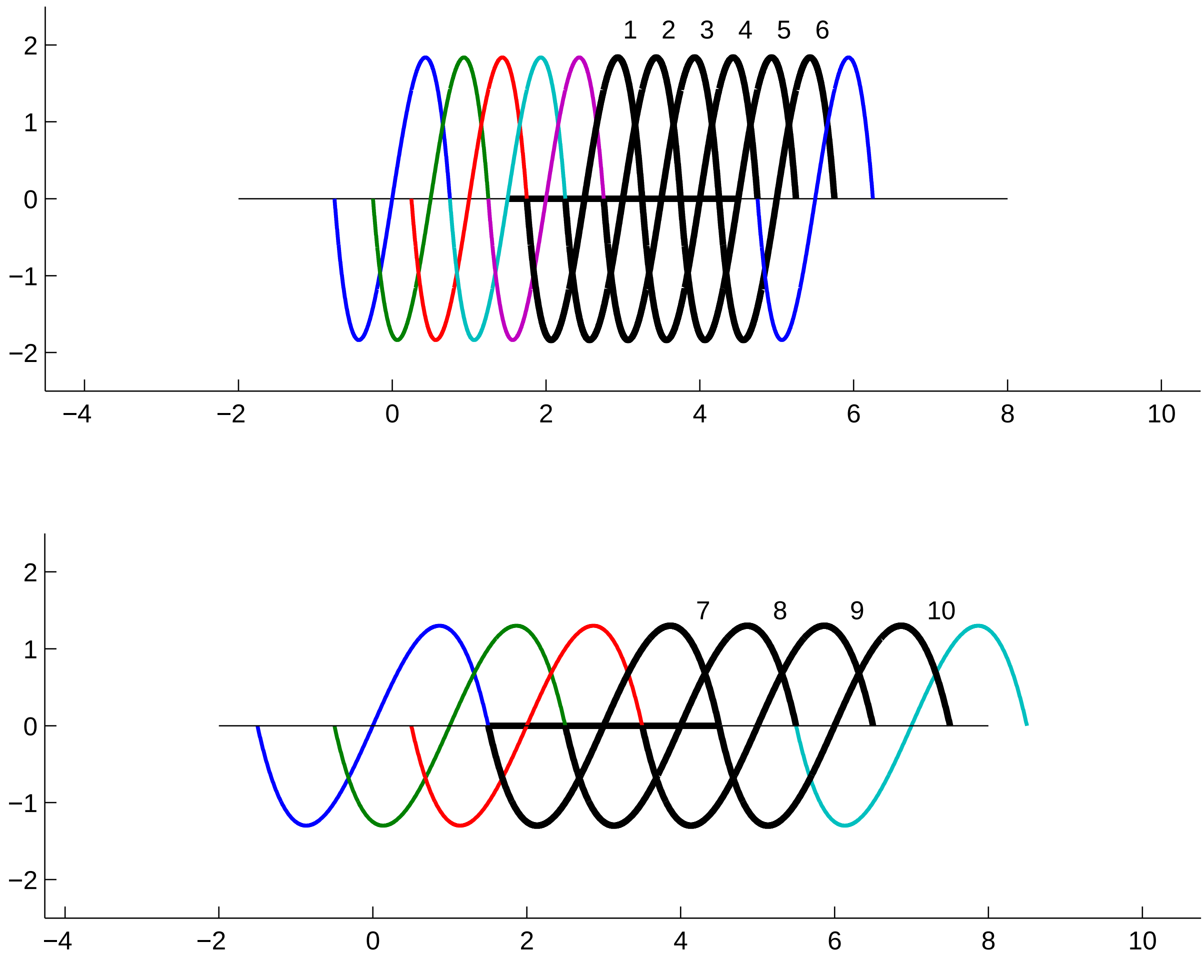

Figure 5.20: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.5, 4.5]$, $|J| = 3$, $\left[\frac{|J|}{2}, |J|\right] = [1.5, 3]$, $l \in \{1.5, 3\}$ $(j \in \{1, 0\})$: $\mathbf{\#\Lambda_J = 4 + 6 = 10 = D_2}$.

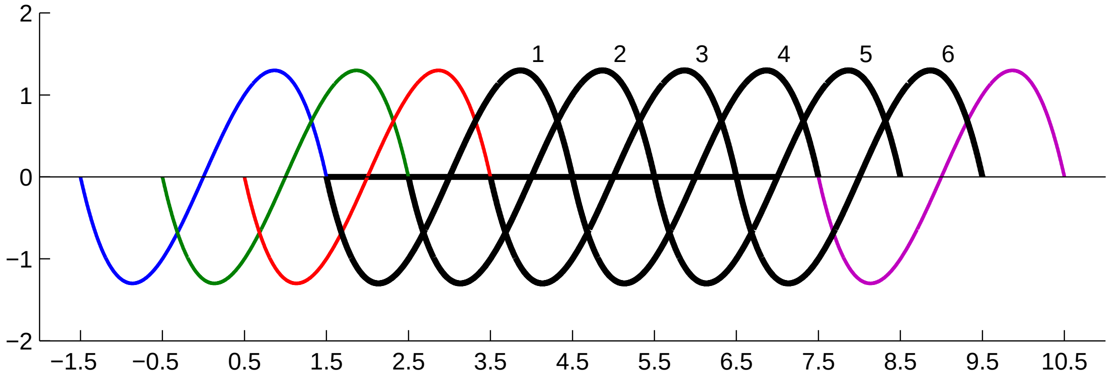

Figure 5.21: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.5, 7)$, $|J| = 5.5$, $\left[\frac{|J|}{2}, |J|\right] = [2.75, 5.5]$, $l = 3$ $(j = 0)$: $\#\Lambda_J = 6$. For $J = [1.5, 7]$ we obtain again $\#\Lambda_J = 6$.

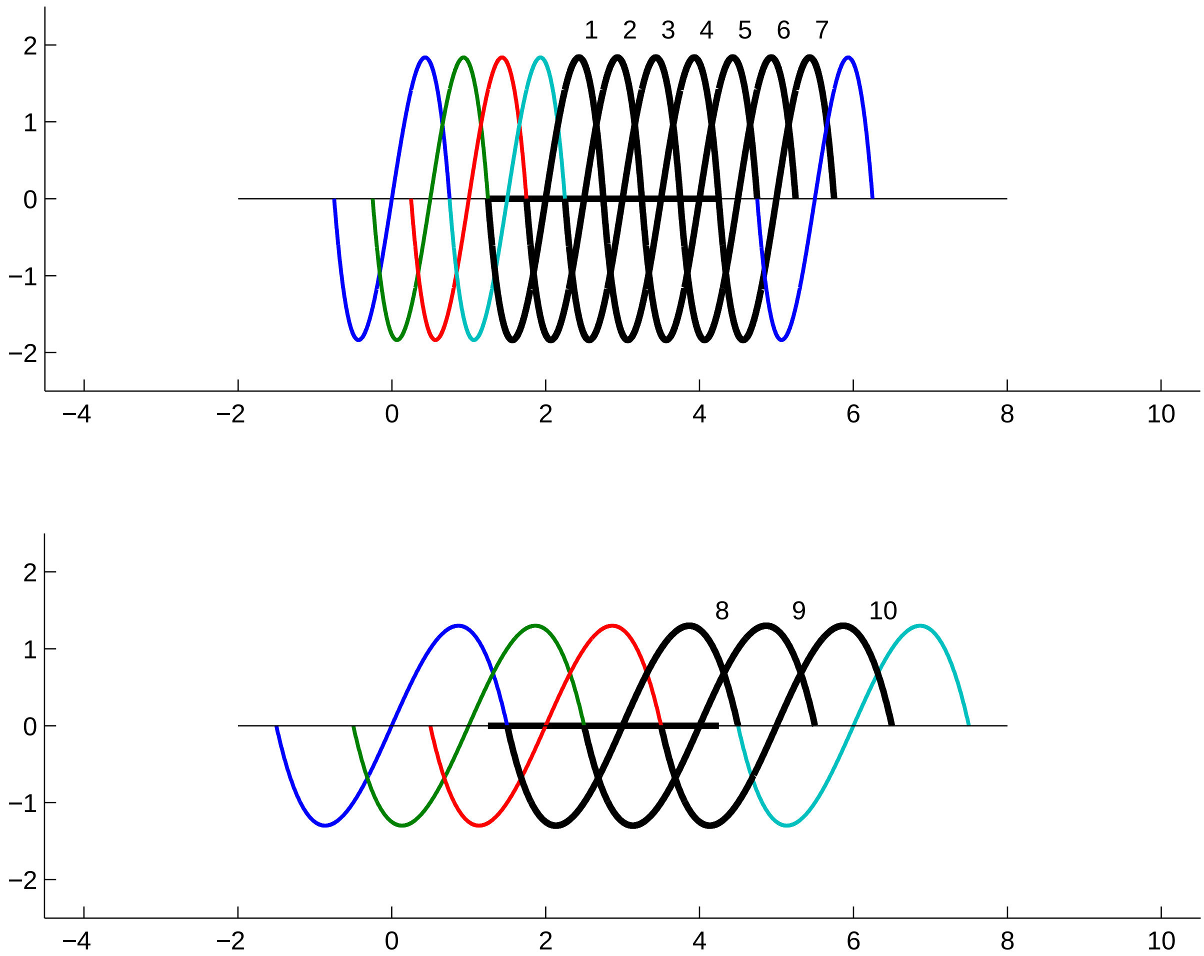

Figure 5.22: Example of overlapping constant for the family of intervals $\mathcal{I} = \{I_{j,k} = [2^{-j}(-1.5+k), 2^{-j}(1.5+k))\}_{(j,k)\in\mathbb{Z}^2}$ for the case $J = [1.25, 4.25]$, $|J| = 3$, $\left[\frac{|J|}{2}, |J|\right] = [1.5, 3]$, $l \in \{1.5, 3\}$ $(j \in \{1, 0\})$: $\#\mathbf{\Lambda_J} = \mathbf{3} + \mathbf{7} = \mathbf{10} = \mathbf{D_2}$.

In the following we prove that for this family of intervals the overlapping constant takes the value $D_2 = 10$.

Proof. *Let* $J = [c_J, b_J] \subset \mathbb{R}$ *be an arbitrary bounded interval with* $|J| \neq 0$. *In the sequel* W *will denote the set* $\{|I| \cdot 2^j \,:\, j \in \mathbb{Z}\}$ *with* $|I| = |[-1.5, 1.5]| = 3$. *We distinguish in the sequel between the following two cases.*

Case I. $|\mathbf{J}| = |\mathbf{I}| \cdot \mathbf{2^p} = \mathbf{3} \cdot \mathbf{2^p}$ **for some fixed** $\mathbf{p} \in \mathbb{Z}$. *We obtain*

$$\left[\frac{|J|}{2}, |J|\right] = [3 \cdot 2^{p-1}, 3 \cdot 2^p].$$

We are interested in counting vaguelettes $f_{j,k}$ *with*

$$length\, supp\, f_{j,k} = |I_{j,k}| = |[2^{-j}(-1.5 + k), 2^{-j}(1.5 + k)]| = 3 \cdot 2^{-j} \in \left[\frac{|J|}{2}, |J|\right]$$

and $c_{j,k} = 2^{-j}(-1.5 + k) \in J$. *This means that*

$$length\, supp\, f_{j,k} \in [3 \cdot 2^{p-1}, 3 \cdot 2^p] \cap W = \{3 \cdot 2^{p-1}, 3 \cdot 2^p\},$$

i.e., $j \in \{-p + 1, -p\}$. *Further there exists a unique* $k_0 := k_0(J) \in \mathbb{Z}$ *such that*

$$c_J \in [2^p k_0, 2^p(k_0 + 1))\,.$$

Because $|J| = 3 \cdot 2^p$ *it follows that*

$$b_J \in (2^p(k_0 + 3), 2^p(k_0 + 4))\,.$$

In this setting there exist <u>at most</u> 4 *functions* $f_{j,k}$ *with* $j = -p$, *support length* $3 \cdot 2^p$ *and corner* $2^p(-1.5 + k) \in J$, *corresponding to consecutive* k*'s from the set*

$$\{k_0 + 2, k_0 + 3, k_0 + 4, k_0 + 5\},$$

and <u>at most</u> 7 *functions* $f_{j,k}$ *with* $j = -p+1$, *support length* $3 \cdot 2^{p-1}$ *and corner* $2^{p-1}(-1.5+k) \in J$, *corresponding to consecutive values*

$$k \in \{2k_0 + 2, 2k_0 + 3, \ldots, 2k_0 + 9\}.$$

One also can observe that because of $|J| = 3 \cdot 2^p$ *it is not possible to attain in both cases the maximum (4 and 7, respectively; see Figures 5.18, 5.20 and 5.22). Thus we have either the combination* $(3; 7)$, *or* $(4; 6)$. *This yields in both cases the number* 10.

Case II. $|\mathbf{J}| \in (|\mathbf{I}| \cdot \mathbf{2^p}, |\mathbf{I}| \cdot \mathbf{2^{p+1}}) = (\mathbf{3} \cdot \mathbf{2^p}, \mathbf{6} \cdot \mathbf{2^p})$ **for some fixed** $\mathbf{p} \in \mathbb{Z}$. *We obtain*

$$\left[\frac{|J|}{2}, |J|\right] \subset (3 \cdot 2^{p-1}, 3 \cdot 2^{p+1}).$$

We are interested again in counting vaguelettes $f_{j,k}$ *with*

$$length\, supp\, f_{j,k} = |I_{j,k}| = |[2^{-j}(-1.5 + k), 2^{-j}(1.5 + k)]| = 3 \cdot 2^{-j} \in \left[\frac{|J|}{2}, |J|\right]$$

and $c_{j,k} = 2^{-j}(-1.5 + k) \in J$. *This means that*

$$length\, supp\, f_{j,k} \in (3 \cdot 2^{p-1}, 3 \cdot 2^{p+1}) \cap W = \{3 \cdot 2^p\},$$

i.e., $j = -p$. *Further there exists a unique* $k_0 := k_0(J) \in \mathbb{Z}$ *such that*

$$c_J \in [2^p k_0, 2^p(k_0+1))\,.$$

Because $|J| \in (3 \cdot 2^p, 6 \cdot 2^p)$ *it follows that*

$$b_J \in (2^p(k_0+3), 2^p(k_0+7))\,.$$

In this setting there exist <u>*at most*</u> 6 *functions* $f_{j,k}$ *with* $j = -p$*, support length* $3 \cdot 2^p$ *and corner* $2^p(-1.5+k) \in J$*, corresponding to consecutive* k*'s from the set*

$$\{k_0+2, k_0+3, k_0+4, k_0+5, k_0+6, k_0+7, k_0+8\}$$

(see Figures 5.17, 5.19 and 5.21).

Comparing the numbers 10 *(obtained in Case I) and* 6 *(obtained in Case II) we get altogether the overlapping constant* $D_2 = 10$*, which is independent of the chosen interval* J. □

5.7 General boundedness result for the non–stationary univariate setting

For the proof of the general result we need the following technical lemma.

Lemma 5.34 *(Riemann–type sum)*
Let $t \in \mathbb{R}$*,* $\gamma > 0$ *and* $I \in \{[a,b),\ (a,b],\ [a,b]\}$ *be a bounded real interval with length* $|I| \leq 1$. *The function* $f_t : \mathbb{R} \to (0,1]$ *is defined through*

$$f_t(x) := \frac{1}{(1+|x-t|)^{1+\gamma}}, \qquad x \in \mathbb{R}.$$

a) The quantity $|I| \cdot f_t(\xi)$ *can be estimated for all* $\xi \in I$ *in the following way:*

$$|I| \cdot f_t(\xi) \leq C(\gamma) \cdot \int_I f_t(x)\,dx$$

with constant

$$C(\gamma) = 2^{1+\gamma}$$

being independent of the quantities t*,* I *and* ξ.

b) In case the intervals $\{I_r\}_{r\in\mathbb{Z}}$ *constitute a partition of* $\mathbb{R}$ *and* ξ_r *denotes an intermediate point in* I_r *we get the following estimate for the corresponding Riemann–type sum of the function* f_t*:*

$$\sum_{r\in\mathbb{Z}} |I_r| \cdot f_t(\xi_r) \leq \frac{2^{2+\gamma}}{\gamma}.$$

Proof. a) Note first that f_t is a strictly increasing function on $(-\infty, t]$ and a strictly decreasing one on $[t, +\infty)$. Therefore, we discuss the following cases separately.

Case A) $\mathbf{I} \subset [\mathbf{t}, +\infty)$. Let $x \in I$ and $x_0 \in (0,1]$ be two arbitrary but fixed points. We obtain

$$\begin{aligned} \frac{f_t(x)}{f_t(x+x_0)} &= \left(\frac{1+x+x_0-t}{1+x-t}\right)^{1+\gamma} = \left(1+\frac{x_0}{1+x-t}\right)^{1+\gamma} \\ &\overset{\gamma>0}{\leq} 2^{1+\gamma}, \end{aligned}$$

which, by the positivity of f_t and translation, implies

$$f_t(x - x_0) \le 2^{1+\gamma} f_t(x), \qquad x_0 \in (0,1], \;\; x \in I. \tag{5.23}$$

Note that $|I| \le 1$ implies $b - a \in (0,1]$. Further we get

$$\begin{aligned}
2^{1+\gamma} \int_I f_t(x)\,dx \quad &\ge \quad 2^{1+\gamma} \cdot |I| \cdot \min_{x \in I} f_t(x) \\
&\overset{\text{monot.}}{=} \quad 2^{1+\gamma} \cdot |I| \cdot f_t(b) \\
&\overset{(5.23)}{\ge} \quad |I| \cdot f_t(b - (b-a)) = |I| \cdot f_t(a) \\
&\overset{\text{monot.}}{\ge} \quad |I| \cdot f_t(\xi), \qquad \forall\, \xi \in I.
\end{aligned}$$

Case B) $\mathbf{I} \subset (-\infty, \mathbf{t}]$. Symmetric arguments to the above ones lead to the desired estimate.

Case C) $\mathbf{I} \cap (\mathbf{t}, +\infty) \neq \phi$ and $\mathbf{I} \cap (-\infty, \mathbf{t}) \neq \phi$. Let $\xi \in I$ be fixed but arbitrary in I. We consider further $\xi \ge t$; the opposite case is similar. With $I_1 := [a,t]$ and $I_2 := [t,b]$ and making use of the symmetry of f_t w.r.t. t we obtain

$$\begin{aligned}
|I| \cdot f_t(\xi) \quad &\le \quad |I_1| \cdot \max\{f_t(a), f_t(t - (\xi - t))\} + |I_2| \cdot f_t(t + (\xi - t)) \\
&= \quad |I_1| \cdot \max\{f_t(a), f_t(2t - \xi)\} + |I_2| \cdot f_t(\xi).
\end{aligned}$$

Applying Case A for the first term and Case B for the second we get the desired result also in this case.

b) We get the desired estimate by applying for each element of the Riemann–type sum the above result, i.e.,

$$\begin{aligned}
\sum_{r \in \mathbb{Z}} |I_r| \cdot f_t(\xi_r) \quad &\le \quad 2^{1+\gamma} \sum_{r \in \mathbb{Z}} \int_{I_r} f_t(x)\,dx \\
&= \quad 2^{1+\gamma} \int_{\mathbb{R}} f_t(x)\,dx = 2^{1+\gamma} \int_{\mathbb{R}} f_0(x)\,dx = 2^{1+\gamma} \cdot \frac{2}{\gamma} \\
&= \quad \frac{2^{2+\gamma}}{\gamma}. \qquad \square
\end{aligned}$$

In the sequel we generalize Theorem 5.17 of Frazier&Jawerth (see [32, Theorem 3.3]) for the univariate non–stationary setting.

Theorem 5.35 *(General boundedness result)*
Every matrix which is almost diagonal w.r.t. a relatively separated family of intervals defines a bounded operator on l_2.

Proof. Let $A = (a_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda}$ be an almost diagonal matrix w.r.t. the relatively separated family of intervals $\{I_\lambda\}_{\lambda \in \Lambda}$, i.e., (5.12) and (5.22) hold. We denote the right hand side in (5.12) by $M_{\lambda,\lambda'}$.

In order to obtain the boundedness we want to apply Schur's Lemma 5.6 with $m_{\lambda,\lambda'} := a_{\lambda,\lambda'}$. If we choose $\omega_\lambda := |I_\lambda|^{1/2}$, we have to check that there exists a constant $a(C, D_2, \varepsilon)$ such that

$$S_\lambda := \sum_{\lambda' \in \Lambda} |I_{\lambda'}|^{1/2} \cdot |a_{\lambda,\lambda'}| \le a(C, D_2, \varepsilon) \cdot |I_\lambda|^{1/2}, \quad \lambda \in \Lambda.$$

The second inequality in the hypothesis of Schur's lemma will then follow by symmetry.

Let λ be fixed. We consider the disjoint partition $\Lambda = A_\lambda^- \sqcup A_\lambda^+$ with

$$A_\lambda^- := \{\lambda' \in \Lambda \,:\, |I_{\lambda'}| \le |I_\lambda|\},$$
$$A_\lambda^+ := \{\lambda' \in \Lambda \,:\, |I_{\lambda'}| > |I_\lambda|\}.$$

Thus we have

$$\begin{aligned} S_\lambda &= \sum_{\lambda'\in\Lambda} |I_{\lambda'}|^{1/2} \cdot |a_{\lambda,\lambda'}| \\ &\overset{(5.12)}{\le} \sum_{\lambda'\in\Lambda} |I_{\lambda'}|^{1/2} \cdot M_{\lambda,\lambda'} \\ &= \sum_{\lambda'\in A_\lambda^-} |I_{\lambda'}|^{1/2} \cdot M_{\lambda,\lambda'} + \sum_{\lambda'\in A_\lambda^+} |I_{\lambda'}|^{1/2} \cdot M_{\lambda,\lambda'} =: S_\lambda^- + S_\lambda^+. \end{aligned}$$

Furthermore, for every $(l,r) \in I\!N_0 \times Z\!\!Z$ we define the index sets

$$R_\lambda^-(l,r) := \Big\{\lambda' \in A_\lambda^- \;:\; |I_{\lambda'}| \in \Big(2^{-(l+1)} \cdot |I_\lambda|, 2^{-l} \cdot |I_\lambda|\Big], \tag{5.24}$$
$$c_{\lambda'} \in \Big[2^{-l} \cdot |I_\lambda| \cdot r, 2^{-l} \cdot |I_\lambda| \cdot (r+1)\Big)\Big\}, \tag{5.25}$$

$$R_\lambda^+(l,r) := \Big\{\lambda' \in A_\lambda^+ \;:\; |I_{\lambda'}| \in \Big[2^{l} \cdot |I_\lambda|, 2^{l+1} \cdot |I_\lambda|\Big), \tag{5.26}$$
$$c_{\lambda'} \in \Big[2^{l+1} \cdot |I_\lambda| \cdot r, 2^{l+1} \cdot |I_\lambda| \cdot (r+1)\Big)\Big\}. \tag{5.27}$$

Thus we get the disjoint partitions

$$\begin{aligned} A_\lambda^- &= \sqcup_{l\in I\!N_0} \sqcup_{r\in Z\!\!Z} R_\lambda^-(l,r), \\ A_\lambda^+ &= \sqcup_{l\in I\!N_0} \sqcup_{r\in Z\!\!Z} R_\lambda^+(l,r). \end{aligned}$$

($\sqcup$ denotes the union of disjoint sets.) If we consider the interval

$$J := \Big[2^{-l} \cdot |I_\lambda| \cdot r, 2^{-l} \cdot |I_\lambda| \cdot (r+1)\Big),$$

then for every function $\psi_{\lambda'}$ with λ' from $R_\lambda^-(l,r)$ we have

$$|I_{\lambda'}| \in \left[\frac{|J|}{2}, |J|\right] \quad \text{and} \quad c_{\lambda'} \in J.$$

Definition 5.32 implies that there exist at most D_2 functions ψ_λ of this type, i.e., for all (λ, l, r) there holds $\#R_\lambda^-(l,r) \le D_2$. Considering

$$J := \Big[2^{l+1} \cdot |I_\lambda| \cdot r, 2^{l+1} \cdot |I_\lambda| \cdot (r+1)\Big)$$

we get in an analogous way $\#R_\lambda^+(l,r) \le D_2$ for all (λ, l, r).

By combining the previous arguments we obtain

$$\begin{aligned} S_\lambda^- &= \sum_{\lambda'\in A_\lambda^-} |I_{\lambda'}|^{1/2} \cdot M_{\lambda,\lambda'} \\ &= C \cdot \sum_{\lambda'\in A_\lambda^-} |I_{\lambda'}|^{1/2} \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{|I_\lambda|}\right)^{-1-\varepsilon} \cdot \left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{(1+\varepsilon)/2} \\ &= C \cdot |I_\lambda|^{1/2} \cdot \sum_{\lambda'\in A_\lambda^-} \left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{1+\frac{\varepsilon}{2}} \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{|I_\lambda|}\right)^{-1-\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&= C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \sum_{r \in \mathbb{Z}} \sum_{\lambda' \in R_\lambda^-(l,r)} \left(\frac{|I_{\lambda'}|}{|I_\lambda|} \right)^{1+\frac{\varepsilon}{2}} \cdot \frac{1}{\left(1 + \left|\frac{c_\lambda}{|I_\lambda|} - \frac{c_{\lambda'}}{|I_\lambda|}\right|\right)^{1+\varepsilon}} \\
&\overset{(5.24)}{\leq} C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \sum_{r \in \mathbb{Z}} \frac{1}{2^l} \sum_{\lambda' \in R_\lambda^-(l,r)} \frac{1}{\left(1 + \left|\frac{c_\lambda}{|I_\lambda|} - \frac{c_{\lambda'}}{|I_\lambda|}\right|\right)^{1+\varepsilon}} \\
&\overset{(5.22)}{\leq} C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \sum_{r \in \mathbb{Z}} \frac{1}{2^l} \cdot D_2 \cdot \max_{\lambda' \in R_\lambda^-(l,r)} \frac{1}{\left(1 + \left|\frac{c_\lambda}{|I_\lambda|} - \frac{c_{\lambda'}}{|I_\lambda|}\right|\right)^{1+\varepsilon}} .
\end{aligned}
$$

For the subsequent computations we can denote $\frac{c_\lambda}{|I_\lambda|}$ by t, because λ is a fixed parameter at this moment. Further we assume that the above maximum is attained for $\lambda_0' \in R_\lambda^-(l,r)$. Let $\xi_{l,r} \in [2^{-l} \cdot r, 2^{-l}(r+1))$ denote the quantity $\frac{c_{\lambda_0'}}{|I_\lambda|}$. We get further

$$
\begin{aligned}
S_\lambda^- &\overset{(5.25)}{\leq} C \cdot D_2 \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \sum_{r \in \mathbb{Z}} \frac{1}{2^l} \cdot \frac{1}{(1 + |t - \xi_{l,r}|)^{1+\varepsilon}} \\
&\overset{L.\ 5.34.b)}{\leq} C \cdot D_2 \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \cdot \frac{2^{2+\varepsilon}}{\varepsilon} \\
&= \frac{2^{2+\varepsilon}}{\varepsilon} \cdot C \cdot D_2 \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^{\varepsilon/2}}\right)^l \\
&= \frac{2^{(4+5\varepsilon)/2}}{\varepsilon(2^{\varepsilon/2} - 1)} \cdot C \cdot D_2 \cdot |I_\lambda|^{1/2} =: a^- \cdot |I_\lambda|^{1/2}.
\end{aligned}
$$

An upper bound for S_λ^+ is given by

$$
\begin{aligned}
S_\lambda^+ &= \sum_{\lambda' \in A_\lambda^+} |I_{\lambda'}|^{1/2} \cdot M_{\lambda,\lambda'} \\
&= C \cdot \sum_{\lambda' \in A_\lambda^+} |I_{\lambda'}|^{1/2} \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{|I_{\lambda'}|}\right)^{-1-\varepsilon} \cdot \left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{(1+\varepsilon)/2} \\
&= C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \sum_{r \in \mathbb{Z}} \sum_{\lambda' \in R_\lambda^+(l,r)} \left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{\frac{\varepsilon}{2}} \cdot \frac{1}{\left(1 + \frac{|I_\lambda|}{|I_{\lambda'}|} \cdot \left|\frac{c_\lambda}{|I_\lambda|} - \frac{c_{\lambda'}}{|I_\lambda|}\right|\right)^{1+\varepsilon}} \\
&\overset{(5.26)}{\leq} C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \sum_{r \in \mathbb{Z}} \sum_{\lambda' \in R_\lambda^+(l,r)} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \cdot \frac{1}{\left(1 + \frac{1}{2^{l+1}} \cdot \left|\frac{c_\lambda}{|I_\lambda|} - \frac{c_{\lambda'}}{|I_\lambda|}\right|\right)^{1+\varepsilon}} \\
&= C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \sum_{r \in \mathbb{Z}} \sum_{\lambda' \in R_\lambda^+(l,r)} \frac{1}{\left(1 + \left|\frac{c_\lambda}{2^{l+1} \cdot |I_\lambda|} - \frac{c_{\lambda'}}{2^{l+1} \cdot |I_\lambda|}\right|\right)^{1+\varepsilon}} \\
&\overset{(5.22)}{\leq} C \cdot |I_\lambda|^{1/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}} \sum_{r \in \mathbb{Z}} D_2 \cdot \max_{\lambda' \in R_\lambda^+(l,r)} \frac{1}{\left(1 + \left|\frac{c_\lambda}{2^{l+1} \cdot |I_\lambda|} - \frac{c_{\lambda'}}{2^{l+1} \cdot |I_\lambda|}\right|\right)^{1+\varepsilon}} .
\end{aligned}
$$

For the subsequent computations we can denote $\frac{c_\lambda}{2^{l+1} \cdot |I_\lambda|}$ by t, because λ and l are fixed parameters at this point where the maximum has to be determined. Further we assume that the above maximum is attained for $\lambda_0' \in R_\lambda^+(l,r)$. Let $\xi_{l,r} \in [r, r+1)$ denote the quantity

$\frac{c_{\lambda_0'}}{2^{l+1}\cdot|I_\lambda|}$. We get further

$$\begin{aligned}
S_\lambda^+ &\overset{(5.27)}{\leq} C\cdot D_2\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}}\sum_{r\in\mathbb{Z}}\frac{1}{(1+|t-\xi_{l,r}|)^{1+\varepsilon}}\\
&\overset{L.\ 5.34.b)}{\leq} C\cdot D_2\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^l}\right)^{\frac{\varepsilon}{2}}\cdot\frac{2^{2+\varepsilon}}{\varepsilon}\\
&= \frac{2^{2+\varepsilon}}{\varepsilon}\cdot C\cdot D_2\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^{\varepsilon/2}}\right)^l\\
&= \frac{2^{(4+5\varepsilon)/2}}{\varepsilon(2^{\,\varepsilon/2}-1)}\cdot C\cdot D_2\cdot|I_\lambda|^{1/2} =: a^+\cdot|I_\lambda|^{1/2}.
\end{aligned}$$

Note the difference in the application of Lemma 5.34 for the case S_λ^+ in comparison to the case S_λ^-. Finally, it follows

$$S_\lambda \leq a(C,D_2,\varepsilon)\cdot|I_\lambda|^{1/2} \qquad \text{for all } \lambda\in\Lambda$$

with

$$\begin{aligned}
a(C,D_2,\varepsilon) &:= a^- + a^+\\
&= \frac{2^{(6+5\varepsilon)/2}}{\varepsilon(2^{\,\varepsilon/2}-1)}\cdot C\cdot D_2.
\end{aligned}$$

This enables us to apply Schur's lemma with $a_1 = a_2 = a(C,D_2,\varepsilon)$. It implies that the matrix operator $M: l^2(\Lambda)\to l^2(\Lambda)$ is well–defined, bounded and its norm is less than or equal to $a(C,D_2,\varepsilon)$. □

In the sequel upper bounds of certain linear operator norms play an important rôle in our considerations. Therefore we formulate the above result more precisely.

Corollary 5.36 *(General bound for the operator norm)*
Let $A = (a_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}$ be an almost diagonal matrix w.r.t. a relatively separated family of intervals $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ which satisfies (5.12) with parameters C and ε. If D_2 denotes the overlapping constant of $\mathcal{I}$, then the matrix operator associated to A is bounded on $l_2(\Lambda)$ with bound

$$a(C,D_2,\varepsilon) = \frac{2^{(6+5\varepsilon)/2}}{\varepsilon(2^{\,\varepsilon/2}-1)}\cdot C\cdot D_2.$$

5.8 Univariate non–stationary vaguelettes with compact support

In this section we focus on function families with compact support from $L_2(X)$, where $X\in\{\mathbb{R},[a,\infty),(-\infty,b],[a,b]\}$. Our motivation is the subsequent construction of sibling spline frames on a compact interval $[a,b]\subset\mathbb{R}$. This section makes essential use of our results in Section 5.5.

Our general strategy for the non–stationary case is thus similar to the one employed by Meyer and Frazier&Jawerth for the stationary situation, as described in Remark 5.19. However, in the non–stationary case it is necessary to introduce a so–called "finite overlapping constant" for the function family in order to prove the desired result. This concept is new and was first introduced in [4]. It implies the separation of the supports of the function family in the sense of Definition 5.32 and enables us to determine Bessel bounds. It should be noted that in the stationary case the existence of such overlapping constant is automatically given (see Proposition 5.42).

We introduce compactly supported vaguelettes families from $L_2(X)$, where $X \in \{\mathbb{R}, [a,\infty), (-\infty,b], [a,b]\}$ which are well adapted to the non–stationary wavelet setting and generalize in this way the families introduced by Meyer (see Definition 5.7).

Meyer's Definition 5.7 of vaguelettes corresponds to the d–dimensional dyadic stationary case for functions without compact support. Following the ideas in (5.6) and (5.7) we propose the following generalization for the one–dimensional non–stationary case and for functions with compact support.

Definition 5.37 *(One–dimensional compactly supported non–stationary vaguelettes family) Let Λ be a countable index set. A family Ψ of continuous and compactly supported functions $\psi_\lambda : \mathbb{R} \to \mathbf{C}$, $\lambda \in \Lambda$, with*

$$\mathrm{supp}\,\psi_\lambda \subseteq [c_\lambda, b_\lambda] =: I_\lambda, \tag{5.28}$$

is called a one–dimensional compactly supported non-stationary vaguelettes family on $X := co\,(\cup_{\lambda\in\Lambda} I_\lambda)$[5], if the following conditions are satisfied:

a) $\displaystyle\int_{I_\lambda} \psi_\lambda(x)\,dx = 0 \qquad \text{for all } \lambda \in \Lambda$

(*vanishing moment*);

b) $$\exists\, C_1 > 0 \quad \forall \lambda \in \Lambda : \quad \|\psi_\lambda\|_\infty \le C_1 \cdot |I_\lambda|^{-1/2} \tag{5.29}$$

(*support–adapted uniform boundedness*);

c) $\exists \beta > 0 \quad \exists\, C_2 > 0 \quad \forall \lambda \in \Lambda \quad \forall x, x' \in I_\lambda \quad (x \ne x') :$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \le C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |x - x'|^\beta \tag{5.30}$$

(*support–adapted Hölder continuity with exponent* β);

d) $$\exists D_2 > 0 \quad \forall J \subset \mathbb{R} \quad \text{bounded interval:} \quad \#\Lambda_J \le D_2, \tag{5.31}$$

$$\text{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \;:\; |I_\lambda| \in \left[\frac{|J|}{2}, |J|\right],\; c_\lambda \in J \right\}$$

(*relatively separated family of supports, i.e., finite overlapping constant*).

C_1, C_2, β and D_2 are the parameters of the family Ψ. The support of the family Ψ, denoted by X, is the convex hull of $\cup_{\lambda\in\Lambda} I_\lambda$. This may be a bounded or unbounded interval. The operator S associated to the family Ψ (see Definition 5.2) is called the vaguelettes operator – in analogy to the frame case. For every g in $L_2(X)$ the sequence $(\langle g, \psi_\lambda \rangle)_{\lambda\in\Lambda}$ is called the 'vaguelettes decomposition' of g w.r.t. Ψ.

Remark 5.38 *Note that condition d) in Definition 5.37 describes a property of the supports I_λ only, namely their distribution on the real line. Other features of the functions ψ_λ are not referred to at this point.*
Recalling the results from Section 5.5, conditions a)–c) in Definition 5.37 are sufficient to prove the almost diagonality of the Gramian associated to the function family. We emphasize

[5] *co* denotes the convex hull.

that condition d) is the key to carry out the second step in the scheme described earlier (see Remark 5.19), namely to prove the boundedness of the matrix operator associated to the Gramian, as we already have seen in Section 5.7.

In the sequel we generalize Theorem 5.17 of Frazier&Jawerth (see [32, Theorem 3.3]) by giving a boundedness criterion for one–dimensional compactly supported non–stationary vaguelettes families.

Theorem 5.39 *(Boundedness criterion)*
Let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a one–dimensional compactly supported non–stationary vaguelettes family with parameters C_1, C_2, β and D_2.
Then $M := Gram(\Psi)$ defines a bounded linear operator on $l_2(\Lambda)$ and

$$\|M\|_{l_2(\Lambda)\to l_2(\Lambda)} \le D_2 \cdot C_1 C_2 \cdot \frac{2^\beta}{\beta+1}\left(\frac{3}{2^\beta-1}+\frac{2}{2^{\beta+1}-1}\right).$$

Proof. Let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a vaguelettes family with parameters C_1, C_2, β, D_2. In the sequel I_λ denotes the interval in (5.28) and c_λ its left endpoint.
From Theorem 5.26 we get the estimate

$$|\langle\psi_\lambda,\psi_{\lambda'}\rangle| \le \frac{C_1C_2}{\beta+1}\cdot\min\left\{\frac{|I_{\lambda'}|}{|I_\lambda|},\frac{|I_\lambda|}{|I_{\lambda'}|}\right\}^{(1+2\beta)/2} =: \mathbf{M}_{\lambda,\lambda'} \qquad \text{if } \mathring{I}_\lambda\cap\mathring{I}_{\lambda'}\neq\emptyset;$$

in the opposite case we have $|\langle\psi_\lambda,\psi_{\lambda'}\rangle| = 0$.
We want to apply Schur's Lemma 5.6 with $m_{\lambda,\lambda'} := \langle\psi_\lambda,\psi_{\lambda'}\rangle$. If we choose $\omega_\lambda := |I_\lambda|^{1/2}$, we have to check that there exists a constant $a(C_1,C_2,\beta,D_2)$ such that

$$S_\lambda := \sum_{\lambda'\in\Lambda}|I_{\lambda'}|^{1/2}\cdot|\langle\psi_\lambda,\psi_{\lambda'}\rangle| \le a(C_1,C_2,\beta,D_2)\cdot|I_\lambda|^{1/2}, \quad \lambda\in\Lambda.$$

The second inequality in the hypothesis of Schur's lemma will then follow by symmetry.
Let λ be fixed and

$$A_\lambda := \{\lambda'\in\Lambda : \mathring{I}_\lambda\cap\mathring{I}_{\lambda'}\neq\emptyset\}.$$

We consider the disjoint partition $A_\lambda = A_\lambda^-\sqcup A_\lambda^+$ with

$$\begin{aligned} A_\lambda^- &:= \{\lambda'\in A_\lambda : |I_{\lambda'}|\le|I_\lambda|\},\\ A_\lambda^+ &:= \{\lambda'\in A_\lambda : |I_{\lambda'}|>|I_\lambda|\}.\end{aligned}$$

Thus we have

$$\begin{aligned} S_\lambda &= \sum_{\lambda'\in A_\lambda}|I_{\lambda'}|^{1/2}\cdot|\langle\psi_\lambda,\psi_{\lambda'}\rangle|\\ &\overset{(5.17)}{\le} \sum_{\lambda'\in A_\lambda}|I_{\lambda'}|^{1/2}\cdot\mathbf{M}_{\lambda,\lambda'}\\ &= \sum_{\lambda'\in A_\lambda^-}|I_{\lambda'}|^{1/2}\cdot\mathbf{M}_{\lambda,\lambda'} + \sum_{\lambda'\in A_\lambda^+}|I_{\lambda'}|^{1/2}\cdot\mathbf{M}_{\lambda,\lambda'} =: S_\lambda^- + S_\lambda^+.\end{aligned}$$

Furthermore, for every $(l,r)\in\mathbb{N}_0\times\mathbb{Z}$ we define the index sets

$$\begin{aligned} R_\lambda^-(l,r) := \Big\{\lambda'\in A_\lambda^- : \quad & |I_{\lambda'}|\in\left(2^{-(l+1)}\cdot|I_\lambda|, 2^{-l}\cdot|I_\lambda|\right],\\ & c_{\lambda'}\in\left[2^{-l}\cdot|I_\lambda|\cdot r, 2^{-l}\cdot|I_\lambda|\cdot(r+1)\right)\Big\},\\ R_\lambda^+(l,r) := \Big\{\lambda'\in A_\lambda^+ : \quad & |I_{\lambda'}|\in\left[2^{l}\cdot|I_\lambda|, 2^{l+1}\cdot|I_\lambda|\right),\\ & c_{\lambda'}\in\left[2^{l+1}\cdot|I_\lambda|\cdot r, 2^{l+1}\cdot|I_\lambda|\cdot(r+1)\right)\Big\}.\end{aligned}$$

Thus we get the disjoint partitions

$$\begin{aligned} A_\lambda^- &= \sqcup_{l\in\mathbb{N}_0} \sqcup_{r\in\mathbb{Z}} R_\lambda^-(l,r), \\ A_\lambda^+ &= \sqcup_{l\in\mathbb{N}_0} \sqcup_{r\in\mathbb{Z}} R_\lambda^+(l,r). \end{aligned}$$

We observe that, for fixed λ and l, at most 2^l+1 intervals of the type

$$\left[2^{-l}\cdot|I_\lambda|\cdot r, 2^{-l}\cdot|I_\lambda|\cdot(r+1)\right)$$

intersect the interval I_λ. Let us denote by $H_{\lambda,l}^-$ the index set of those $r\in\mathbb{Z}$ for which this intersection property holds. Note further that we have at most 2 intervals of type

$$\left[2^{l+1}\cdot|I_\lambda|\cdot r, 2^{l+1}\cdot|I_\lambda|\cdot(r+1)\right)$$

which intersect I_λ. The index set of the corresponding r's will be denoted by $H_{\lambda,l}^+$. It follows that our partitions are

$$\begin{aligned} A_\lambda^- &= \sqcup_{l\in\mathbb{N}_0} \sqcup_{r\in H_{\lambda,l}^-} R_\lambda^-(l,r), \\ A_\lambda^+ &= \sqcup_{l\in\mathbb{N}_0} \sqcup_{r\in H_{\lambda,l}^+} R_\lambda^+(l,r). \end{aligned}$$

If we consider the interval

$$J := \left[2^{-l}\cdot|I_\lambda|\cdot r, 2^{-l}\cdot|I_\lambda|\cdot(r+1)\right),$$

then for every function $\psi_{\lambda'}$ with λ' from $R_\lambda^-(l,r)$ we have

$$|I_{\lambda'}| \in \left[\frac{|J|}{2}, |J|\right] \quad \text{and} \quad c_{\lambda'}\in J.$$

Definition 5.37.d) implies that there exist at most D_2 functions ψ_λ of this type, i.e.,

$$\#R_\lambda^-(l,r) \le D_2 \quad \text{for all } (\lambda,l,r).$$

Considering

$$J := \left[2^{l+1}\cdot|I_\lambda|\cdot r, 2^{l+1}\cdot|I_\lambda|\cdot(r+1)\right)$$

we get in an analogous way $\#R_\lambda^+(l,r)\le D_2$ for all (λ,l,r).

By combining the previous arguments we obtain

$$\begin{aligned} S_\lambda^- &= \sum_{\lambda'\in A_\lambda^-} |I_{\lambda'}|^{1/2}\cdot \mathbf{M}_{\lambda,\lambda'} \\ &= \frac{C_1C_2}{\beta+1}\cdot\sum_{l\in\mathbb{N}_0}\sum_{r\in H_{\lambda,l}^-}\sum_{\lambda'\in R_\lambda^-(l,r)} |I_{\lambda'}|^{1/2}\cdot\left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{(1+2\beta)/2} \\ &= \frac{C_1C_2}{\beta+1}\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\sum_{r\in H_{\lambda,l}^-}\sum_{\lambda'\in R_\lambda^-(l,r)}\left(\frac{|I_{\lambda'}|}{|I_\lambda|}\right)^{1+\beta} \\ &\le \frac{C_1C_2}{\beta+1}\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^l}\right)^{1+\beta}\sum_{r\in H_{\lambda,l}^-}\#R_\lambda^-(l,r) \\ &\le D_2\cdot\frac{C_1C_2}{\beta+1}\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^l}\right)^{1+\beta}\cdot\#H_{\lambda,l}^- \\ &\le D_2\cdot\frac{C_1C_2}{\beta+1}\cdot|I_\lambda|^{1/2}\cdot\sum_{l\in\mathbb{N}_0}\left(\frac{1}{2^l}\right)^{1+\beta}\cdot(2^l+1) \\ &= D_2\cdot\frac{2^\beta C_1C_2}{\beta+1}\cdot\left(\frac{1}{2^\beta-1}+\frac{2}{2^{\beta+1}-1}\right)\cdot|I_\lambda|^{1/2} =: a^-\cdot|I_\lambda|^{1/2}. \end{aligned}$$

Similarly, an upper bound for S^+_λ is given by

$$\begin{aligned}
S^+_\lambda &= \sum_{\lambda'\in A^+_\lambda} |I_{\lambda'}|^{1/2}\cdot \mathbf{M}_{\lambda,\lambda'} \\
&= \frac{C_1C_2}{\beta+1}\cdot \sum_{l\in I\!N_0}\sum_{r\in H^+_{\lambda,l}}\sum_{\lambda'\in R^+_\lambda(l,r)} |I_{\lambda'}|^{1/2}\cdot\left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{(1+2\beta)/2} \\
&= \frac{C_1C_2}{\beta+1}\cdot |I_\lambda|^{1/2}\cdot \sum_{l\in I\!N_0}\sum_{r\in H^+_{\lambda,l}}\sum_{\lambda'\in R^+_\lambda(l,r)} \left(\frac{|I_\lambda|}{|I_{\lambda'}|}\right)^{\beta} \\
&\le \frac{C_1C_2}{\beta+1}\cdot |I_\lambda|^{1/2}\cdot \sum_{l\in I\!N_0}\left(\frac{1}{2^l}\right)^{\beta}\sum_{r\in H^+_{\lambda,l}} \#R^+_\lambda(l,r) \\
&\le D_2\cdot\frac{C_1C_2}{\beta+1}\cdot |I_\lambda|^{1/2}\cdot \sum_{l\in I\!N_0}\left(\frac{1}{2^l}\right)^{\beta}\cdot \#H^+_{\lambda,l} \\
&\le D_2\cdot\frac{C_1C_2}{\beta+1}\cdot |I_\lambda|^{1/2}\cdot \sum_{l\in I\!N_0}\left(\frac{1}{2^l}\right)^{\beta}\cdot 2 \\
&= D_2\cdot\frac{2^{\beta+1}C_1C_2}{(\beta+1)(2^\beta-1)}\cdot |I_\lambda|^{1/2} =: a^+\cdot |I_\lambda|^{1/2}.
\end{aligned}$$

Finally, it follows

$$S_\lambda \le a(C_1,C_2,\beta,D_2)\cdot |I_\lambda|^{1/2} \qquad \text{for all } \lambda\in\Lambda$$

with

$$\begin{aligned}
a(C_1,C_2,\beta,D_2) &:= a^- + a^+ \\
&= D_2\cdot\frac{2^\beta C_1C_2}{\beta+1}\cdot\left(\frac{3}{2^\beta-1}+\frac{2}{2^{\beta+1}-1}\right).
\end{aligned}$$

This enables us to apply Schur's lemma with $a_1 = a_2 = a(C_1,C_2,\beta,D_2)$. It implies that the matrix operator $M : l^2(\Lambda)\to l^2(\Lambda)$ is well–defined, bounded and its norm is less than or equal to $a(C_1,C_2,\beta,D_2)$. □

Combining Theorems 5.39 and 5.4 we get the following result for function families with support $X\in\{I\!R,\ [a,\infty),\ (-\infty,b],\ [a,b]\}$.

Theorem 5.40 *(Univariate Bessel families)*
Every one–dimensional compactly supported non–stationary vaguelettes family with support X is a Bessel family in $L_2(X)$. Moreover, if C_1, C_2, β and D_2 are the parameters of the vaguelettes family, then a Bessel bound is given by

$$B_2 := \sqrt{D_2\cdot C_1C_2\cdot\frac{2^\beta}{\beta+1}\left(\frac{3}{2^\beta-1}+\frac{2}{2^{\beta+1}-1}\right)}. \tag{5.32}$$

Corollary 5.41 *With X defined as above, let Ψ be a one–dimensional compactly supported non–stationary vaguelettes family with support X and parameters C_1, C_2, β, D_2.*
The corresponding analysis operator $T_\Psi : L_2(X)\to l_2(\Lambda)$, synthesis operator $T^*_\Psi : l_2(\Lambda)\to L_2(X)$ and vaguelettes operator $S_\Psi : L_2(X)\to L_2(X)$ are linear and continuous. Moreover, we have

$$\|T_\Psi\| = \|T^*_\Psi\| \le B_2 \quad \text{and} \quad \|S_\Psi\| \le B_2^2$$

with B_2 given by (5.32).

One natural question is: "What happens with condition d) of Definition 5.37 in Meyer's dyadic stationary case for compactly supported functions defined on $I\!R$? " We will give an answer for a special case (disjoint compact supports on every level) in the following result. The case of non–disjoint supports has been detailed for a special case in Example 5.33.

Proposition 5.42 *(Overlapping constants in the dyadic stationary case with disjoint compact supports on every level)*
In the one–dimensional compactly supported dyadic stationary case – where with the above notations holds

$$\psi_\lambda := f_{j,k} \quad \textit{and} \tag{5.33}$$
$$\textit{supp}\, \psi_\lambda \subseteq I_\lambda := Q_{j,k} = [2^{-j}k, 2^{-j}(k+1)) =: [c_{j,k}, b_{j,k}) \tag{5.34}$$

for $\lambda \in \Lambda$ *and* $(j,k) \in Z\!\!Z^2$ *– we have an overlapping constant* $D_2 = 6$ *and thus a Bessel bound*

$$B_2 = \sqrt{C_1 C_2} \cdot \sqrt{\frac{6 \cdot 2^\beta}{\beta+1}\left(\frac{3}{2^\beta - 1} + \frac{2}{2^{\beta+1}-1}\right)}. \tag{5.35}$$

Proof. Let $J = [c_J, b_J] \subset I\!R$ be an arbitrary bounded interval with $|J| \neq 0$. In the sequel W will denote the set $\{2^j : j \in Z\!\!Z\}$.
There exists a unique $l := l(J) \in Z\!\!Z$ such that $|J| \in [2^l, 2^{l+1})$. We are interested in counting vaguelettes $f_{j,k}$ with

$$\textit{length supp}\, f_{j,k} \leq |Q_{j,k}| = 2^{-j} \in \left[\frac{|J|}{2}, |J|\right] \quad \text{and} \quad c_{j,k} \in J.$$

This means that

$$\textit{length supp}\, f_{j,k} \leq 2^{-j} \in [2^{l-1}, 2^{l+1}) \cap W = \{2^{l-1}, 2^l\},$$

i.e., $j \in \{-l+1, -l\}$. Further there exists a unique $k_0 := k_0(J) \in Z\!\!Z$ such that

$$c_J \in \left[2^{l+1}k_0, 2^{l+1}(k_0+1)\right).$$

Because $|J| < 2^{l+1}$ it follows that

$$b_J \in \left(2^{l+1}k_0, 2^{l+1}(k_0+2)\right).$$

In this setting there exist at most 2 functions $f_{j,k}$ with $j = -l$, support length 2^l and corner $2^l k \in J$, corresponding to two consecutive k's from the set

$$\{2k_0, 2k_0+1, 2k_0+2, 2k_0+3\},$$

and at most 4 functions $f_{j,k}$ with $j = -l+1$, support length 2^{l-1} and corner $2^{l-1}k \in J$, corresponding to four consecutive values

$$k \in \{4k_0, 4k_0+1, \ldots, 4k_0+7\}.$$

This yields the overlapping constant $D_2 = 2 + 4 = 6$, which is independent of the chosen interval J. □

Because of the intervals $\left[\frac{|J|}{2}, |J|\right]$ which appear in condition d) of Definition 5.37 we will call this the 'dyadic' non–stationary case. The reader may have noticed that at this point it was important to introduce an interval describing a 'neighborhood' of $|J|$. That is why we can consider any interval of the type $\left[\frac{|J|}{n}, |J|\right]$ with $n \in I\!N$ and $n \geq 2$.

If we replace in Definition 5.37 condition d) by the more general condition d') finite overlapping constant, i.e.,

$$\exists n \in \mathbb{N} \setminus \{1\} \quad \exists D_n > 0 \quad \forall J \subset \mathbb{R} \quad \text{bounded interval:} \quad \#\Lambda_J \leq D_n,$$
$$\text{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \ : \ |I_\lambda| \in \left[\frac{|J|}{n}, |J| \right], \ c_\lambda \in J \right\},$$

then we can prove the following.

Theorem 5.43 *(Univariate Bessel families)*
Every one–dimensional compactly supported non–stationary vaguelettes family with support X is a Bessel family in $L_2(X)$. Moreover, if C_1, C_2, β and D_n are the parameters of the vaguelettes family, then a Bessel bound is given by

$$B_n := \sqrt{D_n \cdot C_1 C_2 \cdot \frac{n^\beta}{\beta+1} \cdot \left(\frac{3}{n^\beta - 1} + \frac{n}{n^{\beta+1} - 1} \right)}. \tag{5.36}$$

Proof. With straightforward modifications one gets

$$\begin{aligned} a^- &= D_n \cdot \frac{C_1 C_2}{\beta+1} \sum_{l \geq 0} \left(\frac{1}{n^l} \right)^{1+\beta} \cdot (n^l + 1), \\ a^+ &= D_n \cdot \frac{C_1 C_2}{\beta+1} \sum_{l \geq 0} \left(\frac{1}{n^l} \right)^{\beta} \cdot 2, \end{aligned}$$

and thus the mentioned bound. □

For a fixed interval J the nesting of the interval sequence $\left(\left[\frac{|J|}{n}, |J| \right] \right)_{n \geq 2}$ implies

Proposition 5.44 *For a given compactly supported non–stationary vaguelettes family the constants D_n with $n \geq 2$ form an increasing sequence.*

Extending Meyer's dyadic vaguelettes $f_{j,k}$ in a natural fashion to the n–adic case, still using the notation $f_{j,k}$ with $(j,k) \in \mathbb{Z} \times \mathbb{Z}^d$ for the members of the family, we get in the case $d = 1$ the following result.

Proposition 5.45 *(Overlapping constants in the n–adic stationary case with disjoint compact supports on every level)*
In the one–dimensional compactly supported n–adic stationary case – where with the above notations holds

$$\psi_\lambda := f_{j,k} \quad \text{and} \tag{5.37}$$
$$\text{supp}\, \psi_\lambda \subseteq I_\lambda := I_{j,k} = [n^{-j}k, n^{-j}(k+1)) =: [c_{j,k}, b_{j,k}) \tag{5.38}$$

for $\lambda \in \Lambda$ and $(j,k) \in \mathbb{Z}^2$ – we have the overlapping constant

$$D_n = n(n+1) \quad \text{and thus} \quad D_n = \mathcal{O}(n^2).$$

Furthermore, we obtain in this case the Bessel bound

$$B_n = \sqrt{C_1 C_2} \cdot \sqrt{\frac{n^{\beta+1}(n+1)}{\beta+1} \cdot \left(\frac{3}{n^\beta - 1} + \frac{n}{n^{\beta+1} - 1} \right)}. \tag{5.39}$$

Proof. Let $J = [c_J, b_J] \subset \mathbb{R}$ be an arbitrary bounded interval with $|J| \neq 0$ and $n \in \mathbb{N}$ be fixed such that $n \geq 2$. In the sequel W will denote the set $\{n^j : j \in \mathbb{Z}\}$.
There exists a unique $l := l(J) \in \mathbb{Z}$ such that $|J| \in [n^l, n^{l+1})$. We are interested in counting vaguelettes $f_{j,k}$ with

$$\mathit{length\,supp}\, f_{j,k} \leq |I_{j,k}| = n^{-j} \in \left[\frac{|J|}{n}, |J|\right] \quad \text{and} \quad c_{j,k} \in J.$$

This means that

$$\mathit{length\,supp}\, f_{j,k} \leq n^{-j} \in \left[n^{l-1}, n^{l+1}\right) \cap W = \{n^{l-1}, n^l\},$$

i.e., $j \in \{-l+1, -l\}$. Further there exists a unique $k_0 := k_0(J) \in \mathbb{Z}$ such that

$$c_J \in \left[n^{l+1}k_0, n^{l+1}(k_0+1)\right).$$

Because $|J| < n^{l+1}$ it follows that

$$b_J \in \left(n^{l+1}k_0, n^{l+1}(k_0+2)\right).$$

In this setting there exist at most n functions $f_{j,k}$ with $j = -l$, support length n^l and corner $n^l k \in J$, corresponding to n consecutive k's from the set

$$\{nk_0, nk_0+1, nk_0+2, \ldots, nk_0+(2n-1)\},$$

and at most n^2 functions $f_{j,k}$ with $j = -l+1$, support length n^{l-1} and corner $n^{l-1}k \in J$, corresponding to n^2 consecutive values

$$k \in \{n^2k_0, n^2k_0+1, \ldots, n^2k_0+(2n^2-1)\}.$$

This yields the overlapping constant $D_n = n + n^2 = n(n+1)$, which is independent of the chosen interval J. □

5.9 Univariate non–stationary vaguelettes with infinite support

In this section we focus on function families without compact support from $L_2(\mathbb{R})$. The motivation for this section is to demonstrate that our theory also covers the non–compact case. However, we will not present explicit constructions in the sequel, but only give the theoretical results to round up our work. This section makes essential use of our results in Sections 5.5 and 5.7.

We introduce vaguelettes families without compact support from $L_2(\mathbb{R})$ which are well adapted to the non–stationary wavelet setting and fully generalize the families introduced by Meyer (see Definition 5.7) for the univariate case.

Definition 5.46 *(One–dimensional non–stationary vaguelettes family)*
Let Λ be a countable index set. A family Ψ of continuous functions $\psi_\lambda : \mathbb{R} \to \mathbf{C}$, $\lambda \in \Lambda$, with

$$\text{(geometrical or abstract) essential supp}\ \psi_\lambda \subseteq [c_\lambda, b_\lambda] =: I_\lambda, \tag{5.40}$$

is called a one–dimensional non-stationary vaguelettes family, if the following conditions are satisfied:

a) $\int_{I\!R} \psi_\lambda(x)\, dx = 0$ *for all* $\lambda \in \Lambda$

(*vanishing moment*);

b) $\exists\, \alpha > 0 \quad \exists\, C_1 > 0 \quad \forall \lambda \in \Lambda \quad \forall x \in I\!R:$

$$|\psi_\lambda(x)| \le C_1 \cdot |I_\lambda|^{-1/2} \left[1 + |I_\lambda|^{-1} \cdot |x - c_\lambda|\right]^{-(1+\alpha)}$$

(*support–adapted boundedness and decay*);

c) $\exists \beta \in (0, \alpha) \quad \exists\, C_2 > 0 \quad \forall \lambda \in \Lambda \quad \forall x, x' \in I\!R \quad (x \ne x'):$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \le C_2 \cdot |I_\lambda|^{-(1+2\beta)/2} \cdot |x - x'|^\beta$$

(*support–adapted Hölder continuity with exponent* β);

d) $\exists D_2 > 0 \quad \forall J \subset I\!R$ *bounded interval:* $\#\Lambda_J \le D_2$,

$$\text{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \; : \; |I_\lambda| \in \left[\frac{|J|}{2}, |J|\right], \; c_\lambda \in J \right\}$$

(*relatively separated family of essential supports, i.e., finite overlapping constant*).

C_1, C_2, α, β and D_2 are the parameters of the family Ψ. The operator S associated to the family Ψ (see Definition 5.2) is called the vaguelettes operator – in analogy to the frame case. For every g in $L_2(X)$ the sequence $(\langle g, \psi_\lambda\rangle)_{\lambda\in\Lambda}$ is called the 'vaguelettes decomposition' of g w.r.t. Ψ.

Theorem 5.47 *(Boundedness criterion)*
Let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a one–dimensional non–stationary vaguelettes family with parameters C_1, C_2, α, β and D_2.
Then $M := Gram(\Psi)$ defines a bounded linear operator on $l_2(\Lambda)$ and

$$\|M\|_{l_2(\Lambda)\to l_2(\Lambda)} \le 2C_1 \cdot D_2 \cdot \left[\left(1 + 2^{2+\alpha}\right) \frac{C_1}{\alpha}\right]^{\frac{1+2\beta}{1+\alpha+2\beta}} \left(\frac{C_2}{\alpha - \beta}\right)^{\frac{\alpha}{1+\alpha+2\beta}} \cdot \frac{2^{(6+5\varepsilon)/2}}{\varepsilon(2^{\,\varepsilon/2} - 1)}$$

with

$$\varepsilon = \frac{2\alpha\beta}{1 + \alpha + 2\beta}. \tag{5.41}$$

Proof. Theorem 5.24 implies the almost diagonality of $Gram(\Psi)$ on $l_2(\Lambda)$ w.r.t. the essential supports $\{I_\lambda\}_{\lambda\in\Lambda}$, i.e., with the notation $I_\lambda = [c_\lambda, b_\lambda]$ we get the estimate

$$|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \le C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{|I_\lambda|, |I_{\lambda'}|\}}\right)^{-1-\varepsilon} \cdot \min\left\{\frac{|I_\lambda|}{|I_{\lambda'}|}, \frac{|I_{\lambda'}|}{|I_\lambda|}\right\}^{(1+\varepsilon)/2}$$

with exponent ε from (5.41) and constant

$$C = 2C_1 \cdot \left[\left(1 + 2^{2+\alpha}\right) \frac{C_1}{\alpha}\right]^{\frac{1+2\beta}{1+\alpha+2\beta}} \left(\frac{C_2}{\alpha - \beta}\right)^{\frac{\alpha}{1+\alpha+2\beta}}.$$

Applying further the general boundedness result from Corollary 5.36 we obtain the boundedness of the matrix operator M associated to $Gram(\Psi)$ and the following upper bound for the operator norm:

$$\|M\|_{l_2(\Lambda)\to l_2(\Lambda)} \le \frac{2^{(6+5\varepsilon)/2}}{\varepsilon(2^{\varepsilon/2}-1)} \cdot C \cdot D_2.$$

The insertion of C from above gives the desired result. □

Theorem 5.48 *(Univariate Bessel families)*
Every one–dimensional non–stationary vaguelettes family is a Bessel family in $L_2(\mathbb{R})$. Moreover, if C_1, C_2, α, β and D_2 are the parameters of the vaguelettes family, then a Bessel bound is given by

$$B_2 := \sqrt{2C_1 \cdot D_2 \cdot \left[(1+2^{2+\alpha})\,\frac{C_1}{\alpha}\right]^{\frac{1+2\beta}{1+\alpha+2\beta}} \left(\frac{C_2}{\alpha-\beta}\right)^{\frac{\alpha}{1+\alpha+2\beta}} \cdot \frac{2^{(6+5\varepsilon)/2}}{\varepsilon(2^{\varepsilon/2}-1)}}\,. \qquad (5.42)$$

with ε from (5.41).

5.10 Almost diagonality in the non–stationary multivariate case

In the remainder of this chapter we present results for the d–variate case. The basic tool is a tensor product approach, and thus many of our previous univariate results carry over in a very natural fashion.

For a cuboid (generalized rectangle) $I = [c, b] \subset \mathbb{R}^d$, with $c = (c_1, \ldots, c_d)$ and $b = (b_1, \ldots, b_d)$, we will use the following geometric quantities:

- Euclidian volume: $vol(I) := |I| := \prod_{i=1}^{d}(b_i - c_i)$;
- length of diagonal: $diam(I) := diag(I) := \left(\sum_{i=1}^{d}(b_i - c_i)^2\right)^{1/2}$;
- lower left corner: c.

By a cube from $\mathbb{R}^d$ we will mean in the sequel a regular cuboid from $\mathbb{R}^d$. The side length of a cube $I \subset \mathbb{R}^d$ will be denoted by $l(I)$.

For an extension of our concepts from the one–dimensional to the d–dimensional case we choose as measure for our cuboids the length of their diagonal.

Next we introduce and discuss the concept of an almost diagonal matrix in the non–stationary multivariate case.

Definition 5.49 *(Almost diagonal matrix on $l_2(\Lambda)$)*
Let Λ be a countable index set and $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ a collection of cuboids in $\mathbb{R}^d$. The diagonal of I_λ will be denoted by $diam(I_\lambda)$ and its left corner by c_λ.

a) A matrix $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ is called almost diagonal on $l_2(\Lambda)$ w.r.t. $\{I_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^d$ if there exist $C > 0$ and $\varepsilon > 0$ such that

$$|a_{\lambda,\lambda'}| \le C \cdot \left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{diam(I_\lambda), diam(I_{\lambda'})\}}\right)^{-d-\varepsilon} \cdot$$
$$\cdot \min\left\{\frac{diam(I_\lambda)}{diam(I_{\lambda'})}, \frac{diam(I_{\lambda'})}{diam(I_\lambda)}\right\}^{(d+\varepsilon)/2} \qquad (5.43)$$

holds for all possible choices of $(\lambda, \lambda') \in \Lambda^2$. *Here* $|c_\lambda - c_{\lambda'}|$ *denotes the Euclidian distance between* c_λ *and* $c_{\lambda'}$ *in* $\mathbb{R}^d$.

b) *A linear operator* $A : l_2(\Lambda) \to l_2(\Lambda)$ *is called almost diagonal if its associated matrix possesses this property.*

Remark 5.50 *Notice that the first pair of parentheses on the right–hand side in (5.43) describes the decay of* $|a_{\lambda,\lambda'}|$ *when* I_λ *and* $I_{\lambda'}$ *get apart from each other (such that the difference* $|c_\lambda - c_{\lambda'}|$ *becomes big). The second one explains the decay in the case that* $\frac{diam(I_\lambda)}{diam(I_{\lambda'})}$ *goes to* 0 *or* ∞. *Therefore, if one of the mentioned decay properties happens to be intrinsic for a particular matrix, one drops the corresponding pair of parentheses and obtains a simpler condition to be proved for the entries of the matrix.*

In the sequel we will encounter the case where for I_λ *and* $I_{\lambda'}$ *sufficiently apart from each other the entries* $a_{\lambda,\lambda'}$ *of the matrix* A *will turn out to be zero. This is always the case when the matrix* A *is the Gram matrix of a function family with compact supports* $I_\lambda \subset \mathbb{R}^d$, $\lambda \in \Lambda$. *In this case we will estimate* $|a_{\lambda,\lambda'}|$ *by*

$$C \cdot 1 \cdot \min\left\{\frac{diam(I_\lambda)}{diam(I_{\lambda'})}, \frac{diam(I_{\lambda'})}{diam(I_\lambda)}\right\}^{(d+\varepsilon)/2}.$$

Although the first pair of parentheses can always be estimated from above by 1, *i.e.,*

$$\left(1 + \frac{|c_\lambda - c_{\lambda'}|}{\max\{diam(I_\lambda), diam(I_{\lambda'})\}}\right)^{-d-\varepsilon} \leq 1,$$

in certain other cases it is necessary to have a better bound. This is always the case when the matrix A *is the Gram matrix of a function family with unbounded supports, with* I_λ *being the corresponding essential supports. In this thesis we do not consider the latter case.*

Under the conditions of Definition 5.49 we get a distance function in $\mathcal{I} \subset \mathbb{R}^d$ by setting

$$d_{\mathcal{I}}(I_\lambda, I_{\lambda'}) := d_{Pc}\left((c_\lambda, diam(I_\lambda)), (c_{\lambda'}, diam(I_{\lambda'}))\right) \qquad \text{for all } (\lambda, \lambda') \in \Lambda^2, \tag{5.44}$$

where d_{P_c} is the Poincaré metric defined in (5.9) and (5.10).

As a direct application of Lemma 5.14 we arrive at the following exponential localization of an almost diagonal matrix in the non–stationary multivariate case.

Proposition 5.51 *(Characterization of almost diagonal matrices on* $l_2(\Lambda)$*)*
Let Λ *be a countable index set and let* $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ *be a system of compact intervals from* $\mathbb{R}^d$.

a) $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ *is an almost diagonal matrix on* $l_2(\Lambda)$ *w.r.t.* $\mathcal{I}$ *in the sense of Definition 5.49 if and only if there exist* $\varepsilon > 0$ *and* $C' > 0$ *such that*

$$|a_{\lambda,\lambda'}| \leq C' \cdot e^{-(d+\varepsilon)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})} \tag{5.45}$$

holds for all $(\lambda, \lambda') \in \Lambda^2$ *with the distance function* $d_{\mathcal{I}}$ *defined in (5.44).*

b) *Moreover, if* $A = (a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda^2}$ *is an almost diagonal matrix on* $l_2(\Lambda)$ *w.r.t.* $\mathcal{I}$ *in the sense of Definition 5.49, then with* ε *and* C *from Definition 5.49.a) we obtain for* C' *in (5.45) the value* $2^{d+\varepsilon}C$.

The proof of Proposition 5.51 is similar to that of Proposition 5.22 due to the fact that Lemma 5.14 was proved for the d–dimensional case.

Theorem 5.52 *(Sufficient conditions for the almost diagonality of a Gram matrix)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(X)$, $X \in \{\mathbb{R}^d, (-\infty, b], [a,\infty), [a,b]\}$, with compact supports

$$\mathit{supp}\, \psi_\lambda \subset [c_\lambda, b_\lambda] =: I_\lambda, \qquad \lambda \in \Lambda,$$

satisfying the following conditions:

a) $\displaystyle\int_{I_\lambda} \psi_\lambda(x)\, dx = 0$ *for all* $\lambda \in \Lambda$; *(vanishing moment);*

b) $\exists\, C_1 > 0 \quad \forall \lambda \in \Lambda : \quad \|\psi_\lambda\|_\infty \le C_1 \cdot \mathit{diam}(I_\lambda)^{-d/2}$ *(boundedness);*

c) $\exists \beta > 0 \quad \exists\, C_2 > 0 \quad \forall \lambda \in \Lambda \quad \forall x, x' \in I_\lambda \quad (x \ne x')$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \le C_2 \cdot \mathit{diam}(I_\lambda)^{-(d+2\beta)/2} \cdot |x - x'|^\beta \qquad \text{(Hölder continuity).}$$

On the right hand side $|\cdot|$ stands for the Euclidian distance in $\mathbb{R}^d$.

Then $Gram(\Psi)$ is an almost diagonal matrix on $l_2(\Lambda)$ w.r.t. $\{I_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^d$ in the sense of Definition 5.49 with exponent $\varepsilon = 2\beta$ and constant

$$C = \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1 + \frac{d}{2}\right)},$$

where Γ denotes the gamma function defined through $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$.
I.e., for all $(\lambda, \lambda') \in \Lambda^2$ we have

$$|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \le \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1 + \frac{d}{2}\right)} \cdot \min \left\{ \frac{\mathit{diam}(I_\lambda)}{\mathit{diam}(I_{\lambda'})}, \frac{\mathit{diam}(I_{\lambda'})}{\mathit{diam}(I_\lambda)} \right\}^{(d+2\beta)/2}.$$

Proof. Let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a d–dimensional compactly supported function family with parameters C_1, C_2 and β as described above.
Let further the indices $\lambda, \lambda' \in \Lambda$ be arbitrarily fixed, but such that $\mathring{I}_\lambda \cap \mathring{I}_{\lambda'} \ne \emptyset$. For the entries of the Gramian we have

$$\begin{aligned} |\langle \psi_\lambda, \psi_{\lambda'} \rangle| &= \left| \int_{I_\lambda \cap I_{\lambda'}} \psi_\lambda(x) \cdot \overline{\psi_{\lambda'}(x)}\, dx \right| \\ &\overset{a)}{=} \left| \int_{I_\lambda \cap I_{\lambda'}} [\psi_\lambda(x) - \psi_\lambda(c_{\lambda'})] \cdot \overline{\psi_{\lambda'}(x)}\, dx \right| \\ &\le \int_{I_\lambda \cap I_{\lambda'}} |\psi_\lambda(x) - \psi_\lambda(c_{\lambda'})| \cdot |\psi_{\lambda'}(x)|\, dx \\ &\overset{b),c)}{\le} C_2 \cdot \mathit{diam}(I_\lambda)^{-(d+2\beta)/2} \cdot C_1 \cdot \mathit{diam}(I_{\lambda'})^{-d/2} \int_{I_\lambda \cap I_{\lambda'}} |x - c_{\lambda'}|^\beta\, dx. \end{aligned} \tag{5.46}$$

For the case $\mathit{diam}(I_{\lambda'}) \le \mathit{diam}(I_\lambda)$ we get further the estimate

$$|\langle \psi_\lambda, \psi_{\lambda'} \rangle| \le C_1 C_2 \cdot \mathit{diam}(I_\lambda)^{-(d+2\beta)/2} \cdot \mathit{diam}(I_{\lambda'})^{-d/2} \int_{I_{\lambda'}} |x - c_{\lambda'}|^\beta\, dx. \tag{5.47}$$

Let $\mathbf{B}_{\lambda'} := \mathbf{B}(c_{\lambda'}, \mathit{diam}(I_{\lambda'})) \subset \mathbb{R}^d$ denote the sphere of center $c_{\lambda'}$ and radius equal to $\mathit{diam}(I_{\lambda'})$. If $c_{\lambda'} = (c^1_{\lambda'}, \dots, c^d_{\lambda'})$, let

$$\mathbf{B}^+_{\lambda'} := \mathbf{B}(c_{\lambda'}, \mathit{diam}(I_{\lambda'})) \cap \{(x_1, \dots, x_d) \in \mathbb{R}^d \,:\, x_i - c^i_{\lambda'} \ge 0,\, i = 1, \dots, d\}.$$

With this notation we have $I_{\lambda'} \subset \mathbf{B}^+_{\lambda'}$ and $vol(\mathbf{B}^+_{\lambda'}) = vol(\mathbf{B}_{\lambda'})/2^d$. Thus

$$\begin{aligned}
|\langle\psi_\lambda, \psi_{\lambda'}\rangle| &\leq C_1 C_2 \cdot diam(I_\lambda)^{-(d+2\beta)/2} \cdot diam(I_{\lambda'})^{-d/2} \int_{\mathbf{B}^+_{\lambda'}} |x - c_{\lambda'}|^\beta \, dx \\
&\leq C_1 C_2 \cdot diam(I_\lambda)^{-(d+2\beta)/2} \cdot diam(I_{\lambda'})^{-d/2} \cdot vol(\mathbf{B}^+_{\lambda'}) \cdot \max_{x \in \mathbf{B}^+_{\lambda'}} |x - c_{\lambda'}|^\beta \\
&= \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1 + \frac{d}{2}\right)} \cdot \left(\frac{diam(I_{\lambda'})}{diam(I_\lambda)}\right)^{(d+2\beta)/2} . \qquad (5.48)
\end{aligned}$$

With symmetric arguments in (5.46) and (5.47) we get for the case $diam(I_{\lambda'}) \geq diam(I_\lambda)$ an estimate which is symmetric to (5.48), namely

$$|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \leq \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1 + \frac{d}{2}\right)} \cdot \left(\frac{diam(I_\lambda)}{diam(I_{\lambda'})}\right)^{(d+2\beta)/2} . \qquad (5.49)$$

Combining (5.48) with (5.49) we finally get for all indices λ and λ' the desired estimate

$$\begin{aligned}
&|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \\
&\leq \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1 + \frac{d}{2}\right)} \cdot \min\left\{\frac{diam(I_{\lambda'})}{diam(I_\lambda)}, \frac{diam(I_\lambda)}{diam(I_{\lambda'})}\right\}^{(d+2\beta)/2} . \qquad \square \qquad (5.50)
\end{aligned}$$

Combining Proposition 5.51 with Theorem 5.52 we obtain the following result.

Proposition 5.53 *(Localization property for a multivariate function family with compact supports)*
Let Λ be a countable index set and let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions from $L_2(X)$, $X \in \{\mathbb{R}^d, (-\infty, b], [a, \infty), [a, b]\}$, $a, b \in \mathbb{R}^d$, with compact supports $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$.
If Ψ satisfies conditions a)–c) in Theorem 5.52, then we have

$$|\langle\psi_\lambda, \psi_{\lambda'}\rangle| \leq \frac{2^{2\beta} C_1 C_2 \pi^{d/2}}{\Gamma\left(1 + \frac{d}{2}\right)} \cdot e^{-(d+2\beta)\cdot d_{\mathcal{I}}(I_\lambda, I_{\lambda'})} \qquad \text{for all } (\lambda, \lambda') \in \Lambda^2,$$

with distance function $d_{\mathcal{I}}$ defined in (5.44).

5.11 Multivariate non–stationary vaguelettes functions with compact support

We first extend the definition of a compactly supported non–stationary vaguelettes family to the d–dimensional case by proposing the following.

Definition 5.54 *(Multivariate compactly supported non–stationary vaguelettes family)*
Let Λ be a countable index set. A family Ψ of continuous and compactly supported functions $\psi_\lambda : \mathbb{R}^d \to \mathbf{C}$, $\lambda \in \Lambda$, with

$$supp\, \psi_\lambda \subseteq [c_\lambda, b_\lambda] =: I_\lambda,$$

is called a d–dimensional compactly supported non–stationary vaguelettes family on $X := co\,(\cup_{\lambda\in\Lambda} I_\lambda)$, if the following conditions are satisfied:

a) $\displaystyle\int_{I_\lambda} \psi_\lambda(x)\, dx = 0$ *for all $\lambda \in \Lambda$*

(vanishing moment);

b) $\exists C_1 > 0 \quad \forall \lambda \in \Lambda : \quad \|\psi_\lambda\|_\infty \leq C_1 \cdot diam(I_\lambda)^{-d/2}$

(support–adapted uniform boundedness);

c) $\exists \beta > 0 \quad \exists C_2 > 0 \quad \forall \lambda \in \Lambda \quad \forall x, x' \in I_\lambda \quad (x \neq x') :$

$$|\psi_\lambda(x) - \psi_\lambda(x')| \leq C_2 \cdot diam(I_\lambda)^{-(d+2\beta)/2} \cdot |x - x'|^\beta$$

(support–adapted Hölder continuity);

on the right hand side $|\cdot|$ *stands for the Euclidian distance in* $\mathbb{R}^d$*;*

d) $\exists D_2 > 0 \quad \forall J \subset \mathbb{R}^d$ *cuboid* $\#\Lambda_J \leq D_2$,

$$\text{where} \quad \Lambda_J := \left\{ \lambda \in \Lambda \ : \ diam(I_\lambda) \in \left[\tfrac{diam(J)}{2}, diam(J)\right], \ c_\lambda \in J \right\}$$

(finite overlapping constant).

The constants C_1*,* C_2*,* β *and* D_2 *will be called the parameters of the vaguelettes family* Ψ*. The support of the family* Ψ*, denoted by* X*, is the convex hull of* $\cup_{\lambda \in \Lambda} I_\lambda$*. This may be a bounded or unbounded interval in* $\mathbb{R}^d$*. The operator* S *associated to the family* Ψ *(see Definition 5.2) is called the vaguelettes operator – in analogy to the frame case. For every* g *in* $L_2(X)$ *the sequence* $(\langle g, \psi_\lambda \rangle)_{\lambda \in \Lambda}$ *is called the 'vaguelettes decomposition' of* g *w.r.t.* Ψ*.*

An analogon to Remark 5.38 holds in the multivariate case, too.

Note that in the compactly supported case Meyer's vaguelettes (see Definition 5.7) are included in Definition 5.54, namely for $\lambda = (j, k) \in \Lambda = \mathbb{Z} \times \mathbb{Z}^d$. One can rewrite (5.4) and (5.5) as

$$|f_{j,k}(x)| \leq C \cdot l(I_{j,k})^{-d/2} \cdot [1 + l(I_{j,k})^{-1} \cdot |x - c(I_{j,k})|]^{-(d+\alpha)}, \tag{5.51}$$

$$|f_{j,k}(x) - f_{j,k}(x')| \leq C \cdot l(I_{j,k})^{-(d/2+\beta)} \cdot |x - x'|^\beta. \tag{5.52}$$

Remember that in the stationary case we have $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)) \subset \mathbb{R}^d$, which means that all $I_{j,k}$'s are cubes and thus

$$diam(I_{j,k}) = \sqrt{d} \cdot l(I_{j,k}). \tag{5.53}$$

In Proposition 5.57 we will prove that condition d) in Definition 5.54 is fulfilled in the stationary case. Combining (5.51) and (5.52) with (5.53) and noting that Definition 5.54.b) implies (5.51), one gets the desired relation between Definition 5.7 and Definition 5.54.

Our next result contains a boundedness criterion for multivariate compactly supported non–stationary vaguelettes families and is thus one further generalization of Theorem 5.17 of Frazier&Jawerth (see [32, Theorem 3.3]).

Theorem 5.55 *(Boundedness criterion)*
Let $\Psi = \{\psi_\lambda\}_{\lambda \in \Lambda}$ *be a* d*–dimensional compactly supported non–stationary vaguelettes family with parameters* C_1*,* C_2*,* β *and* D_2*.*
Then $M := Gram(\Psi)$ *defines a bounded linear operator on* $l_2(\Lambda)$ *and*

$$\|M\|_{l_2(\Lambda) \to l_2(\Lambda)} \leq D_2 \cdot C_1 C_2 \cdot \frac{2^{\beta+1}}{2^\beta - 1} \cdot \frac{\pi^{d/2}}{\Gamma\left(1 + \frac{d}{2}\right)}.$$

Proof. Let $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ be a d–dimensional compactly supported vaguelettes family with parameters C_1, C_2, β and D_2. From Theorem 5.52 we get the estimate

$$\begin{aligned} &|\langle\psi_\lambda,\psi_{\lambda'}\rangle| \\ &\leq \frac{C_1C_2\pi^{d/2}}{2^d\cdot\Gamma\left(1+\frac{d}{2}\right)}\cdot\min\left\{\frac{diam(I_{\lambda'})}{diam(I_\lambda)},\frac{diam(I_\lambda)}{diam(I_{\lambda'})}\right\}^{(d+2\beta)/2} =: \mathbf{M}_{\lambda,\lambda'}. \end{aligned} \tag{5.54}$$

We want to apply Schur's lemma for $m_{\lambda,\lambda'} = \langle\psi_\lambda,\psi_{\lambda'}\rangle$. Therefore we have to choose a suitable sequence ω. This will be $\omega(\lambda) := diam(I_\lambda)^{d/2}$. Next we have to check that there exists a constant $a(d,C_1,C_2,\beta,D_2)$ such that for any arbitrarily fixed λ there holds

$$S_\lambda := \sum_{\lambda'\in\Lambda} diam(I_{\lambda'})^{d/2}\cdot|\langle\psi_\lambda,\psi_{\lambda'}\rangle| \leq a(d,C_1,C_2,\beta,D_2)\cdot diam(I_\lambda)^{d/2}.$$

The second inequality in the hypothesis of Schur's lemma will follow then by analogous arguments, due to symmetry.
In the remainder of this proof let λ be arbitrarily fixed. For the index set

$$A_\lambda := \{\lambda'\in\Lambda\ : \mathring{I}_\lambda\cap\mathring{I}_{\lambda'}\neq\emptyset\}$$

we consider again the disjoint partition $A_\lambda = A_\lambda^-\sqcup A_\lambda^+$ with

$$\begin{aligned} A_\lambda^- &:= \{\lambda'\in A_\lambda\ :\ diam(I_{\lambda'})\leq diam(I_\lambda)\},\\ A_\lambda^+ &:= \{\lambda'\in A_\lambda\ :\ diam(I_{\lambda'}) > diam(I_\lambda)\}. \end{aligned}$$

Thus we have

$$S_\lambda \overset{(5.54)}{\leq} \sum_{\lambda'\in A_\lambda^-} diam(I_{\lambda'})^{d/2}\cdot\mathbf{M}_{\lambda,\lambda'} + \sum_{\lambda'\in A_\lambda^+} diam(I_{\lambda'})^{d/2}\cdot\mathbf{M}_{\lambda,\lambda'} =: S_\lambda^- + S_\lambda^+.$$

For every $(l,r)\in I\!N_0\times Z\!\!\!Z$ we define the index sets

$$\begin{aligned} R_\lambda^-(l,r) := \Big\{\lambda'\in A_\lambda^-\ :\quad & diam(I_{\lambda'})\in\left(2^{-(l+1)}\cdot diam(I_\lambda), 2^{-l}\cdot diam(I_\lambda)\right],\\ & c_{\lambda'}\in\left[2^{-l}\cdot diam(I_\lambda)\cdot r, 2^{-l}\cdot diam(I_\lambda)\cdot(r+1)\right)\Big\},\\ R_\lambda^+(l,r) := \Big\{\lambda'\in A_\lambda^+\ :\quad & diam(I_{\lambda'})\in\left[2^{l}\cdot diam(I_\lambda), 2^{l+1}\cdot diam(I_\lambda)\right),\\ & c_{\lambda'}\in\left[2^{l+1}\cdot diam(I_\lambda)\cdot r, 2^{l+1}\cdot diam(I_\lambda)\cdot(r+1)\right)\Big\}. \end{aligned}$$

In analogy with the univariate case we get further the disjoint partitions

$$\begin{aligned} A_\lambda^- &= \sqcup_{l\in I\!N_0}\sqcup_{r\in H_{\lambda,l}^-} R_\lambda^-(l,r),\\ A_\lambda^+ &= \sqcup_{l\in I\!N_0}\sqcup_{r\in H_{\lambda,l}^+} R_\lambda^+(l,r), \end{aligned}$$

where $\#H_{\lambda,l}^-\leq(2^l+1)^d$, $\#H_{\lambda,l}^+\leq 2^d$, $\#R_\lambda^-(l,r)\leq D_2$ and $\#R_\lambda^+(l,r)\leq D_2$.
The arguments from above now entail

$$\begin{aligned} S_\lambda^- &= \sum_{\lambda'\in A_\lambda^-} diam(I_{\lambda'})^{d/2}\cdot\mathbf{M}_{\lambda,\lambda'}\\ &= \frac{C_1C_2\pi^{d/2}}{2^d\cdot\Gamma\left(1+\frac{d}{2}\right)}\cdot diam(I_\lambda)^{d/2}\cdot\sum_{l\in I\!N_0}\sum_{r\in H_{\lambda,l}^-}\sum_{\lambda'\in R_\lambda^-(l,r)}\left(\frac{diam(I_{\lambda'})}{diam(I_\lambda)}\right)^{d+\beta} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{d+\beta} \sum_{r \in H^-_{\lambda,l}} \# R^-_\lambda(l,r) \\
&\leq D_2 \cdot \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{d+\beta} \cdot \# H^-_{\lambda,l} \\
&\leq D_2 \cdot \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{d+\beta} \cdot (2^l+1)^d \\
&\leq D_2 \cdot \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot 2^d \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\beta} \\
&= D_2 \cdot \frac{2^\beta C_1 C_2 \pi^{d/2}}{(2^\beta - 1) \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} =: a^- \cdot diam(I_\lambda)^{d/2}
\end{aligned}
$$

and

$$
\begin{aligned}
S^+_\lambda &= \sum_{\lambda' \in A^+_\lambda} diam(I_{\lambda'})^{d/2} \cdot \mathbf{M}_{\lambda,\lambda'} \\
&= \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \sum_{r \in H^+_{\lambda,l}} \sum_{\lambda' \in R^+_\lambda(l,r)} \left(\frac{diam(I_\lambda)}{diam(I_{\lambda'})}\right)^{\beta} \\
&\leq \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\beta} \sum_{r \in H^+_{\lambda,l}} \# R^+_\lambda(l,r) \\
&\leq D_2 \cdot \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\beta} \cdot \# H^+_{\lambda,l} \\
&\leq D_2 \cdot \frac{C_1 C_2 \pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \cdot \sum_{l \in \mathbb{N}_0} \left(\frac{1}{2^l}\right)^{\beta} \cdot 2^d \\
&= D_2 \cdot \frac{2^\beta C_1 C_2 \pi^{d/2}}{(2^\beta - 1) \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} =: a^+ \cdot diam(I_\lambda)^{d/2}
\end{aligned}
$$

Finally, it follows

$$
S_\lambda \leq a(d, C_1, C_2, \beta, D_2) \cdot diam(I_\lambda)^{d/2}, \qquad \forall \lambda \in \Lambda,
$$

with

$$
\begin{aligned}
a(d, C_1, C_2, \beta, D_2) &:= a^- + a^+ \\
&= D_2 \cdot \frac{2^{\beta+1} C_1 C_2 \pi^{d/2}}{(2^\beta - 1) \cdot \Gamma\left(1+\frac{d}{2}\right)}.
\end{aligned}
$$

Hence we can apply Schur's lemma with $a_1 = a_2 = a(d, C_1, C_2, \beta, D_2)$. An argument similar to the univariate one completes the proof. □

Theorem 5.56 *(Multivariate Bessel families)*
Every d–dimensional compactly supported non–stationary vaguelettes family with support X is a Bessel family in $L_2(X)$. Moreover, if C_1, C_2, β and D_2 are the parameters of the vaguelettes family, then a Bessel bound is given by

$$
B_2 := \sqrt{D_2 \cdot C_1 C_2 \cdot \frac{2^{\beta+1}}{2^\beta - 1} \cdot \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}}. \tag{5.55}
$$

Corollary 5.41 holds in the multivariate case, too, but with B_2 given in (5.55).

Again the question for the significance of condition d) in Definition 5.54 arises, if one deals with Meyer's dyadic stationary case in $\mathbb{R}^d$ (compact case with disjoint supports on each level). The answer is given by the next result.

Proposition 5.57 *(Overlapping constants in the d–dimensional dyadic stationary case with disjoint supports on each level)*
In the d–dimensional compactly supported stationary case with disjoint supports $Q_{j,k} = [2^{-j}k, 2^{-j}(k+1))$, $k \in \mathbb{Z}^d$, *on each level* $j \in \mathbb{Z}$ *we have the overlapping constant*

$$D_2 = 2^d(2^d+1) \quad \textit{and thus} \quad D_2 = \mathcal{O}(2^{2d}).$$

Moreover, we obtain the Bessel bound

$$B_2 = \sqrt{C_1C_2} \cdot \sqrt{\frac{2^\beta}{2^\beta - 1}} \cdot \sqrt{\frac{2^{d+1}(2^d+1)\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}}. \tag{5.56}$$

Some possible situations for D_2 in the two–dimensional case are presented in Figures 5.23, 5.24 and 5.25. For a proof of Proposition 5.57 see the more general Proposition 5.59 below. If we replace in Definition 5.54 condition d) by a more general one, namely

d") finite overlapping constant, i.e.,

$$\exists n \in \mathbb{N} \setminus \{1\} \quad \exists D_n > 0 \quad \forall J \subset \mathbb{R} \text{ bounded interval:} \quad \#\Lambda_J \leq D_n,$$
$$\text{where} \quad \Lambda_J := \left\{\lambda \in \Lambda \;:\; diam(I_\lambda) \in \left[\frac{diam(J)}{n}, diam(J)\right],\; c_\lambda \in J\right\},$$

then we can prove the following result for this n–adic non–stationary case.

Theorem 5.58 *(Multivariate Bessel families)*
Every d–dimensional n–adic compactly supported non–stationary vaguelettes family with support X is a Bessel family in $L_2(X)$*. Moreover, if* C_1*,* C_2*,* β *and* D_n *are the parameters of the vaguelettes family, then a Bessel bound is given by*

$$B_n := \sqrt{D_n \cdot C_1C_2 \cdot \frac{2n^\beta}{n^\beta - 1} \cdot \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}}. \tag{5.57}$$

Proof. With straightforward modifications one gets

$$S_\lambda^- \leq D_n \cdot \frac{C_1C_2\pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \sum_{l\geq 0} \left(\frac{1}{n^l}\right)^{d+\beta} \cdot (n^l+1)^d,$$
$$S_\lambda^+ \leq D_n \cdot \frac{C_1C_2\pi^{d/2}}{2^d \cdot \Gamma\left(1+\frac{d}{2}\right)} \cdot diam(I_\lambda)^{d/2} \sum_{l\geq 0} \left(\frac{1}{n^l}\right)^{\beta} \cdot 2^d,$$

and thus the mentioned bound. □

An analogon to Proposition 5.44 also holds in the multivariate case.

If we extend Meyer's dyadic vaguelettes $f_{j,k}$ in a natural fashion to the n–adic case, thus obtaining functions $f_{j,k}$ with $(j,k) \in \mathbb{Z} \times \mathbb{Z}^d$, then the overlapping constants D_n and Bessel bounds B_n have explicit representations as given in the following result.

Proposition 5.59 *(Overlapping constants in the d–dimensional n–adic stationary case with disjoint supports on each level)*
In the d–dimensional n–adic compactly supported stationary case with disjoint supports $I_{j,k} = [n^{-j}k, n^{-j}(k+1))$, $k \in \mathbb{Z}^d$, *on each level* $j \in \mathbb{Z}$ *we have the overlapping constant*

$$D_n = n^d(n^d+1) \quad \text{and thus} \quad D_n = \mathcal{O}(n^{2d}).$$

Furthermore, we obtain in this case the Bessel bound

$$B_n = \sqrt{C_1 C_2} \cdot \sqrt{\frac{2n^\beta}{n^\beta - 1}} \cdot \sqrt{\frac{n^d(n^d+1)\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}}. \tag{5.58}$$

Proof. Let $J = [c_J, b_J] \subset \mathbb{R}^d$ be an arbitrary cube with $|J| \neq 0$ and $n \in \mathbb{N}$ be fixed such that $n \geq 2$. In the sequel W will denote the set $\{\sqrt{d} \cdot n^j : j \in \mathbb{Z}\}$.
There exists a unique $l := l(J) \in \mathbb{Z}$ such that $diam(J) \in [\sqrt{d} \cdot n^l, \sqrt{d} \cdot n^{l+1})$. We are interested in counting vaguelettes $f_{j,k}$ with

$$diam\, supp\, f_{j,k} \leq diam(I_{j,k}) \in \left[\frac{diam(J)}{n}, diam(J)\right] \quad \text{and} \quad c_{j,k} \in J.$$

This means in the stationary case that

$$diam\, supp\, f_{j,k} \leq diam(I_{j,k}) \in \left[\sqrt{d} \cdot n^{l-1}, \sqrt{d} \cdot n^{l+1}\right) \cap W = \{\sqrt{d} \cdot n^{l-1}, \sqrt{d} \cdot n^l\},$$

i.e., $j \in \{-l+1, -l\}$. Furthermore, there exists a unique $k_0 := k_0(J) \in \mathbb{Z}^d$ such that

$$c_J \in \left[n^{l+1}k_0, n^{l+1}(k_0+1)\right).$$

Because $diam(J) < \sqrt{d} \cdot n^{l+1}$ and $diam(J) = \sqrt{d} \cdot l(J)$ it follows that

$$b_J \in \left(n^{l+1}k_0, n^{l+1}(k_0+2)\right).$$

In this setting there exist at most n^d functions $f_{j,k}$ with $j = -l$, support diameter $\sqrt{d} \cdot n^l$ and corner $n^l k \in J$, corresponding to n^d adjacent k's from the tensor product

$$\begin{aligned}
&\{nk_0, nk_0 + e_1, nk_0 + 2e_1, \ldots, nk_0 + (2n-1)e_1\} \times \\
&\{nk_0, nk_0 + e_2, nk_0 + 2e_2, \ldots, nk_0 + (2n-1)e_2\} \times \ldots \times \\
&\{nk_0, nk_0 + e_d, nk_0 + 2e_d, \ldots, nk_0 + (2n-1)e_d\}.
\end{aligned}$$

Here $\{e_1, e_2, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. For the bi–dimensional case some illustrative examples are given in the left column of the Figures 5.23, 5.24, 5.25, respectively. Furthermore, there exist at most n^{2d} functions $f_{j,k}$ with $j = -l+1$, support diameter $\sqrt{d} \cdot n^{l-1}$ and corner $n^{l-1}k \in J$, corresponding to n^{2d} adjacent values k in

$$\begin{aligned}
&\{n^2k_0, n^2k_0 + e_1, n^2k_0 + 2e_1, \ldots, n^2k_0 + (2n^2-1)e_1\} \times \\
&\{n^2k_0, n^2k_0 + e_2, n^2k_0 + 2e_2, \ldots, n^2k_0 + (2n^2-1)e_2\} \times \ldots \times \\
&\{n^2k_0, n^2k_0 + e_d, n^2k_0 + 2e_d, \ldots, n^2k_0 + (2n^2-1)e_d\}.
\end{aligned}$$

For examples in the two–dimensional case see the right columns of Figures 5.23, 5.24, 5.25. This yields the overlapping constant $D_n = n^d + n^{2d} = n^d(n^d+1)$, independently of the interval J chosen. □

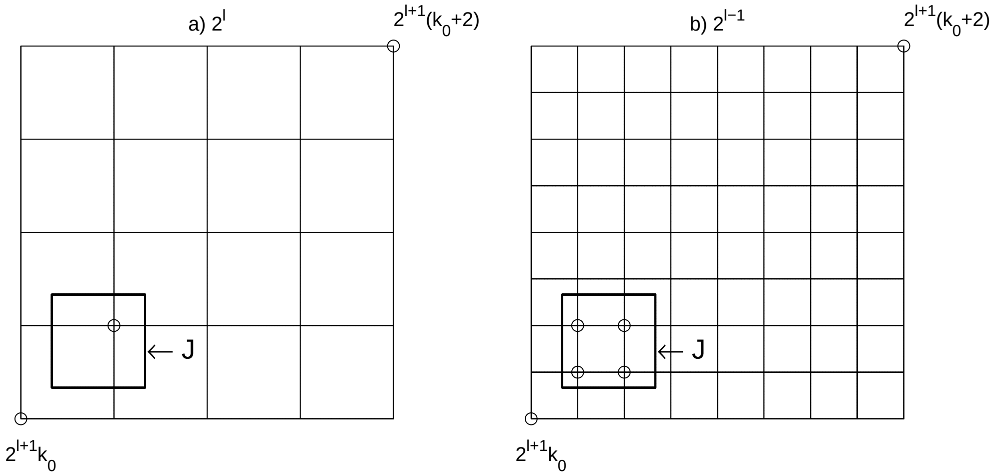

Figure 5.23: $d = 2$, $\#\Lambda_J = 5$

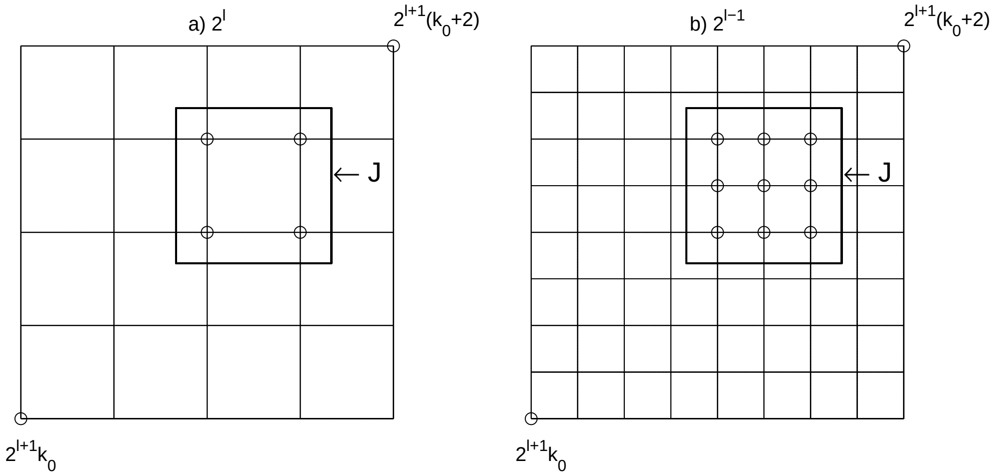

Figure 5.24: $d = 2$, $\#\Lambda_J = 13$

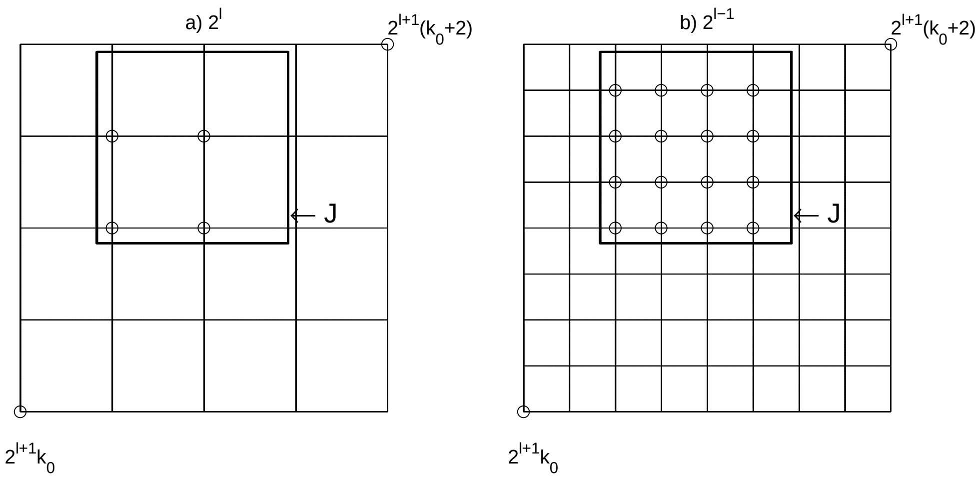

Figure 5.25: $d = 2$, maximal value for $\#\Lambda_J$: $D_2 = 20$

Chapter 6

Sibling Spline Frames

The aim of this chapter is to give concrete examples of sibling spline frames in $L_2[a,b]$ and to demonstrate hereby that the theory developed in Chapter 5 can indeed be used to check the Bessel property for spline function families.

We present a general construction scheme for *the quasi–uniform situation* and propose one for *the case of locally comparable support lengths*. For the quasi–uniform case we prove in detail that our scheme is fulfilling the sufficient conditions formulated in Theorem 5.40. It thus generates univariate Bessel families of spline functions on a compact interval $[a,b]$ which are thus constituting sibling frames.

6.1 Our general ansatz for sibling spline frames

We consider in this chapter the B–spline setting on a bounded interval $[a,b]$, as presented in Chapters 2&3. In the sequel we briefly summarize the most important parameters of this setting.

The natural number m denotes as usual the order of B–splines. The non–stationary spline MRA on the bounded interval $[a,b]$ is generated by the dense sequence of finite admissible knot vectors[1]

$$\mathbf{t}_0 \subset \cdots \subset \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \cdots \subset [a,b], \qquad \text{i.e.,}$$

$$\Pi_{m-1}[a,b] \subset \mathcal{S}_m(\mathbf{t}_0,[a,b]) \subset \cdots \subset \mathcal{S}_m(\mathbf{t}_j,[a,b]) \subset \mathcal{S}_m(\mathbf{t}_{j+1},[a,b]) \subset \cdots \subset L_2[a,b].$$

We assume further that $\mathbf{t}_j$ has N_j interior knots of multiplicity **at most** $(\mathbf{m-1})$ and stacked boundary knots of maximal multiplicity m $(m \geq 2)$. The L_2–normalized B–splines of order m over the knot sequence $\mathbf{t}_j$ are denoted as usual by

$$[N^B_{\mathbf{t}_j;m,k}]_{k\in I\!M_{\mathbf{t}_j;m,0}} := [(d_{\mathbf{t}_j;m,0,k})^{-1/2} \cdot N_{\mathbf{t}_j;m,k}]_{k\in I\!M_{\mathbf{t}_j;m,0}}$$

with index set $I\!M_{\mathbf{t}_j;m,\nu} := \{-m+1,\ldots,N_j-\nu\}$ and weighted knot differences

$$d_{\mathbf{t}_j;m,\nu,k} := \frac{t^{(j)}_{m+k+\nu} - t^{(j)}_k}{m+\nu}, \qquad 0 \leq \nu \leq m.$$

The whole B–spline Riesz basis of the Schoenberg space $\mathcal{S}_m(\mathbf{t}_j,[a,b])$ is denoted by $\mathbf{\Phi}^B_{\mathbf{t}_j;m}$. The refinement relations of the corresponding B–splines are given by

$$\mathbf{\Phi}^B_{\mathbf{t}_j;m} \;=\; \mathbf{\Phi}^B_{\mathbf{t}_{j+1};m} \cdot P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\,,$$

[1] In the sense of Definition 2.1.

with refinement matrix $P_{\mathbf{t}_j,\mathbf{t}_{j+1};m}$ obtained from the Oslo algorithm and

$$P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} := diag\left(d^{1/2}_{\mathbf{t}_{j+1};m,0}\right) \cdot P_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot diag\left(d^{-1/2}_{\mathbf{t}_j;m,0}\right).$$

The integer $L \in \mathbb{N}$, $(1 \leq L \leq m)$, stands for the order of the approximate dual matrix $S_{\mathbf{t}_j;m,L}$ of Chui, He and Stöckler (constructed directly and only from the knot sequences $\mathbf{t}_j$, as presented in Section 3.3), and thus also for the number of vanishing moments of the (dual) framelets $\psi_{j,k}$ (and $\tilde{\psi}_{j,k}$, respectively), as explained below.

In Subsection 4.3.3 we formulated and justified the following general construction principle for sibling spline frames:

> In order to obtain sibling spline frames of $L_2[a,b]$ we have to factorize the matrices
>
> $$S^B_{\mathbf{t}_{j+1};m,L} - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} \cdot S^B_{\mathbf{t}_j;m,L} \cdot \left(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\right)^T \tag{6.1}$$
>
> appropriately into $Q_j \cdot \tilde{Q}_j^T$, i.e., we have to determine coefficient matrices Q_j and $\tilde{Q}_j$ such that the Bessel conditions (4.10) and (4.11) are satisfied.

Taking into account the representation (3.31) for the matrix in (6.1) we are able to formulate next the **general ansatz** for our construction schemes for sibling spline frames:

$$\begin{aligned}
\Psi = \{\Psi_j(x)\}_{j\geq 0} &:= \left\{\frac{d^L}{dx^L}\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m+L}(x) \cdot A_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}\right\}_{j\geq 0} && (6.2)\\
&= \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot A_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}\right\}_{j\geq 0}, && (6.3)
\end{aligned}$$

$$\begin{aligned}
\tilde{\Psi} = \{\tilde{\Psi}_j(x)\}_{j\geq 0} &:= \left\{\frac{d^L}{dx^L}\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m+L}(x) \cdot Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L} \cdot A^{-1}_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}\right\}_{j\geq 0} && (6.4)\\
&= \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L} \cdot A^{-1}_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}\right\}_{j\geq 0}, && (6.5)
\end{aligned}$$

with $E^B_{\mathbf{t_j};m,L}$ from (2.47)–(2.48) and $Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ from (3.32)–(3.34). Thus we consider for $j \in \mathbb{N}_0$ the (dual) frame matrices

$$\begin{aligned}
Q_j &= E^B_{\mathbf{t_{j+1}};m,L} \cdot A_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}, && (6.6)\\
\tilde{Q}_j &= E^B_{\mathbf{t_{j+1}};m,L} \cdot Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L} \cdot A^{-1}_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}. && (6.7)
\end{aligned}$$

We still have to specify the normalization matrices $A_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$. This will be done for every case in part in Sections 6.3 and 6.5. Thus we provide explicit formulations for both the frame and the dual frame elements.

The above structure of the function systems Ψ, $\tilde{\Psi}$ reveals straight away the following feature of the (dual) framelets: every framelet $\psi_{j,k}$ and every dual framelet $\tilde{\psi}_{j,k}$ exhibit L vanishing moments.

For further computations we need to analyze the matrix $Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ more carefully.

6.2 The structure of the matrix Z

In this section we study in detail the structure of the matrix $Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ in order to obtain a useful estimate for its elements. This is needed for our subsequent study of the dual frame, as mentioned already in Section 6.1.

Subsection 6.2.1 presents the general situation, Subsection 6.2.2 details the case of bounded refinement rate between adjacent levels. In Subsection 6.2.3 we develop an example which visualizes the structures of the matrices V, P and Z.

6.2.1 The general situation

For splines of order m with L vanishing moments and for the refinement $\mathbf{t}_j \subset \mathbf{t}_{j+1}$ we obtain the matrix $Z^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ by the following algorithm (see Section 3.5).

Algorithm 6.1 *(Computation steps for the matrix Z)*

- **Input:** m, L, $\mathbf{t}_j$, $\mathbf{t}_{j+1}$.
- **Output:** $Z^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$.
- **Procedure:**

Denote the cardinality of the refinement by $M := \#(\mathbf{t}_{j+1} \setminus \mathbf{t}_j)$ *and the intermediate knot vectors by*

$$\mathbf{t}_j = \mathbf{s_0} \subset \mathbf{s_1} \subset \cdots \subset \mathbf{s_{M-1}} \subset \mathbf{s_M} = \mathbf{t}_{j+1},$$

where $\#(\mathbf{s_{k+1}} \setminus \mathbf{s_k}) = 1$ *for all possible* k. *In this procedure the* M *new knots are inserted one by one, in order, from the left to the right.* $N_{\mathbf{s_k}}$ *denotes the number of inner knots in the sequence* $\mathbf{s_k}$ *and* $s_i^{(k)}$ *some element of the sequence* $\mathbf{s_k}$.

We have

$$\begin{aligned} Z^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} &= diag\left(d_{\mathbf{t}_{j+1};m,L}\right) \cdot Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}, \\ Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} &= \sum_{k=1}^{M} P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}, \\ V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} &= diag\left(v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}\right), \end{aligned} \tag{6.8}$$

where $P_{\mathbf{s_k},\mathbf{s_M};m+L}$ *are the refinement matrices given by the Oslo algorithm and the diagonal matrices* $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$ *have non–negative entries denoted as follows:*

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} = [v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l}]_{l \in \mathbb{M}_{\mathbf{s_k};m,L-1}}.$$

For the knot $\{\tau\} := \mathbf{s_k} \setminus \mathbf{s_{k-1}}$ *and the corresponding index* ρ *defined via*

$$\tau \in \left[s_\rho^{(k-1)}, s_{\rho+1}^{(k-1)}\right)$$

we have the following representation for the entries of $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$*:*

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} = \begin{cases} \dfrac{s^{(k)}_{l+m+L} - s^{(k)}_{\rho+1}}{s^{(k)}_{l+m+L} - s^{(k)}_{l}} \cdot \dfrac{s^{(M)}_{\rho+1} - s^{(M)}_{l}}{m+L-1} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} \quad \textit{for} \\ \qquad \max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}} - \rho + 1), \\ 0 \quad \textit{otherwise} \,, \end{cases}$$

with β–coefficients defined for $l \in \mathbb{M}_{\mathbf{s_{k-1}};m,L-2}$ by

$$\beta_{\mathbf{s_{k-1}};m,L-1,l} = \frac{m!(m-L)!}{(m+L-1)!(m+L-2)!} \cdot \cdot \sum_{\substack{l+1 \le i_1,\ldots,i_{2(L-1)} \le l+m+L-2 \\ i_1,\ldots,i_{2(L-1)} \text{ distinct} \\ i_1 > i_3 > \cdots > i_{2L-3} \\ i_{2j-1} > i_{2j} \text{ for } 1 \le j \le L-1}} \prod_{j=1}^{L-1} \left(s_{i_{2j-1}}^{(k-1)} - s_{i_{2j}}^{(k-1)} \right)^2$$

in case $2 \le L \le m$ and with $\beta_{\mathbf{s_{k-1}};m,L-1,l} = 1$ for $L = 1$. □

In order to study the matrix Z we need to understand in detail the structure of the refinement matrices P.

Discussion 6.2 *(Description of the matrices $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ and $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$)*

a) *Consider $\mathbf{t} \subseteq \tilde{\mathbf{t}}$ two knot sequences with m stacked boundary knots, respectively. The row indices of $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ refer to the basis $\mathbf{\Phi}_{\tilde{\mathbf{t}};m}$ and the column indices refer to the basis $\mathbf{\Phi}_{\mathbf{t};m}$. There holds $\mathbf{\Phi}_{\mathbf{t};m} = \mathbf{\Phi}_{\tilde{\mathbf{t}};m} \cdot P_{\mathbf{t},\tilde{\mathbf{t}};m}$.*

   If $\tilde{\mathbf{t}} = \mathbf{t}$, then $P_{\mathbf{t},\tilde{\mathbf{t}};m} = I_{m+N}$, where N denotes the cardinality of inner knots in $\tilde{\mathbf{t}}$ and I_n the identity matrix of dimension n.

   If $\tilde{\mathbf{t}} \setminus \mathbf{t} \neq \emptyset$, then $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ is a rectangular matrix of dimension $(m+\tilde{N}) \times (m+N)$. The constant $\tilde{N}$ (N) denotes the number of inner knots in $\tilde{\mathbf{t}}$ ($\mathbf{t}$), respectively.

b) *Assume $\mathbf{t} \subseteq \tilde{\mathbf{t}}$ to be two knot sequences with m stacked boundary knots, respectively. Let L be the number of vanishing moments ($1 \le L \le m$).*

   The row indices of $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ refer to the function vector $\mathbf{\Phi}_{\tilde{\mathbf{t}};m+L}$ and the column indices refer to the function vector $\mathbf{\Phi}_{\mathbf{t};m+L}$. These vectors do not represent a whole B–spline basis of order $m+L$, because the boundary knots have only multiplicity m and not $m+L$. Therefore, the first L and the last L basis functions are missing, respectively.

   Nonetheless, $\mathbf{\Phi}_{\mathbf{t};m+L} = \mathbf{\Phi}_{\tilde{\mathbf{t}};m+L} \cdot P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$, if by $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ we denote the refinement matrix between the corresponding bases of order $m+L$ modified in the following way: the first L rows, the first L columns, the last L rows and the last L columns were eliminated.

   If $\tilde{\mathbf{t}} = \mathbf{t}$, then $P_{\mathbf{t},\tilde{\mathbf{t}};m+L} = I_{m-L+N}$, where N denotes the cardinality of inner knots in $\tilde{\mathbf{t}}$ and I_n the identity matrix of dimension n.

   If $\tilde{\mathbf{t}} \setminus \mathbf{t} \neq \emptyset$, then $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ is a rectangular matrix of dimension $(m-L+\tilde{N}) \times (m-L+N)$. The constant $\tilde{N}$ (N) denotes the number of inner knots in $\tilde{\mathbf{t}}$ ($\mathbf{t}$), respectively.

c) *Each element p of a general refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ given by the Oslo algorithm is either equal to 0, or it is positive and its value does not exceed 1 (see [18, p.157] and [51, p. 118–119]). The same property holds for the modified matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$.*

d) *Each row of $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ sums up to 1. The sum of each row from $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ has a value greater than 0 and less than or equal to 1.*

e) *Examples of matrices $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ can be found in Subsection 6.2.3 for $m = 4$, $L = 1$, $\mathbf{t}_j = \mathbf{t}_0$, $\mathbf{t}_{j+1} = \mathbf{t}_1$, $N = 5$ and $\tilde{N} = 23$, where*

$$\begin{aligned} \mathbf{t}_0 &= [0\ 0\ 0\ 0\ 1\ 2\ 3\ 4\ 5\ 6\ 6\ 6\ 6], \\ \mathbf{t}_1 \setminus \mathbf{t}_0 &= [0.25\ 0.5\ 0.75\ 1.25\ 1.5\ 1.75\ 2.25\ 2.5\ 2.75\ 3.25\ 3.5\ 3.75 \\ &\qquad 4.25\ 4.5\ 4.75\ 5.25\ 5.5\ 5.75]. \end{aligned}$$

The elements p of $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ with magnitude $p \in (0,1)$ are just symbolized by the characters $*$. The exact specification of these elements would make the representation of the (rather large) matrices rather unclear.

We focus next our attention on the formulae in Algorithm 6.1. Note the following.

Remark 6.3 *a) The rectangular refinement matrix $P_{\mathbf{s_k},\mathbf{s_M};m+L}$, $k \in \{1,\dots,M\}$, has dimension*

$$\begin{aligned}(\#\mathbb{M}_{\mathbf{s_M};m,L}) \times (\#\mathbb{M}_{\mathbf{s_k};m,L}) &= (N_{j+1}+m-L) \times (N_{\mathbf{s_k}}+m-L) \\ &= (N_{j+1}+m-L) \times (N_j+k+m-L).\end{aligned}$$

b) The square matrix $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$, $k \in \{1,\dots,M\}$, has dimension $N_{\mathbf{s_k}}+m-L = N_j + k+m-L$.

c) Each product matrix $P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$, $k \in \{1,\dots,M\}$, has dimension $N_{j+1}+m-L$.

d) Each matrix $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$ has at most $m+L-1$ non–zero elements on the diagonal.

e) For $k=1$ to $k=M$ the non–zero blocks on the diagonal of $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$ are 'moving' from the upper left corner to the bottom right corner of the matrix.

f) In the product $P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$ only those columns of $P_{\mathbf{s_k},\mathbf{s_M};m+L}$ (and only those rows of $P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$) are 'active', which correspond to the non–zero elements of $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$.

I.e., each element of the product $P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$ is a sum of at most $m+L-1$ terms of the form

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} \cdot p \cdot p',$$

where p is some element of $P_{\mathbf{s_k},\mathbf{s_M};m+L}$ and p' some element of $P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$.

g) The elements of the matrices $V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L}$, $k \in \{1,\dots,M\}$, and $Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ are non–negative.

h) Each element $e^{(k)}_{r,s}$ of the product $P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$ is non–negative and it can be estimated from above by

$$e^{(k)}_{r,s} \le (m+L-1) \cdot \max_{l=\max(\rho+2-m-L,1-m)}^{\min(\rho,N_{\mathbf{s_k}}-\rho+1)} \{v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l}\}. \tag{6.9}$$

In general the constant M <u>is not</u> a fixed constant. It depends on the level j and (in significant applications) its magnitude grows for $j \to \infty$. Therefore we <u>cannot</u> estimate an element $z_{r,s}$ of the matrix $Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ by

$$z_{r,s} \le M \cdot \max_{k=1}^{M} \left\{e^{(k)}_{r,s}\right\}.$$

For refinements with a certain structure a useful estimate (with an absolute constant instead of $M = M(j)$) for the elements $z_{r,s}$ will be given in Subsection 6.2.2.

Further we are interested in an estimate for the elements

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} = \frac{s^{(k)}_{l+m+L} - s^{(k)}_{\rho+1}}{s^{(k)}_{l+m+L} - s^{(k)}_{l}} \cdot \frac{s^{(M)}_{\rho+1} - s^{(M)}_{l}}{m+L-1} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l}\,.$$

Because of $l \le \rho$ and $\mathbf{s_M} = \mathbf{t}_{j+1}$ we obtain

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} \le 1 \cdot \frac{t^{(j+1)}_{\rho+1} - t^{(j+1)}_{l}}{m+L-1} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l}\,.$$

The inequality $l \ge \max(\rho+2-m-L, 1-m)$ yields further

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} \le d_{\mathbf{t}_{j+1};m,L-1,\max(\rho+2-m-L,1-m)} \cdot \beta_{\mathbf{s_{k-1}};m,L-1,l} \tag{6.10}$$

with $\beta_{\mathbf{s_{k-1}};m,L-1,l} = 1$ for $L = 1$ and for $2 \le L \le m$ with β–coefficients of the form

$$\beta_{\mathbf{s_{k-1}};m,L-1,l} \;=\; \frac{m!(m-L)!}{(m+L-1)!(m+L-2)!} \cdot F_{L-1}\left(s^{(k-1)}_{l+1}, \ldots, s^{(k-1)}_{l+m+L-2}\right), \tag{6.11}$$

where $\max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}} - \rho + 1)$.

Formula (5.7) from [18] yields the following estimate for the polynomial F_{L-1}:

$$F_{L-1}\left(s^{(k-1)}_{l+1}, \ldots, s^{(k-1)}_{l+m+L-2}\right) \le \frac{2^{-L+1}\cdot(m+L-2)!}{(L-1)!(m-L)!} \cdot$$

$$\cdot\left(s^{(k-1)}_{l+m+L-2} - s^{(k-1)}_{l+L-1}\right)^2 \cdot \left(s^{(k-1)}_{l+m+L-3} - s^{(k-1)}_{l+L-2}\right)^2 \cdot \ldots \cdot \left(s^{(k-1)}_{l+m} - s^{(k-1)}_{l+1}\right)^2 .$$

We obtain next

$$F_{L-1}\left(s^{(k-1)}_{l+1}, \ldots, s^{(k-1)}_{l+m+L-2}\right) \le \frac{2^{-L+1}\cdot(m+L-2)!}{(L-1)!(m-L)!} \cdot \left(\max_{r=1}^{L-1}\left(s^{(k-1)}_{l+m-1+r} - s^{(k-1)}_{l+r}\right)^2\right)^{L-1} .$$

The ranges for the indices $l+1$ and $l+m+L-2$ are

$$\max(\rho+3-m-L, 2-m) \le l+1 \le \min(\rho+1, N_{\mathbf{s_k}} - \rho + 2) \quad \text{and}$$

$$\max(\rho, L-1) \le l+m+L-2 \le \min(\rho+m+L-2, N_{\mathbf{s_k}} - \rho + m + L - 1),$$

respectively. Considering the construction scheme of the intermediate knot sequences $\mathbf{s_0}$ to $\mathbf{s_M}$, we obtain the following properties:

$$\begin{array}{lclcl}
s^{(k-1)}_{\rho+1} & = & t^{(j)}_{\rho+1-(k-1)} & = & t^{(j)}_{\rho+2-k}\,, \\
s^{(k-1)}_{\rho+3-m-L} & \ge & t^{(j)}_{\rho+3-m-L-(k-1)} & = & t^{(j)}_{\rho+4-m-L-k}\,, \\
 & & & & \\
s^{(k-1)}_{\rho+m+L-2} & = & t^{(j)}_{\rho+m+L-2-(k-1)} & = & t^{(j)}_{\rho+m+L-1-k}\,, \\
s^{(k-1)}_{\rho} & \ge & t^{(j)}_{\rho-(k-1)} & = & t^{(j)}_{\rho+1-k}\,,
\end{array}$$

which finally imply the following conclusions.

Proposition 6.4 *(Estimates for the generalized Marsden coefficients F)*
In the above setting, the homogeneous polynomials F_{L-1} verify for $2 \le L \le m$ and

$$\max(\rho+2-m-L, 1-m) \le l \le \min(\rho, N_{\mathbf{s_k}} - \rho + 1)$$

the following estimates:

$$\begin{aligned}
&F_{L-1}\left(s^{(k-1)}_{l+1}, \ldots, s^{(k-1)}_{l+m+L-2}\right) \\
&\le \frac{2^{-L+1}\cdot(m+L-2)!}{(L-1)!(m-L)!} \cdot (m-1)^{2(L-1)} \cdot \left(\max_{n=\rho+4-m-L-k}^{\rho+L-k}\{d_{\mathbf{t}_j;m-1,0,n}\}\right)^{2(L-1)} &(6.12)\\
&\le \frac{2^{-L+1}\cdot(m+L-2)!}{(L-1)!(m-L)!} \cdot (m-1)^{2(L-1)} \cdot \left(\max_{n\in \mathbb{M}_{\mathbf{t}_j;m-1,0}}\{d_{\mathbf{t}_j;m-1,0,n}\}\right)^{2(L-1)} . &(6.13)
\end{aligned}$$

Note the local character w.r.t. the parameter ρ of (6.12) and the global character of (6.13). This difference will be also encountered in the next three results. Formula (6.11) combined with Proposition 6.4 yields the following.

Proposition 6.5 *(Estimates for the β–coefficients of order greater than or equal to 1) With the aforementioned notations, the β–coefficients verify for $2 \le L \le m$ and for*

$$\max(\rho + 2 - m - L, 1 - m) \le l \le \min(\rho, N_{\mathbf{s_k}} - \rho + 1)$$

the inequalities

$$\begin{aligned}
\beta_{\mathbf{s_{k-1}};m,L-1,l} &\le \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-1)!(L-1)!} \cdot \left(\max_{n=\rho+4-m-L-k}^{\rho+L-k} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)} \\
&\le \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-1)!(L-1)!} \cdot \left(\max_{n \in I\!M_{\mathbf{t}_j;m-1,0}} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)}.
\end{aligned}$$

We obtain next estimates for the non–zero elements of the diagonal matrices V from (6.10) and Proposition 6.5.

Proposition 6.6 *(Estimates for the elements of the matrices V) For $L = 1$ there holds*

$$v_{\mathbf{s_{k-1}},\mathbf{s_k};m,1,l} \le d_{\mathbf{t}_{j+1};m,0,\max(\rho+1-m,1-m)} \le \max_{n \in I\!M_{\mathbf{t}_{j+1};m,0}} \{d_{\mathbf{t}_{j+1};m,0,n}\};$$

and for $2 \le L \le m$ the following:

$$\begin{aligned}
v_{\mathbf{s_{k-1}},\mathbf{s_k};m,L,l} &\le d_{\mathbf{t}_{j+1};m,L-1,\max(\rho+2-m-L,1-m)} \cdot \\
&\quad \cdot \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-1)!(L-1)!} \cdot \left(\max_{n=\rho+4-m-L-k}^{\rho+L-k} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)} \\
&\le \max_{n \in I\!M_{\mathbf{t}_{j+1};m,L-1}} \{d_{\mathbf{t}_{j+1};m,L-1,n}\} \cdot \\
&\quad \cdot \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-1)!(L-1)!} \cdot \left(\max_{n \in I\!M_{\mathbf{t}_j;m-1,0}} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)}.
\end{aligned}$$

In both cases $\max(\rho + 2 - m - L, 1 - m) \le l \le \min(\rho, N_{\mathbf{s_k}} - \rho + 1)$*.*

Proposition 6.6 and (6.9) imply local and global estimates for the elements of the product matrices PVP^T, which enable us to formulate in the sequel estimates for the elements of the matrices Z. This will be done in Subsection 6.2.2 for a special type of refinement.

Proposition 6.7 *(Estimates for the elements of the matrices PVP^T) If $L = 1$, then each non–zero element $e_{r,s}^{(k)}$ of the product $P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$ verifies*

$$e_{r,s}^{(k)} \le m \cdot d_{\mathbf{t}_{j+1};m,0,\max(\rho+1-m,1-m)} \le m \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,0}} \{d_{\mathbf{t}_{j+1};m,0,n}\}.$$

For the cases $2 \leq L \leq m$ the following inequalities hold:

$$e_{r,s}^{(k)} \leq \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-2)!(L-1)!} \cdot$$

$$\cdot d_{\mathbf{t}_{j+1};m,L-1,\max(\rho+2-m-L,1-m)} \cdot \left(\max_{n=\rho+4-m-L-k}^{\rho+L-k} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)}$$

$$\leq \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-2)!(L-1)!} \cdot$$

$$\cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,L-1}} \{d_{\mathbf{t}_{j+1};m,L-1,n}\} \cdot \left(\max_{n \in I\!M_{\mathbf{t}_j;m-1,0}} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)}.$$

6.2.2 Bounded refinement rate

In our subsequent constructions of sibling spline frames we consider the following refinement situation, which from now on will be called *'the bounded refinement rate case'*:

> **From each level j to the next one at most R new knots are inserted between two old ones.**

The constant R will be called in the sequel *'bound of refinement rate'*.

This property of the refinement implies a specific structure of $Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ which will enable us to find useful estimates for the elements of this matrix.

Note first the following three characteristics of the refinement matrices P obtained from the Oslo algorithm.

Proposition 6.8 *(Length of the non–zero blocks in the columns of P)*
Each column of a refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ has at most $(m+L) \cdot R+1$ non–zero elements; m denotes the order of the B–splines, L the order of vanishing moments $(1 \leq L \leq m)$ and R is the bound of the refinement rate between $\mathbf{t}$ and $\tilde{\mathbf{t}}$.

Proof. It is known that only those B–splines from $\mathbf{\Phi}_{\tilde{\mathbf{t}};m+L}$ appear in the refinement relation for a B–spline $N_{\mathbf{t};m+L,k}$ from $\mathbf{\Phi}_{\mathbf{t};m+L}$, whose (entire) support is contained in the support of $N_{\mathbf{t};m+L,k}$.
The support of $N_{\mathbf{t};m+L,k}$ contains $m+L+1$ old knots. In addition we have in $[t_k, t_{k+m+L}]$ at most $(m+L) \cdot R$ new knots. This implies at most $(m+L)(R+1)+1$ knots in $[t_k, t_{k+m+L}]$ and thus at most

$$(m+L)(R+1)+1-(m+L) = (m+L) \cdot R+1$$

B–splines over the knot sequence $\tilde{\mathbf{t}}$ with support contained in $[t_k, t_{k+m+L}]$. □

For $m = 4$, $L = 1$ and $R = 3$ we obtain the maximal length $(m+L) \cdot R+1 = 16$ for a non–zero block in the columns of P. This example is illustrated in Section 6.2.3. See, for example, column number 6 of the matrix $P_{\mathbf{s_1},\mathbf{s_{18}};4+1}$.

Proposition 6.9 *(Length of the non–zero blocks in the rows of P)*

a) <u>The general case.</u>

Each row of a refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ has at most $m+L$ non–zero elements; m denotes the order of the B–splines and L the number of vanishing moments $(1 \leq L \leq m)$.

b) The bounded refinement rate case.

Each row of a refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ *has at most*

$$\min\left\{\left[\frac{m+L+1+R}{2}\right],\ m+L\right\}$$

non–zero elements; m denotes the order of the B–splines, L the number of vanishing moments ($1 \le L \le m$) *and R is the bound of the refinement rate between* $\mathbf{t}$ *and* $\tilde{\mathbf{t}}$*. The brackets* $[\cdot]$ *stand for the integer part function.*

Proof. Considering the elements of an individual row i_0 of $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$, only those are non–zero which correspond to columns j such that

$$supp\ (N_{\tilde{\mathbf{t}};m+L,i_0}) \subseteq supp\ (N_{\mathbf{t};m+L,j}).$$

a) The maximum $m+L$ is attained when all $m+L-1$ inner knots of the B–spline $N_{\tilde{\mathbf{t}};m+L,i_0}$ are new knots (i.e., knots from $\tilde{\mathbf{t}} \setminus \mathbf{t}$), for example. Each old knot from $\mathbf{t}$ which plays the rôle of an inner knot for $N_{\tilde{\mathbf{t}};m+L,i_0}$ reduces the number of non–zero elements by 1.
b) If it is not possible to have all inner knots of $N_{\tilde{\mathbf{t}};m+L,i_0}$ from $\tilde{\mathbf{t}} \setminus \mathbf{t}$, because between two old knots are inserted at most R new knots (and $R < m+L-1$), then we have at most

$$\left[m+L-\frac{m+L-1-R}{2}\right] = \left[\frac{m+L+1+R}{2}\right]$$

splines $N_{\mathbf{t};m+L,j}$ verifying the inclusion $supp\ (N_{\tilde{\mathbf{t}};m+L,i_0}) \subseteq supp\ (N_{\mathbf{t};m+L,j})$. □

For $m = 4$, $L = 1$ and $R = 3$ we obtain at most

$$\min\left\{\left[\frac{m+L+1+R}{2}\right],\ m+L\right\} = \min\{[4.5];\ 4\} = 4$$

non–zero elements per row in the matrix P. All matrices $P_{\mathbf{s_i},\mathbf{s_{18}};4+1}$, $1 \le i \le 18$, from Section 6.2.3 exhibit this property.

In products of the type PVP^T only $m+L-1$ consecutive columns of P are 'active'. Therefore, we count next the number of 'active' rows (i.e., non–zero rows) in such blocks. Proposition 6.8 implies the following result.

Proposition 6.10 *(Dimension of the non–zero blocks determined by* $m+L-1$ *consecutive columns of P)*
Let m denote the order of the B–splines, L the number of vanishing moments ($1 \le L \le m$) *and R the bound of the refinement rate between the knot sequences* $\mathbf{t}$ *and* $\tilde{\mathbf{t}}$*.*

a) The starting rows i_1 *and* i_2 *of the non–zero blocks of two consecutive columns from* $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ *differ at most by* $R+1$ *(i.e.,* $0 \le i_2 - i_1 \le R+1$*).*

b) A block of $m+L-1$ *consecutive columns from* $P_{\mathbf{t},\tilde{\mathbf{t}};m+L}$ *has at most* $(m+L-1)(2R+1)$ *'active' rows.*

Proof. Part a) is obvious. The constant in part b) consists of the maximal length $(m+L)R+1$ of a non–zero column block and $m+L-2$ shifts of dimension $R+1$; this yields

$$(m+L)R+1+(m+L-2)(R+1) = (m+L-1)(2R+1). \qquad \square$$

For $m = 4$, $L = 1$ and $R = 3$ we obtain $R+1 = 4$. Compare column 8 and column 9 of the matrix $P_{\mathbf{s_1},\mathbf{s_{18}};4+1}$ from Section 6.2.3 in order to obtain an example for part a) of Proposition

6.10. Moreover, $(m+L-1)(2R+1) = 28$. The columns 6, 7, 8 and 9 of the aforementioned matrix constitute a block with 19 active rows; considering a longer knot sequence we could attain the maximal number of 28 active rows. Note that in Section 6.2.3 the non–zero blocks of the matrices PVP^T which are relevant for the matrix Z never attain the maximal number of 'active' rows, namely 28.

We are now in the position to specify the dimension of the non–zero block of a product matrix PVP^T appearing on the right hand side of (6.8). Proposition 6.10 implies the following.

Proposition 6.11 *(Dimension of the non–zero block of a matrix PVP^T)*
The quadratic non–zero block of a matrix PVP^T from

$$Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L} = \sum_{k=1}^{M} P_{\mathbf{s_k},\mathbf{s_M};m+L} \cdot V_{\mathbf{s_{k-1}},\mathbf{s_k};m,L} \cdot P^T_{\mathbf{s_k},\mathbf{s_M};m+L}$$

has a dimension less than or equal to

$$(m+L-1)(2R+1).$$

Here again, m denotes the order of the B–splines, L the number of vanishing moments ($1 \leq L \leq m$) and R the bound of the refinement rate between the knot sequences $\mathbf{t}_j$ and $\mathbf{t}_{j+1}$.

In the sequel we need the following properties of the matrix Z^B.

Proposition 6.12 *(Length of the non–zero blocks in the columns of Z and Z^B)*
Each column of the matrices $Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ and $Z^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ has a non–zero block of maximal length

$$2 \cdot (m+L-1)(2R+1) - 1. \tag{6.14}$$

Here m, L and R are as in Proposition 6.11.

Proof. Each time when ρ has to be determined in Algorithm 6.1 it takes at least the last value of ρ incremented by one. This implies the 'movement' of the non–zeros blocks of the matrices PVP (as illustrated in Figure 6.1) and thus the above mentioned maximal length of the non–zero block of every column in Z and Z^B. □

Proposition 6.11 in connection with Remark 6.3.e) yields the following assertion (see also Subsection 6.2.3, especially Figure 6.1 for an illustration).

Proposition 6.13 *(Local structure of Z and Z^B)*
In the bounded refinement rate case the sum with M terms on the right hand side of (6.8) reduces locally to a sum with at most

$$\left[\frac{(m+L-1)(2R+1)}{R+1}\right] \cdot R \tag{6.15}$$

terms. The brackets $[\cdot]$ stand for the integer part function. Here m, L and R are given as in Proposition 6.11.

For $m = 4$, $L = 1$ and $R = 3$ we obtain

$$\left[\frac{(m+L-1)(2R+1)}{R+1}\right] \cdot R = 21$$

and

$$2 \cdot (m + L - 1)(2R + 1) - 1 = 55.$$

Comparing this result with the illustration in Figure 6.1 we detect that in this example the maximal number (21) of terms in the sum is never attained. This can be explained by the fact that the non–zero blocks of the matrices PVP^T which are relevant for the matrix Z never attain the maximal number of 'active' rows 28. This also implies the much shorter non–zero blocks in the columns of Z.

Remark 6.14 *The above "overestimation" in particular cases is not important for our purpose.* **The essential issue is the independence of the constants (6.14) and (6.15) of the knot sequences $\mathbf{t}_j$ and $\mathbf{t}_{j+1}$, in contrary to the constants $N_{j+1} + m - L$ and $M := \#(\mathbf{t}_{j+1} \setminus \mathbf{t}_j)$ designating the dimension of $Z_{\mathbf{t}_j,\mathbf{t}_{j+1};m,L}$ and the number of terms in the sum (6.8), respectively.** *Without this independence we could not prove the Bessel property for the dual frame.*

Proposition 6.13 in connection with Proposition 6.7 enables us to formulate the desired result concerning the elements of the matrix $Z_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$.

Proposition 6.15 *(Estimate for the elements of Z)*
In the bounded refinement rate case each element $z_{r,s}$ of the matrix $Z_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ verifies the inequality

$$\begin{aligned} z_{r,s} \quad \leq \quad & \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot R \cdot \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-2)!(L-1)!} \cdot \\ & \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,L-1}} \{d_{\mathbf{t}_{j+1};m,L-1,n}\} \cdot \left(\max_{n \in I\!M_{\mathbf{t}_j;m-1,0}} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)} \end{aligned}$$

for $2 \leq L \leq m$. For $L = 1$ one has

$$z_{r,s} \quad \leq \quad \left\lceil \frac{m(2R+1)}{R+1} \right\rceil \cdot R \cdot m \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,0}} \{d_{\mathbf{t}_{j+1};m,0,n}\}.$$

Proposition 6.16 *(Estimate for the elements of Z^B)*
In the bounded refinement rate case each element $z^B_{r,s}$ of the matrix $Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ verifies the inequality

$$\begin{aligned} z^B_{r,s} \quad \leq \quad & \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot R \cdot \frac{2^{-L+1} \cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-2)!(L-1)!} \cdot \\ & \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,L-1}} \{d_{\mathbf{t}_{j+1};m,L-1,n}\} \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,L}} \{d_{\mathbf{t}_{j+1};m,L,n}\} \\ & \cdot \left(\max_{n \in I\!M_{\mathbf{t}_j;m-1,0}} \{d_{\mathbf{t}_j;m-1,0,n}\} \right)^{2(L-1)} \end{aligned}$$

for $2 \leq L \leq m$. For $L = 1$ there holds the estimate

$$\begin{aligned} z^B_{r,s} \quad \leq \quad & \left\lceil \frac{m \cdot (2R+1)}{R+1} \right\rceil \cdot R \cdot m \cdot \\ & \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,0}} \{d_{\mathbf{t}_{j+1};m,0,n}\} \cdot \max_{n \in I\!M_{\mathbf{t}_{j+1};m,1}} \{d_{\mathbf{t}_{j+1};m,1,n}\}. \end{aligned}$$

6.2.3 An example

This subsection illustrates the structure of the matrices V, P and Z for the cubic case ($m = 4$) with one vanishing moment ($L = 1$) and a bounded refinement rate ($R = 3$, i.e., maximal number of 3 new knots between two old ones).

We consider the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ with

$$\begin{aligned} \mathbf{t}_0 &= [\,0\ 0\ 0\ 0\ 1\ 2\ 3\ 4\ 5\ 6\ 6\ 6\ 6\,], \\ \mathbf{t}_1 \setminus \mathbf{t}_0 &= [\,0.25\ 0.5\ 0.75\ 1.25\ 1.5\ 1.75\ 2.25\ 2.5\ 2.75\ 3.25\ 3.5\ 3.75 \\ &\qquad 4.25\ 4.5\ 4.75\ 5.25\ 5.5\ 5.75\,]. \end{aligned}$$

Thus we have $N_0 = 5$ inner knots in $\mathbf{t}_0$ and $N_1 = 23$ inner knots in $\mathbf{t}_1$ (5 old knots, 18 new knots). The cardinality of the (in this setting maximal possible) refinement is thus $M := \#(\mathbf{t}_1 \setminus \mathbf{t}_0) = (N_0 + 1) \cdot R = 18 = N_1 - N_0$ and the intermediate knot vectors are

$$\mathbf{t}_0 = \mathbf{s_0} \subset \mathbf{s_1} \subset \cdots \subset \mathbf{s_{14}} \subset \mathbf{s_{18}} = \mathbf{t}_1,$$

where $\#(\mathbf{s_{k+1}} \setminus \mathbf{s_k}) = 1$ for all possible k. The $M = 18$ new knots are inserted one by one, in order, from the left to the right. $N_{\mathbf{s_k}}$ denotes the number of inner knots in the sequence $\mathbf{s_k}$ and $s_i^{(k)}$ some element of the sequence $\mathbf{s_k}$.

We obtain the matrices V presented in Table 6.1.

The refinement matrices P are presented in Tables 6.2–6.10. The elements p of P with magnitude $p \in (0, 1)$ are symbolized by the charactere $*$. Their exact magnitude has no importance for the structure of the matrix Z and it would make the exposition unclear.

Those columns which are 'active' in the product PVP^T are marked. We have also marked those rows which are 'active' in association with the respective columns. We have at most $m + L - 1 = 4$ columns and at most $(m + L - 1)(2R + 1) = 28$ rows.

The structure of the matrix Z is illustrated in Figure 6.1. Locally we have a summation of at most 12 terms.

$$\rho = 0 \quad \Rightarrow V_{\mathbf{s_0},\mathbf{s_1};4,1} \quad = \quad diag([* \ * \ * \ * \ 0\,0\,0\,0\,0])$$

$$\rho = 1 \quad \Rightarrow V_{\mathbf{s_1},\mathbf{s_2};4,1} \quad = \quad diag([0 \ * \ * \ * \ * \, 0\,0\,0\,0\,0])$$

$$\rho = 2 \quad \Rightarrow V_{\mathbf{s_2},\mathbf{s_3};4,1} \quad = \quad diag([0\,0 \ * \ * \ * \ * \, 0\,0\,0\,0\,0])$$

$$\rho = 4 \quad \Rightarrow V_{\mathbf{s_3},\mathbf{s_4};4,1} \quad = \quad diag([0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0\,0])$$

$$\rho = 5 \quad \Rightarrow V_{\mathbf{s_4},\mathbf{s_5};4,1} \quad = \quad diag([0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0\,0])$$

$$\rho = 6 \quad \Rightarrow V_{\mathbf{s_5},\mathbf{s_6};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0\,0])$$

$$\rho = 8 \quad \Rightarrow V_{\mathbf{s_6},\mathbf{s_7};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0])$$

$$\rho = 9 \quad \Rightarrow V_{\mathbf{s_7},\mathbf{s_8};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0])$$

$$\rho = 10 \quad \Rightarrow V_{\mathbf{s_8},\mathbf{s_9};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0\,0])$$

$$\rho = 12 \quad \Rightarrow V_{\mathbf{s_9},\mathbf{s_{10}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0])$$

$$\rho = 13 \quad \Rightarrow V_{\mathbf{s_{10}},\mathbf{s_{11}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0])$$

$$\rho = 14 \quad \Rightarrow V_{\mathbf{s_{11}},\mathbf{s_{12}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0\,0])$$

$$\rho = 16 \quad \Rightarrow V_{\mathbf{s_{12}},\mathbf{s_{13}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0])$$

$$\rho = 17 \quad \Rightarrow V_{\mathbf{s_{13}},\mathbf{s_{14}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0])$$

$$\rho = 18 \quad \Rightarrow V_{\mathbf{s_{14}},\mathbf{s_{15}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ * \, 0])$$

$$\rho = 20 \quad \Rightarrow V_{\mathbf{s_{15}},\mathbf{s_{16}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ *])$$

$$\rho = 21 \quad \Rightarrow V_{\mathbf{s_{16}},\mathbf{s_{17}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ *])$$

$$\rho = 22 \quad \Rightarrow V_{\mathbf{s_{17}},\mathbf{s_{18}};4,1} \quad = \quad diag([0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ * \ * \ * \ *])$$

Table 6.1: Indices ρ and matrices V for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_1},\mathbf{s_{18}};4+1} = \left(\begin{array}{|cccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_2},\mathbf{s_{18}};4+1} = \left(\begin{array}{c|cccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Table 6.2: Matrices $P_{\mathbf{s_1},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_2},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_3},\mathbf{s_{18}};4+1} = \left(\begin{array}{cc|cccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_4},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Table 6.3: Matrices $P_{\mathbf{s_3},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_4},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_5},\mathbf{s_{18}};4+1} = \left(\begin{array}{ccccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_6},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Table 6.4: Matrices $P_{\mathbf{s_5},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_6},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_7},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_8},\mathbf{s_{18}};4+1} = \left(\begin{array}{ccccccccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Table 6.5: Matrices $P_{\mathbf{s_7},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_8},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_9},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_{10}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccc|cccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

Table 6.6: $P_{\mathbf{s_9},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_{10}},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_{11}},\mathbf{s_{18}};4+1} = \left(\begin{array}{ccccccccccccc|cccc|cc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*
\end{array}\right)
$$

$$
P_{\mathbf{s_{12}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccccc|cccc|cc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*
\end{array}\right)
$$

Table 6.7: $P_{\mathbf{s_{11}},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_{12}},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_{13}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccccccc|cccc|c}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ \cline{17-20}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*\\ \cline{17-20}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*
\end{array}\right)
$$

$$
P_{\mathbf{s_{14}},\mathbf{s_{18}};4+1} = \left(\begin{array}{ccccccccccccccccc|cccc|c}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ \cline{18-21}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*\\ \cline{18-21}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*
\end{array}\right)
$$

Table 6.8: $P_{\mathbf{s_{13}},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_{14}},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_{15}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccccccccc|cccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\cline{19-22}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
\cline{19-22}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_{16}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccccccccccc|cccc|}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\cline{21-24}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\cline{21-24}
\end{array}\right)
$$

Table 6.9: $P_{\mathbf{s_{15}},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_{16}},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

$$
P_{\mathbf{s_{17}},\mathbf{s_{18}};4+1} = \left(\begin{array}{ccccccccccccccccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{array}\right)
$$

$$
P_{\mathbf{s_{18}},\mathbf{s_{18}};4+1} = \left(\begin{array}{cccccccccccccccccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Table 6.10: $P_{\mathbf{s_{17}},\mathbf{s_{18}};4+1}$ and $P_{\mathbf{s_{18}},\mathbf{s_{18}};4+1}$ for the refinement $\mathbf{t}_0 \subset \mathbf{t}_1$ from Subsection 6.2.3.

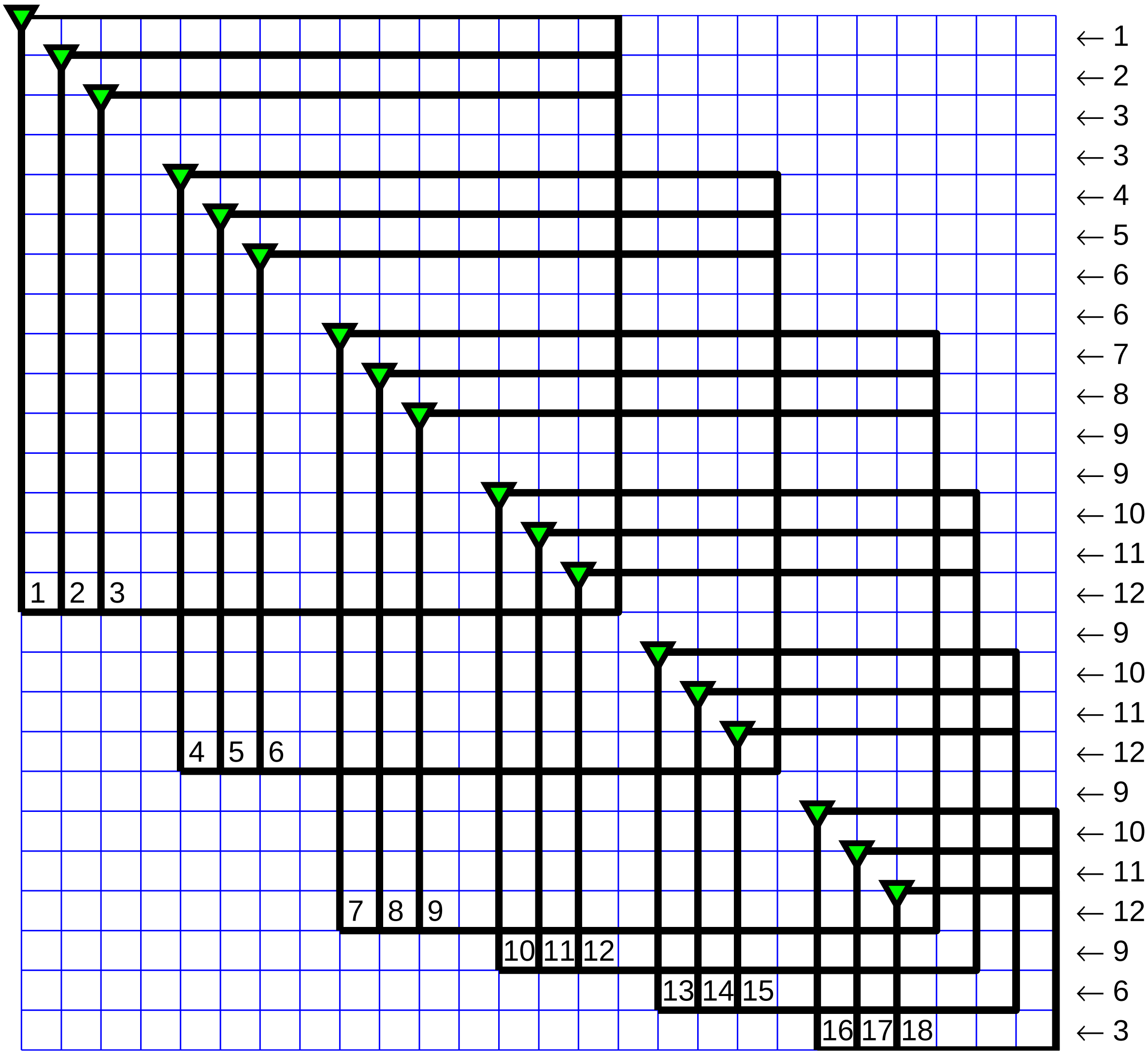

Figure 6.1: Summation of the visualized blocks PVP^T (numbered 1 to 18) yields the matrix Z. Locally we thus have a summation of at most 12 terms. The maximum local number of terms in the sum yielding the matrix Z is indicated on the right hand side for each row of Z in part (just count the number of relevant down arrows).

6.3 Our construction scheme for quasi–uniform sibling spline frames

First we define the exact setting by detailing the conditions on the knot sequences we need in order to prove the Bessel property for the function systems defined below.

Conditions 6.17 *(Conditions on the knot sequences* $\mathbf{t}_j$*)*
At the beginning of this chapter we already mentioned the general conditions on the knot sequences under which we will work, i.e.,

- *dense sequence of finite admissible knot vectors*

$$\mathbf{t}_0 \subset \cdots \subset \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \cdots \subset [a, b];$$

- $\mathbf{t}_j$ *has* N_j *interior knots of multiplicity* **at most** $(\mathbf{m-1})$ *and stacked boundary knots* a *and* b *of maximal multiplicity* m.

The natural number $m \in \mathbb{N}$ $(m \geq 2)$ *denotes the order of B–splines. Furthermore, we require quasi–uniformity of order* $m-1$ *in the following sense:*

- *There exists a perturbation parameter* $\varepsilon \in [0, 1)$ *such that for every level* j *there exists a parameter* h_j *satisfying the properties*

$$(1-\varepsilon)h_j \leq \textit{length}\ (\textit{supp}\ N^B_{\mathbf{t}_j;m-1,k}) \leq (1+\varepsilon)h_j, \quad k \in \mathbb{M}_{\mathbf{t}_j;m-1,0}, \tag{6.16}$$

  i.e., on every level j *the lengths of the supports of the B–spline basis* $\mathbf{\Phi}^B_{\mathbf{t}_j;m-1}$ *are approximately equal (quasi–uniform). Relation (6.16) can be rephrased in terms of knot sequences as follows:*

$$(1-\varepsilon)h_j \leq t^{(j)}_{k+m-1} - t^{(j)}_k \leq (1+\varepsilon)h_j, \qquad k \in \mathbb{M}_{\mathbf{t}_j;m-1,0}, \quad j \in \mathbb{N}_0. \tag{6.17}$$

  By abuse of notation in $\mathbf{\Phi}^B_{\mathbf{t}_j;m-1}$ *and* $\mathbb{M}_{\mathbf{t}_j;m-1,0}$ *the knot sequence* $\mathbf{t}_j$ *is considered with only* $m-1$ *stacked boundary knots* a *and* b.

The existence of a bounded refinement rate R *is also assumed, i.e.,*

- *from each level* j *to the next one at most* R *new knots are inserted between two old ones,*

as well as the existence of

- *constants* $K_1 \geq 1$ *and* $K_2 \geq 2$ *satisfying*

$$\frac{h_{j+K_1}}{h_j} \leq \frac{1}{K_2}, \qquad \textit{for all } j \in \mathbb{N}_0, \tag{6.18}$$

  where h_j *are the parameters defined by (6.16).*

Note that this setting depends on the parameters m, R, ε, K_1, K_2 and $\{h_j\}_{j\geq 0}$, but not on the number of desired vanishing moments L.

Intuitively we can explain the constants R, K_1, K_2 in the following way: R does not allow to refine to much, and (K_1, K_2) take care that one refines enough and without big 'holes'.

Under the above conditions for the knot sequences $\mathbf{t}_j$ we succeed to prove that Ψ and $\tilde{\Psi}$ from (6.2)–(6.5) are Bessel families with L vanishing moments ($1 \leq L \leq m$) and thus are sibling spline frames with the following concrete choice for the matrices $A_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$:

$$\Psi = \{\Psi_j(x)\}_{j\geq 0} \quad := \quad \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot \mathbf{diag}\left(\mathbf{h^L_{j+1}}\right)\right\}_{j\geq 0}, \tag{6.19}$$

$$\tilde{\Psi} = \{\tilde{\Psi}_j(x)\}_{j\geq 0} \quad := \quad \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L} \cdot \mathbf{diag}\left(\mathbf{h^{-L}_{j+1}}\right)\right\}_{j\geq 0}. \tag{6.20}$$

To demonstrate this, below we describe how the sufficient conditions formulated in Theorem 5.40 are verified by the systems Ψ and $\tilde{\Psi}$.

First observe the following.

Proposition 6.18 *For the quasi–uniform refinement with bounded refinement rate one has*

$$\frac{h_j}{h_{j+1}} \leq 1 + (R+1) \cdot \frac{1+\varepsilon}{1-\varepsilon}.$$

Proof. Consider the B–spline $N_{\mathbf{t}_j;m-1,k_0}$ with the smallest possible support on the level j, i.e.,

$$t^{(j)}_{k_0+m-1} - t^{(j)}_{k_0} = (1-\varepsilon)h_j.$$

In addition to the m old knots $t^{(j)}_{k_0}$ to $t^{(j)}_{k_0+m-1}$ we have on level $j+1$ at most $(m-1)\cdot R$ new knots contained in the support of $N_{\mathbf{t}_j;m-1,k_0}$. This implies a maximal number of

$$\begin{cases} \left[\frac{m+(m-1)\cdot R}{m-1}\right] + 1 = \left[R+1+\frac{1}{m-1}\right] + 1 = R+2 & \text{for } m > 2, \\ \left[\frac{2+1\cdot R}{1}\right] = R+2 & \text{for } m = 2, \end{cases}$$

B–splines of order $m-1$ on the level $j+1$ with pairwise disjoint supports intersecting $\left[t^{(j)}_{k_0}, t^{(j)}_{k_0+m-1}\right]$; the leftmost B–spline may start before $t^{(j)}_{k_0}$ and the rightmost one may end after $t^{(j)}_{k_0+m-1}$ for $m > 2$. A spline with smallest possible support on the level $j+1$ (i.e., length of the support equal to $(1-\varepsilon)h_{j+1}$) must therefore verify the following inequality:

$$(1-\varepsilon)h_{j+1} \geq (1-\varepsilon)h_j - (R+1)\cdot(1+\varepsilon)h_{j+1},$$

i.e., its $R+1$ neighbors (in the sense described above) have support lengths less than or equal to $(1+\varepsilon)h_{j+1}$. This implies the above relation between h_j and h_{j+1}. □

Proposition 6.19 *For all possible j and k there hold the following inequalities:*

$$\begin{aligned}
\frac{(1-\varepsilon)h_j}{m-1} &\leq d_{\mathbf{t}_j;m-1,0,k} \leq \frac{(1+\varepsilon)h_j}{m-1}, \\
\frac{(1-\varepsilon)h_j}{m} &\leq d_{\mathbf{t}_j;m,0,k} \leq \frac{2(1+\varepsilon)h_j}{m}, \\
\frac{(1-\varepsilon)h_j}{m+\nu} &\leq d_{\mathbf{t}_j;m,\nu,k} \leq \frac{2(1+\varepsilon)h_j}{m+\nu}, \qquad 1 \leq \nu \leq m-2, \\
\frac{(1-\varepsilon)h_j}{m+\nu} &\leq d_{\mathbf{t}_j;m,\nu,k} \leq \frac{3(1+\varepsilon)h_j}{m+\nu}, \qquad \nu \in \{m-1, m\}.
\end{aligned}$$

We denote the elements of the frame Ψ by $\psi_{j,k}$ and those of the dual frame $\tilde{\Psi}$ by $\tilde{\psi}_{j,k}$. Further let the interval $I_{j,k}$ be the support of $\psi_{j,k}$ and let $\tilde{I}_{j,k}$ be the support of $\tilde{\psi}_{j,k}$.

Proposition 6.20 *For $0 \leq \nu \leq L \leq m$ and all possible j and k there hold the estimates*

$$\max_{k \in I\!M_{\mathbf{t}_{j+1};m,0}} \{d_{\mathbf{t}_{j+1};m,0,k}\} \leq \frac{2(1+\varepsilon)h_{j+1}}{m}, \tag{6.21}$$

$$\max_{k \in I\!M_{\mathbf{t}_{j+1};m,L-1}} \{d_{\mathbf{t}_{j+1};m,L-1,k}\} \leq \frac{3(1+\varepsilon)h_{j+1}}{m+L-1}, \tag{6.22}$$

$$\max_{k \in I\!M_{\mathbf{t}_{j+1};m,L}} \{d_{\mathbf{t}_{j+1};m,L,k}\} \leq \frac{3(1+\varepsilon)h_{j+1}}{m+L}, \tag{6.23}$$

$$\max_{k \in I\!M_{\mathbf{t}_{j};m-1,0}} \{d_{\mathbf{t}_{j};m-1,0,k}\} \leq \frac{(1+\varepsilon)h_{j}}{m-1}, \tag{6.24}$$

$$d_{\mathbf{t}_{j+1};m,\nu,k}^{-1/2} \leq \left[\frac{(1-\varepsilon)h_{j+1}}{m+\nu}\right]^{-1/2}, \tag{6.25}$$

$$[3(1+\varepsilon)h_{j+1}]^{-1/2} \leq |I_{j,k}|^{-1/2}. \tag{6.26}$$

Proof. The estimates (6.22), (6.23), (6.24) and (6.25) follow directly from Proposition 6.19. Relation (6.26) is obtained from

$$(1-\varepsilon)h_{j+1} \leq \mathit{length}\ (\mathit{supp}\ N^B_{\mathbf{t}_{j+1};m+L,k}) \leq 3(1+\varepsilon)h_{j+1},$$

in combination with

$$\mathit{length}\ (\mathit{supp}\ N^B_{\mathbf{t}_{j+1};m+L,k}) = \mathit{length}\ (\mathit{supp}\ \psi_{j,k}) = |I_{j,k}|. \qquad \square$$

In order to be able to analyze the dual framelets $\tilde{\psi}_{j,k}$ we need estimates for the elements of the matrix Z^B which are adapted to the present setting.

Proposition 6.21 *(Estimate for the elements of Z^B)*
In the quasi–uniform case with bounded refinement rate R each element $z^B_{r,s}$ of the matrix $Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L}$ verifies the inequality

$$z^B_{r,s} \leq \left[\frac{(m+L-1)(2R+1)}{R+1}\right] \cdot R \cdot \frac{2^{-L+1}\cdot 9 \cdot m! \cdot (1+\varepsilon)^{2L}}{(m+L)!(L-1)!} \cdot$$

$$\cdot \left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot h^{2L}_{j+1}. \tag{6.27}$$

for $2 \leq L \leq m$. In case $L=1$ we have

$$z^B_{r,s} \leq \left[\frac{m\cdot(2R+1)}{R+1}\right] \cdot R \cdot \frac{6(1+\varepsilon)^2}{m+1} \cdot h^2_{j+1}. \tag{6.28}$$

In all cases

$$z^B_{r,s} = \mathcal{O}\left(h^{2L}_{j+1}\right)$$

with an absolute constant depending on the parameters m, L, R and ε.

Proof. Combining Proposition 6.16, the estimates (6.22)–(6.24) and Proposition 6.18 yields first

$$z^B_{r,s} \leq \left[\frac{(m+L-1)(2R+1)}{R+1}\right] \cdot R \cdot \frac{2^{-L+1}\cdot m! \cdot (m-1)^{2(L-1)}}{(m+L-2)!(L-1)!} \cdot$$

$$\cdot \frac{9\cdot(1+\varepsilon)^{2L}}{(m+L-1)(m+L)\cdot(m-1)^{2(L-1)}} \cdot \left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot h^{2L}_{j+1}$$

and then the desired bound (6.27). The estimate (6.28) can be obtained in an analogous way from Proposition 6.16, (6.21) and (6.23). □

The support–adapted uniform boundedness of the families Ψ and $\tilde{\Psi}$ is proved in the following Proposition 6.22 and Proposition 6.24, respectively.

Proposition 6.22 *(Property (5.29) for Ψ)*
Property (5.29) holds for Ψ defined in (6.19) with constant

$$C_1 = C_1(m, L, \varepsilon) = \left(\frac{2}{1-\varepsilon}\right)^{L+1/2} \cdot \sqrt{\frac{3(1+\varepsilon)}{2(m+L)} \cdot \frac{(m+L)!}{(m-1)!}}. \tag{6.29}$$

Proof. An arbitrary element $\psi_{j,k}$ of the frame Ψ defined in (6.19) is actually a suitably normalized derivative of order L of a B–spline of order $m+L$. It thus has the structure of a sum with 2^L terms (cf. the recurrence relation for derivatives of B–splines) of the form

$$N^B_{\mathbf{t}_{j+1};m,l} \cdot d^{-1/2}_{\mathbf{t}_{j+1};m,0,k_1} \cdot d^{-1}_{\mathbf{t}_{j+1};m,1,k_2} \cdot d^{-1}_{\mathbf{t}_{j+1};m,2,k_3} \cdot \ldots \cdot d^{-1}_{\mathbf{t}_{j+1};m,L-1,k_L} \cdot d^{-1/2}_{\mathbf{t}_{j+1};m,L,k_{L+1}} \cdot h^L_{j+1}$$

(see also the Formulae (2.46), (2.48), (2.45), (2.41)). Further we obtain

$$\begin{aligned}
|\psi_{j,k}(x)| &\overset{(6.25)}{\leq} 2^L \cdot \left(\max_l |N^B_{\mathbf{t}_{j+1};m,l}(x)|\right) \cdot \left[\frac{(1-\varepsilon)h_{j+1}}{m}\right]^{-1/2} \cdot \left[\frac{(1-\varepsilon)h_{j+1}}{m+1} \cdot \right.\\
&\quad \left. \cdot \frac{(1-\varepsilon)h_{j+1}}{m+2} \cdot \ldots \cdot \frac{(1-\varepsilon)h_{j+1}}{m+L-1}\right]^{-1} \cdot \left[\frac{(1-\varepsilon)h_{j+1}}{m+L}\right]^{-1/2} \cdot h^L_{j+1}\\
&\overset{(6.25)}{\leq} 2^L \cdot \left(\left[\frac{(1-\varepsilon)h_{j+1}}{m}\right]^{-1/2} \cdot 1\right) \cdot \frac{(m+L-1)!}{m!} \cdot \frac{\sqrt{m(m+L)}}{(1-\varepsilon)^L}\\
&= 2^L \cdot \frac{(m+L-1)!}{(m-1)!} \cdot \frac{\sqrt{m+L}}{(1-\varepsilon)^{L+1/2}} \cdot \sqrt{3(1+\varepsilon)} \cdot [3(1+\varepsilon)h_{j+1}]^{-1/2}\\
&\overset{(6.26)}{\leq} \left(\frac{2}{1-\varepsilon}\right)^{L+1/2} \cdot \sqrt{\frac{3(1+\varepsilon)}{2(m+L)} \cdot \frac{(m+L)!}{(m-1)!}} \cdot |I_{j,k}|^{-1/2}
\end{aligned}$$

and thus (5.29) holds for Ψ with the constant indicated. □

The proof of Proposition 6.22 also implies the validity of the next result.

Proposition 6.23 *(Estimate for the derivatives of order L)*
For the derivatives of order L ($1 \leq L \leq m$, $m \geq 2$) defined by

$$\frac{d^L}{dx^L}\mathbf{\Phi}^B_{\mathbf{t}_{j+1};m+L} = \mathbf{\Phi}^B_{\mathbf{t}_{j+1};m} \cdot E^B_{\mathbf{t}_{j+1};m,L}$$

for all $x \in [a,b]$ the following estimates hold:

$$\begin{aligned}
&\left|\frac{d^L}{dx^L} N^B_{\mathbf{t}_{j+1};m+L,l}(x)\right|\\
&\leq 2^L \cdot \left(\max_l |N^B_{\mathbf{t}_{j+1};m,l}(x)|\right) \cdot \frac{(m+L-1)!}{m!} \cdot \frac{\sqrt{m(m+L)}}{(1-\varepsilon)^L} \cdot h^{-L}_{j+1} && (6.30)\\
&\leq \left(\frac{2}{1-\varepsilon}\right)^L \cdot \sqrt{\frac{m+L}{1-\varepsilon}} \cdot \frac{(m+L-1)!}{(m-1)!} \cdot h^{-(L+\frac{1}{2})}_{j+1}. && (6.31)
\end{aligned}$$

For the length of the support $[t_l^{(j+1)}, t_{l+m+L}^{(j+1)}]$ *of the derivative of order* L *one has*

$$(1-\varepsilon)h_{j+1} \le t_{l+m+L}^{(j+1)} - t_l^{(j+1)} \le \begin{cases} 2(1+\varepsilon)h_{j+1} & \text{for } 1 \le L \le m-2 \\ 3(1+\varepsilon)h_{j+1} & \text{for } L \in \{m-1, m\}. \end{cases}$$

Proposition 6.24 *(Property (5.29) for* $\tilde{\Psi}$*)*
Property (5.29) holds for $\tilde{\Psi}$ *defined in (6.20) with constant*

$$\begin{aligned} \tilde{C}_1 = \tilde{C}_1(m, L, R, \varepsilon) &= (2(m+L-1)(2R+1)-1)^{3/2} \cdot \\ &\cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{L+1/2} \cdot \frac{18\sqrt{3}\cdot m \cdot R \cdot (1+\varepsilon)^L}{(L-1)! \cdot \sqrt{m+L}} \cdot \\ &\cdot \left(1 + (R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot \\ &\cdot \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \end{aligned} \tag{6.32}$$

in the cases $2 \le L \le m$ *and for* $L = 1$ *with*

$$\begin{aligned} \tilde{C}_1 = \tilde{C}_1(m, 1, R, \varepsilon) &= (2m(2R+1)-1)^{3/2} \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{3/2} \cdot \\ &\cdot \frac{12\sqrt{3}\cdot m \cdot R \cdot (1+\varepsilon)}{\sqrt{m+1}} \cdot \left\lceil \frac{m(2R+1)}{R+1} \right\rceil . \end{aligned} \tag{6.33}$$

Proof. An arbitrary element $\tilde{\psi}_{j,k}$ of the frame $\tilde{\Psi}$ defined in (6.20) has the structure of a sum with at least 1 term and at most $2(m+L-1)(2R+1)-1$ terms (see Proposition 6.12) all of them being of the form

$$\left(\frac{d^L}{dx^L} N^B_{\mathbf{t_{j+1}};m+L,i}\right) \cdot z^B_{i,l} \cdot h_{j+1}^{-L}.$$

Applying (6.27) and Proposition 6.23 we obtain further

$$\begin{aligned} |\tilde{\psi}_{j,k}(x)| &\le (2(m+L-1)(2R+1)-1) \cdot \\ &\cdot \left(\frac{2}{1-\varepsilon}\right)^L \cdot \sqrt{\frac{m+L}{1-\varepsilon}} \cdot \frac{(m+L-1)!}{(m-1)!} \cdot h_{j+1}^{-(L+\frac{1}{2})} \cdot \\ &\cdot \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot R \cdot \frac{2^{-L+1}\cdot 9 \cdot m! \cdot (1+\varepsilon)^{2L}}{(m+L)!(L-1)!} \cdot \\ &\cdot \left(1 + (R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot h_{j+1}^{2L} \\ &\cdot h_{j+1}^{-L} \\ &= \mathcal{O}\left(h_{j+1}^{-1/2}\right) \end{aligned}$$

and

$$length\,(supp\ \tilde{\psi}_{j,k}) = |\tilde{I}_{j,k}| \le 3(1+\varepsilon)h_{j+1} \cdot (2(m+L-1)(2R+1)-1).$$

Considering

$$h_{j+1}^{-1/2} \le \sqrt{3(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1)} \cdot |\tilde{I}_{j,k}|^{-1/2}$$

we obtain finally (5.29) with the above mentioned constant $\tilde{C}_1(m, L, R, \varepsilon)$ for $2 \le L \le m$. For $L = 1$ we get the constant (6.33) by carrying out the same steps like for $2 \le L \le m$ with (6.28) instead of (6.27). □

The following two results prove the support–adapted Hölder continuity with exponent $\beta = 1$ for the function systems Ψ and $\tilde{\Psi}$, respectively.

Proposition 6.25 *(Property (5.30) for Ψ)*
Property (5.30) holds for Ψ defined in (6.19) with $\beta = 1$ and constant

$$C_2 = C_2(m, L, \varepsilon) = 3\sqrt{3} \cdot \sqrt{m+L} \cdot \left(\frac{2}{1-\varepsilon}\right)^L \cdot \frac{(m+L-1)!}{(m-2)!} \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{3/2}. \qquad (6.34)$$

Proof. Following up the ideas in the proof of Proposition 6.22, and the estimate for the derivative (6.30) in particular and combined with ideas from Example 5.28, we obtain

$$\begin{aligned}
&|\psi_{j,k}(x) - \psi_{j,k}(x')| \\
&\le 2^L \cdot \left(\max_l |N^B_{\mathbf{t}_{j+1};m,l}(x) - N^B_{\mathbf{t}_{j+1};m,l}(x')|\right) \cdot \frac{(m+L-1)!}{m!} \cdot \frac{\sqrt{m(m+L)}}{(1-\varepsilon)^L} \\
&\le 2^L \cdot \frac{(m+L-1)!}{m!} \cdot \frac{\sqrt{m(m+L)}}{(1-\varepsilon)^L} \cdot \max_l \left\| \left(N^B_{\mathbf{t}_{j+1};m,l}\right)' \right\|_{\infty,(x,x')} \cdot |x - x'|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\| \left(N^B_{\mathbf{t}_{j+1};m,l}\right)' \right\|_{\infty,(x,x')} \\
&= \left\| d^{-1/2}_{\mathbf{t}_{j+1};m,0,l} \cdot \left(N_{\mathbf{t}_{j+1};m,l}\right)' \right\|_{\infty,(x,x')} \\
&= \left\| d^{-1/2}_{\mathbf{t}_{j+1};m,0,l} \cdot d^{-1}_{\mathbf{t}_{j+1};m-1,0,l} N_{\mathbf{t}_{j+1};m-1,l} - d^{-1/2}_{\mathbf{t}_{j+1};m,0,l} \cdot d^{-1}_{\mathbf{t}_{j+1};m-1,0,l+1} N_{\mathbf{t}_{j+1};m-1,l+1} \right\|_{\infty,(x,x')} \\
&\le \max \left\{ \left\| d^{-1/2}_{\mathbf{t}_{j+1};m,0,l} \cdot d^{-1}_{\mathbf{t}_{j+1};m-1,0,l} N_{\mathbf{t}_{j+1};m-1,l} \right\|_{\infty,(x,x')}, \right. \\
&\qquad\qquad \left. \left\| d^{-1/2}_{\mathbf{t}_{j+1};m,0,l} \cdot d^{-1}_{\mathbf{t}_{j+1};m-1,0,l+1} N_{\mathbf{t}_{j+1};m-1,l+1} \right\|_{\infty,(x,x')} \right\} \\
&\overset{(6.25)}{\le} \left[\frac{(1-\varepsilon)h_{j+1}}{m}\right]^{-1/2} \cdot \left[\frac{(1-\varepsilon)h_{j+1}}{m-1}\right]^{-1} \cdot 1 = \frac{(m-1)\cdot\sqrt{m}}{(1-\varepsilon)^{3/2}} \cdot h^{-3/2}_{j+1} \\
&= \frac{(m-1)\cdot\sqrt{m}}{(1-\varepsilon)^{3/2}} \cdot [3(1+\varepsilon)]^{3/2} \cdot [3(1+\varepsilon)\cdot h_{j+1}]^{-3/2} \\
&\overset{(6.26)}{\le} 3\sqrt{3} \cdot (m-1) \cdot \sqrt{m} \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{3/2} \cdot |I_{j,k}|^{-3/2}
\end{aligned}$$

and thus the assertion claimed. □

Proposition 6.26 *(Property (5.30) for $\tilde{\Psi}$)*
Property (5.30) holds for $\tilde{\Psi}$ defined in (6.20) with $\beta = 1$ and a constant $\tilde{C}_2 = \tilde{C}_2(m, L, R, \varepsilon)$ having the following representation for $2 \le L \le m$:

$$\begin{aligned}
\tilde{C}_2 \;=\;& (2(m+L-1)(2R+1)-1)^{5/2} \cdot \left(1 + (R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot \\
& \cdot \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot \frac{108\sqrt{3}\cdot R \cdot m \cdot (m-1)}{\sqrt{m+L}\cdot(L-1)!} \cdot \\
& \cdot (1+\varepsilon)^{2L+3/2}(1-\varepsilon)^{-(L+3/2)}. \qquad (6.35)
\end{aligned}$$

For $L=1$ the same relation holds with $\beta = 1$ and constant

$$\tilde{C}_2 = \tilde{C}_2(m,1,R,\varepsilon) \;=\; (2m(2R+1)-1)^{5/2} \cdot \left\lceil \frac{m(2R+1)}{R+1} \right\rceil \cdot \cdot \frac{36\sqrt{3}\cdot R\cdot(m-1)\cdot m\cdot(1+\varepsilon)^{7/2}\cdot(1-\varepsilon)^{-5/2}}{\sqrt{m+1}}. \tag{6.36}$$

Proof. Recalling again the proof of Proposition 6.24, combined with (6.27) and ideas from Example 5.28, we obtain

$$\begin{aligned}
&|\tilde{\psi}_{j,k}(x) - \tilde{\psi}_{j,k}(x')| \\
&\quad \le (2(m+L-1)(2R+1)-1) \cdot \\
&\quad\quad \cdot \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot R \cdot \frac{2^{-L+1}\cdot 9\cdot m!\cdot(1+\varepsilon)^{2L}}{(m+L)!(L-1)!} \cdot \\
&\quad\quad \cdot \left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot h_{j+1}^{2L} \cdot \\
&\quad\quad \cdot h_{j+1}^{-L} \cdot \\
&\quad\quad \cdot \max_i \left| \frac{d^L}{dx^L} N^B_{\mathbf{t_{j+1}};m+L,i}(x) - \frac{d^L}{dx^L} N^B_{\mathbf{t_{j+1}};m+L,i}(x') \right| \\
&\quad \le (2(m+L-1)(2R+1)-1) \cdot \left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)} \cdot \\
&\quad\quad \cdot \left\lceil \frac{(m+L-1)(2R+1)}{R+1} \right\rceil \cdot R \cdot \frac{2^{-L+1}\cdot 9\cdot m!\cdot(1+\varepsilon)^{2L}}{(m+L)!(L-1)!} \cdot \\
&\quad\quad h_{j+1}^{L} \cdot \max_i \left\| \frac{d^{L+1}}{dx^{L+1}} N^B_{\mathbf{t_{j+1}};m+L,i} \right\|_{\infty,(x,x')} \cdot |x-x'|.
\end{aligned}$$

Resuming also the ideas from the proof of Proposition 6.22 we further obtain

$$\begin{aligned}
&\left\| \frac{d^{L+1}}{dx^{L+1}} N^B_{\mathbf{t_{j+1}};m+L,i} \right\|_{\infty,(x,x')} \\
&= \left\| \frac{d}{dx} \left(\sum_1^{2^L} N^B_{\mathbf{t}_{j+1};m,k_1} \cdot d^{-1/2}_{\mathbf{t}_{j+1};m,0,k_1} \cdot d^{-1}_{\mathbf{t}_{j+1};m,1,k_2} \cdot \ldots \cdot d^{-1}_{\mathbf{t}_{j+1};m,L-1,k_L} \cdot d^{-1/2}_{\mathbf{t}_{j+1};m,L,k_{L+1}} \right) \right\|_{\infty,(x,x')} \\
&= \left\| \sum_1^{2^{L+1}} N_{\mathbf{t}_{j+1};m-1,l} \cdot d^{-1}_{\mathbf{t}_{j+1};m-1,0,l} \cdot \right. \\
&\quad\quad \left. \cdot d^{-1}_{\mathbf{t}_{j+1};m,0,k_1} \cdot d^{-1}_{\mathbf{t}_{j+1};m,1,k_2} \cdot \ldots \cdot d^{-1}_{\mathbf{t}_{j+1};m,L-1,k_L} \cdot d^{-1/2}_{\mathbf{t}_{j+1};m,L,k_{L+1}} \right\|_{\infty,(x,x')} \\
&\overset{(6.25)}{\le} 2^{L+1} \cdot 1 \cdot \frac{(m+L-1)!}{(m-2)!} \cdot \sqrt{m+L} \cdot (1-\varepsilon)^{-(L+3/2)} \cdot h_{j+1}^{-(L+3/2)}.
\end{aligned}$$

Finally this leads to

$$|\tilde{\psi}_{j,k}(x) - \tilde{\psi}_{j,k}(x')|$$

$$\leq (2(m+L-1)(2R+1)-1)\cdot\left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)}\cdot$$

$$\cdot\left\lceil\frac{(m+L-1)(2R+1)}{R+1}\right\rceil\cdot R\cdot\frac{2^{-L+1}\cdot 9\cdot m!\cdot(1+\varepsilon)^{2L}}{(m+L)!(L-1)!}\cdot$$

$$\cdot h_{j+1}^{L}\cdot 2^{L+1}\cdot\frac{(m+L-1)!}{(m-2)!}\cdot\sqrt{m+L}\cdot(1-\varepsilon)^{-(L+3/2)}\cdot h_{j+1}^{-(L+3/2)}\cdot|x-x'|$$

$$\leq (2(m+L-1)(2R+1)-1)\cdot\left(1+(R+1)\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^{2(L-1)}\cdot(1-\varepsilon)^{-(L+3/2)}\cdot$$

$$\cdot\left\lceil\frac{(m+L-1)(2R+1)}{R+1}\right\rceil\cdot\frac{36\cdot R\cdot m\cdot(m-1)\cdot(1+\varepsilon)^{2L}}{\sqrt{m+L}\cdot(L-1)!}\cdot$$

$$\cdot(3(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1))^{3/2}\cdot|\tilde{I}_{j,k}|^{-3/2}\cdot|x-x'|.$$

This implies the desired result for $2\leq L\leq m$.

An analogous procedure for $L=1$, i.e.,

$$|\tilde{\psi}_{j,k}(x)-\tilde{\psi}_{j,k}(x')|$$

$$\leq (2m(2R+1)-1)\cdot\left\lceil\frac{m(2R+1)}{R+1}\right\rceil\cdot R\cdot\frac{6\cdot(1+\varepsilon)^2}{m+1}\cdot h_{j+1}^{2}\cdot h_{j+1}^{-1}$$

$$\cdot\max_i\left|\frac{d}{dx}N^B_{\mathbf{t_{j+1}};m+1,i}(x)-\frac{d}{dx}N^B_{\mathbf{t_{j+1}};m+1,i}(x')\right|$$

$$\leq (2m(2R+1)-1)\cdot\left\lceil\frac{m(2R+1)}{R+1}\right\rceil\cdot R\cdot\frac{6\cdot(1+\varepsilon)^2}{m+1}\cdot h_{j+1}\cdot$$

$$\cdot\max_i\left\|\frac{d^2}{dx^2}N^B_{\mathbf{t_{j+1}};m+1,i}\right\|_{\infty,(x,x')}\cdot|x-x'|$$

$$\leq (2m(2R+1)-1)\cdot\left\lceil\frac{m(2R+1)}{R+1}\right\rceil\cdot R\cdot\frac{6\cdot(1+\varepsilon)^2}{m+1}\cdot h_{j+1}\cdot$$

$$\cdot 2\cdot(1-\varepsilon)^{-2-1/2}\cdot(m-1)\cdot m\cdot\sqrt{m+1}\cdot h_{j+1}^{-2-1/2}\cdot|x-x'|$$

$$\leq (2m(2R+1)-1)\cdot\left\lceil\frac{m(2R+1)}{R+1}\right\rceil\cdot\frac{12\cdot R\cdot(m-1)\cdot m\cdot(1+\varepsilon)^2\cdot(1-\varepsilon)^{-5/2}}{\sqrt{m+1}}\cdot$$

$$\cdot(3(1+\varepsilon)\cdot(2m(2R+1)-1))^{3/2}\cdot|\tilde{I}_{j,k}|^{-3/2}\cdot|x-x'|,$$

yields again $\beta=1$ and the constant $\tilde{C}_2(m,1,R,\varepsilon)$ from (6.36). □

The existence of finite overlapping constants for Ψ and $\tilde{\Psi}$, respectively, is the last property to be proved in order to be able to assert that the function systems Ψ and $\tilde{\Psi}$ indeed constitute sibling frames with L vanishing moments. This will be done next.

Proposition 6.27 *(Property (5.31) for Ψ)*
Property (5.31) holds for Ψ defined in (6.19) with constant

$$D_2\leq\sum_{p=0}^{i_0K_1}\left(m(R+1)^p+R\sum_{l=0}^{p-1}(R+1)^l\right). \tag{6.37}$$

Here

$$i_0 := \left[\log_{K_2} \frac{6(1+\varepsilon)}{1-\varepsilon}\right] + 1. \tag{6.38}$$

Proof. The supports of the frame elements $\psi_{j,k}$ are denoted by $I_{j,k} = [c_{j,k}, b_{j,k}]$. We want to prove that $\{I_{j,k}\}_{j,k}$ constitutes a relatively separated family of supports, i.e., that there exists a finite overlapping constant $D_2 > 0$ such that

$$\forall J \subset I\!R \quad \text{bounded interval:} \quad \#\Lambda_J \le D_2,$$
$$\text{where} \quad \Lambda_J := \left\{(j,k) \ : \ |I_{j,k}| \in \left[\frac{|J|}{2}, |J|\right], \ c_{j,k} \in J\right\}.$$

Let $J \subset I\!R$ be an arbitrary bounded interval.
Let further $j_0 \in I\!N$ be the greatest possible scale with the following property: the shortest support length $|I_{j_0-1,k}|$ on the precedent level $j_0 - 1$ is greater than $|J|$, i.e., by (6.16)

$$(1-\varepsilon)h_{j_0-1} > |J|.$$

This implies the existence of a k_0 such that for ψ_{j_0,k_0} it holds that

$$\mathit{length}\,(\mathit{supp}\ \psi_{j_0,k_0}) = |I_{j_0,k_0}| \le |J|.$$

Successively we obtain from (6.18)

$$\begin{aligned}
\frac{h_{j_0+K_1}}{|J|} &\le \frac{h_{j_0+K_1}}{|I_{j_0,k_0}|} \le \frac{h_{j_0+K_1}}{(1-\varepsilon)h_{j_0}} \le \frac{1}{1-\varepsilon}\cdot\frac{1}{K_2},\\
\frac{h_{j_0+2K_1}}{|J|} &\le \frac{h_{j_0+2K_1}}{(1-\varepsilon)h_{j_0}} = \frac{1}{1-\varepsilon}\cdot\frac{h_{j_0+2K_1}}{h_{j_0+K_1}}\cdot\frac{h_{j_0+K_1}}{h_{j_0}} \le \frac{1}{1-\varepsilon}\cdot\frac{1}{(K_2)^2},\\
\frac{h_{j_0+i\cdot K_1}}{|J|} &\le \frac{1}{1-\varepsilon}\cdot\frac{1}{(K_2)^i}, \qquad \text{for all } i \in I\!N.
\end{aligned}$$

The longest possible support on the level $j_0 + i\cdot K_1$ thus satisfies by (6.26)

$$|I_{j_0+i\cdot K_1,k}| \le 3(1+\varepsilon)h_{j_0+i\cdot K_1} \le \frac{3(1+\varepsilon)}{1-\varepsilon}\cdot\frac{|J|}{(K_2)^i}.$$

We determine next a concrete (and as small as possible) value for i such that the longest possible support on the level $j_0 + i\cdot K_1$ has a length less than $\frac{|J|}{2}$ by requiring

$$\frac{3(1+\varepsilon)}{1-\varepsilon}\cdot\frac{|J|}{(K_2)^i} < \frac{|J|}{2}.$$

This entails

$$\log_{K_2}\frac{6(1+\varepsilon)}{1-\varepsilon} < i,$$

and finally

$$i_0 = \left[\log_{K_2}\frac{6(1+\varepsilon)}{1-\varepsilon}\right] + 1.$$

In order to count indices (j,k) satisfying $|I_{j,k}| \in \left[\frac{|J|}{2}, |J|\right]$ and $c_{j,k} \in J$ we have to look only at levels

$$j_0,\ j_0+1, \ldots,\ j_0 + i_0K_1 - 1.$$

We next need to know how many starting points $c_{j,k}$ for intervals $I_{j,k}$ with $|I_{j,k}| \in \left[\frac{|J|}{2}, |J|\right]$ exist in J on the aforementioned levels. Recall that

$$(1-\varepsilon)h_{j_0-1} > |J|.$$

This implies a maximal number of m knots belonging to J on the level $j_0 - 1$. On the next level j_0 we thus obtain at most

$$m + (m+1)\cdot R = m(R+1) + R$$

knots belonging to J, and on the level $j_0 + 1$

$$(m(R+1)+R) + (m(R+1)+R+1)\cdot R = m(R+1)^2 + (R+1)R + R,$$

respectively. Iterating further we obtain for the last relevant level $j_0 + i_0K_1 - 1$ a maximal number of

$$m\cdot(R+1)^{i_0K_1} + R\cdot\sum_{l=0}^{i_0K_1-1}(R+1)^l$$

knots belonging to J. Combining all this we obtain the following bound for the desired constant D_2:

$$D_2 \le \sum_{p=0}^{i_0K_1}\left(m(R+1)^p + R\sum_{l=0}^{p-1}(R+1)^l\right). \qquad \square$$

Proposition 6.28 *(Property (5.31) for $\tilde\Psi$)*
Property (5.31) holds for $\tilde\Psi$ defined in (6.20) with constant

$$\tilde D_2 \le \sum_{p=0}^{i_0K_1}\left(m(R+1)^p + R\sum_{l=0}^{p-1}(R+1)^l\right), \tag{6.39}$$

where

$$i_0 := \left[\log_{K_2}\frac{6(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1)}{1-\varepsilon}\right] + 1. \tag{6.40}$$

Proof. The supports of the dual frame elements $\tilde\psi_{j,k}$ are denoted by $\tilde I_{j,k} = [\tilde c_{j,k}, \tilde b_{j,k}]$. We want to prove that $\{\tilde I_{j,k}\}_{j,k}$ constitutes a relatively separated family of supports, i.e., that there exists a finite overlapping constant $\tilde D_2 > 0$ such that

$$\forall J \subset I\!\!R \quad \text{bounded interval:} \quad \#\Lambda_J \le \tilde D_2,$$
$$\text{where} \quad \Lambda_J := \left\{(j,k)\ :\ |\tilde I_{j,k}| \in \left[\frac{|J|}{2}, |J|\right],\ \tilde c_{j,k} \in J\right\}.$$

Let $J \subset I\!\!R$ be an arbitrary bounded interval.
Let further $j_0 \in I\!\!N$ be the greatest possible scale with the following property: the shortest support on the preceding level $j_0 - 1$ has length greater than $|J|$, i.e.,

$$(1-\varepsilon)h_{j_0-1} > |J|.$$

This implies the existence of a k_0 such that for $\tilde\psi_{j_0,k_0}$ holds

$$\textit{length}\,(\textit{supp}\ \tilde\psi_{j_0,k_0}) = |\tilde I_{j_0,k_0}| \le |J|.$$

We further successively obtain as in the proof of Proposition 6.27

$$\frac{h_{j_0+i\cdot K_1}}{|J|} \le \frac{1}{1-\varepsilon}\cdot\frac{1}{(K_2)^i}, \qquad \text{for all } i \in I\!N.$$

The longest possible support on level $j_0 + i \cdot K_1$ thus satisfies (cf. the proof of Proposition 6.24)

$$\begin{aligned} |I_{j_0+i\cdot K_1,k}| &\le 3(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1)\cdot h_{j_0+i\cdot K_1} \\ &\le (2(m+L-1)(2R+1)-1)\cdot\frac{3(1+\varepsilon)}{1-\varepsilon}\cdot\frac{|J|}{(K_2)^i}. \end{aligned}$$

We determine next a concrete (and as small as possible) value for i such that the longest possible support on the level $j_0+i\cdot K_1$ has a length less than $\frac{|J|}{2}$ by solving the following for:

$$(2(m+L-1)(2R+1)-1)\cdot\frac{3(1+\varepsilon)}{1-\varepsilon}\cdot\frac{|J|}{(K_2)^i} < \frac{|J|}{2}.$$

This yields

$$\log_{K_2}\frac{6(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1)}{1-\varepsilon} < i,$$

and finally

$$i_0 = \left[\log_{K_2}\frac{6(1+\varepsilon)\cdot(2(m+L-1)(2R+1)-1)}{1-\varepsilon}\right] + 1.$$

In order to count indices (j,k) satisfying $|I_{j,k}| \in \left[\frac{|J|}{2}, |J|\right]$ we proceed as in the proof of Proposition 6.27 to obtain

$$D_2 \le \sum_{p=0}^{i_0K_1}\left(m(R+1)^p + R\sum_{l=0}^{p-1}(R+1)^l\right). \qquad \square$$

Summing up the assertions proved in this section we state the following result.

Theorem 6.29 *(Quasi–uniform sibling spline frames)*
With knot sequences satisfying Conditions 6.17 the families defined by (6.19) and (6.20) constitute Bessel families and thus sibling frames with L vanishing moments in $L_2[a,b]$.

The constants C_1, C_2, D_2 for Ψ have the representations (6.29), (6.34) and (6.37)–(6.38), respectively. The constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{D}_2$ for $\tilde{\Psi}$ are given by (6.32)–(6.33), (6.35)–(6.36) and (6.39)–(6.40), respectively. For both families the parameter β takes the value 1.

6.4 Examples of quasi–uniform sibling spline frames

Example 6.30 *(Sibling spline frames of order 4 with one, two, three and four vanishing moments)*
Our choices:

- *interval* $[\mathbf{a},\mathbf{b}] = [\mathbf{0},\mathbf{1}]$*;*
- *order of B–splines* $\mathbf{m} = \mathbf{4}$*;*
- *number of vanishing moments* $\mathbf{L} = \mathbf{1}$ *(and* $\mathbf{L}=\mathbf{2}$*,* $\mathbf{L}=\mathbf{3}$*,* $\mathbf{L}=\mathbf{4}$*, respectively);*

- *knot sequences t_0, t_1 defined by*

$$\mathbf{t_0} = [\,\mathbf{0\;0\;0\;0\;0.2\;0.4\;0.4\;0.8\;1\;1\;1\;1}\,],$$

$$\mathbf{t_1} \setminus \mathbf{t_0} = [\,\mathbf{.25\;.65\;.65}\,].$$

With parameters

- $\varepsilon := \mathbf{0.5}$,
- $\mathbf{h_0} := \mathbf{0.40} > \mathbf{h_1} := \mathbf{0.30}$,
- $\mathbf{R} = \mathbf{2}$,
- $\mathbf{K_1} := \mathbf{1}$, $\mathbf{K_2} := \mathbf{1.25}$,

we obtain the following:

- $(1-\varepsilon)\cdot h_0 = 0.2$, $(1+\varepsilon)\cdot h_0 = 0.6$;
- $(1-\varepsilon)\cdot h_1 = 0.15$, $(1+\varepsilon)\cdot h_1 = 0.45$;
- *length* $(\textit{supp}\ N^B_{\mathbf{t}_0;3,k}) \in \{0.2,\ 0.4,\ 0.6\} \subset [0.2, 0.6]$;
- *length* $(\textit{supp}\ N^B_{\mathbf{t}_1;3,k}) \in \{0.2,\ 0.25,\ 0.35,\ 0.4\} \subset [0.15, 0.45]$;
- $\dfrac{h_1}{h_0} = \dfrac{0.3}{0.4} = 0.75 \le 0.8 = \dfrac{1}{K_2}$.

We thus can conclude that Conditions 6.17 on the knot sequences are satisfied.

For $L = 1$ the families Ψ_0, $\tilde{\Psi}_0$ are visualized in Figures 6.2–6.5, for $L = 2$ in Figures 6.6–6.9, for $L = 3$ in Figures 6.10–6.13 and for $L = 4$ in Figures 6.14–6.17, respectively.

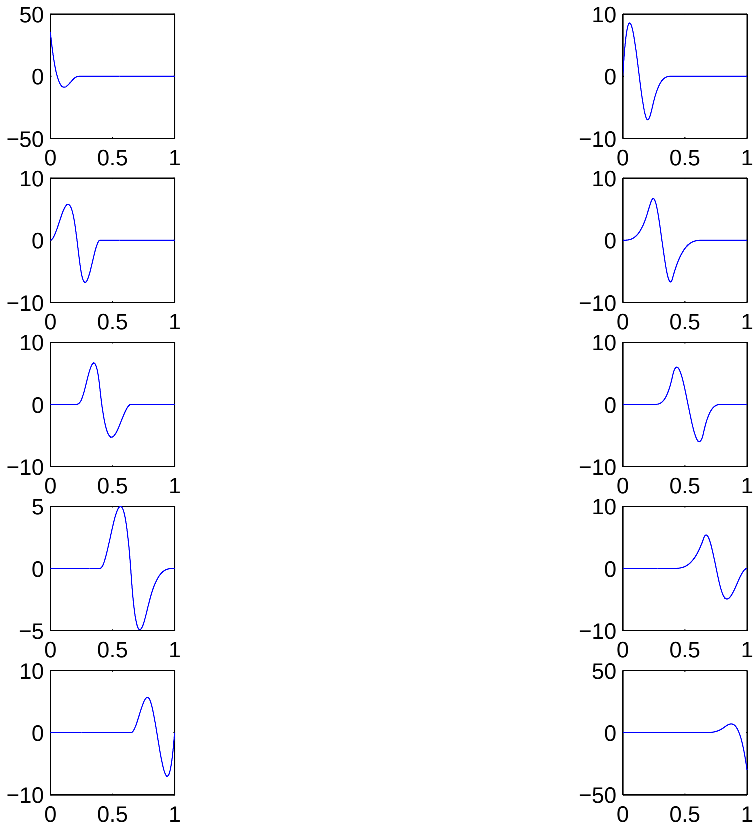

Figure 6.2: Frame elements $\psi_{0,k}$ for $L = 1$ in Example 6.30.

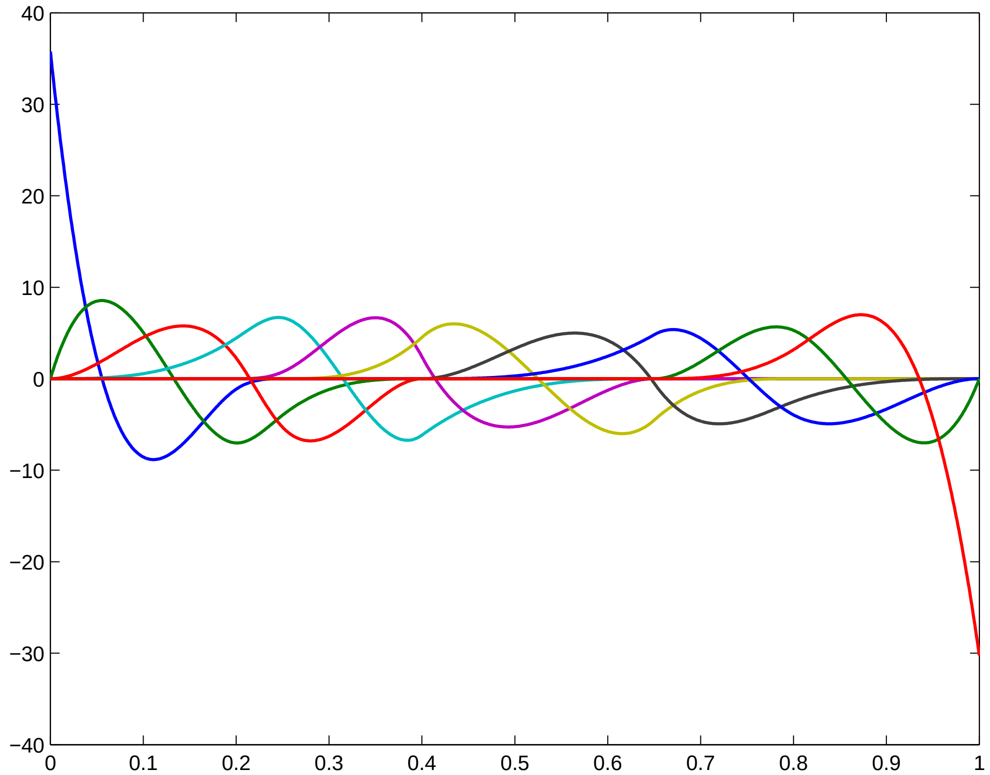

Figure 6.3: Frame elements $\psi_{0,k}$ for $L = 1$ in Example 6.30.

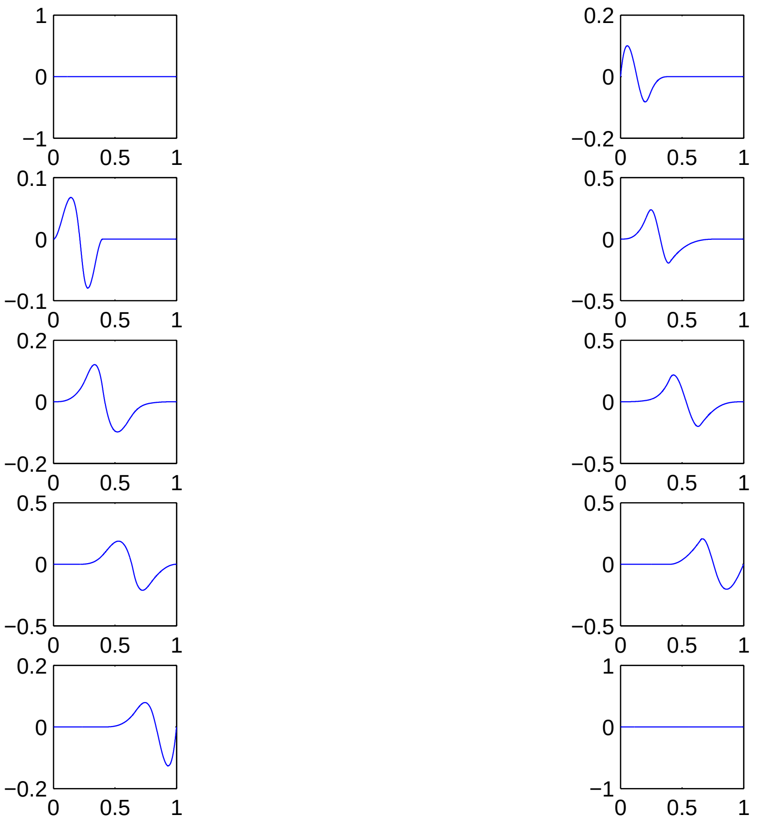

Figure 6.4: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 1$ in Example 6.30.

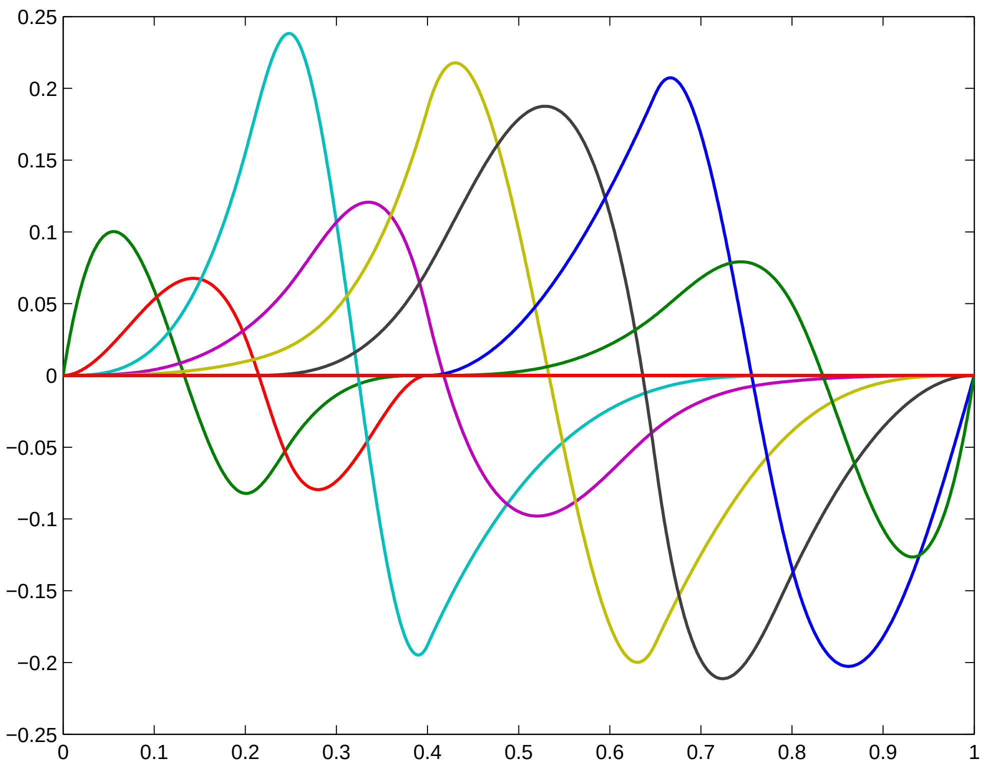

Figure 6.5: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 1$ in Example 6.30.

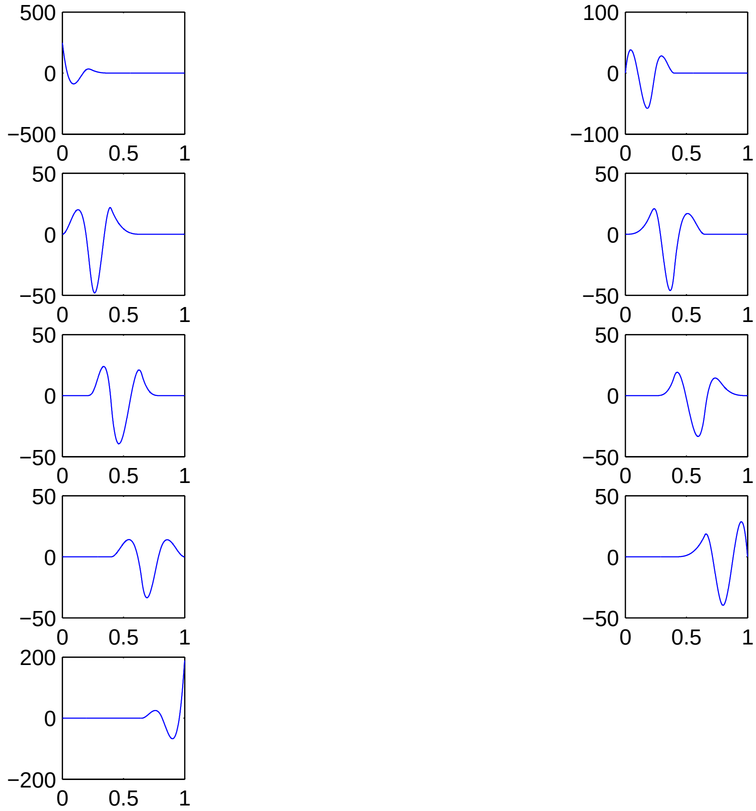

Figure 6.6: Frame elements $\psi_{0,k}$ for $L = 2$ in Example 6.30.

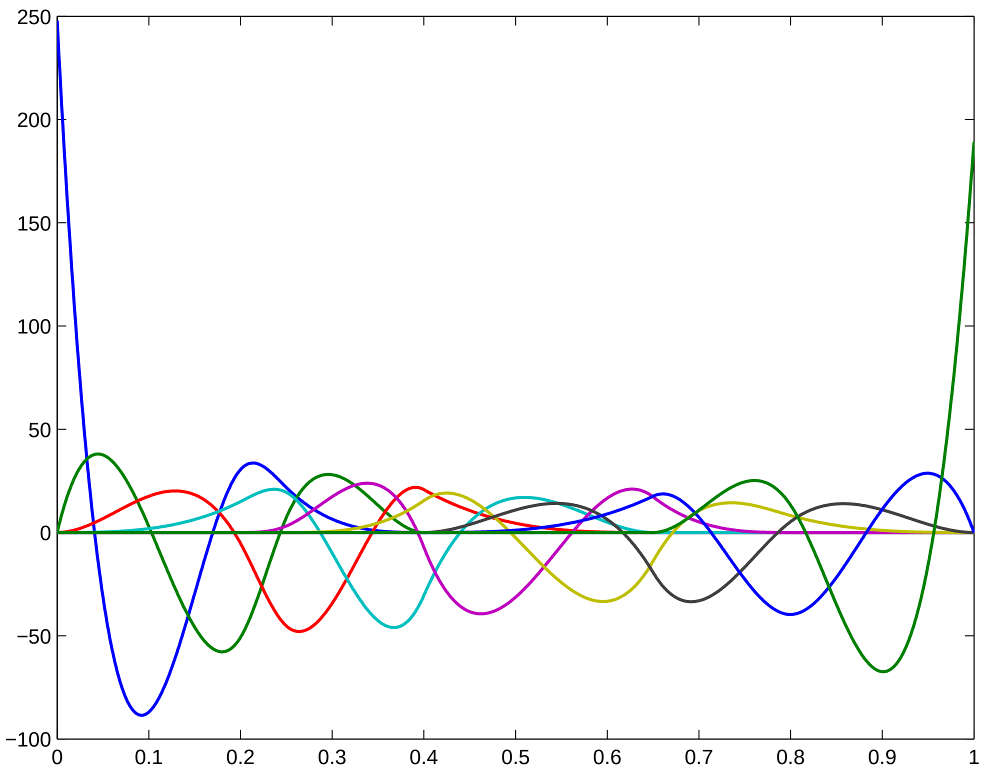

Figure 6.7: Frame elements $\psi_{0,k}$ for $L = 2$ in Example 6.30.

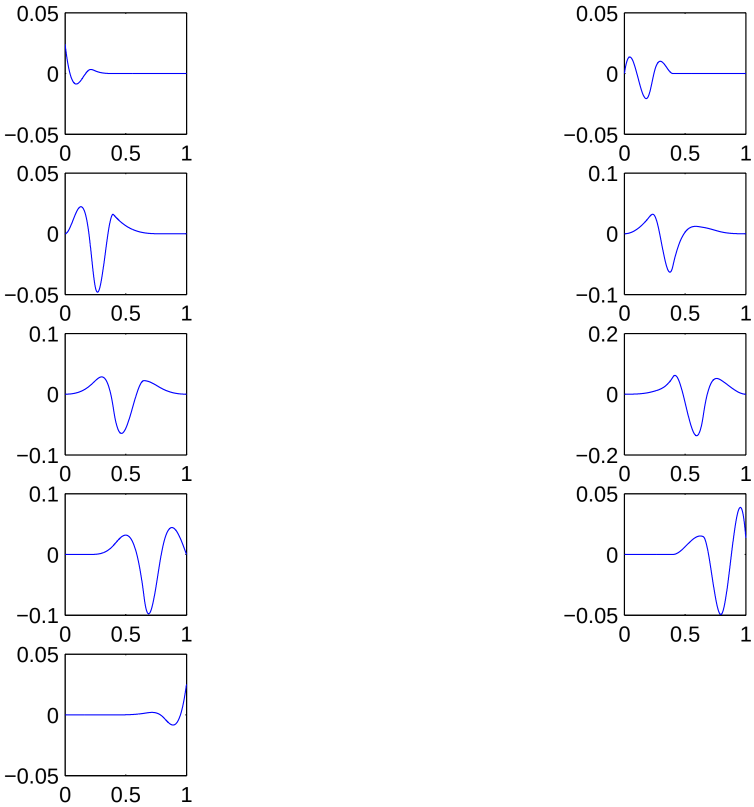

Figure 6.8: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 2$ in Example 6.30.

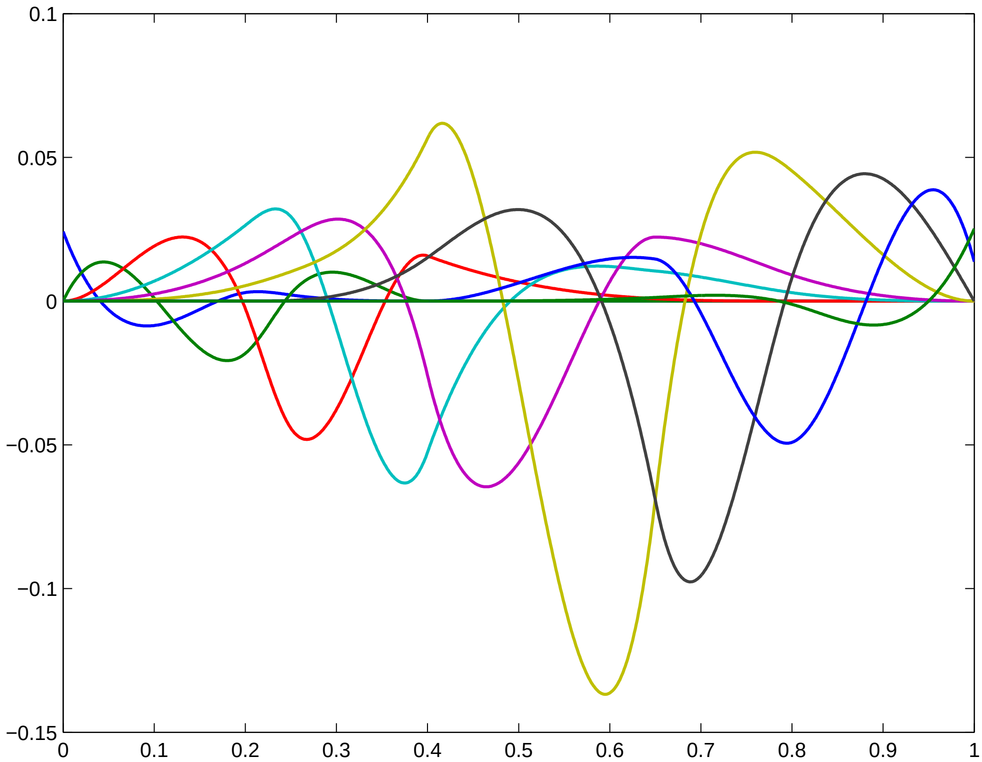

Figure 6.9: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 2$ in Example 6.30.

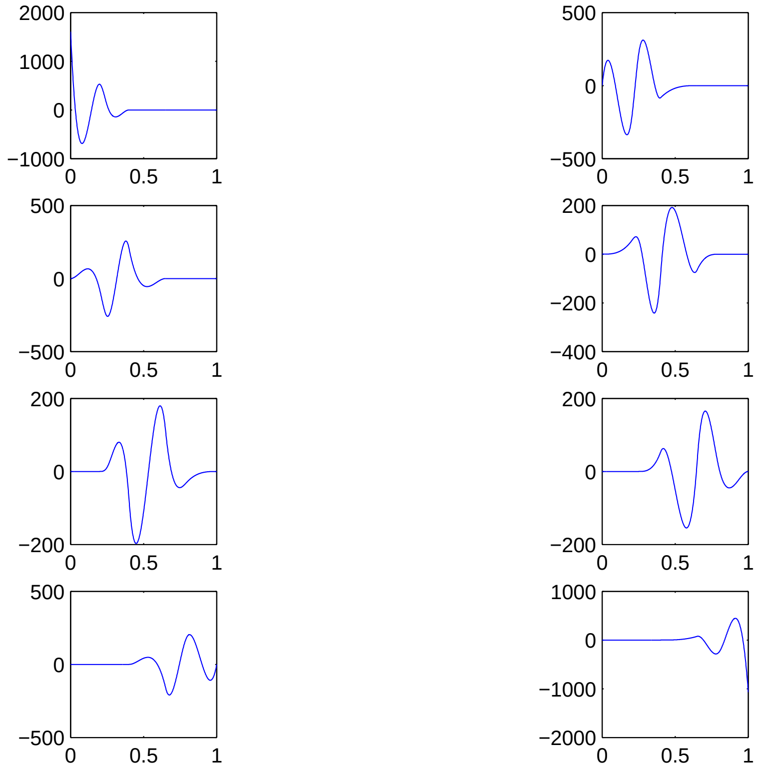

Figure 6.10: Frame elements $\psi_{0,k}$ for $L = 3$ in Example 6.30.

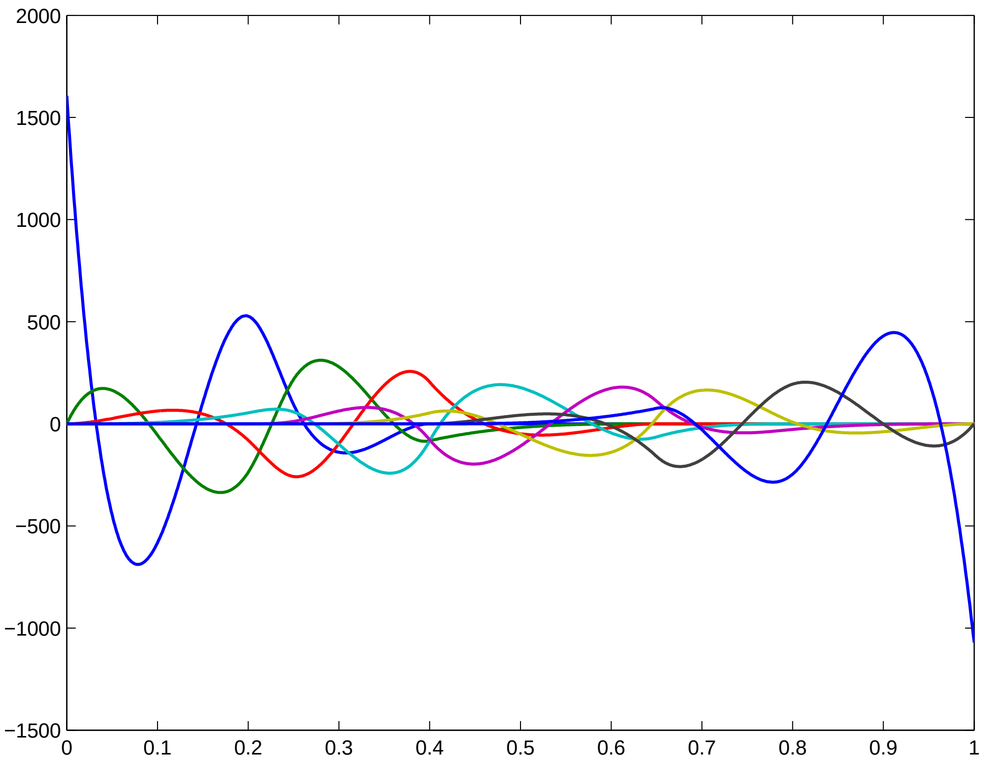

Figure 6.11: Frame elements $\psi_{0,k}$ for $L = 3$ in Example 6.30.

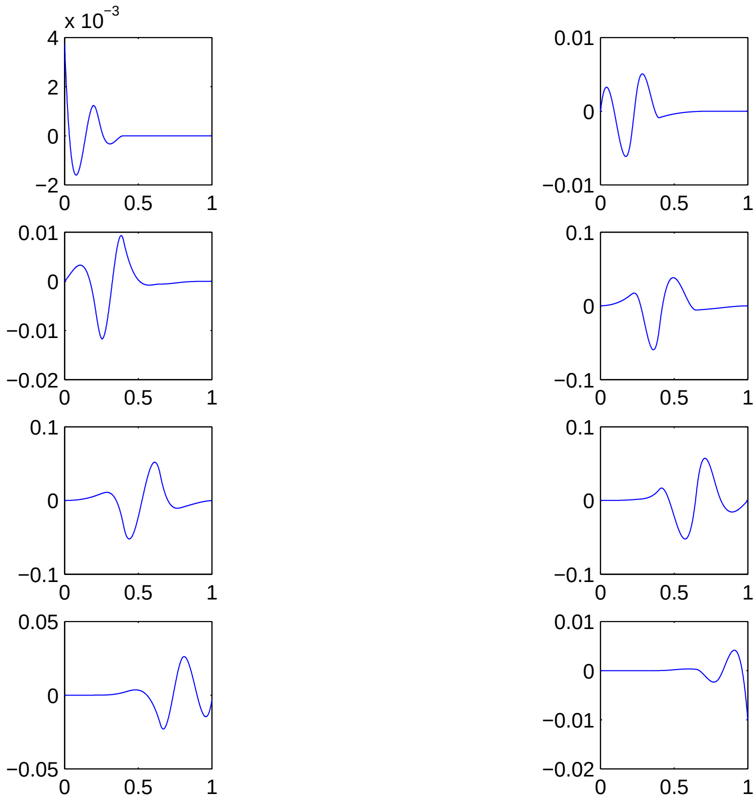

Figure 6.12: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 3$ in Example 6.30.

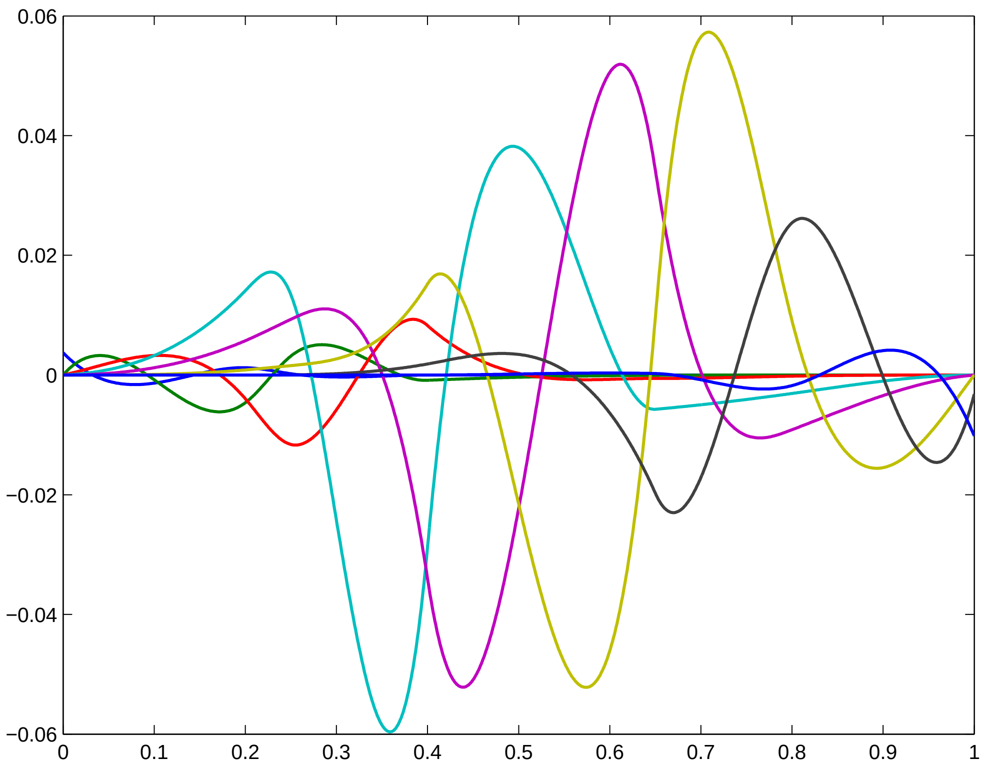

Figure 6.13: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 3$ in Example 6.30.

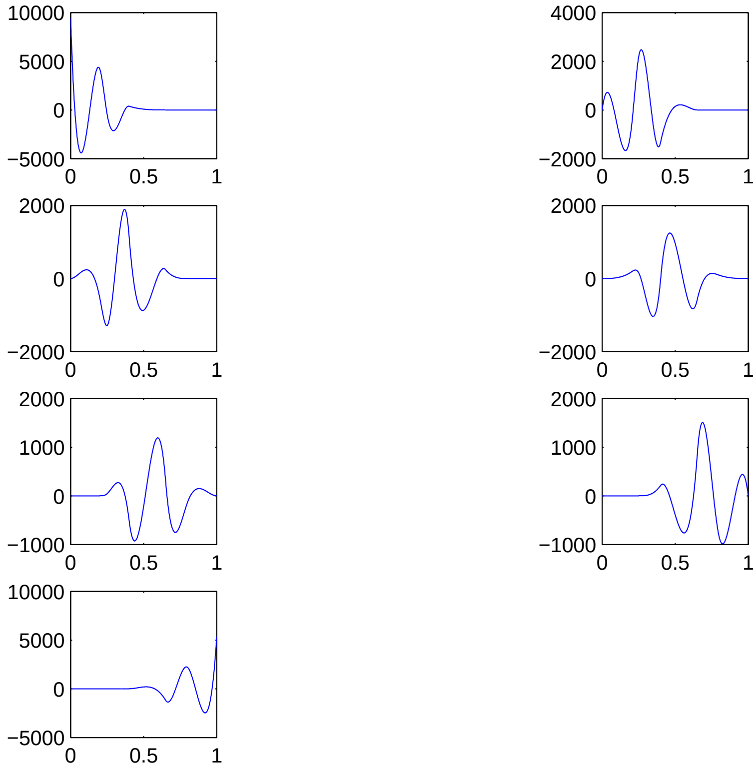

Figure 6.14: Frame elements $\psi_{0,k}$ for $L = 4$ in Example 6.30.

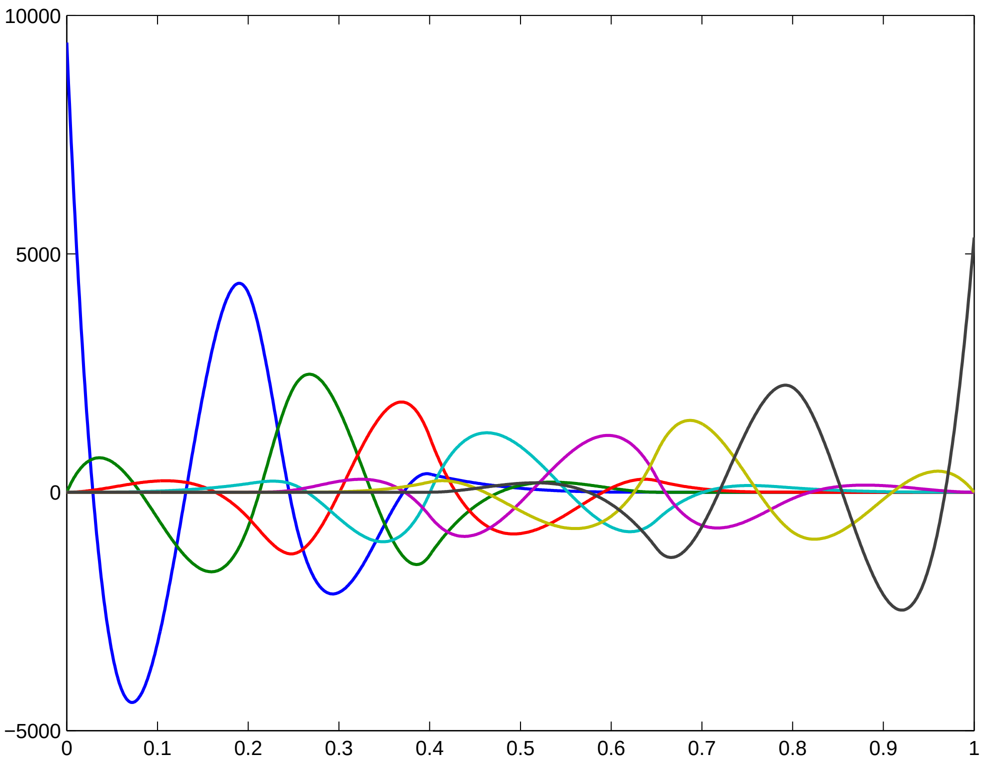

Figure 6.15: Frame elements $\psi_{0,k}$ for $L = 4$ in Example 6.30.

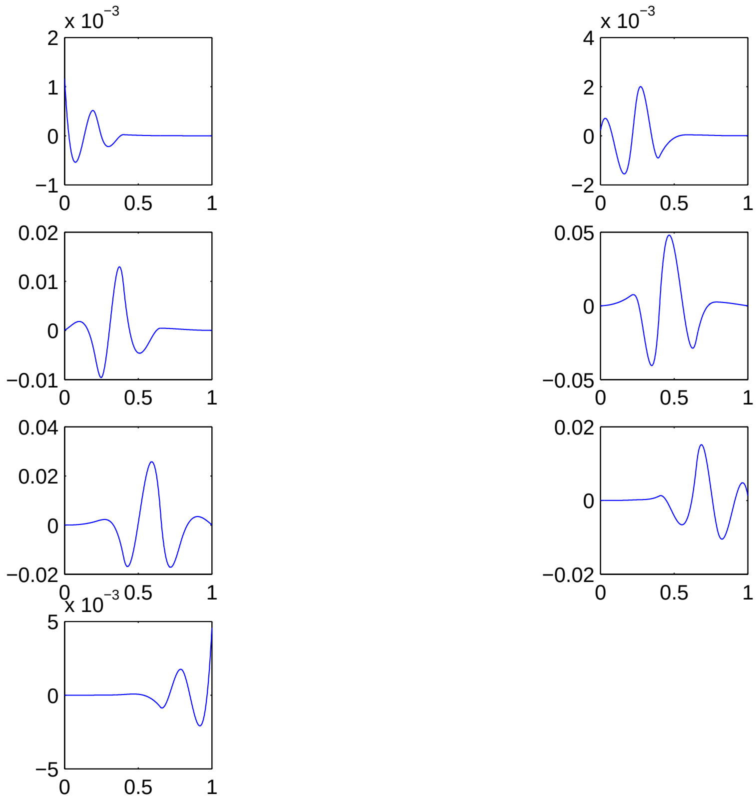

Figure 6.16: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 4$ in Example 6.30.

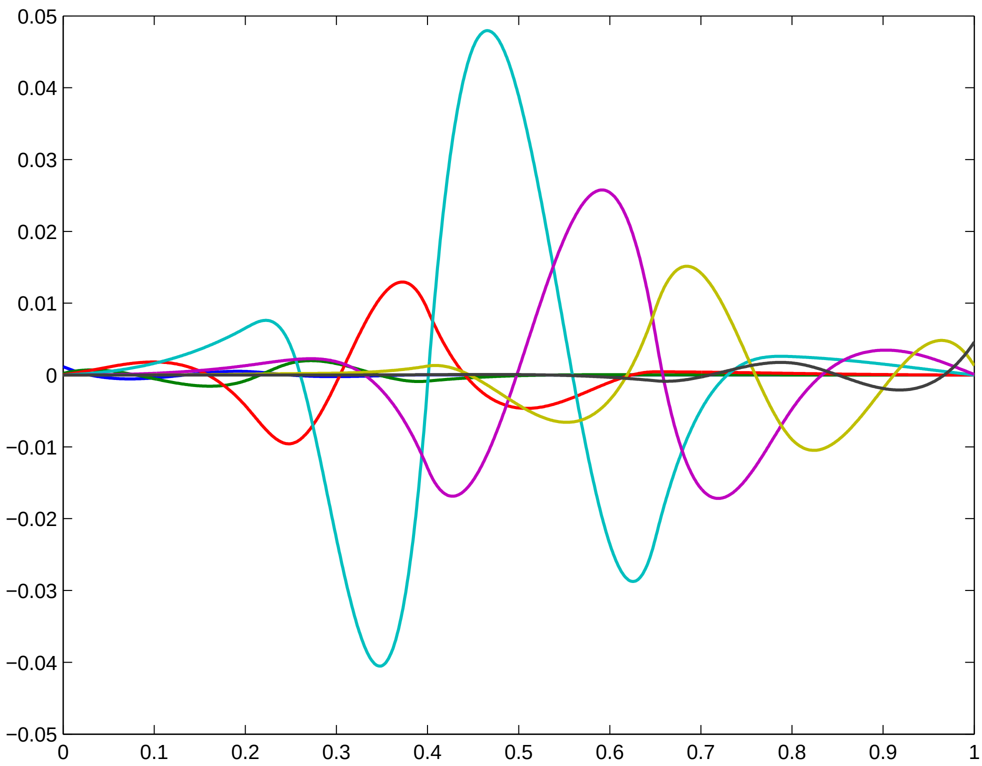

Figure 6.17: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 4$ in Example 6.30.

Example 6.31 *(Sibling spline frames of order 5 with one and two vanishing moments)*

Our choices:

- *interval* $[\mathbf{a}, \mathbf{b}] = [\mathbf{0}, \mathbf{1}]$*;*
- *order of B–splines* $\mathbf{m} = \mathbf{5}$*;*
- *number of vanishing moments* $\mathbf{L} = \mathbf{1}$ *(and* $L = 2$*, respectively);*
- *knot sequences* t_0*,* t_1*,* t_2 *defined by*

$$\begin{aligned}
\mathbf{t_0} &= [\,\mathbf{0\ 0\ 0\ 0\ 0\ 0.32\ \ 0.33\ \ 0.34\ \ 0.36\ \ 0.64\ \ 0.65\ \ 0.67\ \ 0.68\ \ 1\ 1\ 1\ 1\ 1}\,], \\
\mathbf{t_1} \setminus \mathbf{t_0} &= [\,\mathbf{0.15\ \ 0.15\ \ 0.17\ \ 0.20\ \ 0.49\ \ 0.49\ \ 0.52\ \ 0.53\ \ 0.79\ \ 0.80\ \ 0.85\ \ 0.85}\,], \\
\mathbf{t_2} \setminus \mathbf{t_1} &= [\,\mathbf{0.08\ \ 0.08\ \ 0.09\ \ 0.09\ \ 0.22\ \ 0.26\ \ 0.27\ \ 0.29\ \ 0.41\ \ 0.41\ \ 0.41\ \ 0.46} \\
&\qquad \mathbf{0.56\ \ 0.57\ \ 0.60\ \ 0.60\ \ 0.72\ \ 0.73\ \ 0.76\ \ 0.78\ \ 0.87\ \ 0.93\ \ 0.93\ \ 0.93}\,],
\end{aligned}$$

  see Figure 6.18.

With parameters

- $\varepsilon := \mathbf{0.3}$,
- $\mathbf{h_0} := \mathbf{0.40} > \mathbf{h_1} := \mathbf{0.18} > \mathbf{h_2} := \mathbf{0.10}$,
- $\mathbf{R} = \mathbf{4}$,
- $\mathbf{K_1} := \mathbf{1}$, $\mathbf{K_2} := \mathbf{1.6}$,

we obtain the following:

- $(1 - \varepsilon) \cdot h_0 = 0.28$, $(1 + \varepsilon) \cdot h_0 = 0.52$*;*
- $(1 - \varepsilon) \cdot h_1 = 0.126$, $(1 + \varepsilon) \cdot h_1 = 0.234$*;*
- $(1 - \varepsilon) \cdot h_2 = 0.07$, $(1 + \varepsilon) \cdot h_2 = 0.13$*;*
- *length* $(\textit{supp}\ N^B_{\mathbf{t}_0;4,k}) \in \{0.32,\ 0.33,\ 0.34,\ 0.35,\ 0.36\} \subset [0.28, 0.52]$;
- *length* $(\textit{supp}\ N^B_{\mathbf{t}_1;4,k}) \in \{0.15,\ 0.16,\ 0.17,\ 0.18,\ 0.20,\ 0.21\} \subset [0.126, 0.234]$;
- *length* $(\textit{supp}\ N^B_{\mathbf{t}_2;4,k}) \in \{0.07,\ 0.08,\ 0.09,\ 0.10,\ 0.11,\ 0.13\} \subset [0.07, 0.13]$;
- $\dfrac{h_1}{h_0} = \dfrac{0.18}{0.40} = 0.45 \leq 0.625 = \dfrac{1}{K_2}$*;*
- $\dfrac{h_2}{h_1} = \dfrac{0.10}{0.18} = 0.\bar{5} \leq 0.625 = \dfrac{1}{K_2}$.

We thus can conclude that Conditions 6.17 on the knot sequences are satisfied.

For $L = 1$ *the families* Ψ_0*,* $\tilde{\Psi}_0$*,* Ψ_1*,* $\tilde{\Psi}_1$ *are visualized in Figures 6.19–6.22 and for* $L = 2$ *in Figures 6.23–6.26, respectively.*

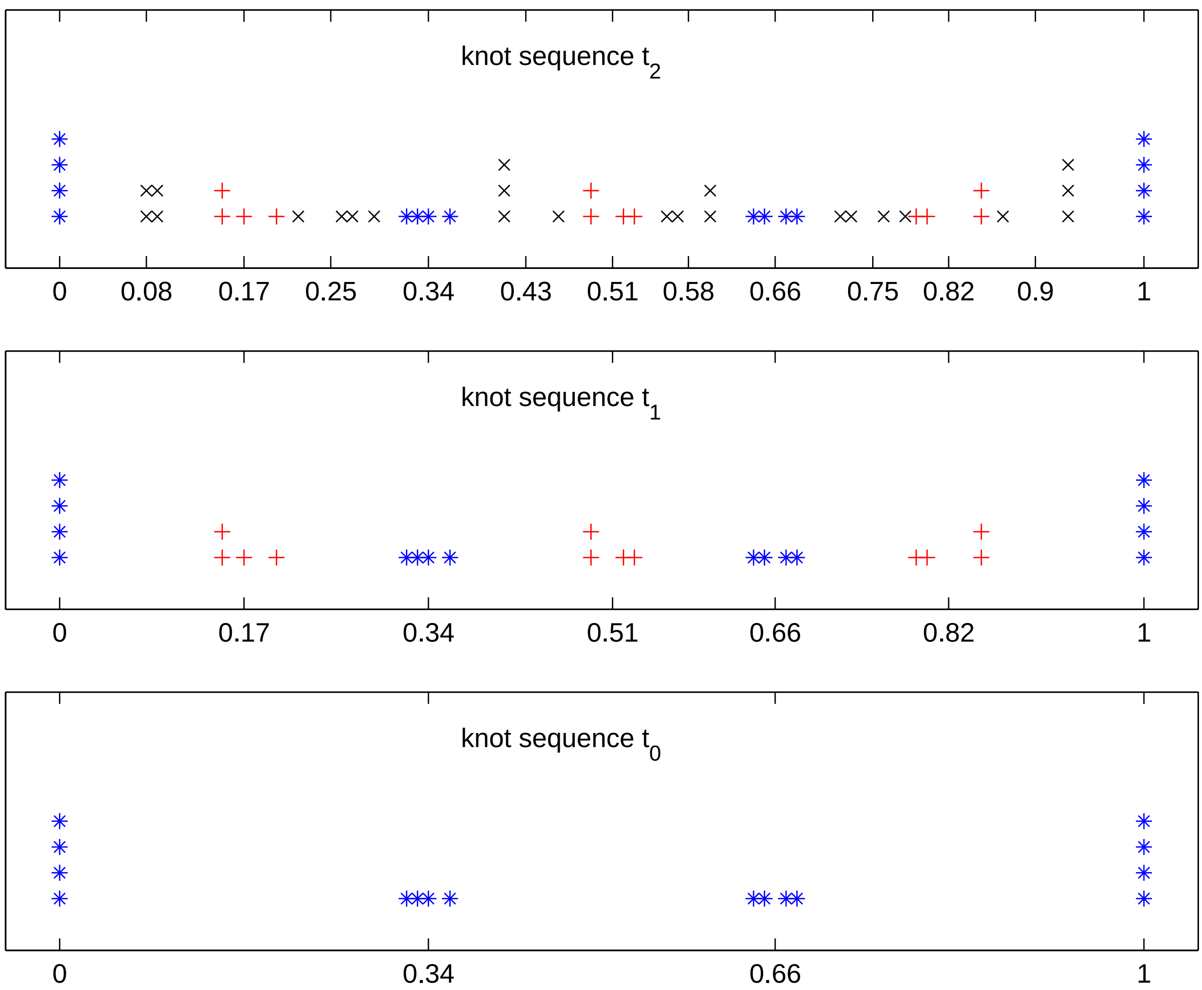

Figure 6.18: Knot sequences t_0, t_1, t_2 for the quasi–uniform case, see Example 6.31.

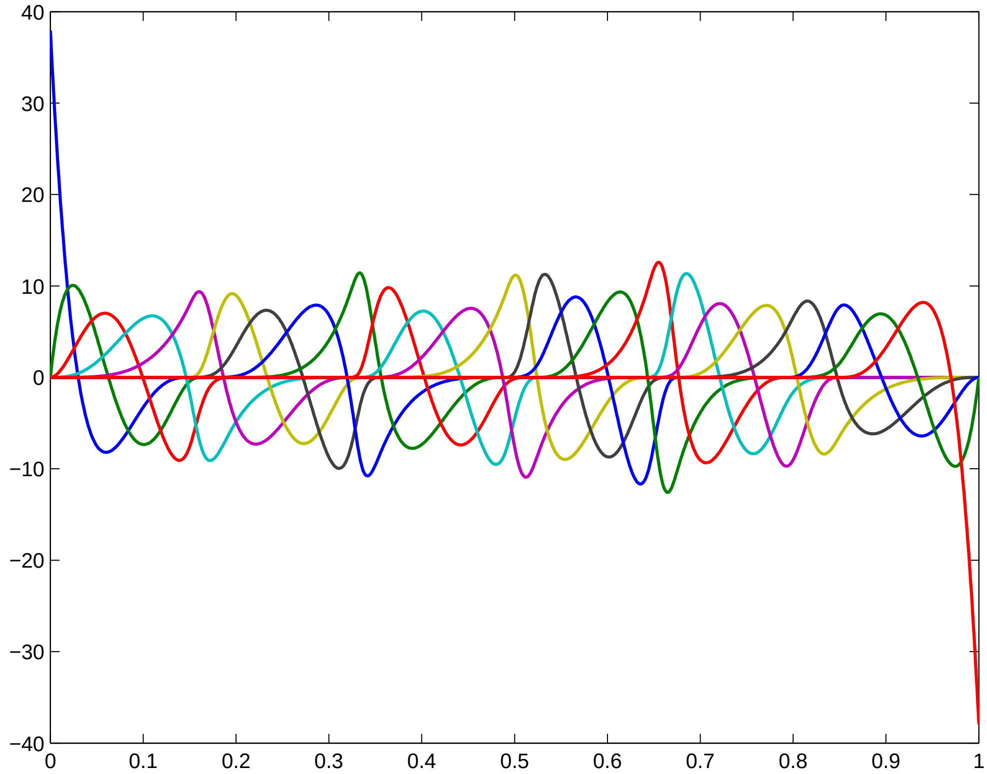

Figure 6.19: Frame elements $\psi_{0,k}$ for $L = 1$ in Example 6.31.

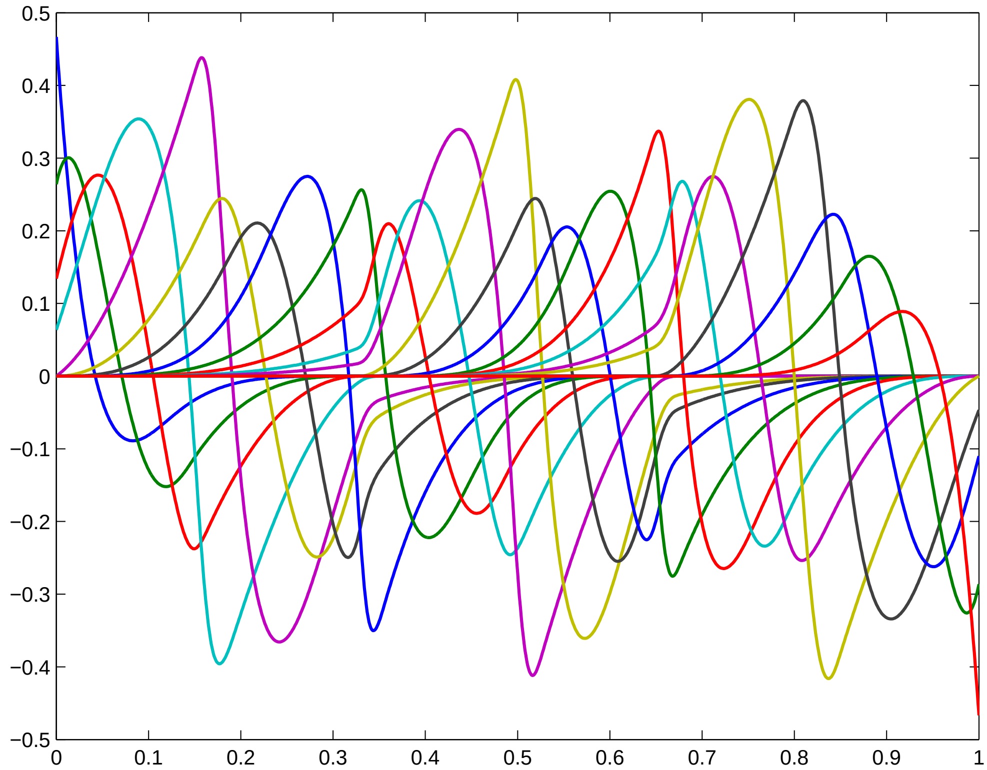

Figure 6.20: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 1$ in Example 6.31.

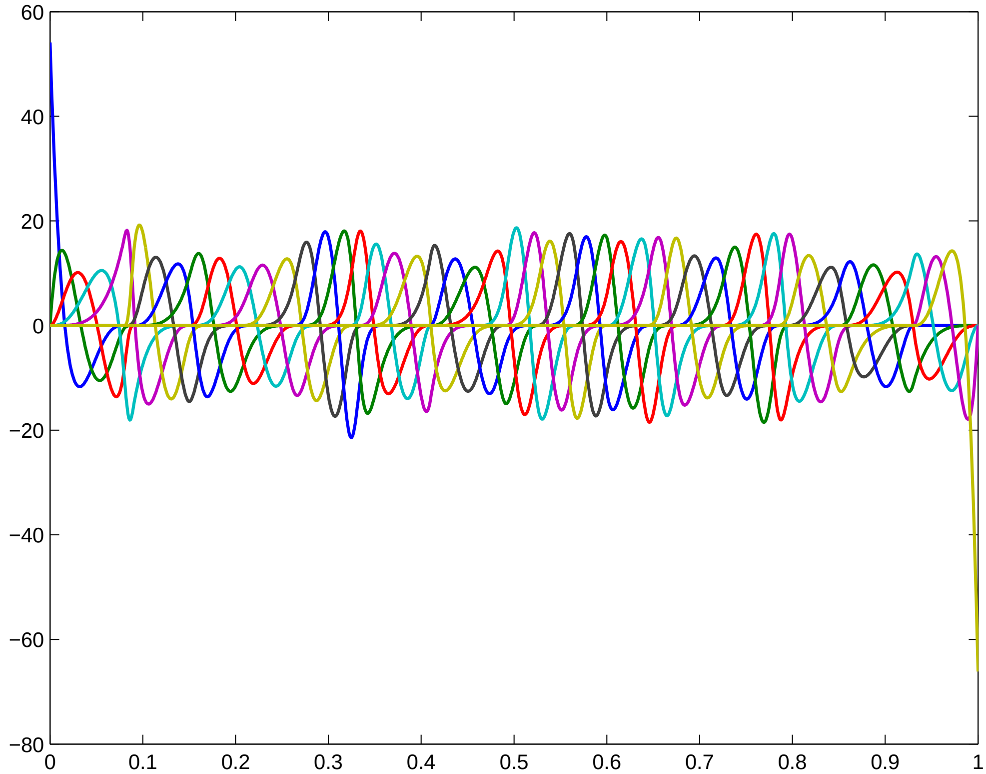

Figure 6.21: Frame elements $\psi_{1,k}$ for $L = 1$ in Example 6.31.

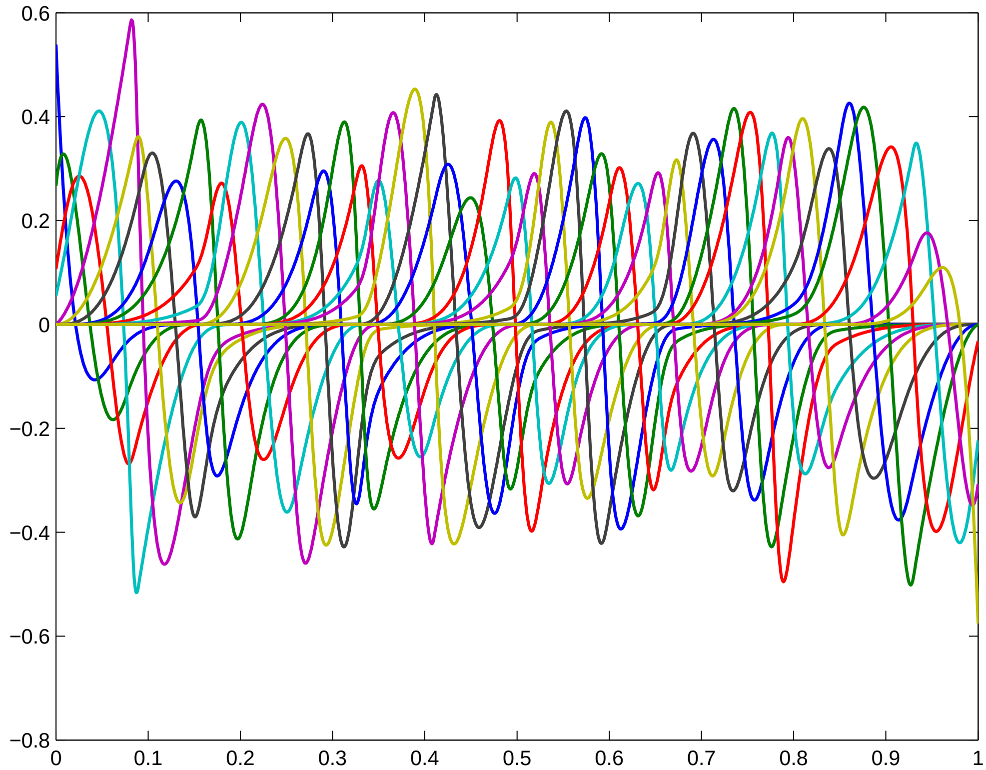

Figure 6.22: Dual frame elements $\tilde{\psi}_{1,k}$ for $L = 1$ in Example 6.31.

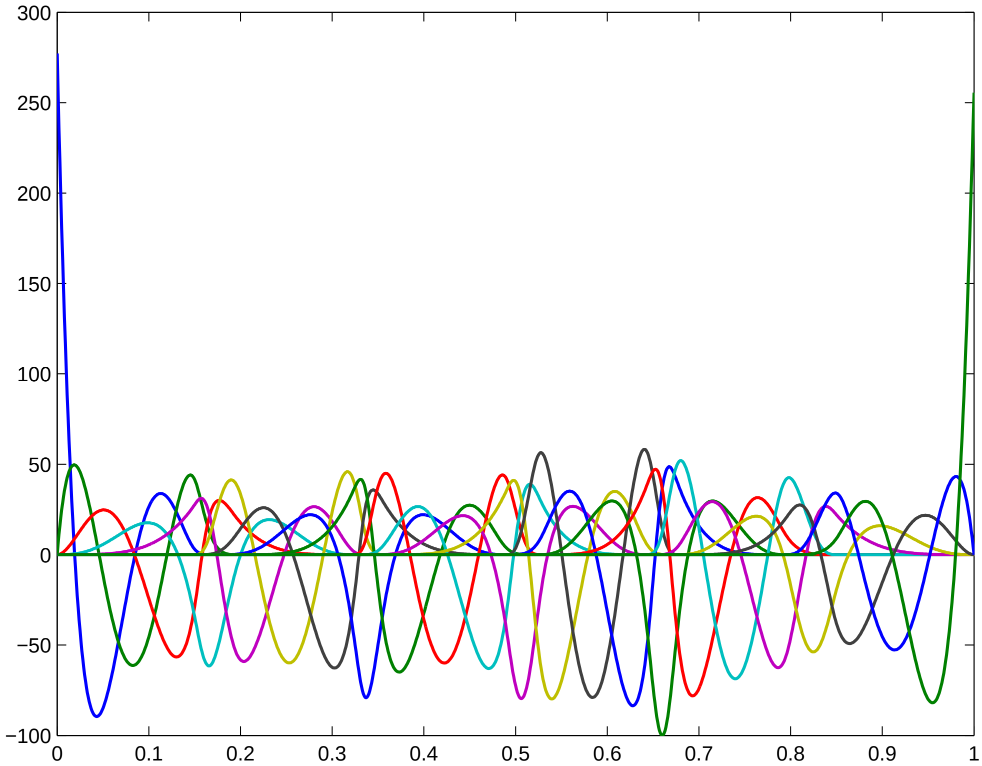

Figure 6.23: Frame elements $\psi_{0,k}$ for $L = 2$ in Example 6.31.

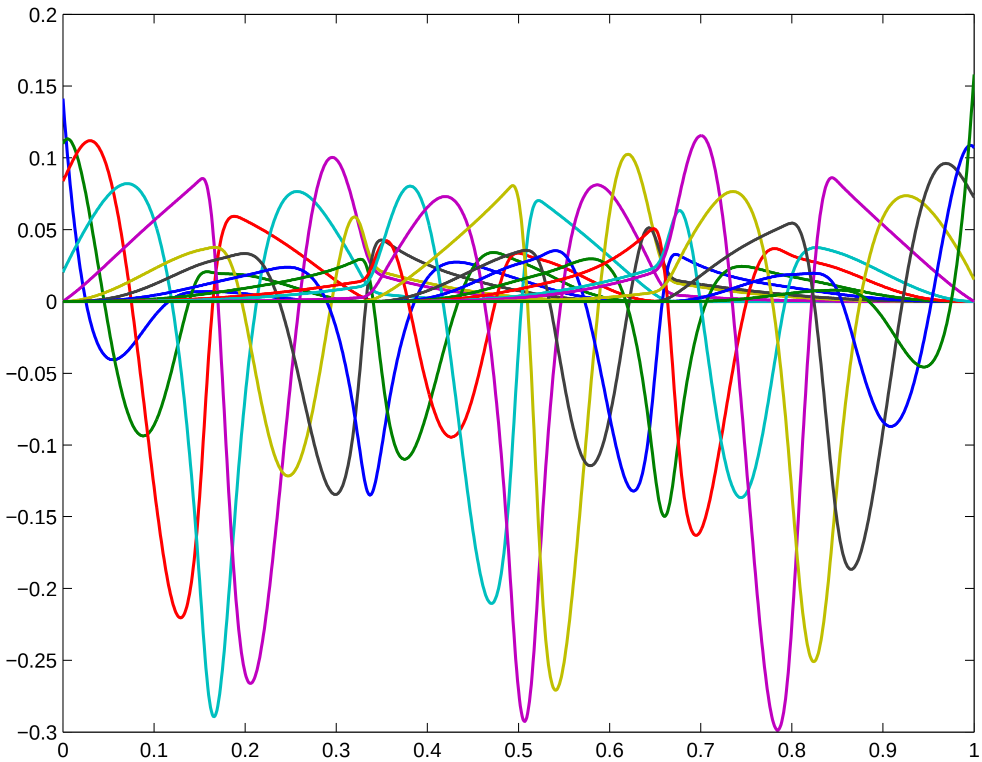

Figure 6.24: Dual frame elements $\tilde{\psi}_{0,k}$ for $L = 2$ in Example 6.31.

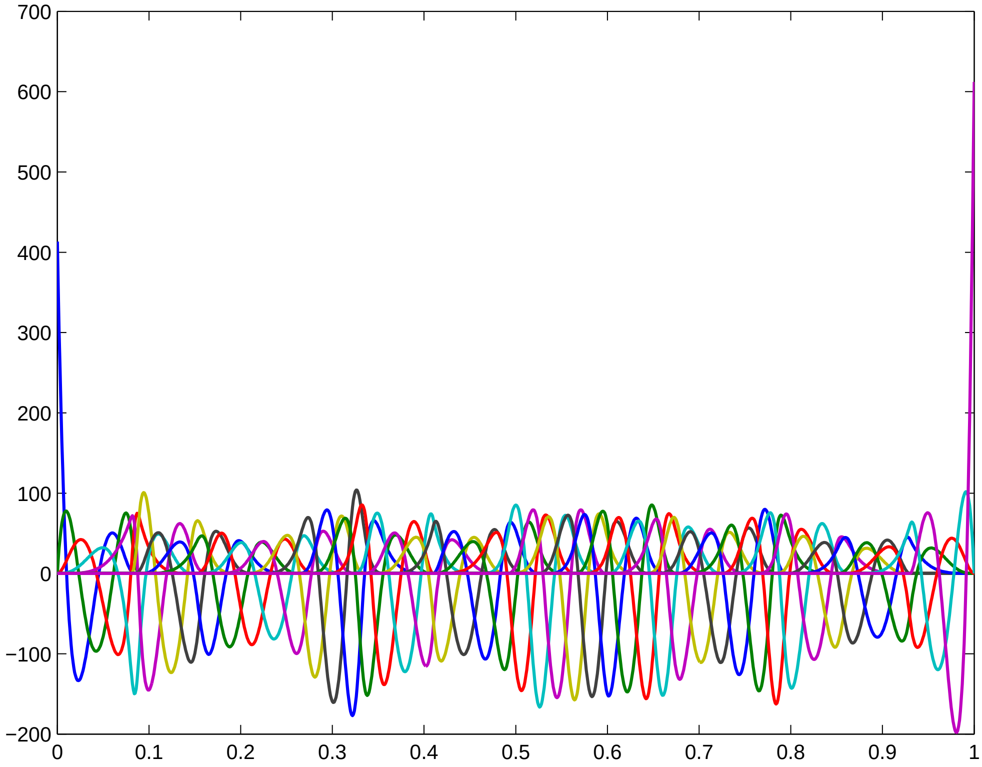

Figure 6.25: Frame elements $\psi_{1,k}$ for $L = 2$ in Example 6.31.

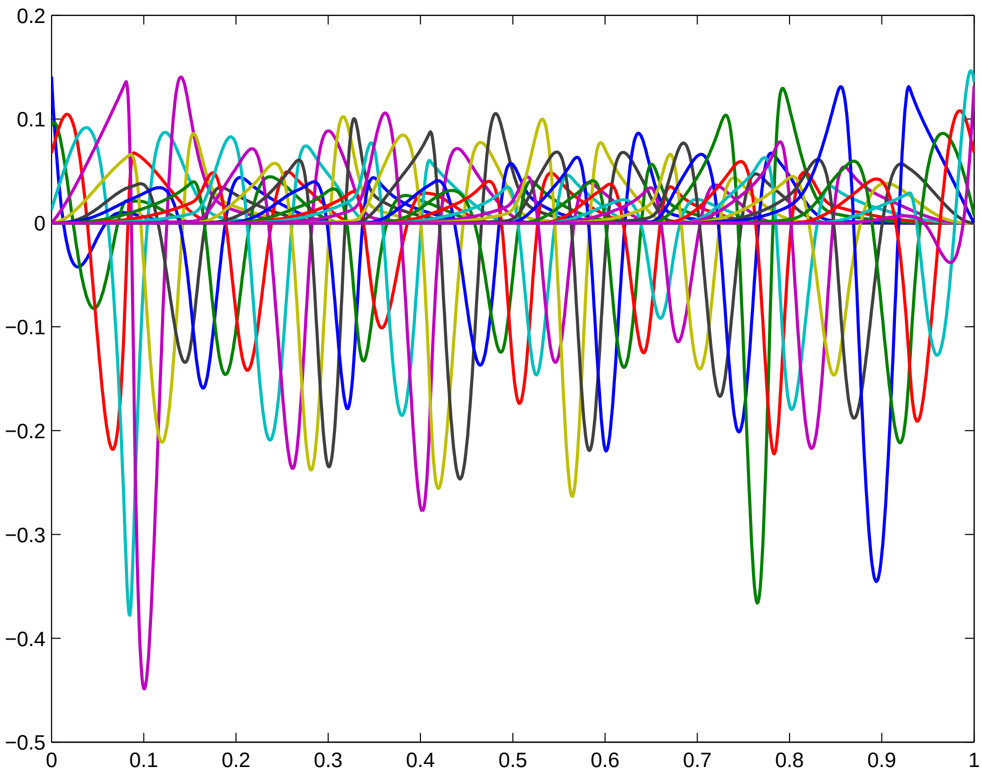

Figure 6.26: Dual frame elements $\tilde{\psi}_{1,k}$ for $L = 2$ in Example 6.31.

6.5 Outlook for further research

For knot sequences satisfying – instead of quasi–uniformity – a property of the type

$$\exists B > 1 \ \forall j \ \forall k \qquad \frac{1}{B} \leq \frac{t^{(j)}_{k+1+(m-1)} - t^{(j)}_{k+1}}{t^{(j)}_{k+(m-1)} - t^{(j)}_{k}} \leq B,$$

(i.e., locally comparable support length of order $m-1$ on the same level), we conjecture that the scheme

$$\Psi = \{\Psi_j(x)\}_{j\geq 0} := \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot \mathbf{diag}\left(\left(\mathbf{t^{(j+1)}_{k+m+L}} - \mathbf{t^{(j+1)}_{k}}\right)^{\mathbf{L}}_{\mathbf{k}}\right)\right\}_{j\geq 0},$$

$$\tilde{\Psi} = \{\tilde{\Psi}_j(x)\}_{j\geq 0} := \left\{\mathbf{\Phi}^B_{\mathbf{t_{j+1}};m}(x) \cdot E^B_{\mathbf{t_{j+1}};m,L} \cdot Z^B_{\mathbf{t_j},\mathbf{t_{j+1}};m,L} \cdot \mathbf{diag}\left(\left(\mathbf{t^{(j+1)}_{k+m+L}} - \mathbf{t^{(j+1)}_{k}}\right)^{-\mathbf{L}}_{\mathbf{k}}\right)\right\}_{j\geq 0}$$

provides sibling spline frames with L vanishing moments. The local structure of Z^B, as detailed in Section 6.2, opens this potential. The maximal multiplicite $m-1$ for the inner knots and the bounded refinement rate R have of course to be kept. Condition (6.18) has to be reformulated.

In order to carry over the local estimates of the elements of the product matrices PVP^T to the elements of Z^B, one has to determine the starting point of the non–zero block in the columns of Z^B in dependency of ρ.

Example 6.32 *For the interval $[a,b] = [0,1]$, for $m = 4$ and $L = 1$, with knot sequences*

$$t_0 = \left[0,\ 0,\ 0,\ 0,\ \frac{1}{10},\ \frac{3}{10},\ \frac{9}{20},\ \frac{13}{20},\ \frac{7}{10},\ \frac{7}{10},\ \frac{3}{4},\ 1,\ 1,\ 1,\ 1\right],$$

$$t_1 \setminus t_0 = \left[\frac{1}{10},\ \frac{3}{5},\ \frac{13}{20},\ \frac{3}{4},\ \frac{4}{5}\right],$$

$$t_2 \setminus t_1 = \left[\frac{1}{20},\ \frac{1}{20},\ \frac{3}{20},\ \frac{3}{20},\ \frac{1}{4}\right],$$

$$t_3 \setminus t_2 = \left[\frac{3}{40},\ \frac{3}{40},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{5}\right],$$

(see Figure 6.27), one obtains the families of functions Ψ_0, $\tilde{\Psi}_0$, Ψ_1, $\tilde{\Psi}_1$, Ψ_2, $\tilde{\Psi}_2$ presented in Figures 6.28–6.33, respectively. Note the relation between the local character of the new inserted knots in $\mathbf{t}_2$ and the local character of the functions $\psi_{1,k}$ and $\tilde{\psi}_{1,k}$. The same phenomenon is visible for $\mathbf{t}_3$ and $\psi_{2,k}$–$\tilde{\psi}_{2,k}$. This locality property will hopefully motivate also other people to work on this (still open) problem.

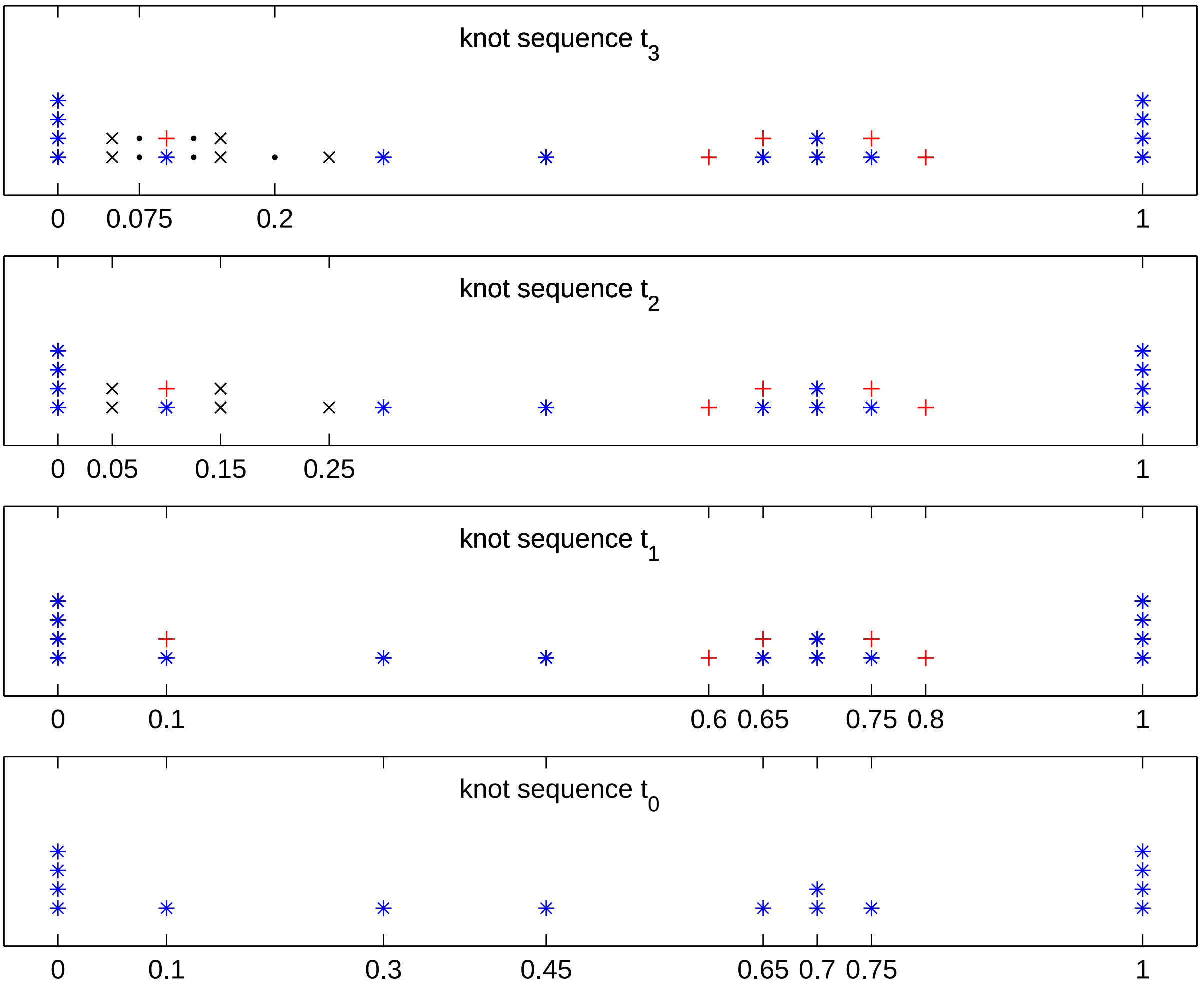

Figure 6.27: Knot sequences $\mathbf{t}_0$, $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ from Example 6.32.

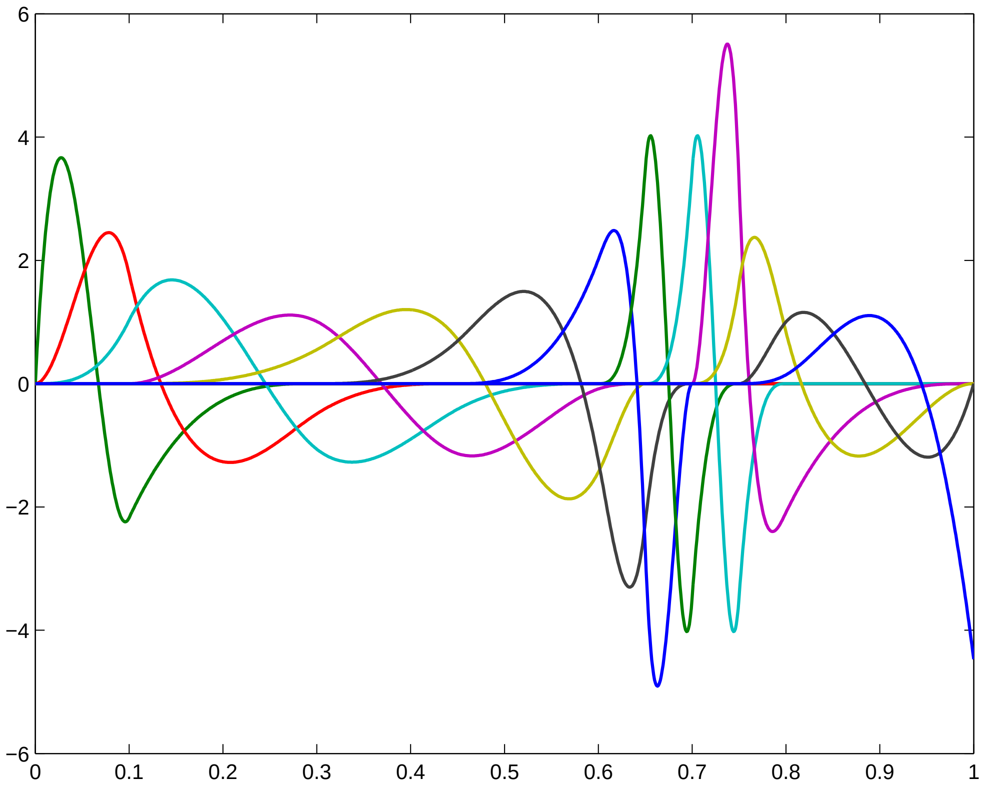

Figure 6.28: Family Ψ_0 from Example 6.32.

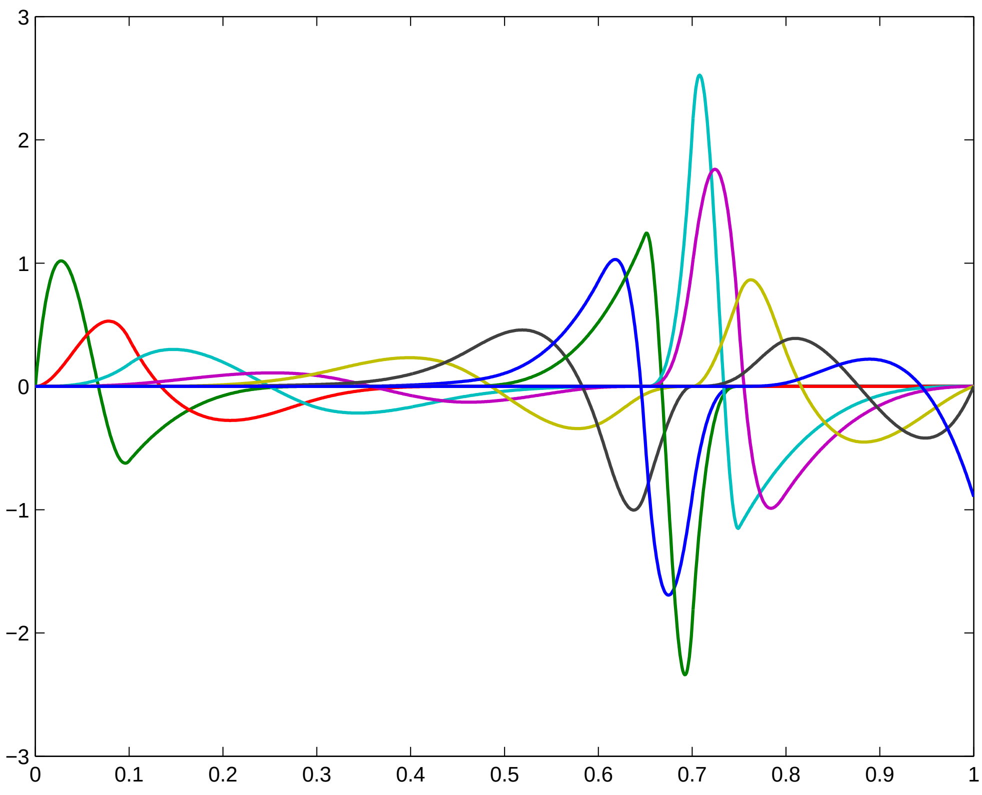

Figure 6.29: Family $\tilde{\Psi}_0$ from Example 6.32.

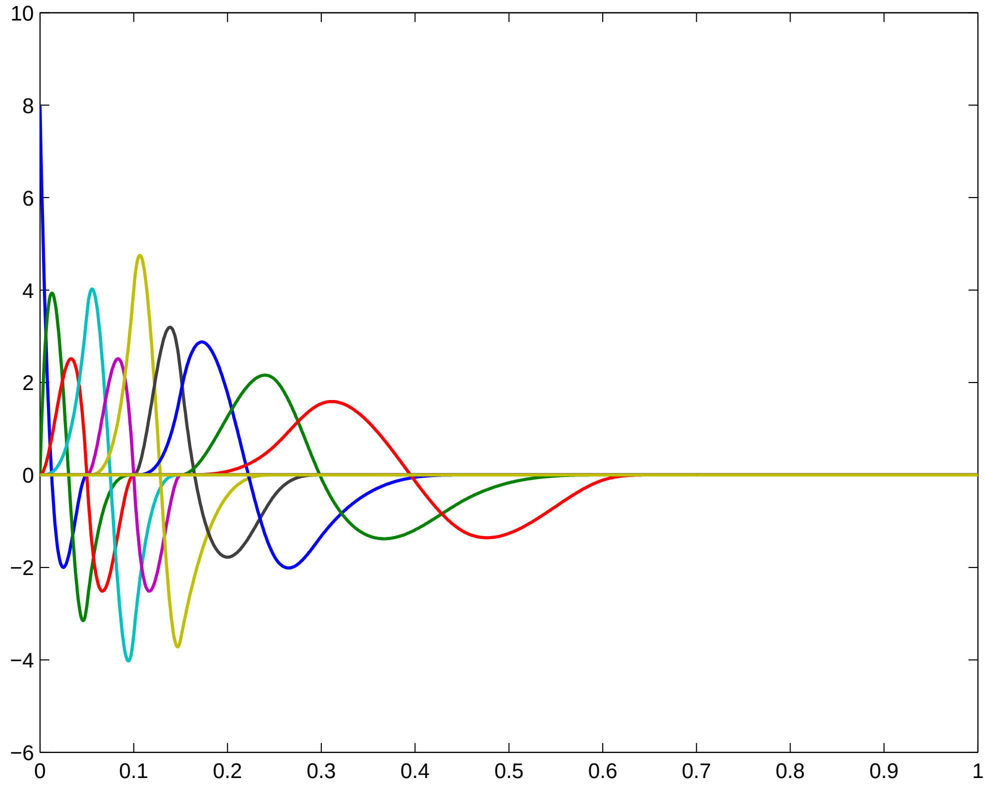

Figure 6.30: Family Ψ_1 from Example 6.32. Note the relation between the local character of the new inserted knots in $\mathbf{t}_2$ and the local character of the functions $\psi_{1,k}$.

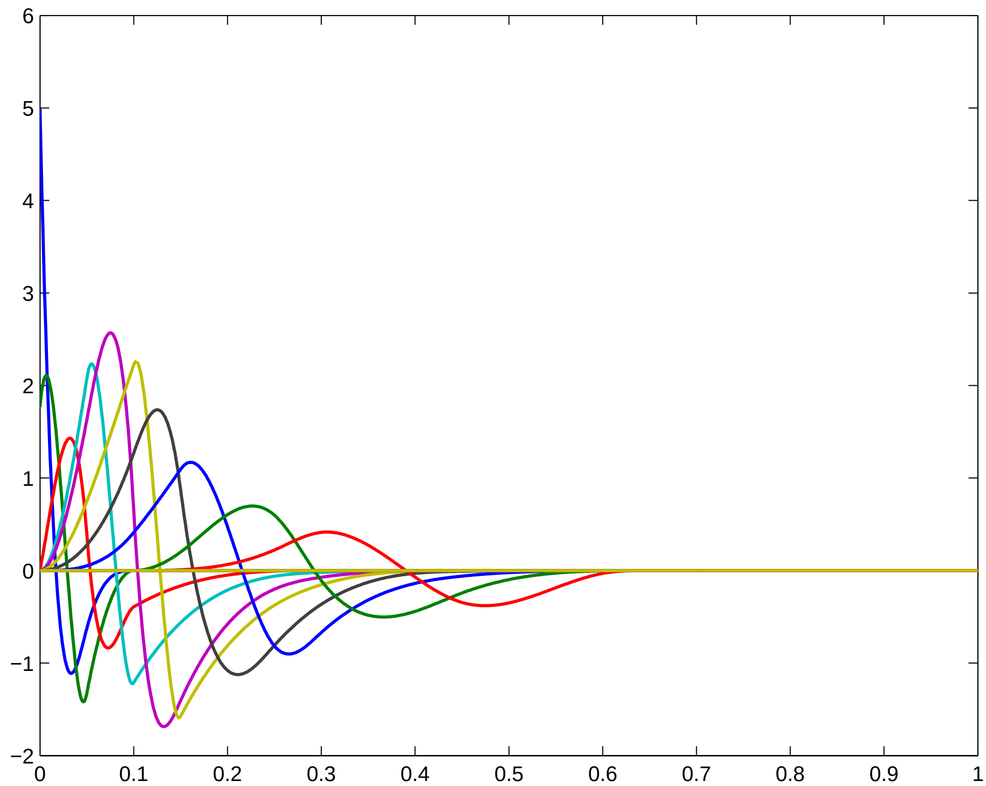

Figure 6.31: Family $\tilde{\Psi}_1$ from Example 6.32. Note the relation between the local character of the new inserted knots in $\mathbf{t}_2$ and the local character of the functions $\tilde{\psi}_{1,k}$.

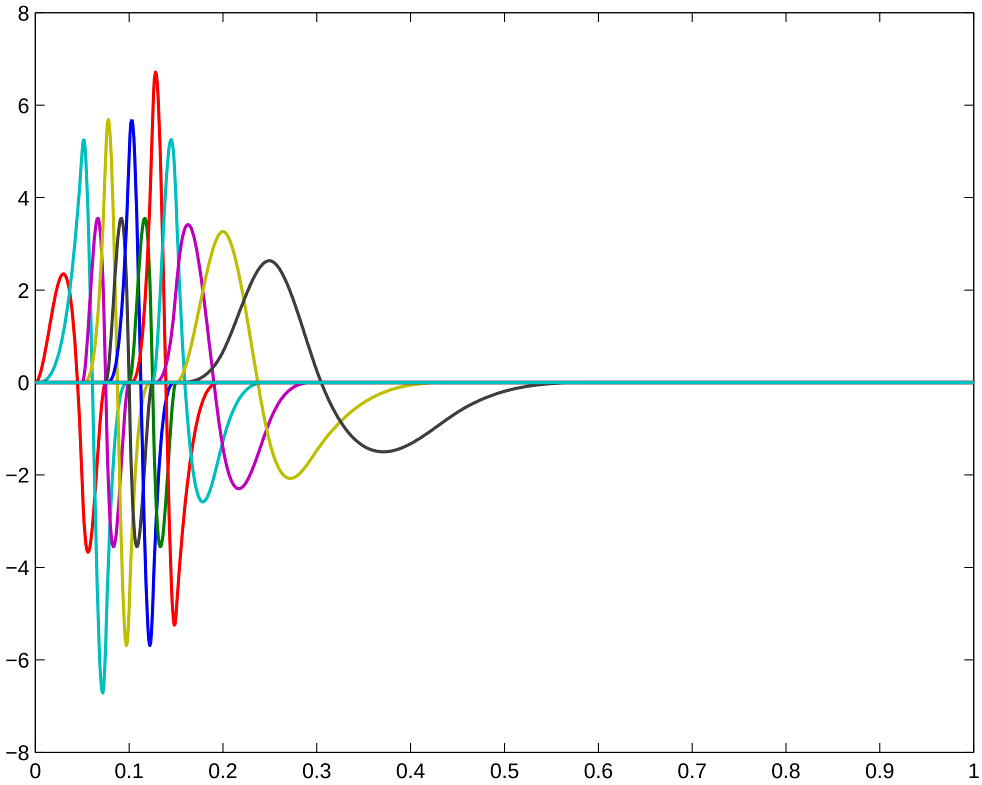

Figure 6.32: Family Ψ_2 from Example 6.32. Note the relation between the local character of the new inserted knots in $\mathbf{t}_3$ and the local character of the functions $\psi_{2,k}$.

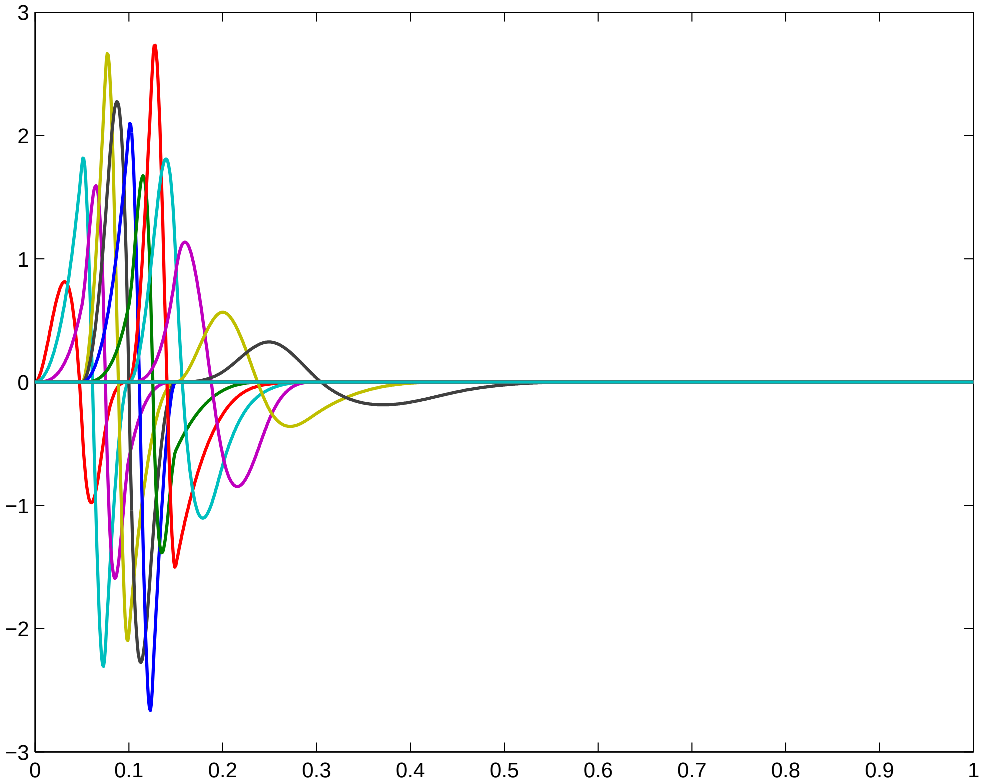

Figure 6.33: Family $\tilde{\Psi}_2$ from Example 6.32. Note the relation between the local character of the new inserted knots in $\mathbf{t}_3$ and the local character of the functions $\tilde{\psi}_{2,k}$.

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