Spaces of Lattices in Equal and Mixed Characteristics

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Dipl.-Ing. Martin Kreidl
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Vorsitz: Prof. Dr. Patrizio Neff (Universität Duisburg-Essen, Campus Essen)

Gutachter:
Prof. Dr. Ulrich Görtz (Universität Duisburg-Essen, Campus Essen)
Prof. Dr. Hélène Esnault (Universität Duisburg-Essen, Campus Essen)
Prof. Dr. Urs Hartl (Universität Münster)

Meiner Familie
To my family

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Introduction

Let k denote an arbitrary field and let $G = \mathrm{Sl}_n$ denote the special linear group over k. The affine Grassmannian for G is constructed as an algebro-geometric model of the quotient $G(k((z)))/G(k[[z]])$. This means that one considers on the category of k-algebras the functor

$$\mathcal{G} = \mathcal{G}_G : R \mapsto G(R((z)))/G(R[[z]])$$

(or rather the fpqc-sheaf associated with this functor) and obtains a description of $\mathcal{G}$ as an ind-scheme over k as follows. Let R be any k-algebra. Recall that a lattice $L \subset R((z))^n$ is a finitely generated projective $R[[z]]$-submodule of $R((z))^n$ which satisfies $L \otimes_{R[[z]]} R((z)) = R((z))^n$. Further, let N be any positive integer, and let $\mathcal{L}att_N^n(R)$ be the set of lattices L with the property that $z^N R[[z]]^n \subset L \subset z^{-N} R[[z]]^n$. Denote by $\mathcal{L}att_N^{n,0}(R) \subset \mathcal{L}att_N^n(R)$ the subset of special lattices – that is, lattices L with the additional property $\wedge^n L = R[[z]]$. In this situation Beauville and Laszlo [**BL94**] prove that $\mathcal{G}(R) = \cup_{N \in \mathbb{N}} \mathcal{L}att_N^{n,0}(R)$ and that the functor $\mathcal{L}att_N^{n,0}$ is represented by a closed subscheme of an ordinary Grassmannian (more precisely, the Grassmannian which parametrizes nN-dimensional k-linear subspaces in k^{2nN}). Hence the functor $\mathcal{G}$ is an ascending union of projective k-schemes, or, in other words, an ind-projective k-ind-scheme. This ind-scheme is the affine Grassmannian for $G = \mathrm{Sl}_n$. The affine Grassmannian, also for other linear algebraic groups than Sl_n, and its variants such as partial or full flag varieties, are well studied as natural objects within the geometric Langlands program (see e.g. Mirkovic-Vilonen [**MV00**], Frenkel [**Fre07**], and others) as well as for example in the theory of local models for certain Shimura varieties: The special fibers of the various local models of Shimura varieties constructed by Rapoport-Zink in [**RZ96**], and by Pappas-Rapoport in [**PR03**] and [**PR05**], are closed subvarieties of affine partial flag varieties. See also Görtz [**Gör01**].

However, from the point of view of number theory it is also natural to look at quotients of the form $G(L)/G(\mathcal{O})$, where G is a linear algebraic group G over some perfect field k of positive characteristic, and

$\mathcal{O} = \mathrm{W}(k)$ denotes the ring of Witt vectors over k and $L = \mathrm{W}(k)[1/p]$. Let us refer to this setting as the 'p-adic case' in the following, while by the 'function field case' we mean the situation discussed in the preceding paragraph.

One motivation for the search for an algebro-geometric structure on $G(L)/G(\mathcal{O})$ is the following. Assume that k is algebraically closed, and let $\mathbb{X}$ be a p-divisible group over k and consider the functor $\mathcal{M}$ defined in [**RZ96**], Def. 2.15. Loosely spoken, this functor parametrizes families, on locally nilpotent $\mathcal{O}$-schemes, of p-divisible groups together with a quasi-isogeny to $\mathbb{X}$ over the locus $\{p = 0\}$. Rapoport and Zink prove that $\mathcal{M}$ is representable by a formal scheme over $\mathcal{O}$ and that its special fiber is a scheme over k whose irreducible components are projective k-schemes. On the other hand, Dieudonné theory provides an anti-equivalence of categories between the category of p-divisible groups and the category of Dieudonné modules which are finitely generated and free as modules over $\mathcal{O}$. Via this anti-equivalence, the set $\mathcal{M}(k)$ is identified with a 'generalized affine Deligne-Lusztig set'

$$X_\mu(b) = \{g \in G(L)/G(\mathcal{O}) \mid g^{-1} b\sigma(g) \in G(\mathcal{O})p^\mu G(\mathcal{O})\},$$

where $b \in G(L)$, and μ is a dominant cocharacter of a maximal torus of G. (For a detailled discussion of generalized affine Deligne-Lusztig sets see e.g. Viehmann [**Vie08**].) In other words, the subset $X_\mu(b) \subset G(L)/G(\mathcal{O})$ carries the structure of a k-scheme, and it would be interesting to see whether this scheme-structure is induced by a similar structure on all of $G(L)/G(\mathcal{O})$.

In his paper [**Hab05**], Haboush attempts to endow the quotient sets $\mathrm{Sl}_n(L)/\mathrm{Sl}_n(\mathcal{O})$ with an ind-scheme structure over k analogous to the one discussed above for the function field case. However, the situation seems to be significantly more complicated in the p-adic case, and what Haboush does in [**Hab05**] seems to be at least problematic. One source of complication in the p-adic case is certainly the simple fact that $\mathrm{W}(R)$ (R any ring) does not carry a structure of R-module, which makes impossible the construction of an analogue of $\mathcal{L}att_N^{n,0}(R)$ inside an ordinary Grassmannian, as described above for the function field case. The natural strategy, pursued by Haboush, is to identify lattices in the p-adic situation with certain subvarieties ('lattice schemes') of the 'affine space' $\mathrm{W}(k)^n$, and parametrize these by a closed subscheme of a (multigraded) Hilbert scheme. However, it is a well-known and very natural phenomenon that certain fibers in flat families of schemes are non-reduced, even if the family as a whole is reduced. And indeed it turns out that Haboush's lattice schemes will in general carry infinitesimal structure, with the consequence that the desired bijection (lattices $\leftrightarrow$ lattice schemes) does not exist. There are simply too many lattice schemes with different infinitesimal structure giving rise to the same lattice.

Before we proceed with a detailled outline of the present work, let us mention that $\mathrm{Sl}_n(L)/\mathrm{Sl}_n(\mathcal{O})$ is the set of vertices of the Bruhat-Tits building $\mathcal{B}_n$ of $\mathrm{Sl}_n(L)$, and we are in fact looking for a geometric structure on this set of vertices. In [**Ber95**] Berkovich describes the construction of an equivariant closed embedding of $\mathcal{B}_n$ into the analytification of the $n-1$-dimensional Drinfeld half plane Ω_n. This induces on $\mathcal{B}_n$ the structure of an analytic space. However, the structure which is induced on the set of vertices of $\mathcal{B}_n$ by this construction is discrete, and this is certainly not what we are looking for. For details on Berkovich's construction see also Berkovich [**Ber90**] and Werner [**Wer04**].

Part 1. In Part 1 of the present work we study the phenomenon of infinitesimal structure on lattice schemes in a simplified setting, which is motivated by the Witt vector situation, but ultimately gives rise to objects which are very close to Demazure resolutions of Schubert varieties in the function field case. Much of the material presented here, and in particular the main result Theorem 0.1 below, has been published by the author in [**Kre10**].

Let k be a field of positive characteristic p. For every dominant cocharacter $\lambda \in \check{\mathrm{X}}_+(T)$ of the standard maximal torus $T \subset \mathrm{Sl}_n$ we construct a projective k-subvariety $\mathcal{D}(\lambda)$ of a multigraded Hilbert scheme. The variety $\mathcal{D}(\lambda)$ will be a parameter space for certain 'lattice schemes'. To the Schubert variety $\mathcal{S}(\lambda) \subset \mathcal{G}_{\mathrm{Sl}_n}$ we can associate a Demazure resolution $\pi(\lambda) : \Sigma(\mu_1, \ldots, \mu_N) \to \mathcal{S}(\lambda)$ (where the μ_i are suitably chosen minuscule dominant cocharacters) such that the following theorem holds.

THEOREM 0.1 (Kreidl, [**Kre10**]). *The k-variety $\mathcal{D}(\lambda)$ is an iterated bundle of ordinary Grassmannians, and there is a universal homeomorphism*

$$\sigma : \mathcal{D}(\lambda) \to \Sigma(\mu_1, \ldots, \mu_N).$$

Furthermore, let $\pi'(\lambda) : \mathcal{D}(\lambda) \to \Sigma(\mu_1, \ldots, \mu_N) \to \mathcal{S}(\lambda)$ denote the composition of σ with the Demazure resolution $\pi(\lambda)$ of $\mathcal{S}(\lambda)$. Then a lattice scheme given by a k-valued point in $\mathcal{D}(\lambda)$ is reduced if and only if it is mapped under $\pi'(\lambda)$ to the big cell of $\mathcal{S}(\lambda)$.

Hence the fiber of the Demazure resolution $\pi(\lambda)$ over a lattice $\mathcal{L}$ in the boundary of $\mathcal{S}(\lambda)$ is universally homeomorphic to a variety of non-trivial infinitesimal structures on $\mathcal{L}$.

Further, we illustrate this result by explicitly calculating the respective objects and morphisms occuring in Theorem 0.1 for the simplest non-trivial situation, given by $n = 2$ and $\lambda = (1, -1) \in \check{\mathrm{X}}_+(T)$. (Surprisingly, this example will reappear in Part 2.) In particular, we obtain

PROPOSITION 0.2 (Kreidl, [**Kre10**]). *In the situation $n = 2$ the variety $\mathcal{D}((1, -1))$ is a bundle of projective lines over $\mathbb{P}^1_k$. More precisely,*

$$\mathcal{D}((1, -1)) \simeq \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p)).$$

The boundary of the big cell is the divisor $\mathbb{P}^1_k \times_k \{\infty\}$.

All the above is treated in Chapter 3. To set the ground, we recall in Chapter 1 the well-known construction by Beauville and Laszlo of the affine Grassmannian for Sl_n in the function field case, while in Chapter 2 we explain two well-known interpretations of Demazure resolutions, on the one hand as quotients of loop groups, and on the other hand as varieties of lattice chains. As a sort of aside, we explain in Chapter 1 the moduli interpretation of the affine Grassmannian in terms of vector bundles on curves, and we give a seemingly new, very elementary proof of the correspondence (R-valued points of $\mathcal{G}_{\mathrm{Sl}_n}$) $\leftrightarrow$ (equivalence classes of pairs (E, ρ), where E is a vector bundle of rank n of trivial determinant on the fixed projective curve X/k, and ρ is a trivialization outside the fixed closed point $p \in X$). In particular, our proof does not refer to the 'descent lemma', proven by Beauville and Laszlo in [**BL95**].

Part 2. Here we return to the original question for a p-adic version of the affine Grassmannian. Again we consider a field k of positive characteristic, and assume furthermore that k is perfect.

Haboush introduces in [**Hab05**] the notion of 'localized Greenberg realization' in the category of topological k-schemes. It seems that this is not the 'correct' notion as we will argue by example in Chapter 4. Further, we discuss Haboush's construction of spaces of (p-adic) lattices, which are meant to play the role of Schubert varieties. However, it turns out that Haboush's constructions parametrize lattice schemes which will in general carry infinitesimal structure. In fact, pursuing his construction for $p > 2$, $n = 2$ of the parameter space of lattice schemes of 'height at most 1' we obtain precisely the variety $\mathcal{D}((1, -1))$ described in Proposition 0.2.

In the subsequent Chapter 5 we explain an alternative construction of 'localized Greenberg realizations' in the category of ind-schemes, and in particular the p-adic loop groups $\mathrm{L}_p \mathrm{Sl}_n$ and $\mathrm{L}_p^+ \mathrm{Sl}_n$, and we introduce the p-adic affine Grassmannian for Sl_n as the fpqc-quotient $\mathcal{G}rass_p = \mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$. Moreover, we prove the following

THEOREM 0.3. *For every dominant cocharacter λ of T there is a projective k-variety D_λ together with a natural action of the positive p-adic loop group $\mathrm{L}_p^+ \mathrm{Sl}_n$, and an $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant morphism $\pi_\lambda : D_\lambda \to \mathcal{G}rass_p$ with the following properties: Let $C_\lambda \subset D_\lambda$ be the open orbit, and let $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$ be the Schubert cell corresponding to λ. Then π_λ induces an isomorphism of functors $C_\lambda \simeq \mathcal{C}_\lambda$, which proves that the Schubert cells are quasi-projective k-schemes. Moreover, the image under $\pi_\lambda(k)$ of $D_\lambda(k)$ is precisely the union of the sets of k-valued points of the Schubert cells indexed by λ' with $\lambda' \leq \lambda$ for the Bruhat-order.*

Unfortunately, the morphisms π_λ are not injective on the level of k-valued points as one might hope. In the case $p > 2$, $n = 2$, $\lambda = (1, -1)$, the variety D_λ is equal to $\mathcal{D}(\lambda)$ from Part 1. This suggests that the D_λ should perhaps better be viewed as an analogue of Demazure resolutions in the p-adic setting.

We also deal with the question whether we can describe not only the k-valued points of the p-adic Grassmannian, but also its R-valued points for more general k-algebras R. In the case of a perfect k-algebra R we introduce a notion of lattice in $\mathrm{W}(R)[1/p]^n$ analogous to the function field case, and we prove that the property of being a lattice is local on the base R - just as in the function field case. We prove different characterizations of lattices and obtain

THEOREM 0.4. *If R is a perfect k-algebra, then the set of R-valued points of $\mathcal{G}rass_p$ is equal to the set of lattices $L \subset \mathrm{W}(R)[1/p]^n$ with $\wedge^n L = \mathrm{W}(R)$.*

Finally, in the Appendix we collect a couple of easy resp. standard results on fpqc-sheaves and fpqc-sheafification which are used throughout the paper. Moreover we discuss very briefly the set-theoretical problems which occur when talking about fpqc-sheafifications, and which are often ignored. Using results of Waterhouse, [**Wat75**], we check that such complications do not occur in our construction of the p-adic affine Grassmannian as an fpqc-sheaf quotient of p-adic loop groups.

Part 1

Equal Characteristic

CHAPTER 1

The Affine Grassmannian

1.1. Spaces and Ind-Schemes

Throughout this section, k denotes a field.

In this work we make extensive use of the language of ind-schemes. Since there are different definitions of the term 'ind-scheme' scattered through the literature let us begin by fixing terminology and giving a brief discussion of our notion of ind-scheme.

Definition 1.1. *Let S be a scheme. An* S-space *is a sheaf on the fpqc-site over S. An* ind-scheme over S *(or* S-ind-scheme*, or simply* ind-scheme*) is the colimit in the category of S-spaces of a direct system of quasi-compact S-schemes. Morphisms of ind-schemes are morphisms of functors.*

If an S-ind-scheme X has the form $X = \varinjlim_{i \in I} X_i$ with all the X_i quasi-compact, then we say that X is represented *by the direct system $(X_i)_{i \in I}$. By abuse of language we will also simply speak of the ind-scheme (X_i). Moreover, an ind-scheme $(X_i)_{i \in I}$ is called* ind-affine (resp. ind-projective,...)*, if all the X_i can be chosen to be affine (resp. projective,...).*

If X is an ind-scheme, then by a sub-ind-scheme $Y \subset X$ *we mean a subfunctor of X which is itself an ind-scheme. A sub-ind-scheme $Y \subset (X_i)_i$ is called* ind-closed*, if it is represented by a system of closed subschemes $Y_i \subset X_i$.*

We will always assume the directed index set I to be countable. In particular, there always exists a cofinal subset $I' \subset I$ which can be identified with the natural numbers. We denote the category of S-spaces by (S-Sp) (the morphisms between two S-spaces being natural transformations of functors), and by (ind-Sch/S) we denote its full subcategory whose objects are the S-ind-schemes. In other words, we have the following fully faithful functors:

$$(\mathrm{Sch}/S) \hookrightarrow (\text{ind-Sch}/S) \hookrightarrow (S\text{-Sp})$$

Remark 1.2. *(1) Our definitions of S-space and S-ind-scheme coincide with those given by Beauville and Laszlo in* [**BL94**] *in the case* *where S=Spec k. There are different definitions for these terms, e.g. by Drinfel'd in* [**Dri03**].

(2) The existence of colimits in the category of S-spaces of a of direct system of S-schemes needs a little justification which is given in the appendix (Corollary A.7). In fact, sheafification of an arbitrary presheaf for the fpqc-topology poses set-theoretical problems, for whose discussion we refer to Waterhouse, [**Wat75**]*, and again to the appendix of this thesis.*

Let us collect a few easy facts about ind-schemes.

LEMMA 1.3. *If T is a quasi-compact scheme and X is an ind-scheme which is represented by a direct system (X_i), then*

$$Mor(T,X) = \varinjlim Mor(T,X_i).$$

PROOF. As we prove in the appendix (Corollary A.7), the ind-scheme X is just the Zariski-sheafification of the presheaf-direct limit $\varinjlim X_i$. Since every Zariski-covering of a quasi-compact T has a finite subcovering, the lemma follows. □

Let X and Y be ind-schemes which are represented by direct systems (X_i) and (Y_i), respectively. Any morphism of direct systems $(X_i) \to (Y_i)$ (i.e. a system of compatible maps $f_i : X_i \to Y_{i'}$) induces a morphism $f : X \to Y$. In this case we say that f is represented by the system (f_i). From the above lemma the following converse is easy to deduce.

LEMMA 1.4. *Let X and Y be ind-schemes which are represented by direct systems (X_i) and (Y_i), respectively. Then every morphism $X \to Y$ is represented by a compatible system of maps $f_i : X_i \to Y_{i'}$.* □

Note that this lemma holds precisely because quasi-compactness of all the X_i is built in the definition of ind-scheme resp. representing direct systems. Moreover, as remarked above, we can always assume that all our index sets are equal to the set of natural numbers, and that compatible systems of maps are of the form $f_i : X_i \to Y_i$ (i.e. preserve the index).

LEMMA 1.5 (Products). *Let X, Y, Z be ind-schemes which are represented by direct systems $(X_i), (Y_i), (Z_i)$, respectively, and let $X \to Z$ and $Y \to Z$ be morphisms represented by compatible systems of maps $X_i \to Z_i$ and $Y_i \to Z_i$. Then the fiber product (in the category of k-spaces) $X \times_Z Y$ is an ind-scheme and is represented by the direct system $(X_i \times_{Z_i} Y_i)$.* □

We will make a further assumption to simplify our presentation. *Throughout this work, all test-schemes which occur will be assumed to be quasi-compact.* In other words, all functors are considered to be functors on categories of quasi-compact schemes. This simplification is justified by the fact that an S-space is determined by its values on quasi-compact (or even affine) S-schemes. Thus we will not further distinguish between the ind-scheme represented by a direct system (X_i) and the presheaf-direct limit $\varinjlim X_i$. This is also the point of view taken by Beauville and Laszlo in [**BL94**].

1.2. The Affine Grassmannian after Beauville and Laszlo

We summarize in this section the construction of the affine Grassmannian for the group Sl_n. Everything discussed here is contained in the paper [**BL94**] by Beauville and Laszlo, however, at some points, as for example in the discussion of the several equivalent definitions of 'lattice', we try to present more details.

Let G be a linear algebraic group over k and consider the following functors on the category of k-algebras:

$$\begin{aligned} \mathrm{L}^+ G :\quad & R \mapsto G(R[[z]]) \quad \text{(the 'positive loop group')},\\ \mathrm{L}\, G :\quad & R \mapsto G(R((z))) \quad \text{(the 'loop group')}. \end{aligned}$$

It is easy to see that the positive loop group is representable by an (infinite dimensional) affine k-scheme, while the loop group is representable by an ind-affine k-ind-scheme. Indeed, the functor which associates to every R the subset $\mathrm{L}\, G^{\geq -N} G(R) \subset G(R((z)))$ of matrices whose entries have pole order at most N, is represented by an (infinite dimensional) affine scheme, $\mathrm{L}^{\geq 0} G$ being $\mathrm{L}^+ G$, of course. The natural inclusions of functors $\mathrm{L}^{\geq -N} G \subset \mathrm{L}^{\geq -N-1} G$ then determine closed immersions of the corresponding affine schemes. The inductive system so obtained is the k-ind-scheme $\mathrm{L}\, G$. Let us note here, that the scheme $\mathrm{L}^+ G$ is an instance of a general construction, the so-called *Greenberg realization* of the $k[[z]]$-group $G \times_k \operatorname{Spec} k[[z]]$. For a detailled explanation of this notion we refer to Chapter 5 in this work, and to the original work by Greenberg [**Gre61**]. On the other hand, the ind-scheme representing the loop group $\mathrm{L}\, G$ will be a special case of the construction of 'localized Greenberg realization', to be developed in Chapter 5, too. However, we do not need these notions here.

Definition 1.6. *The quotient* $\mathrm{L}\, G/\, \mathrm{L}^+ G$ *in the category of* k*-spaces is called the* affine Grassmannian *for* G*. In the sequel we write* $\mathcal{G}_G := \mathrm{L}\, G/\, \mathrm{L}^+ G$ *for the quotient-*k*-space.*

Remark 1.7. *In general there arise set-theoretical problems when one wants to talk about sheafifications for the fpqc-topology (as we do in Definition 1.6). This means, sheafifications of general functors exist only after restricting to a fixed universe, but the sheafification so obtained will depend on this choice. For a discussion of these questions we refer to the Appendix A. However, in our particular situation such problems do not occur. In the cases* $G = \mathrm{Gl}_n$ *and* $G = \mathrm{Sl}_n$ *Theorem 1.9 will give us an explicit description of the desired sheafification of the presheaf-quotient* $\mathrm{L}\,G/\,\mathrm{L}^+\,G$ *in terms of lattices.*

As we have just pointed out, there is a close relationship between the affine Grassmannian and lattices in $R((z))^n$, which we are going to study now. The following definition and theorem are basically due to Beauville and Laszlo, [**BL94**]; for a discussion similar to ours see also Görtz, [**Gör10**].

Definition 1.8. *Let* R *be a* k*-algebra. A* lattice $L \subset R((z))^n$ *is a finitely generated projective* $R[[z]]$*-submodule such that* $L \otimes_{R[[z]]} R((z)) = R((z))^n$. *A lattice* L *is called* special*, if its determinant is trivial, i.e.* $\wedge^n L = R[[z]]$.

Theorem 1.9. *For an* $R[[z]]$*-submodule* $L \subset R((z))^n$ *the following are equivalent:*

(1) *The submodule* L *is a lattice.*
(2) *Zariski-locally on* R*,* L *is a free* $R[[z]]$*-submodule of rank* n *(i.e. there exist* $f_1, \ldots, f_r \in R$ *such that* $(f_1, \ldots, f_r) = R$ *and for all* i*,* $L \otimes_{R[[z]]} R_{f_i}[[z]]$ *is free of rank* n *and* $L \otimes_{R[[z]]} R_{f_i}((z)) = R_{f_i}((z))^n$.
(3) *fpqc-locally on* R*,* L *is a free* $R[[z]]$*-submodule of rank* n *(i.e. there exists a faithfully flat ring homomorphisms* $R \to S$ *such that* $L \otimes_{R[[z]]} S[[z]]$ *is free of rank* n *and* $L \otimes_{R[[z]]} S((z)) = S((z))^n$.
(4) *There exists a positive integer* N *such that* $z^N R[[z]]^n \subset L \subset z^{-N} R[[z]]^n$ *and* $z^{-N} R[[z]]^n / L$ *is a projective* R*-module.*

Proof. Let us check (1) $\Rightarrow$ (2). Note that L is by definition finitely generated and projective over $R[[z]]$, which implies that it is even finitely presented and further that it is Zariski-locally free. In other words, there exist $g_1, \ldots, g_r \in R[[z]]$ which generate the unit ideal and such that $L \otimes_{R[[z]]} R[[z]]_{g_i}$ is free for every i. But if we denote by f_i the constant coefficient of g_i, then $R[[z]]_{g_i} \subset R_{f_i}[[z]]$ and the f_i have the properties required in (2).

The implication (2) $\Rightarrow$ (3) is trivial. Let us check (3) $\Rightarrow$ (4). We write $F_R = R[[z]]^n$ for any k-algebra R. If $R \to S$ is a faithfully flat homomorphism of k-algebras such that $L \otimes_{R[[z]]} S[[z]]$ is free of rank n and $L \otimes_{R[[z]]} S((z)) = S((z))^n$, then there exists a positive integer N such that

$$z^N F_S \subset L \otimes_{R[[z]]} S[[z]] \subset z^{-N} F_S.$$

This implies that the homomorphism of modules

$$L \to S((z))^n / z^{-N} F_S = \varinjlim_i z^{-N-i} F_S / F_S$$

is zero. On the other hand, this morphism is obtained as the composition

$$L \to R((z))^n / z^{-N} F_R \hookrightarrow S((z))^n / z^{-N} F_S,$$

the right hand map being injective since $R \to S$ is flat. This implies $L \subset z^{-N} F_R$. Similarly, the homomorphism

$$z^N F_R / z^{N+i} F_R \to z^{-N} F_R / (L + z^{N+i} F_R)$$

is zero after base change from R to S, and hence is itself zero, which proves that $z^N F_R \subset L$ by passage to the limit over i. This proves the first part of (4). To check the second part, note that clearly $(L/z^{2N}L) \otimes_{R[[z]]} S[[z]] = (L/z^{2N}L) \otimes_R S$ is projective (even free) over S and thus, by faithful flatness of $R \to S$, $L/z^{2N}L$ is projective over R. We consider the short exact sequence

$$0 \to F_R / z^N L \hookrightarrow z^{-N} L / z^N L \twoheadrightarrow z^{-N} L / F_R \to 0, \tag{1.2.1}$$

and observe that this sequence is split by the retraction $z^{-N}L/z^N L \to F_R/z^N L$ which is induced by $z^{-2N} F_R \to F_R$. This shows that $z^{-N} F_R / L$ is a direct summand of the projective R-module $z^{-N}L/z^N L$, i.e. it is itself R-projective.

In order to check (4) $\Rightarrow$ (1), we apply a similar argument as just before: Clearly, a module L as in (4) is finitely generated over $R[[z]]$ and satisfies $L \otimes_{R[[z]]} R((z)) = R((z))^n$. Let us consider the short exact sequence

$$0 \to L / z^N F_R \hookrightarrow z^{-N} F_R / z^N F_R \twoheadrightarrow z^{-N} F_R / L \to 0, \tag{1.2.2}$$

and observe that by R-projectivity of the right hand module this sequence splits. In particular, $L/z^N F_R$ is projective over R. This implies that in the sequence (1.2.1) the two outer modules are finitely presented and projective over R, whence the same is true for $L/z^{2N}L$. It now follows from Lemma 1.10 that L is projective as an $R[[z]]$-module. This finishes the proof. $\square$

LEMMA 1.10. *Let $(A_i)_{i \in \mathbb{N}}$ be an inverse system of rings, with all the connecting maps $A_i \to A_{i-1}$ surjective, and let $\hat{A}$ be its limit. Let M be a finitely generated $\hat{A}$-module, write $M_i := M \otimes_{\hat{A}} A_i$ and assume* *that $M = \varprojlim M_i$. If all the M_i are projective A_i-modules, then M is a projective $\hat{A}$-module.*

Proof. Consider a surjective $\hat{A}$-homomorphism $\pi : \hat{A}^n \twoheadrightarrow M$. We shall show that it splits by constructing a system of compatible splittings of the induced maps $\pi_i : A_i^n \twoheadrightarrow M_i$.

Of course, the maps π_i split, since the M_i are projective by assumption. Our strategy will be to construct *compatible* splittings by induction on i. So assume we have a compatible system of splittings $s_i : M_i \to A_i^n$ up to a certain index i. By tensoring the sequence $0 \to \ker \to A_{i+1} \to A_i \to 0$ with $\pi_{i+1} : (A_{i+1})^n \to M_{i+1}$ we obtain the following diagram of A_{i+1}-modules, with exact rows (L and K being the kernels by definition) and all vertical maps surjective:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & A_{i+1}^n \arrow[r] \arrow[d, "\pi_{i+1}"] & A_i^n \arrow[r] \arrow[d, "\pi_i"] & 0 \\ 0 \arrow[r] & K \arrow[r] & M_{i+1} \arrow[r] & M_i \arrow[r] & 0. \end{tikzcd}$$

Note that the map $L \to K$ is surjective, for the following reason: Projectivity of M_{i+1} yields a decomposition $A_{i+1}^n = M_{i+1} \oplus C_{i+1}$, C_{i+1} being the kernel of π_{i+1}. By tensoring with A_i we obtain an analogous decomposition $A_i^n = M_i \oplus C_i$, with C_{i+1} surjecting onto $C_i = \ker(\pi_i)$. Now the 5-lemma shows that indeed L surjects onto K.

By induction, for the map $\pi_i : A_i^n \to M_i$ we already have a splitting s_i. By A_{i+1}-projectivity of M_{i+1} we may lift the composition $M_{i+1} \to M_i \to A_i^n$ in order to obtain a map $\widetilde{s_{i+1}} : M_{i+1} \to A_{i+1}$, rendering the right square in the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & A_{i+1}^n \arrow[r] & A_i^n \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] & M_{i+1} \arrow[r] \arrow[u, "\widetilde{s_{i+1}}"] & M_i \arrow[r] \arrow[u, "s_i"] & 0. \end{tikzcd}$$

In general, $\widetilde{s_{i+1}}$ will not be a splitting of π_{i+1}, but it can be properly adjusted: a diagram chase shows that the difference $\delta_i := (\pi_{i+1} \circ \widetilde{s_{i+1}} - 1)$ is a map $M_{i+1} \to \ker(M_{i+1} \to M_i) = K$. Again by projectivity, we can lift δ_i to $\Delta_i : M_{i+1} \to L \to A_{i+1}^n$ (as remarked above, $L \to K$ is surjective). If we set $s_{i+1} := \widetilde{s_{i+1}} - \Delta_i$, we get indeed a splitting of π_{i+1} which still forms a commutative square

$$\begin{tikzcd} A_{i+1}^n \arrow[r] & A_i^n \\ M_{i+1} \arrow[r] \arrow[u, "s_{i+1}"] & M_i. \arrow[u, "s_i"] \end{tikzcd}$$

Inductively applying this construction, we end up with a projective system of splittings, the limit of which is the desired splitting of π. Thus we are done. $\square$

We may interpret the equivalent characterizations in Theorem 1.9 as follows. Let $\mathcal{L}att^n$ (resp. $\mathcal{L}att^{n,0}$) be the functor which associates to every k-algebra R the set of (special) lattices in $R((z))^n$. Then from Theorem 1.9 (4) it follows immediately that $\mathcal{L}att^n$ (resp. $\mathcal{L}att^{n,0}$) is a sheaf for the fpqc-topology. From Theorem 1.9 (2) (resp. (3)), we deduce then that $\mathcal{L}att^n$ (resp. $\mathcal{L}att^{n,0}$) is the Zariski-sheafification (resp. fpqc-sheafification) of the functor which associates to R the set of *free* (special) lattices in $R((z))^n$. (Here we invoke the characterization of sheafifications given in Proposition A.2 in the appendix). This latter functor is nothing but the presheaf-quotient $\mathrm{L}\,\mathrm{Gl}_n \,/\, \mathrm{L}^+\,\mathrm{Gl}_n$ (resp. $\mathrm{L}\,\mathrm{Sl}_n \,/\, \mathrm{L}^+\,\mathrm{Sl}_n$). Thus we obtain

COROLLARY 1.11. *The action of* $\mathrm{L}\,\mathrm{Gl}_n$ *on* $\mathcal{L}att^n$ *induces isomorphisms*

$$\mathcal{G}_{\mathrm{Gl}_n} \simeq \mathcal{L}att^n; \quad \mathcal{G}_{\mathrm{Sl}_n} \simeq \mathcal{L}att^{n,0}.$$

In the following we consider the case $G = \mathrm{Sl}_n$, and we abbreviate $\mathcal{G} = \mathcal{G}_{\mathrm{Sl}_n}$. Let $T \subset B \subset G$ be the standard maximal torus (of diagonal matrices) and the standard Borel subgroup (of upper triangular matrices) of Sl_n. Moreover, denote by $\check{\mathrm{X}}(T)$ the group of cocharacters of T, and by $\check{\mathrm{X}}_+(T)$ the subset of dominant cocharacters (with respect to the fixed Borel B). Further, we identify $\check{\mathrm{X}}(T)$ with $\mathbb{Z}^n$ and hence obtain the injection

$$\check{\mathrm{X}}(T) \to \mathrm{L}\,G; \quad \lambda \mapsto z^\lambda := \mathrm{diag}(z^{\lambda_1}, \ldots, z^{\lambda_n}). \tag{1.2.3}$$

Thus we may regard the group of cocharacters of T as a subset of $\mathrm{L}\,G(R)$ for any R.

On $\mathcal{G}$ we have a natural action of $\mathrm{L}^+\,G$ by multiplication on the left. On the level of k-valued points, this action induces a well-known double coset decomposition (the *Cartan decomposition* for $\mathrm{L}\,G(k)$, namely

$$\mathrm{L}\,G(k) = \cup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathrm{L}^+\,G(k) z^\lambda\, \mathrm{L}^+\,G(k). \tag{1.2.4}$$

This shows that the $\mathrm{L}^+\,G(k)$-orbits in $\mathcal{G}(k)$ are parametrized by the dominant cocharacters of T. These orbits are called the *Schubert cells* of $\mathcal{G}$, and denoted in the sequel by $\mathcal{C}(\lambda)$ for $\lambda \in \check{\mathrm{X}}_+(T)$.

THEOREM 1.12 (Beauville, Laszlo, [**BL94**]). *The affine Grassmannian* $\mathcal{G}$ *is isomorphic to the functor, which associates to every* k*-algebra* R *the set of special lattices in* $R((z))^n$*. Moreover, it is representable by* *an inductive limit of closed subschemes of usual Grassmannians, i.e. it is an ind-projective k-ind-scheme.*

Sketch of proof. Details can be found in [**BL94**]. The idea of the proof is to write $\mathcal{G} = \cup_{N\in\mathbb{N}}\mathcal{G}^{(N)}$, where

$$\mathcal{G}^{(N)}(R) = \{ \text{ special lattices } \mathcal{L} \text{ with } z^N R[[z]]^n \subset \mathcal{L} \subset z^{-N}R[[z]]^n \}.$$

One can then show that for any N the inclusion

$$\mathcal{G}^{(N)}(R) \hookrightarrow \mathrm{Grass}(nN, z^{-N}k[[z]]^n/z^N k[[z]]^n)(R); \quad \mathcal{L} \mapsto \mathcal{L}/z^N R[[z]]^n$$

defines an isomorphism of functors from $\mathcal{G}^{(N)}$ to a closed subscheme of the Grassmannian $\mathrm{Grass}(nN, z^{-N}k[[z]]^n/z^N k[[z]]^n)$ of nN-dimensional k-linear subspaces of $z^{-N}k[[z]]^n/z^N k[[z]]^n$. $\square$

It is easy to see that $\mathrm{L}^+ G$ acts algebraically on every $\mathcal{G}^{(N)}$, whence the Schubert cells are quasi-projective k-schemes, each lying in a suitable $\mathcal{G}^{(N)}$. For any $\lambda \in \check{\mathrm{X}}_+(T)$, the closure of the Schubert cell $\mathcal{C}(\lambda) \subset \mathcal{G}$, denoted by $\mathcal{S}(\lambda)$ in the sequel, is called a *Schubert variety.*

Remark 1.13. *An analogous result of course holds in the case $G = \mathrm{Gl}_n$ (see* [**BL94**]*): The affine Grassmannian $\mathcal{G}_{\mathrm{Gl}_n}$ is an ind-scheme over k which is representable by an inductive limit of closed subschemes of usual Grassmannians.*

1.3. Vector Bundles on Projective Curves

There is a well-known correspondence between points of the affine Grassmannian for G and vector bundles on a projective curve together with certain trivializations (where the precise meaning of 'certain' dependes on the choice of G). In this section we will mainly focus on the case $G = \mathrm{Gl}_n$, and give a brief account on the case $G = \mathrm{Sl}_n$ at the end of the section. Let us recall the above mentioned correspondence, as Beauville and Laszlo describe it in [**BL94**].

Let X be a smooth projective curve over k, $p \in X$ be a closed point, and choose a uniformizer $z \in \mathcal{O}_{X,p}$. We fix these data for the rest of this section. For every k-algebra R we set

$$\begin{aligned} &X_R := X \otimes_{\operatorname{Spec} k} \operatorname{Spec} R, \quad X_R^* := \operatorname{Spec}(\mathcal{O}_X(X - \{p\}) \otimes_k R), \\ &D_R := \operatorname{Spec} R[[z]], \quad D_R^* := R((z)). \end{aligned} \tag{1.3.1}$$

These data determine a cartesian diagram of schemes

(1.3.2)
$$\begin{tikzcd} D_R^* \arrow[r, "\psi"] \arrow[d, "i"'] & X_R^* \arrow[d, "j"] \\ D_R \arrow[r, "f"] & X_R. \end{tikzcd}$$

Beauville and Laszlo prove the following

Proposition 1.14 ([**BL94**], Proposition 1.4). *The functor*

$$\mathrm{L}\,\mathrm{Gl}_n : R \mapsto \mathrm{Gl}_n(R((z)))$$

on the category of k-algebras is isomorphic to the functor which associates to R the set of isomorphism classes of triples (E, ρ, σ), where E is a vector bundle of rank n over X_R, and ρ and σ are trivializations of E over X_R^* and D_R, respectively.

As a consequence they obtain

Proposition 1.15 ([**BL94**], Proposition 2.1 and Remark 2.2). *The affine Grassmannian for Gl_n, by definition the fpqc-sheafification of the functor $R \mapsto \mathrm{Gl}_n(R((z)))/\,\mathrm{Gl}_n(R[[z]])$, is isomorphic to the functor which associates to R the set of isomorphism classes of pairs (E, ρ), where E is a vector bundle of rank n over X_R, and ρ is a trivialization of E over X_R^*.*

The interesting part in the proof of Proposition 1.14 is to see why the data of trivial vector bundles of rank n on D_R and X_R^*, respectively, together with a transition function over X_R^*, determine a vector bundle on X_R. This is not a classical descent situation, since if R is not Noetherian, D_R is in general not flat over X_R. In [**BL95**] Beauville and Laszlo prove that descent holds nonetheless.

In the present section we present an alternative proof of Proposition 1.14 using the following strategy. We define the subring $A_R \subset R[[z]]$ as a certain localization of $\mathcal{O}_{X,p} \otimes_k R$, which depends functorially on R and determines a flat neighborhood of the locus $z = 0$ in X_R. Let us write $\Delta_R = \operatorname{Spec} A_R$ and $\Delta_R^* = \operatorname{Spec} A_R[1/z]$. Then $\Delta_R \coprod X_R^* \to X_R$ is an fppf-covering, and if we could replace D_R by Δ_R and D_R^* by Δ_R^* in the formulation of Proposition 1.14, then this proposition would immediately follow by faithfully flat descent. Indeed, we will show below how to arrive at this situation using a simple approximation argument. Moreover, the concrete situation will turn out to be not only fppf-local, but even Zariski-local, so that descent of vector bundles holds trivially.

1.3.1. Vector bundles with trivializations. The choice of a uniformizer $z \in \mathcal{O}_{X,p}$ determines an inclusion $(R \otimes_k \mathcal{O}_{X,p}) \subset R[[z]]$, $R[[z]]$ being the completion with respect to the z-adic valuation. For each $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$ we define $S_{R,f} := (R \otimes_k \mathcal{O}_{X,p})_f \subset R[[z]]$. The union of all these rings, for varying f, will be denoted A_R. Writing $\Delta_R := \operatorname{Spec} A_R$ and $\Delta_R^* := \operatorname{Spec} A_R[1/z]$ we have a cartesian diagram

$$\begin{tikzcd} \Delta_R^* \arrow[r, "\psi"] \arrow[d, "\iota"] & X_R^* \arrow[d, "j"] \\ \Delta_R \arrow[r, "\varphi"] & X_R. \end{tikzcd}$$

Moreover we set $U_{R,f} := \operatorname{Spec} S_{R,f}$.

LEMMA 1.16. *The morphism $D_R \coprod X_R^* \to X_R$ is surjective. Thus $\Delta_R \coprod X_R^* \to X_R$ is an fppf-, and $U_{R,f} \coprod X_R^* \to X_R$ is a Zariski-covering for each $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$.*

PROOF. Let P be a point of X_R and let $A = (\mathcal{O}_X \otimes R)_P$ be the local ring at P. Either z is invertible in A – then $P \in X_R^*$ – or z is in the maximal ideal $\mathfrak{p} \subset A$. In the latter case we consider $can : A \to \hat{A} = \varprojlim(A/z^N)$ and the ideal $\hat{\mathfrak{p}} = \varprojlim(\mathfrak{p}/z^N)$. Passing to the inverse limit over the short exact sequences

$$0 \to \mathfrak{p}/(z^N) \to A/(z^N) \to A/\mathfrak{p} \to 0$$

we obtain $can^{-1}(\hat{\mathfrak{p}}) = \mathfrak{p}$, and the commutative square

$$\begin{tikzcd} \operatorname{Spec} \hat{A} \arrow[r] \arrow[d] & \operatorname{Spec} R[[z]] = D_R \arrow[d] \\ \operatorname{Spec} A \arrow[r] & X_R. \end{tikzcd}$$

shows that $\hat{\mathfrak{p}} \cap R[[z]] \subset R[[z]]$ is a preimage of P in D_R. $\square$

Let T be the functor on the category of k-algebras, which associates to a k-algebra R the set of isomorphisms classes of triples (E, ρ, σ), where E is a vector bundle of rank n on X_R, and

$$\rho : \mathcal{O}^n_{X_R^*} \xrightarrow{\simeq} E_{|X_R^*},$$
$$\sigma : \mathcal{O}^n_{\Delta_R} \xrightarrow{\simeq} E_{|\Delta_R}$$

are trivializations. To each isomorphism class $[(E, \rho, \sigma)] \in T(R)$ we may assign the respective 'transition matrix over Δ_R^*'. This is independent of the actual representative of $[(E, \rho, \sigma)]$ and hence determines a morphism of functors

$$\Phi(R) : T(R) \to \mathrm{Gl}_n(A_R[1/z]); \quad (E, \rho, \sigma) \mapsto \Gamma(X_R, (\rho|_{\Delta_R^*}) \circ (\sigma^{-1}|_{\Delta_R^*})).$$

PROPOSITION 1.17. *The morphism $\Phi(R)$ defined above is an isomorphism of functors.*

PROOF. We have to construct an inverse for $\Phi(R)$. To this end, we choose a matrix $g \in \mathrm{Gl}_n(A_R[1/z])$ and consider the following diagram of quasi-coherent sheaves on X_R,

$$\begin{tikzcd}
E \arrow[rr] \arrow[d] & & \mathcal{O}^n_{X^*_R} \arrow[d, "can"] \\
\mathcal{O}^n_{\Delta_R} \arrow[r, "can"] & \mathcal{O}^n_{\Delta^*_R} \arrow[r, "g"] & \mathcal{O}^n_{\Delta^*_R},
\end{tikzcd}$$

where E is uniquely determined up to isomorphism by requiring that the diagram be cartesian. (By abuse of notation we do not indicate the obvious push-forwards to X_R in this diagram.) It is easy to check (by pullback to Δ_R and X^*_R, respectively) that this diagram determines trivializations of E over Δ_R and X^*_R. The transition function for these two trivializations is equal to g by construction.

To see that this construction indeed gives an inverse for $\Phi(R)$ it remains to check that E is a vector bundle. This is immediate by Lemma 1.16 together with faithfully flat descent, or by the following elementary argument: the matrix g involves only finitely many elements of $A_R[1/z]$, whence in fact $g \in S_{R,f}[1/z]$ for some $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$. This shows that E can as well be obtained by gluing trivial bundles over $U_{R,f}$ and over X^*_R, respectively. Now, since $U_{R,f} \subset X_R$ is Zariski-open, this shows that E is a vector bundle. $\square$

1.3.2. 'Formal' descent of vector bundles. Let us now consider the situation introduced at the beginning in diagram (1.3.2), where we consider the formal neighborhood $D_R = \operatorname{Spec} R[[z]]$ of the closed subscheme $\operatorname{Spec} R \times \{p\} \subset X_R$.

By $\hat{T}$ we denote the functor, which associates to every k-algebra R the set of isomorphism classes of triples (E, ρ, σ), where E is a vector bundle of rank n over X_R and

$$\rho : \mathcal{O}^n_{X^*_R} \xrightarrow{\simeq} E_{|X^*_R},$$
$$\sigma : \mathcal{O}^n_{D_R} \xrightarrow{\simeq} E_{|D_R}$$

are trivializations.

As in the previous section, we obtain a functorial morphism $\hat{\Phi}(R) : \hat{T}(R) \to \mathrm{Gl}_n(R((z)))$ by assigning to each triple (E, ρ, σ) the corresponding transition function over D^*_R.

THEOREM 1.18 ([**BL94**], Proposition 1.4). *The morphism $\hat{\Phi}$ is an isomorphism of functors.*

Proof. In order to construct an inverse for $\hat{\Phi}$, i.e. to construct a triple (E, ρ, σ) from a given $\gamma \in \mathrm{Gl}_n(R((z)))$, we proceed exactly as in the proof of Proposition 1.17. The only non-trivial thing to check is that the quasi-coherent sheaf E, defined so to make the diagram

$$\begin{tikzcd} E \arrow[rr] \arrow[d] & & \mathcal{O}^n_{X^*_R} \arrow[d, "can"] \\ \mathcal{O}^n_{D_R} \arrow[r, "can"] & \mathcal{O}^n_{D^*_R} \arrow[r, "\gamma"] & \mathcal{O}^n_{D^*_R}, \end{tikzcd} \tag{1.3.3}$$

cartesian, is a vector bundle over X_R. We do this by reducing to a situation where Proposition 1.17 applies. More precisely, Lemma 1.19 below shows that every $\gamma \in \mathrm{Gl}_n(R((z)))$ can be written as a product $\gamma = g \cdot \delta$, where $g \in \mathrm{Gl}_n(A_R[1/z])$ and $\delta \in \mathrm{Gl}_n(R[[z]])$.

Thus diagram (1.3.3) 'decomposes' likewise, and yields the big diagram

$$\begin{tikzcd} E \arrow[r, equal] \arrow[dd] & E \arrow[rr] \arrow[d] & & \mathcal{O}^n_{X^*_R} \arrow[d, "can"] \\ & \mathcal{O}^n_{\Delta_R} \arrow[r, "can"] \arrow[d] & \mathcal{O}^n_{\Delta^*_R} \arrow[r, "g"] \arrow[d] & \mathcal{O}^n_{\Delta^*_R} \arrow[d] \\ \mathcal{O}^n_{D_R} \arrow[r, "\simeq", "\delta"'] & \mathcal{O}^n_{D_R} \arrow[r, "can"] & \mathcal{O}^n_{D^*_R} \arrow[r, "g"'] & \mathcal{O}^n_{D^*_R}. \end{tikzcd}$$

The two small squares in this diagram are trivially cartesian, while the big rectangle coincides with the square (1.3.3), and is thus cartesian by definition of E. Consequently, the upper rectangle is cartesian, which proves that E is nothing but the vector bundle corresponding to the transition matrix $g \in \mathrm{Gl}_n(A_R[1/z])$ under the correspondence of proposition 1.17. □

Lemma 1.19. $\mathrm{Gl}_n(R((z))) = \mathrm{Gl}_n(A_R[1/z]) \cdot \mathrm{Gl}_n(R[[z]])$.

Proof. We set $B := \cup_{P \in R[z] \cap R[[z]]^\times} R[z, z^{-1}, P^{-1}] \subset R((z))$ (Note that the ring $B \cap R[[z]]$ is equal to the ring A_R in the case $X = \mathbb{P}^1_k$.). Since $B \subset A_R[1/z]$, it suffices to check that $\mathrm{Gl}_n(R((z))) = \mathrm{Gl}_n(B) \cdot \mathrm{Gl}_n(R[[z]])$. First we note that $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Gl}_n(R((z)))$ is open: Namely, $\det : \mathrm{Mat}_n(R[[z]]) \to R[[z]]$ is continuous and R carries the discrete topology, whence $R^\times \subset R$ is open. This shows that $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Mat}_n(R[[z]]) \subset \mathrm{Mat}_n(R((z)))$ are two open inclusions, so $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Gl}_n(R((z)))$ is as well open. As a second step we deduce from Lemma 1.20 below that $\mathrm{Gl}_n(B) = \mathrm{Gl}_n(R((z))) \cap \mathrm{Mat}_n(B)$. Now, as $\mathrm{Mat}_n(B) \subset \mathrm{Mat}_n(R((z)))$ is dense, so is $\mathrm{Gl}_n(B) \subset \mathrm{Gl}_n(R((z)))$.

These two statements together imply that $\mathrm{Gl}_n(B) \cdot \mathrm{Gl}_n(R[[z]])$ is dense and closed in $\mathrm{Gl}_n(R((z)))$, whence the claimed equality. □

LEMMA 1.20. *The subring $B \subset R((z))$ defined above satisfies $B^\times = R((z))^\times \cap B$.*

PROOF. We consider $f \in R((z))^\times \cap B$. By multiplying with a suitable $P \in R[z] \cap R[[z]]^\times$, we may reduce to the case $f \in R((z))^\times \cap R[z, z^{-1}]$. Such an f has the form $f = -N + Q$, where $N \in R[z, z^{-1}]$ is a nilpotent Laurent polynomial and the leading coefficient of $Q \in R((z))^\times$ is a unit in R. Using the formula $(-N + Q)(N^i + N^{i-1}Q + \cdots + Q^i) = (-N^i + Q^i)$ we may assume that $f = Q^i$, i.e. has a leading coefficient in $R^\times$. Multiplying with z^m for a suitable $m \in \mathbb{Z}$ we obtain $z^m f \in R[z] \cap R[[z]]^\times$, which is invertible in B by construction. □

The property of the ring B which is exhibited in the last lemma is crucial for our strategy of approximation to work. This is what forces us to consider the, at first glance, rather artificial rings A_R instead of for example just $\mathcal{O}_{X,p} \otimes R$. The latter would not contain the ring B, and in particular would not have the property of Lemma 1.20.

To conclude this section let us briefly explain how one obtains the the description of the affine Grassmannian for Gl_n of Proposition 1.15, and how one deduces an analogous description in the case of Sl_n.

PROOF OF PROPOSITION 1.15. Let $\bar{T}$ denote the functor which associates to every k-algebra R the set of pairs (E, ρ), where E is a vector bundle on X_R which is trivial on D_R and where ρ is a trivialization of E over X_R^*. It is easy to see that the isomorphism $\hat{\Phi}(R) : \hat{T}(R) \to \mathrm{Gl}_n(R((z)))$ induces a diagram

$$\begin{array}{ccc} \hat{T} & \xrightarrow{\hat{\Phi}} & \mathrm{Gl}_n(R((z))) \\ \downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle can} \\ \bar{T} & \xrightarrow[\bar{\Phi}]{\simeq} & \mathrm{Gl}_n(R((z)))/\,\mathrm{Gl}_n(R[[z]]), \end{array}$$

where the quotient on the lower right hand side is the presheaf-quotient. The fpqc-sheafification of $\bar{T}$ (which is equal to its Zariski-sheafification) is the functor $\bar{T}^{sheaf}$, which by definition associates to R the set of pairs (E, ρ) where E is a rank n vector bundle on X_R, and ρ is a trivialization over X_R^*. Indeed, it is easy to check the conditions of Proposition A.2 in the appendix. First, $\bar{T}^{sheaf}$ is an fpqc-sheaf, since the property of being a vector bundle is local in the fpqc-topology, and since a trivialization, which is given fpqc-locally on X_R^* and satisfies the cocycle condition, descends to a global trivialization over X_R^*. Secondly, every vector bundle on D_R becomes trivial after Zariski-localization on R(!) by the same argument as in the proof of (1) $\Rightarrow$ (2) of Theorem 1.9. This is (1) of Proposition A.2, and (2) is trivial. Hence $\bar{T}^{sheaf}$ is indeed the fpqc-sheafification of $\bar{T}$ and is isomorphic to the affine Grassmannian for Gl_n. $\square$

So far we have only discussed the situation where the group is Gl_n. However, it is easy to deduce from this a description of the affine Grassmannian for Sl_n in terms of vector bundles. Let us say that a vector bundle E of rank n on the curve X has trivial determinant if $\wedge^n E$ is the structure sheaf. Note that, if any vector bundle E on a scheme Y is trivialized over open subsets U and V of Y, with transition function $g \in \mathrm{Gl}_n(\mathcal{O}_Y(U \cap V))$, then $\det g \in \mathcal{O}_Y(U \cap V)$ is the transition function for the corresponding trivializations of $\wedge^n E$. This gives us a description of $\mathrm{L}\,\mathrm{Sl}_n \,/\, \mathrm{L}^+\,\mathrm{Sl}_n$ as follows.

We denote by $\hat{T}'(R) \subset \hat{T}(R)$ the subset of triples (E, ρ, σ), where E is a vector bundle of rank n with trivial determinant. Clearly, $\hat{T}'$ is a subfunctor of $\hat{T}$. Moreover, let $\bar{T}' \subset \bar{T}$ resp. $\bar{T}'^{sheaf} \subset \bar{T}^{sheaf}$ be the subfunctors parametrizing those pairs (E, ρ) where E is a vector bundle on X_R with trivial determinant and ρ is a trivialization over X_R^*.

Lemma 1.21. *The functor $\bar{T}'^{sheaf}$ is a sheaf for the fpqc-topology.*

Proof. First we check that $\bar{T}'^{sheaf}$ is a Zariski-sheaf. For any faithfully flat homomorphism of k-algebras $R \to S$ we look at the diagram

$$\begin{tikzcd} \bar{T}(R) \arrow[r] & \bar{T}(S) \arrow[r, shift left] \arrow[r, shift right] & \bar{T}(S \otimes_R S) \\ \bar{T}'(R) \arrow[r] \arrow[u, hook] & \bar{T}'(S) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, hook] & \bar{T}'(S \otimes_R S) \arrow[u, hook] \end{tikzcd}$$

The upper line is an equalizer by the proof of Proposition 1.15, which we have just presented. In order to see that the lower line is an equalizer, too, we just have to observe that the left hand square is cartesian. So let $(E, \rho) \in \bar{T}(R)$. The condition that $\wedge^n \rho|_{X_R^*} : E^n|_{X_R^*} \to \mathcal{O}_{X_R}|_{X_R^*}$ extends to a trivialization $E^n \simeq \mathcal{O}_{X_R}$ is equivalent to the condition that the preimage of the 1-section, $\omega = (\wedge^n \rho|_{X_R^*})^{-1}(1)$, extends to a nowhere vanishing section of $\wedge^n E$. Using a trivialization of E in a neighborhood U_P for each point $P \in \{z = 0\}$, this is equivalent to asking whether $\omega_P \in \mathcal{O}_{X_R}(U_P^*)^\times$ extends to $\omega_P \in \mathcal{O}_{X_R}(U_P)^\times$. We assume that this holds after basechange to S, i.e. $\omega_{X_S,P} \in \mathcal{O}_{X_S}(U_{P,S})^\times$.

Let us fix P and drop the suffix P from the notation. As both lines in the following diagram are equalizers,

$$\begin{tikzcd} \mathcal{O}_{X_R}(U^*)^\times \arrow[r] & \mathcal{O}_{X_S}(U^*_S)^\times \arrow[r, shift left] \arrow[r, shift right] & \mathcal{O}_{X_{S\otimes_R S}}(U^*_{S\otimes_R S})^\times \\ \mathcal{O}_{X_R}(U)^\times \arrow[r] \arrow[u, hook] & \mathcal{O}_{X_S}(U_S)^\times \arrow[r, shift left] \arrow[r, shift right] \arrow[u, hook] & \mathcal{O}_{X_{S\otimes_R S}}(U_{S\otimes_R S})^\times \arrow[u, hook] \end{tikzcd}$$

we see that the left hand square is cartesian. Our assumptions on ω say precisely that $\omega \in \mathcal{O}_{X_R}(U)^\times$, which is what we wanted to see. An analogous argument shows that $\bar{T}'^{sheaf}$ is also a Zariski-sheaf, so that by Theorem A.4 it is indeed an fpqc-sheaf. □

COROLLARY 1.22. *The isomorphism* $\hat{\Phi}(R) : \hat{T}(R) \to \mathrm{Gl}_n(R((z)))$ *restricts to an isomorphism* $\hat{T}'(R) \simeq \mathrm{Sl}_n(R((z)))$. *By passage to the quotient and sheafification for the fpqc-topology one obtains an isomorphism*

$$\bar{T}'^{sheaf} \xrightarrow{\simeq} \mathrm{L}\,\mathrm{Sl}_n \,/\, \mathrm{L}^+\,\mathrm{Sl}_n\,.$$

PROOF. In the lemma before we have shown that $\bar{T}'^{sheaf}$ is an fpqc-sheaf. Now the proof of the corollary is completely analogous to the proof of Proposition 1.15. □

CHAPTER 2

Demazure Resolutions of Schubert Varieties

In this chapter we describe the classical notion of Demazure-Hansen-Bott-Samelson desingularization (for short: Demazure resolution) of Schubert varieties in the affine Grassmannian for Sl_n. We explain the 'standard'-definition in terms of quotients of loop groups, as well as an alternative description in terms of lattice chains. We continue to write $G = \mathrm{Sl}_n$.

2.1. Iwahori and Parahoric Subgroups

Let $\epsilon : G(k[[z]]) \to G(k)$ be the map induced by $z \mapsto 0$. The *standard Iwahori subgroup* $\mathcal{I} \subset G(k((z)))$ is by definition the preimage under ϵ of the standard Borel subgroup $B \subset G$. A preimage under ϵ of a standard parabolic subgroup $B \subset P \subset G$ is called a *standard parahoric subgroup* of $G(k((z)))$. In general, an *Iwahori-(resp. parahoric) subgroup* of $G(k((z)))$ is a $G(k((z)))$-conjugate of $\mathcal{I}$ (resp. a standard parahoric subgroup). Note that Iwahori- (resp. parahoric) subgroups are exactly the stabilizers of complete (resp. partial) lattice chains

$$z\mathcal{L} \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_l \subset \mathcal{L}, \quad l \leq n-1.$$

In particular, the standard Iwahori subgroup is the stabilizer of the *standard lattice chain*

$$zk[[z]]^n \subset k[[z]] \oplus zk[[z]]^{n-1} \subset \cdots \subset k[[z]]^{n-1} \oplus zk[[z]] \subset k[[z]]^n,$$

while the standard parahoric subgroups are the stabilizers of its respective subflags. A maximal standard parahoric subgroup is the stabilizer of a single lattice in the standard lattice chain.

We are going to relate the set of maximal parahoric subgroups of Sl_n to the set of minuscule dominant cocharacters of the standard maximal torus of its adjoint $\mathbb{P}\,\mathrm{Gl}_n$: Let $A \subset \mathbb{P}\,\mathrm{Gl}_n$ be the standard maximal torus and let $\check{\mathrm{X}}(A) \simeq \mathbb{Z}^n/(1,\ldots,1)\mathbb{Z}$ be the set of cocharacters of A. There is an obvious *inclusion*

$$\iota : \check{\mathrm{X}}(T) \hookrightarrow \check{\mathrm{X}}(A),$$

induced by the canonical map $T \to A$. With the identifications $\check{\mathrm{X}}(T) \subset \mathbb{Z}^n$ and $\check{\mathrm{X}}(A) \simeq \mathbb{Z}^n/(1,\ldots,1)\mathbb{Z}$ the map ι is given by the quotient map

$\mathbb{Z}^n \to \mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$. Moreover, the image of ι is precisely the kernel of the map

$$\mathbb{Z}^n/(1,\dots,1)\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}; \quad (z_1,\dots,z_n) \mapsto z_1 + \cdots + z_n. \tag{2.1.1}$$

If $\lambda = (z_1,\dots,z_n)$ then we will also write $|\lambda| = z_1 + \cdots + z_n$. We denote by $\check{\mathrm{X}}_+(T) \subset \check{\mathrm{X}}(T)$ the subset of *dominant* cocharacters of T, and analogously for A, and note that ι sends $\check{\mathrm{X}}_+(T)$ to $\check{\mathrm{X}}_+(A)$. Furthermore we denote by $\check{\mathrm{X}}_{+,\min}(A)$ the set of minuscule dominant cocharacters of A - these are the classes in $\mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$ of the vectors of the form $(1,\dots,1,0,\dots,0)$. Then we have the bijection

$$\begin{aligned} \check{\mathrm{X}}_{+,\min}(A) &\to \{ \text{ maximal standard parahorics} \subset \mathrm{Sl}_n \ \} \\ \mu &\mapsto P_\mu := z^{-\tilde{\mu}} G(k[[z]]) z^{\tilde{\mu}}, \end{aligned} \tag{2.1.2}$$

where $\tilde{\mu} \in \mathbb{Z}^n$ is a representative of μ. To abbreviate notation in the following section, we set $P_{\mu_1,\mu_2} := P_{\mu_1} \cap P_{\mu_2}$, the stabilizer of the two respective lattices in the standard flag, for μ_i minuscule dominant cocharacters of A.

2.2. Demazure Resolutions as Varieties of Lattice Chains

The presentation of the material in this section follows roughly the exposition by Gaussent and Littelmann in [**GL03**], and complements it with an alternative description of Demazure-Hansen-Bott-Samelson varieties in terms of lattice chains. The results are essentially due to Contou-Carrère [**CC83**]. Note that instead of 'Demazure-Hansen-Bott-Samelson variety' we will often simply speak of 'Demazure variety'. This is common in the literature.

Definition 2.1 (Demazure variety). *Let $\nu_1,\dots,\nu_m$ be a sequence of minuscule dominant cocharacters of $\mathbb{P}\,\mathrm{Gl}_n$. The* Demazure-Hansen-Bott-Samelson variety *$\Sigma(\nu_1,\dots,\nu_m)$ is defined as*

$$\Sigma(\nu_1,\dots,\nu_m) = P_0 \times^{P_{0,\nu_1}} P_{\nu_1} \times^{P_{\nu_1,\nu_2}} \cdots \times^{P_{\nu_{m-1},\nu_m}} P_{\nu_m}/P_{\nu_m,0},$$

i.e. the variety $P_0 \times P_{\nu_1} \times \cdots \times P_{\nu_m}$ modulo the right-action of the subgroup $P_{0,\nu_1} \times \cdots \times P_{\nu_{m-1},\nu_m} \times P_{\nu_m,0}$ given by

$$(g_0,\dots,g_m)\cdot(q_0,\dots,q_m) = (g_0q_0, q_0^{-1}g_1q_1,\dots,q_{m-1}^{-1}g_mq_m).$$

There is a natural morphism

$$\begin{aligned} \pi : \Sigma(\nu_1,\dots,\nu_m) &\to \mathcal{G} \\ [g_0,\dots,g_m] &\mapsto g_0\cdots g_m G(k[[z]])/G(k[[z]]), \end{aligned} \tag{2.2.1}$$

whose image is a closed $G(k[[z]])$-invariant subvariety of $\mathcal{G}$.

Let us remark here that in [**GL03**] Gaussent and Littelmann define a more general notion of Demazure variety in terms of sequences of 'types' (admitting not only maximal, but any standard parahoric subgroups in the above definition). Ideed, any minuscule dominant cocharacter gives rise to a type in the sense of [**GL03**].

We are now going to describe Demazure-Hansen-Bott-Samelson varieties in terms of descending lattice chains.

Choose a sequence $\mu_1, \dots, \mu_{m+1} \in \{0,1\}^n$ such that every μ_i represents a minuscule dominant cocharacter of A and $\mu := \mu_1 + \cdots + \mu_{m+1}$ represents a dominant cocharacter in $\iota(\check{\mathrm{X}}_+(T))$. We denote by $\lambda \in \check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ its preimage under ι. Then $\frac{1}{n}|\lambda|$ is an integer by the description of $\iota(\check{\mathrm{X}}_+(T))$ in (2.1.1).

Given the sequence $\mu_1, \dots, \mu_{m+1}$ we consider the following subset

$$\{(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{m+1}) \mid \mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) = \mu_{i+1}\} \subset \prod_{j=1}^{m+1} \mathcal{G}_{\mathrm{Gl}_n}(k) \tag{2.2.2}$$

of a product of affine Grassmannians for Gl_n, where $\mathcal{L}_0 = k[[z]]^n$ and by $\mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1})$ we denote the vector of elementary divisors of $\mathcal{L}_{i+1}$ relative to $\mathcal{L}_i$, ordered by decreasing size. It is easy to see that this is a closed subset, and we denote by $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}) \subset \prod_{j=1}^{m+1} \mathcal{G}_{\mathrm{Gl}_n}$ the subscheme carrying the reduced induced scheme-structure.

REMARK 2.2. *This definition implies that for any i we have $z\mathcal{L}_i \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i$. The points of $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$ thus correspond to sequences of vertices in the affine building of Sl_n, where two subsequent vertices are joint by a 1-dimensional face in the building.*

Again there is a morphism to the affine Grassmannian for Sl_n,

$$\begin{aligned} \tilde{\pi} : \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}) &\to \mathcal{G} \\ (\mathcal{L}_0, \dots, \mathcal{L}_{m+1}) &\mapsto \mathcal{L}_{m+1} z^{-\frac{1}{n}|\lambda|} \end{aligned} \tag{2.2.3}$$

(recall that $-\frac{1}{n}|\lambda|$ is an integer). It is easy to check that the lattice $\mathcal{L}_{m+1} z^{-\frac{1}{n}|\lambda|}$ is special by looking at the elementary divisors of the corresponding lattices.

In order to relate the previous two constructions, let $\nu_i, i = 1, \dots, m$, be as in the definition of Demazure variety, and for $i = 1, \dots, m$ choose μ_i to be a representative of the unique minuscule dominant cocharacter lying in the W-orbit of $\nu_{i-1} - \nu_i$ ($W = \mathrm{S}_n$, the Weyl group of $\mathbb{P}\,\mathrm{Gl}_n$). Moreover, set $\nu_{m+1} = 0$, and for every $i = 1, \dots, m+1$ choose a representative $\tilde{\nu}_i$ of ν_i such that $\tilde{\nu}_{i-1} - \tilde{\nu}_i \in \{0,1\}^n$. Of course, this construction can be reversed: to any sequence of μ_i's we can find the corresponding ν_i's. With this notation we have the following

PROPOSITION 2.3. *Let $[g_0, \dots, g_m] \in \Sigma(\nu_1, \dots, \nu_m)$. The assignment*

$$\mathcal{L}_i := g_0 \cdots g_{i-1} z^{-\tilde{\nu}_i} \mathcal{L}_0, \quad i = 1, \dots, m+1, \tag{2.2.4}$$

defines an isomorphism of varieties

$$\varphi : \Sigma(\nu_1, \dots, \nu_m) \xrightarrow{\cong} \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1}).$$

Furthermore, we have $\varphi \circ \tilde{\pi} = \pi$, and the morphisms (2.2.1) *and* (2.2.3) *are desingularizations of the Schubert variety $\mathcal{S}(\lambda)$ in the affine Grassmannian of $G = \mathrm{Sl}_n$.*

PROOF. Let $(g_0, \dots, g_m)$ be a representative of an R-valued point in $\Sigma(\nu_1, \dots, \nu_m)$ (R any k-algebra), and let $\mathcal{L}_i$ be as in the statement of the proposition, i.e. the lattice generated by the columns of the matrix $g_0 \cdots g_{i-1} z^{-\tilde{\nu}_i}$, for $i = 1, \dots, m+1$. The calculation

$$\begin{aligned}\mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) &= \mathrm{inv}(\mathcal{L}_0, z^{\tilde{\nu}_i} g_i z^{-\tilde{\nu}_{i+1}} \mathcal{L}_0) = \\ &= \mathrm{inv}(\mathcal{L}_0, h z^{\tilde{\nu}_i - \tilde{\nu}_{i+1}} \mathcal{L}_0) \text{ (for some } h \in G(k[[z]])) \ = \mu_{i+1}\end{aligned}$$

shows that (2.2.4) indeed gives a point in $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$. It is independent of the representative $(g_0, \dots, g_m)$, and indeed $\varphi \circ \tilde{\pi} = \pi$. We are left with the construction of an inverse map, which we will first explain on the level of lattices in $k((z))^n$. Let $(\mathcal{L}_0, \dots, \mathcal{L}_{m+1}) \in \tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})(k)$, and assume we have constructed matrices $g_0 \in P_{\nu_0}, \dots, g_{i-1} \in P_{\nu_{i-1}}$ up to the right action of $P_{0,\nu_1} \times \cdots \times P_{\nu_{i-1},\nu_i}$, such that $\mathcal{L}_j = g_0 \cdots g_{j-1} z^{-\tilde{\nu}_j} \mathcal{L}_0$ for $j \leq i$. Choose $h_i \in \mathrm{Sl}_n(k((z)))$ such that $\mathcal{L}_{i+1} = h_i \mathcal{L}_0$, and set $g_i = g_{i-1}^{-1} \cdots g_0^{-1} h_i z^{\tilde{\nu}_{i+1}}$. Then the equation

$$\mathrm{inv}(\mathcal{L}_0, z^{\tilde{\nu}_i} g_i z^{-\tilde{\nu}_{i+1}} \mathcal{L}_0) = \mathrm{inv}(\mathcal{L}_i, \mathcal{L}_{i+1}) = \mu_{i+1}$$

shows that $g_i \in P_{\nu_i} P_{\nu_{i+1}}$. Moreover, for an appropriate choice of representative h_i we even get $g_i \in P_{\nu_i}$, which determines g_i up to right action by $P_{\nu_i,\nu_{i+1}}$. Hence, by induction, we get a unique preimage for any sequence of lattices in $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$. In order to obtain a true morphism, we have to describe the map on the level of R-valued points for any local k-algebra R. However, the quotient $\mathrm{L}\,\mathrm{Gl}_n \to \mathrm{L}\,\mathrm{Gl}_n / \mathrm{L}^{\geq 0}\,\mathrm{Gl}_n$ is locally trivial for the Zariski-topology (see e.g. Faltings [**Fal03**]), which shows that the above construction can indeed be carried out for R-valued points, R any local k-algebra. This proves the first claim. Finally note that, by Remark 2.2, $\tilde{\Sigma}(\mu_1, \dots, \mu_{m+1})$ is a twisted product of ordinary Grassmannians and thus smooth and projective. Since the dimensions of source and target of the morphisms in question are equal and the involved schemes are projective over k, the second claim follows. $\square$

For later use we state explicitly a well-known formula for the dimension of $\Sigma(\nu_1,\dots,\nu_m)$ (see e.g. Gaussent and Littelmann, [**GL03**]).

PROPOSITION 2.4. *Let $\mu_1,\dots,\mu_{m+1}$ and λ be determined by the ν_i as before, and let ρ be half the sum of the positive roots $\{\alpha_{i,j}; 1 \le i < j \le n\}$ of $T \subset \mathrm{Sl}_n$. Then*

$$\dim \tilde{\Sigma}(\mu_1,\dots,\mu_{m+1}) = \dim \Sigma(\nu_1,\dots,\nu_m) = 2\langle \lambda, \rho\rangle. \tag{2.2.5}$$

PROOF. First observe that each μ_l has the form $(1,\dots,1,0,\dots,0)$, where the number of 1's in this vector is of course $|\mu_l|$. Identifying the group of characters of T with $\mathbb{Z}^n$ in the usual way, we have $\alpha_{i,j} = e_i - e_j$ and thus

$$\langle \mu_l, \alpha_{i,j}\rangle = \begin{cases} 1 & \text{if } i \le l < j, \\ 0 & \text{else.} \end{cases}$$

Consequently, if we sum over all positive roots $\alpha_{i,j}$, we obtain

$$2\langle \mu_l, \rho\rangle = |\{(i<j) \ | \ i \le l < j\}| = |\mu_l|(n - |\mu_l|).$$

This latter number is the dimension of the Grassmannian $\mathrm{Grass}(|\mu_l|, n)$, which parametrizes $|\mu_l|$-dimensional subspaces of k^n. Summing over all $l = 1,\dots,m+1$ we obtain the desired formula

$$2\langle \lambda, \rho\rangle = \sum_{i=1}^{m+1} \dim \mathrm{Grass}(|\mu_i|, n) = \dim \tilde{\Sigma}(\mu_1,\dots,\mu_{m+1}).$$

(Again, the last equality holds since $\tilde{\Sigma}(\mu_1,\dots,\mu_{m+1})$ is a twisted product of the respective ordinary Grassmannians.) $\square$

CHAPTER 3

Varieties of Lattices with Infinitesimal Structure

Besides the two interpretations of Demazure resolutions which we discussed in the preceding chapter (varieties of lattice chains, and quotients of loop groups, respectively) there is a third one: If the ground field k has positive characteristic p, then Demazure resolutions of Schubert varieties can be identified, up to a Frobenius twist, with certain varieties which parametrize lattices with infinitesimal structure. In the present chapter we will construct these varieties and finally relate them to the constructions of the previous chapter.

Throughout the whole chapter, k denotes a field of positive characteristic p, and R denotes a k-algebra. We consider the group $G = \mathrm{Sl}_n$.

3.1. Frobenius-Twisted Power Series Rings

Let $\mathrm{W}(R)$ be the ring of Witt vectors over R and identify it as a set with $R^{\mathbb{N}}$. On $\mathrm{W}(R)$ we consider the filtration $\mathcal{I} : \mathrm{W}(R) = I_0 \supset I_1 \supset I_2 \supset \cdots$, where, for every $n \in \mathbb{N}$, I_n is the ideal with underlying set $\{0\}^n \times R^{\mathbb{N}} \subset R^{\mathbb{N}}$.

Definition 3.1. *(1) The* ring of Frobenius-twisted power series *over R is the completion of the graded ring $\mathrm{gr}_{\mathcal{I}}\,\mathrm{W}(R) = \oplus_{i\in\mathbb{N}} I_i/I_{i+1}$ with respect to the filtration given by the ideals $\oplus_{i\geq N} I_i/I_{i+1}$. We denote this ring by $R[[z]]^F$.*
(2) The ring of (truncated) Frobenius-twisted power series of length N *is the quotient of $\mathrm{gr}_{\mathcal{I}}\,\mathrm{W}(R)$ by the ideal $\oplus_{i\geq N} I_i/I_{i+1}$. We denote it by $R[[z]]^F_N$.*

Note that the ring $R[[z]]^F$ contains $R = \mathrm{W}(R)/I_1$ as a subring, and its underlying additive group is isomorphic to $R^{\mathbb{N}}$ with addition given componentwise. Furthermore, if we write an element $(a_0, a_1, a_2, \dots) \in R[[z]]^F$ as $a_0 + a_1 z + a_2 z^2 + \cdots$, then multiplication in $R[[z]]^F$ is as follows:

$$(3.1.1)\quad (a_0 + a_1 z + a_2 z^2 + \dots) \cdot (b_0 + b_1 z + b_2 z^2 + \dots) = \\ = a_0 b_0 + (a_0^p b_1 + a_1 b_0^p) z + (a_0^{p^2} b_2 + a_1^p b_1^p + a_2 b_0^{p^2}) z^2 + \cdots .$$

By the way this gives us a morphism of rings

$$\begin{aligned}\mathcal{F} = 1 \times F \times F^2 \times \cdots : R[[z]] &\to R[[z]]^F \\ a_0 + a_1 z + a_2 z^2 + \cdots &\mapsto a_0 + a_1^p z + a_2^{p^2} z^2 + \cdots ,\end{aligned} \tag{3.1.2}$$

which is an isomorphism if and only if R is perfect. Note furthermore, that $R[[z]]_N^F = R[[z]]^F / z^N R[[z]]^F$ is only true if R is perfect.

On $\mathbb{A}_k^N = \operatorname{Spec} k[x_0, x_1, \ldots, x_{N-1}]$ the ring structure of $k[[z]]_N^F$ induces a structure of ring scheme which we denote by $\mathfrak{P}_N$, such that for any k-algebra R we have $\mathfrak{P}_N(R) = R[[z]]_N^F$. We call this the *scheme of (truncated) Frobenius-twisted power series over* k. Similarly, we obtain a structure of $k[[z]]_N^F$-module scheme on $\mathfrak{P}_N^n \simeq (\mathbb{A}_k^N)^n$.

A grading on the coordinate ring of $\mathfrak{P}_N^n$. If we fix the grading

$$\deg x_{i,j} = p^j, \quad i = 1, \ldots, n; j = 0, \ldots, N-1$$

on the coordinate ring of $\mathfrak{P}_N^n \simeq \mathbb{A}_k^{nN}$, then addition,

$$a : \mathfrak{P}_N^n \times_{\operatorname{Spec} k} \mathfrak{P}_N^n \to \mathfrak{P}_N^n,$$

as well as multiplication by a scalar $\in k[[z]]_N^F$, are given by *graded* homomorphisms of the respective coordinate rings (see eq. (3.1.1)). *From now on, we will consider the coordinate ring of $\mathfrak{P}_N^n$ endowed with this grading.* In the following section we are going to construct a moduli space for closed subschemes $V \subset \mathfrak{P}_N^n$ with the property that addition and scalar multiplication restrict to V; that is, V inherits the module structure from $\mathfrak{P}_N^n$.

3.2. A Moduli Space for Lattice Schemes

3.2.1. Multigraded Hilbert schemes. We first recall a result by Haiman and Sturmfels ([**HS04**]) on the representability of the multigraded Hilbert functor.

Let R be any ring, and let $\mathbb{A}_R^n = \operatorname{Spec} R[x_1, \ldots, x_n]$ be the n-dimensional affine space over R, and identify $u \in \mathbb{N}^n$ with the monomial $x_1^{u_1} \cdots x_n^{u_n}$. Then a multigrading of $R[x_1, \ldots, x_n]$ by a semigroup A is given by a semigroup homomorphism $\deg : \mathbb{N}^n \to A$. This induces a decomposition

$$R[x_1, \ldots, x_n] = \oplus_{a \in A} R[x_1, \ldots, x_n]_a,$$

where $R[x_1, \ldots, x_n]_a$ is the R-span of the monomials of degree a.

A homogeneous ideal $I \subset R[x_1, \ldots, x_n]$ is called *admissible* (over R), if $(R[x_1, \ldots, x_n]/I)_a$ is a locally free module of constant finite rank on $\operatorname{Spec} R$, for all $a \in A$. Every admissible ideal $I \subset R[x_1, \ldots, x_n]$ has then a well-defined *Hilbert function*, given by

$$h_I : A \to \mathbb{N}, \quad a \mapsto \operatorname{rk}(R[x_1, \ldots, x_n]/I)_a.$$

An R-subscheme $V \subset \operatorname{Spec} R[x_1, \ldots, x_n]$ which is defined by an admissible ideal will also be called admissible, and by the Hilbert function of such a V we will mean the Hilbert function of its defining ideal.

Let $h : A \to \mathbb{N}$ be any function supported on $\deg(\mathbb{N}^n)$, and define the *Hilbert functor* $\mathcal{H}^h_R : (R\text{-Alg}) \to (\text{Set})$ by

$$\mathcal{H}^h_R(S) = \{\text{admissible ideals } I \subset S[x_1, \ldots, x_n] \mid \\ \operatorname{rk}(S[x_1, \ldots, x_n]/I) = h(a) \text{ for all } a \in A\}.$$

THEOREM 3.2 (Haiman, Sturmfels). *There exists a quasiprojective scheme H^h_R over R which represents the functor $\mathcal{H}^h_R$. If the grading of $R[x_1, \ldots, x_n]$ is positive, i. e. 1 is the only monomial with degree 0, then this scheme is even projective over R.* □

The scheme H^h_R is called the 'multigraded Hilbert scheme' for the Hilbert function h. In the sequel, if we do not specify a Hilbert function h, then by the term 'multigraded Hilbert scheme', or just Hilbert scheme, we refer to the union of the multigraded Hilbert schemes for all possible Hilbert functions. We denote this scheme by H_R, or simply by H if the ring R is fixed.

3.2.2. Lattice schemes. Let R be a ring. For any ring scheme $\mathfrak{R}$ over R we have the obvious notion of an $\mathfrak{R}$-module scheme over R. In particular, we have the free $\mathfrak{R}$-module scheme of rank n, denoted $\mathfrak{R}^n$. $\mathfrak{R}$-submodule schemes of an $\mathfrak{R}$-module scheme M are closed R-subschemes of M which are 'stable under the morphisms defining the module operations on M'. This means that for a closed R-subscheme $V \subset M$ we require the following diagrams to exist:

$$\begin{tikzcd} M \times M \arrow[r, "add."] & M \\ V \times V \arrow[u, hook] \arrow[r, dotted] & V \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{R} \times M \arrow[r, "mult."] & M \\ \mathfrak{R} \times V \arrow[u, hook] \arrow[r, dotted] & V \arrow[u, hook] \end{tikzcd}$$

Analogous diagrams are required to exist for the zero-section and additive inverses.

In the sequel, we always assume that $\mathfrak{R}$ is a ring scheme which is isomorphic as an R-scheme to $\mathbb{A}^N_R$ ($0 \le N < \infty$). Let us furthermore fix a grading over R of the structure sheaf of $\mathfrak{R} \simeq \mathbb{A}^N_R$ so that the ring operations on $\mathfrak{R}$ are defined by graded homomorphisms on the structure sheaf. Then also the structure sheaf of $\mathfrak{R}^n$ is graded.

DEFINITION 3.3. *We call a submodule scheme $V \subset \mathfrak{R}^n$ a* lattice-scheme *if the defining ideal of V is admissible.*

PROPOSITION 3.4. *The set of lattice schemes in $\mathfrak{R}^n$ is parametrized by a closed subscheme $\mathbb{L}_{\mathfrak{R}^n}$ of the multigraded Hilbert scheme of $\mathfrak{R}^n$ over R. Let us denote by $\mathbb{L}^h_{\mathfrak{R}^n}$ the component of $\mathbb{L}_{\mathfrak{R}^n}$ corresponding to the Hilbert function h. Then the R-scheme $\mathbb{L}^h_{\mathfrak{R}^n}$ is quasi-projective over R, and it is projective over R if the grading of $\mathfrak{R}$ is positive.*

PROOF. Let $H \to \operatorname{Spec} R$ be the multigraded Hilbert scheme of $\mathfrak{R}^n$ and let $U \to H$ be the universal family. We have to show that there exists a closed subscheme $\mathbb{L}_{\mathfrak{R}^n} \subset H$ such that for any morphism $Y \to H$, $V = Y \times_H U \subset Y \times_{\operatorname{Spec} R} \mathfrak{R}^n$ is a submodule scheme if and only if $Y \to H$ factors through $\mathbb{L}_{\mathfrak{R}^n} \subset H$. It suffices to check this locally, i.e. for an affine open subscheme $H' = \operatorname{Spec} A \subset H$ instead of H itself. Then also $U' := H' \times_H U$ is affine, and U' is given by an ideal $I \subset A[x_{i,j} \mid i = 1, \ldots, n; j = 0, 1, \ldots, N]$ with A-locally free quotient $A[x_{i,j}]/I$. Now for any morphism $Y' = \operatorname{Spec} B \to H'$ the condition that $V' = Y' \times_{H'} U' \subset U'$ be stable under the module operations on $\mathfrak{R}^n$ translates into the condition that the image of I under the comorphism of addition vanishes in $B[x_{i,j}]/I \otimes_B B[x_{i,j}]/I$, besides analogous vanishing conditions concerning scalar multiplication, units and additive inverses. Since $A[x_{i,j}]/I$ is locally free over A, these vanishing conditions can be expressed by equations with coefficients in A, which then define a closed subscheme $Z' \subset H' = \operatorname{Spec} A$. By construction, V' is stable under the module operations if and only if $Y' \to H'$ factors throuth Z'. By gluing all the $Z' \subset H$ we obtain the closed subscheme $\mathbb{L}_{\mathfrak{R}^n} \subset H$ which possesses the desired universal property. $\square$

In situations where the ring scheme $\mathfrak{R}$ and the dimension n are fixed or clear from the context, we will usually drop the index $\mathfrak{R}^n$ and write $\mathbb{L} = \mathbb{L}_{\mathfrak{R}^n}$, and $\mathbb{L}^h = \mathbb{L}^h_{\mathfrak{R}^n}$, respectively.

PROPOSITION 3.5 (Group actions on H). *Let $\Gamma / \operatorname{Spec} R$ be an algebraic group acting algebraically on $\mathfrak{R}^n$, and assume that this action respects the grading on the structure sheaf of $\mathfrak{R}^n$. Then Γ acts on the Hilbert scheme H^h of $\mathfrak{R}^n$ for any Hilbert function h. If furthermore the action of Γ on $\mathfrak{R}^n$ is by automorphisms of $\mathfrak{R}$-module schemes, then the action of Γ on H^h restricts to an action on $\mathbb{L}^h$.*

PROOF. This is a formal consequence of the universal properties of H^h and $\mathbb{L}^h$ and the fact that the action of Γ on $\mathfrak{R}^n$ is algebraic, i.e. functorial. $\square$

3.2.3. The case of Frobenius-twisted power series. We consider now the situation, where the ring scheme $\mathfrak{R} = \mathfrak{P}_N$ is the scheme of (truncated) Frobenius-twisted power series over k. Recall that $G = \mathrm{Sl}_n$ over k.

It is easy to see that the (Set)-valued functor on the category of k-algebras

$$\mathrm{L}_F^+ G : R \mapsto G(R[[z]]^F)$$

is representable by an infinite dimensional affine group scheme over k. Similarly, the functor

$$\mathrm{L}_{F,N}^+ G : R \mapsto G(R[[z]]_N^F)$$

is representable by a (finite dimensional) algebraic group over k. There are canonical morphisms of k-groups

$$\mathrm{L}^+ G \to \mathrm{L}_F^+ G, \quad \mathrm{L}_F^+ G \to \mathrm{L}_{F,N}^+ G, \tag{3.2.1}$$

where the first of these morphisms is induced by the map (3.1.2). Moreover, there is an obvious algebraic operation of the k-group $\mathrm{L}_F^+ G$ on $\mathfrak{P}_N^n$ given by multiplication of an $n \times n$-matrix and an $n \times 1$-vector over the ring of twisted truncated power series of length N. This operation factors through $\mathrm{L}_{F,N}^+ G$ and respects the grading of the structure sheaf of $\mathfrak{P}_N^n$.

Remark 3.6. *In fact, all the objects and morphisms in this paragraph (except for the morphism $\mathrm{L}^+ G \to \mathrm{L}_F^+ G$) can be understood as instances of the general construction of Greenberg realizations, applied to the ring scheme $\mathfrak{R} = \mathfrak{P}_N$. For a discussion of Greenberg realizations we refer to Chapter 4.*

By Proposition 3.5 we obtain an operation

$$\mathrm{L}_{F,N}^+ G \times_{\mathrm{Spec}\, k} \mathbb{L}^h \to \mathbb{L}^h.$$

We are interested in the orbits and orbit-closures of this action.

The standard lattice scheme for a dominant cocharacter. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \check{\mathrm{X}}_+(T)$, set $\tilde{\lambda} = \lambda - (\lambda_n, \dots, \lambda_n) \in \mathbb{N}^n$ and set $N := \tilde{\lambda}_1$. Then $\tilde{\lambda}$ defines a lattice scheme in $\mathfrak{P}_N^n$, given by the ideal $I(\lambda) \subset k[x_{i,j}; i = 1, \dots, n; j = 0, \dots, N-1]$, where

$$I(\lambda) = (x_{1,0}, \dots, x_{1,\tilde{\lambda}_1 - 1}, \dots, x_{n-1,0}, \dots, x_{n-1,\tilde{\lambda}_{n-1}-1}). \tag{3.2.2}$$

Let us call this lattice scheme the *standard lattice scheme* associated with λ. We denote its $\mathrm{L}_F^+ G$-orbit in $\mathbb{L}^h$ by $\mathcal{O}(\lambda)$, and its orbit-closure by $\mathcal{D}(\lambda)$. The latter will turn out to be closely related to a Demazure resolution of the Schubert variety $\mathcal{S}(\lambda)$ with respect to the 'standard decomposition of λ' into minuscule dominant cocharacters, which we describe below.

The standard decomposition of a dominant cocharacter. For $\lambda \in \check{\mathrm{X}}_+(T)$ and for $i = 1, \dots, N$ choose $\mu_i = (1, \dots, 1, 0, \dots, 0)$ such that the number of 1's in this expression equals the number of entries in $\tilde{\lambda}$ which are $\geq i$. This defines a decomposition of λ into minuscule dominant cocharacters $\bar{\mu}_i$ as described in Section 2.2. Obviously we have $\tilde{\lambda} = \mu_1 + \dots + \mu_N$, and the μ_i's are ordered 'by size':

$$|\mu_1| \geq \dots \geq |\mu_N|.$$

We call this decomposition of λ into minuscule dominant cocharacters the *standard decomposition* of λ. For the rest of the chapter we will assume λ, $\tilde{\lambda}$ and the μ_i's chosen in this way.

3.3. Twisted Linear Ideals and Flatness Results

To give an idea of the relation of $\mathcal{D}(\lambda)$ to Demazure resolutions and thereby motivate the subsequent technical section, consider

Example 3.7. *Choose $n = 2$ and $\lambda = (1, -1) \in \check{\mathrm{X}}_+(T)$. Then $N = 2$, $\tilde{\lambda} = (2, 0)$, and the standard lattice scheme associated with λ is given by $I(\lambda) = \langle x_{1,0}, x_{1,1} \rangle$. For convenience we rename the variables $x_{1,j} \mapsto x_j, x_{2,j} \mapsto y_j$, whence $I(\lambda) = \langle x_0, x_1 \rangle$.*
Claim: *The $\mathrm{L}_F^+ G$-orbit of $I(\lambda)$ consists of all ideals of the form*

$$I := A \cdot I(\lambda) = \langle a_0 x_0 + b_0 y_0, a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p \rangle, \qquad \text{with } a_0 \neq 0 \text{ or } b_0 \neq 0.$$

To see this, one calculates the effect of a matrix

$$A = \begin{pmatrix} a_0 + a_1 z & b_0 + b_1 z \\ c_0 + c_1 z & d_0 + d_1 z \end{pmatrix} \in \mathrm{Sl}_2(k[[z]]^F) = \mathrm{L}_F^+ \mathrm{Sl}_2(k)$$

on the vector $x = \begin{pmatrix} x_0 + x_1 z \\ y_0 + y_1 z \end{pmatrix} \in (k[x_i, y_i][[z]]^F)^2$. We obtain

$$A \cdot x \equiv \begin{pmatrix} (a_0 x_0 + b_0 y_0) + (a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p) z \\ (c_0 x_0 + d_0 y_0) + (c_0^p x_1 + c_1 x_0^p + d_0^p y_1 + d_1 y_0^p) z \end{pmatrix} \mod z^2.$$

Since the operation of $\mathrm{L}_F^+ G$ on $k[x_i, y_i]$ is given by the transpose of this action, we see that the images of x_0 resp. x_1 are of the form $a_0 x_0 + b_0 y_0$ resp. $a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p$, with $a_0 \neq 0$ or $b_0 \neq 0$. Thus the claim.
Claim: *the ideal I is graded, and denoting the i-th graded component by I_i, we have*

$$I_1 \cap \langle x_0, y_0 \rangle = \langle a_0 x_0 + b_0 y_0 \rangle$$

and

$$I_p \cap \langle x_0^p, y_0^p, x_1, y_1 \rangle = \langle a_0^p x_0^p + b_0^p y_0^p, a_1 x_0^p + b_1 y_0^p + a_0^p x_1 + b_0^p y_1 \rangle.$$

Only the latter equation requires an argument: we have to verify that, if $(a_0x_0+b_0y_0)\cdot P(x_0,y_0) \in \langle x_0^p, y_0^p\rangle$, *then* $P(x_0,y_0) = \gamma(a_0x_0+b_0y_0)^{p-1}$, *for* $\gamma \in k$. *Assume* $a_0 \neq 0$ *and consider the linear transformation of variables* $x_0 \mapsto (1/a_0)x_0-(b_0/a_0)y_0$, *which stabilizes* $\langle x_0^p, y_0^p\rangle$. *Hence* $(a_0x_0+b_0y_0)\cdot P(x_0,y_0) \in \langle x_0^p, y_0^p\rangle$ *if and only if* $x_0 \cdot P((1/a_0)x_0-(b_0/a_0)y_0, y_0) \in \langle x_0^p, y_0^p\rangle$. *Thus we must have* $P((1/a_0)x_0 - (b_0/a_0)y_0, y_0) = \gamma' x_0^{p-1}$ *for some* $\gamma' \in k$, *whence* $P(x_0,y_0) = \gamma(a_0x_0+b_0y_0)^{p-1}$. *The case* $b_0 \neq 0$ *is similar, which proves the claim.*

Applying the absolute Frobenius morphism to I_1 *yields a submodule of* I_p, *and identifying* x_0^p, y_0^p, x_1, y_1 *with the standard basis of* $k^{2\times 2}$, *these define a descending sequence of lattices in* $k^{2\times 2}$ *in the sense of Section 2.2:*

$$\begin{aligned}\mathcal{L}_0 = \mathrm{Hom}_k(k^{2\times 2}, k) \supset \mathcal{L}_1 = \mathrm{Hom}_k(k^{2\times 2}/\langle a_0^p x_0^p + b_0^p y_0^p\rangle, k) \supset \\ \supset \mathcal{L}_2 = \mathrm{Hom}_k(k^{2\times 2}/\langle a_0^p x_0^p + b_0^p y_0^p, a_1 x_0^p + b_1 y_0^p + a_0^p x_1 + b_0^p y_1\rangle, k).\end{aligned}$$

This corresponds to a point in the Demazure resolution of $\mathcal{S}(\lambda)$ *(in the situation of this example there is only one Demazure resolution, since there is only one decomposition of* λ *into minuscule dominant cocharacters).*

Remark 3.8. *From the example it is clear that we will have to deal with certain Frobenius twists when relating points of* $\mathcal{D}(\lambda)$ *to points of a Demazure variety.*

In the following we are going to prove auxiliary results, which will later be used to study the general case $G = \mathrm{Sl}_n$. As always, let R be an arbitrary k-algebra, denote by $\mathfrak{X} = \{x_1, \ldots, x_m\}$ a set of indeterminates with $\deg x_i = p^{d_i}$ for any i, and set

$$F_{p^l}(R) = \oplus_{d_i \leq l} R x_i^{p^{l-d_i}}.$$

To carry out the construction of the example for any (R-valued) point in $\mathcal{D}(\lambda)$, possibly meeting the boundary of $\mathcal{O}(\lambda)$, we need to know that the corresponding ideals are well behaved, in a sense to be made precise, with respect to intersections with the R-submodules $F_{p^l}(R)$ of $R[x_{i,j}]$. To show this is the goal of this section.

The crucial property that the ideals which we consider will turn out to have, is subject of the following

Definition 3.9. *Let* R *be any* k*-algebra. We call an element* $f \in R[\mathfrak{X}]$ twisted-linear, *if* $f \in \cup_l F_{p^l}(R)$. *We call an ideal* $I \subset R[\mathfrak{X}]$

twisted-linear, *if I is generated by a finite subset consisting of twisted-linear elements. Equivalently, a finitely generated ideal I is twisted-linear, if and only if it is generated by $\cup_l (F_{p^l}(R) \cap I)$. Such an ideal is obviously graded.*

In the following, if M is a graded R-module, we denote by M_d its degree d-part. Note that $F_{p^N}(R)$ is a direct summand of $R[\mathfrak{X}]_{p^N}$. Note furthermore that, if I is finitely generated, then $R[\mathfrak{X}]/I$ is finitely presented as an R-algebra, whence every homogeneous component of $R[\mathfrak{X}]/I$ is finitely presented as an R-module. Thus every such homogeneous component is R-flat if and only if it is R-projective if and only if it is locally free over R. In this case, the same properties hold for the homogeneous components of I.

LEMMA 3.10. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is R-flat. Assume that I is generated by twisted-linear elements f_j with $\deg f_j = p^{e_j}$ for each j. Then we have for any N*

$$\sum_{e_j \leq N} R f_j^{p^{N-e_j}} = F_{p^N}(R) \cap I.$$

Before proving this lemma, we state two corollaries:

COROLLARY 3.11. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is flat over R. Then for any R-algebra S:*

$$(F_{p^N}(R) \cap I) \cdot S = F_{p^N}(S) \cap (I \otimes_R S)$$

(where the left expression denotes the image of $(F_{p^N}(R) \cap I) \otimes_R S$ in $S[\mathfrak{X}]$).

PROOF. The ideal $I \otimes_R S \subset S[\mathfrak{X}]$ satisfies the assumptions of the lemma, with R replaced by S. In particular, it is generated by the images of the f_j in $S[\mathfrak{X}]$. Hence we have

$$F_{p^N}(S) \cap (I \otimes_R S) = \sum_{e_j \leq N} S f_j^{p^{N-e_j}} = (F_{p^N}(R) \cap I) \cdot S,$$

where the first equality results from the lemma applied to the ideal $I \otimes_R S$, while the second is a consequence of the lemma applied to I itself. $\square$

COROLLARY 3.12. *Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal such that $R[\mathfrak{X}]/I$ is flat over R. Then the R-module $F_{p^N}(R)/(F_{p^N}(R) \cap I)$ is R-flat, as well as the quotient of degree p^N-components $R[\mathfrak{X}]_{p^N}/(F_{p^N}(R) + I_{p^N})$.*

PROOF. The second module in question is the cokernel of the injection of R-modules

$$F_{p^N}(R)/(F_{p^N}(R) \cap I) \hookrightarrow (R[\mathfrak{X}]/I)_{p^N}.$$

Corollary 3.11 states that this map is still injective after tensoring with any R-algebra S, so the long exact sequence of Tor's tells us that the cokernel is flat. And thus also $F_{p^N}(R)/(F_{p^N}(R) \cap I)$. $\square$

Hence we see that we can in fact restate the equation in Corollary 3.11 in the form

$$(F_{p^N}(R) \cap I) \otimes_R S = F_{p^N}(S) \cap (I \otimes_R S).$$

We interpret this equation as follows: *cutting out of a twisted-linear ideal the twisted-linear part of given degree is functorial with respect to base change.*

PROOF OF LEMMA 3.10. Since $R[\mathfrak{X}]_1/I_1$ is R-flat and of finite presentation, and since our claim is local on R, we can assume that $R[\mathfrak{X}]_1/I_1$ is free. Since invertible linear transformations of the degree 1-variables do not affect twisted-linearity nor flatness, we can assume without loss of generality that the f_j of degree 1 are in fact in $\mathfrak{X}$, i.e. variables of degree 1. If all the f_j have degree 1, we are done. Otherwise, we apply induction on n, where p^n is the maximum degree of a generator f_j. By our assumption on I_1, we have $R[\mathfrak{X}]/I \simeq R[\mathfrak{X}']/I'$, where $\mathfrak{X}'$ is $\mathfrak{X}$ minus the variables generating I_1, and I' is the image of I under the quotient map $R[\mathfrak{X}] \to R[\mathfrak{X}']$. Hence, I' is twisted-linear and generated by those $g_j = \text{image}(f_j) \in R[\mathfrak{X}']$ which have degree $\geq p$.

Set $\mathfrak{X}'' = \{x \in \mathfrak{X}'; \deg x > 1\} \cup \{x^p \in \mathfrak{X}'; \deg x = 1\}$. Then the g_j already lie in $R[\mathfrak{X}'']$ and generate an ideal $I'' \subset R[\mathfrak{X}'']$. We obtain $R[\mathfrak{X}']/I' \simeq (R[\mathfrak{X}'']/I'')[p\text{-th roots of some variables in } \mathfrak{X}'']$. By faithful flatness of extension by p-th roots, $R[\mathfrak{X}'']/I''$ is flat over R. Now we only deal with variables of degree $\geq p$, and I'' is still twisted-linear, so for the moment we can think of all degrees divided by p and apply the induction hypotheses in order to obtain the claim of the lemma for $I'' \subset R[\mathfrak{X}'']$:

$$\sum_{1 \leq e_j \leq N} R g_j^{p^{N-e_j}} = F_{p^N}(R)'' \cap I''.$$

(Here, by abuse of notation, we denote by g_j also the (unique) preimages of the g_j in $R[\mathfrak{X}'']$. As for $R[\mathfrak{X}]$, we denote by $F_{p^N}(R)''$ and $F_{p^N}(R)'$ the modules of twisted-linear monomials of degree p^N in the algebras $R[\mathfrak{X}'']$ and $R[\mathfrak{X}']$, respectively.) Since

$$R[\mathfrak{X}']/I' \simeq (R[\mathfrak{X}'']/I'')[p\text{-th roots}],$$

we have $I'' = I' \cap R[\mathfrak{X}'']$, $F_{p^N}(R)' = F_{p^N}(R)'' \subset R[\mathfrak{X}'']$ and thus obtain

$$\sum_{1 \leq e_j \leq N} R g_j^{p^{N-e_j}} = F_{p^N}(R)' \cap I'.$$

Now let $\varphi : R[\mathfrak{X}'] \to R[\mathfrak{X}]$ be the canonical splitting of the quotient map $R[\mathfrak{X}] \to R[\mathfrak{X}']$. Then $I \subset \varphi(I') \oplus \sum_{m \neq 1} mR[\mathfrak{X}]$, where the sum runs over all monomials m in variables in $\mathfrak{X} - \mathfrak{X}'$ not equal to 1. Since similarly $F_{p^N}(R) = \varphi(F_{p^N}(R)') \oplus \bigoplus_{x \in \mathfrak{X} - \mathfrak{X}'} R x^{p^N}$, we have

$$F_{p^N}(R) \cap I \subset \varphi(F_{p^N}(R)' \cap I') \oplus \bigoplus_{x \in \mathfrak{X} - \mathfrak{X}'} R x^{p^N} = \sum_{e_j \leq N} R f_j^{p^{N-e_j}}.$$

The opposite inclusion is trivial. $\square$

By a similar induction argument we prove

Lemma 3.13. *Assume that the twisted-linear ideal $I \subset R[\mathfrak{X}]$ is generated in degrees $d \leq p^n$ and that the graded components $(R[\mathfrak{X}]/I)_d$ are flat over R for $d \leq p^n$. Then $R[\mathfrak{X}]/I$ is R-flat.*

Proof. We proceed by induction on n. Since the graded components of $R[\mathfrak{X}]/I$ are of finite presentation, $(R[\mathfrak{X}]/I)_d$ is even locally free for $d \leq p^n$. Thus, since our claim is local on R, we can assume that $R[\mathfrak{X}]/\langle I_1 \rangle$ is isomorphic to a polynomial ring $R[\mathfrak{X}']$, with $\mathfrak{X}' \subset \mathfrak{X}$. In case $n = 0$ we are already done. Otherwise, the image of $I/\langle I_1 \rangle$ in $R[\mathfrak{X}']$ is a twisted-linear ideal $I' \subset R[\mathfrak{X}']$, generated by the images g_j of those generators of I which have degree $\geq p$. Set $\mathfrak{X}'' = \{x \in \mathfrak{X}'; \deg x > 1\} \cup \{x^p \in \mathfrak{X}'; \deg x = 1\}$. Then the g_j of degree > 1 lie in $R[\mathfrak{X}'']$ and generate an ideal $I'' \subset R[\mathfrak{X}'']$. We have

$$R[\mathfrak{X}]/I \simeq R[\mathfrak{X}']/I' \simeq (R[\mathfrak{X}'']/I'')[p\text{-th roots of some variables in } \mathfrak{X}''].$$

Still, I'' is twisted-linear and $R[\mathfrak{X}'']/I''$ is R-flat in degrees $\leq p^n$, since adjoining p-th roots is faithfully flat. In $\mathfrak{X}''$ there only occur variables of degrees p^e with $e \geq 1$, so we can divide all degrees by p and arrive at a situation where we can use the induction hypotheses: $R[\mathfrak{X}'']/I''$ is R-flat in *all* degrees. But then so is $R[\mathfrak{X}']/I' \simeq R[\mathfrak{X}]/I$. $\square$

Remark 3.14. *(1) For any ideal $I \subset R[\mathfrak{X}]$, we denote by $I^{\leq n}$ the ideal generated by the graded components of I which have degree $\leq n$. Then, if $R[\mathfrak{X}]/I$ is R-flat and I is twisted-linear, Lemma 3.13 tells us that the same properties hold for $R[\mathfrak{X}]/I^{\leq p^n}$. In particular, we obtain from Corollary 3.12 that $R[\mathfrak{X}]_{p^n}/(F_{p^n}(R) + (I^{\leq p^m})_{p^n})$ is R-flat for all m and n.*

(2) Let $I \subset R[\mathfrak{X}]$ be a twisted-linear ideal generated in degrees $\leq p^n$. For the quotient $R[\mathfrak{X}]/I$ to be R-flat it is sufficient that $(R[\mathfrak{X}]/I)_{p^i}$ is

R-flat for $i = 0, \dots, n$. This follows from Lemma 3.13 by induction on n.

Combining these two remarks, we obtain a stronger version of the preceding lemma:

COROLLARY 3.15. *Assume that the twisted-linear ideal $I \subset R[\mathfrak{X}]$ is generated in degrees $d \le p^n$. Then $R[\mathfrak{X}]/I$ is R-flat if and only if the R-modules*

$$F_{p^m}(R)/(F_{p^m}(R) \cap I)$$

are flat for $m = 0, \dots, n$.

PROOF. The 'only if'-part was proved in Corollary 3.12. For the 'if'-part, we proceed by induction on n, the case $n = 0$ being a consequence of Lemma 3.13, since $F_{p^0}(R)/(F_{p^0}(R) \cap I) = (R[\mathfrak{X}]/I)_1$. To verify the statement for $n > 0$ assume $R[\mathfrak{X}]/I^{\le p^{n-1}}$ is R-flat, and consider the exact sequence

$$0 \to F_{p^n}(R)/(F_{p^n}(R) \cap I) \to R[\mathfrak{X}]_{p^n}/I_{p^n} \to \\ \to R[\mathfrak{X}]_{p^n}/(F_{p^n}(R) + (I^{\le p^{n-1}})_{p^n}) \to 0.$$

Since the right-hand module is R-flat by Corollary 3.12, and the left-hand module is R-flat by hypotheses, the same holds for the middle module. Now use (2) of Remark 3.14. □

We are now prepared to show that twisted-linearity is a closed condition on the Hilbert scheme H^h of admissible ideals in $R[\mathfrak{X}]$. In order to give this statement a precise meaning and a neat formulation, we generalize the notion of twisted-linearity to sheaves of ideals:

DEFINITION 3.16. *Let X be any k-scheme, and let $\mathcal{I} \subset \mathcal{O}_X[\mathfrak{X}]$ be a sheaf of ideals. We say $\mathcal{I}$ is twisted-linear, if for every open affine subscheme $Y = \operatorname{Spec} R$ of X, $\Gamma(Y, \mathcal{I}) \subset R[\mathfrak{X}]$ is a twisted-linear ideal.*

REMARK 3.17. *Twisted linearity can be tested on an affine open covering: A sheaf of ideals $\mathcal{I}$ as in the previous definition is twisted-linear, if and only if there exists a covering by open affines, $X = \cup Y_i, Y_i = \operatorname{Spec} R_i$, such that every $I_i = \Gamma(Y_i, \mathcal{I}) \subset R_i[\mathfrak{X}]$ is a twisted-linear ideal. The 'only if'-part is of course trivial. To verify the 'if'-part, we can assume that $X = \operatorname{Spec} R$ is affine. Then $I = \Gamma(X, \mathcal{I}) \subset R[\mathfrak{X}]$ is twisted-linear if and only if*

$$\langle I \cap F_{p^n}(R), n \in \mathbb{N} \rangle = I.$$

(That the property of being finitely generated, which we require for twisted-linear ideals, is local for the Zariski-topology on $\operatorname{Spec} R$, even

for the faithfully flat topology, is a well-known fact.) This equation holds, by flatness of $R \to R_i$, if and only if it holds after localizing in R_i for every i. But, using Corollary 3.11, this is twisted-linearity on each Y_i, which was our assumption.

PROPOSITION 3.18. *Let $p : X \to H^h$ be a morphism of k-schemes such that the corresponding sheaf of ideals on X is twisted-linear. Assume that $p_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{H^h_k}$-module and let Y be the scheme-theoretic image of p. Then the corresponding sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y[\mathfrak{X}]$ is twisted-linear.*

PROOF. The defining ideal-sheaf of $Y \subset H^h_k$ is equal to $\ker(\mathcal{O}_H \to p_*\mathcal{O}_X)$. Hence to the map $f : X \to Y$ corresponds an *injective* map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and, in particular, on an affine open subset $\operatorname{Spec} R \subset Y$ we get $R \hookrightarrow \mathcal{O}_X(f^{-1}(\operatorname{Spec} R))$. Covering $f^{-1}(\operatorname{Spec} R) = \cup \operatorname{Spec} S_i$ by open affines, we obtain an injective map of rings

$$R \hookrightarrow \mathcal{O}_X(f^{-1}(\operatorname{Spec} R)) \hookrightarrow \prod S_i =: S.$$

By assumption, the ideal J_i corresponding to

$$\operatorname{Spec} S_i \hookrightarrow f^{-1}(\operatorname{Spec} R) \to X \to H^h$$

is twisted-linear. We have to verify that the same holds for the ideal I corresponding to $\operatorname{Spec} R \hookrightarrow Y \to H^h$. Then the claim follows from the remark, as Y can be covered by open affines like $\operatorname{Spec} R$.

We inductively construct a generating system for I consisting of twisted-linear elements. First note that any element of degree 1 is trivially twisted-linear, i.e. $I^{\leq 1}$ is twisted linear. For the inductive step, assume that $I^{\leq p^{n-1}}$ is twisted linear. Then, by the remarks following Lemma 3.13, $M := R[\mathfrak{X}]_{p^n}/(F_{p^n}(R) + (I^{\leq p^{n-1}})_{p^n})$ is projective over R, whence, in the line below, the middle map is injective:

$$I_{p^n} \to M \hookrightarrow M \otimes_R S \xrightarrow{\simeq} \prod (M \otimes_R S_i). \tag{3.3.1}$$

To justify that the right hand map is an isomorphism, note that, if M is even free, it is a finite product of copies of R, and the above map is indeed an isomorphism since arbitrary products commute. In general, M is a direct summand (and hence also a direct factor) of a finitely generated free R-module $M \oplus M'$. This shows that $(M \otimes_R S) \oplus (M' \otimes_R S) \simeq (\prod M \otimes_R S_i) \oplus (\prod M' \otimes_R S_i)$. Since the composition of the maps in (3.3.1) is zero ($I \otimes_R S_i = J_i$ being twisted-linear), so is the left-hand map. This shows that, in order to obtain a generating system of $I^{\leq p^n}$, we can extend a generating system of $I^{\leq p^{n-1}}$ by twisted-linear elements living in $F_{p^n}(R)$. This proves the inductive step. $\square$

COROLLARY 3.19. *The sheaf of ideals corresponding to $\mathcal{D}(\lambda) \subset H^h$ is twisted-linear.*

PROOF. The ideal $I(\lambda)$ is twisted-linear. By definition of $R[[z]]^F$, the operation of $\mathrm{L}_F^+ \mathrm{Sl}_n$ on H^h preserves twisted-linearity, whence the sheaf of ideals corresponding to the inclusion $p : \mathcal{O}(\lambda) \subset H^h$ is twisted-linear. Now the claim follows from Proposition 3.18, since $\mathcal{D}(\lambda)$ is the closure (=scheme-theoretic image) of $\mathcal{O}(\lambda)$ in H^h. □

3.4. Grassmann-Bundles

From now on, we write $\mathfrak{X} = \{x_{i,j}; i = 1, \dots, n; j = 0, \dots, N-1\}$, i.e. $R[\mathfrak{X}]$ is the affine coordinate ring of $\mathfrak{P}^n_{N,R}$ with the grading $\deg x_{i,j} = p^j$, introduced in Section 3.1. Moreover, we denote by $z^\#$ the comorphism of multiplication by z on $\mathfrak{P}^n_{N,R}$, i.e. the ring homomorphism

$$z^\# : R[\mathfrak{X}] \to R[\mathfrak{X}]; \quad x_{i,j} \mapsto x^p_{i,j-1}.$$

In the previous section we introduced the ideals $I^{\leq p^i}$, $i = 0, 1, \dots$, associated with a twisted linear admissible ideal $I \subset R[\mathfrak{X}]$. If we consider ideals I with the property that $z^\# I^{\leq p^i} \subset I^{\leq p^{i-1}}$ (which is the case for R-valued points of $\mathcal{D}(\lambda)$), the assignments $I \to I^{\leq p^i}$ have a nice geometric interpretation, which we are going to describe in the sequel.

Denote by h_m the Hilbert function of $k[\mathfrak{X}]/I(\lambda)^{\leq p^{m-1}}$ for $m = 0, \dots, N$. In particular, h_0 is the Hilbert function of $k[\mathfrak{X}]$ (namely, $I(\lambda)$ contains no elements of degree $\leq p^{-1}$ whence $I^{\leq p^{-1}} = 0$), and $h_N = h$, the Hilbert function of $k[\mathfrak{X}]/I(\lambda)$. For $m = 0, \dots, N$ denote by $\mathcal{T}_m(\lambda)$ the following (set)-valued functor on the category of k-algebras:

$$\begin{aligned}(3.4.1)\quad \mathcal{T}_m(\lambda)(R) = \{&\text{admissible twisted linear ideals } I \subset R[\mathfrak{X}] \\ &\text{with Hilbert function } h_m \text{ and such that } z^\# I^{\leq i} \subset I^{\leq i-1}\}.\end{aligned}$$

Note that these functors are sheaves for the Zariski-topology by Remark 3.17. Note further that $\mathcal{T}_0(\lambda) = \operatorname{Spec} k$ and $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$ as functors. Moreover, we have morphisms of functors

$$\mathcal{T}_{m+1}(\lambda) \to \mathcal{T}_m(\lambda); \quad I \mapsto I^{\leq p^{m-1}},$$

once we know that the Hilbert function of $R[\mathfrak{X}]/I$, for I admissible, twisted linear and generated in degrees $\leq p^{m-1}$, is completely determined by its values on $\{0, \dots, p^{m-1}\}$. But this follows from

PROPOSITION 3.20. *The Hilbert function of an admissible twisted linear ideal $I \subset R[\mathfrak{X}]$, which is generated in degrees $\leq p^m$, determines the ranks of the R-modules $F_{p^0}(R) \cap I, \dots, F_{p^m}(R) \cap I$ and vice versa.*

PROOF. By Corollary 3.12 the R-modules $F_{p^i}(R)/F_{p^i}(R) \cap I$ are flat, whence the statement of the proposition makes sense. Consider first ideals $I \subset k[\mathfrak{X}]$ having a *monomial* twisted linear generating system that is, a generating system consisting of monomials $\xi_1, \ldots, \xi_l$, each a power of a variable $x_{i,j}$ and of degree $\leq p^m$. Hence, if the ξ_i form a minimal generating system of I (i.e. none of them can be omitted), they also form a regular sequence in I. Moreover, the number of elements in $\{\xi_i\}$ having a given degree is determined by the dimensions of the vector spaces $F_{p^i}(k) \cap I, i = 0, \ldots, m$. On the other hand Hilbert functions are additive with respect to short exact sequences as

$$0 \to k[\mathfrak{X}]/\langle \xi_1, \ldots, \xi_{i-1} \rangle \xrightarrow{\cdot \xi_i} k[\mathfrak{X}]/\langle \xi_1, \ldots, \xi_{i-1} \rangle \to k[\mathfrak{X}]/\langle \xi_1, \ldots, \xi_i \rangle \to 0.$$

Thus the Hilbert function of $R[\mathfrak{X}]/I$ is determined by the dimensions of $F_{p^i}(k) \cap I, i = 0, \ldots, m$, and vice versa, as claimed. Now we show that the general case reduces to the case just studied: By flatness of $R[\mathfrak{X}]/I$ and by corollaries 3.12 and 3.11 we may assume $R = k$. Consider the action on $H_k^{h'}$ (where h' denotes the Hilbert function of I) of the 1-parameter subgroup

$$\{D(t) = \operatorname{diag}(t^{-n+1}, t^{-n+3}, \ldots, t^{n-1}); t \in k\} \subset T \subset \mathrm{Sl}_n(k).$$

By properness of $H_k^{h'}$ the orbit of I extends to a closed curve $C \subset H_k^{h'}$, and by Proposition 3.18 the ideal I' corresponding to $t = 0$ is generated by twisted linear homogeneous elements of degrees $\leq p^m$. Let $\sum_{i=1}^{n} P_i(x_{i,0}, \ldots, x_{i,N-1})$ be such an element, the $P_i(x_{i,0}, \ldots, x_{i,N-1})$ denoting polynomials of the same degree d over k in N variables, or the zero polynomial. Then, by construction, the multiplicative group acts (via $D(t)$) on P_i with weight $(-n - 1 + 2i)d$. Since I' is a fixed point under the action of $D(t)$, this shows that each of the summands $P_i(x_{i,0}, \ldots, x_{i,N-1})$ is itself contained in I'. But from this it follows easily that I' is even a monomial ideal generated by elements of the form $x_{i,j}^{p^{e_{i,j}}}$ for some non-negative integers $e_{i,j}$. Namely, if $P(x_{i,0}, \ldots, x_{i,N-1}) = a_e x_{i,0}^{p^{d+e}} + \cdots + a_0 x_{i,e}^{p^d}$ with $a_0 \neq 0$, then we have $x_{i,j}^{p^{d+e-j}} \in I'$ for every $0 \leq j \leq e$, since I' is stable under the map $z^{\#} : x_{i,j} \mapsto x_{i,j-1}^p$. Of course, the Hilbert functions of $k[\mathfrak{X}]/I$ and $k[\mathfrak{X}]/I'$ coincide, and the application of Corollary 3.11 to the specializations $\operatorname{Spec} k \to C$ given by $t = 0$ and $t = 1$, respectively, shows that also the dimensions of $F_{p^i}(k) \cap I$ and $F_{p^i}(k) \cap I'$ coincide for $i = 0, \ldots, m$. Thus we are indeed reduced to the special case of monomial ideals. $\square$

For the following recall the standard decomposition of λ defined at the end of Section 3.2.

THEOREM 3.21. *For each $m = 0, \ldots, N$ the morphism $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is relatively representable by a bundle of ordinary Grassmannians* $\mathrm{Grass}(|\mu_m|, n)$*. In particular, it is smooth of relative dimension* $|\mu_m|(n - |\mu_m|)$.

COROLLARY 3.22. *The functor $\mathcal{T}_N(\lambda)$ is representable by a smooth, connected, projective k-scheme of dimension $\sum_{m=1}^{N}|\mu_m|(n - |\mu_m|) = \dim \tilde{\Sigma}(\mu_1, \ldots, \mu_N)$. The functorial map $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$ is thus a closed immersion of k-schemes.*

PROOF. Recall that $\mathcal{T}_0(\lambda) = \operatorname{Spec} k$ and use induction on N. □

PROOF OF THE THEOREM. Let I be any R-valued point of $\mathcal{T}_{m-1}(\lambda)$ and consider the exact sequence of flat R-modules

$$0 \to K := \ker(z^{\#}) \to F_{p^{m-1}}(R)/(F_{p^{m-1}}(R) \cap I) \xrightarrow{z^{\#}}$$
$$\to F_{p^{m-1}}(R)/(F_{p^{m-1}}(R) \cap I) \to \sum_{i=1}^{n} Rx_{i,m-1} \to 0.$$

Then K is flat over R, and hence locally free since it is of finite presentation, and of rank n. By Proposition 3.20 and Corollary 3.15 the fiber of $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is in functorial bijection with the set of R-submodules $L \subset K$ such that K/L is flat over R and of constant rank

(3.4.2)
$$\operatorname{rk}(I(\lambda)^{\le p^{m-1}}/I(\lambda)^{\le p^{m-2}})_{p^{m-1}} = \operatorname{rk} I(\lambda)_{p^{m-1}} - \operatorname{rk} I(\lambda)_{p^{m-2}} = |\mu_m|.$$

But these are exactly the R-valued points of a $\mathrm{Grass}(|\mu_m|, n)$-bundle over $\operatorname{Spec} R$. In particular, the relative dimension of $\mathcal{T}_m(\lambda) \to \mathcal{T}_{m-1}(\lambda)$ is equal to $(|\mu_m|)(n - |\mu_m|)$. □

3.5. Demazure Varieties Reviewed

3.5.1. The relation with Demazure varieties. Recall the ideal $I(\lambda)$ of the standard lattice scheme associated to λ and its orbit closure $\mathcal{D}(\lambda)$. Denote by $h : \mathbb{N} \to \mathbb{N}$ the Hilbert function of $I(\lambda)$. Furthermore, let N be as in the definition of the standard lattice scheme, and, as in the previous section, let $\mathfrak{X} = \{x_{i,j}; i = 1, \ldots, n; j = 0, \ldots, N-1\}$.

Denote by $\mathrm{Frob}_R : R[\mathfrak{X}] \to R[\mathfrak{X}]$ the relative Frobenius morphism over R. Let F^* be the pullback-functor on the category of R-modules along the absolute Frobenius morphism $R \to R$, whose effect on a submodule of $R[\mathfrak{X}]$ is raising the coefficients of its elements to the p-th power. Note that applying Frob_R to a submodule of $R[\mathfrak{X}]$ and then pulling the image back via F^* yields the image of the original submodule under the *absolute* Frobenius. The pullback functor F^* induces an $\mathbb{F}_p$-morphism $\varphi : \mathcal{G}^{(N)}_{\mathrm{Gl}_n} \to \mathcal{G}^{(N)}_{\mathrm{Gl}_n}; \mathcal{L} \mapsto F^*(\mathcal{L})$. We set

$$\Phi := \varphi^{N-1} \times \cdots \times \varphi \times \mathrm{id} : \prod_{i=1}^{N} \mathcal{G}^{(N)}_{\mathrm{Gl}_n} \to \prod_{i=1}^{N} \mathcal{G}^{(N)}_{\mathrm{Gl}_n}.$$

Furthermore set $M := F_{p^{N-1}}(R)$ and note that on M we have an endomorphism $z^\#$ which is defined by restriction of scalar multiplication with z on $R[\mathfrak{X}]$: it sends $x_{i,j}^{p^{N-1-j}}$ to $x_{i,j-1}^{p^{N-j}}$ if $j > 0$ and to 0 otherwise. Let $\{e_{i,j}\}$ be the dual basis of $\{x_{i,j}^{p^{N-1-j}}\}$ and consider the dual map of $z^\#$, which we again denote by z:

$$z : V := \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, R) \simeq V; e_{i,j-1} \mapsto e_{i,j}.$$

Note that $V \simeq R^{nN}$, as well as its R-submodules with projective quotient which are stable under z, are lattices in the sense of Section 2.2 (they correspond to precisely those lattices $\mathcal{L} \subset R[[z]]^n$ which contain $z^N R[[z]]^n$).

We make an additional remark before stating the main result of this section: On $\mathcal{D}(\lambda)$ there is an action of the positive loop group $\mathrm{L}^+\mathrm{Sl}_n$ given as the pullback of the natural $\mathrm{L}^+_F\mathrm{Sl}_n$-action via the morphism in eq. (3.2.1). On the other hand, the morphism $\mathrm{L}^+\mathrm{Sl}_n \times_k k \to \mathrm{L}^+\mathrm{Sl}_n$, where the fiber product ist taken over $k \to k; x \mapsto x^{p^{N-1}}$, defines by composition an action of $\mathrm{L}^+\mathrm{Sl}_n \simeq \mathrm{L}^+\mathrm{Sl}_n \times_k k$ on the affine Grassmannian.

Theorem 3.23. *Let $\Sigma(\lambda) = \tilde{\Sigma}(\mu_1, \ldots, \mu_N)$ denote the variety of lattice chains defined in Section 2.2, with $\mu_1, \ldots, \mu_N$ the standard decomposition of λ. Then there is a closed immersion $\iota : \mathcal{D}(\lambda) \to \prod_{i=1}^N \mathcal{G}^{(N)}_{\mathrm{Gl}_n}$, such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{D}(\lambda) \arrow[r, "\iota"] \arrow[d, "\sigma"] & \prod_{i=1}^N \mathcal{G}^{(N)}_{\mathrm{Gl}_n} \arrow[d, "\Phi"] \\
\Sigma(\lambda) \arrow[r, hook] & \prod_{i=1}^N \mathcal{G}^{(N)}_{\mathrm{Gl}_n}.
\end{tikzcd}$$

The map σ is a universal homeomorphism and the map from $\mathcal{D}(\lambda)$ to the fiber product of $\Sigma(\lambda)$ with $\prod_{i=1}^N \mathcal{G}^{(N)}_{\mathrm{Gl}_n}$ is a nil-immersion. Moreover, σ is equivariant for the above defined actions of $\mathrm{L}^+\mathrm{Sl}_n$.

Proof. Recall that every R-valued point of $\mathcal{D}(\lambda)$ is a twisted-linear ideal I by Corollary 3.19. For $l = 1, \ldots, N$ we set $L_l := \mathrm{Frob}_R^{N-l}(F_{p^{l-1}}(R) \cap I)$ and $\mathcal{L}_l := \mathrm{Hom}_R(M/L_l, R)$ (Compare Example 3.7 at the beginning of the previous section!). Then by Corollary

3.12, the modules L_l are projective. By Lemma 3.10 this assignment is functorial, and finally, since any twisted-linear ideal I is generated by the sets $F_{p^l}(R) \cap I$, we have a functorial *injection.* Since split exact sequences are preserved by both Frob_R and $\mathrm{Hom}_R(-, R)$, $V/\mathcal{L}_l$ is again projective, whence the map $\iota : I \mapsto (\mathcal{L}_1, \dots, \mathcal{L}_N)$ is well-defined. Note furthermore the equation $F^* L_{l-1} = \mathrm{Frob}_R^{N-l} \mathrm{Frob}(F_{p^{l-2}}(R) \cap I) \subset L_l$. It implies

(3.5.1)

$$F^* \mathcal{L}_{l-1} = F^* \mathrm{Hom}_R(M/L_{l-1}, R) = \mathrm{Hom}_R(M/F^* L_{l-1}, R) \supset \mathcal{L}_l,$$

which proves that the image of $\Phi \circ \iota$ indeed consists of descending lattice chains. Again by projectivity, the rank of successive quotients $F^* \mathcal{L}_{l-1}/\mathcal{L}_l$ is constant on $\mathrm{Spec}\, R$, i.e. equal to $|\mu_l|$ by eq. (3.4.2). Hence the map σ. It is easily seen that σ is equivariant for the above-mentioned actions.

It remains to check that the immersion

$$\alpha : \mathcal{D}(\lambda) \to S := \Sigma(\lambda) \times_{\prod_{i=1}^N \mathcal{G}_{Gl_n}^{(N)}} \prod_{i=1}^N \mathcal{G}_{Gl_n}^{(N)}$$

of k-varieties is indeed a nil-immersion, i.e. that it is surjective. (Since $S \to \Sigma(\lambda)$ is a universal homeomorphism, this will imply that σ is a universal homeomorphism as well.) But $\mathrm{L}^+ \mathrm{Sl}_n$-equivariance shows that the image of $\mathcal{O}(\lambda)$ is dense in $\Sigma(\lambda)$, and in particular both have the same dimension. By finiteness of Φ we see that α has dense image as well, and is therefore surjective. □

Let us recall the iterated bundle of Grassmannians $\mathcal{T}_N(\lambda)$ defined in Section 3.4. In Corollary 3.22 we saw that there is a natural closed immersion $\mathcal{D}(\lambda) \hookrightarrow \mathcal{T}_N(\lambda)$, where the dimension of the latter equals the dimension of $\Sigma(\lambda)$. But in Theorem 3.23 we have now seen that this is also the dimension of $\mathcal{D}(\lambda)$. Thus we have proved

COROLLARY 3.24. *The varieties $\mathcal{D}(\lambda)$ and $\mathcal{T}_N(\lambda)$ are equal. In particular, $\mathcal{D}(\lambda)$ is an iterated bundle of ordinary Grassmannians.* □

EXAMPLE 3.25. *We illustrate the theorem by calculating explicitly the smallest nontrivial example: Let $n = 2$ and $N = 2$. We are thus dealing with ideals in the polynomial ring $R[x_0, x_1, y_0, y_1]$ for some k-algebra R. Let $\lambda = (1, -1) \in \check{\mathrm{X}}_+(T)$ whence $\tilde{\lambda} = (2, 0)$, $\mu_1 = \mu_2 = (1, 0)$ and $I(\lambda) = (x_0, x_1)$. (Note that this is a continuation of Example 3.7.)*

PROPOSITION 3.26. *In the above situation the variety* $\mathcal{D}(\lambda)$ *is isomorphic to the projective space bundle* $X = \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p))$. *The boundary of the big cell is the divisor* $\mathbb{P}^1_k \times_k \{\infty\}$.

Note that X *can be explicitly constructed by gluing two copies of* $U = \mathbb{A}^1_k \times \mathbb{P}^1_k$ *via the self-inverse isomorphism*

$$\chi : (\mathbb{A}^1_k - \{0\}) \times \mathbb{P}^1_k \xrightarrow{\simeq} (\mathbb{A}^1_k - \{0\}) \times \mathbb{P}^1_k; (a, (c : d)) \mapsto (1/a, (c : a^{2p}d)).$$

We will use this description in the

PROOF OF THE PROPOSITION. We define two maps on the level of R-valued points:

$$\begin{aligned} \varphi : U(R) &\to \mathcal{D}(\lambda)(R) \\ (a, (c : d)) &\mapsto (ax_0 + y_0, cx_0^p + da^p x_1 + dy_1) \\ \psi : U(R) &\to \mathcal{D}(\lambda)(R) \\ (a, (c : d)) &\mapsto (x_0 + ay_0, -cy_0^p + da^p y_1 + dx_1) \end{aligned}$$

The maps $\varphi(R)$ and $\psi(R)$ are clearly functorial in R, so they constitute morphisms of k-schemes. Furthermore, observe that $\varphi \circ \chi = \psi$ on $(\mathbb{A}^1_k - \{0\}) \times \mathbb{P}^1_k$. This shows that φ and ψ give a morphism $f : X \to \mathcal{D}(\lambda)$. To show that this is an isomorphism, we have to find an inverse $f(R)^{-1}$ which is functorial in R. But by Proposition 3.19 every R-valued point of $\mathcal{D}(\lambda)$ is twisted-linear, i.e. locally of the form $(ax_0 + by_0, cx_0^p + da^p x_1 + db^p y_1)$ or $(ax_0 + by_0, -cy_0^p + da^p x_1 + db^p y_1)$ (depending on whether $b \neq 0$ or $a \neq 0$) with a or b a unit in R and c or d a unit in R. Such an ideal defines in a functorial way an R-valued point of X. Obviously, the divisor $\{d = 0\} = \mathbb{P}^1_k \times_k \{\infty\}$ maps bijectively to the boundary of the big cell: it parametrizes the ideals of the form $(ax_0 + y_0, x_0^p)$ or $(x_0 + ay_0, y_0^p)$. □

Similarly, we compute the Demazure-variety $\Sigma(\lambda)$. *Its* k*-valued points are descending chains of subspaces* $k^{2\times 2}_{x_0,x_1,y_0,y_1} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2$ *which are stable under multiplication by* z: $x_0 \mapsto x_1, y_0 \mapsto y_1$ *and of codimension* 1 *and* 2, *respectively. Hence one sees like in the proposition above that* $\Sigma(\lambda) \simeq \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2))$, *using two charts on* R*-valued points*

$$\begin{aligned} \varphi' : U(R) &\mapsto \mathcal{D}_\mu(R) \\ (a, (c : d)) &\mapsto (ax_0 + y_0, cx_0 + dax_1 + dy_1) \\ \psi' : U(R) &\mapsto \mathcal{D}_\mu(R) \\ (a, (c : d)) &\mapsto (x_0 + ay_0, -cy_0 + day_1 + dx_1), \end{aligned}$$

and gluing them via $(a, (c : d)) \mapsto (1/a, (c : a^2 d))$.

Since the Frobenius morphism commutes with dualization, to compute the map $\sigma : \mathcal{D}(\lambda) \to \Sigma(\lambda)$ of Theorem 3.23 explicitly, we may compute in terms of defining relations of lattices, instead of generators of lattices. Hence, using the charts from above, we immediately obtain the picture

$$\begin{tikzcd}
\mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p)) \arrow[r, "\sigma"] \arrow[d] & \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2)) \arrow[d] \\
\mathbb{P}^1_k \arrow[r, "Frob"] \arrow[d] & \mathbb{P}^1_k \arrow[d] \\
\operatorname{Spec} k \arrow[r, "Frob"] & \operatorname{Spec} k,
\end{tikzcd}$$

where the map σ is given on the respective charts by

$$(a,(c:d)) \mapsto (a^p,(c:d)).$$

It is elementary to check that the upper square is cartesian, by looking at the transition functions defining the respective line bundles. This is in agreement with Theorem 3.23: namely, a diagram relating this with the square of Theorem 3.23 has the form

$$\begin{tikzcd}
\mathcal{D}(\lambda) \arrow[r, "\iota"] \arrow[d, "\sigma"] \arrow[rrr, bend left] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \times \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[r, "\pi_1"] \arrow[d, "\varphi\times\mathrm{id}"] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[d, "\varphi"] & \mathbb{P}^1 \arrow[l] \arrow[d, "\mathrm{Frob}"] \\
\Sigma(\lambda) \arrow[r, hook] \arrow[rrr, bend right] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \times \mathcal{G}^{(2)}_{\mathrm{Gl}_2} \arrow[r, "\pi_1"] & \mathcal{G}^{(2)}_{\mathrm{Gl}_2} & \mathbb{P}^1. \arrow[l]
\end{tikzcd}$$

Here the horizontal maps on the right are given by identifying $\mathbb{P}^1_k$ with the space of lattices $\mathcal{L}_1$ with $z\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_0$. Note that the middle square is trivially cartesian, while the right hand square is not. Thus also the left hand square fails to be cartesian and the map α of Theorem 3.23 is indeed a nontrivial nil-immersion.

3.5.2. Demazure varieties as schemes of lattices with infinitesimal structure. The k-valued points of $\mathcal{D}(\lambda)$ are lattice schemes in $\mathfrak{P}^n_N$ in the sense of Definition 3.3, which, in general, will not be reduced. For instance, in the situation of Example 3.25, the set of points in $\mathcal{D}((1,-1)) \simeq \mathrm{Proj}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2p))$ which have non-reduced fibers in the universal family, is exactly the divisor $d = 0$ (the boundary of the big cell $\mathcal{O}(\lambda)$). The corresponding lattice schemes have the form $(ax_0 + by_0, x_0^p, y_0^p)$ with a and b not both zero. In general, we obtain

COROLLARY 3.27. *If k is a perfect field of characteristic p, then the map $\sigma : \mathcal{D}(\lambda) \to \Sigma(\lambda)$ defines a bijection of k-valued points. In particular, for any lattice $\mathcal{L} \in \mathcal{S}(\lambda)$, the fiber of $\mathcal{L}$ in $\Sigma(\lambda)$ can be interpreted as a variety of infinitesimal structures on $\mathcal{L}$. The lattice schemes in $\mathcal{D}(\lambda)$ which have non-trivial infinitesimal structure are exactly those lying in the boundary of the big cell $\mathcal{O}(\lambda)$.*

PROOF. Only the very last assertion requires a proof: by construction, the lattices in $\mathcal{O}(\lambda)$ are reduced (since $I(\lambda)$ is). On the other hand, take any lattice $\mathcal{L}$ in the boundary of $\mathcal{O}(\lambda)$. Such a lattice maps to the boundary of $\mathcal{S}(\lambda)$ under $\mathcal{D}(\lambda) \to \Sigma(\lambda) \to \mathcal{S}(\lambda)$ by the $\mathrm{L}^+ \mathrm{Sl}_n$-equivariance of σ (see proof of Theorem 3.23). Thus its reduced structure corresponds to a point in $\mathcal{O}(\lambda')$ for some $\lambda' < \lambda$ (Bruhat-order), whence it has a Hilbert function different from that of $I(\lambda)$. Thus, by constancy of the Hilbert function on $\mathcal{D}(\lambda)$, the reduced structure of $\mathcal{L}$ cannot belong to $\mathcal{D}(\lambda)$. $\square$

Finally, let us briefly study the invariants of lattices in $\mathbb{L}^h$. (If $d_1, \ldots, d_n$ denote the elementary divisors of a lattice $\mathcal{L} \subset k((z))^n$, then by the *invariants* of $\mathcal{L}$ we mean the vector $(\mathrm{val}_z\, d_1, \ldots, \mathrm{val}_z\, d_n)$ with entries ordered by decreasing size.) We will need the following purely combinatorial lemma about partitions of a given positive integer. For an n-tuple $\sigma \in \mathbb{N}^n$ we write $|\sigma| = \sum_{i=1}^n \sigma_i$.

LEMMA 3.28. *Let $\sigma, \sigma' \in \mathbb{N}^n$ with $|\sigma| = |\sigma'|$ and such that $\sigma_i \geq \sigma_{i+1}$ and $\sigma'_i \geq \sigma'_{i+1}$ for all $1 \leq i \leq n$. Let N be the maximum of all the σ_i and σ'_i and set*

$$\tau_i := \textit{number of entries in } \sigma \textit{ which are} \geq i,$$
$$\tau'_i := \textit{number of entries in } \sigma' \textit{ which are} \geq i,$$

for $i = 1, \ldots, N$ (the 'dual partitions' for σ and σ'). Then, with respect to the Bruhat order, $\sigma \geq \sigma'$ if and only if $\tau \leq \tau'$.

PROOF. By definition, $\tau \leq \tau'$ if and only if for all $1 \leq i \leq N$, $0 \leq \sum_{j=1}^i (\tau'_j - \tau_j)$. Since $|\tau| = |\tau'|$, this is equivalent to $0 \leq \sum_{j=i}^N (\tau_j - \tau'_j) = \sum_{j=i}^N (\#\{l \mid \sigma_l \geq j\} - \#\{l \mid \sigma'_l \geq j\}) = \sum_{l=1}^n (\max\{\sigma_l, i\} - \max\{\sigma'_l, i\})$. We show that this holds if $\sigma \geq \sigma'$: Let $1 \leq r \leq n$ be the largest index such that both $\sigma_r \geq i$ and $\sigma'_r \geq i$, and let $1 \leq s \leq n$ be the largest index such that σ_s or σ'_s is $\geq i$. If $\sigma_{r+1} < i$, then we have

$$\sum_{j=i}^N (\tau_j - \tau'_j) = \sum_{l=1}^r (\sigma_l - \sigma'_l) + \sum_{l=r+1}^s (i - \sigma'_l) \geq \sum_{l=1}^s (\sigma_l - \sigma'_l).$$

On the other hand, if $\sigma'_{r+1} < i$ we obtain

$$\sum_{j=i}^{N}(\tau_j - \tau'_j) = \sum_{l=1}^{r}(\sigma_l - \sigma'_l) + \sum_{l=r+1}^{s}(\sigma_l - i) \geq \sum_{l=1}^{r}(\sigma_l - \sigma'_l).$$

Both expressions are thus non-negative if $\sigma \geq \sigma'$, which proves one direction. The other implication holds by duality: τ (resp. τ') can be regarded as the dual partition for σ (resp. σ'). Thus all the arguments remain valid if we interchange σ and τ' (resp. σ' and τ). □

Let $h : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of $I(\lambda)$ and let $V \subset \mathfrak{P}^n_N$ be a lattice scheme corresponding to a k-valued point in $\mathbb{L}^h$. Since k is assumed to be perfect, the map in (3.1.2) for $\bar{R} = k$ is bijective, whence $k[[z]]^F$ is by definition isomorphic to $k[[z]]$. Via this isomorphism we can regard the set $V(k)$ of k-valued points as a lattice $z^N k[[z]]^n \subset \mathcal{L} \subset z^{-N} k[[z]]^n$ (after multiplication by a suitable power of z). From Lemma 3.28 we obtain

Corollary 3.29. *Assume that the defining ideal of V is twisted-linear, and let λ' denote the invariants (ordered by decreasing size) of the corresponding lattice $\mathcal{L} \subset k((z))^n$. Then $\lambda' \leq \lambda$.*

Proof. Let $I(V)$ denote the defining ideal of V. Multiplying with a suitable matrix in $\mathrm{Sl}_n(k[[z]]^F)$ we may assume that V_{red} is given by the ideal $I(V)_{red} = \langle x_{1,0}, \ldots, x_{1,\tilde{\lambda}'_1-1}, \ldots, x_{n,0}, \ldots, x_{n,\tilde{\lambda}'_n-1}\rangle$ for some $\tilde{\lambda}' \in \mathbb{N}^n$ with $\tilde{\lambda}'_i \geq \tilde{\lambda}'_{i+1}$. Since $I(V)$ is twisted linear, its Hilbert function h determines the dimension of the k-vector space $I(V) \cap F_{p^N}(k) = I(V)_{red} \cap F_{p^N}(k)$: it is therefore equal to the dimension of $I(\lambda) \cap F_{p^N}(k)$, whence the determinants of the respective lattices conincide. In other words: $|\tilde{\lambda}| = |\tilde{\lambda}'|$. Now we have to show that $\tilde{\lambda} \geq \tilde{\lambda}'$, or equivalently (by Lemma 3.28) $\mu \leq \mu'$, where μ and μ' are the respective duals in the sense of Lemma 3.28. Since $\mu_i = \dim_k I(V)_{p^{i-1}} \cap F_{p^{i-1}}(k)$, while $\mu'_i = \dim_k (I(V)_{red})_{p^{i-1}} \cap F_{p^{i-1}}(k)$, the claim follows. □

Thus the k-valued points in $\mathbb{L}^h$ which are given by twisted linear ideals correspond to lattices with invariants $\leq \lambda$. (I don't know, if every k-valued valued point in $\mathbb{L}^h$ is twisted-linear.) Thinking of the analogous situation for Schubert varieties in the affine Grassmannian (where the Schubert variety $\mathcal{S}(\lambda)$ parametrizes exactly those lattices with invariants $\leq \lambda$) one could be tempted to think that $\mathbb{L}^h$ and $\mathcal{D}(\lambda)$ coincide. However, looking once more at the simple situation of Example 3.25, we see that this is in general not the case:

Let $P = k[x_0, x_1, y_0, y_1]$, $\lambda = (2, 0)$ and set $I = (y_0, x_0^p) \in \mathcal{T} := \mathcal{T}_2$, $\mathbb{L}^h := \mathbb{L}^h_{k[[z]]^2_1}$. Note that an infinitesimal deformation $\tilde{I} \in \mathbb{L}^h(k[\epsilon]/\epsilon^2)$

of I is given by 2 generators with indeterminate coefficients $a, b, c \in k$:

$$g_1 := y_0 + \epsilon a x_0,$$
$$g_2 := x_0^p + \epsilon(b y_1 + c x_1).$$

Hence $\dim \mathrm{T}_I \mathbb{L}^h = 3$, while the dimension of $\mathrm{T}_I \mathcal{T}$ is 2, e.g. by Theorem 3.21. (In a more elementary way, we could argue that on $\mathrm{T}_I \mathcal{T}$ we have the additional condition that $z^{\#} g_2 \subset (g_1)$, which means that $\epsilon(b y_0^p + c x_0^p)$ must be a multiple of $g_1^p = y_0^p$. This forces $c = 0$, whence again $\dim \mathrm{T}_I \mathcal{T} = 2$).

Of course, this exhibits only a difference in the infinitesimal structures at the point $I(\lambda)$. But also the topological spaces of $\mathbb{L}^h$ and $\mathcal{D}(\lambda)$ differ in general: Let $n = 4, \lambda = (1, 1, -1, -1)$, whence $N = 2$ and $\tilde{\lambda} = (2, 2, 0, 0)$. Then $I(\lambda) = \langle x_0, x_1, y_0, y_1 \rangle \subset k[x_i, y_i, z_i, w_i; i = 0, 1]$. On the other hand, consider the twisted-linear ideal $I = \langle x_0, y_0, z_0^p, z_1 \rangle$ in the same polynomial ring. Certainly, it has the same Hilbert function as $I(\lambda)$, and it defines a lattice scheme with invariants $(1, 0, 0, -1)$. Thus it is a k-valued point of $\mathbb{L}^h$. However, it violates the condition $z^{\#} I = z^{\#} I^p \subset I^{\leq 1}$ which, by (3.4.1), is satisfied by points of $\mathcal{T}_2(\lambda) = \mathcal{D}(\lambda)$.

Part 2

Mixed Characteristic

CHAPTER 4

Discussion of Haboush's Approach

In this chapter we will discuss Haboush's paper [**Hab05**], in which he proposes a construction in mixed characteristic analogous to the affine Grassmannian. We will start by recalling Greenberg's classical definition of realizations ([**Gre61**]) in Section 4.1. The subsequent sections are devoted to the discussion of Haboush's generalization of this notion in [**Hab05**], which he calls a 'localized Greenberg realization', and to Haboush's construction of spaces of p-adic lattices.

4.1. Greenberg Realizations

Our reference for this is Greenberg [**Gre61**]. We will stay close to Greenberg's notation, and in particular *in this section we use the letter R to denote a ring scheme.* So let S be a scheme and $R \to S$ a ring scheme over S. Hence R represents a sheaf of rings on the Zariski-site over S, and thus defines a covariant functor

$$\begin{aligned} G_R : (\mathrm{Sch}/S) &\to (\text{Ringed spaces}/\operatorname{Spec} R(S)) \\ (X, \mathcal{O}_X) &\mapsto G_R(X) = (X, \mathcal{O}_{G_R(X)}), \end{aligned}$$

where $\mathcal{O}_{G_R(X)}(U) := R(U)$, the set of S-morphisms from U to R. The ring scheme R is called a *local* ring scheme, if the functor G_R takes values in the category of locally ringed spaces.

Example 4.1. *Let $R = \mathrm{W}_N$ be the scheme of Witt-vectors of length N over $S = \operatorname{Spec} k$, with $0 \leq N \leq \infty$. We claim that W_N is a local ring scheme. Namely, for any S-scheme X the stalk of $G_{\mathrm{W}_N} X$ at $x \in X$ is given by $\mathcal{O}_{G_{\mathrm{W}_N}(X),x} = \varinjlim \mathrm{W}_N(U)$, and $f = (f_0, f_1, \dots) \in \mathcal{O}_{G_{\mathrm{W}_N}(X),x}$ is invertible if and only if $f_0 \in \mathcal{O}_{X,x}$ is invertible. The 'only if'-part is trivial, and the 'if'-part can be seen as follows. Whenever f_0 is invertible in $\mathcal{O}_{X,x}$, then there exists an open neighbourhood U of x such that f_0 is invertible in $\mathcal{O}_X(U)$. But then f is invertible in $\mathrm{W}_N(U)$ and a fortiori in $\mathcal{O}_{G_{\mathrm{W}_N}(X),x}$.*

The situation of this example, R being the scheme of Witt vectors of finite or infinite length over a perfect field k, will be the most interesting for us in this and the following.

Example 4.2. *In the same way one checks that the scheme of (truncated) power series over a field k, as well as the scheme of (truncated) Frobenius-twisted power series over k, are local ring schemes. As for W_N in the previous example, both are isomorphic to an affine space over k. For the definition of Frobenius-twisted power series see Chapter 3.*

In the following let R be a local ring scheme over S.

Definition 4.3 (Greenberg, [**Gre61**]). *Let X be a scheme over the ring $R(S)$. A* (Greenberg) realization *of X* over *S is an S-scheme $F_R X$ which represents the functor*

$$Y \mapsto \mathrm{Hom}_{(l.r.sp./R(S))}(G_R(Y), X).$$

In the sequel, to simplify notation, we will occasionally drop the index indicating the ring scheme R. The following proposition and its corollary are purely formal consequences of the universality of representing objects. However, since these are especially interesting for our later applications in the construction of loop groups, we state them explicitly:

Proposition 4.4. *Realizations commute with fiber products. More precisely, if X, X', T are $R(S)$-schemes with realizations FX, FX', FT over S, then $FX \times_{FT} FX'$ is a realization over S of $X \times_T X'$.* $\square$

Corollary 4.5. *Let X be a group scheme over $R(S)$ having a realization FX over S. Then FX is a group scheme over S.* $\square$

Let us now explicitly describe realizations in situations which are of particular interest for us. For detailled proofs we refer to Greenberg, [**Gre61**].

Proposition 4.6 (Greenberg, [**Gre61**]). *Assume that there is an isomorphism of S-schemes*

$$\varphi = (\varphi_1, \ldots, \varphi_N) : R \to \mathbb{A}^N_S,$$

where $0 \leq N \leq \infty$. Then $\mathbb{A}^d_{R(S)}$ has as a Greenberg realization the S-scheme $F(\mathbb{A}^d_{R(S)}) = (\mathbb{A}^N_S)^d$ together with the universal arrow $\lambda : GF(\mathbb{A}^d_{R(S)}) \to (\mathbb{A}^d_{R(S)})$, which is given in terms of global sections by the ring homomorphism

$$\lambda^\# : R(S)[T_1, \ldots, T_d] \to (S[t_{1,1}, \ldots, t_{1,N}, \ldots, t_{d,1}, \ldots, t_{d,N}])^N$$
$$T_i \mapsto (t_{i,1}, \ldots, t_{i,N}).$$

If $f : \mathbb{A}^d_{R(S)} \to \mathbb{A}^e_{R(S)}$ is a morphism of $R(S)$-schemes and $P_1, \ldots, P_e$ are the polynomials in $R(S)[T_1, \ldots, T_d]$ defining f, then the morphism

Ff between the respective Greenberg realizations is given in terms of global sections by

$$t'_{i,j} \mapsto \varphi_j(\lambda^{\#}(P_i)).$$

Here, the $t_{i,j}$ are the coordinates on $F(\mathbb{A}^d_{R(S)})$, while the $t'_{i,j}$ are the coordinates on $F(\mathbb{A}^e_{R(S)})$. In other words, to calculate the image of $t'_{i,j}$, we have to substitute $T_l \mapsto (t_{l,j})_j$ in the polynomial P_i and then take the j-th component of the result under the isomorphism φ.

Proof. This is proved by Greenberg in [**Gre61**] in the case where N is finite. However, his proof works literally also the situation $N = \infty$. $\square$

Proposition 4.7 (Greenberg, [**Gre61**]). *Let R be a local ring scheme over S, being isomorphic as an S-scheme to an N-dimensional affine space over S (recall that we allow $N = \infty$). Let moreover X be an affine scheme of finite type over $R(S)$ having a realization by an affine scheme FX over S. Then every closed subscheme of X has a realization over S by a closed subscheme of FX.*

Proof. This is proved in [**Gre61**]. The strategy is as follows. First we may, by universality of realizations, assume that X itself is an affine space over $R(S)$, whence X has a realization FX isomorphic to an affine space over S (Proposition 4.6). We obtain a realization of X as follows. Let $X = \mathbb{A}^d_{R(S)}$ and choose a set of defining equations $f_i(X_1, \dots, X_d)$ for a closed subscheme $Y \subset X$. Then each X_j can be understood as a vector of coordinates $X_i = (x_{i,0}, \dots, x_{i,N})$ (according to the isomorphism $R(S) \simeq \mathbb{A}^N_S(S)$). Plugging these into the equations $f_i = 0$ yields 'coordinate-wise' equations in the variables $x_{i,j}$. These are the defining equations of $FY \subset FX$. $\square$

Let us consider for instance the case $R = \mathrm{W}_N$. Let X be the affine space $\mathbb{A}^d_{\mathrm{W}_N(S)} = \operatorname{Spec} \mathrm{W}_N(S)[T_1, \dots, T_d]$. Then a closed subscheme $X \subset \mathbb{A}^d_{\mathrm{W}_N(S)}$ is given by a set of equations, say

$$\{f_1(T_1, \dots, T_d), f_2(T_1, \dots, T_d) \dots\}.$$

The equations of the realization $FX \subset \operatorname{Spec} S[t_{i,j}]$ are then obtained by plugging the Witt vectors

$$(t_{i,0}, t_{i,1}, \dots) \in W_N(S[t_{i,0}, t_{i,1}, \dots])$$

into the equations f_m. The components of the Witt vectors

$$f_m(t_{i,0}, t_{i,1}, \dots) \in W_N(S[t_{i,0}, t_{i,1}])$$

for varying m are the defining equations of the realization FX.

4.2. Haboush's Localized Greenberg Realizations

We are now going to discuss Haboush's notion of 'localized Greenberg Realization', which he introduces in [**Hab05**]. In this section, k is a perfect field, fixed once and for all, $\mathrm{W}(k)$ denotes the ring of Witt vectors over k, and K denotes its fraction field $\mathrm{W}(k)[1/p]$. Though Haboush discusses localized Greenberg realizations for more general discrete valuations rings than just $\mathrm{W}(k)$ (for example his discussion also includes the case of power series rings), we will stick to this particular situation, because it makes our presentation easier to read, without changing any of the conclusions.

To be consistent with Haboush's notation, we will henceforth write $\mathfrak{W}$ for the functor G_R ($R := \mathrm{W}$) of the previous section, and we will write $F = \mathcal{G}_0$ from now on for Greenberg realization. As we have seen in the previous section, Greenberg realizations exist for schemes X which are of finite type over $\mathrm{W}(k)$. The Greenberg realization of X is an (infinite dimensional) k-scheme $\mathcal{G}_0 X$ with the property

$$\mathrm{Hom}_{\mathrm{W}}(\mathfrak{W}(Z), X) = \mathrm{Hom}_k(Z, \mathcal{G}_0 X) \tag{4.2.1}$$

for every k-scheme Z.

In the first part of [**Hab05**], Haboush proposes a construction of a functor $\mathfrak{W}_p$ from the category of k-schemes to the category of ringed spaces over K, and for certain K-schemes X he describes a topological k-scheme $\mathcal{G}X$ such that the analogous equation to (4.2.1) holds:

$$\mathrm{Hom}_K(\mathfrak{W}_p(Z), X) = \mathrm{Hom}_k(Z, \mathcal{G}X), \tag{4.2.2}$$

for every k-scheme Z. The topological k-scheme $\mathcal{G}X$ is what he calls a 'localized Greenberg realization'. However, both the construction of $\mathfrak{W}_p$ as well as the construction of $\mathcal{G}X$ present certain problems in the way Haboush deals with them.

4.2.1. Haboush's functor $\mathfrak{W}_p$. We adopt Haboush's notation, i.e. for any k-algebra A we denote by $A^{p^{-\infty}}$ its perfection. That is, $A^{p^{-\infty}} = \varinjlim_{n\in\mathbb{N}} A$, where the transition map from step n to step $n+1$ is the absolute Frobenius.

Definition 4.8 (Haboush, [**Hab05**], Definition 13). *The functor $\mathfrak{W}'_p$ is the functor which associates to every k-algebra A the K-algebra $\mathrm{W}(A^{p^{-\infty}})[1/p]$. The functor $\mathfrak{W}_p$ is the functor from the category of k-schemes to the category of K-ringed spaces which associates to every k-scheme X the K-ringed space $\mathfrak{W}_p(X)$, with underlying topological space $|X|$ and its sheaf of rings given by*

$$U \mapsto \mathfrak{W}'_p(\mathcal{O}_X(U)). \tag{4.2.3}$$

Note that Haboush uses the same symbol $\mathfrak{W}_\pi$ for both functors which we denote by $\mathfrak{W}_p$ and $\mathfrak{W}'_p$, respectively.

In his Lemma 2 he gives a short argument which proves that the presheaf given by (4.2.3) is indeed a sheaf. Further, he claims that the ringed space $\mathfrak{W}_p(X)$ is in fact a *locally ringed space*. Unfortunately, this is not true in general, as we see by the following

EXAMPLE 4.9. *Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[T]$, and consider $\mathfrak{p} = (T) \in |X|$. A basis of neighborhoods of $\mathfrak{p}$ is given by the open subsets $U_f = \operatorname{Spec} k[T]_f$ for every $f \notin (T)$. We are going to calculate the stalk of $\mathfrak{W}_p(X)$ at $\mathfrak{p}$ in order to show that it is* not *a local ring. This stalk is by definition*

$$S := \mathcal{O}_{\mathfrak{W}_p(X),\mathfrak{p}} = \varinjlim_f \mathcal{O}_{\mathfrak{W}_p(X)}(U_f) = \varinjlim_f \mathrm{W}(k[T]_f^{p^{-\infty}})_p, \tag{4.2.4}$$

where the index f runs through all polynomials $f \in k[T]$ with $f(0) \neq 0$. By square brackets $[\cdot]$ we denote Teichmüller representatives. Now we just have to observe that $[T]+p, [T] \in S$ are not invertible, but their difference, p, is. Thus S cannot be a local ring.

This causes 'pathologies' which one does not encounter when dealing with locally ringed spaces. For example, a morphism from $\mathbb{A}^1_K$ to $\mathfrak{W}_p(\mathbb{A}^1_k)$ is not uniquely determined by a global section of $\mathfrak{W}_p(\mathbb{A}^1_k)$, as it was if the target were a locally ringed space ([**EGAI**]). In any case, though Haboush makes the following Definition 4.10, he seems to work in the sequel with the different(!) Definition 4.11.

DEFINITION 4.10 (Haboush, [**Hab05**], Definition 14). *Let X be a K-scheme. Then a* localized Greenberg scheme *associated to X is a topological k-scheme $\mathcal{G}X$, satisfying the functorial equation* (4.2.2) *for every k-scheme Z. This is,*

$$\operatorname{Hom}_K(\mathfrak{W}_p(Z), X) = \operatorname{Hom}_k(Z, \mathcal{G}X).$$

DEFINITION 4.11 (Alternative definition). *Let X be a K-scheme. Then a* localized Greenberg scheme *associated to X is a topological k-scheme $\mathcal{G}X$, satisfying the functorial equation*

$$\operatorname{Hom}_K(\operatorname{Spec} \mathfrak{W}'_p(R), X) = \operatorname{Hom}_k(R, \mathcal{G}X),$$

for any k-algebra R.

This latter definition is better behaved. Namely, since $\mathcal{G}X$ is by requirement a topological scheme, and thus in particular a *sheaf* for the big Zariski-site on $\operatorname{Spec} k$, it is determined by its values on the category of k-algebras. This is checked by Haboush in [**Hab05**], Lemma 4. On the other hand, $\operatorname{Spec} \mathfrak{W}'_p(R)$ is of course a nice locally ringed space.

And, in fact, it is evident that Haboush works with this latter definition in the proof of Lemma 5, which we are going to discuss next.

4.2.2. Haboush's localized Greenberg realizations. Here we discuss the following lemma, which is stated by Haboush in [**Hab05**] in order to prove that his notion of generalized Greenberg realization makes sense and that the schemes $\mathcal{G}X$ exist in the situations where one expects them to exist.

LEMMA 4.12 (Haboush, [**Hab05**], Lemma 5). *Let X be a locally closed subset of $\mathbb{A}^n_K$. Then there exists a localized Greenberg functor $\mathcal{G}X$.*

4.2.2.1. *A counterexample.* This lemma seems to be false. At least Haboush's construction of the topolgical k-scheme $\mathcal{G}X$ does not yield what he claims. Let us go through his proof in the situation $n = 1$, $X = \mathbb{A}^1_K$ and $R = k[T]$.

Haboush considers the topological k-algebra $k^+\langle X_{\mathbb{Z}}\rangle_\infty$, which he defines as follows (see [**Hab05**], pp. 71-73): Take $k[x_i; i \in \mathbb{Z}]$ together with the linear topology defined by the fundamental system of neighborhoods $J_\nu = \langle x_i; i < \nu\rangle$ of 0 in $k[x_i; i \in \mathbb{Z}]$. Then $k^+\langle X_{\mathbb{Z}}\rangle_\infty$ is defined to be the completion (with respect to this fundamental system of neighborhoods) of the perfect closure $k[x_i; i \in \mathbb{Z}]^{p^{-\infty}}$.

Now Haboush claims to construct a topological k-scheme $\mathcal{G}\mathbb{A}^1_K$ with the property that

$$\mathfrak{W}'_p(R) = \mathrm{Hom}_K(\mathrm{Spec}\,\mathfrak{W}'_p(R), \mathbb{A}^1_K) = \mathrm{Hom}_k(R, \mathcal{G}\mathbb{A}^1_K), \tag{4.2.5}$$

and, more precisely, he claims that $\mathcal{G}\mathbb{A}^1_K = \mathrm{Spf}\, k^+\langle X_{\mathbb{Z}}\rangle_\infty$ does the job. Let us check this. The right hand side of equation (4.2.5) is equal to the set of continuous k-homomorphisms $\mathrm{Hom}_{cont}(k^+\langle X_{\mathbb{Z}}\rangle_\infty, k[T])$. But since $k^+\langle X_{\mathbb{Z}}\rangle_\infty$ is *perfect* by construction, the image of every homomorphism $\varphi \in \mathrm{Hom}_{cont}(k^+\langle X_{\mathbb{Z}}\rangle_\infty, k[T])$ is a perfect subring of $k[T]$. Hence each such φ factors through the maximal perfect subring of $k[T]$, which is k. Consequently, the right hand side of equation (4.2.5) is in bijective correspondence with $\mathrm{Spf}\, k^+\langle X_{\mathbb{Z}}\rangle_\infty(k) = \mathrm{W}(k)[1/p] = K$, which is certainly different from the left hand side $\mathfrak{W}'_p(R) = \mathrm{W}(k[T]^{p^{-\infty}})[1/p]$.

4.2.2.2. *Tangent spaces.* There is also a more conceptual reason why Haboush's notion of localized Greenberg realization is problematic. Let us discuss the desired functorial equation

$$\mathrm{Hom}_K(\mathrm{Spec}\,\mathfrak{W}'_p(R), X) = \mathrm{Hom}_k(\mathrm{Spec}\, R, \mathcal{G}X) \tag{4.2.6}$$

of Definition 4.11. Recall that $\mathfrak{W}'_p(R) = \mathrm{W}(R^{p^{-\infty}})[1/p]$. Recall further that, in the ring of Witt vectors, multiplication by p is the composition of Frobenius and Verschiebung, in either order, whence the

Frobenius morphism is an automorphism of $\mathrm{W}(R^{p^{-\infty}})[1/p]$ whatever k-algebra R we choose. In other words, if we choose any perfect field extension $k \subset l$ and set $D := l[\epsilon]/(\epsilon^2)$, the map $\varphi : D \to D; x \mapsto x^p$ induces an automorphism of $\mathrm{W}(D^{p^{-\infty}})[1/p]$ and hence an automorphism of the set $\mathrm{Hom}_K(\mathrm{Spec}\,\mathfrak{W}'_p(D), X)$. Now, assuming the functorial isomorphism (4.2.6), φ had to induce also an automorphism on the set $\mathrm{Hom}_k(\mathrm{Spec}\,l[\epsilon]/(\epsilon^2), \mathcal{G}X)$, which is only possible if

$$\mathrm{Hom}_k(\mathrm{Spec}\,l[\epsilon]/(\epsilon^2), \mathcal{G}X) = \mathrm{Hom}_k(l, \mathcal{G}X).$$

In other words, all tangent spaces of $\mathcal{G}X$ are trivial, which implies that $\mathcal{G}X$ is a disjoint union of points. But this seems absurd.

The same reasoning of course applies to Haboush's original definition of localized Greenberg realization using his functor $\mathfrak{W}_p$, i.e. Definition 4.10. Namely, φ induces an isomorphism on the ringed space $\mathfrak{W}_p(X)$, too.

Remark 4.13. *(1) Of course, this argument can be significantly simplified by observing that $D^{p^{-\infty}} = l$, and consequently that $\mathfrak{W}'_p(D) = \mathfrak{W}'_p(l)$, thereby showing that all tangent spaces of a possible $\mathcal{G}X$ must be trivial. However, the above argument has the 'advantage' of showing that there is no chance of repairing this defect by replacing $R^{p^{-\infty}}$ by any other ring derived from R. (2) The defect of 'non-existence of tangent spaces' is observed by Haboush in the introduction to his paper, but not explicitly pursued.*

All in all, the category of topological schemes does not seem to be well-suited for the construction of a 'localized' analogon of Greenberg realizations. An alternative approach via constructing these analoga in the category of k-ind-schemes will be presented in Chapter 5, along with the explicit description of p-adic loop groups. Note here in advance that the concept of ind-scheme (which we will use) is a generalization of the concept of topological scheme. Namely, as Haboush notes at the beginning of the proof of his Lemma 4, any topological k-scheme Z can be represented as an inductive limit $Z = \varinjlim_{i\in I} Z_i$, where the Z_i are the closed subschemes which are defined by open ideals of the structure sheaf.

Philosophically, the reason behind this failure is ultimately due to the fact that multiplication by p in the Witt ring involves p-th powers in the components of Witt vectors. This will also be the source of other difficulties occuring in the next section.

4.3. Haboush's Spaces of Lattices

In Section 3 of [**Hab05**], Haboush attempts to construct spaces of special lattices over $\mathrm{W}(k)$ as geometric objects over k, analogous to the construction of the affine Grassmannian, and further to study these spaces. In other words, he aims at constructing a p-adic affine Grassmannian for the group Sl_n. We are now going to discuss certain aspects of his constructions, found in [**Hab05**], pp. 85-91.

Let k be the algebraic closure of $\mathbb{F}_p$, let $\mathcal{O} = \mathrm{W}(k)$ be the ring of Witt vectors over k, and let K be its fraction field. We denote by $F = \mathcal{O}^n \subset K^n$ the free $\mathcal{O}$-submodule of rank n spanned by the standard basis. A *lattice* in K^n is a free $\mathcal{O}$-submodule of rank n; it is called *special* if its n-th exterior power is $\mathcal{O} \subset K$. A lattice $L \subset K^n$ is of *height at most* r if $L \subset p^{-r}F$.

In fact, in all what follows Haboush speaks, instead of special lattices, more generally of lattices of index q, the special lattices being those of index 0. However, since all aspects that we want to discuss are present in the case $q = 0$, we will stick to this case and discuss only special lattices. Whenever we cite statements from [**Hab05**], we will specialize them to this case without further comment.

Notation 4.14 ([**Hab05**], Definition 15). *The set of special lattices in K^n of height at most r will be denoted $\mathbb{L}at_r^n(K)$.*

Haboush observes that the $\mathrm{W}(k)$-module $p^{-r}F/p^{(n-1)r}F$ is an algebraic group over k ([**Hab05**], Lemma 7). In fact, the set $p^{-r}F/p^{(n-1)r}F$ can be identified with the k-valued points of the Greenberg realization of the $\mathrm{W}(k)$-module $\mathrm{W}_{nr}(k)^n$, which is explicitly given by $M := \operatorname{Spec} k[x_{i,j}; i = 1, \dots, n; j = -r, \dots, (n-1)r-1]$. The idea to construct a p-adic Grassmannian is now to look at closed subschemes of M which are stable under the operations induced by the $\mathrm{W}(k)$-module structure of $p^{-r}F/p^{(n-1)r}F$, and parametrize them by a certain subscheme of a Hilbert scheme. Note that there is a natural choice of a grading on the coordinate ring of M, namely

$$\deg x_{i,j} = p^j.$$

The reason for this is that the action of Teichmüller representatives on Witt vectors in $p^{-r}\,\mathrm{W}(k)$ is given by the well-known formula

$$[\alpha] \cdot (x_{-r}, x_{-r+1}, \dots) = (\alpha^{p^{-r}} x_{-r}, \alpha^{p^{-r+1}} x_{-r+1} \dots, \alpha x_0, \alpha^p x_1, \dots).$$

In detail, Haboush claims

Lemma 4.15 ([**Hab05**], Lemma 7). *The group $p^{-r}F/p^{(n-1)r}F$ is a unipotent algebraic group of dimension rn^2 and the special lattices* *of rank n and height at most r are in bijective correspondence with the k^*-stable (for the action via Teichmüller representatives) connected subgroup schemes of dimension $(n-1)nr$ of M.*

As it stands, this lemma is incorrect, because it does not account for the possibility of many different non-reduced structures on one and the same closed subset of M. Of course, the statement can easily be repaired by replacing 'connected subgroup schemes' by 'reduced connected subgroup schemes'. However, it will later turn out that non-reduced structures on connected subgroup schemes arise in Haboush's construction very naturally and in a seemingly unavoidable manner. This is a source of errors, which we are going to analyze in detail below. For the same reason, the following statement is not correct:

PROPOSITION 4.16 (Haboush, [**Hab05**], Proposition 6). *There is a k-scheme $\mathbb{L}$ which is projective and of finite type over k, and a flat commutative group scheme $\mathbb{U} \subset M \times_{\operatorname{Spec} k} \mathbb{L}$ which is a universal family of flat subschemes of M of dimension $(n-1)nr$.*

Haboush constructs the scheme $\mathbb{L}$ as a closed subscheme of a Hilbert scheme of the weighted projective space associated with M, i.e. the quotient scheme $(M - \{0\})/k^*$. This does not seem to present any problems, however, the $\mathbb{L}$ so obtained will not be of finite type. Again, the reason is that apart from trivial cases one can endow a closed subgroup scheme of M with infinitely many different infinitesimal structures, all giving rise again to sub*group* schemes of M, with pairwise distinct Hilbert functions. Hence $\mathbb{L}$ will have infinitely many connected components.

We turn to one of the main theorems of this part of Haboush's paper, namely [**Hab05**], Theorem 4. In the statement of this theorem Haboush claims that there exists a projective k-variety $X_r \subset \mathbb{L}$ which represents the functor classifying 'flat group subschemes of special lattices of height at most r'. He does not make precise what he means by the term 'flat group subschemes of lattices', but from the proof of this theorem it becomes clear that X_r should be taken to be the component of $\mathbb{L}$ which contains the k-point corresponding to the ideal $\langle x_{i,j}; i = 1, \ldots, n; j < 0\rangle$. This is the ideal of the reduced subscheme of M corresponding to the standard lattice $F/p^{r(n-1)}F \subset p^{-r}F/p^{r(n-1)}F$.

In the penultimate paragraph on p. 90, [**Hab05**], Haboush tries to prove that the fibers of the universal family $\mathbb{U}$ over X_r are reduced. Evidently this is supposed to establish, by applying the repaired version of Lemma 4.15, a 1-1 correspondence between k-valued points of X_r

and special lattices of height at most r. Unfortunately, his argument is circular, and indeed what he claims is false. We will check this by

EXAMPLE 4.17. *We set $r = 1, n = 2$, hence $F = \mathrm{W}(k)^2$. Thus we are interested in special lattices of height at most 1, that is, lattices $\subset p^{-1}\,\mathrm{W}(k)^2$, or equivalently, certain $\mathrm{W}(k)$-submodules of $p^{-1}F/pF$. Moreover, $M = \operatorname{Spec} k[x_{-1}, x_0, y_{-1}, y_0]$, and X_1 and $\mathbb{L}$ are subvarieties of the Hilbert scheme of $\operatorname{Proj} k[x_{-1}, x_0, y_{-1}, y_0]$, with the grading*

$$\deg x_{-1} = \deg y_{-1} = p^{-1}, \quad \deg x_0 = \deg y_0 = 1.$$

Consider the following flat $\mathbb{G}_m$-family of graded ideals, where $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$,

$$I_t = \langle x_{-1}, x_0 + t y_{-1}^p \rangle. \tag{4.3.1}$$

Since X_1 is projective, this family extends over 0 and infinity, where the fibers are given by the ideals

$$I_0 = \langle x_{-1}, x_0 \rangle, \ \text{and } I_\infty = \langle x_{-1}, y_{-1}^p \rangle,$$

respectively. In particular, the fiber over $t = \infty$ in this family is not *reduced. On the other hand, it is obvious that the corresponding reduced fiber has a different Hilbert polynomial than the fiber over $t = 0$, whence they cannot both lie in X_1.*

It may be worth mentioning that X_1 must contain the above constructed $\mathbb{P}^1$-family of subschemes of M, as soon as one wants X_1 to contain the orbit closure of the lattice $\langle pe_1, p^{-1}e_2 \rangle \subset p^{-1}F$ (which corresponds to I_0) under the natural action of $\mathrm{Sl}_n(\mathrm{W}(k))$ on the Hilbert scheme of $\operatorname{Proj} k[x_{-1}, x_0, y_{-1}, y_0]$. Namely, the ideal I_t is the image under this action of the matrix

$$\begin{pmatrix} 1 & 1 \\ p[t^{1/p}] & 1 + p[t^{1/p}] \end{pmatrix}^{-1} \in \mathrm{Sl}_n(\mathrm{W}(k[t^{\pm 1/p}])).$$

Note further that the matrix

$$\begin{pmatrix} p[t^{1/p}] & 1 + p[t^{1/p}] \\ 1 & 1 \end{pmatrix}^{-1} \in \mathrm{Sl}_n(\mathrm{W}(k[t^{\pm 1/p}])).$$

gives rise to a different orbit of I_0, namely $J_t = \langle y_{-1}, y_0 + t x_{-1}^p \rangle$. The fiber over $t = \infty$ is then given by the ideal

$$J_\infty = \langle y_{-1}, x_{-1}^p \rangle.$$

The presence of infinitesimal structures in the fibers of $\mathbb{U}$ over points of X_r is the reason for two serious problems:

(1) In general there are many fibers in the family $\mathbb{U}$ over X_r which differ only by their infinitesimal structure. As examples take the ideals

I_∞ and J_∞ above. They are different, but both correspond to the reduced ideal $\langle x_{-1}, y_{-1}\rangle$ and thus to the standard lattice $F \subset p^{-1}F$. In other words, *there is no bijection between k-valued points of X_r and special lattices of height at most r.*

(2) *There does not exist a natural direct system of schemes $X_{r-1} \to X_r$*, as Haboush claims in [**Hab05**], Definition 17. The existence already fails at the level $r = 1$, and to see this, we may again invoke the above example. Namely, X_0 is the one-point scheme $\operatorname{Spec} k$ representing the unique lattice of height at most 0, i.e. the standard lattice $F \subset K^n$. On the other hand, in X_1 there are several (to be more precise, a whole $\mathbb{P}^1$) points whose reduced fibers correspond to the standard lattice. None of them is a natural choice for defining a map $X_0 \to X_1$, and this phenomenon continues to arise for every $r > 0$.

REMARK 4.18. *Of course, Example 4.17 is closely related to Example 3.25. In fact, the orbit closures in both the equal- and mixed-characteristic situations conincide if $p \neq 2$. We will discuss this in more detail in the Chapter 5, Example 5.16. The common phenomenon of infinitesimal structures on lattice schemes in both situations leads one to suspect that one should think of Haboush's construction as a sort of Demazure resolution of Schubert varieties, rather than the Schubert varieties themselves, in the p-adic Grassmannian.*

It seems to follow from our discussion that [**Hab05**], Definition 17, where Haboush defines the 'space of special lattices' as an ind-scheme, does not make sense. In Chapter 5 we will analyze what we can still say about the existence of such a space in whatever sense.

CHAPTER 5

Spaces of p-adic Lattices

5.1. p-adic Loop Groups

5.1.1. Localized Greenberg realizations. Let R be a local ring scheme over a *quasi-compact* scheme S. In this section we will generalize Greenberg's notion of realization (Section 4.1) to the situation where X is a scheme over $R(S)[1/a]$, for $a \in R(S)$. This is meant as an alternative to the approach by Haboush, which is discussed in the previous chapter. Localized Greenberg realizations in our sense will be objects in the category of S-ind-schemes in the sense of Section 1.1. Again, we remind the reader that the situation of interest for us will be the case where $S = \operatorname{Spec} k$ is the spectrum of a perfect field and $R = \mathrm{W}$ is the scheme of Witt vectors over S, and $a = p$ is a uniformizer.

Observe that the ring $R(S)[1/a]$ is the colimit of the inductive system of rings

$$R(S) \xrightarrow{\cdot a} R(S) \xrightarrow{\cdot a} R(S) \xrightarrow{\cdot a} \dots .$$

Assume again that R is isomorphic as an S-scheme to $\mathbb{A}^N_S$, i.e. that the affine line over $R(S)$ can be realized in the sense of Greenberg, Definition 4.3, by $\mathbb{A}^N_S$. Passing to Greenberg realizations we obtain the inductive system

$$\mathbb{A}^N_S \xrightarrow{F(\cdot a)} \mathbb{A}^N_S \xrightarrow{F(\cdot a)} \mathbb{A}^N_S \xrightarrow{F(\cdot a)} \dots .$$

If we denote the corresponding S-ind-scheme by $F_a\mathbb{A}^1_{R(S)}$, then for any S-scheme Y we obtain natural bijections

$$\begin{aligned} \mathrm{Hom}_{(\text{ind-Sch}/S)}(Y, F_a\mathbb{A}^1_{R(S)}) &\simeq \varinjlim(\mathbb{A}^N_S(Y)) = \\ =\varinjlim \mathrm{Hom}_{(\text{l.r.sp.}/R(S))}(G(Y), \mathbb{A}^1_{R(S)}) &= \varinjlim R(Y) = R(Y)[1/a]. \end{aligned}$$

In other words, the functor $Y \mapsto R(Y)[1/a]$ is represented by the S-ind-scheme $F_a\mathbb{A}^1_{R(S)}$. This motivates the following definition.

Definition 5.1. *Let X be a $R(S)[1/a]$-scheme. A* localized Greenberg realization *of X over S is an S-ind-scheme which represents the functor $Y \mapsto X(R(Y)[1/a])$ on the category of (quasi-compact) S-schemes.*

Since the category of ind-schemes has fiber products, and by the universal property of Greenberg realizations, we obtain:

(1) Let $X \to T$ and $X' \to T$ be morphisms of $R(S)[1/a]$-schemes which admit localized Greenberg realizations (F_iX), (F_iX') and (F_iT) over S. Then the fiber product $(F_iX \times_{F_iT} F_iX')$ is a localized Greenberg realization over S of $X \times_T X'$.

(2) If X is a group scheme over $R(S)[1/a]$ having a localized Greenberg realization (F_iX) over S, then (F_iX) is a group object in the category of ind-schemes over S.

Let us gather a few observations which we will use to prove the existence of localized Greenberg realizations in certain cases. First note that the existence of a localized Greenberg realization of the affine line $\mathbb{A}^1_{R(S)}$ is already proven by our remarks before Definition 5.1. Now let X be any affine scheme of finite type over $R(S)[1/a]$ and fix a closed immersion $X \subset \mathbb{A}^d_{R(S)[1/a]}$. Let moreover

$$\varphi_n : \mathbb{A}^d_{R(S)[1/a]} \to \mathbb{A}^d_{R(S)[1/a]} = \operatorname{Spec} R(S)[1/a][T_1, \dots, T_d]$$

be the automorphism given by $T_i \mapsto a^n T_i$ for $i = 1, \dots, d$. This yields a diagram of the form

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{A}^d_{R(S)[1/a]} \arrow[r, "\varphi_1"] & \mathbb{A}^d_{R(S)[1/a]} \arrow[r, "\varphi_1"] & \cdots \\ \cdots \arrow[r] & \varphi_n(X) \arrow[r] \arrow[u] & \varphi_{n+1}(X) \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

where all the horizontal maps are isomorphisms of schemes over $R(S)[1/a]$. Then define X_n to be the scheme-theoretic image of

$$\varphi_n(X) \hookrightarrow \mathbb{A}^d_{R(S)[1/a]} \hookrightarrow \mathbb{A}^d_{R(S)}.$$

In terms of ideals this means: if $I \subset R(S)[1/a][T_1, \dots, T_d]$ is the defining ideal of $X \subset \mathbb{A}^d_{R(S)[1/a]}$, then $X_n \subset \mathbb{A}^d_{R(S)}$ is defined by the ideal $R(S)[T_1, \dots, T_d] \cap (I|_{T_i \mapsto a^{-n} T_i})$. We obtain the $R(S)$-ind-scheme $(X_n)_n$.

In the sequel we write for any $R(S)$-scheme Y:

$$Y[1/a] := Y \times_{\operatorname{Spec} R(S)} \operatorname{Spec} R(S)[1/a].$$

With this notation we have $X_n[1/a] \simeq \varphi_n(X) \simeq_{\varphi_n^{-1}} X$ for all $n \in \mathbb{N}$.

Lemma 5.2. *The $R(S)$-ind-scheme $(X_n)_n$ represents the functor*

$$L : Y \mapsto \operatorname{Hom}_{R(S)[1/a]}(Y[1/a], X)$$

on the category of (quasi-compact) $R(S)$-schemes.

PROOF. A morphism of functors $\psi_n : X_n \to L$ is given by the functorial map

$$X_n(Y) = \mathrm{Hom}_{R(S)}(Y, X_n) \to \mathrm{Hom}_{R(S)[1/a]}(Y[1/a], X_n[1/a]) \\ \simeq_{\varphi_n^{-1}} \mathrm{Hom}_{R(S)[1/a]}(Y[1/a], X).$$

Obviously the morphisms ψ_n are compatible, so we obtain a morphism of functors $\psi : (X_n)_n \to L$. Since every $Y[1/a]$-valued point P of X is given by a d-tuple p in

$$\Gamma(Y[1/a])^d = (\Gamma(Y) \otimes_{R(S)} R(S)[1/a])^d,$$

there exists some $n \in \mathbb{N}$ such that $a^n \cdot p \in \Gamma(Y)^d$ and thus $\varphi_n(P)$ extends to a Y-valued point of X_n. This shows that $\psi(Y)$ is surjective for every $Y/R(S)$. To check injectivity, take $P, Q \in X_n(Y)$ such that P and Q have the same image in $L(Y)$. This means in particular, that the corresponding morphisms $P', Q' : Y[1/a] \to X_n[1/a] = \varphi_n(X)$ are equal, and consequently the respective $R(S)$-morphisms $P'', Q'' : Y[1/a] \to Y \to X_n$ are equal. But both P and Q are given by d-tuples p, q of sections in $\Gamma(Y)$, and for these the equality $P'' = Q''$ says that there exists an $m \in \mathbb{N}$ such that $a^m p = a^m q$. This means that the compositions

$$Y \xrightarrow{P,Q} X_n \xrightarrow{\varphi_m} X_{n+m}$$

coincide, whence a fortiori P and Q coincide as elements of $(X_n)_n(Y)$. □

It is now easy to construct localized Greenberg realizations for affine $R(S)[1/a]$-schemes which are of finite type.

PROPOSITION 5.3. *Let X be an affine scheme of finite type over $R(S)[1/a]$, and assume that R is isomorphic as an S-scheme to some affine space over S. Then there exists an S-ind-scheme which represents the functor $Y \mapsto X(R(Y)[1/a])$ on the category of (quasi-compact) S-schemes.*

PROOF. Fix a closed immersion $X \subset \mathbb{A}^d_{R(S)[1/a]}$ and let $(X_n)_n$ be as above. Now apply Greenberg realization to the $R(S)$-schemes X_n and their transition maps. I claim that the resulting S-ind-scheme $(FX_n)_n$ has the desired form. Indeed, we have

$$\mathrm{Hom}(Y, (FX_n)_n) = \varinjlim \mathrm{Hom}(Y, FX_n) = \\ = \varinjlim \mathrm{Hom}_{R(S)}(R(Y), X_n) = \mathrm{Hom}(R(Y)[1/a], X),$$

where the second equality is by definition of Greenberg realization, and the third one follows from the previous lemma. □

EXAMPLE 5.4. *Let us illustrate this in our standard situation of Witt vectors of infinite length over a perfect field k. Let $X = \mathbb{A}^d_{\mathrm{W}(k)[1/p]}$. Then the k-ind-scheme which is the localized Greenberg realization of X is represented by the inductive system*

$$\operatorname{Spec} k[x_{i,j}; i = 1, \dots, d; j = 0, 1, \dots] \xrightarrow{\cdot p} \operatorname{Spec} k[x_{i,j}] \xrightarrow{\cdot p} \dots,$$

where the transition maps $\cdot p$ are defined by $x_{i,j} \mapsto x^p_{i,j-1}$ (for $j = 1, \dots, \infty$) and $x_{i,0} \mapsto 0$.

5.1.2. Construction of generalized loop groups. From now on we will consider the following situation: Let $\mathfrak{D}$ be a local ring scheme over a field k such that $D = \mathfrak{D}(k)$ is a discrete valuation ring with uniformizer $u \in D$. Moreover we assume that $\mathfrak{D}$ is isomorphic to $\mathbb{A}^{\mathbb{N}}_k$ as a scheme over k. Typical special cases are

(1) the scheme of power series in one variable over k,
(2) the scheme of Frobenius-twisted power series in one variable over a field k of positive characteristic, and
(3) the scheme of Witt vectors over a perfect field k of positive characteristic.

By K we denote the fraction field of D. Moreover, we now return to usual practice and use the letter R to denote a ring, usually a k-algebra. Let X be a scheme over $\operatorname{Spec} D$. The functors on the category of k-algebras

$$\mathrm{L}\, X : R \mapsto X(\mathfrak{D}(R)[1/u])$$

and

$$\mathrm{L}^+ X : R \mapsto X(\mathfrak{D}(R))$$

will be called the (generalized) loop space, resp. positive loop space, associated with X. Obviously there is a natural morphism of functors $\mathrm{L}^+ X \to \mathrm{L}\, X$. If in addition $X = G$ is a group scheme over D, then we call $\mathrm{L}\, G$ and $\mathrm{L}^+ G$ the (generalized) *loop group* and the (generalized) *positive loop group*, respectively, associated with G.

Note that if $\mathfrak{D}$ is the k-scheme of power series in one variable over k, we recover the usual notions of loop space, loop group etc., as described by Beauville and Laszlo, [**BL94**], by Pappas and Rapoport, [**PR08**], and also in Chapter 1 of the present work. On the other hand, if $\mathfrak{D}$ is the k-scheme of Frobenius-twisted power series and $G = \mathrm{Sl}_n$ over D, then this construction produces $\mathrm{L}^+ G = \mathrm{L}^+_F G$, and the similar objects discussed in Section 3.2.

The following proposition is an immediate consequence of our discussion on Greenberg realizations:

PROPOSITION 5.5. *If X is an affine scheme over D then the functor $\mathrm{L}^+ X$ is representable by an affine scheme over k, namely the Greenberg realization over k of X. If X is affine and of finite type over K, then $\mathrm{L}\, X$ is representable by the localized Greenberg realization over k of X.*

In fact, in all situations that we are going to consider the affine scheme X comes together with a 'natural' embedding into some affine space: $\iota : X \subset \mathbb{A}^d_K$. With respect to this embedding, the construction of the localized Greenberg realization LX described in the preceeding section produces an explicit direct system $(FX_i)_{i\in\mathbb{N}}$ of k-schemes which represents LX. We will refer to this direct system as the 'natural representation' of LX. Explicitly, the i-th step of the natural representation of LX parametrizes the K-points of X whose coordinates (with respect to the embedding ι) have 'poles' of order at most i.

5.1.3. Operations. Let G be a linear algebraic group over $D = \mathfrak{D}(k)$ and fix a closed immersion $G \subset \mathrm{Gl}_{n,D} \subset \mathbb{A}^{n\times n}_D$. The natural action of G on $\mathbb{A}^n_D$ induces, by functoriality of L and L^+, a commutative diagram

$$\begin{array}{ccc} \mathrm{L}\, G \times_k \mathrm{L}\, \mathbb{A}^n_K & \longrightarrow & \mathrm{L}\, \mathbb{A}^n_K \\ \uparrow & & \uparrow \\ \mathrm{L}^+ G \times_k \mathrm{L}^+ \mathbb{A}^n_D & \longrightarrow & \mathrm{L}^+ \mathbb{A}^n_D \end{array} \cdot$$

It is easy to describe the action in the upper line explicitly in terms of the natural representations of the loop spaces involved. In fact, this action is described by the compatible system of maps

$$F(\mathbb{A}^{n\times n}_D)_m \times_k F(\mathbb{A}^n_D)_{m'} \to F(\mathbb{A}^n_D)_{m+m'},$$

where each of these maps is nothing but the usual 'multiplication of a matrix and a vector'. More precisely one could say that it is the (usual) Greenberg realization of the map $\mathbb{A}^{n\times n}_D \times \mathbb{A}^n_D \to \mathbb{A}^n_D$ given by multiplication of matrix and vector. The indices $m, m', m+m'$ may be explained as follows: if M is a k-point of $F(\mathbb{A}^{n\times n}_D)_m$ and v is a k-point of $F(\mathbb{A}^n_D)_{m'}$, then these two objects represent the elements $u^{-m}M \in G(K)$ and $u^{-m'}v \in \mathbb{A}^n(K)$, respectively. Their product is $u^{-m-m'}M \cdot v \in \mathbb{A}^n(K)$, which is thus represented by the product $M \cdot v$ – viewed as a k-point of $F(\mathbb{A}^n_D)_{m+m'}$.

Let us look at the $\mathrm{L}\, G$-action which is thereby induced on sub-ind-schemes of $\mathrm{L}\, \mathbb{A}^n_K$: For any k-scheme T let $\mathcal{S}(T)$ be the set of ind-closed T-sub-ind-schemes of $\mathcal{A}_T := (\mathrm{L}\, \mathbb{A}^n_K) \times_k T$, i.e. $\mathcal{S}(T)$ is the set of classes of commutative diagrams

$$\begin{tikzcd} \cdots \arrow[r] & (\mathbb{A}_T^{\mathbb{N}})^n \arrow[r] & (\mathbb{A}_T^{\mathbb{N}})^n \arrow[r] & \cdots, \\ \cdots \arrow[r] & L_i \arrow[r] \arrow[u, hook] & L_{i+1} \arrow[r] \arrow[u, hook] & \cdots \end{tikzcd}$$

where the vertical maps are closed immersions. Clearly, the assignment $T \mapsto \mathcal{S}(T)$ is a functor on the category of k-schemes. From the $\mathrm{L}\,G$-operation on $\mathrm{L}\,\mathbb{A}_K^n$ we obtain an $\mathrm{L}\,G$-operation (on the right) on $\mathcal{S}$ by pulling back T-sub-ind-schemes along the morphism $\mathcal{A}_T \to \mathcal{A}_T$ given by a T-valued point of $\mathrm{L}\,G$. More precisely, we consider the natural morphism of functors

$$\rho'(T) : \mathrm{L}\,G(T) \times_T \mathcal{S}(T) \to \mathcal{S}(T); \quad (g, (L_i)) \mapsto (L_i) \times_{\mathcal{A}_T, g} \mathcal{A}_T.$$

Combining with the inverse map $\mathrm{L}\,G \to \mathrm{L}\,G, g \mapsto g^{-1}$, we can make this into a left-operation, which we denote by

$$\rho : \mathrm{L}\,G \times_k \mathcal{S} \to \mathcal{S}.$$

The action ρ can be described explicitly as follows: Let $g \in \mathrm{L}\,G(T) \subset \mathrm{L}\,\mathbb{A}_K^{n\times n}(T)$ be represented by $M \in F(G)_m(T) \subset F(\mathbb{A}_D^{n\times n})_m(T)$. Then the map $\mathcal{A}_T \to \mathcal{A}_T$ which is induced by g is represented by the system of maps

$$F(\mathbb{A}_D^{n\times n})_{m'} \to F(\mathbb{A}_D^{n\times n})_{m'+m}; \quad (x_{i,j})_{1\le i\le n} \mapsto M \cdot (x_{i,j})_{1\le i\le n}$$

for any $m' \in \mathbb{N}$. Closed T-subschemes are pulled back as usual by plugging the defining polynomials of this morphism into their equations.

5.1.4. The quotient $\mathrm{L}\,G/\mathrm{L}^+\,G$. In this subsection we construct the quotient $\mathrm{L}\,G/\mathrm{L}^+\,G$ as k-space by considering the 'standard lattice', a certain sub-ind-scheme of $\mathrm{L}\,\mathbb{A}_D^n$, which has $\mathrm{L}^+\,G$ as its stabilizer.

DEFINITION 5.6. *The standard lattice $\mathbb{S} \subset \mathrm{L}\,\mathbb{A}_K^n$ is the fpqc-sheaf image of the natural map $\mathrm{L}^+\,\mathbb{A}_D^n \to \mathrm{L}\,\mathbb{A}_K^n$.*

LEMMA 5.7. *The standard lattice $\mathbb{S} \subset \mathrm{L}\,\mathbb{A}_K^n$ over k is the k-sub-ind-scheme represented by the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & F\mathbb{A}_D^n \arrow[r, "F(\cdot u)"] & F\mathbb{A}_D^n \arrow[r] & \cdots \\ \cdots \arrow[r] & \mathbb{S}_i \arrow[r] \arrow[u, hook] & \mathbb{S}_{i+1} \arrow[r] \arrow[u, hook] & \cdots, \end{tikzcd}$$

where $\mathbb{S}_i$ is the scheme-theoretic image of $F\mathbb{A}_D^n = (\mathbb{A}_k^{\mathbb{N}})^n$ under $F(\cdot u^i) = (F(\cdot u))^i$.

PROOF. This is obvious. $\square$

Let us consider the two standard situations for which these constructions are significant: (1) Let $D = k[[z]]$ be the power series ring in the variable z over k. Then for every k-algebra R our constructions yield:

$$\mathrm{L}^+ G(R) = G(R[[z]]), \quad \mathrm{L}\, G(R) = G(R((z))),$$
$$\mathrm{L}^+ \mathbb{A}^n_D(R) = \mathbb{S}(R) = R[[z]]^n, \quad \mathrm{L}\, \mathbb{A}^n_D(R) = R((z))^n.$$

(2) If $\mathfrak{D} = \mathrm{W}$ is the scheme of Witt vectors over k, then the situation is slightly more complicated, regarding the standard lattice: We have, for any k-algebra R,

$$\mathrm{L}^+ G(R) = G(\mathrm{W}(R)), \quad \mathrm{L}\, G(R) = G(\mathrm{W}(R)[1/p]),$$
$$\mathrm{L}^+ \mathbb{A}^n_D(R) = \mathrm{W}(R)^n, \quad \mathrm{L}\, \mathbb{A}^n_D(R) = \mathrm{W}(R)[1/p]^n.$$

The standard lattice $\mathbb{S}$ in this case is not the same as $\mathrm{L}^+ \mathbb{A}^n_D$: namely, multiplication by p in $\mathrm{W}(R)$ involves p-th roots of elements of R, which has the effect that the presheaf-image of $\mathrm{L}^+ \mathbb{A}^n_D$ is not an fpqc-sheaf and sheafification really produces a different object $\mathbb{S} \neq \mathrm{L}^+ \mathbb{A}^n_D$. For example, if $n = 1$ and $R = k[T]$, then $(T^{1/p}, 0, \dots)$ is not in $\mathrm{L}^+ \mathbb{A}^1_D(k[T])$. But it is in $\mathrm{L}^+ \mathbb{A}^1_D(k[T^{1/p}])$, and $k[T] \to k[T^{1/p}]$ is a faithfully flat extension. Thus $(T^{1/p}, 0, \dots) \in \mathbb{S}(R)$.

Theorem 5.8. *The stabilizer of the standard lattice under the action ρ is precisely the fpqc-sheaf-image of* $\mathrm{L}^+ G \to \mathrm{L}\, G$.

Proof. Let R be a k-algebra, and let $g \in \mathrm{L}\, G(R) = G(\mathfrak{D}(R)[1/u])$ be in the stabilizer of $\mathbb{S}(R) \subset \mathfrak{D}(R)[1/u]^n$. Consider the 'standard basis' $e_1, \dots, e_n \in \mathbb{S}(R)$. Then there exists a faithfully flat homomorphism of k-algebras $R \to R'$ such that the $g \cdot e_i$ induce R'-valued points of $\mathbb{S}$ which actually come from points in $\mathrm{L}^+ \mathbb{A}^n_D(R')$. In other words, g considered as an element of $G(\mathfrak{D}(R')[1/u])$ stabilizes $\mathfrak{D}(R')^n$, which shows that indeed $g \in G(\mathfrak{D}(R')) = \mathrm{L}^+ G(R')$. On the other hand, clearly every R-point of the sheaf-image of $\mathrm{L}^+ G \to \mathrm{L}\, G$ stabilizes the standard lattice. $\square$

5.2. The p-adic affine Grassmannian

In this section we apply the considerations of the previous paragraphs to the case where $G = \mathrm{Sl}_n$, the special linear group over a perfect field k, and where $\mathfrak{D} = \mathrm{W}$ is the scheme of Witt vectors over k, $u = p \in W(k)$. To make this situation also visible in our notation, we will henceforth write $\mathrm{L}_p\, G$ (instead of $\mathrm{L}\, G$) and $\mathrm{L}^+_p\, G$ (instead of $\mathrm{L}^+ G$), and call it the *p-adic loop group* and the *positive p-adic loop group*, respectively. Analogously we write $\mathrm{L}_p\, \mathbb{A}^n_K$ etc.

Denote by T the standard maximal torus contained in the standard Borel subgroup $B \subset \mathrm{Sl}_n$ of upper triangular matrices, and let $\check{\mathrm{X}}(T)$ and $\check{\mathrm{X}}_+(T)$ be the sets of cocharacters and dominant cocharacters, respectively. Identify $\check{\mathrm{X}}(T)$ with the subset of $\mathbb{Z}^n$ consisting of those vectors whose coordinates sum up to 0. Then $\check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ consists of the vectors whose coordinates *decrease* and sum up to 0.

5.2.1. Lattice schemes in the Witt vector setting. The situation we discuss here is completely analogous to Section 3.2. We consider the ring scheme $\mathfrak{R} = \mathrm{W}_N = \operatorname{Spec} k[\alpha_0, \ldots, \alpha_{N-1}]$ and endow

$$\mathfrak{R}^n = \mathrm{W}_N^n = \operatorname{Spec} k[x_{i,j} \mid i = 1, \ldots, n; j = 0, \ldots, N-1]$$

with the respective module operations. Again we consider the grading

$$\deg x_{i,j} = \deg \alpha_j = p^j, \quad x_{i,j} \in \Gamma(\mathrm{W}^n, \mathcal{O}) \tag{5.2.1}$$

on the respective affine rings, and by H we denote the multigraded Hilbert scheme of W_N^n with respect to this grading. Then the morphisms defining the module operations on the scheme W_N^n are defined by *graded* homomorphisms of the affine rings, as follows from the definition of Witt vector arithmetics. (Note that the standard grading $\deg x_{i,j} = 1$ is not respected by the module-operations and is hence not suited for our construction.) Hence, if we let $\Gamma = \mathrm{L}_p^+ \mathrm{Sl}_n$ be the positive p-adic loop group of Sl_n, then the assumptions of Proposition 3.5 are satisfied and we obtain

Corollary 5.9. *The positive loop group $\mathrm{L}_p^+ \mathrm{Sl}_n$ operates on the Hilbert scheme H of W_N^n, and if we denote by $\mathbb{L}^h = \mathbb{L}^h_{W_N^n}$ the subscheme which parametrizes lattice schemes in W_N^n with Hilbert function h, then the action of $\mathrm{L}_p^+ \mathrm{Sl}_n$ restricts to an action on $\mathbb{L}^h$.* $\square$

The standard lattice scheme for a dominant cocharacter. For any dominant cocharacter $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \in \check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ set $\tilde{\lambda} = (\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$ and define the ideal

$$I_\lambda = \langle x_{1,0}, \ldots, x_{1,\tilde{\lambda}_1 - 1}, \ldots, x_{n-1,0}, \ldots, x_{n-1,\tilde{\lambda}_{n-1}-1} \rangle.$$

Choose $N > \tilde{\lambda}_1$. Then this ideal determines a lattice scheme $V_\lambda \subset \mathrm{W}_N^n$. We denote by C_λ the orbit of $V_\lambda \in H$ under the action of $\mathrm{L}_p^+ G$ on H, and by D_λ its orbit-closure in H. Theorem 3.2 asserts in particular that D_λ is a projective k-variety, which contains C_λ as an open subvariety.

5.2.2. The p-adic affine Grassmannian and its Schubert cells. We consider the embedding

$$\iota' : \check{\mathrm{X}}(T) \hookrightarrow G(K); \quad \lambda = (\lambda_1, \dots, \lambda_n) \mapsto \mathrm{diag}(p^{\lambda_1}, \dots, p^{\lambda_n}).$$

This embedding determines an embedding of $\check{\mathrm{X}}(T)$ into the loop group of G,

$$\iota : \check{\mathrm{X}}(T) \hookrightarrow \mathrm{L}_p\, G(k).$$

Hence, we may associate to every $\lambda \in \check{\mathrm{X}}_+(T)$ the k-sub-ind-scheme

$$\mathbb{S}_\lambda = \iota(\lambda) \cdot \mathbb{S} \in \mathcal{S}(k),$$

which can be explicitly described as follows: Let $N > \tilde{\lambda}_1$, and let

$$F(\mathbb{A}^n_K)_{-\lambda_n} \simeq \mathrm{W}^n$$

be the $-\lambda_n$-th step in the natural representation of the k-ind-scheme $\mathrm{L}_p\, \mathbb{A}^n_K$. We consider the projection $\pi : F(\mathbb{A}^n_K)_{-\lambda_n} \to \mathrm{W}^n_N$ which represents the truncation map $\mathrm{W}(k)^n \to \mathrm{W}_N(k)^n$. These data together form the diagram

$$\begin{tikzcd}
\pi^{-1}(V_\lambda) \arrow[r, hook] \arrow[d] & F(\mathbb{A}^n_K)_{-\lambda_n} \arrow[r] \arrow[d, "\pi"] & \mathrm{L}_p\, \mathbb{A}^n_K \\
V_\lambda \arrow[r, hook] & \mathrm{W}^n_N &
\end{tikzcd}$$

Then $\mathbb{S}_\lambda$ is the fpqc-sheaf-image of the composition $\pi^{-1}V_\lambda \to \mathrm{L}_p\, \mathbb{A}^n_K$. Clearly, we have $\mathbb{S} = \mathbb{S}_0$.

Definition 5.10. *The* p-adic Grassmannian $\mathcal{G}rass_p$ *is by definition the fpqc-sheaf-image of the map* $\mathrm{L}_p\, G \to \mathcal{S}$*, given by operation on the standard lattice* $\mathbb{S}$*. Moreover, the* Schubert cell $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$ *is by definition the fpqc-sheaf-image of the map* $\mathrm{L}^+_p\, G \to \mathcal{S}$ *given by operation of* $\mathrm{L}^+_p\, G$ *on the sub-ind-scheme* $\mathbb{S}_\lambda$*.*

Remark 5.11. *In general the process of fpqc-sheafification presents set-theoretical problems, with the consequence that in certain cases one cannot speak of such sheafifications without making further restrictions (i.e. specifying a universe one wants to work in, causing the sheafification to depend on this choice). In the appendix we present an argument in order to prove that these problems do not occur in the present situation (Theorem A.10).*

The k-valued points of the p-adic affine Grassmannian have a similar description as in the function field case.

PROPOSITION 5.12. *The set of k-valued points of the p-adic Grassmannian is in bijective correspondence with the set of lattices in $\mathrm{W}(k)^n$. This correspondence is given by*

$$f : \mathcal{G}rass_p(k) \xrightarrow{\simeq} \{\mathcal{L} \subset \mathrm{W}(k)^n \mid \mathcal{L} \text{ a lattice}\} \qquad (5.2.2)$$
$$L \mapsto L(k),$$

i.e. the k-ind-scheme L is mapped to its set of k-valued points.

PROOF. Observe that the set of k-valued points of the standard lattice $\mathbb{S}$ is equal to $\mathrm{W}(k)^n \subset \mathrm{W}(k)[1/p]^n = \mathrm{L}_p \, \mathbb{A}^n_{\mathrm{W}(k)[1/p]}(k)$. Now let $L = g \cdot \mathbb{S} \in \mathcal{G}rass_p(k)$, where $g \in \mathrm{L}_p \, \mathrm{Sl}_n(k) = \mathrm{Sl}_n(\mathrm{W}(k)[1/p])$. This means that L is the pullback of $\mathbb{S}$ under $g^{-1} : \mathrm{L}_p \, \mathbb{A}^n_{\mathrm{W}(k)[1/p]} \to \mathrm{L}_p \, \mathbb{A}^n_{\mathrm{W}(k)[1/p]}$. Then $L(k) \subset \mathrm{W}(k)[1/p]^n$ is mapped bijectively onto $\mathbb{S}(k) = \mathrm{W}(k)^n$ by g^{-1}. In other words, $L(k) = g \cdot \mathrm{W}(k)^n$, i.e. a lattice. Since every lattice $\mathcal{L} \subset \mathrm{W}(k)[1/p]^n$ has the form $g \cdot \mathrm{W}(k)^n$, the map f is clearly surjective. Finally, if two k-sub-ind-schemes $g \cdot \mathbb{S}$ and $h \cdot \mathbb{S}$ have the same set of k-valued points, then the matrix $h^{-1}g \in \mathrm{Sl}_n(\mathrm{W}(k)[1/p])$ must actually lie in the stabilizer of $\mathrm{W}(k)^n$, i.e. in $\mathrm{Sl}_n(\mathrm{W}(k))$. But this implies $(h^{-1}g) \cdot \mathbb{S} = \mathbb{S}$, or equivalently, $h \cdot \mathbb{S} = g \cdot \mathbb{S}$, which proves injectivity. □

By the elementary divisors-theorem, we see that

$$\mathcal{G}rass_p(k) = \cup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathcal{C}_\lambda(k).$$

We will see in the following sections that, for any $\lambda \in \check{\mathrm{X}}_+(T)$, $\mathcal{C}_\lambda$ is representable by a quasi-projective k-scheme C_λ. Moreover, this affine k-scheme comes together with an open embedding into a projective k-scheme D_λ which maps naturally (as an fpqc-sheaf) to $\mathcal{G}rass_p$, thereby inducing the isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$ as well as a surjection $D_\lambda(k) \twoheadrightarrow \cup_{\lambda' \leq \lambda} \mathcal{C}_{\lambda'}(k)$.

5.2.3. Construction of a morphism $D_\lambda \to \mathcal{G}rass_p$. The purpose of this section is to relate the constructions of the two preceding sections, i.e. the construction of the p-adic affine Grassmannian on the one hand, and the orbit-closure $D_\lambda \subset H$ on the other hand, by a morphism of fpqc-sheaves

$$D_\lambda \to \mathcal{G}rass_p.$$

Fix $\lambda \in \check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ and let $D_\lambda \subset H$ be the orbit-closure constructed in Section 5.2.1. Let moreover $\mathbb{U}_\lambda$ be the universal family obtained by pull-back from the universal family over H:

$$\begin{tikzcd} \mathbb{U}_\lambda \arrow[r] \arrow[d] & \mathbb{U} \arrow[d] \\ D_\lambda \arrow[r] & H. \end{tikzcd}$$

We consider the determinant

$$\det : (\mathrm{W}_N(k)^n)^n = \mathrm{W}_N(k)^{n\times n} \to \mathrm{W}_N(k)$$

and its Greenberg realization

$$\Delta = (\Delta_0, \ldots, \Delta_{N-1}) = F(\det) : (\mathbb{A}_k^{N\times n})^n \to \mathbb{A}_k^N.$$

We obtain the commutative diagram
$$\begin{tikzcd} (\mathbb{U}_\lambda)^n \arrow[r, hook] \arrow[dr] & (\mathbb{A}_{D_\lambda}^{N\times n})^n \arrow[r, "\Delta"] \arrow[d] & \mathbb{A}_{D_\lambda}^N \arrow[dl] \\ & D_\lambda & \end{tikzcd}.$$

Let us set $\Lambda = -n\lambda_n (= \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n)$ and observe, that in fact we have a factorization

$$\Delta : (\mathbb{U}_\lambda)^n \to 0 \times \cdots \times 0 \times \mathbb{A}_{D_\lambda}^{N-\Lambda} \hookrightarrow \mathbb{A}_{D_\lambda}^N,$$

which can be interpreted as follows: The set of k-valued points of the fiber in U_λ over any point in D_λ is a submodule of $\mathrm{W}_N(k)^n$ whose determinant is precisely the ideal $(p^\Lambda) \subset \mathrm{W}_N(k)$. This factorization is due to the fact that Δ factorizes in this way over the point $V_\lambda \in D_\lambda$ and hence over its $\mathrm{L}_p^+ \mathrm{Sl}_n$ orbit, since the determinant map is invariant under the $\mathrm{L}_p^+ \mathrm{Sl}_n$-operation on $\mathbb{U}_\lambda \to D_\lambda$. Consequently, Δ factorizes also through the orbit-closure D_λ.

We set $Y := 0 \times \cdots \times 0 \times \mathbb{A}_{D_\lambda}^{N-\Lambda}$, and further we denote by $Y' \subset Y$ the open subvariety $Y' = 0 \times \cdots \times 0 \times (\mathbb{A}_{D_\lambda} - \{0\}) \times \mathbb{A}_{D_\lambda}^{N-\Lambda-1}$. While the k-valued points of Y correspond to the Witt vectors in $(p^\Lambda) \subset \mathrm{W}_N(k)$, the k-valued points of Y' correspond to $(p^\Lambda) - (p^{\Lambda+1}) \subset \mathrm{W}_N(k)$.

We define X so to make the following diagram cartesian:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y' \arrow[d, "\varphi"] \\ (\mathbb{U}_\lambda)^n \arrow[r, "\Delta"] & Y. \end{tikzcd}$$

Thus X is an open subvariety of $(\mathbb{U}_\lambda)^n$, and since $\mathbb{U}_\lambda \to D_\lambda$ is flat by construction, also the morphism $X \to D_\lambda$ is *flat*. We even have

PROPOSITION 5.13. *The morphism $X \to D_\lambda$ is faithfully flat and quasi-compact.*

PROOF. Quasi-compactness of $X \to D_\lambda$ is trivial, and we have already argued that it is flat. So it remains to check its surjectivity. But this is also easy: Take any point $x \in D_\lambda$, and let $\kappa(x)$ be its residue field and $\overline{\kappa(x)}$ its algebraic closure. Then the fiber $(\mathbb{U}_\lambda)^n \times_{D_\lambda} \overline{\kappa(x)}$ admits a family of $\overline{\kappa(x)}$-valued points with determinant $\equiv p^\Lambda \mod p^{\Lambda+1}$, i.e. a section which factors through the subscheme $X \subset (\mathbb{U}_\lambda)^n$. $\square$

This gives us an fpqc-covering $X \to D_\lambda$ with the property that 'locally on X' the family $(\mathbb{U}_\lambda)^n \to D_\lambda$ has an algebraic family of sections $s : X \to (\mathbb{U}_\lambda)^n \times_{D_\lambda} X$ whose determinant is non-zero $\mod p^{\Lambda+1}$. Namely, we can take

$$\begin{tikzcd} X \times_{D_\lambda} (\mathbb{U}_\lambda)^n \arrow[r] \arrow[d] & (\mathbb{U}_\lambda)^n \arrow[d] \\ X \arrow[r] \arrow[u, bend left, "s"] & D_\lambda, \end{tikzcd}$$

the section s on the left being given by the product of the identity and the open immersion $X \hookrightarrow (\mathbb{U}_\lambda)^n$. In other words, this family provides, when pulled back to any $x \in X$, a basis of the free $\mathrm{W}_N(\overline{\kappa(x)})$-module $\mathbb{U}_\lambda(\overline{\kappa(x)})$. We will use this section s to give an $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant morphism

$$X \to \mathrm{L}_p \mathrm{Sl}_n \to \mathcal{G}rass_p.$$

To this end we consider the following closed embedding of Greenberg realizations:

(5.2.3)
$$F(\mathrm{Mat}_n(\mathrm{W}_N(k))) \simeq (\mathbb{A}_k^N)^{n\times n} \hookrightarrow (\mathbb{A}_k^\infty)^{n \times n} = F(\mathrm{Mat}_n(\mathrm{W}(k))),$$

given by $(x_{(i,l),j=0,\dots,N-1}) \mapsto (x_{(i,l),0}, \dots, x_{(i,l),N-1}, 0, 0, \dots)$. It induces a map

$$X \to F(\mathrm{Gl}_{n,K})_{-\lambda_n} \subset F(\mathrm{Mat}_{n,K})_{-\lambda_n} = (\mathbb{A}_k^\infty)^{n\times n},$$

($F(\cdot)_{-\lambda_n}$, as usual, denotes the $-\lambda_n$-th scheme in the natural representation of the respective k-ind-schemes) and thus a morphism $X \to \mathrm{L}_p \mathrm{Gl}_{n,K}$. Composing with the morphism $\mathrm{L}_p(\mathrm{Gl}_{n,K} \to \mathrm{Sl}_{n,K})$ which divides the first column of any invertible matrix by its determinant, we obtain a morphism of k-ind-schemes

$$\Phi : X \to \mathrm{L}_p \mathrm{Sl}_n,$$

and hence $\bar{\Phi} : X \to \mathcal{G}rass_p$. In order to show that this morphism is $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant we have to check that $\bar{\Phi}$ does not 'depend on the 0's' in the map in (5.2.3), or, in other words, that putting any other sections of X in place of the 0's in (5.2.3) would not change $\bar{\Phi}$. This will follow from the next lemma.

For the formulation and proof of this lemma we denote by $\mathrm{Sl}_n(\mathrm{W}(R))'$

the image of the morphism

$$\mathrm{Sl}_n(\mathrm{W}(R)) \to \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$$

(which is not the same if the ring R is non-reduced), and analogously for $\mathrm{Mat}_n(\mathrm{W}(R))$ and $\mathrm{Gl}_n(\mathrm{W}(R))$.

LEMMA 5.14. *Let $m \in \mathbb{N}$, and let $A \in \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$ such that $p^m A^{-1} \in \mathrm{Mat}_n(\mathrm{W}(R))'$. If $B \in \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$ such that $A - B \in p^{m+1}\,\mathrm{Mat}_n(\mathrm{W}(R))'$, then $A^{-1}B \in \mathrm{Sl}_n(\mathrm{W}(R))'$.*

PROOF. By the hypotheses of the lemma we have $1 - A^{-1}B = A^{-1}(A-B) \in p\,\mathrm{Mat}_n(\mathrm{W}(R))'$. Using the geometric series one sees that $A^{-1}B$ is invertible in $\mathrm{Mat}_n(\mathrm{W}(R))'$, i.e. is an element of $\mathrm{Gl}_n(\mathrm{W}(R))'$. As both A and B have determinant 1, so has $A^{-1}B$. $\square$

In particular, A and B as in the lemma induce the same morphism $\mathrm{Spec}\, R \to \mathrm{L}_p\,\mathrm{Sl}_n \,/\, \mathrm{L}_p^+\,\mathrm{Sl}_n \to \mathcal{G}rass_p$. Now recall that we chose $N > \tilde{\lambda}_1 = \lambda_1 - \lambda_n$. Thus changing the morphism of (5.2.3) in those coordinates with $j \geq N$ amounts to changing the morphism Φ by something in $p^{\lambda_1+1}\,\mathrm{Mat}_n(\mathrm{W}(R))$. So the lemma tells us that in any case we get the same $\bar{\Phi} : X \to \mathcal{G}rass_p$, and $X \to \mathcal{G}rass_p$ is thus $\mathrm{L}_p^+\,\mathrm{Sl}_n$-equivariant.

By $\mathcal{X} \in \mathcal{G}rass_p(X)$ we denote the X-valued point corresponding to $\bar{\Phi}$.

THEOREM 5.15. *The X-valued point $\mathcal{X} \in \mathcal{G}rass_p(X)$ descends to a D_λ-valued point of $\mathcal{G}rass_p$. The corresponding morphism $\pi_\lambda : D_\lambda \to \mathcal{G}rass_p$ is equivariant for the (left-)action of $\mathrm{L}_p^+\,\mathrm{Sl}_n$ and sends the lattice scheme V_λ to the lattice $\mathbb{S}_\lambda = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}) \cdot \mathbb{S}$. Moreover, this map restricts to an isomorphism of the respective Schubert cells: $C_\lambda \simeq \mathcal{C}_\lambda$.*

PROOF. Since by definition $\mathcal{G}rass_p$ is an fpqc-sheaf and by Proposition 5.13 $X \to D_\lambda$ is faithfully flat, we have an exact sequence

$$\mathcal{G}rass_p(D_\lambda) \hookrightarrow \mathcal{G}rass_p(X) \rightrightarrows \mathcal{G}rass_p(X \times_{D_\lambda} X).$$

So we have to show that $\mathcal{X}$ is in the difference kernel of the maps $\mathcal{G}rass_p(X) \rightrightarrows \mathcal{G}rass_p(X \times_{D_\lambda} X)$. First observe that $X \to D_\lambda$ is an affine morphism, and that the descent problem is Zariski-local on D_λ. We may thus replace D_λ by an affine open subset $\mathrm{Spec}\, R \subset D_\lambda$, and X by $\mathrm{Spec}\, S = \mathrm{Spec}\, R \times_{D_\lambda} X$, and ask whether $\mathcal{X}_S : \mathrm{Spec}\, S \to X \to \mathcal{G}rass_p$ is in the difference kernel of the maps $\mathcal{G}rass_p(S) \rightrightarrows \mathcal{G}rass_p(S \otimes_R S)$. In other words, we have to check the following: Let

$$\Phi_1, \Phi_2 \in \mathrm{L}_p\,\mathrm{Sl}_n(S \otimes_R S) = \mathrm{Sl}_n(\mathrm{W}(S \otimes_R S)[1/p])$$

be the two compositions $\operatorname{Spec}(S \otimes_R S) \rightrightarrows \operatorname{Spec} S \xrightarrow{\Phi} \mathrm{L}_p \mathrm{Sl}_n$. Then we require $\Phi_1^{-1} \cdot \Phi_2$ to be in $\mathrm{Sl}_n(\mathrm{W}(S \otimes_R S))$ (possibly after faithfully flat base change). Let $k \subset \kappa$ be an algebraically closed field extension. By construction, a κ-valued point of $\operatorname{Spec}(S \otimes_R S)$ corresponds to a pair of bases of one and the same lattice (given by the corresponding κ-valued point of $\operatorname{Spec} R \subset D_\lambda$). Thus the map $\mathrm{Sl}_n(\mathrm{W}(S \otimes_R S)[1/p]) \to \mathrm{Sl}_n(\mathrm{W}(\kappa)[1/p])$ sends $\Phi_1^{-1} \cdot \Phi_2$ to $\mathrm{Sl}_n(\mathrm{W}(\kappa))$. This means, that $\Phi_1^{-1} \cdot \Phi_2 = \Psi + \Omega$, where $\Psi \in \mathrm{Sl}_n(\mathrm{W}(S \otimes_R S))$ (possibly after adjoining p-th roots, which is faithfully flat) and Ω has only nilpotent coefficients. Since multiplication by p-powers in $\mathrm{W}[1/p]$ kills nilpotent coefficients, we obtain that $\Phi_1^{-1} \cdot \Phi_2 = \Psi$ is in the image of $\mathrm{Sl}_n(\mathrm{W}(S \otimes_R S))$ in $\mathrm{Sl}_n(\mathrm{W}(S \otimes_R S)[1/p])$, which concludes the first part of the proof.

It is immediate from the definition of $\mathcal{X}$ that the induced morphism $\pi_\lambda : D_\lambda \to \mathcal{G}rass_p$ sends the lattice scheme V_λ to the lattice $\mathbb{S}_\lambda$. In order to see that π_λ induces an isomorphism of fpqc-sheaves $C_\lambda \simeq \mathcal{C}_\lambda$, let $\varphi_\lambda : \operatorname{Spec} k \to C_\lambda$ be the morphism corresponding to V_λ, and consider the diagram

$$\begin{tikzcd}
\mathrm{L}_p^+ \mathrm{Sl}_n \arrow[r, hook, "(\mathrm{id},\varphi_\lambda)"] \arrow[d, equal] & \mathrm{L}_p^+ \mathrm{Sl}_n \times C_\lambda \arrow[r] \arrow[d, "(\mathrm{id},\pi_\lambda)"] & C_\lambda \arrow[d, "\pi_\lambda"] \\
\mathrm{L}_p^+ \mathrm{Sl}_n \arrow[r, hook, "(\mathrm{id},\varphi_\lambda)"] & \mathrm{L}_p^+ \mathrm{Sl}_n \times \mathcal{C}_\lambda \arrow[r] & \mathcal{C}_\lambda,
\end{tikzcd}$$

where the right hand horizontal maps are the maps defining the respective left actions on C_λ and $\mathcal{C}_\lambda$, respectively. Both horizontal compositions are surjective maps of fpqc-sheaves, and to check that $\pi_\lambda : C_\lambda \to \mathcal{C}_\lambda$ is an isomorphism it suffices to check that the stabilizers, i.e. the respective preimages of V_λ and $\mathbb{S}_\lambda$ in $\mathrm{L}_p^+ \mathrm{Sl}_n$, are equal. But considering an R-valued point $P \in \mathrm{L}_p^+ \mathrm{Sl}_n$ as an element in $\mathrm{Sl}_n(\mathrm{W}(R))$ and looking at the definitions of V_λ and $\mathbb{S}_\lambda$, it is immediate that P stabilizes $\mathbb{S}_\lambda$ if and only if it stabilizes V_λ. $\square$

Note, that for the last line in this proof it is essential that V_λ is reduced. An example for how the situation looks like in the case where V_λ is non-reduced is presented in the following subsection.

5.2.4. Properties of the morphism $D_\lambda \to \mathcal{G}rass_p$. It would be desirable that the isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$ extended to a closed immersion of functors $D_\lambda \to \mathcal{G}rass_p$, in order to obtain 'Schubert varieties' in the p-adic setting. Unfortunately, this is not the case, for the reason that the final assertion on the equality of stabilizers in the preceeding proof does not hold for points in $D_\lambda - C_\lambda$.

EXAMPLE 5.16. *Let $n = 2$ and let $\lambda = (-1,1) \in \mathbb{Z}^2$, $N = 3$. Then $V_\lambda \subset \mathbb{A}_k^{2\times 3} = \operatorname{Spec} k[x_0,x_1,x_2,y_0,y_1,y_2]$ is defined by the ideal $I_\lambda = \langle x_0, x_1\rangle$. By exactly the same calculation as in Example 3.7 we check the effect of a matrix*

$$A = \begin{pmatrix} (a_0,a_1) & (b_0,b_1) \\ (c_0,c_1) & (d_0,d_1) \end{pmatrix} \in \mathrm{Sl}_2(\mathrm{W}(k)) = \mathrm{L}_p^+ \mathrm{Sl}_n(k)$$

on the ideal I_λ. We obtain

$$A \cdot I_\lambda = \langle a_0x_0 + b_0y_0, a_0^p x_1 + a_1 x_0^p + b_0^p y_1 + b_1 y_0^p + S\rangle, \qquad \text{with } a_0 \neq 0 \text{ or } b_0 \neq 0,$$

*where $S = -\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}(a_0x_0)^i(b_0y_0)^{p-i}$. As soon as $p > 2$, we see that $S \in \langle a_0x_0 + b_0y_0\rangle$, which shows that **the $\mathrm{Sl}_2(\mathrm{W}(k))$-orbit of $I_\lambda \in H$ is exactly the same as the $\mathrm{Sl}_2(k[[z]]^F)$-orbit of $I(\lambda) \in H$, defined in Chapter 3. Thus $D_{(1,-1)}$ and $\mathcal{D}((1,-1))$ of Chapter 3 coincide.***

It is easy to see that the boundary $D_\lambda - C_\lambda$ parametrizes the ideals of the form $\langle a_0x_0 + b_0y_0, x_0^p, y_0^p \mid a_0 \neq 0 \text{ or } b_0 \neq 0\rangle$ (also if $p = 2$). Hence $D_\lambda - C_\lambda \simeq \mathbb{P}_k^1$, and this whole $\mathbb{P}_k^1$ maps to the standard lattice $\mathbb{S} \in \mathcal{G}rass_p(k)$. In particular, the isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$ does not extend to an immersion $D_\lambda \to \mathcal{G}rass_p$. In terms of stabilizers we may state that the standard lattice $\mathbb{S}_\lambda$ is fixed e.g. by the matrix which swaps the x- and y-coordinates, while the points of $\mathbb{P}_k^1$ are in general not fixed.

There seems to be no way out of this situation. Namely, the reason for the phenomenon that there are in general many different points in D_λ mapping to the same point in $\mathcal{G}rass_p$ is the following: The subschemes of affine space which correspond to points in $D_\lambda - C_\lambda$ carry infinitesimal structure, which is forgotten by the map $D_\lambda \to \mathcal{G}rass_p$. On the other hand, these infinitesimal structures cannot be avoided as soon as we represent lattices by points in a Hilbert scheme, since we are then forced to use a Hilbert scheme for a non-standard grading as described in (5.2.1). E.g. in the before-mentioned example, the ideals $\langle y_0, y_1\rangle$ and $\langle x_0, y_0\rangle$ can never have the same Hilbert function, whence the latter cannot lie in the orbit-closure in H of the former. However, e.g. $\langle x_0^p, y_0\rangle$ will be in the orbit-closure of $\langle y_0, y_1\rangle$.

Examples 3.7 and 5.16, and the phenomenon of lattice schemes with infinitesimal structure in general, suggest that there is a close relationship between the varieties $\mathcal{D}(\lambda)$ (constructed in Chapter 3) and the varieties D_λ of the present chapter. They also suggest that one should think of D_λ as some sort of Demazure resolution of a Schubert variety in $\mathcal{G}rass_p$, but I do not know at present how to make this a precise statement. At least, it seems reasonable to expect that the following holds true.

CONJECTURE 5.17. *The projective k-varieties D_λ are smooth.*

Though we have seen that the morphism $D_\lambda \to \mathcal{G}rass_p$ is in general not injective on the level of k-valued points, its image behaves as expected.

THEOREM 5.18. *We have $\mathcal{G}rass_p(k) = \cup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathcal{C}_\lambda(k)$, and on the level of k-valued points, $D_\lambda \to \mathcal{G}rass_p$ induces a surjection*

$$\pi_\lambda : D_\lambda(k) \twoheadrightarrow \cup_{\lambda' \leq \lambda} \mathcal{C}_{\lambda'}(k).$$

The symbol $\leq$ here refers to the Bruhat-order on $\check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$.

PROOF. The first claim follows from the elementary divisors theorem, as we have already explained at the end of section 5.1. In order to see that π is well-defined, we have to argue that none of the $\mathbb{S}_{\lambda'}$ with $\lambda' > \lambda \in \check{\mathrm{X}}_+(T)$ is in the image of π. So choose a $\lambda' > \lambda$. Then an easy combinatorial argument shows that the Hilbert function h' of the lattice scheme $V_{\lambda'}$ is bigger than that of V_λ itself, denoted h (where for two functions $h, h' : \mathbb{N} \to \mathbb{N}$ we say $h' > h$ if and only if $h'(n) \geq h(n)$ for all n and $h \neq h'$). But since V_λ is reduced, it has already the smallest possible Hilbert function among those lattice schemes which possibly map to $\mathbb{S}_{\lambda'}$. As D_λ contains only lattice schemes with the same Hilbert function as V_λ, $V_{\lambda'} \notin D_\lambda(k)$.

In order to prove surjectivity of π, we use an argument similar to the one given by Beauville and Laszlo in [**BL94**], Proposition 2.6: For integers $e > d$ consider the following equation of matrices over $\mathrm{W}(k((t)))[1/p]$.

(5.2.4)
$$\begin{pmatrix} 0 & t \\ -t^{-1} & t^{-1}p \end{pmatrix} \begin{pmatrix} p^e & 0 \\ 0 & p^d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ t^{-1}p^{e-d-1} & t \end{pmatrix} = \begin{pmatrix} p^{e-1} & t^2p^d \\ 0 & p^{d+1} \end{pmatrix}.$$

If we assume $e + d = 0$, it follows that the right hand matrix gives rise to a lattice scheme $V \in D_{(e,d)}(k((t)))$, which corresponds to a $k((t))$-point of $C_{(e,d)}$. Since $D_{(d,e)}$ is projective, this $k((t))$-valued point extends to a lattice scheme $\bar{V}$ over $k[[z]]$, whose fiber over $t = 0$ maps to $\mathbb{S}_{(d+1,e-1)}$. The case for a general n and λ is proved likewise. $\square$

5.3. p-adic Lattices

In Section 5.2 we have described the set of k-valued points of the p-adic affine Grassmannian $\mathcal{G}rass_p$. The purpose of the present section is to describe the R-valued points of $\mathcal{G}rass_p$ for more general k-algebras R.

Let us remind the reader of the description in terms of lattices of the R-valued points of the affine Grassmannian in the function field case, as we gave it in Section 1.2. Our goal is to obtain a similar notion of 'lattice', with analogous characterizations, in the Witt vector setting where R is a perfect k-algebra. As a corollary we will then obtain a description in terms of lattices of the R-valued points of $\mathcal{G}rass_p$ for R perfect. Recall that a ring R of characteristic $p > 0$ is called perfect, if the Frobenius homomorphism $x \mapsto x^p$ is an isomorphism.

DEFINITION 5.19. *Let R be any perfect k-algebra. A* lattice $L \subset \mathrm{W}(R)[1/p]^n$ *(or simply: a $\mathrm{W}(R)$-lattice of rank n) is a finitely generated, projective $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)[1/p]^n$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$. Further, a lattice $L \subset \mathrm{W}(R)$ is called* special, *if $\wedge^n L = \mathrm{W}(R)$. By $\mathcal{L}att_p^n(R)$ we denote the set of lattices of rank n over $\mathrm{W}(R)$, and $\mathcal{L}att_p^{n,0}(R) \subset \mathcal{L}att_p^n(R)$ is the subset of special lattices.*

If $R = k$ is a field, then we recover the usual notion of lattice over $\mathrm{W}(k)$. Let us note furthermore that for a finitely generated $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)^n$ the condition $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$ is equivalent to the existence of a natural number N such that $p^N \mathrm{W}(R)^n \subset L \subset p^{-N} \mathrm{W}(R)^n$.

First we want to see that the assignment $R \mapsto \mathcal{L}att_p^n(R)$ is a functor on the category of perfect k-algebras. To this end, we prove

LEMMA 5.20. *Let $R \to S$ be a homomorphism of perfect rings, and let $p^N \mathrm{W}(R)^n \subset L \subset p^{-N} \mathrm{W}(R)^n$ be a flat $\mathrm{W}(R)$-submodule. Then we have*

$$\mathrm{Tor}_1^{W(R)}(\mathrm{W}(R)^r/L, \mathrm{W}(S)) = 0.$$

Equivalently, this means $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \subset p^{-N} \mathrm{W}(S)^n \subset \mathrm{W}(S)[1/p]^n$.

PROOF. Let $F = p^{-N} \mathrm{W}(R)^n$ and consider the exact sequence

$$0 \to \mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S)) \to L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to \\ \to \mathrm{W}(S)^n \to F/L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to 0.$$

Since multiplication by p^{2N} is the zero-map on F/L, we see that p acts nilpotently on $\mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S))$. On the other hand, $(L \xrightarrow{p} L) \otimes \mathrm{W}(S) = L \otimes (\mathrm{W}(S) \xrightarrow{p} \mathrm{W}(S))$ is injective, since L is flat. Hence, p acts faithfully on $\mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S))$, which therefore vanishes. $\square$

PROPOSITION 5.21. *The assignment $R \mapsto \mathcal{L}att_p^n(R)$ defines a functor from the category of perfect rings to the category of sets. Namely, to any homomorphism $R \to S$ assign the map*

$$\mathcal{L}att_p^n(R) \to \mathcal{L}att_p^n(S); \quad L \mapsto L \otimes_{\mathrm{W}(R)} \mathrm{W}(S).$$

The assignment $R \mapsto \mathcal{L}att_p^{n,0}(R)$ is a subfunctor. □

The rest of this section is devoted to the study of the Zariski-/fpqc-sheaf properties of $\mathcal{L}att_p^n$ resp. $\mathcal{L}att_p^{n,0}$.

THEOREM 5.22. *(1) The functor $\mathcal{L}att_p^n$ is the Zariski-sheafification of the functor on the category of perfect k-algebras, which associates to any perfect k-algebra R the set of* free *rank-n lattices over* $\mathrm{W}(R)$.

(2) Moreover, $\mathcal{L}att_p^n$ is even an fpqc-sheaf on the category of perfect k-algebras. Together with (1) this says that $\mathcal{L}att_p^n$ is also the fpqc-sheafification of the functor which associates to any perfect k-algebra R the set of free rank-n lattices over $\mathrm{W}(R)$.

(3) The analogous assertions hold if we replace $\mathcal{L}att_p^n$ by $\mathcal{L}att_p^{n,0}$ and 'free lattices' by 'free special lattices'.

PROOF. It suffices to prove the first two parts of the theorem, part (3) will then follow. The first part of the theorem is easy: Since by definition $L \in \mathcal{L}att_p^n(R)$ is projective and finitely generated as a $\mathrm{W}(R)$-module, it is even finitely presented and (Zariski-)locally free over $\mathrm{W}(R)$. This means that there exist Witt vectors $f_1, \ldots, f_m \in \mathrm{W}(R)$ which generate the unit ideal in $\mathrm{W}(R)$ and such that for each $1 \leq i \leq m$ the localization $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/f_i]$ is free over $\mathrm{W}(R)[1/f_i]$. Denote by $g_i \in R$ the class mod p of f_i. Then the g_i generate the unit ideal in R, and I claim that the $\mathrm{W}(R[1/g_i])$-module $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R[1/g_i])$ is free for each i. Namely, if we denote by $[g_i]$ the Teichmüller representative of g_i we may write

$$f_i = [g_i] \cdot \alpha, \quad \alpha \in 1 + p\,\mathrm{W}(R) \subset \mathrm{W}(R)^\times$$

since we assumed R to be perfect. Thus we have

$$\mathrm{W}(R)[1/f_i] = \mathrm{W}(R)[1/[g_i]] \subset \mathrm{W}(R[1/g_i]),$$

and we may choose $\coprod_{i=1}^m \operatorname{Spec} R[1/g_i] \to \operatorname{Spec} R$ as a Zariski-covering on which L becomes free.

The proof of part (2) requires more work and will occupy us for the rest of this section.

LEMMA 5.23. *Let $R \to S$ be a homomorphism of* perfect *rings. Then*

$$\mathrm{W}_N(S) \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) = \mathrm{W}_N(S \otimes_R S).$$

PROOF. The ring $\mathrm{W}(S \otimes_R S)$ carries a natural structure of $\mathrm{W}(R)$-module, and for this module structure we have a linear map $\mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to \mathrm{W}(S \otimes_R S)$. We will show by induction on N that this map reduces to an isomorphism modulo p^N for every N, the case $N = 1$ being trivial. Assume now that $N > 1$, that the induced map $\mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S)/p^{N-1} \to \mathrm{W}(S \otimes_R S)/p^{N-1}$ is an isomorphism and consider the commutative diagram

$$\begin{tikzcd}
p^{N-1}\,\mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r, hook] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r, two heads] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^{N-1} \arrow[d] \\
p^{N-1}\,\mathrm{W}(S \otimes_R S)/p^N \arrow[r, hook] & \mathrm{W}(S \otimes_R S)/p^N \arrow[r, two heads] & \mathrm{W}(S \otimes_R S)/p^{N-1}.
\end{tikzcd}$$

Since $\mathrm{W}(S) \otimes \mathrm{W}(S)$ has no p-torsion, this diagram is isomorphic to

$$\begin{tikzcd}
S \otimes_R S \arrow[r, hook] \arrow[d, "\mathrm{id}"] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r, two heads] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^{N-1} \arrow[d, "\simeq"] \\
S \otimes_R S \arrow[r, hook] & \mathrm{W}(S \otimes_R S)/p^N \arrow[r, two heads] & \mathrm{W}(S \otimes_R S)/p^{N-1}.
\end{tikzcd}$$

On applying the 5-lemma we see that

$$\mathrm{W}_N(S) \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) = \mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S)/p^N = \mathrm{W}(S \otimes_R S)/p^N,$$

which finishes the induction step. □

LEMMA 5.24. *Let $R \to S$ be a homomorphism of perfect rings. Then for every $N \geq 1$:*

(1) $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ *is flat if and only if* $R \to S$ *is flat,*
(2) $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ *is faithful if and only if* $R \to S$ *is faithful.*

(A homomorphism of rings is said to be faithful iff it induces a surjective map on the associated spectra.)

PROOF. Let $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ be flat, and let $M \hookrightarrow M'$ be an injection of R-modules. Since every R-module is also a $\mathrm{W}_N(R)$-module via the residue map $\mathrm{W}_N(R) \to R$, we obtain

$$\begin{tikzcd}
M \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) \arrow[r, hook] \arrow[d, "\mathrm{id}"] & M' \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) \arrow[d, "\mathrm{id}"] \\
M \otimes_R S \arrow[r, hook, dotted] & M' \otimes_R S.
\end{tikzcd}$$

Thus also $R \to S$ is flat. To prove the converse, we use the following theorem of Govorov and Lazard ([**Eis95**] Theorem A6.6): An R-module is flat if and only if it is the colimit of a *filtered* direct system of free modules. Moreover we note that in this situation the colimit in the category of sets coincides with the colimit in the category of R-modules.

So let $(F_i \simeq R^{d_i})_i$ be a filtered direct system having S as its colimit (the d_i may be infinite). I claim that $\mathrm{W}_N(S)$ is the filtered colimit of the induced filtered direct system $(\mathrm{W}_N(F_i) := (\mathrm{W}_N(R)^{d_i})_i$. As noted before, the filtered direct limit of $(\mathrm{W}_N(F_i))_i$ can be calculated in the category of sets, and there we have $\mathrm{W}_N(F_i) = (R^N)^{d_i}$. But since filtered direct limits commute with finite products we obtain

$$\varinjlim \mathrm{W}_N(F_i) = \varinjlim (R^{d_i})^N = (\varinjlim R^{d_i})^N = S^N = \mathrm{W}_N(S).$$

In other words, the $\mathrm{W}_N(R)$-module $\mathrm{W}_N(S)$ is the colimit of a direct system of free $\mathrm{W}_N(R)$-modules, hence it is flat.

To prove the second statment, we just note that for every ring R the reduction mod p, $\mathrm{W}_N(R) \to R$, induces a bijection between the associated spectra:

$$\operatorname{Spec} R \xrightarrow{\simeq} \operatorname{Spec} \mathrm{W}_N(R).$$

Namely, since p is nilpotent in $\mathrm{W}_N(R)$ it is contained in every prime ideal of $\mathrm{W}_N(R)$. □

We are now able to prove that the functor $R \mapsto \mathcal{L}att_p^n(R)$ is a sheaf for the fpqc-topology on the category of perfect rings. To begin with, note that for any perfect ring R and any $\mathrm{W}(R)$-submodule $M \subset \mathrm{W}(R)[1/p]^n$ satisfying $p^N \mathrm{W}(R)^n \subset M \subset p^{-N} \mathrm{W}(R)^n$ for some N, we have

$$\varprojlim (M \otimes \mathrm{W}(R)/p^i \mathrm{W}(R)) = \varprojlim M/p^j \mathrm{W}(R)^n = M. \tag{5.3.1}$$

Here the first equality holds since the respective inverse systems are coinitial, while the second equality follows from the short exact sequence

$$0 \to M/p^j \mathrm{W}(R)^n \to p^{-N} \mathrm{W}(R)^n/p^j \mathrm{W}(R)^n \to \\ \to p^{-N} \mathrm{W}(R)^n/M \to 0 \quad (j >> 0)$$

upon passage to the inverse limit.

Since we already know that $\mathcal{L}att_p^n$ is a Zariski-sheaf, it suffices by Theorem A.4 to consider a faithfully flat homomorphism $R \to S$ of perfect rings, and show that the sequence

$$\mathcal{L}att_p^n(R) \to \mathcal{L}att_p^n(S) \rightrightarrows \mathcal{L}att_p^n(S \otimes_R S) \tag{5.3.2}$$

is an equalizer.

(1) $\mathcal{L}att_p^n(R) \to \mathcal{L}att_p^n(S)$ is injective: Take $L, L' \in \mathcal{L}att_p^n(R)$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) = L' \otimes_{\mathrm{W}(R)} \mathrm{W}(S)$. By Lemma 5.24 we know that $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ is faithfully flat for every N, which tells us that $L \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R) = L' \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R)$. Using (5.3.1) this proves $L = L'$.

(2) The difference kernel of $\mathcal{L}att_p^n(S) \rightrightarrows \mathcal{L}att_p^n(S \otimes_R S)$ is equal to $\mathcal{L}att_p^n(R)$: Clearly, $\mathcal{L}att_p^n(R)$ is contained in the difference kernel. Conversely, choose $L \in \mathcal{L}att_p^n(S)$, such that $L' = L \otimes_{\mathrm{W}(S),1} \mathrm{W}(S \otimes_R S)$ equals $L'' = L \otimes_{\mathrm{W}(S),2} \mathrm{W}(S \otimes_R S)$. Note that by the indices 1 and 2, respectively, at the $\otimes$-symbol we indicate which module structure on $\mathrm{W}(S \otimes_R S)$ is under consideration. Then

$$\begin{aligned}(L \otimes \mathrm{W}_i(S)) \otimes_{\mathrm{W}_i(S),1} (\mathrm{W}_i(S) \otimes_{\mathrm{W}_i(R)} \mathrm{W}_i(S)) = \\ = (L \otimes \mathrm{W}_i(S)) \otimes_{\mathrm{W}_i(S),2} (\mathrm{W}_i(S) \otimes_{\mathrm{W}_i(R)} \mathrm{W}_i(S)),\end{aligned} \tag{5.3.3}$$

and similarly

$$\begin{aligned}(L/p^i(\mathrm{W}(S))^n) \otimes_{\mathrm{W}_{N+i}(S),1} (\mathrm{W}_{N+i}(S) \otimes_{\mathrm{W}_{N+i}(R)} \mathrm{W}_{N+i}(S)) = \\ = (L/p^i(\mathrm{W}(S))^n) \otimes_{\mathrm{W}_{N+i}(S),2} (\mathrm{W}_{N+i}(S) \otimes_{\mathrm{W}_{N+i}(R)} \mathrm{W}_{N+i}(S))\end{aligned} \tag{5.3.4}$$

for i big enough. (here we use Lemma 5.23).

For $i > 2N$ we consider now the diagram of $\mathrm{W}_{i+N}(S)$-modules

$$\begin{tikzcd}
(p^{-N}\mathrm{W}(S)^n)/(p^{i+N}\mathrm{W}(S)^n) \arrow[r] & p^{-N}\mathrm{W}_i(S)^n \arrow[r, equal] & (p^{-N}\mathrm{W}(S)^n)/(p^{i-N}\mathrm{W}(S)^n) \\
L/p^{N+i}\mathrm{W}(S)^n \arrow[u, hook] \arrow[r, two heads] & L \otimes_{\mathrm{W}(S)} \mathrm{W}_i(S) \arrow[u] \arrow[r, two heads] & L/p^{i-N}\mathrm{W}(S)^n \arrow[u, hook] \\
(p^{N}\mathrm{W}(S)^n)/(p^{i+N}\mathrm{W}(S)^n) \arrow[u, hook] \arrow[r, equal] & p^{N}\mathrm{W}_i(S)^n \arrow[u] \arrow[r] & (p^{N}\mathrm{W}(S)^n)/(p^{i-N}\mathrm{W}(S)^n) \arrow[u, hook]
\end{tikzcd}$$

Now (5.3.3) and (5.3.4) together with Lemma 5.24 say that this diagram descends to a diagram of $\mathrm{W}_{i+N}(R)$-modules, i.e. we obtain

$$\begin{tikzcd}
(p^{-N}\mathrm{W}(R)^n)/(p^{i+N}\mathrm{W}(R)^n) \arrow[r] & p^{-N}\mathrm{W}_i(R)^n \arrow[r, equal] & (p^{-N}\mathrm{W}(R)^n)/(p^{i-N}\mathrm{W}(R)^n) \\
P_{N+i} \arrow[u, hook] \arrow[r, two heads] & M_i \arrow[u] \arrow[r, two heads] & P_{i-N} \arrow[u, hook] \\
(p^{N}\mathrm{W}(R)^n)/(p^{i+N}\mathrm{W}(R)^n) \arrow[u, hook] \arrow[r, equal] & p^{N}\mathrm{W}_i(R)^n \arrow[u] \arrow[r] & (p^{N}\mathrm{W}(R)^n)/(p^{i-N}\mathrm{W}(R)^n) \arrow[u, hook]
\end{tikzcd}$$

We thus have two cofinal systems of $\mathrm{W}(R)$-modules, (M_i) and (P_i), whose inverse limit is a $\mathrm{W}(R)$-module M. I claim that this is the desired $\mathrm{W}(R)$-lattice. First observe that for N big enough we have an exact sequence

$$0 \to p^N\,\mathrm{W}(R)^n \hookrightarrow M \to P_N \to 0,$$

as we see by taking the inverse limit over $i > N$ of the sequence

$$0 \to p^N\,\mathrm{W}(R)^n/p^i\,\mathrm{W}(R)^n \hookrightarrow P_i \to P_i/p^N\,\mathrm{W}(R)^n = P_N \to 0.$$

Since P_i is finitely generated (by faithfully flat descent) as well as $p^N\,\mathrm{W}(R)^n$, also M is finitely generated. On the other hand, since

$$0 \to p^N L \otimes_{\mathrm{W}(S)} \mathrm{W}_i(S) \to L \otimes_{\mathrm{W}(S)} \mathrm{W}_{i+N}(S) \to L \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S) \to 0$$

is exact, we obtain by faithfully flat descent a short exact sequence

$$0 \to p^N M_i \to M_{i+N} \to M_N \to 0.$$

Passing to the inverse limit over i we obtain

$$0 \to p^N M \to M \to M_N \to 0,$$

and thus $M \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R) = M_N$, which is a projective $\mathrm{W}_N(R)$-module, by faithfully flat descent. Hence we have arrived at a situation where Lemma 1.10 applies, proving that $M = \varprojlim(M \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R))$ is a $\mathrm{W}(R)$-lattice. Clearly, $(M \otimes_{\mathrm{W}(R)} \mathrm{W}(S)) \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S) = M_N \otimes \mathrm{W}_N(S) = L \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S)$. Taking the limit over N we obtain $M \otimes_{\mathrm{W}(R)} \mathrm{W}(S) = L$, which finishes the proof. □

As a corollary we obtain the desired analogon of Theorem 1.9 in the Witt vector setting.

Corollary 5.25. *Let R be a perfect k-algebra. For any $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)[1/p]^n$, the following are equivalent:*

(1) *The submodule L is a lattice.*
(2) *Zariski-locally on R, L is a free $\mathrm{W}(R)$-submodule of rank n (i.e. there exist $f_1, \dots, f_r \in R$ such that $(f_1, \dots, f_r) = \mathrm{W}(R)$ and for all i, $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R_{f_i})$ is free of rank n and $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$.*
(3) *fpqc-locally on R, L is a free $\mathrm{W}(R)$-submodule of rank n (i.e. there exists a faithfully flat ring homomorphisms $R \to S$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S)$ is free of rank n and $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$.*

Proof. This follows immediately from Theorem 5.22. □

It is not clear to me whether there is a good translation of condition (4) of Theorem 1.9 to the Witt vector setting. The obvious obstacle is the fact that $\mathrm{W}(R)$ does not carry a structure of R-module.

Corollary 5.26. *The fpqc-sheaf $\mathcal{L}att_p^{n,0}$ is equal to the restriction of the p-adic affine Grassmannian $\mathcal{G}rass_p$ to the category of perfect k-algebras.*

Proof. The presheaf $R \mapsto \mathrm{Sl}_n(\mathrm{W}(R)[1/p])/\,\mathrm{Sl}_n(\mathrm{W}(R))$ coincides with the presheaf $R \mapsto \{$ free special lattices of rank n over $\mathrm{W}(R)$ $\}$ on the category of perfect k-algebras. Thus it suffices to prove that for any presheaf F on the fpqc-site over k the processes of 'sheafification' and 'restriction to the category of perfect k-algebras' commute. Let R be a perfect k-algebra and let $\{U_i \to \operatorname{Spec} R\}$ be a covering (on the fpqc-site over k). Refining the covering we may assume that the U_i are all affine. For every i denote by U_i^{perf} the perfection of U_i. Then the morphisms $U_i^{perf} \to \operatorname{Spec} R$ are still flat and jointly surjective and thus define a refinement of $\{U_i \to \operatorname{Spec} R\}$, which is by definition also a covering in the fpqc-site on the category of perfect k-algebras. Now the claim follows from Lemma A.3 in the appendix. $\square$

APPENDIX A

FPQC-Sheaves and Sheafifications

Definition A.1. *(Sheafification) Let $\mathcal{C}$ be an arbitrary site and let F be a (Set)-valued functor on the underlying category, i.e. a presheaf on $\mathcal{C}$ with values in (Set). A* sheafification *of F is a morphism of presheaves $\varphi : F \to F^a$ where F^a is a sheaf and such that φ induces a natural isomorphism* $\mathrm{Hom}_{(Sh)}(F^a, S) \simeq \mathrm{Hom}_{(Presh)}(F, S)$ *for every sheaf S on $\mathcal{C}$.*

Proposition A.2 (Characterization of sheafification). *Let F be a presheaf on the site $\mathcal{C}$. A morphism of presheaves $\varphi : F \to G$ is a sheafification if and only if G is a sheaf and the following two conditions hold:*

(1) *If $\xi, \xi' \in F(X)$ have the same image in $G(X)$, then there exists a covering $\pi : \mathcal{U} \to X$ such that $\pi^*(\xi) = \pi^*(\xi')$, and*
(2) *for every $\xi \in G(X)$ there exists a covering $\pi : \mathcal{U} \to X$ and $\eta \in F(\mathcal{U})$ such that $\pi^*(\xi) = \varphi_*(\eta)$.*

Proof. Let S be a sheaf on $\mathcal{C}$ and let $\psi : F \to S$ be a morphism of presheaves. Assume $\varphi : F \to G$ as in the proposition. Then for every $X \in \mathcal{C}$ we have to define $\psi'(X) : G(X) \to S(X)$ in a functorial way. Let $\mathcal{U} \to X$ and $\mathcal{V} \to \mathcal{U} \times_X \mathcal{U}$ be coverings in $\mathcal{C}$, and consider the diagram

$$\begin{tikzcd}
G(X) \arrow[r, hook] & G(\mathcal{U}) \arrow[r, shift left] \arrow[r, shift right] & G(\mathcal{U} \times_X \mathcal{U}) \arrow[r, hook] & G(\mathcal{V}) \\
F(X) \arrow[r] \arrow[u, "\varphi"] \arrow[d, "\psi"'] & F(\mathcal{U}) \arrow[r] \arrow[u] \arrow[d] & F(\mathcal{U} \times_X \mathcal{U}) \arrow[r] \arrow[u] \arrow[d] & F(\mathcal{V}) \arrow[u] \arrow[d] \\
S(X) \arrow[r, hook] & S(\mathcal{U}) \arrow[r, shift left] \arrow[r, shift right] & S(\mathcal{U} \times_X \mathcal{U}) \arrow[r, hook] & S(\mathcal{V}).
\end{tikzcd}$$

The proposition is obtained by chasing this diagram, as we explain now. Given $\xi \in G(X)$ we may choose $\mathcal{U}$ 'fine enough' (by (2)) so that $\xi|_{\mathcal{U}} = \varphi_*(\eta)$ for some $\eta \in F(\mathcal{U})$. I claim that $\psi(\eta) \in S(\mathcal{U})$ descends to an element $\psi'(X)(\xi) \in S(X)$. Indeed, since $\xi \in G(X)$, both images of η in $F(\mathcal{U} \times_X \mathcal{U})$ map to the same element in $G(\mathcal{U} \times_X \mathcal{U})$. Hence, by (1), we may choose $\mathcal{V}$ fine enough so that both images of η in $F(\mathcal{V})$ coincide.

This means in turn that the two images of η in $S(\mathcal{V})$, and hence those in $S(\mathcal{U}\times_X\mathcal{U})$ conincide. Thus, since S is a sheaf, $\psi(\eta)$ descends to element $\psi'(X)(\xi) \in S(X)$, as claimed. Moreover, $\psi'(X)(\xi)$ does not depend on the choice of $\mathcal{U}$ and η. Namely, given $\eta \in F(\mathcal{U})$ and $\eta' \in F(\mathcal{U}')$, we may assume, after replacing $\mathcal{U}$ and $\mathcal{U}'$ by a common refinement, that $\mathcal{U} = \mathcal{U}'$. Since both η and η' map to $\xi|_{\mathcal{U}} \in G(\mathcal{U})$, we can, by (1), replace $\mathcal{U}$ by a refinement so that $\eta = \eta'$. Then $\psi(\eta) = \psi(\eta')$ is trivial, and hence $\psi'(X)(\xi) \in S(X)$ does not depend on $\mathcal{U}$ and η.

Hence we obtain a morphism of functors $\psi' : G \to S$ which satisfies $\psi = \psi' \circ \varphi$. By (1) this is the unique morphism with this property, which establishes the desired isomorphism $\mathrm{Hom}_{(\mathrm{Sh})}(G, S) \simeq \mathrm{Hom}_{(\mathrm{Presh})}(F, S)$. □

LEMMA A.3. *Let $\mathcal{D} \subset \mathcal{C}$ be an inclusion of sites, such that fiber products in $\mathcal{D}$ are mapped to fiber products in $\mathcal{C}$. Assume that for every covering $\mathcal{U} = \{U_i \to X\}$ in $\mathcal{C}$ of an object $X \in \mathcal{D}$ there exists a refinement $\mathcal{V} = \{V_i \to X\}$ of $\mathcal{U}$ with $V_i \in \mathcal{D}$ such that $\mathcal{V}$ is also a covering of X in $\mathcal{D}$.*

Claim: if F has a sheafifcation F^a, then $F^a|_{\mathcal{D}}$ is a (the) sheafification of $F|_{\mathcal{D}}$.

PROOF. Let F^a be the sheafification of F. Clearly, $F^a|_{\mathcal{D}}$ is a sheaf on $\mathcal{D}$, whence the canonical map $(F|_{\mathcal{D}})^a \to (F^a)|_{\mathcal{D}}$. To prove that this is an isomorphism, we check that the morphism $F|_{\mathcal{D}} \to (F^a)|_{\mathcal{D}}$ is a sheafification on $\mathcal{D}$. More precisely, we check the two conditions of Proposition A.2. Thus let $X \in \mathcal{D}$ and let $\xi, \eta \in F(X)$ such that their images in $F^a(X)$ coincide. By definition of sheafification there exists a covering (in $\mathcal{C}$) of X on which ξ and η coincide. But by assumption this covering can be refined so to obtain a covering of X in $\mathcal{D}$ on which ξ and η coincide a fortiori. This is condidition (1). On the other hand, every element $\xi \in F^a(X)$ can be represented locally (on a covering in $\mathcal{C}$) by sections of F. Refining this covering, we see that ξ can be represented on a covering in $\mathcal{D}$ by sections of F. This is (2). □

THEOREM A.4. *Let F be a presheaf on the fpqc-site over the category $\mathcal{C}$ of schemes. Assume that F is a sheaf for the Zariski topology. Then F is an fpqc-sheaf on $\mathcal{C}$ if and only if for every faithfully flat homomorphism of affine schemes $Y \to X$ the sequence*

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y) \tag{A.0.5}$$

is an equalizer.

PROOF. See Vistoli [**Vis08**]. □

PROPOSITION A.5. *Let F be a functor (= fpqc-presheaf) on the category of schemes. Assume that F satisfies the following two conditions:*

(1) *for every faithfully flat morphism of* affine *schemes $Y \to X$ the sequence*

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y)$$

is an equalizer, and

(2) *for every finite collection of affine schemes $Y_1, \dots, Y_n$ we have*

$$F(Y_1 \coprod \cdots \coprod Y_n) = F(Y_1) \times \cdots \times F(Y_n).$$

Then the Zariski-sheafification F^a of F is an fpqc-sheaf. In particular, F^a is an fpqc-sheafification of F. Moreover, the natural transformation $F \to F^a$ restricts to an isomorphism on the category of affine schemes.

PROOF. In view of Theorem A.4 we only have to prove that the condition in (1) of the present proposition remains valid after Zariski-sheafification. Thus it will suffice to prove the last assertion, namely that the natural map $F(X) \to F^a(X)$ is indeed an isomorphism for every affine X. To this end, for an arbitrary scheme X and any Zariski-covering $\mathcal{U}$ of X let $K(\mathcal{U})$ be the difference kernel of $F(\mathcal{U}) \rightrightarrows F(\mathcal{U} \times_X \mathcal{U})$. If we set $F'(X) = \varinjlim_{\mathcal{U}} K(\mathcal{U})$, where the colimit is taken over all Zariski-coverings of X, then F' will be a separated presheaf. Applying this procedure twice, i.e. forming F'', will yield a *sheaf*, and indeed F'' is equal to the Zariski-sheafification F^a of F. Now observe the following: if X is affine, there is a cofinal subsystem of all Zariski coverings of X given by those coverings which consist of only *finitely many affines.* Thus, using assumption (2),

$$F'(X) = \varinjlim_{Y \to X} \ker(F(Y) \rightrightarrows F(Y \times_X Y)),$$

where now the limit is taken over a certain family of faithfully flat morphisms $Y \to X$ of affine schemes. But by assumption (1) for every such $Y \to X$ we have $F(X) = \ker(F(Y) \rightrightarrows F(Y \times_X Y))$, whence $F'(X) = F(X)$. This implies $F^a(X) = F(X)$, as desired. □

COROLLARY A.6. *Let F be as in the proposition. Then the restriction of F to the site of affine schemes (with arbitrary covering families consisting of affine schemes) is a sheaf for the fpqc-topology.*

COROLLARY A.7. *A functor which is represented by an inductive system of schemes admits an fpqc-sheafification. Indeed, it suffices to take its Zariski-sheafification, which is then automatically an fpqc-sheaf(ification). Moreover, the restriction of this sheafification to the* *category of affine schemes coincides with the original presheaf defined by the inductive system of schemes.*

Proof. We have to check that such a functor satisfies the assumptions (1) and (2) of Proposition A.5. To this end, let (X_i) be a direct system of schemes and let $\varinjlim X_i$ be its colimit in the category of presheaves. Let $T_1, \ldots, T_n$ be affine schemes. Then we have

$$(\varinjlim X_i)(T_1 \coprod \cdots \coprod T_n) = \varinjlim(X_i(T_1 \coprod \cdots \coprod T_n)) = \\ = \varinjlim(X_i(T_1) \times \cdots \times X_i(T_n)) = (\varinjlim X_i)(T_1) \times \cdots \times (\varinjlim X_i)(T_n),$$

which is condition (2). It remains to check exactness of the sequence

$$(\varinjlim X_i)(R) \to (\varinjlim X_i)(S) \rightrightarrows (\varinjlim X_i)(S \otimes_R S),$$

where $R \to S$ is a faithfully flat homomorphism of rings. Thus let $P \in (\varinjlim X_i)(S)$ such that both images of P in $(\varinjlim X_i)(S \otimes_R S)$ coincide. Assume that P is represented by an element $P' \in X_i(S)$. By definition of the inductive limit, there exists some $i \leq j \in I$ such that that the induced objects in $X_j(S \otimes_R S)$ coincide. Now we can use the exactness of the sequence

$$X_j(R) \to X_j(S) \rightrightarrows X_j(S \otimes_R S)$$

to obtain an R-valued point of X_j, and hence an R-valued point of $\varinjlim X_i$ which induces P. This shows that the difference kernel of the right hand maps is precisely the image of the left hand map. Injectivity of the left hand map is proved likewise, which shows that condition (1) holds as well. □

In other words, if we restrict the functor direct-limit $\varinjlim X_i$ to the category of affine schemes (or more generally: quasi-compact schemes), then it is already a sheaf for the fpqc-topology. This is Beauville and Laszlo's point of view.

Contrary to what Vistoli claims in [**Vis08**] Theorem 2.64, arbitrary functors on the category of k-schemes do not in general admit an fpqc-sheafification. An example of such a functor is described by Waterhouse in [**Wat75**]. As Waterhouse explains, the general problem with constructing an fpqc-sheafification of an arbitrary functor is that one is forced to consider direct limits over 'all' fpqc-coverings of a given scheme. However, the entirety of 'all' fpqc-coverings will not be a set, but a proper class. One way out of this problem would be to restrict attention to a fixed universe, which will have the drawback that sheafifications depend on the particular choice of the universe. On the other hand, Waterhouse proves that for 'basically bounded' functors it suffices to look at direct limits over certain *sets* of fpqc-coverings, which resolves the above described set-theoretical problems. The purpose of this section is to check that the quotient-functor $\mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$ is basically bounded, and thus has a well-defined fpqc-sheafification.

Let m be a cardinal number not less than the cardinality of k, fix a set S of cardinality m, and let (k-Alg(m)) be the category of k-algebras whose underlying set is contained in S. Let (k-Alg) denote the category of 'all' k-algebras, and let $j : (k\text{-Alg(m)}) \hookrightarrow (k\text{-Alg})$ be the inclusion. For any set-valued functor on the category of k-algebras, let j^* denote the restriction to (k-Alg(m)). Right-adjoint to j^* is the Kan extension j_* along $(k\text{-Alg(m)}) \hookrightarrow (k\text{-Alg})$.

Definition A.8. *A functor F on the category of k-algebras is m-*based *if it has the form j_*G for some functor G on (k-Alg(m)). A functor is* basically bounded *if there exists a cardinal m such that it is m-based.*

Theorem A.9 ([**Wat75**], Corollary 5.2). *If a functor F on the category of k-algebras is m-based, then it has an fpqc-sheafification. More precisely, if $j^*F \to j^*G$ is a sheafification on the fpqc-site over (k-Alg(m)), then $F = j_*j^*F \to j_*G$ is an fpqc-sheafification on (k-Alg).* □

We use the following two observations by Waterhouse: (1) A functor which is represented by an affine scheme whose underlying ring has cardinality $\leq m$ is m-based. (2) The Kan extension j_* preserves colimits, and in particular, the colimit over a system of basically bounded functors is again basically bounded.

Theorem A.10. *The functor-quotient $\mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$ is basically bounded, and hence has a well-defined fpqc-sheafification. Thus the p-adic affine Grassmannian in our sense exists.*

Proof. By (2) above, $\mathrm{L}_p \mathrm{Sl}_n$ as well as $\mathrm{L}_p^+ \mathrm{Sl}_n$ are basically bounded functors on the category of k-algebras. Since $\mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$ is the colimit of a system

$$\mathrm{L}_p \mathrm{Sl}_n \times \mathrm{L}_p^+ \mathrm{Sl}_n \rightrightarrows \mathrm{L}_p \mathrm{Sl}_n,$$

it is basically bounded, too. By Waterhouse's theorem, it thus has an fpqc-sheafification $\mathcal{G}rass'_p$ on the category of k-algebras. Moreover, since the functor-quotient $\mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$ satisfies condition (1) of Proposition A.5, so does $\mathcal{G}rass'_p$: namely, the set of fpqc-covers inside (k-Alg(m)) of $\coprod T_i$ (finite disjoint union) is in natural bijection with the product $\prod\{$ fpqc-covers of T_i $\}$, and direct limits (used to compute sheafifications) commute with finite products. All in all, $\mathcal{G}rass'_p$ satisfies the hypotheses of Proposition A.5, and its Zariski-sheafification $\mathcal{G}rass_p$ will be the desired fpqc-sheafification of $\mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$ on the category of k-schemes. $\square$

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