On Koszul Algebras

Dem Fachbereich 6
Mathematik und Informatik
der Universität Essen
zur Erlangung des
Doktorgrades (Dr. rer. nat.)

vorgelegt von

Stefan Blum

aus
Essen, Deutschland

2001

Diese Dissertation lag dem Promotionsausschuss des Fachbereichs 6 - Mathematik und Informatik der Universität Essen vor. Als Gutachter haben mitgewirkt Prof. Dr. Jürgen Herzog (Universität Essen), Prof. Dr. Victor Reiner (Universität Minneapolis) und Prof. Dr. Volkmar Welker (Universität Marburg). Die Disputation fand am 8.6.2001 statt.

Acknowledgements

I wish to express my deep thanks to my teacher and thesis advisor Jürgen Herzog, who opened the gates to commutative algebra for me and supported me with an enormous amount of inspiring discussions and hints during the time of my studies.

I am grateful to Victor Reiner for many helpful comments on combinatorics and discussions on matroid theory. Moreover, I thank the School of Mathematics, University of Minnesota, for the warm hospitality when a part of Chapter 2 was developed.

I thank Volkmar Welker for giving useful comments on my questions about combinatorics and Hidefumi Ohsugi for sharing his profound knowledge about Gröbner bases with me during his stay in Essen.

My thanks go to Tim Römer, whose friendship has always been a reliable basis during the whole time of my studies in mathematics.

Last but not least I would like to thank my parents, who gave me the possibility to study, and my girlfriend Melanie.

Contents

Introduction

Standard graded algebras over a field K are called Koszul if they have the nice homological property that the residue class field, considered as a graded module over the algebra, has a linear free resolution. These algebras have been introduced by Priddy in 1970 (see [**44**]) and occur in many research areas of commutative and non-commutative algebra, such as algebraic geometry and combinatorial commutative algebra. Many classical results which are known for polynomial rings have been extended to commutative Koszul algebras: All finitely generated modules have finite Castelnuovo-Mumford regularity [**7**],[**8**] and the Poincaré series of K is a rational function [**39**]. A comprehensive survey on this subject is given by Fröberg in [**29**].

This thesis is concerned with the Koszul property of commutative, standard graded algebras, in other words, algebras of the form $R = K[X_1, \dots, X_n]/I$ where $K[X_1, \dots, X_n]$ is the polynomial ring over K with standard grading $\deg(X_i) = 1$ and I a graded ideal.

In the first chapter we introduce notation and recall some well-known facts about Koszul algebras, Gröbner bases, simplicial complexes, matroids, integral polymatroids and semigroup rings.

Chapter 2 is devoted to the study of a specific class of Koszul algebras which admit a certain Koszul filtration. These filtrations have been introduced by Conca, Trung and Valla in [**21**] and form an effective concept to find classes of Koszul algebras. We call a K-algebra R initially Koszul (i-Koszul for short) with respect to a sequence $x_1, \dots, x_n$ of 1-forms in R if the flag $\mathcal{F} = \{(x_1, \dots, x_i) \colon i = 0, \dots, n\}$ forms a Koszul filtration for R. Conca, Rossi and Valla have proved that i-Koszulness implies that the defining ideal has a quadratic Gröbner basis with respect to the reverse lexicographic order on $K[X_1, \dots, X_n]$ induced by $X_1 < \dots < X_n$ (see [**20**]). We give a condition on the initial ideal $\text{in}(I)$ which characterizes the i-Koszul property with respect to the sequence $X_1 + I, \dots, X_n + I$ in R (see Theorem 2.2.1). Using this criterion we consider some examples of i-Koszul algebras. We show that, for an algebra R, generic flags form a Koszul filtrations if and only if the defining ideal I has a 2-linear resolution (see Proposition 2.4.1). Furthermore we discuss algebras which are i-Koszul with respect to every K-basis of R_1. This leads to the notion of universally initially Koszul algebras. We classify these algebras in Theorem 2.4.4 for algebraically closed fields K with $\text{char}(K) \neq 2$ showing that $I = (0)$ or $I = (X_1, \dots, X_n)^2$ or $I = (g^2)$ for some linear form g. A homogeneous semigroup ring is said to be i-Koszul if it is i-Koszul with respect to an ordering of the semigroup generators. Using a lemma by Hibi we obtain that an i-Koszul semigroup ring has shellable divisor posets. We observe that a semigroup ring which is i-Koszul for all permutations of the semigroup generators is already a polynomial ring.

In the third chapter we study Koszul algebras which arise from combinatorics. The bases of a matroid M with ground set $[d] = \{1, \dots, d\}$ define a standard graded semigroup ring $R_{\mathcal{B}(M)} \subset K[T_1, \dots, T_d]$ which is generated by those square-free monomials whose support forms a basis of M. $R_{\mathcal{B}(M)}$ is called the basis monomial ring of M and was introduced in 1977 by N. White, who showed that for every matroid the ring $R_{\mathcal{B}(M)}$ is normal and thus Cohen-Macaulay (see [**51**]). Motivated by the question whether $R_{\mathcal{B}(M)}$ is Koszul, we introduce the notion of sortability for a system $\mathcal{V}$ of equicardinal subsets of $[d]$. To such a system $\mathcal{V}$ we associate a graded semigroup ring $R_{\mathcal{V}}$, which is generated by the square-free monomials in $K[T_1, \dots, T_d]$ with support in $\mathcal{V}$. As a tool for studying rings of the form $R_{\mathcal{V}}$ we generalize matroid operations such as deletion, contraction, duality and parallel extension to systems $\mathcal{V}$ of equicardinal sets. This extends the concept of combinatorial pure subrings introduced by Herzog, Hibi and Ohsugi in [**42**]. We use a result by Sturmfels (see [**49**, Theorem 13.4]) to observe that the toric ideal of $R_{\mathcal{V}}$ has a quadratic Gröbner basis provided $\mathcal{V}$ is sortable. This is a sufficient condition for $R_{\mathcal{V}}$ to be Koszul and gives rise to the introduction of the class of base-sortable matroids, which are matroids for which the basis monomial ring is sortable. We obtain that this class contains all matroids of rank less or equal to 2 and characterize all base-sortable rank-3 matroids by an infinite list of excluded deletions in Theorem 3.3.11. Moreover, it is shown that the class of graphic base-sortable matroids consists of direct sums of parallel-series networks, whose excluded minors are $M(K_4)$ and $U_{2,4}$ (see Theorem 3.4.1). Here $M(K_4)$ is the graphic matroid of the complete graph K_4 on four vertices. Let C_d denote the regular d-gon in the plane whose vertices are labeled clockwise from 1 to d. We prove in Theorem 3.4.2 that a transversal matroid M on $[d]$ is base-sortable if M has a presentation $\mathcal{A} = (A_1, \dots, A_r)$ such that every set A_i labels a consecutive set of vertices of C_d.

We classify the class of matroids for which the basis monomial ring is strongly Koszul. This Koszul property has been introduced by Herzog, Hibi and Restuccia in [**32**]. A matroid $M = M_1 \oplus \ldots \oplus M_k$ belongs to this class if every connected component M_i is either isomorphic to $U_{1,k}$ for some $k \geq 1$, to $U_{2,4}$ or to a parallel extension at a single point of $U_{r,r+1}$ for some $r \geq 2$ (see Theorem 3.6.5).

Chapter 4 is concerned with the study of standard bigraded Koszul algebras. Let $R = S/J$ where $S = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ is a polynomial ring with standard bigrading $\deg(X_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$ and $J \subset S$ a bigraded ideal. For such an algebra we consider two kinds of subalgebras. Let $a, b \geq 0$ be two integers with $(a, b) \neq (0, 0)$. Then the (a, b)-diagonal subalgebra is the standard graded subring $R_\Delta = \bigoplus_{i \geq 0} R_{(ia,ib)}$ where $R_{(i,j)}$ denotes the $(i, j)^{th}$ bigraded component of R. Moreover, a generalized bigraded Veronese subring $R_{\tilde{\Delta}} = \bigoplus_{i,j \geq 0} R_{(ia,jb)}$ is defined by Römer in [**46**]. During the last years, diagonal subalgebras have been intensively studied. In [**22**] Conca, Herzog, Trung and Valla discussed many algebraic properties. In particular, they proved that for an arbitrary bigraded algebra R, the diagonals R_Δ are Koszul, provided one chooses a and b large enough. They asked two questions in this article, of which one has been positively answered by Aramova, Crona and De Negri, who showed that the defining ideal of R_Δ has a quadratic Gröbner basis for $a, b \gg 0$ (see [**4**]). This is a stronger property than Koszulness. The main result of Chapter 4 is the positive answer to the second question posed in [**22**]: Suppose R is a Koszul algebra, then all diagonal subalgebras R_Δ are Koszul. Moreover, we prove that all generalized Veronese subrings $R_{\tilde{\Delta}}$ inherit the Koszul property (see Theorem 4.2.1). In the proof we generalize techniques used by Aramova, Barcanescu and Herzog in [**3**], where they obtain upper bounds for rates of modules over arbitrary Veronese algebras. For a finitely generated bigraded R-module and two integers $c, d \geq 0$ we define a sidediagonal module $M_\Delta^{(c,d)}$ as the R_Δ-module with graded components $(M_\Delta^{(c,d)})_i = M_{(ia+c,ib+d)}$ and similarly modules $M_{\tilde{\Delta}}^{(c,d)}$. Provided R is Koszul, we get upper bounds for the Castelnuovo-Mumford regularity of these modules, which become small for $a, b \gg 0$ (see Theorem 4.2.6).

There are several applications to symmetric algebras and Rees algebras. Let A be a standard graded K-algebra and M be a finitely generated graded A-module. Provided the symmetric algebra $S(M)$ is Koszul, we show that all symmetric powers of M have linear resolutions. For the graded maximal ideal $\mathfrak{m}$ of A, we prove that $S(\mathfrak{m})$ is a Koszul algebra if the defining ideal of A has a 2-linear resolution. Under the weaker assumption that A is Koszul, we obtain that all symmetric powers of $\mathfrak{m}$ have a linear A-resolution. Let $I \subset A$ be a graded ideal generated in one degree. Provided the Rees ring $R(I)$ is Koszul, all powers of I have linear A-resolutions. Generalizing the notion of matroidal ideals in [**24**] we study the class of polymatroidal ideals for which all powers and all symmetric powers have linear resolutions. Moreover, we recover some well-known results that were first proved by Backelin and Fröberg in [**10**] saying that the Koszul property is preserved under tensor products over K, Segre products and Veronese subrings. Interpreting our result for bigraded semigroup rings we observe that the Cohen-Macaulay property of certain divisor posets (see [**35**] and [**45**]) is compatible with taking diagonals and generalized Veronese subrings.

The results in Chapter 2 have been published in [**12**], most of the contents of Chapter 3 and 4 is submitted (see [**13**],[**14**]).

Stefan Blum
February 2001

CHAPTER 1

Background

In this chapter we introduce notation, give basic definitions and recall some well-known results about Koszul algebras, matroid theory, integral polymatroids and graded semigroup rings.

A detailed exposition on the fundamental facts in Sections 1.1, 1.2 and 1.5 can be found in the book by Bruns and Herzog [**15**] or in Eisenbud's book [**26**]. Section 1.3, which is concerned with several results about Koszul algebras, is based on Fröberg's survey given in [**29**]. In Section 1.4 we consider the powerful tool of Gröbner bases (see e.g. [**26**]). The aim of Section 1.6 is to recall some basic facts about matroid theory. For a detailed introduction to matroids refer to Oxley's book [**41**]. Integral polymatroids, which we consider in Section 1.7, have been studied by Welsh in [**50**]. Finally we describe the fundamental properties of graded semigroups rings for which Sturmfels provides further details in [**49**].

1.1. Standard graded algebras

We start by giving central definitions and notation. Let K be a field and $S = K[X_1, \dots, X_n]$ the polynomial ring with standard grading $\deg(X_i) = 1$ for $i = 1, \dots, n$. We have a decomposition $S = \bigoplus_{i \geq 0} S_i$ (as a $\mathbb{Z}$-module) where S_i is the K-vector space spanned by all monomials of degree i. A polynomial $f \in S_i$ is called *homogeneous of degree* i or said to be an i*-form*. We write $\deg(f) = i$. An ideal $I \subset S$ which has a system of homogeneous generators is called a *graded ideal*. Let I_i denote the K-vector space spanned by all i-forms of I. Then the quotient ring $R = S/I$ has a natural decomposition $R = \bigoplus_{i \geq 0} R_i$ where $R_i = S_i/I_i$. Clearly, each graded component R_i is a finite dimensional K-vector space and $R_0 = K$. We have $R_iR_j \subset R_{i+j}$ for all integers $i, j \geq 0$ and R is finitely generated as a K-algebra by elements of R_1.

Definition 1.1.1. A K-algebra R is called *standard graded* if it is of the form $R = S/I$ where $I \subset S$ is a graded ideal.

Let R be a standard graded algebra. A finitely generated R-module M is said to be $\mathbb{Z}$*-graded* (or simply *graded*) if it has a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (as a $\mathbb{Z}$-module) such that $R_iM_j \subset M_{i+j}$ for all integers i, j with $i \geq 0$.

Similarly, every M_i forms a finite dimensional K-vector space, which we call the i^{th} *graded component* of M. An element $x \in M_i$ is said to be *homogeneous of degree* i.

Notation 1.1.2. Throughout this thesis R will always denote a standard graded K-algebra of the form $R = S/I$ where $S = K[X_1, \dots, X_n]$ is the polynomial ring and $I \subset S$ a graded ideal. Let $\mathfrak{m} \subset R$ be the graded maximal ideal of R, that is the ideal generated by the residue classes of the variables $X_1, \ldots, X_n$. We will use $\mathcal{M}_{\mathbb{Z}}(R)$ to denote the collection of finitely generated graded R-modules.

Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Since every graded component M_i has a finite K-vector space dimension, the following generating function is well-defined.

Definition 1.1.3. Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The formal power series

$$H_M(t) = \sum_{i \in \mathbb{Z}} (\dim_K M_i) t^i$$

is called the *Hilbert series* of M.

It is well-known that the Hilbert series H_M of a non-zero module M is always a rational function of the form $H_M(t) = Q_M(t)/(1-t)^d$ where $Q_M \in \mathbb{Z}[t, t^{-1}]$, $Q_M(1) \neq 0$ and d denotes the Krull-dimension of M (see [**15**, Corollary 4.1.8]). We consider an example.

Example 1.1.4. The set of all monomials of degree i forms a K-basis for S_i. Thus $\dim_K S_i = \binom{i+n-1}{n-1}$ and the Hilbert series of S is given by

$$H_S(t) = \sum_{i \geq 0} \binom{i+n-1}{n-1} t^i.$$

Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Then, for $a \in \mathbb{Z}$, the *twisted module* $M(a)$ is defined as the graded R-module with components $M(a)_i = M_{a+i}$ for all $i \in \mathbb{Z}$. For two modules $M, N \in \mathcal{M}_{\mathbb{Z}}(R)$ an R-module homomorphism $\varphi : M \to N$ is said to be *homogeneous of degree* a if $\varphi(M_i) \subset N_{i+a}$ is satisfied for all $i \in \mathbb{Z}$. We call a complex

$$C_{\bullet}: \quad \ldots \to C_r \overset{\partial_r}{\to} C_{r-1} \to \ldots \to C_1 \overset{\partial_1}{\to} C_0 \to 0$$

of finitely generated, graded R-modules C_r *graded,* provided every homomorphism $\partial_r : C_r \to C_{r-1}$ is homogeneous of degree 0. Note that $\mathcal{M}_{\mathbb{Z}}(R)$ together with the homogeneous homomorphisms of degree 0 forms a category. For every $j \in \mathbb{Z}$, the restriction to the j^{th} graded component defines a complex of finite dimensional K-vector spaces

$$(C_{\bullet})_j: \quad \ldots \to (C_r)_j \overset{(\partial_r)_j}{\to} (C_{r-1})_j \to \ldots \to (C_1)_j \overset{(\partial_1)_j}{\to} (C_0)_j \to 0.$$

1.2. The minimal graded free resolution

Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Then every module $\operatorname{Tor}_i^R(M, K)$ is naturally standard graded and M has a *minimal graded free resolution* $F_{\bullet}$ which is unique up to a base change. It has the form

$$F_{\bullet}: \ldots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{ij}^R(M)} \to \ldots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1j}^R(M)} \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0j}^R(M)} \to M \to 0$$

where the $\beta_{ij}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$ are non-negative integers and, for every fixed $i \geq 0$, there are only finitely many j such that $\beta_{ij}^R(M) \neq 0$.

Definition 1.2.1. The numbers $\beta_{ij}^R(M)$ are called the *graded Betti numbers* of M.

The resolution $F_{\bullet}$ can be constructed in the following way. Let $m_1, \ldots, m_r$ be a minimal system of homogeneous generators for M with $\deg(m_j) = a_j$ for $j = 1, \ldots, r$. We define the graded free module $F_0 = \bigoplus_{j=1}^{r} R(-a_j)$ with basis elements $e_1, \ldots, e_r$ such that $\deg(e_j) = a_j$. Let $\varphi_0 : F_0 \to M$ be the homomorphism with $\varphi(e_j) = m_j$ for $j = 1, \ldots, r$. Then φ_0 is surjective and homogeneous of degree 0. We take φ_0 to be the first map of the resolution. The module $\Omega_1(M) = \ker(\varphi)$ is called the *first syzygy module of M*. It is a graded submodule of F_0 and, since $m_1, \ldots, m_r$ is a minimal system of generators of M, we have $\ker(\varphi_0) \subset \mathfrak{m}F_0$. In a similar way we find a finitely generated graded free module F_1 and an epimorphism $\varphi_1 : F_1 \to \ker(\varphi_0)$. Let $i : \ker(\varphi_0) \hookrightarrow F_0$ be the inclusion map. Then $i \circ \varphi_0 : F_1 \to F_0$ is the second map of the minimal graded free resolution $F_{\bullet}$ and this construction can be continued inductively.

Example 1.2.2. Let S be the standard graded polynomial ring and $K = S/\mathfrak{m}$ the residue class field. It is well-known that the Koszul complex gives the (finite) minimal graded free S-resolution of K (see [**15**, Section 1.6] for more details):

$$F_{\bullet}: \quad 0 \to S(-n)^{\binom{n}{n}} \to S(-n+1)^{\binom{n}{n-1}} \to \ldots \to S(-1)^{\binom{n}{1}} \to S \to K \to 0.$$

Thus we have $\beta_{ii}^{S}(K) = \binom{n}{i}$ and $\beta_{ij}^{S}(K) = 0$ for $i \neq j$.

The data given by the graded Betti numbers $\beta_{ij}^{R}(M)$ in the minimal graded free resolution is collected in a generating function, the Poincaré series.

Definition 1.2.3. Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The *graded Poincaré series $P_R^M(t,z)$* of M is defined as the formal power series

$$P_R^M(t,z) = \sum_{i,j} \beta_{ij}^{R}(M) t^j z^i.$$

Let $M \neq 0$. We set $t_i(M) = \sup\{j \colon \beta_{ij}^{R}(M) \neq 0\}$ with $t_i(M) = -\infty$ if $\beta_{ij}^{R}(M) = 0$ for all $j \in \mathbb{Z}$. The *Castelnuovo-Mumford regularity* is given by

$$\operatorname{reg}_R(M) = \sup\{t_i(M) - i \colon i \geq 0\}.$$

The integer $\min\{i \colon M_i \neq 0\}$ is called the *initial degree* $\operatorname{indeg}(M)$ of M. Moreover, the module M has an *i-linear* (or just *linear*) *resolution* if $\operatorname{reg}_R(M) = \operatorname{indeg}(M) = i$.

Note that the initial degree of M equals the least degree of a homogeneous generator of M, i.e. $\operatorname{indeg}(M) = \min\{j \colon \beta_{0j}^{R}(M) \neq 0\}$. In a way the regularity measures the complexity of the minimal graded free resolution. In the most simple case the resolution is linear. Then the entries of the maps in the resolution are linear forms.

Remark 1.2.4. Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$ with $\operatorname{indeg}(M) = d$. From the construction of the minimal graded free resolution it is easy to see that $\beta_{ij}^{R}(M) = 0$ for all $j \in \mathbb{Z}$ such that $j < i + d$.

We conclude this section with an example.

Example 1.2.5. By 1.2.2 the graded Poincaré series of K equals

$$P_S^K(t,z) = \sum_{i=0}^{n} \binom{n}{i} t^i z^i = (1+tz)^n.$$

The field $K = S/\mathfrak{m}$ has a 0-linear resolution over S and $\operatorname{reg}_S(K) = 0$.

In general, for an arbitrary K-algebra R, the Poincaré series P_R^K of the residue class field is not a rational function. A concrete example can be found in [**47**].

1.3. The Koszul property

In this section we define the Koszul property for standard graded K-algebras and recall some well-known facts. Note that the notion of Koszulness exists in a more general context for associative K-algebras which need not to be commutative. A good survey is given by Fröberg in [**29**]. We always assume that R is standard graded.

Definition 1.3.1. A K-algebra R is said to be *Koszul* if the R-module $K = R/\mathfrak{m}$ has a linear minimal free resolution over R.

The definition can be formulated in several equivalent ways. We collect them in the following proposition, which is a direct consequence of the definitions above and Remark 1.2.4.

Proposition 1.3.2. *The following statements are equivalent.*

(a) *R is a Koszul algebra.*
(b) $\operatorname{reg}_R(K) = 0$.
(c) $\operatorname{Tor}_i^R(K,K)_j = 0$ *for all* $i \neq j$.

There is a further characterization of the Koszul property. Let R be Koszul and $F_{\bullet}$ be the minimal graded free R-resolution of K. For every $j \geq 0$ the restriction $(F_{\bullet})_j$ to the j^{th} graded components is a finite exact complex of finite dimensional K-vector spaces. Since the alternating sum of the K-vector space dimensions in $(F_{\bullet})_j$ equals 0, we get $H_R(t)P_R^K(-1,t) = 1$. In fact, this equality is equivalent to R being Koszul (see [**39**]). We summarize.

Theorem 1.3.3. *A K-algebra R is Koszul if and only if* $H_R(t)P_R^K(-1,t) = 1$.

This theorem immediately implies that the Poincaré series of K over a Koszul algebra is always a rational function. We return to our favorite example.

Example 1.3.4. The polynomial ring S is Koszul as we observed in Example 1.2.5. We have already computed the Poincaré series $P_S^K(t,z) = (1+tz)^n$ and the Hilbert series $H_S(t) = \sum_{i\geq 0} \binom{i+n-1}{n-1} t^i$. By Theorem 1.3.3 we see that $H_S(t) = 1/(1-t)^n$ which can also be checked directly.

Let $R = S/I$ be standard graded. The condition that $\operatorname{Tor}_2^R(K,K)$ is concentrated in degree 2 is equivalent to I being generated by forms of degree ≤ 2. Therefore, we get the following observation.

Proposition 1.3.5. *If $R = S/I$ is Koszul, then the ideal I is generated by forms of degree* ≤ 2.

In fact, we may always assume that I does not contain linear forms. Then I is generated by quadrics. In this case the algebra R is called *quadratic*. The converse of Proposition 1.3.5 is false. We take a concrete example from [**29**].

Example 1.3.6. The K-algebra $K[X_1, X_2, X_3]/(X_1^2, X_2X_3, X_1X_3 + X_2^2)$ is quadratic, but not Koszul.

We recall some constructions which naturally occur in algebraic geometry. Let R and R' be two standard graded K-algebras. For an integer $d \geq 1$ the d^{th} Veronese subring $R^{(d)}$ of R is the subalgebra

$$R^{(d)} = \bigoplus_{i \geq 0} R_{id}.$$

The Segre product which we denote with $R * R'$ is defined as the graded algebra

$$R * R' = \bigoplus_{i \geq 0} R_i \otimes_k R'_i.$$

Let $S = K[X_1, \ldots, X_n]$ and $T = K[Y_1, \ldots, Y_m]$ be two polynomial rings and let $R = S/(f_1, \ldots, f_r)$ and $R' = T/(g_1, \ldots, g_s)$ be two standard graded algebras. The tensor product $R \otimes_K R'$ is naturally standard graded with components

$$(R \otimes_K R')_i = \bigoplus_{k+l=i} R_k \otimes_K R_l.$$

It has a presentation of the form $R \otimes_K R' = K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/Q$ where $Q = (f_1, \ldots, f_r, g_1, \ldots, g_s)$. We define the *fiber product* $R \circ R'$ as the quotient ring $K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/Q'$ where

$$Q' = (f_1, \ldots, f_r, g_1, \ldots, g_s, X_i Y_j \colon i = 1, \ldots, n \quad j = 1, \ldots, m).$$

An i-form $f \in R$ is called a *non-zerodivisor* of R if the multiplication map $R(-i) \xrightarrow{f} R$ is injective. The following results were proved by Backelin and Fröberg in [**10**].

Theorem 1.3.7. *Let R and R' be two K-algebras.*

(a) *R is Koszul if and only if the Veronese subalgebra $R^{(d)}$ is Koszul for all $d \geq 1$.*
(b) *If R and R' are Koszul, then $R * R'$ is Koszul.*
(c) *$R \otimes_K R'$ is Koszul if and only if R and R' are Koszul.*
(d) *$R \circ R'$ is Koszul if and only if R and R' are Koszul.*
(e) *Let $f \in R$ be a homogeneous element of degree 1 or 2. The algebra R is Koszul if and only if $R/(f)$ is Koszul.*

We will recover the statements of (a),(b) and (c) in Chapter 4 when we discuss bigraded algebras.

Example 1.3.8. Let $R = S/(f_1, \ldots, f_r)$ be a complete intersection where $f_1, \ldots, f_r$ are homogeneous elements of degree 1 or 2. By Theorem 1.3.7(e) R is Koszul.

At the end of this section we quote a result from [**29**] which is important for the role of Gröbner bases in the discussion of Koszul algebras.

Theorem 1.3.9. *Let $I \subset S$ be an ideal which is generated by monomials of degree 2. Then $R = S/I$ is Koszul.*

1.4. Gröbner bases

An effective method to prove the Koszul property of a K-algebra $R = S/I$ is to compute a quadratic Gröbner basis for the defining ideal I. Therefore, we give a brief introduction to the theory of Gröbner bases and recall some tools which we will need for the forthcoming chapters. For more details [**26**] serves as a reference.

In this section $F = \bigoplus_{i=1}^{r} Se_i$ denotes a graded free S-module of rank r with homogeneous basis elements $e_1, \dots, e_r$. A *monomial* $m \in F$ is an element of the form ue_i where $u \in S$ is a monomial in the usual sense and $i \in \{1, \dots, r\}$. Every $f \in F$ has a unique presentation as a K-linear combination of monomials. In other words, the set of monomials forms a K-basis for F.

Definition 1.4.1. A total order $<$ on the set of monomials in F is called a (degree refining) *monomial order* on F if two conditions are satisfied. For all monomials $ue_i, ve_j \in F$ and every monomial $w \in S$ we have:

(a) If $\deg(ue_i) > \deg(ve_j)$, then $ue_i > ve_j$.
(b) If $ue_i > ve_j$, then $wue_i > wve_j$.

Note that one can define monomial orders which do not respect the partial order given by the degree. Since we only consider the graded case, we restrict to degree refining monomial orders.

A *term* in F is an element of the form $aue_j \in F$ where $a \in K \setminus \{0\}$ is a non-zero scalar and ue_j a monomial. Every monomial order $<$ on F extends naturally to a total order on the terms of F by neglecting the scalars, i.e. we set $aue_i \geq bve_j$ if $ue_i \geq ve_j$. Such an order is called a *term order.*

Definition 1.4.2. Let $f \in F$ and $<$ be a monomial order on F. Then the largest term of f with respect to $<$ is called the *initial term* of F. We denote it with $\mathrm{in}_<(f)$. For a graded submodule $U \subset F$ the *initial module* $\mathrm{in}_<(U)$ is the module generated by the set $\{\mathrm{in}_<(f) \colon f \in U\}$.

We simply write $\mathrm{in}(f)$ and $\mathrm{in}(U)$ when the given monomial order $<$ is obvious from the context. The following well-known result was first proved by Macaulay.

Theorem 1.4.3. *Let U be a graded submodule of F and $<$ a monomial order on F. Then the residue classes of the monomials which do not belong to* $\mathrm{in}_<(U)$ *form a K-basis for F/U.*

The monomials which do not belong to $\mathrm{in}_<(U)$ are called *standard* with respect to the order $<$. Theorem 1.4.3 has the consequence that the modules F/U and $F/\mathrm{in}(U)$ have the same Hilbert series. In particular, for a graded ideal $I \subset S$ we obtain

$$H_{S/I}(t) = H_{S/\mathrm{in}(I)}(t).$$

There are many possible monomial orders on F. We give two important examples for the case $F = S$. To simplify notation we use multi-indices. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a vector. Then we write X^α for the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. It will be useful in the forthcoming chapters to identify the vector α the corresponding multiset, that is for every $i \in \{1, 2, \dots, n\}$ the component α_i counts the multiplicity of the element i. The cardinality of α is given by $|\alpha| = \sum_{i=1}^{d} \alpha_i$. The multiset

α is called the *support* of X^α, which we denote with $\operatorname{supp}(X^\alpha)$. If the monomial X^α is square-free, then α is just a subset of $\{1,\dots,n\}$.

Examples 1.4.4. For a fixed order $X_1 > X_2 > \cdots > X_n$ of the variables and two monomials $X^\alpha, X^\beta \in S$ we have

(a) the *degree reverse lexicographic order:* We set $X^\alpha <_{\text{rlex}} X^\beta$ if the last non-zero component of the vector $(\beta_1-\alpha_1,\dots,\beta_n-\alpha_n,|\alpha|-|\beta|)$ is negative.
(b) the *degree lexicographic order:* We set $X^\alpha <_{\text{lex}} X^\beta$ if the first non-zero component of the vector $(|\beta|-|\alpha|,\beta_1-\alpha_1,\dots,\beta_n-\alpha_n)$ is positive.

There are some characteristic properties of these term orders which turn out to be useful for computing the corresponding initial ideals. They follow directly from the definitions.

Proposition 1.4.5. *With the notation of* 1.4.4 *we have:*

(a) *If f is homogeneous and $\operatorname{in}_{<_{\text{lex}}}(f) \in K[X_s,\dots,X_n]$ for some $s \le n$, then $f \in K[X_s,\dots,X_n]$.*
(b) *If f is homogeneous and $\operatorname{in}_{<_{\text{rlex}}}(f) \in (X_s,\dots,X_n)$ for some $s \le n$, then $f \in (X_s,\dots,X_n)$.*
(c) *Let $I \subset S$ be a graded ideal. Then $\operatorname{in}_{<_{\text{rlex}}}(I+(X_n)) = \operatorname{in}_{<_{\text{rlex}}}(I)+(X_n)$ and $\operatorname{in}_{<_{\text{rlex}}}(I)\colon (X_n)) = (\operatorname{in}_{<_{\text{rlex}}}(I)\colon (X_n))$.*

We come to the central definition of this section.

Definition 1.4.6. Let $U \subset F$ be a graded submodule and $<$ a monomial order on F. A set $G=\{g_1,\dots,g_t\} \subset F$ is said to be a *Gröbner basis* for U if $\operatorname{in}_<(U)$ is generated by the set $\{\operatorname{in}(g_1),\dots,\operatorname{in}(g_t)\}$.

A Gröbner basis G is said to be *reduced* if for any two distinct elements $g, g' \in G$, no term of g' is divided by $\operatorname{in}_<(g)$. To decide whether a set G forms a Gröbner basis the Buchberger criterion serves as an important computational tool. We will describe it in the sequel.

Let $f, g \in F$ be homogeneous elements and $<$ a monomial order on F. We write $\operatorname{in}(f) = ue_i$ and $\operatorname{in}(g) = ve_j$ where $u, v \in S$ are two monomials and $i, j \in \{1,\dots,r\}$. If the monomials $\operatorname{in}(f)$ and $\operatorname{in}(g)$ involve the same basis element, i.e. $i = j$, then the *S-pair* $S(f,g)$ is defined as

$$S(f,g) = \frac{u}{\gcd(u,v)}f - \frac{v}{\gcd(u,v)}g$$

where gcd denotes the greatest common divisor of two monomials in S. If $i \ne j$, we set $S(f,g)=0$.

Theorem 1.4.7. (Buchberger)
Let $<$ be a monomial order on F. The following statements are equivalent.

(a) *$g_1,\dots,g_t \in F$ form a Gröbner basis (for the submodule that they generate).*
(b) *For all $1 \le i < j \le t$ the S-pair $S(g_i,g_j)$ has a presentation of the form*

$$S(g_i,g_j) = \sum_{k=1}^{t} h_k g_k$$

with polynomials $h_k \in S$ *such that* $\text{in}(S(g_i, g_j)) \geq \text{in}(h_k g_k)$ *for all* $k = 1, \dots, t$.

A sum satisfying the condition in (b) is called a *standard expression* of f with respect to $g_1, \dots, g_t$. The following is well-known (see e.g. [**30**]).

Proposition 1.4.8. *Let* $U \subset F$ *be a graded submodule and* $<$ *a monomial order on* F. *Then we have a coefficientwise inequality of graded Poincaré series*

$$P_S^{F/U}(t, s) \leq P_S^{F/\text{in}(U)}(t, s).$$

We use this to prove a lemma which we will need in a forthcoming chapter.

Lemma 1.4.9. *Let* $\deg(e_i) = d$ *for* $i = 1, \dots, r$ *and* $<$ *be a monomial order on* F. *If* $U \subset F$ *is a graded submodule such that* $\text{in}(U) = L_{k_1}e_{k_1} + L_{k_2}e_{k_2} + \ldots + L_{k_t}e_{k_t}$ *where* $L_{k_j} \subset S$ *are ideals generated by linear forms, then the* S*-module* U *has a* d*-linear resolution.*

We sketch the proof of this proposition.

Proof. By Proposition 1.4.8 it suffices to show that $F/\text{in}(U)$ has a linear resolution. The hypothesis implies that $\text{in}(U) \cong \bigoplus_{j=1}^{t} L_{k_j}(-d)$. Since every L_{k_j} is generated by linear forms, the Koszul complex is a (linear) minimal graded free S-resolution for L_{k_j}. Therefore $\text{in}(U)$ has a d-linear resolution. □

We return to the Koszul property. Let $<$ be a monomial order on S and $I \subset S$ a graded ideal. There is a relationship of Poincaré series (see [**1**] or [**16**])

$$P_{S/I}^{K}(s, t) \leq P_{S/\text{in}(I)}^{K}(s, t).$$

If the initial ideal $\text{in}(I)$ is generated in degree 2, then Proposition 1.3.9 implies that $S/\text{in}(I)$ is Koszul. These two results have a well-known consequence.

Proposition 1.4.10. *Let* $R = S/I$ *and let* $<$ *be a monomial order on* S. *If* I *has a quadratic Gröbner basis, then* R *is Koszul.*

The converse of Proposition 1.4.10 is false. We give a concrete example which is taken from [**27**].

Example 1.4.11. Let $R = K[X_1, X_2, X_3]/I$ where

$$I = (X_1^2 + X_1X_2, X_2^2 + X_2X_3, X_3^2 + X_1X_3).$$

Since R is a complete intersection, R is a Koszul algebra. In [**27**] it is shown that for any ordering, even after any linear change of coordinates, I does not have a quadratic Gröbner basis.

Let R be a standard graded K-algebra. In [**9**] Backelin showed that the d^{th} Veronese subring $R^{(d)}$ is Koszul provided $d \gg 0$. Even stronger is the result of Eisenbud, Reeves and Totaro in [**27**]:

Theorem 1.4.12. *The defining ideal of the* d^{th} *Veronese subring* $R^{(d)}$ *has a quadratic Gröbner basis for* $d \gg 0$.

In the following we discuss a class of ideals which have a special quadratic Gröbner basis. For this, we recall the definition of the generic initial ideal. Refer to [**26**] for a detailed exposition. Let $\mathrm{GL}(n, K)$ be the general linear group of invertible $n \times n$ matrices over K.

Theorem and Definition 1.4.13. *Let K be an infinite field and $<_{\mathrm{rlex}}$ denote the reverse lexicographic term order on S induced by $X_1 > \ldots > X_n$. For every graded ideal $I \subset S$ there exists a non-empty Zariski open subset $\mathcal{U} \subset \mathrm{GL}(n, K)$ and a monomial ideal $\mathrm{gin}(I)$ such that $\mathrm{in}_{<_{\mathrm{rlex}}}(gI) = \mathrm{gin}(I)$ for all $g \in \mathcal{U}$.*

The ideal $\mathrm{gin}(I)$ is called the *generic initial ideal* of I with respect to $>_{\mathrm{rlex}}$.

To collect some fundamental properties of $\mathrm{gin}(I)$ we introduce a partial order on the natural numbers. Let p be a prime number. For $a, b \in \mathbb{N}$ we set $a \prec_p b$ if each digit in the p-base expansion of a is less or equal to the corresponding digit of b. Moreover, we denote the usual order on $\mathbb{N}$ with $\prec_0$.

Proposition 1.4.14. *Let $\mathrm{char}(K) = p$ and $I \subset S$ be a graded ideal. We have*

(a) *The ideal $\mathrm{gin}(I)$ is p-borel: If u is a monomial generator of $\mathrm{gin}(I)$ which is divisible by X_j^t but by no higher power of X_j, then $(X_i/X_j)^s u \in \mathrm{gin}(I)$ for all $i < j$ and $s \prec_p t$.*

(b) *It is $\mathrm{reg}_S(\mathrm{gin}(I)) = \mathrm{reg}_S(I)$.*

We recall the stable property for set of monomials.

Definition 1.4.15. Let $m \in S$ be a monomial with $m \neq 1$. We write $\max(m)$ for the largest index i such that X_i divides m.

(a) A set $\mathcal{M}$ of monomials is called (combinatorially) *stable* if for every $m \in \mathcal{M}$ and $i < \max(m)$ the monomial $X_i m / X_{\max(m)} \in \mathcal{M}$.

(b) We call a set $\mathcal{M}$ of monomials *strongly stable* if for every $m \in \mathcal{M}$ and $i < j \leq \max(m)$ the monomial $X_i m / X_j \in \mathcal{M}$.

If $I \subset S$ is a monomial ideal, then there exists a unique minimal system of generators for I. We denote it with $G(I)$. The ideal I is said to be *stable* if the set of monomials which belong to I is stable.

As a consequence of Proposition 1.4.14 we obtain a result for ideals with a 2-linear resolution.

Lemma 1.4.16. *Let K be an infinite field with $\mathrm{char}(K) \neq 2$, and $I \subset S$ a graded ideal. If I has a 2-linear resolution, then $\mathrm{gin}(I)$ is generated by monomials of degree 2 and $G(\mathrm{gin}(I))$ is stable, i.e.*

If $X_i X_j \in \mathrm{gin}(I)$ with $i \leq j$ and $k < j$, then $X_k X_i \in \mathrm{gin}(I)$.

In particular, $R = S/I$ is a Koszul algebra.

We note here that an algebra R is said to be *Golod* if the Poincaré series P_R^K satisfies a certain equality (see [**2**, p. 44]) and it is well-known that I has a 2-linear resolution if and only if $R = S/I$ is Koszul and Golod.

An often used technique to compute Gröbner bases of subalgebras is the following.

Lemma 1.4.17. *Let $S = K[X_1, \ldots, X_n]$ be the polynomial ring, $<$ a monomial order on S, $I \subset S$ a graded ideal and $A \subset \{1, \ldots, n\}$ a subset. We set $S' =$*

$K[X_a\colon a \in A]$. *Suppose that I has a Gröbner basis G with the property:*

$$\textit{If } g \in G \textit{ and } \mathrm{in}_<(g) \in S', \textit{ then } g \in S'.$$

Then $G \cap S'$ is a Gröbner basis for $I \cap S'$ with respect to restricted order $<$.

Proof. Let $f \in I \cap S'$. Then there exists a $g \in G$ such that $\mathrm{in}_<(g)$ divides $\mathrm{in}_<(f)$. In particular, $\mathrm{in}_<(g) \in S'$. By hypothesis we get $g \in S'$ which concludes the proof. □

Definition 1.4.18. With the notation and hypothesis of Lemma 1.4.17, $S'/I \cap S'$ is said to be a *consistent subalgebra* of S/I.

Note that a Gröbner basis $g_1, \dots, g_t$ for an ideal I is said to *square-free* if $\mathrm{in}(g_i)$ is a square-free monomial for $i = 1, \dots, r$.

1.5. Simplicial complexes

This section is devoted to a short introduction to simplicial complexes. We need some basic properties for the study of semigroup rings in the forthcoming chapters. For a detailed exposition on simplicial complexes refer to [**15**]. To simplify notation we write $[n]$ for the set $\{1, \dots, n\}$.

Definition 1.5.1. A *simplicial complex* Γ on $[n]$ is a collection of subsets of $[n]$ which satisfies: If $F \in \Gamma$ and $G \subset F$, then $G \in \Gamma$.

The elements of Γ are called *faces* of Γ, the *dimension* of a face F, denoted with $\dim F$, equals $|F| - 1$ and the dimension of Γ is the maximum of all numbers $\dim F$ with $F \in \Gamma$.

Let Γ be a simplicial complex. We call the maximal faces under inclusion *facets* of Γ. Moreover, Γ is said to be *pure* if all facets of Γ have the same dimension. We consider an example.

Example 1.5.2. Let P be a finite set which is partially ordered by $\prec$. The order complex $\Gamma(P)$ is the simplicial complex whose faces are the totally ordered subsets of P, i.e. a set $F = \{p_1, \dots, p_t\} \subset P$ is a face of $\Gamma(P)$ provided $p_1 \prec p_2 \prec \cdots \prec p_t$. If the chain F is unrefinable, then F forms a facet of $\Gamma(P)$.

To every simplicial complex we associate a ring $R[\Gamma] = S/I_\Gamma$, where I_Γ is the ideal generated by the square-free monomials $X_{i_1} \cdots X_{i_s}$ such that $\{i_1, \dots, i_s\} \notin \Gamma$. We call $R[\Gamma]$ the *Stanley-Reisner ring* of Γ.

Definition 1.5.3. A simplicial complex Γ is said to be *Cohen-Macaulay* over K, if the associated Stanley-Reisner ring $R[\Gamma] = S/I_\Gamma$ is Cohen-Macaulay.

Note that this property depends on the base field of the polynomial ring $S = K[X_1, \dots, X_n]$. It is known that a Cohen-Macaulay simplicial complex is always pure.

Definition 1.5.4. Let Γ be a pure simplicial complex with facets $F_1, \dots, F_t$. Assume that the facets are linearly ordered by $<$ such that $F_1 < F_2 < \ldots < F_t$. Then $<$ is said to be a *shelling order for* Γ if for all $i, j \in [t]$ with $i < j$ there exists a $k < i$ such that $F_j \cap F_i \subset F_k \cap F_i$ and $|F_k \cap F_i| = |F_i| - 1$.

If there exists a shelling order for Γ, then Γ is called *shellable.*

Shellability is a stronger property than Cohen-Macaulayness as the next theorem shows (see [**15**, Theorem 5.1.13]).

Theorem 1.5.5. *A shellable simplicial complex is Cohen-Macaulay over every field.*

1.6. Matroids

In this section we give a brief introduction to the theory of matroids. Matroids form a special class of simplicial complexes which occur in several fields of combinatorics. They will be the main topic of Chapter 3. A detailed exposition can be found in Oxley's book [**41**]. We start with basic definitions.

Definition 1.6.1. *A matroid* M on $[d] = \{1, \dots, d\}$ is a pair $([d], \mathcal{I})$ where $\mathcal{I}$ is a pure simplicial complex on $[d]$ whose facets, called *bases* of M, satisfy the basis exchange property: If B, B' are two bases of M, then for every $x \in B \setminus B'$ there exists an element $y \in B' \setminus B$ such that $(B - x) \cup y$ is also a basis of M.

The equal cardinality of all bases of M is said to be the *rank* of M. We denote it with $\operatorname{rank}(M)$. We write $\mathcal{B}(M) = \{B_1, \dots, B_n\}$ for the collection of bases of M. The faces of the complex $\mathcal{I}$ are called the *independent sets* of the matroid M. A subset $A \subset [d]$ is said to be *dependent* if A is not an independent set of M.

There are several equivalent axiom systems for matroids. We restate two of them.

Proposition 1.6.2. *We have:*

(a) (Axiom for independent sets): *A collection $\mathcal{I}$ of subsets of $[d]$ forms the collection of independent sets of a matroid if and only if $\mathcal{I}$ is a simplicial complex and: For all $I, I' \in \mathcal{I}$ with $|I| < |I'|$ there exists an element $x \in I' \setminus I$ such that $I \cup x \in \mathcal{I}$.*

(b) (Dual exchange property): *Let $\mathcal{B}$ be a collection of subsets of $[d]$ which all have the same cardinality. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if the following exchange property holds: For $B, B' \in \mathcal{B}$ and every $x \in B \setminus B'$ there exists an element $y \in B' \setminus B$ such that $(B' - y) \cup x \in \mathcal{B}$.*

Examples 1.6.3. We give two examples.

(a) Let r, d be two integers such that $0 \leq r \leq d$ and let $\mathcal{I}$ be the collection of all sets $I \subset [d]$ with $|I| \leq r$. Then $U_{r,d} = ([d], \mathcal{I})$ is a matroid. A set $A \subset [d]$ is a basis of $U_{r,d}$ if $|A| = r$. It is a dependent set provided $|A| > r$. $U_{r,d}$ is called the uniform matroid on $[d]$ of rank r. A matroid M is said to be *uniform* if $M = U_{r,d}$ for some $0 \leq r \leq d$.

(b) Take the matrix A with rational entries

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and let V be the column $\mathbb{Q}$-vector space of A. We label the columns of A with the numbers $1, \dots, 6$ from the left to the right. We set $\mathcal{I}$ to be the collection of sets $\{i_1, \dots, i_t\} \subset \{1, 2, \dots, 6\}$ such that the column vectors with labels $i_1, i_2, \dots, i_t$ form a $\mathbb{Q}$-linear independent set in V. Then $\mathcal{W}^3 = ([6], \mathcal{I})$ is a matroid on [6] and the bases of $\mathcal{W}^3$ correspond to the bases of V which consist of column vectors of A. Here the basis exchange property of $\mathcal{W}^3$ corresponds to the Steinitz exchange theorem for vector spaces. In general, matroids which are defined by matrices over a field are called *representable.*

A *circuit* C of M is a dependent set such that $C-x$ is independent for all $x \in C$. We recall that every matroid M has a unique rank function $\mathrm{rk}_M : 2^{[d]} \to \mathbb{R}$ which counts, for every $A \subset [d]$, the cardinality of a maximal independent set contained in A. If it is clear from the context which matroid M is meant we simplify the notation by using rk instead of rk_M. In Chapter 3 we will need the following basic fact.

Lemma 1.6.4. *The rank function* rk_M *of a matroid* M *on* $[d]$ *satisfies the condition: If* $X, Y \subset [d]$ *are two subsets, then*

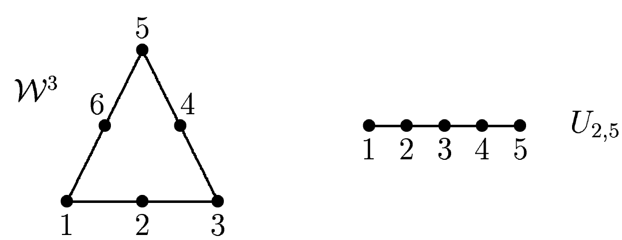
$$\mathrm{rk}_M(X \cup Y) + \mathrm{rk}_M(X \cap Y) \leq \mathrm{rk}_M(X) + \mathrm{rk}_M(Y).$$

A subset $F \subset [d]$ is called a *flat* of M if it is closed under the rank function of M, i.e. it is $F = \{x \in [d] \colon \mathrm{rk}(F \cup x) = \mathrm{rk}(F)\}$. We call a flat F with $\mathrm{rk}(F) = \mathrm{rank}(M)-1$ a *hyperplane* of M and F is *proper* if $F \neq \emptyset, [d]$. For a flat of rank k we simply write *k-flat.*

Moreover an element $i \in [d]$ is said to be a *loop* of M if it is not contained in any basis of M. Dually, an *isthmus* of M is an element $i \in [d]$ which is contained in every basis of M. We call two non-loops $i, j \in [d]$ *parallel* if $\mathrm{rk}(\{i, j\}) = 1$. Being parallel defines an equivalence relation on the set of non-loops of M, the equivalence classes of this relation are called *parallel classes.* We call a parallel class trivial if it consists of only one element.

Note that every matroid M of small rank has a *geometric* or *affine representation,* which we describe now. Let $r = \mathrm{rank}(M)$ and $r \leq 4$. We draw a picture in the affine $(r-1)$-space, where the ground set of M is represented as points. A set of points is joined by a possibly curved line if the corresponding subset in the ground set is dependent. Affinely independent sets of points which are not explicitly joined by a line correspond to independent sets of M. Parallel classes are visualized by multiple points.

Example 1.6.5. We return to the matroids from Example 1.6.3. $\mathcal{W}^3$ and $U_{2,5}$ have the geometric representations:



The matroid $\mathcal{W}^3$ has dependent 2-flats $\{1,2,3\}, \{3,4,5\}, \{1,5,6\}$. All these sets are also circuits.

There are some standard operations for matroids which we will shortly recall (see [**41**] for more details). Let M be a matroid on $[d]$ with independent sets $\mathcal{I}(M)$ and A a subset of $[d]$. Then $\{I \in \mathcal{I}(M) \colon I \subset [d] \setminus A\}$ forms the collection of independent sets of a matroid on $[d] \setminus A$ which we denote with $M \setminus A$. It is called the *deletion of* A or the *restriction of* M *to* $[d] \setminus A$. In a geometric representation this operation corresponds to the deletion of the points which are labeled with elements of A. If $A = \{p\}$ only consists of one element, then we simply write $M \setminus p$. The *dual matroid* M^* *of* M is the matroid on $[d]$ with bases $\mathcal{B}(M^*) = \{[d] - B\colon B \in \mathcal{B}(M)\}$. Clearly, we have $\operatorname{rank}(M^*) = d - \operatorname{rank}(M)$. We define now the operation which is dual to the deletion. For $A \subset [d]$ the matroid $M/A = (M^* \setminus A)^*$ is called the *contraction of* M *by* A. A matroid N which can be obtained from M by a finite sequence of contractions and deletions of M is a *minor of* M. To illustrate the definitions we give some examples.

Examples 1.6.6. Let $U_{r,d}$ and $\mathcal{W}^3$ be as in Example 1.6.3.

(a) For $p \in [d]$ it is $U_{r,d} \setminus p = U_{r,d-1}$ and $U_{r,d}/p = U_{r-1,d-1}$.
(b) The matroids $\mathcal{W}^3 \setminus 1$ and $\mathcal{W}^3/1$ have geometric representations of the form:

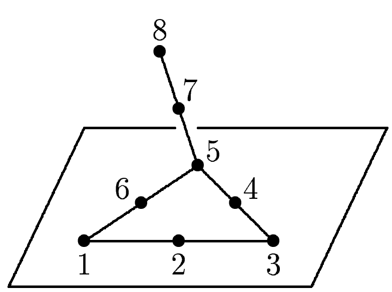
Let M be a matroid on $[d]$. Then $\overline{M}$ denotes the *underlying simple matroid of* M, that is the matroid which we obtain by deleting parallel points such that every parallel class is trivial and by deleting all loops. We call a matroid simple if $M = \overline{M}$.

Example 1.6.7. The matroids $\mathcal{W}^3$ and $U_{2,5}$ are simple.

A matroid $N = M +_F p$ is said to be the *principal extension of* M *along a proper flat* F if there exists an element p in the ground set of N such that N has bases $\mathcal{B}(N) = \mathcal{B}(M) \,\dot\cup\, \{(B - i) \cup p\colon B \in \mathcal{B}(M)$ and $i \in B \cap F\}$. In case that M is representable over the reels this operation corresponds to placing a new point generically into the subspace spanned by the elements of F. More specific, if F is a flat of rank 1 containing a non-loop $i \in [d]$, N is called a *parallel extension* of M at the point i. We write $N = M +_i p$. We use $M +_i A$ for the matroid which is obtained by the iterated parallel extension of the elements in A at the point i. The corresponding dual operation of a parallel extension is a *series extension.*

Let M_1 and M_2 be two matroids with disjoint ground sets. The matroid with bases $\{B_1 \,\dot\cup\, B_2\colon B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$ is called the *direct sum of* M_1 *and* M_2. We denote it with $M_1 \oplus M_2$.

Example 1.6.8. The matroid $M = (\mathcal{W}^3 \oplus \{8\}) +_{\{5,8\}} 7$ has a geometric representation of the form:



M

We will need the concept of connectivity for matroids. We first recall the definition.

Definition 1.6.9. A matroid M on $[d]$ is said to be *connected* if, for all $p, q \in [d]$, there exists a circuit of M containing both p and q.

For every matroid there is a direct sum decomposition into connected matroids:

Theorem 1.6.10. *Let M be a matroid. Then M has a decomposition of the form $M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$ where every matroid M_i is connected. The decomposition is unique up to the numbering of the summands M_i.*

The summands M_i are called *connected components* of M. We state a further well-known fact.

Theorem 1.6.11. *Let M be a matroid and $p \in [d]$. If M is connected, then M/p or $M \setminus p$ is connected.*

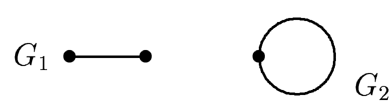
We call a set $A \subset [d]$ a *separator* of M if $M = (M \setminus A) \oplus (M \setminus ([d] - A))$. We collect some observations.

Lemma 1.6.12. *Let M be matroid on $[d]$.*

(a) *A set $A \subset [d]$ is a separator of M if and only if $\mathrm{rk}_M(A) + \mathrm{rk}_M([d] - A) = d$.*
(b) *M is connected if and only if the underlying simple matroid $\overline{M}$ is connected.*

At the end of this section we recall two classes of matroids which we study in Chapter 3. For this, we need some standard notation from graph theory.

Let G be a graph. A closed path of minimal length in G is called a *cycle.* A connected subgraph T of G is said to be a *tree* if T has no cycles. A graph G which is a union of trees is called a *forest.* Let G be a graph with edge set $[d]$. Then the cycles of G correspond to the circuits of a matroid $M(G)$. Equivalently, the spanning forests in G form the bases of $M(G)$. The matroid $M(G)$ is called the *cycle matroid* of G. We call a matroid M on $[d]$ *graphic* if $M = M(G)$ for some graph G. The class of graphic matroids is closed under minors. A graphic matroid $M(G)$ is called a *series-parallel network* if the graph G can be obtained from one of the two connected single-edge graphs G_1 or G_2

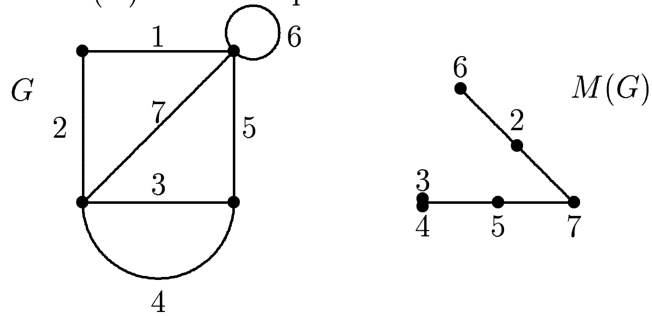


by a sequence of operations, each of which is either a series or a parallel extension. For a graphic matroid $M(G)$ these operations correspond to adding vertices of degree 2 in G or adding parallel edges in G.

Examples 1.6.13. To illustrate we give two examples.

(a) A parallel and a series extension of G_1:

(b) The matroid $M(G)$ is a series-parallel network.



We recall the definition of transversal matroids. Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a finite sequence of non-empty subsets $A_i \subset [d]$. Note that the members of the family $\mathcal{A}$ need not to be distinct. A *transversal* or *system of distinct representatives* of $(A_1, \ldots, A_r)$ is a subset $\{i_1, \ldots, i_r\} \subset [d]$ such that $i_j \in A_j$ for all $j = 1, \ldots, r$ and $i_j \neq i_k$ for all $j \neq k$. The transversals of $\mathcal{A}$ form the collection of bases of a matroid denoted with $M[\mathcal{A}]$. We call $\mathcal{A}$ a *presentation* of $M[\mathcal{A}]$. A matroid M on $[d]$ is said to be *transversal* if there exists a family $\mathcal{A} = (A_1, \ldots, A_r)$ of subsets of $[d]$ such that $M = M[\mathcal{A}]$. We note that the class of transversal matroids is not closed under contraction. The smallest class of matroids containing all minors of transversal matroids is the class of gammoids (see [**41**]). We conclude this section with an example.

Example 1.6.14. Let $A_1 = \{1, 2, 6\}$, $A_2 = \{2, 3, 4\}$ and $A_3 = \{4, 5, 6\}$. Since $\mathcal{W}^3 = M[(A_1, A_2, A_3)]$, the matroid $\mathcal{W}^3$ is transversal.

1.7. Integral polymatroids

The axiom system for the independent sets of a matroid (Proposition 1.6.2(a)) can be generalized to multisets. This leads to the notion of an integral polymatroid. There are several equivalent axiom systems for polymatroids. We recall the definition first introduced by Edmonds in [**25**]. A detailed exposition is given by Welsh in [**50**, Chapter 18].

As before we identify multisets with vectors. Let $A = (A(1), \ldots, A(d)) \in \mathbb{N}^d$ where $A(i)$ is the multiplicity of an element $i \in [d]$ in the multiset A. The cardinality of A is given by $|A| = \sum_{i=1}^{d} A(i)$. For two vectors $A, A' \in \mathbb{N}^d$ we set $A \preceq A'$ if $A(i) \leq A'(i)$ for all $i \in [d]$. Moreover, let $A \vee A'$ be the vector in $\mathbb{N}^d$ such that $(A \vee A')(i) = \max\{A(i), A'(i)\}$ for all $i \in [d]$.

Definition 1.7.1. An *integral polymatroid* on $[d]$ is a pair $P = ([d], \mathcal{I})$ such that $\mathcal{I} \subset \mathbb{N}^d$ is a finite collection of multisets satisfying:

(a) If $I' \in \mathcal{I}$ and $I \preceq I'$, then $I \in \mathcal{I}$.
(b) If $I, I' \in \mathcal{I}$ are two multisets such that $|I'| < |I|$, then there exists a $J \in \mathcal{I}$ such that $I \prec J \prec I \vee I'$.

Analogously to the matroid case the multisets in $\mathcal{I}$ are called *independent*. Elements of maximal cardinality are called *bases* of P. It follows from the property (b) that all bases have the same cardinality. If all elements of $\mathcal{I}$ are 0-1-vectors, then P is simply a matroid on $[d]$.

Let $A \in \mathbb{N}^d$ be a vector. To simplify notation we set $A + i = A + \varepsilon_i$ where ε_i denotes the i^{th} unit vector in $\mathbb{N}^d$. There is an analogon to the exchange property for matroid bases.

Proposition 1.7.2. *Let $\mathcal{B}(P)$ denote the collection of bases of an integral polymatroid. If $B, B' \in \mathcal{B}(P)$ are two bases such that $B(i) > B'(i)$, then there exists an element $j \in [d]$ such that $B'(j) > B(j)$ and $B - i + j \in \mathcal{B}(P)$.*

Proof. Apply condition (b) of Definition 1.7.1 to the multisets $I = B - i$ and $I' = B'$. Then the assertion follows immediately. □

The following is proved in [**40**, Theorem 3].

Theorem and Definition 1.7.3. *Let $P_1, \ldots, P_n$ be integral polymatroids on $[d]$ and let $\mathcal{I} = \{I_1 + \ldots + I_n \colon I_j \in \mathcal{I}(P)\}$. Then $\mathcal{I}$ is the collection of independent multisets of an integral polymatroid P.*

P is called the *polymatroid sum* of $P_1, \ldots, P_n$. We write $P = P_1 \vee \ldots \vee P_n$.

Example 1.7.4. Let $M_1, \ldots, M_n$ be matroids on $[d]$ and $\mathcal{I}(M_1), \ldots, \mathcal{I}(M_n)$ denote their independent sets. We identify a subset of $[d]$ with the corresponding incidence vector in $\mathbb{N}^d$. Then by Theorem 1.7.3

$$\mathcal{I} = \{I_1 + \ldots + I_n \colon I_j \in \mathcal{I}(M_j)\}$$

is the collection of independent multisets of an integral polymatroid.

1.8. Graded semigroup rings

A specific class of standard graded K-algebras which we study in this thesis are semigroup rings. We introduce the corresponding terminology and state some well-known facts.

Definition 1.8.1. Let $\Lambda \subset \mathbb{N}^d$ be a finitely generated semigroup. We call Λ *graded* if the following conditions are satisfied:

(a) Λ is the disjoint union $\bigcup_{i \geq 0} \Lambda_i$.
(b) $\Lambda_0 = 0$, $\Lambda_i + \Lambda_j \subset \Lambda_{i+j}$ for all integers $i, j \geq 0$.
(c) Λ is generated by elements of Λ_1.

We call the elements of Λ_i homogeneous of degree i. Let Λ be a standard graded semigroup which is minimally generated by $\alpha_1, \ldots, \alpha_n \in \Lambda_1$ and let $K[T_1, \ldots, T_d]$ denote the polynomial ring. To every semigroup element $\lambda = (a_1, \ldots, a_d) \in \Lambda$ we associate the monomial $T^\lambda = T_1^{a_1} T_2^{a_2} \cdots T_d^{a_d}$. Recall that the semigroup ring $K[\Lambda]$ is the K-algebra generated by the monomials T^{α_i} for $i = 1, \ldots, n$. Consider the presentation

$$\varphi : S \to K[\Lambda] \qquad X_i \mapsto T^{\alpha_i}.$$

Then $I_\Lambda = \ker(\varphi)$ is called the *toric ideal* of the semigroup ring $K[\Lambda]$. It is easy to show that I_Λ is generated by all binomials of the form $u - u' \in S$ such that $\varphi(u) = \varphi(u')$. If Λ is graded, then $K[\Lambda] = S/I$ is a standard graded K-algebra. We restate a criterion for Λ to be graded (see [**45**]).

Lemma 1.8.2. *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}^d$ and Λ be the semigroup which they generate. The following statements are equivalent:*

(a) Λ *is graded.*
(b) *The elements* $\alpha_1, \dots, \alpha_n$ *lie in an affine hyperplane of* $\mathbb{R}^d$.

The divisibility relation of the monomials in $K[\Lambda]$ defines a partial order $\preceq$ on Λ: For $\mu, \lambda \in \Lambda$ we set $\mu \preceq \lambda$ if $\lambda = \sigma + \mu$ for some $\sigma \in \Lambda$. Then the open and closed intervals

$$(\mu, \lambda) = \{\sigma \in \Lambda \colon \mu \prec \sigma \prec \lambda\}, \qquad [\mu, \lambda] = \{\sigma \in \Lambda \colon \mu \preceq \sigma \preceq \lambda\}$$

are partially ordered with the induced ordering. For a finite partially ordered set $(P, \preceq)$ let $\Gamma(P)$ be the corresponding order complex (see Example 1.5.2). For $\lambda \in \Lambda$ we denote the order complex of the interval $(0, \lambda)$ with Γ_λ. The following is stated in [**45**, Corollary 2.2] and [**35**].

Proposition 1.8.3. *$K[\Lambda]$ is Koszul if and only if Γ_λ is Cohen-Macaulay over K for all $\lambda \in \Lambda$.*

This result together with Theorem 1.5.5 implies:

Proposition 1.8.4. *If Γ_λ is shellable for all $\lambda \in \Lambda$, then $K[\Lambda]$ is Koszul.*

At the end of this section we give some examples of graded semigroup rings.

Example 1.8.5. Let $S = K[T_1, T_2, \dots, T_n]$ be the polynomial ring in n variables. Then the d^{th}-Veronese subring $S^{(d)}$ is the semigroup ring generated by all monomials of degree d in S. The corresponding semigroup Λ is generated by all $\alpha \in \mathbb{N}^n$ which lie in the affine hyperplane

$$H = \{x \in \mathbb{R}^n \colon \sum_{i=1}^{n} x_i = d\}.$$

By Lemma 1.8.2 the algebra $S^{(d)}$ is standard graded. In [**32**] it is shown that all divisor posets Γ_λ with $\lambda \in \Lambda$ are shellable.

In [**49**, Chapter 14] Sturmfels studies a certain class of semigroup rings. Fix an integer $d > 0$ and numbers $s_1, \dots, s_n \geq 0$ and let

$$\mathcal{A} = \{(i_1, i_2, \dots, i_n) \in \mathbb{N}^n : i_1 + \ldots + i_n = d, \quad i_1 \leq s_1, \dots, i_n \leq s_n\}.$$

The semigroup Λ which is generated by the elements in $\mathcal{A}$ is graded and the semigroup ring $R_\mathcal{A} = K[\Lambda]$ is said to be of *Veronese type.* Let $I_\mathcal{A}$ denote the toric ideal of $R_\mathcal{A}$. The following is proved in [**49**].

Theorem 1.8.6. *The toric ideal $I_\mathcal{A}$ has a quadratic Gröbner basis. In particular, every algebra of Veronese type is Koszul.*

Refer also to [**43**] where Hibi and Ohsugi generalize this concept to algebras of Segre-Veronese type.

1.9. Standard bigraded algebras

In Chapter 4 we will study K-algebras of the form $R = S/J$ where $S = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ is the polynomial ring with standard bigrading $\deg(X_i) =$

$(1,0)$, $\deg(Y_j) = (0,1)$ and $J \subset S$ is a bigraded ideal of S. Then R has a decomposition of the from

$$R = \bigoplus_{i,j \geq 0} R_{(i,j)}$$

where $R_{(i,j)} = T_{(i,j)}/J_{(i,j)}$ is a finite dimensional K-vector space. We call such an algebra *standard bigraded.* In analogy to the graded case one defines a bigraded, finitely generated R-module M. Then the $\operatorname{Tor}_i^R(M,K)$-groups are standard bigraded, and the *bigraded Poincaré series of* M is given by

$$P_R^M(s,t,z) = \sum_{i,j,k} \beta_{i,(j,k)}^R(M) s^j t^k z^i.$$

where $\beta_{i,(j,k)}^R(M) = \dim_K \operatorname{Tor}_i^R(M,K)_{(j,k)}$ are the *bigraded Betti numbers of* M. Every bigraded K-algebra R is also naturally graded with i^{th} component $R_i = \bigoplus_{j+k=i} R_{(j,k)}$. Similarly, every bigraded R-module M can be considered as graded. Therefore all definitions which we have given for graded rings and modules are also valid for the bigraded objects.

Notation 1.9.1. Let R be a standard bigraded K-algebra. Then $\mathcal{M}_{\mathbb{Z}^2}(R)$ denotes the collection of finitely generated, bigraded R-modules.

CHAPTER 2

Initially Koszul algebra

This chapter is devoted to a certain class of K-algebras, the initially Koszul algebras. To give the definition we recall the concept of Koszul filtrations (see [**21**]). Then we characterize these algebras in terms of Gröbner bases, discuss applications to graded semigroup rings and study algebras which are initially Koszul after a generic choice of coordinates. This leads to the notion of universally initially Koszulness.

In the sequel $R = S/I$ always denotes a standard graded algebra where $S = K[X_1, \dots, X_n]$ is the standard graded polynomial ring and I a graded ideal which does not contain linear forms, that is $I \subset (X_1, \dots, X_n)^2$.

2.1. Koszul filtrations

Conca, Trung and Valla have introduced an effective way to show that a K-algebra is Koszul (see [**21**]). We recall the definition.

Definition 2.1.1. Let R be a graded K-algebra. A family $\mathcal{F}$ of ideals in R is called a *Koszul filtration* of R, if

(a) every ideal $J \in \mathcal{F}$ is generated by linear forms,
(b) the ideal (0) and the graded maximal ideal of R belong to $\mathcal{F}$ and
(c) for every $J \in \mathcal{F}$, $J \neq 0$, there exists an ideal $L \in \mathcal{F}$ such that $L \subset J$, J/L is cyclic and $L\colon J \in \mathcal{F}$.

The naming Koszul filtration is justified with the following proposition which is stated in [**21**].

Proposition 2.1.2. *Let $\mathcal{F}$ be a Koszul filtration of R. Then $\operatorname{Tor}_i^R(R/J, K)_j = 0$ for $i \neq j$ and for all $J \in \mathcal{F}$. In particular, the graded maximal ideal of R has a system of generators $x_1, \dots, x_n$ such that all ideals $(x_1, \dots, x_j)$ with $j = 1, \dots, n$ have a linear R-free resolution and R is Koszul.*

Koszul filtrations have been studied in various contexts (see [**5**], [**17**], [**20**], [**21**], and [**32**]). We recall some concepts which we will discuss in the forthcoming sections.

Examples 2.1.3. There are several concepts of specific Koszul filtrations. We consider some of them. Let R be a K-algebra.

(a) Let $\mathcal{L}(R)$ be the collection of ideals in R which are generated by linear forms. According to Conca's definition in [**17**] the algebra R is called *universally Koszul* if $\mathcal{L}(R)$ forms a Koszul filtration for R.
(b) Herzog, Hibi and Restuccia have introduced the class of *strongly Koszul algebras* in [**32**]. Let $x_1, \dots, x_n \in R_1$ be a minimal system of generators for

R. Then R is said to be *strongly Koszul,* if the collection

$$\mathcal{F} = \{(x_{j_1}, \ldots, x_{j_r}) \colon \{j_1, \ldots, j_r\} \subset [n]\}$$

is a Koszul filtration for R.

(c) In [**5**] Aramova, Herzog and Hibi have defined *sequentially Koszul* algebras. Such an algebra R has a Koszul filtrations which consists of (not necessarily all) ideals of the form $(x_{j_1}, \ldots, x_{j_r})$ where $\{j_1, \ldots, j_r\} \subset [n]$ and $x_1, \ldots, x_n \in R_1$ denotes a minimal system of generators.

In a sense a universally Koszul algebra has the largest possible Koszul filtration, which is $\mathcal{L}(R)$. We consider the opposite case.

Definition 2.1.4. Let $x_1, \ldots, x_n \in R_1$. We call R *initially Koszul* (i-Koszul for short) with respect to $x_1, \ldots, x_n$ if

$$\mathcal{F} = \{(x_1, \ldots, x_i) \colon i = 0, \ldots, n\}$$

forms a Koszul filtration for R.

In order to simplify notation we say that $R = S/I$ is i-Koszul if R is initially Koszul with respect to $X_1 + I, \ldots, X_n + I$. Koszul filtrations as in Definition 2.1.4 which are generated by a flag of linear subspaces of R, first considered in [**20**], are called *Gröbner flags.* The reason for this naming is the following result.

Theorem 2.1.5. [**20**, Conca, Rossi, Valla] *Let $R = S/I$ be i-Koszul. Then I has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $X_1 < X_2 < \ldots < X_n$.*

By Proposition 2.1.2 any Koszul filtration of R contains a flag. Thus i-Koszulness is equivalent to the existence of a Koszul filtration which is as small as possible.

2.2. Characterization of i-Koszulness

Throughout this section $<$ denotes the reverse lexicographic order induced by $X_1 < X_2 < \ldots < X_n$. The following result, which was shown independently in [**20**], characterizes i-Koszulness in terms of initial ideals.

Theorem 2.2.1. *The following statements are equivalent:*

(a) *$R = S/I$ is i-Koszul.*
(b) *$R' = S/\mathrm{in}_<(I)$ is i-Koszul.*
(c) *I has a quadratic Gröbner basis with respect to $<$ and if $X_iX_j \in \mathrm{in}_<(I)$ for some $i < j$, then $X_iX_k \in \mathrm{in}_<(I)$ for all $i \leq k < j$.*

For the proof of Theorem 2.2.1 we need a property of the chosen reverse lexicographic term order $<$.

Lemma 2.2.2. *Let $I \subset S$ be a graded ideal and set $\bar{S} = K[X_2, \ldots, X_n]$. Let $\sigma : S \to \bar{S}$ be the K-algebra homomorphism with $X_1 \mapsto 0$ and $X_i \mapsto X_i$ for $i > 1$. Suppose that $g_1, \ldots, g_t$ is a Gröbner basis for I with respect to $<$ such that X_1 does not divide $\mathrm{in}(g_i)$ for $i = 1, \ldots, r$ and X_1 divides $\mathrm{in}(g_i)$ for $i = r + 1, \ldots, t$. Then $\sigma(g_1), \ldots, \sigma(g_r)$ is a Gröbner basis for the ideal $\bar{I} = (\sigma(f) \colon f \in I)$ with respect to the induced order on $\bar{S}$. In particular, we have $\overline{\mathrm{in}(I)} = \mathrm{in}(\bar{I})$.*

Proof. We use the Buchberger criterion (see Theorem 1.4.7). Since $<$ is a reverse lexicographic order, we may apply Proposition 1.4.5. Thus if $f \in S$ is a polynomial and X_1 divides $\text{in}(f)$, then X_1 divides f. This implies that $S(\sigma(g_i), \sigma(g_j)) = \sigma(S(g_i, g_j))$ for all $i, j \in \{1, \dots, r\}$ with $i \neq j$, and the assertion follows immediately. □

We return now to the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. We prove the equivalence of (a) and (b) by induction on n. The case $n = 1$ is trivial. Let $x_i = X_i + I$ and $x'_i = X_i + \text{in}(I)$ for $i = 1, \dots, n$. Note that R is i-Koszul if and only if

(i) R/x_1R is i-Koszul, and
(ii) $0 : x_1 = (x_1, \dots, x_k)$ for some k.

Using $\text{in}(X_1 + I) = (X_1) + \text{in}(I)$ (see Proposition 1.4.5) and Lemma 2.2.2 we see that (i) is equivalent to R'/x'_1R' being i-Koszul. Since $\text{in}(I : X_1) = \text{in}(I) : X_1$ (see Proposition 1.4.5), we get $0 : x'_1 = (x'_1, \dots, x'_k)$ if and only if (ii) holds. This shows the equivalence of (a) and (b). For the equivalence of (b) and (c) we need the following proposition. □

Proposition 2.2.3. *Let $R = S/I$ where $I = (m_1, \dots, m_r)$ is generated by monomials of degree 2. Then the following statements are equivalent:*

(a) *R is i-Koszul.*
(b) *If $X_iX_j \in I$ for some $j > i$, then $X_iX_k \in I$ for all $i \leq k < j$.*

Proof. Let $x_k = X_k + I$ for $k = 1, \dots, n$ and $J_i = (x_1, \dots, x_i)$ for $i = 0, \dots, n$.

Let us assume (a). If $X_iX_j \in I$ with $i < j$, then $x_ix_j = 0$ and so $x_j \in J_{i-1} : J_i$. Since R is i-Koszul, we have $J_{i-1} : J_i = J_l$ for some $l \geq i - 1$. But then for each $i \leq k < j$ we get $x_ix_k \in J_{i-1}$. Therefore $X_iX_k - X_lX_s \in I$ for some $l \leq i - 1$ and some s. Since I is a monomial ideal, this implies $X_iX_k \in I$. This is condition (b).

Conversely, we assume (b). Then we have to show that $J_{i-1} : J_i = (x_1, \dots, x_{k(i)})$ for each $i = 1, \dots, n$. Let $u \in J_{i-1} : (x_i)$, $u \neq 0$. Since I is a monomial ideal, we may assume that u is a monomial. It is clear that $J_{i-1} \subset J_{i-1} : J_i$. So we assume $u \notin J_{i-1}$. It follows that $ux_i = 0$. There are indices $k \leq l$ such that $X_kX_l \in I$ and $X_kX_l \mid uX_i$. If $i \neq k$ and $i \neq l$, we have $u = 0$ which is a contradiction. Since $u \notin J_{i-1}$, it follows that $i = k$ and $u \in (x_l)$. Condition (b) implies that $(x_i, \dots, x_l) \subset J_{i-1} : x_i$ which yields the assertion. □

2.3. Applications and examples

In this section we use the criterion of Section 2 to show that certain algebras are i-Koszul. We start with algebras whose defining ideal is generated by monomials. From Proposition 2.2.3 we see immediately the following fact.

Corollary 2.3.1. *Let I be generated by monomials of degree 2 and $G(I)$ be the set of minimal generators for I. If $G(I)$ is stable, then $R = S/I$ is i-Koszul.*

Moreover, we observe that i-Koszulness is compatible with tensor products.

Proposition 2.3.2. *If $R = K[X_1, \dots, X_n]/I$ and $R' = K[Y_1, \dots, Y_m]/J$ are i-Koszul algebras, then $R \otimes_K R'$ is also i-Koszul.*

Proof. By Theorem 2.2.1 there are Gröbner bases $\{f_1, \dots, f_k\}$ for I and $\{g_1, \dots, g_l\}$ for J, such that $\text{in}(I)$ and $\text{in}(J)$ satisfy the condition 2.2.1(c). The tensor product $R \otimes R'$ has a presentation of the form T/Q where $T = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ and $Q = IT + JT$. We take the reverse lexicographic order on T induced by the ordering $X_1 < \dots < X_n < Y_1 < \dots < Y_m$. It follows immediately from the Buchberger criterion (see Theorem 1.4.7) that $\{f_1, \dots, f_k, g_1, \dots, g_l\}$ forms a Gröbner basis of Q. Thus condition (b) of 2.2.3 is satisfied for $\text{in}(Q)$. By Theorem 2.2.1 we get the assertion. □

The i-Koszul property is preserved under taking Veronese subrings of algebras whose defining ideal is generated by monomials.

Theorem 2.3.3. *Let I be generated by monomials. If $R = S/I$ is i-Koszul, then the d^{th} Veronese subring $R^{(d)}$ is i-Koszul for every $d > 0$.*

Proof. We first consider the case $R = S$. Let $\mathcal{M}$ be the set of all monomials of degree d in S. Let $<_{\text{lex}}$ denote the lexicographic term order on S induced by $X_1 > \dots > X_n$. We order the elements of $\mathcal{M}$ such that $m_1 >_{\text{lex}} m_2 >_{\text{lex}} \dots >_{\text{lex}} m_t$. Writing $S^{(d)} \cong K[Y_1, \dots, Y_t]/J$ each monomial m_l can be identified with a residue class $y_l = Y_l + J$. Thus we define $J_l = (m_1, \dots, m_l)$ for $l = 0, \dots, t$. We have to show that for every $l = 1, \dots, t$ the ideal $J_{l-1} \colon J_l$ is generated by an initial sequence of the m_i's. We set

$$\mathcal{M}_l = \{m \in \mathcal{M} \colon X_r | m \text{ for some } r \le l\}$$

for $l = 1, \dots, t$ and $\mathcal{M}_0 = \emptyset$. The elements of each $\mathcal{M}_l$ form an initial sequence $m_1, m_2, \dots, m_{i_l}$. As in Chapter 1 we write $\max(m) = \max\{i \colon X_i \text{ divides } m\}$. We claim that

$$J_{l-1} \colon (m_l) = (\mathcal{M}_{\max(m_l)-1})$$

which yields the assertion. For the case $l = 1$ there is nothing to prove, thus we may assume $l > 1$. Let $s = \max(m_l) - 1$. We write $m_l = X_{i_1} \cdots X_{i_d}$ with $i_1 \le \dots \le i_d = s + 1$. Let $u \in J_{l-1} \colon (m_l)$. We may assume that u is a monomial. Then we have $u m_l = w m_r$ for some monomial w and $r \in \{1, \dots, l-1\}$. We write $m_r = X_{j_1} \cdots X_{j_d}$ with $j_1 \le \dots \le j_d$. Since $m_r >_{\text{lex}} m_l$, there exists an integer $q \in \{1, \dots, d\}$ such that $j_m = i_m$ for all $m < q$ and $j_q < i_q \le s + 1$. The equation $u m_l = w m_r$ implies

$$u X_{i_q} \cdots X_{i_d} = w X_{j_q} \cdots X_{j_d}$$

and thus we have $X_{j_q} | u$ which yields $u \in (\mathcal{M}_s)$. Conversely, let $u \in \mathcal{M}_s$. Then there exists a number $r \in \{1, \dots, s\}$ such that $X_r | u$. We define $w = X_{i_1} \cdots X_{i_{d-1}} X_r$. It follows $w >_{\text{lex}} m_l$ and hence $w \in J_{l-1}$. Since

$$u m_l = \Big(\frac{u}{X_r} X_{i_d}\Big) w,$$

we obtain $u \in J_{l-1} : (m_l)$.

We now consider the general case $R = S/I$. Let $x_i = X_i + I$ for $i = 1, \dots, n$. Since I is a monomial ideal, the set of all monomials which do not belong to I forms a K-basis of R. Thus each monomial $u = x_{j_1} x_{j_2} \dots x_{j_r} \in R$ is either 0 or has a unique presentation $u = X_{j_1}X_{j_2}\dots X_{j_r} + I$. Therefore we may identify each monomial with its residue class. We have the following relations:

($*$) For any two non-zero monomials $m, m' \in R$ we have $mm' = 0$ if and only if there are $i, j \in \{1, \dots, n\}$ such that $X_i \mid m$, $X_j \mid m'$ and $X_iX_j \in I$.

$R^{(d)}$ is generated as a K-algebra by the set $\mathcal{M}$ of all non-zero monomials of degree d in R. As in the first case we order the monomials of $\mathcal{M}$ by $m_1 >_{\text{lex}} m_2 >_{\text{lex}} \dots >_{\text{lex}} m_t$ and set $J_i = (m_1, \dots, m_i)$ for $i = 0, \dots, t$. We define

$$\mathcal{N}(m_l) = \{m \in \mathcal{M} \colon \text{there exists } i \le j \text{ with } X_i|m_l\ ,\ X_j|m \text{ and } X_iX_j \in I\}$$

and assert that

$$J_{l-1} : (m_l) = (\mathcal{M}_{\max(m_l)-1}, \mathcal{N}(m_l))$$

for $l = 1, \dots, t$. Let $a \in J_{l-1} : (m_l)$, $a \neq 0$. We may assume that a is a monomial. There are two cases to consider:

(a) $am_l = 0$. We have a relation as in ($*$). If $a \notin (\mathcal{M}_{\max(m_l)-1})$, then, for each index t with $X_t \mid a$, it holds that $t \ge \max(m_l)$. Thus, if $X_i \mid m_l$ and $X_j \mid a$ with $X_iX_j \in I$, it follows that $i \le j$ which yields $a \in \mathcal{N}(m_l)$.

(b) $am_l \neq 0$. We have $am_l = bm_i$ for some monomial $b \in R^{(d)}$ and some $i < l$. There is a K-linear, injective map $\sigma : R = S/I \to S$ with $m + I \mapsto m$ for all non-zero monomials $m \in R$. If $mm' \neq 0$ for two monomials $m, m' \in R$, we get that $\sigma(m)\sigma(m') = \sigma(mm')$. Let $\pi : S \to R = S/I$ be the natural epimorphism. Then $\pi \circ \sigma = \mathrm{id}_R$ holds. Since σ and π respect the standard grading, these maps restrict to $R^{(d)}$ and $S^{(d)}$ respectively. We apply σ to the equation above and, since $am_l \neq 0$, obtain that $\sigma(a)\sigma(m_l) = \sigma(b)\sigma(m_i)$ in $S^{(d)}$. The case $R = S$ yields $\sigma(a) \in (\mathcal{M}_{\max(\sigma(m_l))-1})$. Applying π we get $a \in (\mathcal{M}_{\max(m_l)-1})$.

The converse inclusion $(\mathcal{M}_{\max(m_l)-1}, \mathcal{N}(m_l)) \subset J_{l-1} : (m_l)$ follows immediately from the case $R = S$ and the relations in ($*$).

It remains to show that for all $l = 1, \dots, t$ the ideal $J_{l-1} : (m_l)$ is generated by an initial sequence $m_1, \dots, m_{k(l)}$. Since the elements of $\mathcal{M}_{\max(m_l)-1}$ form already an initial sequence, it suffices to prove the following: If $m_s \in \mathcal{N}(m_l)$ for some s, then $m_{s-1} \in \mathcal{M}_{\max(m_l)-1} \cup \mathcal{N}(m_l)$. Let $m_l = X_{i_1}\dots X_{i_d}$ with $i_1 \le \dots \le i_d$. It is $i_d = \max(m_l)$. Since $m_s \in \mathcal{N}(m_l)$, there are $i \le j$ with $X_i|m_l$, $X_j|m_s$ and $X_iX_j \in I$. By the chosen order we have $m_{s-1} >_{\text{lex}} m_s$. Thus there exists a k with $X_k \mid m_{s-1}$ and $k \le j$. If $k < i_d$, we have $m_{s-1} \in \mathcal{M}_{\max(m_l)-1}$. Otherwise we have $i \le k \le j$. Since R is i-Koszul, we have $X_iX_k \in I$ by Proposition 2.2.3. This yields $m_{s-1} \in \mathcal{N}(m_l)$. □

In [**6**] Aramova, Herzog and Hibi study algebras which arise from lattices. Recall that a lattice is said to be distributive if the two operations join and meet satisfy the distributivity rule.

Definition 2.3.4. Let L be a finite, distributive lattice and $K[\{X_\alpha\}_{\alpha\in L}]$ the polynomial ring over K. Consider the ideal $I_L = (X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta} \colon \alpha, \beta \in L)$ of $K[\{X_\alpha\}_{\alpha\in L}]$. The quotient algebra

$$R_K[L] = K[\{X_\alpha\}_{\alpha\in L}]/I_L$$

is called the *Hibi ring* of L over K.

Hibi has shown that I_L has a quadratic Gröbner basis for any term order which selects, for any two incomparable elements $\alpha, \beta \in L$, the monomial $X_\alpha X_\beta$ as the initial term of $X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta}$ (see [**37**]). Such a term order $<$ is, for example, the reverse lexicographic term order induced by a total ordering of the variables satisfying $X_\alpha < X_\beta$, if $\text{rank}(\alpha) > \text{rank}(\beta)$ (see [**6**]) where here $\text{rank}(\alpha)$ denotes the rank of α in the lattice L. We get the following characterization.

Remark 2.3.5. Let L be a finite distributive lattice and $<$ be a term order on $S = K[\{X_\alpha\}_{\alpha\in L}]$ as above. Then the Hibi ring $R = S/I_L$ is i-Koszul if and only if R is a polynomial ring.

Proof. If $I_L \neq (0)$, we have $X_\alpha X_\beta \in \text{in}(I)$ where α and β are some elements of L, say $X_\alpha \leq X_\beta$. Since R is i-Koszul, it follows that $X_\alpha^2 \in \text{in}(I)$ by Theorem 2.2.1. This yields a contradiction because both monomials in a relation $X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta}$ are square-free. □

2.4. u-i-Koszulness

Let R be i-Koszul. In Proposition 2.1.2 we have seen that K has a linear R-free resolution. If we consider $R = S/I$ as an S-module, we can study the minimal S-free resolution of R. In the next statement $\text{gin}(I)$ denotes the generic initial ideal with respect to the reverse lexicographic order induced by $X_1 > \ldots > X_n$ (see Section 1.4).

Proposition 2.4.1. *Let K be an infinite field,* $\text{char}(K) \neq 2$*, $I \subset S$ a graded ideal and $I \neq (0)$. The following statements are equivalent:*

(a) *I has a 2-linear S-resolution*
(b) *$S/\text{gin}(I)$ is i-Koszul.*
(c) *$S/\text{gin}(I)$ is Koszul.*

Proof. Let us assume (a). By Lemma 1.4.16 $\text{gin}(I)$ is generated by quadratic monomials and the set $G(\text{gin}(I))$ is stable. Using Corollary 2.3.1 we obtain that $S/\text{gin}(I)$ is i-Koszul. This is condition (b) which implies (c) by Proposition 2.1.2.

Assuming (c) it follows from Proposition 1.3.5 that $\text{gin}(I)$ is generated in degree 2. Since by hypothesis K is an infinite field and $\text{char}(K) \neq 2$, the set of monomials in $\text{gin}(I)_2$ is stable by Proposition 1.4.14. We apply [**27**, Proposition 10] which yields that $\text{gin}(I)$ is 2-regular. Since we assume that $I \subset (X_1, \ldots, X_n)^2$, we obtain $\text{reg}(\text{gin}(I)) = 2 = \text{reg}(I)$. Thus I has a 2-linear resolution. □

Proposition 2.4.1 can be interpreted as follows:

Corollary 2.4.2. *I has a 2-linear resolution if and only if all generic flags are Gröbner flags.*

We may now ask for which algebras all flags are Gröbner flags. This leads us to a new definition.

Definition 2.4.3. A K-algebra $R = S/I$ is called *universally initially Koszul* (for short u-i-Koszul) if R is i-Koszul with respect to every K-basis $x_1, \ldots, x_n \in R_1$.

If an algebra R is u-i-Koszul, then the i-Koszul property is preserved under any change of coordinates in R_1. Since this is a strong condition, we can classify all u-i-Koszul algebras in the following case:

Theorem 2.4.4. *Let K be algebraically closed,* $\operatorname{char}(K) \neq 2$ *and $I \subset (X_1, \dots, X_n)^2$. Then $R = S/I$ is u-i-Koszul if and only if $I = (g^2)$ for some linear form $g \in S_1$ or $I = (X_1, \dots, X_n)^2$.*

We divide the proof of Theorem 2.4.4 into several lemmata.

Lemma 2.4.5. *Let R be u-i-Koszul and $x \in R_1 \backslash \{0\}$. Then R/xR is also u-i-Koszul.*

Proof. Let $\bar{R} = R/xR$ and $x_2, \dots, x_n \in \bar{R}_1$ be an arbitrary K-basis of $\bar{R}_1$. We have to show that $\bar{R}$ is i-Koszul with respect to this sequence. Since R is u-i-Koszul, R is i-Koszul with respect to $x, x_2, \dots, x_n$. This yields the assertion. □

Lemma 2.4.6. *Let R be u-i-Koszul,* $\operatorname{char}(K) \neq 2$ *and let $\mathcal{N} \subset R_1$ denote the set of all zerodivisors in R_1. Then $\mathcal{N}$ is a linear subspace of R_1 and $\mathcal{N}^2 = 0$.*

Proof. Since R is u-i-Koszul, we have $(x) \subset 0 : (x)$ for all $x \in \mathcal{N}$. This implies $x^2 = 0$ for all $x \in \mathcal{N}$. Thus for $x, y \in \mathcal{N}$ we have $(x + y)(x - y) = x^2 - y^2 = 0$ and therefore $x + y \in \mathcal{N}$. Since $\operatorname{char}(K) \neq 2$, it follows that $\mathcal{N}^2 = 0$. □

Note that $\dim_K R_1 = n$ because we always assume that I does not contain linear forms.

Lemma 2.4.7. *Let $I = (L^2)$ for some linear subspace L of S_1. Then $R = S/I$ is u-i-Koszul if and only if* $\dim_K L \in \{0, 1, n\}$.

Proof. Let R be u-i-Koszul. After a change of coordinates we may assume that $L = (X_1, \dots, X_i)$ with $i = \dim_K L$ and $I = (X_1, \dots, X_i)^2$. If $i \notin \{0, 1, n\}$, we interchange X_i and X_{i+1}. We obtain a new defining ideal J with $X_1X_{i+1} \in J$, but $X_1X_i \notin J$ which is a contradiction to the i-Koszulness of S/J by Proposition 2.2.3.

Conversely, let $i = \dim_K L \in \{0, 1, n\}$. If $i = 0$, there is nothing to prove. If $i = 1$, then $I = (g^2)$ for some $g \in S_1$. For any transformation we obtain a new defining ideal $J = (h^2)$ with $h \in S_1$. We observe that $\operatorname{in}(h^2)$ is a square. The assertion follows from Theorem 2.2.1. If $i = n$, we have $I = (X_1, \dots, X_n)^2$. In this case the defining ideal does not change for any transformation and we get the claim by Theorem 2.2.1. □

Lemma 2.4.8. *Let K be algebraically closed,* $\operatorname{char}(K) \neq 2$ *and $R = S/I$. If $I \subset (X_1, \dots, X_n)^2$ is a principal ideal, then R is u-i-Koszul if and only if $I = (g^2)$ for some $g \in S_1$.*

Proof. If $I = (g^2)$ for some $g \in S_1$, then R is u-i-Koszul by Lemma 2.4.7. Let R be u-i-Koszul. Since K is algebraically closed and $\operatorname{char}(K) \neq 2$, there exists a K-basis $Y_1, \dots, Y_n$ of S_1 such that the generator of I is of the form $Y_1^2 + \ldots + Y_i^2$ for some $i \leq n$ (see [**38**]). We claim that $i = 1$ and argue by contradiction.

If $i > 1$, we apply $Y_{i-1} \mapsto Y_{i-1} + \sqrt{-1}Y_i$ and $Y_j \mapsto Y_j$ for $j \neq i - 1$. Then the generator f in the new coordinates has $\operatorname{in}(f) = -2\sqrt{-1}Z_{r-1}Z_r$ and thus R is not i-u-Koszul by Theorem 2.2.1. Therefore we have $i = 1$, and $f = Y_1^2$. □

Remark 2.4.9. Let $I \subset S$ have a quadratic Gröbner basis and let $f_1, \dots, f_k$ be a minimal system of generators of I. Then there exists a minimal Gröbner basis of I which consists of K-linear combinations of $f_1, \dots, f_k$.

Proof of Theorem 2.4.4. In Lemma 2.4.7 and Lemma 2.4.8 we have already observed that R is u-i-Koszul, if $I = (g^2)$ or $I = (X_1, \dots, X_n)^2$.

Conversely, let R be u-i-Koszul. By Lemma 2.4.6 the set $\mathcal{N}$ of all zerodivisors in R_1 is a linear subspace of R_1 and $\mathcal{N}^2 = 0$. Thus in the case that $\dim(R) = 0$ we have $\mathcal{N} = R_1$ and therefore $I = (X_1, \dots, X_n)^2$.

Let now $\dim(R) > 0$. We have to show that $I = (g^2)$ for some $g \in R_1$. We use induction on $d = \dim(R)$. Let $d = 1$. We have two cases:

(a) $\mathcal{N} = 0$. In this case R is a 1-dimensional Cohen-Macaulay ring with minimal multiplicity and every $l \in R_1$ is a non-zerodivisor. Suppose that $I \neq (0)$. We show that R must be a domain and deduce a contradiction.

Since $\dim(R) = 1$ and $I \neq (0)$, we have $\operatorname{emb}\dim(R) > 1$. Let $x_i = X_i + I$ for $i = 1, \dots, n$. x_1 is a non-zerodivisor of R. Since $\dim(R/x_1R) = 0$ and R/x_1R is u-i-Koszul by Lemma 2.4.5, we get that $R/x_1R = K[X_2, \dots, X_n]/(X_2, \dots, X_n)^2$ as we have already observed above. Since x_1 is a non-zerodivisor of R, we have $X_1^2 \notin I$. By Theorem 2.2.1 the algebra $S/\mathrm{in}(I)$ is i-Koszul. The term order of Theorem 2.2.1 implies that $X_1^2 \notin \mathrm{in}(I)$. By Proposition 2.2.3 we get $\mathrm{in}(I) = (X_2, \dots, X_n)^2$. It is a general fact that the set of monomials which do not belong to $\mathrm{in}(I)$ forms a K-basis of R. In our case $x_1^i x_1, \dots, x_1^i x_n$ forms a K-basis of R_{i+1} for all $i \geq 0$. If $a \in R_i$, $i \geq 2$, is a homogeneous element, we have $a \in (x_1)^{i-1}$.

Suppose that $ar = 0$ for some $r \in R$. We can write $a = x_1^{i-1} l$ with some linear form $l \in R_1$. It is $ar = x_1^{i-1} lr = 0$. Since x_1 and l are non-zerodivisors by the assumption, it follows that $r = 0$. Thus every homogeneous element of R is a non-zerodivisor which implies that R is a domain. Since K is algebraically closed and I is graded, R is a polynomial ring in one variable. This is a contradiction to $\operatorname{emb}\dim(R) > 1$.

(b) $\mathcal{N} \neq 0$. It is $I \neq (0)$. We start induction on $n = \operatorname{emb}\dim(R)$. Let $n = 2$. By Theorem 2.2.1 and Remark 2.4.9 the ideal $I \subset K[X_1, X_2]$ has a minimal system of generators $f_1, \dots, f_k$ which forms a minimal Gröbner basis. Since we are in the case that $d = \dim(R) = 1$, we have $I \neq (X_1, \dots, X_n)^2$ and thus $k \leq 2$. If $k = 0$, then R is a polynomial ring. For $k = 1$ we get the assertion by Lemma 2.4.8. If $k = 2$, we deduce a contradiction. Since R is i-Koszul, we obtain by Theorem 2.2.1 that $\mathrm{in}(I) = (X_1^2, X_1X_2)$ with respect to the term order of 2.2.1. It follows that $I = (X_1^2, X_1X_2)$ because X_1^2, X_1X_2 are the smallest two monomials of degree two. Thus, by interchanging X_1 and X_2 we get the defining ideal $J = (X_1X_2, X_2^2)$. By Theorem 2.2.1 S/J is not i-Koszul which is a contradiction to R being u-i-Koszul.

Let $n > 2$. We choose $x \in \mathcal{N}$, $x \neq 0$. We may assume that $x = x_1 = X_1 + I$. Since $x_1^2 = 0$ by Lemma 2.4.6, we have that $\dim(R/x_1R) = 1$ and $\operatorname{emb}\dim(R/x_1R) = n-1$. R/x_1R is u-i-Koszul by Lemma 2.4.5. Let $\bar{\mathcal{N}}$ be the set of all zerodivisors of R/x_1R. If $\bar{\mathcal{N}} \neq 0$, by induction hypothesis on n, if $\bar{\mathcal{N}} = 0$, by case (a), it follows that R/x_1R is a hypersurface ring of the form $R/x_1R = K[X_2, \dots, X_n]/(g^2)$ for some $g \in K[X_2, \dots, X_n]_1$.

Let $L \subset S_1$ be the linear subspace with $(I : X_1)_1 = L$. Then we have $I = (X_1 L, g^2 + X_1 l)$ for some linear form $l \in S_1$. By Lemma 2.4.6 we get $X_1 \in L$ and thus $X_1 \in \operatorname{Rad}(I)$. It follows that $g \in \operatorname{Rad}(I)$ which implies $g + I \in \mathcal{N}$. Again by Lemma 2.4.6 we get $g^2 \in I$ and $X_1 g \in I$. This implies that $g, l \in L$ and therefore $I = (L^2)$. Since $d = 1$, we have $(L^2) = I \neq (X_1, \ldots, X_n)^2$. By Lemma 2.4.7 we get the assertion.

We finish now the induction on d. Let $d > 1$. Then we have $\mathcal{N} \neq R_1$. Thus there exists an element $x \in R_1 \setminus \mathcal{N}$, $x \neq 0$. We may assume $x = x_1 = X_1 + I$. By Lemma 2.4.5 R/x_1R is u-i-Koszul. We have $\dim(R/x_1R) = \dim(R) - 1 \geq 1$ and thus by induction hypothesis $R/x_1R = K[X_2, \ldots, X_n]/(g^2)$. It follows that $I = (g^2 + X_1 l)$ for some $l \in R_1$. If $I \neq (0)$, we obtain the assertion by Lemma 2.4.8. □

For K-algebras which are defined by monomial ideals we can classify the u-i-Koszul property also for base fields K of characteristic 2.

Proposition 2.4.10. *Let $I \subset S$ be a proper monomial ideal. $R = S/I$ is u-i-Koszul if and only if $I = (X_1, \ldots, X_n)^2$ or I has the form* $\begin{cases} (X_i^2) & \text{if } \operatorname{char}(K) \neq 2 \\ (X_{i_1}^2, \ldots, X_{i_r}^2) & \text{if } \operatorname{char}(K) = 2 \end{cases}$

Proof. In the case that $I = (X_1, \ldots, X_n)^2$ or I is of the form (X_i^2) for some i the algebra R is u-i-Koszul by Lemma 2.4.7. Now let $\operatorname{char}(K) = 2$ and $I = (X_{i_1}^2, \ldots, X_{i_r}^2)$ for some indices $i_1 < \ldots < i_r$. For any transformation $X_i \mapsto \sum_{j=1}^n a_{ji} X_j$ with $i = 1, \ldots, n$ we obtain a new defining ideal $J = (g_1, \ldots, g_r)$ with $g_k = \sum_{j=1}^n a_{ji_k}^2 X_j^2$ for $k = 1, \ldots, r$. Then J has a minimal system of generators which forms a Gröbner basis of J. In the term order of Theorem 2.2.1 $\operatorname{in}(J)$ is of the form $(X_{j_1}^2, \ldots, X_{j_s}^2)$ for some indices $j_1 < \ldots < j_s$. By Theorem 2.2.1 S/J is i-Koszul and thus $R = S/I$ is u-i-Koszul. Conversely, let us assume that R is u-i-Koszul. There are two cases:

If $\operatorname{char}(K) \neq 2$, then by Lemma 2.4.6 and Lemma 2.4.7 we get $I = (X_i^2)$ for some i or $I = (X_1, \ldots, X_n)^2$.

Let $\operatorname{char}(K) = 2$ and let $G(I)$ be the set of the minimal generators of I. We need some facts which follow immediately from Proposition 2.2.3. R is not u-i-Koszul if

(i) $X_i X_j \in G(I)$ with $i < j$ and $X_i X_k \notin G(I)$ for some $k > j$ or if
(ii) $X_i X_j \in G(I)$ with $i < j$ and $X_k^2 \notin G(I)$ for some $k > i$ or if
(iii) $X_i^2, X_i X_{i+1}, X_i X_n \in G(I)$, $X_1^2, \ldots, X_n^2 \in G(I)$ and $X_{i+1} X_{i+2} \notin G(I)$ for some $i < n - 1$ or if
(iv) $X_i^2, X_i X_{i+1}, X_{i+1}^2 \in G(I)$ and $X_{i-1} X_i \notin G(I)$ for some $1 < i < n$.

We have to show the following: If $I \neq (X_1, \ldots, X_n)^2$ and I is not of the form $(X_{i_1}^2, \ldots, X_{i_r}^2)$, then R is not u-i-Koszul. Under this assumption we have $X_i X_j \in G(I)$ for some $i < j$. By Proposition 2.2.3 and (i) we have $X_i^2, \ldots, X_i X_n \in G(I)$. By (ii) we get $X_{i+1}^2, \ldots, X_n^2 \in G(I)$. Then (iv) implies that $X_{i-1} X_i \in G(I)$. By iteration and using (iii) we obtain that $I = (X_1, \ldots, X_n)^2$, which is a contradiction. □

As a direct consequence from Theorem 2.4.4 and Proposition 2.4.10 we observe.

Corollary 2.4.11. *Let $\operatorname{char}(K) \neq 2$ and K be algebraically closed. If $R = S/I$ is u-i-Koszul, then $R' = S/\operatorname{in}(I)$ is also u-i-Koszul.*

The converse of Corollary 2.4.11 is not true. For example, take $n = 3$ and $I = (X_1X_3 - X_2^2)$. Since $\mathrm{in}(I) = (X_2^2)$, the algebra R' is u-i-Koszul by Proposition 2.4.10, but R is not u-i-Koszul by Theorem 2.4.4. We also get from Theorem 2.4.4:

Corollary 2.4.12. *Let K be algebraically closed,* $\mathrm{char}(K) \neq 2$ *and $R = S/I$ an u-i-Koszul domain. Then $I = (0)$.*

The statements in Theorem 2.4.4 and Corollary 2.4.12 are not true for more general base fields. Take, for example,

$$R = \mathbb{Q}[X_1, X_2]/(X_1^2 - \frac{1}{2}X_2^2).$$

Then $X_1^2 - \frac{1}{2}X_2^2$ is not a square in $\mathbb{Q}[X_1, X_2]$ and R is an u-i-Koszul domain. Moreover,

$$R = \mathbb{Z}/2\mathbb{Z}[X_1, \dots, X_4]/(X_1^2 + X_2^2, X_3^2 + X_4^2)$$

is u-i-Koszul. Therefore the hypothesis $\mathrm{char}(K) \neq 2$ of Theorem 2.4.4 cannot be omitted. The concept of universally Koszul algebras in [**17**] (see Examples 2.1.3(a)) has no direct relation to i-Koszulness. Since on the one hand the algebra

$$K[X_1, X_2]/(X_1X_2)$$

is u-Koszul by [**17**, 1.5.], but not i-Koszul by Proposition 2.2.3. On the other hand

$$K[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$$

is i-Koszul due to Proposition 2.2.3, but not u-Koszul if $n > 3$ and $\mathrm{char}(K) \neq 2$ (see [**17**]).

2.5. i-Koszulness of semigroup rings

In this section we want to study i-Koszul semigroup rings. Let Λ be a graded semigroup of $\mathbb{N}^d$ which is minimally generated by the set $G = \{\alpha_1, \dots, \alpha_n\}$. We identify here a monomial X^λ with the corresponding exponent $\lambda \in \mathbb{N}^d$. A semigroup ring $R = K[\Lambda]$ (see section 1.8 for more details) is called *i-Koszul* if R is i-Koszul with respect to the sequence $\alpha_1, \dots, \alpha_n$. R is said to be *u-i-Koszul* if R is i-Koszul with respect to $\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}$ for any permutation π of $\{1, 2, \dots, n\}$. We will see that i-Koszulness implies a certain shellability of the finite intervals in the divisor poset of R.

The set Σ of all monomials in R is partially ordered by divisibility. If there is an injective map $\gamma : G \to \Pi$ where Π is a totally ordered set, then all unrefineable finite divisor chains

$$C : \lambda_0 \overset{\cdot\alpha_{i_1}}{\to} \lambda_1 \to \cdots \overset{\cdot\alpha_{i_r}}{\to} \lambda_r$$

are labeled by $\gamma(C) = (\gamma(\alpha_{i_1}), \dots, \gamma(\alpha_{i_r})) \in \Pi^r$. Let $<$ denote the lexicographic order on Π^r induced by the order on Π.

Definition 2.5.1. (see [**5**],[**11**]) R is called *naturally shellable* if for every monomial $\lambda \in \Sigma$ the order complex $\Gamma([1, \alpha])$ is shellable with order $\gamma(C_1) < \cdots < \gamma(C_r)$ where $\{C_1, \dots, C_r\}$ is the set of all unrefineable chains in the interval $[1, \lambda]$.

Let $R = S/I$ where I is the toric ideal of the semigroup ring R. Natural shellability can be translated into a condition on $\text{in}(I)$ with respect to the reverse lexicographic order induced by $X_1 < \cdots < X_n$.

Proposition 2.5.2. (T. Hibi) *The following statements are equivalent:*

(a) *R is naturally shellable.*
(b) *$\text{in}(I)$ is quasi-poset, i.e. if $i < k < j$ and $X_iX_j \in \text{in}(I)$, then it follows that $X_iX_k \in \text{in}(I)$ or $X_kX_j \in \text{in}(I)$.*

Consequently, by Theorem 2.2.1 the following is evident:

Corollary 2.5.3. *An i-Koszul semigroup ring is naturally shellable.*

By Proposition 1.8.4 the corollary above gives us an alternative proof for the statement that an i-Koszul semigroup ring is Koszul. We have seen in Proposition 2.3.2 that i-Koszulness is preserved under tensor products. This is not true for Segre products of semigroup rings. For example,

(1) $$R = K[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2] \cong K[Z_1, Z_2, Z_3, Z_4]/(Z_1Z_4 - Z_2Z_3)$$

is not i-Koszul with respect to any permutation of the semigroup generators by Theorem 2.2.1. But it can be shown that R is naturally shellable (see [**11**]). Therefore the converse of Corollary 2.5.3 is not true in general.

We now compare i-Koszulness with the strongly Koszul and the sequentially Koszul properties (see Examples 2.1.3). We recall a characterization for strongly Koszul semigroup rings from [**32**].

Proposition 2.5.4. *Let $R = K[\alpha_1, \dots, \alpha_n]$ be a standard graded semigroup ring. The following statements are equivalent:*

(a) *R is strongly Koszul.*
(b) *The ideal $(\alpha_i) \cap (\alpha_j) \subset R$ is generated in degree 2 for all $i \neq j$.*

It is also proved in [**32**] that the divisor posets Γ_λ (see Section 1.8 for the definition) of a strongly Koszul semigroup ring are shellable for all λ and the strongly Koszul property is preserved under Segre products. Thus the ring $R = K[X_1, X_2] * K[Y_1, Y_2]$ in Example (1) is strongly Koszul. Evidently a strongly Koszul algebra is sequentially Koszul. It is obvious from the definition that:

Remark 2.5.5. *Any i-Koszul algebra R is sequentially Koszul.*

The example in (1) shows that the converse is not true in general. Furthermore i-Koszulness does not imply the strongly Koszul property. Take, for example,

$$T = K[X_1^3, X_1^2X_2, X_1X_2^2, X_1X_2X_3, X_2^2X_3, X_2X_3^2].$$

If we order the generators lexicographically decreasing, we get by a computation with MACAULAY2 [**31**] that T is i-Koszul. However, T is not strongly Koszul by Proposition 2.5.4 because

$$(X_1X_2X_3) :_T (X_2^3) = (X_2^3, X_1^3X_2X_3^2).$$

We observe that the only u-i-Koszul semigroup rings are polynomial rings.

Proposition 2.5.6. *Let $R = K[\alpha_1, \dots, \alpha_n] \subset S$ be a graded semigroup ring. If R is u-i-Koszul, then R is a polynomial ring.*

Proof. We may assume that $\alpha_1 >_{\text{lex}} \ldots >_{\text{lex}} \alpha_n$ where $>_{lex}$ is the lexicographic order on $\mathbb{N}^d$. Let $R = S/I$ with $\alpha_i = X_i + I$ for $i = 1, \ldots, n$. By hypothesis R is i-Koszul with respect to this sequence. We argue by contradiction. If $I \neq (0)$, we get by Theorem 2.2.1 that I has a quadratic Gröbner basis with respect to the reverse lexicographic term order induced by $X_1 < \ldots < X_n$. The toric ideal I is minimally generated by binomials of degree 2. By Remark 2.4.9 the ideal I has a quadratic Gröbner basis G which consists of binomials. The chosen order of the semigroup generators implies that every $f \in G$ is of the form $f = X_iX_j - X_kX_l$ with $k < i \leq j < l$ where $\text{in}(f) = X_iX_j$. We choose the smallest index i such that $X_iX_j \in \text{in}(I)$ for some $j \geq i$. Since R is i-Koszul, we have $X_i^2 \in \text{in}(I)$ by Theorem 2.2.1. Thus there exists $f \in G$ such that $f = X_i^2 - X_{i-r}X_{i+s}$ for some $r, s > 0$. Interchanging X_i and X_{i-r} we get a new defining ideal J and an element $g = X_{i-r}^2 - X_iX_{i+s} \in J$. Taking the same term order on S/J we get that $\text{in}(g) = X_iX_{i+s}$. Since S/J is i-Koszul, it follows that $X_i^2 \in \text{in}(J)$ by Theorem 2.2.1. Thus there exists a binomial $h \in J$ such that $h = X_i^2 - X_aX_b$ for some $a, b \in \{1, \ldots, n\}$. But then, there is a relation $u = X_{i-r}^2 - X_cX_d \in G$ for some $c, d \in \{1, \ldots, n\}$ and the order of the semigroup generators implies $c < i - r < d$. Thus we have $\text{in}(u) = X_{i-r}^2 \in \text{in}(I)$, which is a contradiction to the choice of i. $\square$

CHAPTER 3

Sortable semigroup rings

In this chapter we study a class of graded semigroup rings with a minimal system of generators which satisfies a certain sorting condition. A result by Sturmfels implies that the defining ideals of these algebras have a quadratic Gröbner basis (see [**49**, Theorem 14.2 case $s = (1, 1, \dots, 1)$]). Thus sortable semigroup rings are Koszul. The concept of sortability is motivated by the conjecture of N. White that basis monomial rings, which are defined by the bases of a matroid, have quadratic relations (see [**52**]). Since sortability is compatible with several matroid operations, we can determine some classes of matroids for which the basis monomial ring is even Koszul. Moreover, we classify the matroids which define strongly Koszul semigroup rings.

3.1. Sortability

We start by introducing some notation.

Notation 3.1.1. Let $\mathcal{V} = \{V_1, \dots, V_n\}$ be a collection of subsets of $[d]$ such that $|V_i| = r$ for all $i = 1, \dots, n$. Then $R_\mathcal{V} \subset K[T_1, \dots, T_d]$ denotes the semigroup ring generated by the n square-free monomials

$$\alpha_i = \prod_{j \in V_i} T_j$$

in $K[T_1, \dots, T_d]$ of degree r for $i = 1, \dots, n$. In other words, it is $R_\mathcal{V} = K[\alpha_1, \dots, \alpha_n]$ where $\operatorname{supp}(\alpha_i) = V_i$ for $i = 1, \dots, n$.

Let $S_\mathcal{V} = K[X_{V_1}, \dots, X_{V_n}]$ be the polynomial ring in the variables indexed by the sets in $\mathcal{V}$. Then $R_\mathcal{V}$ has a presentation $\varphi : S_\mathcal{V} \to R_\mathcal{V}$ with $X_{V_i} \mapsto \alpha_i$ for $i = 1, \dots, n$. We write $I_\mathcal{V} = \ker(\varphi)$ for the toric ideal of $R_\mathcal{V}$. Since all sets in $\mathcal{V}$ have cardinality r, the K-algebra $R_\mathcal{V}$ is standard graded by Proposition 1.8.2.

To simplify notation we identify a set $V_i = \{i_1, \dots, i_r\} \subset [d]$, $i_1 < i_2 \dots < i_r$, with the ordered tuple $V_i = (i_1, \dots, i_r)$ in $\mathbb{N}^r$. We may always assume that $V_1 >_{\text{lex}} V_2 >_{\text{lex}} \cdots >_{\text{lex}} V_n$ where $>_{\text{lex}}$ is the lexicographic order on $\mathbb{N}^r$.

Let Σ be the set of monomials in $S_\mathcal{V}$. For every $u \in \Sigma$, $\deg(u) = k$ and $u = X_{V_{i_1}} \cdots X_{V_{i_k}}$, we order the indeterminates such that $i_1 \le i_2 \le \cdots \le i_k$ is satisfied. To u we associate a $k \times r$ matrix $A(u) = \left[V_{i_1}, V_{i_2}, \cdots, V_{i_k}\right]^t$, whose rows are the corresponding vectors $V_{i_1}, \dots, V_{i_k}$ (cf. [**23**]). Since $A(u)$ is unique, we may identify Σ with the corresponding set of matrices $\{A(u) \colon u \in \Sigma\}$. Moreover, we define $\operatorname{supp}(A)$ to be the multiset consisting of all entries of $A(u)$. For all binomial generators $u - u' \in I_\mathcal{V}$ in the toric ideal we have $\operatorname{supp}(A(u)) = \operatorname{supp}(A(u'))$ (see Section 1.8).

Definition 3.1.2. Let $A = (a_{ij})$ be a $k \times r$ matrix with entries in $[d]$ and $<_\omega$ a linear order on $[d]$. Then A is said to be *sorted* with respect to $<_\omega$ if

$$a_{11} \leq_\omega a_{21} \leq_\omega \cdots \leq_\omega a_{k1} \leq_\omega a_{12} \leq_\omega a_{22} \leq_\omega \cdots \leq_\omega a_{2r} \leq_\omega \cdots \leq_\omega a_{kr}.$$

For an arbitrary matrix A, we define $\operatorname{sort}_\omega(A)$ to be the unique sorted matrix of the same size such that $\operatorname{supp}(\operatorname{sort}_\omega(A)) = \operatorname{supp}(A)$.

In case that $<_\omega$ is the usual order $1 < 2 < \ldots < d$ we simply use $<$ to denote the order and write $\operatorname{sort}(A)$ instead of $\operatorname{sort}_\omega(A)$. To illustrate the sorting operator we give an example. Let $1 < 2 < \cdots < 7 < 8$ be the usual linear order and

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

Then we have $\operatorname{supp}(A) = \{1, 2, 3, 4, 6, 7, 8, 8\}$ and

$$\operatorname{sort}(A) = \begin{bmatrix} 1 & 3 & 6 & 8 \\ 2 & 4 & 7 & 8 \end{bmatrix}.$$

Definition 3.1.3. Let $\mathcal{V} = \{V_1, \ldots, V_n\}$ be a system of r-element subsets of the ground set $[d]$. A linear order $<_\omega$ on $[d]$ is called a *sorting order* for the pair $(\mathcal{V}, <_\omega)$ if the following condition is satisfied:

($\mathcal{S}$) Let $V_i, V_j \in \mathcal{V}$ with $i \leq j$ and let $A = \left[V_i, V_j\right]^t$ denote the corresponding matrix. Then the row vectors of the matrix $\operatorname{sort}_\omega(A)$ are also elements of the system $\mathcal{V}$.

$\mathcal{V}$ is called a *sortable* system if there exists a linear order $<_\omega$ on $[d]$ such that $<_\omega$ is a sorting order for $(\mathcal{V}, <_\omega)$.

Note that sortability can be defined more generally for systems of multisets. Since we mainly study rings arising from matroids, we only consider sortable sets. The following definition first appeared in [**51**].

Definition 3.1.4. Let M be a matroid on $[d]$ of rank r and $\mathcal{B}(M)$ its bases. Then the semigroup ring $R_{\mathcal{B}(M)}$ is called the *basis monomial ring* of M. We denote the toric ideal of $R_{\mathcal{B}(M)}$ with $I_{\mathcal{B}(M)}$.

In other words, if $\mathcal{B}(M) = \{B_1, \ldots, B_n\}$, then the $R_{\mathcal{B}(M)} = K[\alpha_1, \ldots, \alpha_n]$ is generated as an algebra by the monomials $\alpha_i = \prod_{j \in B_i} T_j$ for $i = 1, \ldots, n$.

Example 3.1.5. Let $U_{r,d}$ be the uniform matroid of rank r on $[d]$. Then $R_{\mathcal{B}(U_{r,d})}$ is the semigroup ring which is generated by all square-free monomials of degree r in $K[T_1, \ldots, T_d]$. In particular, $R_{\mathcal{B}(U_{r,d})}$ is of Veronese type (see Theorem 1.8.6).

N. White has studied some algebraic properties of basis monomial rings in [**51**]. We summarize his results:

Theorem 3.1.6. *Let M be a matroid on $[d]$ and let $\varkappa(M)$ denote the number of connected components of M. Then the Krull dimension of the basis monomial ring $R_{\mathcal{B}(M)}$ equals $d - \varkappa(M) + 1$ and the algebra $R_{\mathcal{B}(M)}$ is normal. In particular, $R_{\mathcal{B}(M)}$ is a Cohen-Macaulay ring.*

In [**52**, Conjecture 12] N. White formulated a conjecture in terms of exchange properties for matroids which encourages us to consider the Koszul property of basis monomial rings. We rephrase it.

Conjecture 3.1.7. (N. White) *Let M be a matroid. Then the toric ideal $I_{\mathcal{B}(M)}$ of the basis monomial ring of M is generated by quadrics.*

The study of basis monomial rings motivates to the extension of matroid operations to the systems $\mathcal{V}$. Note that the induced operations on the semigroup rings $R_{\mathcal{V}}$ generalize the concept of combinatorial pure subrings which has been introduced by Herzog, Hibi and Ohsugi in [**42**].

We always assume that $\mathcal{V} = \{V_1, \dots, V_n\}$ is a system of r-element subsets on $[d]$. In case that $\mathcal{V}$ is the collection of bases of a matroid M on $[d]$, the following operations coincide with those defined for matroids (see Section 1.3).

Let $i \in [d]$. We define *the deletion of* i, denoted with $\mathcal{V} \setminus i$, to be the following collection of subsets in $\{1, \dots, i-1, i+1, \dots, d\}$

$$\mathcal{V} \setminus i = \begin{cases} \{V_{j_1}, \dots, V_{j_k}\} & \text{if } i \notin V_j \text{ for some } j \in [n], \\ \{V_1 - i, \dots, V_n - i\} & \text{if } i \in V_j \text{ for all } j = 1, \dots, n \end{cases}$$

where in the first case $\{V_{j_1}, \dots, V_{j_k}\}$ is the collection of subsets in $\mathcal{V}$ such that $V_{j_l} \subset [d] - i$. For an arbitrary set $A \subset [d]$ with $|A| > 1$ we define inductively $\mathcal{V} \setminus A = (\mathcal{V} \setminus (A - a)) \setminus a$ for some element $a \in A$. In the case that there is a $V_j \in \mathcal{V}$ such that $V_j \subset [d] - A$, the semigroup ring $R_{\mathcal{V} \setminus A}$ is a combinatorial pure subring of $R_{\mathcal{V}}$, as defined in [**42**]. We call $\mathcal{V}^* = \{[d] - V_1, \dots, [d] - V_n\}$ *the dual* of $\mathcal{V}$. If $R_{\mathcal{V}} = K[\alpha_1, \dots, \alpha_n]$ is the associated homogeneous semigroup ring, then $R_{\mathcal{V}^*}$ is generated by the n square-free monomials α_i^* of degree $d-r$ in $K[T_1, \dots, T_d]$ such that $\operatorname{supp}(\alpha_i^*) = [d] - V_i$, i.e

$$\alpha_i^* = \frac{T_1 T_2 \cdots T_d}{\alpha_i}$$

for $i = 1, \dots, n$. Since all monomials have the same degree $d-r$, $R_{\mathcal{V}^*}$ is also standard graded.

Remark 3.1.8. The semigroup rings $R_{\mathcal{V}}$ and $R_{\mathcal{V}^*}$ are isomorphic as K-algebras.

Proof. Let $R_{\mathcal{V}} = K[\alpha_1, \dots, \alpha_n]$. Then the map $\psi : R_{\mathcal{V}} \to R_{\mathcal{V}^*}$ with $\psi(\alpha_i) = \alpha_i^*$ extends to an isomorphism of K-algebras. □

Let $A \subset [d]$. We define $\mathcal{V}/A = (\mathcal{V}^* \setminus A)^*$ to be the *contraction of* $\mathcal{V}$ *at* A. We call a system $\mathcal{V}'$ of r'-element subsets of $[d']$ a *minor* of the system $\mathcal{V}$ if $\mathcal{V}'$ can be obtained from $\mathcal{V}$ by a finite sequence of contractions and deletions. Let $i \in [d]$ such that $i \in V_j$ for some $j \in \{1, \dots, n\}$. We define

$$\mathcal{V} +_i (d+1) = \{V_1, \dots, V_n\} \,\dot{\cup}\, \{V_j - i \cup (d+1) \colon i \in V_j\}$$

to be a *parallel extension* of $\mathcal{V}$ at i. Moreover, a system of the form $(\mathcal{V}^* +_i (d+1))^*$ is called a *series extension* of $\mathcal{V}$ at i. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two systems of r_1-subsets of $[d_1]$ and r_2-subsets of $[d_2]$ respectively. We identify the subsets of $[d_2]$ with the subsets of set $\{d_1+1, d_1+2, \dots, d_1+d_2\}$. Then the *direct sum*

$$\mathcal{V}_1 \oplus \mathcal{V}_2 = \{V_1 \,\dot{\cup}\, V_2 \colon V_1 \in \mathcal{V}_1, V_2 \in \mathcal{V}_2\}$$

is a system of subsets of $[d_1+d_2]$. We note that the associated semigroup ring $R_{\mathcal{V}_1\oplus\mathcal{V}_2}$ is the Segre product $R_{\mathcal{V}_1} * R_{\mathcal{V}_2}$ of the rings $R_{\mathcal{V}_1}$ and $R_{\mathcal{V}_2}$.

3.2. The class of sortable systems and their semigroup rings

In this section we study the class $\mathcal{S}$ of sortable systems $\mathcal{V}$. Considering this class form the algebraic point of view we first observe that the associated semigroup ring $R_\mathcal{V}$ is Koszul. Then we show that $\mathcal{S}$ is closed under the generalized matroid operations which have been defined in the preceding section.

Proposition 3.2.1. *If $\mathcal{V}$ is a sortable system, then the toric ideal $I_\mathcal{V}$ of the semigroup ring $R_\mathcal{V}$ has a square-free quadratic Gröbner basis. In particular, $R_\mathcal{V}$ is Koszul.*

For the proof of Proposition 3.2.1 we use a result from [**49**] (see also [**23**]). As we have observed in Example 3.1.5, the algebra $R_{\mathcal{B}(U_{r,d})}$ coincides with the r^{th} square-free Veronese subring of $K[T_1,\dots,T_d]$. We reformulate [**49**, Theorem 14.2 case $s=(1,1,\dots,1)$].

Theorem 3.2.2. (Sturmfels) *Let $S_{\mathcal{B}(U_{r,d})}$ be the polynomial ring with indeterminates indexed by the bases of $U_{r,d}$ and Σ be the set of monomials in $S_{\mathcal{B}(U_{r,d})}$. Then there exists a term order $<_\tau$ on $S_{\mathcal{B}(U_{r,d})}$ such that $I_{\mathcal{B}(U_{r,d})}$ has a square-free quadratic Gröbner basis*

$$G=\{u-u'\colon u,u'\in\Sigma, A(u')=\operatorname{sort}(A(u))\}.$$

Moreover, the standard monomials with respect to $<_\tau$ correspond to the sorted matrices of Σ.

Proof of Proposition 3.2.1. Let $\mathcal{V}=\{V_1,\dots,V_n\}$ be a sortable system of subsets of $[d]$ with $|V_i|=r$ for all $i=1,\dots,n$. We consider the semigroup ring $R_\mathcal{V}$ as a K-subalgebra of $R_{\mathcal{B}(U_{r,d})}$. Renumbering the set $[d]$ we may assume that $1<2<\dots<d$ is a sorting order for $\mathcal{V}$. By Theorem 3.2.2 the ideal $I_{\mathcal{B}(U_{r,d})}\subset S_{\mathcal{B}(U_{r,d})}$ has a square-free quadratic Gröbner basis G with respect to a term order $<_\tau$ such that $\operatorname{sort}(\Sigma)$ forms the set of standard monomials. If $<$ is a sorting order for $\mathcal{V}$, then Lemma 1.4.17 implies that $G\cap S_\mathcal{V}$ is a Gröbner basis for the toric ideal $I_\mathcal{V}$. □

As a direct consequence of the results above we observe:

Remark 3.2.3. Let $\mathcal{V}$ be a sortable system. Then $R_\mathcal{V}$ is a consistent subalgebra (see Definition 1.4.18) of $R_{U_{r,d}}$. Moreover, the sorting property extends to arbitrary matrices $A\in\Sigma$, i.e. every row of $\operatorname{sort}_\omega(A)\in\Sigma$ is the support of a monomial generator of $R_\mathcal{V}$.

We study now the class $\mathcal{S}$ of sortable systems with respect to the generalized matroid operations as defined in Section 3.1.

Proposition 3.2.4. *The class $\mathcal{S}$ of sortable systems is closed under the following operations:*

(a) *duality,*
(b) *contraction and deletion,*
(c) *parallel and series extension,*
(d) *direct sums.*

Proof. Let $\mathcal{V} = \{V_1, \dots, V_n\}$ be a sortable system of subsets of $[d]$. By renumbering we may assume that $1 < 2 < \cdots < d$ is a sorting order for $\mathcal{V}$.

(a): Let $V_i^* = [d] - V_i$ for $i = 1, \dots, n$. We show that $1 < 2 < \ldots < d$ is a sorting order for the dual $\mathcal{V}^* = \{V_1^*, \dots, V_n^*\}$ verifying condition $\mathcal{S}$ of Definition 3.1.3. Let $A = \left[V_i, V_j\right]^t$ where $V_i, V_j \in \mathcal{V}$ and $i \leq j$. Since $\mathcal{V}$ has sorting order $<$, the row vectors c_1 and c_2 of the matrix $C = \operatorname{sort}(A)$ are again elements of $\mathcal{V}$. Thus $[d] - c_1$ and $[d] - c_2$ belong to $\mathcal{V}^*$. We set $\operatorname{sort}(A)^* = \left[[d] - c_2, [d] - c_1\right]^t$ and claim that $\operatorname{sort}(A)^* = \operatorname{sort}(\left[V_j^*, V_i^*\right]^t)$ which yields condition $(\mathcal{S})$ for $\mathcal{V}^*$.

By definition we have $\operatorname{supp}(\operatorname{sort}(A)) = V_i \cup V_j$ and therefore $\operatorname{supp}(\operatorname{sort}(A)^*) = V_i^* \cup V_j^*$. Since $\operatorname{sort}(A)$ is sorted, we observe that $\operatorname{sort}(A)^*$ is also sorted which concludes the proof.

(b): Sortability is trivially preserved under deletion. By (a) the same holds for the contraction.

(c): Let $i \in [d]$ be an element such that $i \in V_j$ for some $j \in \{1, \dots, n\}$. Let $\mathcal{V}' = \mathcal{V} +_i (d+1)$ denote a parallel extension at i. We extend the old ordering of the ground set $[d]$ to an ordering of the ground set $[d+1]$ in such a way that $d+1$ occurs between the elements previously labeled i and $i+1$ and show that $1 \prec 2 \prec \ldots \prec i \prec d+1 \prec i+1 \prec \ldots \prec d$ is a sorting order for $\mathcal{V}'$.

Let $V_i', V_j' \in \mathcal{V}'$ with $i \leq j$ be two vectors and $A' = \left[V_i', V_j'\right]^t$ the corresponding matrix. We replace each entry $d+1$ by i. By definition of V' we obtain a matrix A whose row vectors belong to $\mathcal{V}$. Since $<$ is a sorting order for $\mathcal{V}$, the rows of $\operatorname{sort}(A)$ are again elements of $\mathcal{V}$. Without losing the sorting property we can replace the corresponding number of i-entries by $d+1$, this gives us $\operatorname{sort}_\prec(A')$. The row vectors belong to $\mathcal{V}'$ which yields condition $(\mathcal{S})$ for $\mathcal{V}'$. Since parallel and series extension are dual operations, (a) implies the assertion for series extensions.

(d): Let $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are sortable systems on $[d_1]$ and $[d_2]$ respectively. We may assume that $1 < 2 < \ldots < d_i$ a sorting order for $\mathcal{V}_i$. We extend the given linear orders to a linear order $<_\omega$ on $[d_1 + d_2]$ in a way that $i <_\omega j$ for all i, j where i is an element of the ground set of $[d_1]$ and j an element of the ground set of $[d_2]$. By the chosen order we obtain immediately that $(\mathcal{S})$ is satisfied for $\mathcal{V}$. □

Example 3.2.5. In [**23**] the following semigroup rings are considered. Let Σ_r be the set of monomials of degree r in $K[T_1, \dots, T_d]$. For a monomial $v \in \Sigma_r$ let $B(v)$ denote the smallest strongly stable (see Definition 1.4.15) subset of Σ which contains v. We write $B_{sf}(v) = \{\alpha_1, \dots, \alpha_n\}$ for the set of square-free monomials in $B(v)$. Suppose that $\mathcal{V} = \{V_1, \dots, V_n\}$ where $V_i = \operatorname{supp}(\alpha_i)$ for $i = 1, \dots, n$. De Negri has shown in [**23**] that $\mathcal{V}$ belongs to the class $\mathcal{S}$. All semigroup rings which can be obtained by the operations in Proposition 3.2.4 have sortable generators and thus a quadratic Gröbner basis.

3.3. Base-sortable matroids of rank less or equal to 3

In this section we study the class of matroids M for which the collection of bases is a sortable system. We give necessary and sufficient conditions for a matroid to belong to this class and classify all base-sortable rank 3 matroids by excluded minors.

An *ordered matroid* (M, ω) consists of a matroid M and a linear order $<_\omega$ on the ground set $[d]$. As in Section 3.1 we denote the usual linear order $1 < 2 < \cdots < d$ with $<$. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be the collection of bases of a matroid M. In the following we always identify a basis $B_i = \{b_1, \ldots, b_r\}$ with the ordered tuple $B_i = (b_1, \ldots, b_r)$ where $b_1 < b_2 < \ldots < b_r$. Moreover, we may always assume that $B_1 >_{\text{lex}} B_2 >_{\text{lex}} \cdots >_{\text{lex}} B_n$ where $>_{\text{lex}}$ denotes the lexicographic order in $\mathbb{N}^r$.

Definition and Notation 3.3.1. Let (M, ω) be an ordered matroid on $[d]$ with bases $\mathcal{B}$. The linear order $<_\omega$ is called a *base-sorting order* for M if $<_\omega$ is a sorting order for $\mathcal{B}$. We call an arbitrary matroid M *base-sortable* if there exists a linear order $<_\omega$ of the ground set $[d]$ of M which is a sorting order for $\mathcal{B}$ and denote the class of all base-sortable matroids with $\mathcal{BS}$.

Proposition 3.2.1 implies that, for a base-sortable matroid M, the basis monomial ring $R_{\mathcal{B}(M)}$ is Koszul. By Proposition 3.2.4 we have:

Corollary 3.3.2. *The class $\mathcal{BS}$ is closed under taking minors, parallel and series extensions and direct sums of matroids. In particular, $\mathcal{BS}$ forms a hereditary class.*

The following corollary is a direct consequence of Corollary 3.3.2.

Corollary 3.3.3. *A matroid M is base-sortable if and only if the underlying simple matroid $\overline{M}$ has this property. In particular, every matroid of rank 2 is base-sortable.*

Proof. The first assertion is clear since loops do not effect base-sortability, and parallel elements can be created and deleted by Proposition 3.3.2. Every matroid of rank 2 is a parallel extension of a uniform matroid which is base-sortable by Theorem 3.2.2. □

The class of rank-2 matroids coincides with the class of complete multipartite graphs. Therefore Corollary 3.3.3 was first observed by Hibi and Ohsugi in [**43**, Theorem 1.1 and Corollary 1.3].

For the next statement we need some notation. Let C_d be the regular d-gon in the plane whose vertices are labeled clockwise from 1 to d. A set F is said to be *consecutive modulo d* if the elements of F label a consecutive set of vertices of C_d. The class $\mathcal{BS}$ is closed under certain principal extensions along flats of rank 2.

Proposition 3.3.4. Let $(M, <)$ be a base-sortable matroid and F be a proper independent flat of rank 2 which is consecutive modulo d. Then the principal extension $N = M +_F (d+1)$ is base-sortable.

Proof. Using Lemma 3.3.6 we may assume that $F = \{1, 2\}$. Then a similar argument as in the proof of Proposition 3.2.4(c) shows that $1 \prec d+1 \prec 2 \prec 3 \prec \cdots \prec d$ is a sorting order for N. □

In the following we study necessary and sufficient conditions for base-sortability. We note that loops and isthmes do not effect base-sortability.

Proposition 3.3.5. *Let $(M, <)$ be an ordered matroid of rank r without loops and isthmes. If all proper dependent flats of M are consecutive modulo d, then $<$ is a base-sorting order for M.*

For the proof we use the following lemma.

Lemma 3.3.6. *Let* $(M, <)$ *be an ordered matroid and* $\sigma = (1 \dots d)$ *denote the cyclic permutation of size* d. *For a vector* $V = (i_1, i_2, \dots, i_r)$ *with* $i_1 < i_2 < \dots < i_r$ *we set* $\sigma V = (\sigma(i_{j_1}), \dots, \sigma(i_{j_r}))$ *where* $\sigma(i_{j_1}) < \dots < \sigma(i_{j_r})$.

(a) *Suppose that* B_i, B_j *with* $i \le j$ *are two bases of* M *and* $C = \mathrm{sort}([B_i, B_j]^t)$ *is the corresponding sorted matrix with rows* c_1 *and* c_2. *Then the matrix* $[\sigma c_1, \sigma c_2]$ *has the same set of rows as the matrix* $\mathrm{sort}([\sigma B_i, \sigma B_j]^t)$. *In particular, if* $(M, <)$ *is base-sortable, then the same is true for* $(\sigma M, <)$.

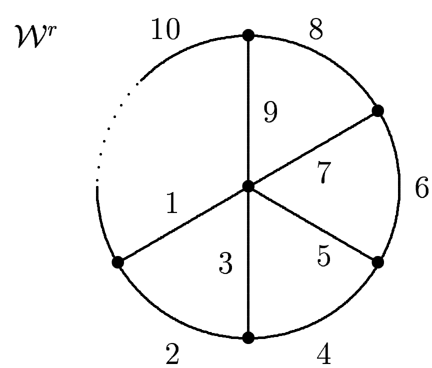
(b) *The reversed order* $1 > 2 > \dots > d$ *is also a base-sorting order for* M.

Proof. (a): Let M be of rank r. Since C is sorted, we have $c_{1r} \le c_{2r}$. If both entries are either strictly less than d or both are equal to d, then $\left[\sigma c_1, \sigma c_2\right]^t$ is sorted, otherwise $\left[\sigma c_2, \sigma c_1\right]^t$ is sorted. Both matrices have support $\sigma B_i \cup \sigma B_j$.

(b): We get the sorted matrices of A with respect to $1 > 2 > \dots > d$ by reversing the order of the entries in $\mathrm{sort}(A)$. □

Proof of Proposition 3.3.5. Suppose that all proper dependent flats are consecutive modulo d. We argue by contradiction and assume that M is not base-sortable with respect to $<$. Then there are bases $B_i, B_j \in \mathcal{B}$ with $i < j$ such that at least one of the rows of $C = \mathrm{sort}\left[B_i, B_j\right]^t$ is not a basis of M. This row is contained in some proper dependent flat F with $\mathrm{rk}(F) \le r - 1$. F is consecutive modulo d and, by Lemma 3.3.6, we may assume the following situation: $1 \in F$, $d \notin F$ and $c_1 \subset F$. We have $c_{11} \in B_i$ because $B_i >_{\mathrm{lex}} B_j$ by our general assumption. Since F is consecutive modulo d, we get $B_j \subset F$ if $c_{2r} \in B_i$, and $B_i \subset F$ otherwise. This is a contradiction because both B_i and B_j are bases of M. □

Example 3.3.7. We consider the rank r whirl $\mathcal{W}^r$.



This matroid is not graphic, but has the edges of the r-spoked wheel $\mathcal{W}_r$ as ground set (see the picture above). The collection of bases of $\mathcal{W}^r$ consist of the rim and all edge sets which form spanning trees in $\mathcal{W}_r$. To be precise, $\mathcal{W}^r$ is the unique relaxation of the graphic matroid $M(\mathcal{W}_r)$, i.e. we obtain $\mathcal{W}^r$ from $M(\mathcal{W}_r)$ by removing the circuit-hyperplane which consists of all edges of the rim (see [**41**, p. 293] for more details). We label the edges of $\mathcal{W}_r$ as shown in the figure. Then all proper dependent flats of $\mathcal{W}^r$ are consecutive modulo d. Thus by Proposition 3.3.5 the matroid $\mathcal{W}^r$ is base-sortable.

We give a necessary condition for the property of base-sortability.

Proposition 3.3.8. *Let M be a matroid without loops and isthmes. If $<$ is a base-sorting order for M, then all circuit-hyperplanes of M are consecutive modulo d and the same holds for all minors of M which have no loops and isthmes.*

Proof. We argue by contradiction. Since $\mathcal{BS}$ is closed under taking minors, we may assume that M has a circuit-hyperplane C which is not consecutive modulo d. Let $\operatorname{rank}(M) = r$. Applying Lemma 3.3.6(a) we may assume that $1 \in C$ and $d \notin C$. Since C is not consecutive modulo d, there exists a $j < d$ such that $\{1, \dots, j-1\} \subset C$ and $j \notin C$. Moreover, we have that $|C| \geq r$ because C is dependent. We define the matrix

$$A = \begin{bmatrix} 1 & 2 & \dots & j-2 & j-1 & j & i_1 & \dots & i_{r-j} \\ 1 & 2 & \dots & j-2 & i_1 & i_2 & \dots & i_{r-j-1} & d \end{bmatrix}$$

where $j < i_1, \dots, i_{r-j} < d$ and $\{i_1, i_2, \dots, i_{r-j}\} \subset F$. Since C is a circuit-hyperplane, the rows of A are bases of M. We get

$$\operatorname{sort}(A) = \begin{bmatrix} 1 & 2 & \dots & j-2 & j-1 & i_1 & \dots & i_{r-j-1} & i_{r-j} \\ 1 & 2 & \dots & j-2 & j & i_1 & \dots & i_{r-j-1} & d \end{bmatrix}.$$

But the first row of $\operatorname{sort}(A)$ is contained in F, thus it is not a basis of M which is a contradiction. □

We believe that Proposition 3.3.8 gives also a sufficient condition for base-sortability:

Conjecture 3.3.9. *Let $(M, <)$ be an ordered matroid without loops and isthmes. Then $<$ is a sorting order for M if and only if all circuit-hyperplanes of M are consecutive modulo d and the same holds for all minors of M, which have no loops and isthmes, with respect to the restricted order.*

We know that Conjecture 3.3.9 is true for matroids of rank 2 or 3 by the following Proposition 3.3.10, and can prove it for rank 4 matroids in a brutal case by case computation.

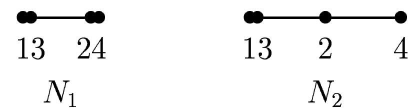
In the specific case of rank 3 matroids, the sufficient condition in Proposition 3.3.5 is also necessary.

Proposition 3.3.10. *Let $<$ be the usual linear order on $[d]$ and $(M, <)$ an ordered matroid of rank 2 or 3. Suppose that M has no loops and isthmes. Then the following statements are equivalent:*

(a) *$<$ is a base-sorting order for M.*
(b) *All proper dependent flats are consecutive modulo d.*
(c) *The circuit-hyperplanes of M are consecutive modulo d and the same hold for all minors of M in the restricted order.*

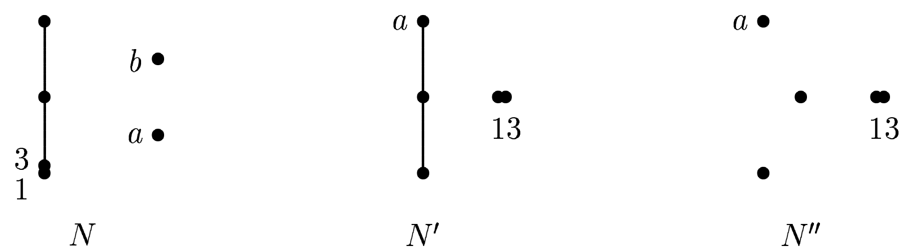
Proof of Proposition 3.3.10. By Proposition 3.3.5 we already know that (b) implies (a) and by Proposition 3.3.8 that (a) implies (c). It remains to show the following: If M has a proper dependent flat F which is not consecutive modulo d, then M or a minor of M has a non-consecutive circuit-hyperplane. Applying σ as in Lemma 3.3.6, we may assume that $\{1, 3\} \subset F$ and $2 \notin F$.

We first consider the case that $\mathrm{rank}(M) = 2$. Then $\mathrm{rk}(F) = 1$. By a suitable deletion one has one of the following minors on $\{1, 2, 3, 4\}$ which both have a non-consecutive circuit-hyperplane:



This is a contradiction.

Let now $\mathrm{rank}(M) = 3$. We have two cases: $\mathrm{rk}(F) = 1$ and $\mathrm{rk}(F) = 2$. We consider the case $\mathrm{rk}(F) = 1$ first. By a suitable deletion we get one of the following minors with vertex set $\{1, 2, 3, 4, 5\}$:



Deleting vertices a and b of N we get N_2. If we contract the matroids N' and N'' at a, then we obtain N_1 and N_2 respectively.

Now let $\mathrm{rk}(F) = 2$. We may assume that all dependent 1-flats of M are consecutive modulo d in the restricted order. Then by a suitable deletion we obtain one of the following matroids on $\{1, 2, 3, 4, 5\}$:

In both cases there is a non-consecutive circuit-hyperplane, a contradiction. □

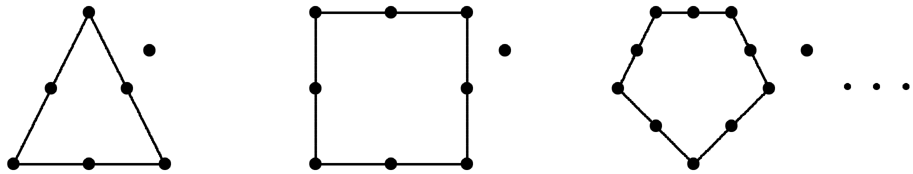
We can now use this result to characterize all base-sortable matroids of rank 3.

Theorem 3.3.11. *Let M be a matroid on $[d]$ of rank 3. Then M is base-sortable if and only if the underlying simple matroid $\overline{M}$ has a geometric realization which either consists of a k-gon, $k \geq 3$, whose lines are formed by the dependent 2-flats of $\overline{M}$, or of a collection of paths, which consist of the dependent 2-flats of $\overline{M}$, and generic points.*

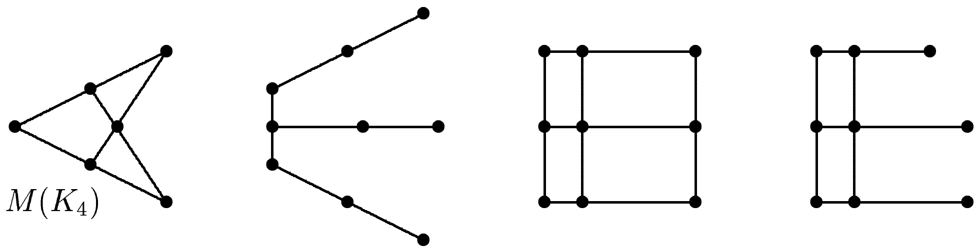
We reformulate Theorem 3.3.11 in terms of excluded deletions.

Corollary 3.3.12. *Let M be a matroid on $[d]$ of rank 3. Then M is base-sortable if and only if M has not a deletion N with geometric representation:*

(a) *A k-gon, $k \geq 3$, whose edges are rank 2 circuits, and an additional generic point, i.e.*

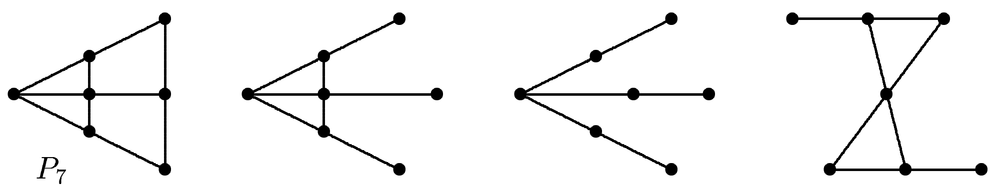


(b) *One of the following:*



$M(K_4)$

where $M(K_4)$ is the graphic matroid defined by K_4, the complete graph on 4 vertices.



P_7

Proof of Theorem 3.3.11 *and Corollary* 3.3.12. If M has an isthmus, then M is isomorphic to a matroid of lower rank. Thus by Corollary 3.3.3 we may assume that M has no isthmes and, additionally, that M is a simple matroid. According to Proposition 3.3.10, $1 < 2 < \cdots < d$ is a base-sorting order for $(M, <)$ if and only if all proper dependent 2-flats are consecutive modulo d.

It is tedious, but straightforward to check that, for any labeling of the ground set $[d]$, the matroids in (a) and (b) do not satisfy this condition. Thus they are excluded deletions for base-sortability.

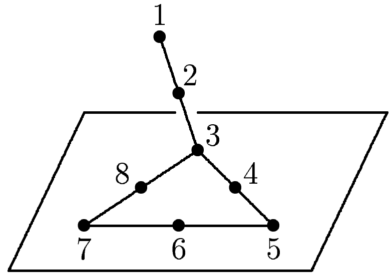
Conversely, if M has no deletion listed in (a) and (b), then one can check out that M has a geometric representation which consists either of a k-gon, $k \geq 3$, whose lines are formed by proper dependent 2-flats, or a collection of paths consisting of proper dependent 2-flats and generic points. In both cases, we can label the vertex set such that all proper dependent 2-flats are consecutive modulo $[d]$. □

The preceding result implies:

Remark 3.3.13. *The hereditary class $\mathcal{BS}$ has infinitely many excluded minors.*

Proposition 3.3.10 is not true for matroids of higher rank. Take, for example, the matroid, $M = (\mathcal{W}^3 \oplus \{1\}) +_{\{1,3\}} 2$ which has base-sorting order $1 < 2 < \cdots <$

$7 < 8$ by Example 3.3.7, Proposition 3.3.2 and Proposition 3.3.4. However, the flat $\{1, 5, 6, 7\}$ is not consecutive modulo d.



M

3.4. Classes of base-sortable matroids

In this section we characterize the class of graphic base-sortable matroids and show that transversal matroids with certain presentations are base-sortable. Compared to the general case, it turns out that for graphic matroids base-sortability is much easier to characterize.

We recall some notation from graph theory. Let G be a graph. A subgraph H of G said to be *homeomorphic from* G if H can be obtained from G by removing vertices of degree 2. We have the following classification:

Theorem 3.4.1. *Let M be a matroid on $[d]$ of rank r and K_4 the complete graph on four vertices. Then the following conditions are equivalent:*

(a) *M is base-sortable and $M = M(G)$ is graphic for some graph G.*
(b) *M has no minor isomorphic to $M(K_4)$ or to $U_{2,4}$.*
(c) *M is a direct sum of series-parallel networks.*
(d) *$M = M(G)$ is graphic and G has no subgraph homeomorphic from K_4.*

Proof. The equivalence of (b),(c) and (d) is stated in [**41**, Theorem 13.4.9]. By Corollary 3.3.2 we get that any direct sum of series-parallel networks is base-sortable, which shows that (c) implies (a). Let us assume (a). Since M is graphic, M has no minor isomorphic to $U_{2,4}$. Moreover, by Theorem 3.3.12 $M(K_4)$ is one of the excluded minors for $\mathcal{BS}$. This yields (b). □

Now we study the class of transversal matroids (see Section 1.6). Let C_d be the regular d-gon in the plane whose vertices are labeled clockwise from 1 to d. We recall that a set $A \subset [d]$ is said to be *consecutive modulo* d if the elements of A label a consecutive set of vertices of C_d.

Theorem 3.4.2. *Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a family of subsets of $[d]$. If all sets A_i are consecutive modulo d, then the matroid $M[\mathcal{A}]$ is base-sortable.*

To prove Theorem 3.4.2 we show that $1 < 2 < \ldots < d$ is a sorting order for $M[\mathcal{A}]$. For this, we introduce some notation.

Let A be a non-empty subset of $[d]$ which is consecutive modulo d and $a, a' \in A$. If $A \neq [d]$, then we write $(a, a')_A$ for the subset of A whose elements label the consecutive vertices of C_d which lie strictly between a and a'. If $A = [d]$, then there are two consecutive sets of vertices of C_d which lie strictly between a and a'. We use the convention that $(a, a')_A$ denotes the set of minimal cardinality or, if both sets have the same cardinality, we choose the set containing the position $\min\{a, a'\} + 1$. Let $[a, a']_A = (a, a')_A \cup \{a, a'\}$. We define the *distance* between $a, a' \in A$ with respect to A by $\operatorname{dist}_A(a, a') = |[a, a']_A| - 1$.

A *marking* is an element of the set $\mathcal{M} = \{\circ, \times\} \times \{A_1, A_2, \dots, A_r\}$ where $\circ$ and $\times$ are considered as different symbols. We call a subset $C \subset \mathcal{M} \times [d]$ a *configuration* of markings which are placed at the vertices of C_d. If $(m, p) \in C$, we say that the marking m is placed at the position p of C_d. Let $[B, B']^t$ with $B \geq_{\text{lex}} B'$ be a matrix whose row vectors B, B' are bases of $M[\mathcal{A}]$. We write the two bases $B = \{b_1, \dots, b_r\}$ and $B' = \{b'_1, \dots, b'_r\}$ so that b_i and b'_i respectively are representatives for the set A_i. Note that $b_i < b_{i+1}$ resp. $b'_i < b'_{i+1}$ does not hold necessarily. We associate a configuration of markings by

$$C(B, B') = \{((\times, A_i), b_i), ((\circ, A_i), b'_i) \colon i = 1, \dots, r\}.$$

Such a configuration $C = C(B, B')$ of markings satisfies the following conditions:

- (a) For every set A_i with $i \in \{1, \dots, r\}$ there is a unique element $((\circ, A_i), p_1) \in C$ and a unique element $((\times, A_i), p_2) \in C$ where $p_1, p_2 \in A_i$ are some positions.
- (b) If (m, p) and (m', q) are two elements in C such that the markings m and m' have the same first component, $\times$ or $\circ$, then it follows that $p \neq q$, i.e. m and m' are placed at different positions p and q.

We call C *valid* if it satisfies (a) and (b). Conversely, for every valid configuration C, we have $C = C(B, B')$ for some matrix $[B, B']^t$ of bases in $M[\mathcal{A}]$ with $B >_{\text{lex}} B'$. We call a valid configuration $C(B, B')$ *sorted* if the matrix $[B, B]^t$ is sorted with respect to $1 < 2 < \dots < d$. If $C = \{(m_1, p_1), \dots, (m_{2r}, p_{2r})\}$ is a configuration, then the multiset $\{p_1, p_2, \dots, p_{2r}\}$ is said to be the *support* of C. In case that $C = C(B, B')$ is valid the support of C coincides with the multiset $\operatorname{supp}([B, B']) = B \cup B'$.

It is our aim to show that $1 < 2 < \dots < d$ is a sorting order for $M[\mathcal{A}]$. Using the notation defined above we formulate this goal as follows:

Lemma 3.4.3. *For every valid configuration C there exists a (valid) sorted configuration C' such that C and C' have the same support.*

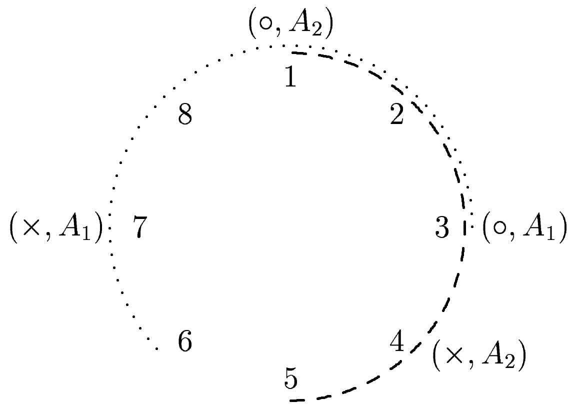
In order to prove Lemma 3.4.3 we give an algorithm which is defined for configurations of markings. Unfortunately it is not enough to consider valid configurations. A configuration C is called *almost valid* if it satisfies:

- (a) For every set A_i with $i \in \{1, \dots, r\}$ there is a unique element $((\circ, A_i), p_1) \in C$ and a unique element $((\times, A_i), p_2) \in C$ for some positions $p_1, p_2 \in A_i$.
- (b') For every $p \in [d]$ there are at most two markings placed at the position p, i.e. $|\{(m, p) \in C \colon m \in \mathcal{M}\}| \leq 2$.

Clearly, every valid configuration is also almost valid. We say that an almost valid configuration C has *twisted markings,* if the following condition is satisfied:

- (T) There are elements $i, j \in [r]$ with $i \neq j$ such that the corresponding markings $((\circ, A_i), p_1)$, $((\times, A_i), p_2)$, $((\circ, A_j), q_1)$, $((\times, A_j), q_2)$ in C satisfy: One of the positions q_1, q_2 belongs to $(p_1, p_2)_{A_i}$ and the other one of the positions q_1, q_2 does not belong to $[p_1, p_2]_{A_i}$. In this case C is said to be (i, j)*-twisted.*

Example 3.4.4. To illustrate the notation we give an example for $d = 8$ with two consecutive sets $A_1 = \{1, 2, 3, 6, 7, 8\}$ and $A_2 = \{1, 2, 3, 4, 5\}$.



The configuration $C = \{((\circ, A_2), 1), ((\times, A_1), 7), ((\circ, A_1), 3), ((\times, A_2), 4)\}$ is valid. The marking $(\circ, A_2)$ is placed at 1 which belongs to $(3, 7)_{A_1} = \{1, 2, 8\}$ and the marking $(\times, A_2)$ is placed at 4 which does not lie in $[3, 7]_{A_1} = \{1, 2, 3, 7, 8\}$. Thus the markings are twisted.

The next statement is a first step to prove Lemma 3.4.3.

Lemma 3.4.5. *For every valid configuration C there exists an almost valid configuration C' without twisted markings such that C and C' have the same support.*

For the proof we define the following exchange operation for an almost valid configuration C:

(E) Let $i \in \{1, \ldots, r\}$ and $((\circ, A_i), p_1)$, $((\times, A_i), p_2)$ in C be the two elements with component A_i. Then the configuration

$$C' = (C - \{((\circ, A_i), p_1), ((\times, A_i), p_2)\}) \cup \{((\circ, A_i), p_2), ((\times, A_i), p_1)\}$$

is also an almost valid configuration and C and C' have the same support.

The operation (E) exchanges the positions of marking $(\circ, A_i)$ and $(\times, A_i)$ for some $i \in [r]$. Let C be an almost valid configuration. For every $i = 1, \ldots, r$, we set $d_i(C) = \mathrm{dist}_{A_i}(p_1, p_2)$ where p_1 and p_2 are the positions of the elements $((\circ, A_i), p_1)$, $((\times, A_i), p_2) \in C$. If C is the configuration in the figure of Example 3.4.4, we have $d_1(C) = \mathrm{dist}_{A_1}(3, 7) = 4$ and $d_2(C) = \mathrm{dist}_{A_2}(1, 4) = 3$.

Proof of Lemma 3.4.5. We define an algorithm for almost valid configurations. We start with the configuration $C = C(B, B')$.

(1): If C does not have twisted markings, then we are done. Otherwise, C is (i, j)-twisted for some indices $i \neq j$. Let $((\circ, A_i), p_1)$, $((\times, A_i), p_2)$, $((\circ, A_j), q_1)$, $((\times, A_j), q_2)$ in C be the corresponding markings. Using (E) we may assume that $q_1 \in (p_1, p_2)_{A_i}$ and $q_2 \notin [p_1, p_2]_{A_i}$. Then exactly one of the markings $(\circ, A_i)$, $(\times, A_i)$ lies in $(q_1, q_2)_{A_j}$, i.e. p_1 or p_2 belongs to $(q_1, q_2)_{A_j}$. Using (E) we may assume that this position is p_1. Since both sets A_i and A_j are consecutive modulo d, we can change the positions of $(\circ, A_i)$ and $(\circ, A_j)$ and get an almost valid configuration C' which is not (i, j)-twisted, that is

$$\begin{aligned} C' &= C - \{((\circ, A_i), p_1), ((\times, A_i), p_2), ((\circ, A_j), q_1), ((\times, A_j), q_2)\} \\ &\quad \cup \{((\circ, A_i), q_1), ((\times, A_i), p_2), ((\circ, A_j), p_1), ((\times, A_j), q_2)\}. \end{aligned}$$

We have $d_i(C') < d_i(C)$, $d_j(C') < d_j(C)$ and $d_k(C') = d_k(C)$ for all $k \neq i, j$. The support of C' is $B \cup B'$. We repeat step (1) with the configuration C'.

The loop in (1) terminates because in every repetition two of the non-negative entries in the tuple $(d_1(C), \dots, d_r(C))$ are strictly decreased. □

Example 3.4.6. To apply the algorithm of Lemma 3.4.5 to Example 3.4.4 we exchange the positions of the markings $(\circ, A_1)$ and $(\circ, A_2)$. Thus we get a configuration C' without twisted markings and $d_1(C') = 2$, $d_2(C') = 1$.

To conclude the proof of Lemma 3.4.3 it remains to show the following:

Lemma 3.4.7. *For every almost valid configuration C without twisted markings there is a (valid) sorted configuration C' such that C and C' have the same support.*

Proof. Let $C = \{(m_1, p_1), \dots, (m_{2r}, p_{2r})\}$ be an almost valid configuration without twisted markings. We assume that $p_j \leq p_{j+1}$ is satisfied for all $j = 1, \dots, 2r-1$. Let $S = \{p_1, \dots, p_{2r}\}$ denote the support of C. Since C is almost valid, there are at most two markings at a fixed position p_j.

If all markings m_j, m_{j+1} with succeeding indices have different first components $\times$ or $\circ$, then $C = C(B_1, B_2)$ is valid and the matrix $[B_1, B_2]^t$ of bases in $M[\mathcal{A}]$ is sorted. In this case we are done.

Otherwise, we possibly have to modify the given numbering. In case that the two markings m_1 and m_2 with second components A_i and A_j respectively are both placed at the least position p_1, we choose m_1 so that $d_i(C) \geq d_j(C)$. Let m_k with $k > 1$ be the marking such that the markings m_1 and m_k have the same second component A_i. We need that the numbering satisfies the following condition:

($*$) If m_s, m_t are two markings in C which have the same second component A_j for some $j \neq i$, then it follows that either $1 < s, t < k$ or $1 < k < s, t$.

If the given numbering does not satisfy ($*$), then there are markings m_s, m_t with the same second component A_j, $j \neq i$ and $1 < s < k < t$. Since C does not have twisted markings, we have $p_1 = p_s$, $p_s = p_k$ or $p_k = p_t$. By the choice of m_1 it follows that $p_s = p_k$ or $p_k = p_t$. The numbering implies that either $s = k-1$ or $t = k+1$. We exchange either the indices of m_{k-1} and m_k or the indices of m_k and m_{k+1}. Then condition ($*$) is satisfied for all markings in C.

Let $symb \in \{\circ, \times\}$. We use induction on $r = \operatorname{rank}(M[\mathcal{A}])$ to show that there exists a sorted configuration $C' = \{(m'_1, p_1), \dots, (m'_{2r}, p_{2r})\}$ for the given support S so that the marking m'_1 has the first component *symb*. If $r = 1$, then every almost valid configuration $C = \{(m_1, p_1), (m_2, p_2)\}$ is trivially sorted. If the first component of m_1 is not *symb*, we use (E) and get the desired sorted configuration C'. Let now $r > 1$ and set

$\mathcal{A}_1 = (A_i\colon A_i$ is the second component of m_s for some $1 < s < k)$ and

$\mathcal{A}_2 = (A_i\colon A_i$ is the second component of m_s for some $k < t < 2r)$.

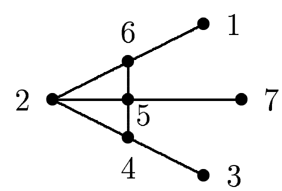
By $(*)$ the families $\mathcal{A}_1$ and $\mathcal{A}_2$ are presentations of transversal matroids $M[\mathcal{A}_1]$ and $M[\mathcal{A}_2]$ on $[d]$ such that $\operatorname{rank}(M[\mathcal{A}_1])$, $\operatorname{rank}(M[\mathcal{A}_2]) < r$. The configurations $C_1 = \{(m_2, p_2), \dots, (m_{k-1}, p_{k-1})\}$ and $C_2 = \{(m_{k+1}, p_{k+1}), \dots, (m_{2r}, p_{2r})\}$ are almost valid with support $S_1 = \{p_2, \dots, p_{k-1}\}$ and $S_2 = \{p_{k+1}, \dots, p_{2r}\}$. For simplicity we may assume that the first component of m_1 is $\circ$. By induction hypothesis there exist configurations C_1' and C_2' with support S_1 and S_2 such that the first marking of C_1' has $\times$ as the first component and the first marking of C_2' has $\circ$ as the first component. Then

$$C' = \{(m_1, p_1)\} \cup C_1' \cup \{(m_k, p_k)\} \cup C_2'$$

is a sorted configuration with support S. □

We cannot generalize Theorem 3.4.2 to arbitrary transversal matroids. Take, for example, the family $\mathcal{A} = (\{1,2,6\}, \{2,3,4\}, \{4,5,6\}, \{2,5,7\})$. Then the set $\{1,2,6\}$ is not consecutive modulo 7 and the matroid $M[\mathcal{A}]$ is not base-sortable, because by Corollary 3.3.12 its dual $M[\mathcal{A}]^*$ is one of the minimal excluded rank-3 minors for base-sortable matroids:

$M[\mathcal{A}]^*$



The result above cannot be generalized to truncations of transversal matroids. For a matroid M on $[d]$ with bases $\mathcal{B}$ the *principal truncation* of M is defined as the matroid $T(M)$ with bases $\{B - i \colon B \in \mathcal{B}, i \in B\}$. Take the presentation $\mathcal{A} = (\{2,3\}, \{1,2,3,4\}, \{5,6\}, \{7,8,9,10\}, \{8,9\})$ of consecutive subsets of $\{1,2,\dots,10\}$. By Theorem 3.4.2 the matroid $M[\mathcal{A}]$ has base-sorting order $1 < 2 < \dots < 10$. This is not true for $T(M[\mathcal{A}])$ because the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \end{bmatrix},$$

whose rows are bases of $T(M[\mathcal{A}])$, does not satisfy the sorting condition with respect to $1 < 2 < \dots < d$. This example also shows that the class $\mathcal{BS}$ of base-sortable matroids is not naturally closed under principal truncation. Moreover, if the sets A_i in Theorem 3.4.2 are not all consecutive modulo d, then $1 < 2 < \dots < d$ can fail to be a base-sorting order for $M[\mathcal{A}]$. Take, for example, the sets $A_1 = \{1,2,3,4\}$ and $A_2 = \{2,4\}$. The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not sortable with respect to $1 < 2 < 3 < 4$. We can interpret the corresponding rank 2 matroid $M[\mathcal{A}]$ as a matroid union of the two rank 1 matroids M_1 and M_2 whose bases are the 1-element subsets of A_1 and A_2. These two matroids have base-sorting order $1 < 2 < 3 < 4$, but not the union $M[\mathcal{A}]$. Thus the class of base-sortable matroids is not naturally closed under matroid union.

Example 3.4.8. Theorem 3.4.2 generalizes the result in Example 3.2.5. Let $v = T_{i_1} T_{i_2} \dots T_{i_r}$ with $i_1 < i_2 < \cdots < i_r$ be a square-free monomial in $K[T_1, \dots, T_d]$.

We set $A_j = \{1, \dots, i_j\}$ for $j = 1, \dots, r$ and $\mathcal{A} = (A_1, \dots, A_r)$. Then the basis monomial ring $R_{M[\mathcal{A}]}$ is generated by the monomials in $B_{sf}(v)$. By Theorem 3.4.2 $M[\mathcal{A}]$ is base-sortable.

3.5. Some open questions

The observations in the preceding sections give motivation to study matroid operations in a more general context for graded semigroup rings R which are generated by square-free monomials of the same degree. Let I denote the toric ideal of R. We introduce some notation.

Let $\mathcal{QG}$ be the collection of all graded semigroup rings such that I has a quadratic Gröbner basis for some term order, $\mathcal{K}$ be the collection of Koszul semigroup rings, and $\mathcal{Q}$ the collection of those rings for which I is generated by quadrics. The inclusions $\mathcal{QG} \subset \mathcal{K} \subset \mathcal{Q}$ are known in general (see Chapter 1). In [**42**, Section 1] Herzog, Hibi and Ohsugi have observed that the classes $\mathcal{QG}$, $\mathcal{K}$ and $\mathcal{Q}$ are closed under deletion. We have:

Proposition 3.5.1. *The classes $\mathcal{QG}$, $\mathcal{K}$ and $\mathcal{Q}$ are closed under taking minors. $\mathcal{K}$ and $\mathcal{Q}$ are also closed under Segre products.*

Proof. Remark 3.1.8 implies immediately that the classes $\mathcal{QG}$, $\mathcal{K}$ and $\mathcal{Q}$ are closed under taking minors. Koszulness is preserved under Segre products for arbitrary homogeneous K-algebras (see Theorem 1.3.7). A straightforward computation shows that the same holds for the class $\mathcal{Q}$. □

Finally, we give a survey on the preceding observations:

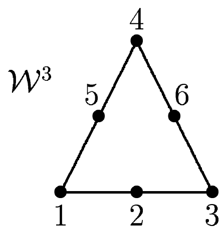
$\mathcal{S}$		$\mathcal{QG}$		$\mathcal{K}$		$\mathcal{Q}$
closed under		*closed under*		*closed under*		*closed under*
deletion		deletion		deletion		deletion
contraction	$\subset$	contraction	$\subset$	contraction	$\subset$	contraction
duality		duality		duality		duality
Segre product				Segre product		Segre product

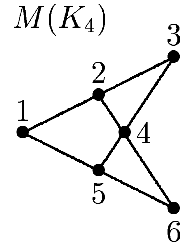
The class of basis monomial rings of matroids which belong to $\mathcal{Q}$ bas been considered by N. White in [**52**]. He has shown that this class is also closed under taking minors and direct sums. So far, we do not know of any counter example to his Conjecture 3.1.7 that all basis monomial rings of matroids belong to $\mathcal{Q}$.

It seems to be difficult to determine all basis monomial rings of matroids in $\mathcal{QG}$. We discuss some examples. By Theorem 3.3.12 $M(K_4)$ is the smallest matroid which is not base-sortable. Naturally the question arises whether there exists some term order such that $I_{M(K_4)}$ has a quadratic Gröbner basis. There is a positive answer. For this, we consider the toric ideal $I_{U_{3,6}}$. Let $S_{\mathcal{B}(U_{3,6})}$ be the polynomial ring with variables indexed by the bases of $U_{3,6}$ and let $<_{\text{lex}}$ denote the lexicographic term order on $S_{\mathcal{B}(U_{3,6})}$ induced by the following ordering. Here we use $a_1a_2a_3$ for the set $\{a_1, a_2, a_3\}$.

$$X_{123} > X_{156} > X_{345} > X_{246} > X_{236} > X_{126} > X_{346} > X_{136} > X_{234} > X_{256}$$
$$> X_{125} > X_{134} > X_{146} > X_{235} > X_{135} > X_{456} > X_{145} > X_{356} > X_{124} > X_{245}.$$

With the help of the computer program MACAULAY2 (see [**31**]) we obtain that $I_{\mathcal{B}(U_{3,6})}$ has a lexicographic Gröbner basis with respect to $<_{\text{lex}}$. We label the ground set of $M(K_4)$ and $\mathcal{W}^3$ in the following way:





By eliminating the three largest variables we get a quadratic Gröbner basis for $I_{\mathcal{B}(\mathcal{W}^3)}$ (see Proposition 1.4.5 and Lemma 1.4.17). If we eliminate the next largest variable, we obtain a quadratic lexicographic Gröbner basis for $I_{\mathcal{B}(M(K_4))}$.

Remark 3.5.2. The defining ideals of the basis monomial rings $R_{\mathcal{B}(U_{3,6})}, R_{\mathcal{B}(M(K_4))}$ and $R_{\mathcal{B}(\mathcal{W}^3)}$ have a lexicographic Gröbner basis. In particular, $M(K_4) \in \mathcal{QG}$.

It seems to be difficult to answer one of the open questions:

Questions 3.5.3. Let M be a matroid on $[d]$.

(a) Has the toric ideal $I_{\mathcal{B}(M)}$ a quadratic Gröbner basis with respect to some term order or even with respect to a lexicographic one?
(b) Is the basis monomial ring $R_{\mathcal{B}(M)}$ Koszul?

Note that a positive answer to one of the questions would imply N. White's Conjecture 3.1.7.

3.6. Strongly Koszul basis monomial rings

In this section we give a classification of the matroids M for which the basis monomial ring $R_{\mathcal{B}(M)}$ is strongly Koszul. We denote this class with $\mathcal{SK}$.

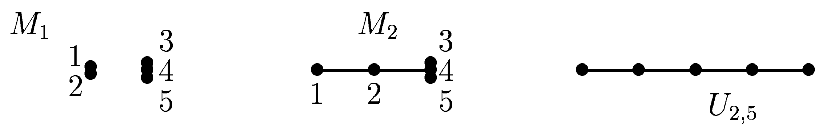
In Example 2.1.3(b) we have recalled the concept of strongly Koszul algebras (introduced in [**32**]). By Proposition 2.5.4 the algebra $R_{\mathcal{B}(M)} = K[\alpha_1, \dots, \alpha_n]$ is strongly Koszul if and only if the ideals $(\alpha_i) \cap (\alpha_j) \subset R_{\mathcal{B}(M)}$ are generated in degree 2 for all $i \neq j$. Since this is a relatively strong condition, one might expect that only few basis monomial rings are strongly Koszul. In [**43**] Hibi and Ohsugi have determined all strongly Koszul edge rings of complete multipartite graphs. Since the class of complete multipartite graphs coincides with the class of rank-2 matroids, we can reformulate their result in the following way.

Theorem 3.6.1. *Let M be a matroid of rank 2. Then M belongs to $\mathcal{SK}$ if and only if M is isomorphic to one of the matroids:*

(a) $U_{1,p} \oplus U_{1,q}$ *for some natural numbers* $p, q \geq 1$.
(b) $U_{2,3}$ *or* $U_{2,3} +_3 \{4, 5, \dots, p\}$ *for some* $p \geq 4$.
(c) $U_{2,4}$.

Here $U_{r,d}$ denotes the rank-r uniform matroid on [d] (see Examples 1.6.3), and we write $M +_i \{r+1,\dots,r+p\}$ for the matroid which is obtained from M by an iterated parallel extension of the elements in $\{r+1,\dots,r+p\}$ at the point i (see Section 1.6). To illustrate the result we give an example.

Example 3.6.2. The matroids $M_1 = U_{1,2} \oplus U_{1,3}$ and $M_2 = U_{2,3} +_3 \{4,5\}$ belong to $\mathcal{SK}$, while $U_{2,5}$ is an excluded minor of $\mathcal{SK}$.



Using Theorem 3.6.1 we determine the class $\mathcal{SK}$. We start with a first observation.

Proposition 3.6.3. *The class $\mathcal{SK}$ is closed under taking minors and direct sums.*

Proof. By Proposition 2.5.4 it is evident that $\mathcal{SK}$ is closed under taking minors. The strongly Koszul property is preserved under Segre products of semigroup rings (see [**32**]). Since we have $R_{\mathcal{B}(M_1)} * R_{\mathcal{B}(M_2)} = R_{\mathcal{B}(M_1 \oplus M_2)}$ for two matroids M_1 and M_2, the class $\mathcal{SK}$ is closed under direct sums. □

Recall that every matroid has a unique decomposition $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ into its connected components M_i (see Theorem 1.6.10). The previous proposition implies that a matroid M belongs to $\mathcal{SK}$ if and only if every connected component of M belongs to $\mathcal{SK}$. Therefore the following statement is crucial for our classification.

Proposition 3.6.4. *Let M be a connected matroid with* $\operatorname{rank}(M) = r \geq 3$ *having no loops. Then M belongs to $\mathcal{SK}$ if and only if M is isomorphic to $U_{r,r+1}$ or to $U_{r,r+1} +_{r+1} \{r+2,\dots,r+p\}$ for some $p \geq 2$.*

The main result of this section follows immediately from Proposition 3.6.3, Proposition 3.6.4 and Theorem 3.6.1.

Theorem 3.6.5. *A matroid M without loops belongs to $\mathcal{SK}$ if and only if every connected component of M is isomorphic to one of the following matroids:*

(a) *$U_{1,l}$ for some $l \geq 1$.*
(b) *$U_{2,4}$.*
(c) *$U_{r,r+1}$ or $U_{r,r+1} +_{r+1} \{r+2,\dots,r+p\}$ for some $p \geq 2$.*

In other words, M has a decomposition $M = M_1 \oplus \cdots \oplus M_k$ where every matroid M_i is isomorphic to one of the matroids in (a),(b) *or* (c).

For the proof of Proposition 3.6.4 we need several technical lemmata. In the following we write $p(M)$ for the number of parallel classes of M.

Lemma 3.6.6. *Let M be a simple matroid on $[d]$ with* $\operatorname{rank}(M) = r \geq 3$*. If $p(M) = r+1$ and M has a dependent k-flat for some $1 < k < r$, then M is not connected.*

Proof. To deduce a contradiction we assume that F is a dependent k-flat with $1 < k < r$. We have $d = r + 1 = p(M)$ because M is simple. Since $k < r$, there exists an element $i \in [d] - F$ such that $\mathrm{rk}_M(F \cup i) = k + 1$. Moreover, we have

$$\mathrm{rk}([d]-i) = \mathrm{rk}(F \cup ([d]-(F\cup i))) \le \mathrm{rk}(F) + \mathrm{rk}([d]-(F\cup i)) \le k + (r+1) - (k+2).$$

Therefore, we get that $\mathrm{rk}([d]-i) = r-1$. Since M is simple, we have $\mathrm{rk}(i) = 1$. Thus i is a separator by Lemma 1.6.12, that means $M = (M \setminus i) \oplus (M \setminus ([d]-i))$. This is a contradiction to the connectedness of M. □

Lemma 3.6.7. *Let M be a simple matroid on $[d]$ with* $\mathrm{rank}(M) \ge 3$ *and $i \in [d]$. Then*

$$p(M/i) = (d-1) - \sum_{i=1}^{k} (|F_i| - 1) + l$$

where $\{F_1, \dots, F_k\}$ is the collection of dependent 2-flats of M which contain the point i.

Proof. Let $A \subset [d]$ be a subset. The formula $\mathrm{rk}_{M/i}(A) = \mathrm{rk}_M(A \cup i) - \mathrm{rk}_M(i)$ is a general fact (see [**41**]). Since M is simple, we observe that

$$\mathrm{rk}_{M/i}(A) = \mathrm{rk}_M(A \cup i) - 1 \tag{2}$$

for all $A \subset [d]$. Let $\{P_1, \dots, P_l\}$ be the collection of parallel classes of M/i. We may assume that $P_1, \dots, P_k$ are non-trivial while $P_{k+1}, \dots, P_l$ are trivial parallel classes. We want to determine l. Since M is simple, M/i has no loops and $d-1 = \sum_{j=1}^{k} |P_j|$. By the definition a parallel class $P \in \{P_1, \dots, P_k\}$ is a maximal subset of $[d] - i$ such that $\mathrm{rk}_{M/i}(P) = 1$. By (2) this is equivalent to $P \cup i$ being a 2-flat of M and P is trivial if and only if $P \cup i$ is an independent set of M. Therefore, we obtain

$$d - 1 = \sum_{j=1}^{l} |P_j| = \sum_{j=1}^{k} |P_j| + (k - l).$$

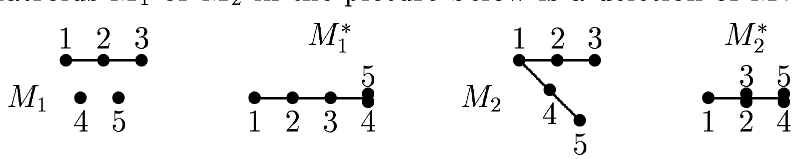
This concludes the proof. □

For a matroid M, we recall that $\overline{M}$ denotes the underlying simple matroid of M (see Section 1.6).

Lemma 3.6.8. *Let M be a connected matroid of rank $r \ge 3$ without loops. If M belongs to $\mathcal{SK}$, then $\overline{M}$ is isomorphic to $U_{r,d}$ for some d.*

Proof. Let $M \in \mathcal{SK}$. We have to show that $\overline{M}$ has no dependent k-flats for $1 < k < r$. By Lemma 1.6.12 we may assume that M is simple, i.e. $M = \overline{M}$. Since M is connected, we have $p(M) = d \ge r+1$. We use induction on d. If $d = r+1$, the assertion follows from Lemma 3.6.6.

Let now $d > r+1$. We use induction on k. Let $k = 2$. Since M is connected and $d \ge 5$, there exist at least two elements of $[d]$ which do not belong to F. Thus one of the matroids M_1 or M_2 in the picture below is a deletion of M.



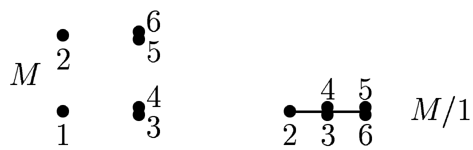
By Theorem 3.6.1 the matroids M_1^* and M_2^* do not belong to $\mathcal{SK}$. Applying Proposition 3.6.3 M_1 and M_2 are not in $\mathcal{SK}$. Therefore, the matroid M has a minor which is not in $\mathcal{SK}$, a contradiction to Proposition 3.6.3.

Let now $2 < k < r$. To deduce a contradiction we suppose that F is a dependent k-flat. By induction on k we may assume that M has no dependent flat of smaller rank than k. Since $k < r$, there is an element $i \in [d] - F$. By Theorem 1.6.11 M/i or $M \setminus i$ is connected. If M/i is connected, we have $p(M/i) \geq r$ using Lemma 3.6.7. We apply the induction hypothesis on d to $\overline{M/i}$. Thus M has a minor which does not belong to $\mathcal{SK}$, a contradiction. Otherwise, if $M \setminus i$ is connected, the induction hypothesis implies that $M \setminus i$ does not belong to $\mathcal{SK}$, a contradiction. This concludes the proof. □

Lemma 3.6.9. *Let M be a connected matroid with $r = rank(M) \geq 3$ without loops. If $M \in \mathcal{SK}$, then $\overline{M} \cong U_{r,r+1}$ and M has at most one non-trivial parallel class.*

Proof. By Lemma 3.6.8 we have $\overline{M} \cong U_{r,d}$ for some $d \geq r$. If $d \geq r+2$, then $\overline{M}$ has $U_{r,r+2}$ as a deletion. Using $r \geq 3$ and Theorem 3.6.1 the matroid $U^*_{r,r+2} = U_{2,r+2}$ does not belong to $\mathcal{SK}$. This is a contradiction to Proposition 3.6.3.

Therefore we have $\overline{M} \cong U_{r,r+1}$. To get a contradiction we suppose that there are two distinct parallel classes P_1, P_2 of M such that $|P_1|, |P_2| > 1$. After a suitable deletion we may assume that $\overline{M} = U_{3,4}$, $|P_1| = |P_2| = 2$ and all other parallel classes are trivial. Then M is of the form



By Theorem 3.6.1 the matroid $M/1$ does not belong to $\mathcal{SK}$, a contradiction to Proposition 3.6.3. □

Finally, we use the preceding lemmata to show:

Proof of Proposition 3.6.4. Let $M \in \mathcal{SK}$ be a connected matroid of rank $r \geq 3$ without loops. Lemma 3.6.9 implies that M is isomorphic to $U_{r,r+1}$ or to $U_{r,r+1} +_{r+1} \{r+2, \ldots, r+p\}$ for some $p \geq 2$.

It remains to show that the matroids $U_{r,r+1}$ and $U_{r,r+1} +_{r+1} \{r+2, \ldots, r+p\}$ belong to $\mathcal{SK}$. First we observe that $U^*_{r,r+1} = U_{1,r+1}$. Thus the basis monomial ring $R_{\mathcal{B}(U_{r,r+1})}$ is isomorphic to $R_{\mathcal{B}(U_{1,r+1})}$, which is a polynomial ring. Therefore $U_{r,r+1}$ belongs to $\mathcal{SK}$.

We consider the matroid $M = U_{r,r+1} +_{r+1} \{r+2, \ldots, r+p\}$ where $p \geq 2$. Let N be the rank-1 matroid with bases $\mathcal{B}(N) = \{\{r+1\}, \ldots, \{r+p\}\}$. Then the basis monomial ring of the matroid $M' = U_{r-1,r} \oplus N$ is strongly Koszul by Proposition 3.6.3. The collection of bases of M has the form $\mathcal{B}(M) = \mathcal{B}(M') \mathbin{\dot\cup} \{\{1, 2, \ldots, r\}\}$. In other words, we have $R_{\mathcal{B}(M)} = R_{\mathcal{B}(M')}[T_1 T_2 \cdots T_r]$. It is straightforward to check that the monomial $T_1 \cdots T_r$ is an indeterminate over $R_{\mathcal{B}(M')}$. Thus $R_{\mathcal{B}(M)}$ is strongly Koszul, and therefore $M \in \mathcal{SK}$. □

CHAPTER 4

Subalgebras of bigraded Koszul algebras

Bigraded algebras form the topic of this chapter. As the main result we show that diagonals and generalized bigraded Veronese subalgebras of a bigraded Koszul algebra inherit the Koszul property. Moreover, we obtain upper bounds for the regularity of sidediagonal and bigraded Veronese modules. Notice that our results also hold with similar proofs if one considers multigraded K-algebras and the corresponding multigraded subalgebras.

In the last three sections we consider several applications of which some appear in the study of Rees rings and symmetric algebras. Polymatroidal ideals form a class of monomial ideals for which all powers and symmetric powers have linear resolutions. These ideals extend the notion of matroidal ideals in [**24**]. We also recover some well-known results for standard graded algebras (see Theorem 1.3.7) and interpret our main theorem for bigraded semigroup rings.

4.1. Diagonals and bigraded Veronese subrings

In the sequel we always assume that $S = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ is the polynomial ring with standard bigrading $\deg(X_i) = (1,0)$ and $\deg(Y_j) = (0,1)$ and that R denotes a bigraded K-algebra of the form $R = S/J$ where J is a bigraded ideal of S. We recall that for two integers $a, b \geq 0$ with $(a,b) \neq (0,0)$ the (a,b)-*diagonal* is the subset $\Delta = \{(sa, sb) \colon s \in \mathbb{Z}\}$ of $\mathbb{Z}^2$. As in [**22**] the *diagonal subalgebra of R along* Δ is defined as the positively graded algebra

$$R_\Delta = \bigoplus_{i \geq 0} R_{(ia,ib)}$$

where $R_{(i,j)}$ denotes the $(i,j)^{th}$ bigraded component of R. The algebra R_Δ is generated by the residue classes of all monomials which have degree (a,b) in S. Therefore, R_Δ admits a standard grading. For two integers $a, b \geq 0$ with $(a,b) \neq (0,0)$ we define, according to [**46**], the *bigraded generalized Veronese subring of R along the set* $\tilde{\Delta} = \{(sa, tb) \colon s, t \in \mathbb{Z}\}$ by

$$R_{\tilde{\Delta}} = \bigoplus_{i,j \geq 0} R_{(ia,jb)}.$$

Here the bigraded components are $(R_{\tilde{\Delta}})_{(i,j)} = R_{(ia,jb)}$. The algebra $R_{\tilde{\Delta}}$ is generated by the residue classes of all monomials which have degree $(a,0)$ or $(0,b)$ in S. Thus $R_{\tilde{\Delta}}$ has the structure of a standard bigraded algebra. Note that $R = R_{\tilde{\Delta}}$ for $(a,b) = (1,1)$. In the case that $n = 0$ or $m = 0$, the algebra R is simply standard graded and the subrings R_Δ and $R_{\tilde{\Delta}}$ are the ordinary Veronese subrings of R. We also observe that the $(1,1)$-diagonal of $R_{\tilde{\Delta}}$ equals R_Δ.

Let $M \in \mathcal{M}_{\mathbb{Z}^2}(R)$ and Δ be the (a,b)-diagonal. For two integers $c,d \geq 0$ we define $M_\Delta^{(c,d)}$ to be the finitely generated, $\mathbb{Z}$-graded R_Δ-module with components $(M_\Delta^{(c,d)})_i = M_{(ia+c,ib+d)}$. For $(c,d) = (0,0)$ we simply use M_Δ instead of $M_\Delta^{(0,0)}$. We call $M_\Delta^{(c,d)}$ the (c,d)-*sidediagonal module* of M. Similarly, we write $M_{\tilde{\Delta}}^{(c,d)}$ for the bigraded $R_{\tilde{\Delta}}$-module with components $(M_{\tilde{\Delta}}^{(c,d)})_{(i,j)} = M_{(ia+c,jb+d)}$ and call it the *relative* (c,d)-*Veronese module* of M. If $n = 0$ or $m = 0$, then these modules coincide with the relative Veronese modules defined in [**3**]. We need two index sets

$$\mathcal{I}(a,b) = \{(c,d) \in \mathbb{N}^2 : c < a \text{ or } d < b\} \quad \text{and}$$

$$\tilde{\mathcal{I}}(a,b) = \begin{cases} \{(c,d) \in \mathbb{N}^2 : c < a \text{ and } d < b\} & \text{if } a,b \geq 1, \\ \{(c,0) \in \mathbb{N}^2 : c < a\} & \text{if } a \geq 1 \text{ and } b = 0, \\ \{(0,d) \in \mathbb{N}^2 : d < b\} & \text{if } a = 0 \text{ and } b \geq 1. \end{cases}$$

Note that the index set $\mathcal{I}(a,b)$ is infinite while $\tilde{\mathcal{I}}(a,b)$ is a finite set. For $(c,d) \in \mathcal{I}(a,b)$ the module $R_\Delta^{(c,d)}$ is generated in degree 0 and, for arbitrary $c,d \geq 0$, it is $R_\Delta^{(c,d)} = R_\Delta^{(c',d')}(-l)$ for some integer $l \geq 0$ and some $(c',d') \in \mathcal{I}(a,b)$. An analogous fact holds for the modules $R_{\tilde{\Delta}}^{(c,d)}$. We have the decomposition

$$R = \bigoplus_{(c,d) \in \mathcal{I}(a,b)} R_\Delta^{(c,d)}.$$

Analogously, if $a,b \geq 1$, then R is the finite direct sum of the $R_{\tilde{\Delta}}^{(c,d)}$ with $(c,d) \in \tilde{\mathcal{I}}(a,b)$.

The maps $M \mapsto M_\Delta^{(c,d)}$ and $M \mapsto M_{\tilde{\Delta}}^{(c,d)}$ define exact functors from $\mathcal{M}_{\mathbb{Z}^2}(R)$ to $\mathcal{M}_{\mathbb{Z}}(R_\Delta)$ and to $\mathcal{M}_{\mathbb{Z}^2}(R_{\tilde{\Delta}})$ respectively. In particular, we consider a bigraded free resolution of a module M:

$$F_\bullet : \quad \ldots \to F_i \to \ldots \to F_1 \to F_0 \to M \to 0$$

where every free module F_i decomposes into a finite direct sum $\bigoplus_{p,q} R(-p,-q)^{b_{i,(p,q)}}$. Here, $R(-p,-q)$ denotes the bigraded R-module with components $R(-p,-q)_{(i,j)} = R_{(i-q,j-q)}$. Then we get an exact complex of R_Δ-modules

$$(F_\bullet)_\Delta^{(c,d)} : \quad \ldots \to (F_i)_\Delta^{(c,d)} \to \ldots \to (F_1)_\Delta^{(c,d)} \to (F_0)_\Delta^{(c,d)} \to M_\Delta^{(c,d)} \to 0$$

with $(F_i)_\Delta^{(c,d)} = \bigoplus_{p,q} (R(-p,-q)_\Delta^{(c,d)})^{b_{i,(p,q)}}$. Analogous statements are true for the functor $-_{\tilde{\Delta}}^{(c,d)}$. It will be important for the main result of this chapter to write every module $R(-p,-q)_\Delta^{(c,d)}$ as a shifted sidediagonal module of the form $R_\Delta^{(c',d')}$ for some $(c',d') \in \mathcal{I}(a,b)$. For a reel number α we use $\lceil \alpha \rceil$ for the smallest integer z such that $z \geq \alpha$. We observe:

Remark 4.1.1. Let Δ be the (a,b)-diagonal. For $z \in \mathbb{Z}$ let $\alpha(z) \in \{0,\ldots,a-1\}$ be the integer such that $\alpha(z) \equiv z \mod a$ and $\beta(z) \in \{0,\ldots,b-1\}$ with $\beta(z) \equiv z \mod b$.

(a) (i) Let $a > 0$, $b = 0$ and $(c,d) \in \mathcal{I}(a,b)$. Then

$$R(-p,-q)_{\Delta}^{(c,d)} = \begin{cases} 0, & \text{if } q > d, \\ R_{\Delta}^{(\alpha(c-p),d-q)}(-l), & \text{if } q \leq d, \end{cases}$$

where $l = \max\{0, \lceil \frac{p-c}{a} \rceil\}$.

(ii) Let $a = 0$, $b > 0$ and $(c,d) \in \mathcal{I}(a,b)$. Then

$$R(-p,-q)_{\Delta}^{(c,d)} = \begin{cases} 0, & \text{if } p > c, \\ R_{\Delta}^{(c-p,\beta(d-q))}(-l), & \text{if } p \leq c, \end{cases}$$

where $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$.

(iii) Let $a, b \geq 1$ and $(c,d) \in \mathcal{I}(a,b)$. Then

$$R(-p,-q)_{\Delta}^{(c,d)} = R_{\Delta}^{(la+c-p,lb+d-q)}(-l),$$

where $l = \max\{0, \lceil \frac{p-c}{a} \rceil, \lceil \frac{q-d}{b} \rceil\}$.

(b) (i) Let $a > 0$, $b = 0$ and $(c,0) \in \tilde{\mathcal{I}}(a,b)$. Then

$$R(-p,-q)_{\tilde{\Delta}}^{(c,0)} = \begin{cases} 0, & \text{if } q > 0, \\ R_{\tilde{\Delta}}^{(\alpha(c-p),0)}(-k,0), & \text{if } q = 0, \end{cases}$$

where $k = \max\{0, \lceil \frac{p-c}{a} \rceil\}$.

(ii) Let $a = 0$, $b > 0$ and $(0,d) \in \tilde{\mathcal{I}}(a,b)$. Then

$$R(-p,-q)_{\tilde{\Delta}}^{(0,d)} = \begin{cases} 0, & \text{if } p > 0, \\ R_{\tilde{\Delta}}^{(0,\beta(d-q))}(0,-l), & \text{if } p = 0, \end{cases}$$

where $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$.

(iii) Let $a, b \geq 1$ and $(c,d) \in \tilde{\mathcal{I}}(a,b)$. Then

$$R(-p,-q)_{\tilde{\Delta}}^{(c,d)} = R_{\tilde{\Delta}}^{(\alpha(c-p),\beta(d-q))}(-k,-l)$$

where $k = \max\{0, \lceil \frac{p-c}{a} \rceil\}$ and $l = \max\{0, \lceil \frac{q-d}{b} \rceil\}$.

Recall that every bigraded K-algebra R is also naturally $\mathbb{N}$-graded with i^{th} component $R_i = \bigoplus_{j+k=i} R_{(j,k)}$. Similarly, every bigraded R-module M can be considered as $\mathbb{Z}$-graded. We say that M has a *bigraded a-linear resolution* if $\operatorname{Tor}_i^R(M,K)_{(j,k)} = 0$ for all i, j, k such that $j + k \neq i + a$.

4.2. The Koszul property

In this section we prove the main result of this chapter. We fix a pair (a,b) together the corresponding sets Δ and $\tilde{\Delta}$.

Theorem 4.2.1. If R is a Koszul algebra, then every diagonal subalgebra R_Δ and every generalized Veronese subring $R_{\tilde{\Delta}}$is a Koszul algebra.

For the proof we need several lemmata. Let $\mathfrak{n}_x = (x_1, \ldots, x_n) \subset R$ and $\mathfrak{n}_y = (y_1, \ldots, y_m) \subset R$ be the ideal generated by the residue classes of all X_i and all Y_j respectively.

Lemma 4.2.2. If R is Koszul, then the ideals $\mathfrak{n}_x$ and $\mathfrak{n}_y$ have bigraded 1-linear R-resolutions.

Proof. By symmetry it is enough to show that $\mathfrak{n}_y$ has a bigraded linear resolution. The residue class field K has a 0-linear minimal free R-resolution $F_{\bullet}$ because R is Koszul. Let $\tilde{\Delta}$ the $(1,0)$-Veronese set. Applying the functor $-_{\tilde{\Delta}}$ we get the exact complex $(F_{\bullet})_{\tilde{\Delta}} \to K \to 0$. By Remark 4.1.1(b) the i^{th} module $(F_i)_{\tilde{\Delta}}$ is a direct sum of copies of $R_{\tilde{\Delta}}$ shifted by $(-i,0)$. Thus $R_{\tilde{\Delta}}$ is a standard bigraded Koszul algebra.

Let $p : R \to R_{\tilde{\Delta}}$ be the projection map and $i : R_{\tilde{\Delta}} \to R$ be the inclusion. Note that both maps p and i are bigraded homomorphisms. Via p we consider $R_{\tilde{\Delta}} = R/\mathfrak{n}_y$ as a bigraded R-module. Since p is a ring epimorphism and $p \circ i = \mathrm{id}_{R_{\tilde{\Delta}}}$, the map i is a bigraded algebra retract. We may apply a result from [**33**] to the bigraded case. It yields that $P_K^R = P_{R_{\tilde{\Delta}}}^R P_K^{R_{\tilde{\Delta}}}$. Since R and $R_{\tilde{\Delta}}$ are Koszul, the equality of bigraded Poincaré series implies that $\mathfrak{n}_y$ has a bigraded 1-linear R-resolution. This concludes the proof. □

Proposition 4.2.3. *Let $c, d \geq 0$ be two integers. If $\mathfrak{n}_x$ and $\mathfrak{n}_y$ have bigraded linear resolutions, then*

(a) *the sidediagonal module $R_{\Delta}^{(c,d)}$ has a linear R_{Δ}-resolution.*
(b) *the relative Veronese module $R_{\tilde{\Delta}}^{(c,d)}$ has a bigraded linear $R_{\tilde{\Delta}}$-resolution.*

For the proof of the proposition we need a fact which is stated in [**22**].

Lemma 4.2.4. *Let A be a standard graded K-algebra and let*

$$\ldots \to N_r \to N_{r-1} \to \ldots \to N_1 \to N_0 \to M \to 0$$

be an exact complex of finitely generated graded A-modules. We have:

(a) *Let $h \in \mathbb{N}$ and $a \in \mathbb{Z}$ such that $t_s(N_r) \leq a + r + s$ for all $0 \leq r \leq h$ and $0 \leq s \leq h - r$. Then $t_h(M) \leq a + h$.*
(b) $\operatorname{reg}_A(M) \leq \sup\{\operatorname{reg}_A(N_i) - i \colon i \in \mathbb{N}\}$.

Proof of Proposition 4.2.3. Since the proofs of (a) and (b) are similar, we only consider part (a). Moreover, it is enough to show that all modules $R_{\Delta}^{(c,d)}$ with $(c,d) \in \mathcal{I}(a,b)$ have linear resolutions. Let $G_{\bullet}$ be the minimal bigraded free R-resolution of $\mathfrak{n}_x$. Since $\mathfrak{n}_x$ has a bigraded 1-linear resolution by hypothesis, every free module G_r is of the form

$$G_r = \bigoplus_{p+q=r+1,\ p\geq 1} R(-p,-q)^{\beta_{r,(p,q)}}$$

where $\beta_{r,(p,q)}$ are the bigraded Betti numbers of $\mathfrak{n}_x$. Observe that for $c \geq 1$ and $(c,d) \in \mathcal{I}(a,b)$ we have $(\mathfrak{n}_x)^{(c,d)} = R_{\Delta}^{(c,d)}$. Applying the functor $-_{\Delta}^{(c,d)}$ we obtain an acyclic complex $(G_{\bullet})_{\Delta}^{(c,d)} \to R_{\Delta}^{(c,d)} \to 0$ where

$$(G_r)_{\Delta}^{(c,d)} = \bigoplus_{p+q=r+1,\ p\geq 1} R_{\Delta}^{(c_{p,q},d_{p,q})}(-l_{p,q})^{\beta_{r,(p,q)}}$$

By Remark 4.1.1(a) all occurring shifts $l_{p,q}$ are at most r. Similarly, let $H_{\bullet}$ be the minimal bigraded free resolution of $\mathfrak{n}_y$. Then we observe that for $d \geq 1$ and $(c,d) \in \mathcal{I}(a,b)$ it is $(\mathfrak{n}_y)_\Delta^{(c,d)} = R_\Delta^{(c,d)}$, and the shifts in $(H_r)_\Delta^{(c,d)}$ are bounded by r.

To conclude the proof we show by induction that $t_h(R_\Delta^{(c,d)}) \leq h$ for all $h \in \mathbb{N}$ and $(c,d) \in \mathcal{I}(a,b)$. First we use induction on h. The modules $R_\Delta^{(c,d)}$ are generated in degree 0, thus $t_0(R_\Delta^{(c,d)}) = 0$. Let now $h \geq 1$. We apply the induction hypothesis on $c+d$ where $(c,d) \in \mathcal{I}(a,b)$. For $c+d=0$ it follows that $c=0=d=0$ and therefore trivially $t_h(R_\Delta) \leq h$. Let now $c+d>0$. Then $c \geq 1$ or $d \geq 1$.

We discuss the case $c \geq 1$ first. In order to apply Lemma 4.2.4(a) to the exact complex $(G_{\bullet})_\Delta^{(c,d)} \to R_\Delta^{(c,d)} \to 0$ we show that $t_s((G_r)_\Delta^{(c,d)}) \leq r+s$ for all $0 \leq r \leq h$ and $0 \leq s \leq h-r$. Observe that $(G_0)_\Delta^{(c,d)}$ is a direct sum of n copies of $(R_\Delta^{(c-1,d)})$. Since $(c-1,d) \in \mathcal{I}(a,b)$, the induction hypothesis on $c+d$ implies that $t_s((G_0)_\Delta^{(c,d)}) \leq s$ for all $0 \leq s \leq h$. For $1 \leq r \leq h$ and $0 \leq s \leq h-r$, we have

$$t_s((G_r)_\Delta^{(c,d)}) \leq t_s\Bigg(\bigoplus_{p+q=i+1,\ p\geq 1} R_\Delta^{(c_{p,q},d_{p,q})}\Bigg) + r \leq s+r$$

where the first inequality holds because $l_{p,q} \leq r$ for all occurring p,q and the second inequality holds by induction on h. Now Lemma 4.2.4 implies that $t_h(R_\Delta^{(c,d)}) \leq h$.

If $c=0$ and $d \geq 1$, then the argument above can similarly be applied to the complex $(H_{\bullet})_\Delta^{(c,d)} \to R_\Delta^{(c,d)} \to 0$. □

As a direct consequence of Lemma 4.2.2 and Lemma 4.2.3 we obtain:

Corollary 4.2.5. *Let $c,d \geq 0$ be two integers. If R is Koszul, then all sidediagonal modules $R_\Delta^{(c,d)}$ have linear R_Δ-resolutions and all relative Veronese modules $R_{\tilde\Delta}^{(c,d)}$ have bigraded linear $R_{\tilde\Delta}$-resolutions.*

We use this corollary to get upper bounds for the regularity of sidediagonal and relative Veronese modules.

Theorem 4.2.6. *Let R be a bigraded Koszul algebra and $M \in \mathcal{M}_{\mathbb{Z}^2}(R)$ such that $\operatorname{reg}_R(M) = r$ and $\operatorname{indeg}(M) = 0$.*

(a) *Let $(c,d) \in \mathcal{I}(a,b)$. Then*

$$\operatorname{reg}_{R_\Delta}(M_\Delta^{(c,d)}) \leq \begin{cases} \max\{0, \lceil \frac{r-c}{a} \rceil\}, & \text{if } b=0 \text{ and } a>0,\\ \max\{0, \lceil \frac{r-d}{b} \rceil\}, & \text{if } a=0 \text{ and } b>0,\\ \max\{0, \lceil \frac{r-c}{a} \rceil, \lceil \frac{r-d}{b} \rceil\}, & \text{if } a,b \geq 1. \end{cases}$$

(b) *Let $(c,d) \in \tilde{\mathcal{I}}(a,b)$. Then*

$$\operatorname{reg}_{R_{\tilde\Delta}}(M_{\tilde\Delta}^{(c,d)}) \leq \begin{cases} \max\{0, \lceil \frac{r}{a} \rceil\}, & \text{if } b=0 \text{ and } a>0,\\ \max\{0, \lceil \frac{r}{b} \rceil\}, & \text{if } a=0 \text{ and } b>0,\\ \min\{r, \lceil \frac{r-c}{a} - \frac{d}{b} + 1 \rceil\}, & \text{if } 1 \leq a \leq b. \end{cases}$$

In particular, if $1 \leq a \leq b$, then $\operatorname{reg}_{R_{\tilde\Delta}}(M) \leq \min\{r, \lceil \frac{r}{a} + 1 \rceil\}$.

Proof. Let $F_{\bullet}$ be the minimal bigraded free R-resolution of M. Since $\operatorname{reg}_R(M) = r$, we have $F_i = \bigoplus_{i\leq p+q\leq i+r} R(-p,-q)^{\beta_{i,(p,q)}}$ where $\beta_{i,(p,q)}$ are the bigraded Betti numbers of M. For the proof of part (a) we restrict to the case $a, b \geq 1$. The other cases follow similarly. By Remark 4.1.1(a) and Corollary 4.2.5 we observe that

$$\operatorname{reg}_{R_\Delta}(F_{i\,\Delta}^{(c,d)}) \leq \max\{0, \lceil \tfrac{i+r-c}{a}\rceil, \lceil \tfrac{i+r-d}{b}\rceil\} \leq \max\{0, \lceil \tfrac{r-c}{a}\rceil, \lceil \tfrac{r-d}{b}\rceil\} + i.$$

Now Lemma 4.2.4(b) yields the claim. For part (b) we also restrict to the case $a \geq 1$ and $b \geq 1$. Use Remark 4.1.1(b) and Corollary 4.2.5 to observe that

$$\operatorname{reg}_{R_{\tilde\Delta}}(F_{i\,\tilde\Delta}^{(c,d)}) \leq \max\{\max\{0, \lceil \tfrac{p-c}{a}\rceil\} + \max\{0, \lceil \tfrac{q-d}{b}\rceil\} \colon i \leq p+q \leq i+r\}.$$

The claim follows from an easy case by case computation using $1 \leq a \leq b$ and Lemma 4.2.4(b). Since M decomposes into the finite direct sum $M = \bigoplus_{(c,d)\in \tilde I(a,b)} M_{\tilde\Delta}^{(c,d)}$, we obtain the upper bound for $\operatorname{reg}_{R_{\tilde\Delta}}(M)$. □

As a direct consequence of Theorem 4.2.6 the modules M_Δ and $M_{\tilde\Delta}$ have small regularities for $a, b \gg 0$. More concrete, we have:

Corollary 4.2.7. *Let $M \in \mathcal{M}_{\mathbb{Z}^2}(R)$.*

(a) *If* $\max\{a,b\} \geq \operatorname{reg}_R(M)$, *then* $\operatorname{reg}_{R_\Delta}(M_\Delta) \leq \min\{1, \operatorname{reg}_R(M)\}$.
(b) *Let $a, b \geq 1$. If* $\min\{a,b\} \geq \operatorname{reg}_R(M)$, *then* $\operatorname{reg}_{R_{\tilde\Delta}}(M_{\tilde\Delta}) \leq \min\{2, \operatorname{reg}_R(M)\}$ *and* $\operatorname{reg}_{R_{\tilde\Delta}}(M) \leq \min\{2, \operatorname{reg}_R(M)\}$.

Theorem 4.2.1 is an immediate consequence of the results above.

Proof of Theorem 4.2.1. We note that a graded K-algebra A is Koszul if and only if $\operatorname{reg}_A(K) = 0$. Since $K_\Delta = K$ and $K_{\tilde\Delta} = K$, the claim follows from Theorem 4.2.6. □

Note that the converse of Theorem 4.2.1 is false for diagonals. Take, for example, the algebra $R = K[X_1, Y_1]/(X_1Y_1^2)$. Since the defining ideal of R is generated in degree 3, R is not Koszul. But every diagonal R_Δ is Koszul because R_Δ is either isomorphic to the field K, to the polynomial ring $K[T]$ or to the Koszul algebra $K[T]/(T^2)$.

4.3. Rees algebras

An intensively studied class of bigraded algebras are Rees algebras. In the following sections we set $S_x = K[X_1, \dots, X_n]$ and $S_y = K[Y_1, \dots, Y_m]$ to be standard graded polynomial rings. A will always denote a standard graded algebra of the form $A = S_x/Q$ where $Q \subset S_x$ is a graded ideal. We write $\mathfrak{m}$ for the graded maximal ideal of A.

Let $I \subset A$ be a graded ideal which is minimally generated by homogeneous elements $f_1, f_2, \dots, f_m$ of the same degree d. Recall that the Rees ring $R(I) = A[IT]$ of I admits a standard bigrading by assigning the degree $(1,0)$ to the generators of $\mathfrak{m} \subset A$ and by setting $\deg(f_iT) = (0,1)$ for $i = 1, \dots, m$. As a consequence of Theorem 4.2.1 we observe:

Corollary 4.3.1. *If $R(I)$ is Koszul, then A is Koszul and the ideal I^j has a linear A-resolution for all $j \geq 0$.*

Proof. Let Δ be the $(1,0)$-diagonal. Then $R(I)_\Delta = A$ and $I^j = R(I)_\Delta^{(0,j)}(-dj)$. Thus, by Corollary 4.2.2 the ideal I^j has a linear A-resolution. $\square$

The converse of Corollary 4.3.1 is not true. To give a counter example we use the well-known fact that an ideal $I \subset S_x$ which is generated in one degree by a stable set of monomials has a linear resolution (see [**28**]). Since all powers I^j are also generated by a stable set of monomials, the results in [**28**] imply:

Proposition 4.3.2. *Let $I \subset S_x$ be an ideal which is generated by a stable set $G(I)$ of monomials which all have the same degree. Then I^j has a linear resolution for all $j \geq 1$.*

The semigroup ring $K[G(I)]$ is the $(0,1)$-diagonal of the Rees algebra $R(I)$. Conca and De Negri have communicated several examples where I has the form as in Proposition 4.3.2, but the defining ideal of $K[G(I)]$ is not quadratic and therefore $K[G(I)]$ is not Koszul. In this case the algebra $R(I)$ is not Koszul by Theorem 4.2.1. A concrete example is ([**18**]):

Example 4.3.3. Let $S_x = K[X_1, X_2, \dots, X_5]$ and

$$\begin{aligned} I \;=\; & (X_1^3, X_1^2X_2, X_1X_2^2, X_2^3, X_1^2X_3, X_1X_2X_3, X_2^2X_3, X_1X_3^2, X_2X_3^2, X_3^3, X_1^2X_4, \\ & X_1X_2X_4, X_2^2X_4, X_1X_3X_4, X_2X_3X_4, X_1^2X_5, X_1X_2X_5, X_2^2X_5, X_1X_4^2, X_1X_3X_5). \end{aligned}$$

Then $R(I)$ is not Koszul, but I^j has a linear resolution for all $j \geq 1$.

If $\mathfrak{m} \subset A$ is the graded maximal ideal of a Koszul algebra A, then the Rees algebra $R(\mathfrak{m})$ is always Koszul because it is a Segre product of two Koszul algebras. In [**34**] Herzog, Popescu and Trung have proved that the defining ideal of $R(\mathfrak{m})$ has a quadratic Gröbner basis provided the defining ideal Q has a quadratic Gröbner basis.

We study a class of ideals for which all powers have linear resolutions. These ideals arise from integral polymatroids (see Section 1.7). Extending the notion of matroidal ideals in [**24**] and [**48**], we set:

Definition 4.3.4. Let P be an integral polymatroid on $[n]$ with bases $\mathcal{B}(P)$. Then $I_{\mathcal{B}(P)} \subset S_x$ denotes the ideal minimally generated by the monomials whose support forms a basis of P, that is

$$I_{\mathcal{B}(P)} = (u \in S_x \colon \operatorname{supp}(u) \in \mathcal{B}(P)).$$

A monomial ideal $I \subset S_x$ is said to be *polymatroidal* if $I = I_{\mathcal{B}(P)}$ for some integral polymatroid P.

In the case that I is generated by square-free monomials the definition above is equivalent to I being a matroidal. It is well-known that matroidal ideals have linear resolutions (see [**24**, Proposition 7]). The minimal graded free resolution of matroidal ideals is studied in [**48**]. We obtain the follwoing statement.

Proposition 4.3.5. *If $I \subset S_x$ is a polymatroidal ideal, then I^j has a linear resolution for all $j \geq 1$.*

For the proof we need the following lemma. Let $1 \neq u \in S_x$ be a monomial. We set $u(i) = \max\{l \colon X_i^l \text{ divides } u\}$.

Lemma 4.3.6. *Let $I \subset S_x$ be a monomial ideal minimally generated by $G(I) = \{u_1, \dots, u_m\}$ where $\deg(u_i) = d$ for $i = 1, \dots, m$. Moreover, assume that $F = S_x(-d)^m$ is the free module with basis $e_1, \dots, e_m$ and that $\varphi : F \to I$ is the presentation of I with $\varphi(e_i) = u_i$ for $i = 1, \dots, m$. Suppose that $G(I)$ satisfies the condition:*

(∗) *For $u, u' \in G(I)$ with $u(j) > u'(j)$ there exists an element $i \in [n]$ such that $u'(i) > u(i)$ and $uX_i/X_j \in G(I)$.*

Then $G = \{X_i e_s - X_j e_t \colon X_i u_s = X_j u_t \text{ for some } s, t \in [m]\}$ is a Gröbner basis for $\ker(\varphi) = \Omega_1(I)$ with respect to some monomial order. In particular, I has a linear S_x-resolution.

Proof. The first syzygy module $\Omega_1(I) = \ker(\varphi)$ is the submodule of F generated by all relations of the form

$$f = X_{i_1} \cdots X_{i_r} e_s - X_{j_1} \cdots X_{j_r} e_t \tag{3}$$

where $X_{i_1} \cdots X_{i_r} u_s = X_{j_1} \cdots X_{j_r} u_t$ for some $s, t \in [m]$ (see [**26**]). We may always assume that $X_{i_1} \cdots X_{i_r}$ and $X_{j_1} \cdots X_{j_r}$ have no common factor and that $i_1 \le \dots \le i_r$ and $j_1 \le \dots \le j_r$. Let $<_{\text{rlex}}$ denote the reverse lexicographic term order on S_x induced by $X_1 < X_2 < \dots < X_n$. We define a term order $<$ on F by $ve_i > we_j$ if $v >_{\text{rlex}} w$ or if $v = w$ and $i < j$. We will show that the set

$$G = \{X_i e_s - X_j e_t \colon X_i u_s = X_j u_t \text{ for some } s, t \in [m]\}$$

is a Gröbner basis for $\Omega_1(I)$ with respect to $<$. Then $\text{in}(\Omega_1(I))$ satisfies the hypothesis of Lemma 1.4.9 which gives the second assertion.

Let $f \in \Omega_1(I)$ be a relation as in (3) with $\text{in}(f) = X_{i_1} \cdots X_{i_r} e_s$. By the chosen monomial order we have $i_l > j_1$ for all $l = 1, \dots, r$. Since $X_{i_1} \cdots X_{i_r} u_s = X_{j_1} \cdots X_{j_r} u_t$, we get $u_s(j_1) > u_t(j_1)$. By condition (∗) there exists an element $i \in [n]$ such that $u_t(i) > u_s(i)$ and a generator $u_p \in G(I)$ with $u_s X_i = u_p X_{j_1}$. It follows that $i = i_l$ for some $l \in [r]$. Then $g = X_{i_l} e_s - X_{j_1} e_p$ belongs to G and $\text{in}(g) = i_l e_s$ divides $\text{in}(f)$. Thus G is a Gröbner basis for $\Omega_1(I)$. □

Proof of Proposition 4.3.5. By Proposition 1.7.2 the generators of I^j satisfy the hypothesis of Lemma 4.3.6 which gives the assertion. □

As a direct consequence of Theorem 1.7.3 we obtain:

Corollary 4.3.7. *Let $I = I_{\mathcal{B}(M_1)} I_{\mathcal{B}(M_2)} \cdots I_{\mathcal{B}(M_k)}$ be a product of matroidal ideals. Then I is polymatroidal and I^j has a linear resolution for all $j \ge 1$.*

To illustrate the result above we give an example.

Example 4.3.8. Let $I_1, \dots, I_k \subset S_x$ be ideals which are generated by a subset of the variables. Then every ideal $I_j = I_{\mathcal{B}(M_j)}$ is a matroidal ideal where M_j is a rank-1 matroid for $j = 1, \dots, k$. By Corollary 4.3.7 we get that all powers of the ideal $I = I_1 I_2 \cdots I_k$ have a linear resolution.

Conca and Herzog have shown with different methods that the same statement holds provided all ideals I_j are generated by arbitrary linear forms (see [**19**]).

We study a class of monomial ideals for which the Rees algebra is Koszul.

Theorem 4.3.9. *Let M be a base-sortable matroid on $[n]$ and $I_{\mathcal{B}(M)} \subset S_x$ the corresponding matroidal ideal. Then the defining ideal of the Rees algebra $R(I_{\mathcal{B}(M)})$ has a quadratic Gröbner basis. In particular, $R(I_{\mathcal{B}(M)})$ is Koszul.*

Proof. Let $\mathcal{B}(M) = \{B_1, \dots, B_m\}$ be the collection of bases of M and let $S = K[X_1, \dots, X_n, Y_{B_1}, \dots, Y_{B_m}]$ be standard bigraded polynomial ring. Define $\varphi : S \to R(I_{\mathcal{B}(M)})$ as the epimorphism with $\varphi(X_i) = X_i$ for $i = 1, \dots, n$ and $\varphi(Y_{B_j}) = \alpha_j T$ for $j = 1, \dots, m$, where $\alpha_j = \prod_{k \in B_j} X_k$ is the square-free monomial associated to the basis B_j of M. The ideal $J = \ker(\varphi)$ is generated by bihomogeneous binomials of the form $u_1 v_1 - u_2 v_2$ with monomials $u_1, u_2 \in S_x$ and $v_1, v_2 \in S_y$ such that $\varphi(u_1 v_1) = \varphi(u_2 v_2)$ (see Section 1.8). Since M is base-sortable, there exists a term order $<_{\text{sort}}$ on S_y such that $J \cap S_y$ has a quadratic Gröbner basis G_1 (see Proposition 3.2.1). Let $<_{\text{lex}}$ denote the lexicographic term order on S_x induced by $X_1 > \dots > X_n$. We define a term order on S by $u_1 v_1 > u_2 v_2$ if $u_1 >_{\text{lex}} u_2$ or if $u_1 = u_2$ and $v_1 >_{\text{sort}} v_2$. We set

$$G_2 = \{X_i Y_{B_s} - X_j Y_{B_t} \colon B_s \cup i = B_t \cup j \text{ and } i \neq j\}$$

and show that $G = G_1 \cup G_2$ is a Gröbner basis for J with respect to $<$.

Let $f = u_1 v_1 - u_2 v_2$ be a binomial such that $\varphi(u_1 v_1) = \varphi(u_2 v_2)$. It follows that $s = \deg(v_1) = \deg(v_2) > 0$. We may assume that u_1 and u_2 (resp. v_1 and v_2) have no common factor. Let $\text{in}(f) = u_1 v_1$. We write $v_1 = Y_{B_{k_1}} Y_{B_{k_2}} \cdots Y_{B_{k_s}}$ with $k_1 \leq \dots \leq k_s$ and $v_2 = Y_{B_{l_1}} Y_{B_{l_2}} \cdots Y_{B_{l_s}}$ with $l_1 \leq \dots \leq l_s$. If the matrix $A = [B_{k_1}, \dots, B_{k_s}]^t$, which is associated to v_1, is not sorted, then $v_1 \in \text{in}(J \cap S_y)$ and there exists an element $g_1 \in G_1$ such that $\text{in}(g_1)$ divides v_1 (see Proposition 3.2.1) and therefore also $\text{in}(f)$.

We may now assume that the matrices associated to v_1 and v_2 are both sorted and that $r = \deg(u_1) = \deg(u_2) > 0$. We write $u_1 = X_{i_1} X_{i_2} \cdots X_{i_r}$ with $i_1 \leq \dots \leq i_r$ and $u_2 = X_{j_1} X_{j_2} \cdots X_{j_r}$ with $j_1 \leq \dots \leq j_r$. By the chosen term order we have $i_1 < j_l$ for all $l = 1, \dots, r$. There are indices $p, q \in [s]$ such that $i_1 \in B_{l_p} \setminus B_{k_q}$. Since both associated matrices $[B_{k_1}, \dots, B_{k_s}]^t$ and $[B_{l_1}, \dots, B_{l_s}]^t$ are sorted, i_1 is the least element of the symmetric difference $B_{l_p} \Delta B_{k_q}$. The dual exchange property for matroids (see Proposition 1.6.2) implies that there exists an element $j \in B_{k_q} \setminus B_{l_p}$ such that $B' = (B_{k_q} - j) \cup i_1$ is a basis of M. Thus the relation $g = X_{i_1} Y_{B_{k_q}} - X_j Y_{B'}$ belongs to G_2 and $\text{in}(g) = X_{i_1} Y_{B_{k_q}}$ because $i_1 < j$. Therefore $\text{in}(g)$ divides $\text{in}(f)$ which concludes the proof. □

4.4. Symmetric algebras

In this section we present some applications which arise naturally in the study of symmetric algebras. Let M be a graded A-module with homogeneous generators $f_1, \dots, f_m$ and let (a_{ij}) be the corresponding relation matrix of size $t \times m$. The symmetric algebra

$$S(M) = \bigoplus_{j \geq 0} S^j(M)$$

of M has a presentation of the form $S(M) = A[Y_1, \dots, Y_m]/J$ where $J = (g_1, \dots, g_t)$ and $g_i = \sum_{j=1}^m a_{ij} Y_j$ for $i = 1, \dots, t$. If $f_1, \dots, f_m$ have the same degree, then $S(M)$

is standard bigraded by assigning the degree $(1,0)$ to the residue class of X_i for $i = 1, \dots, n$ and by setting $\deg(Y_i) = (0,1)$. Note that $S^j(M)$ is a graded A-module. As an application of the main result of this chapter we obtain:

Corollary 4.4.1. *If $S(M)$ is Koszul, then A is Koszul and the module $S^j(M)$ has a linear resolution for all $j \geq 0$.*

Proof. Let Δ be the $(1,0)$-diagonal. Then $S(M)_\Delta = A$, and $S^j(M) = S(M)_\Delta^{(0,j)}$. Thus A is Koszul and, by Corollary 4.2.5 the module $S^j(M)$ has a linear A-resolution. □

As one might expect it seems to be a strong condition that the symmetric algebra $S(M)$ is Koszul. In a more specific case however, when $M = \mathfrak{m}$ is the graded maximal ideal of a Koszul algebra, we have a sufficient condition.

Theorem 4.4.2. *Let $A = S_x/Q$ and K be an infinite field with* $\operatorname{char}(K) \neq 2$. *If Q has a 2-linear resolution over S_x, then the defining ideal of $S(\mathfrak{m})$ has a quadratic Gröbner basis with respect to a reverse lexicographic term order. In particular, $S(\mathfrak{m})$ is Koszul.*

To show that the defining ideal of $S(\mathfrak{m})$ has a quadratic Gröbner basis we need some notation taken from [**34**]. Let $S = K[X_1, \dots, X_n, Y_1, \dots, Y_n]$ be the standard bigraded polynomial ring and $f = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_d \leq n} a_{i_1 i_2 \cdots i_d} X_{i_1} X_{i_2} \cdots X_{i_d}$ a form of degree $(d, 0)$. We set

$$f^{(k)} = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_d \leq n} a_{i_1 i_2 \cdots i_d} X_{i_1} X_{i_2} \cdots X_{i_{d-k}} Y_{i_{d-k+1}} \cdots Y_{i_d}$$

for $k = 0, \dots, d$. Note that $f^{(k)}$ is bihomogeneous of degree $(d-k, k)$. Moreover, let $\delta_{ij} = X_i Y_j - X_j Y_i$ for $i \neq j$ and $L = \{\delta_{ij} \colon i \neq j\}$. We need the following lemma.

Lemma 4.4.3. *Let $<$ denote the reverse lexicographic term order on S induced by $X_1 > X_2 > \dots > X_n > Y_1 > \dots > Y_n$ and $\varphi : S \to S$ be the homomorphism with $\varphi(X_i) = X_i$ and $\varphi(Y_i) = X_i$ for $i = 1, \dots, n$. Assume that $f \in S$ is a bihomogeneous polynomial of degree (s,t) such that* $\operatorname{in}(f) = X_{i_1} X_{i_2} \cdots X_{i_s} Y_{j_1} \cdots Y_{j_t}$ *satisfies $i_1 \leq i_2 \leq \dots \leq i_s \leq j_1 \leq \dots \leq j_t$. Then* $\operatorname{in}(\varphi(f)) = \varphi(\operatorname{in}(f))$ *and* $\operatorname{in}(f) = \varphi(\operatorname{in}(f))^{(t)}$.

Proof. With the condition $i_1 \leq i_2 \leq \dots \leq i_s \leq j_1 \leq \dots \leq j_t$ it is easy to see that $\varphi(\operatorname{in}(f)) > \varphi(v)$ for all monomials v of f with $v < \operatorname{in}(f)$. □

Proof of Theorem 4.4.2. By Lemma 1.4.16 we may assume that the defining ideal Q of A has a quadratic Gröbner basis $g_1, \dots, g_t$ with respect to the reverse lexicographic term order induced by $X_1 > X_2 > \dots > X_n$ and that $\operatorname{in}(Q)$ satisfies the condition in 1.4.16. It is easy to see that $S(\mathfrak{m})$ has a presentation $S(\mathfrak{m}) = S/J$ where $J = (g_1, \dots, g_t, g_1^{(1)}, \dots, g_t^{(1)}, L)$.

Let $<$ denote the reverse lexicographic term order on S induced by $X_1 > X_2 > \dots > X_n > Y_1 > \dots > Y_n$. We will show that the set $G = \{g_1, \dots, g_t, g_1^{(1)}, \dots, g_t^{(1)}\} \cup L$ is a Gröbner basis for J with respect to $<$ which concludes the proof of the theorem.

Let $f \in J$ be a bihomogeneous polynomial of degree (s,t). Then $s \geq 1$. We show that $\mathrm{in}(f)$ is divided by some $\mathrm{in}(g)$ with $g \in G$. Let $\mathrm{in}(f) = X_{i_1}X_{i_2}\cdots X_{i_s}Y_{j_1}\cdots Y_{j_t}$ where $i_1 \leq i_2 \leq \ldots \leq i_s$ and $j_1 \leq j_2 \leq \ldots \leq j_t$. If there exist indices p,q such that $i_p > j_q$, then $\mathrm{in}(\delta_{i_pj_q})$ divides $\mathrm{in}(f)$ which is the claim.

Otherwise we have $i_1 \leq i_2 \leq \ldots \leq i_s \leq j_1 \leq j_2 \leq \ldots \leq j_t$. Let φ denote the homomorphism from Lemma 4.4.3. Since $f \in J$, it follows that $\varphi(f) \in Q$. By Lemma 4.4.3 we have $\mathrm{in}(f) = \mathrm{in}(\varphi(f))^{(t)}$ where $\mathrm{in}(\varphi(f)) = X_{i_1}X_{i_2}\cdots X_{i_s}X_{j_1}\cdots X_{j_t}$. Since $g_1,\ldots,g_t$ is a Gröbner basis for Q, there exists a a polynomial $g \in \{g_1,\ldots,g_t\}$ such that $\mathrm{in}(g)$ divides $\mathrm{in}(\varphi(f))$. By the condition in Proposition 1.4.16 we may assume that $\mathrm{in}(g) = X_{i_1}X_{i_2}$ if $s > 1$, or $\mathrm{in}(g) = X_{i_1}X_{j_1}$ if $s = 1$. Now $\mathrm{in}(g)$ or $\mathrm{in}(g^{(1)})$ divides $\mathrm{in}(f)$. □

Under the strong assumption of Theorem 4.4.2 it follows from Corollary 4.4.1 that $S^j(\mathfrak{m})$ has a linear resolution for all $j \geq 1$. Actually we have:

Proposition 4.4.4. Let $j \geq 1$. If A is Koszul, then $S^j(\mathfrak{m})$ has a linear A-resolution.

In the proof we use results from [**36**] and some basic facts about the Koszul complex (see [**15**, Section 1.6] for details).

Proof. Let $A = S_x/Q$. We may assume that the defining ideal Q of A does not contain linear forms. Then Q is generated in degree 2. Let $\mathfrak{m} = (x_1,\ldots,x_n) \subset A$ be the graded maximal ideal of A. We denote the Koszul complex of the sequence $x_1,\ldots,x_n \in A$ with $\mathcal{K}_{\bullet}$. Let $H_1(\mathcal{K}_{\bullet})$ be the first homology group of this complex. Recall that $S(\mathfrak{m}) = A[Y_1,\ldots,Y_n]/J$ for some bihomogeneous ideal J and that $S^j(\mathfrak{m})$ is generated by the residue classes of all monomials in degree $(0,j)$. We consider $S^j(\mathfrak{m})$ as an A-module generated in degree j. For $j \geq 1$, there exists the downgrading homomorphism $\alpha_j : S^j(\mathfrak{m}) \to \mathfrak{m}S^{j-1}(\mathfrak{m})$ which maps a residue class of $Y_{i_1}Y_{i_2}\ldots Y_{i_j}$ to the residue class of $X_{i_1}Y_{i_2}\ldots Y_{i_j}$ (see [**36**, Section 2]). Note that it does not matter which of the factors Y_{i_l} is replaced by X_{i_l}.

To show that $S^j(\mathfrak{m})$ has a linear resolution for all $j \geq 1$ we use induction on j. For $j = 1$, we have $S^1(\mathfrak{m}) = \mathfrak{m}$ which has a linear resolution because A is Koszul. Let now $j > 1$. We have the short exact sequence

$$0 \to U \to S^j(\mathfrak{m}) \overset{\alpha_j}{\to} \mathfrak{m}S^{j-1}(\mathfrak{m}) \to 0 \tag{4}$$

where $U = \ker \alpha_j$. By [**36**, Lemma 2.2] U is a subquotient of the module $N = H_1(\mathcal{K}_{\bullet}) \otimes_{A/\mathfrak{m}} [(A/\mathfrak{m})(-j+2)]^s$ for some integer $s \geq 1$. The module N is annihilated by $\mathfrak{m}$. Since Q is generated in degree 2, it follows that $H_1(\mathcal{K}_{\bullet}) \cong \mathrm{Tor}_1^{S_x}(A,K)$ is generated in degree 2. Therefore, $U \cong K(-j)^t$ for some integer $t \geq 0$ and U has a j-linear A-resolution because A is Koszul. By the induction hypothesis $S^{j-1}(\mathfrak{m})$ has a $(j-1)$-linear A-resolution. Thus by [**22**, Lemma 6.4] the module $\mathfrak{m}S^{j-1}(\mathfrak{m})$ has a j-linear A-resolution. The assertion follows when we apply the long exact sequence of the functor $\mathrm{Tor}_{\bullet}^A(-,K)$ to the sequence (4). □

The hypothesis of Theorem 4.4.2 cannot be weakened to the assumption that A is only Koszul. A counter example is the algebra $A = K[X_1,X_2]/(X_1^2,X_2^2)$. As a complete intersection A is Koszul, but with the help of the program MACAULAY2 we find that $S(\mathfrak{m})$ is not Koszul. This example shows also that the converse of Corollary 4.4.1 is false because by Proposition 4.4.4 all symmetric powers $S^j(\mathfrak{m})$ have a linear resolution.

We study the symmetric powers of polymatroidal ideals. We observe:

Proposition 4.4.5. *If $I \subset S_x$ is a polymatroidal ideal, then $S^p(I)$ has a linear resolution for all $p \geq 1$.*

Proof. Let $G(I) = \{u_1, \dots, u_m\}$. With the notation of Lemma 4.3.6 we know that

$$G = \{X_i e_s - X_j e_t \colon X_i u_s = X_j u_t \text{ for some } s, t \in [m]\}$$

is a Gröbner basis for $\Omega_1(I)$ with respect to some monomial order. Therefore $S(I)$ has the form $S(I) = S/J$ where $S = K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ and

$$J = (X_i Y_s - X_j Y_t \colon X_i u_s = X_j u_t \text{ for some } s, t \in [m]).$$

Let $F = \bigoplus_{|\alpha|=p} S_x e^\alpha$ be the free S-module with the basis consisting of all monomials e^α in the variables $e_1, \dots, e_m$ with $\deg(e^\alpha) = p$ and $\varphi : F \to S^p(I)$ be the presentation with $\varphi(e^\alpha) = Y^\alpha + J$. Then the syzygy module $\Omega_1(S^p(I)) \subset F$ is generated by the set

$$H = \{X_i e^\alpha e_s - X_j e^\alpha e_t \colon X_i u_s = X_j u_t \text{ with } s, t \in [m] \text{ and } |\alpha| = p - 1\}.$$

Let $<_{\text{rlex}}$ denote the reverse lexicographic term order on S_x induced by $X_1 < X_2 < \dots < X_n$ and $\prec$ be a degree refining monomial order on $K[e_1, \dots, e_m]$ such that $e_1 > \dots > e_m$. Extending the order in the proof of Lemma 4.3.6 we define a term order on F by $ve^\alpha > we^\beta$ if $v >_{\text{rlex}} w$ or if $v = w$ and $e^\alpha >_y e^\beta$.

We will show that H is a Gröbner basis for $\Omega_1(S^p(I))$ which concludes the proof by Lemma 1.4.9.

We already know this for $S^1(I) = I$. For $p > 1$ we use the Buchberger criterion (see Theorem 1.4.7). Let $h_1 = X_{i_1} e^\alpha e_s - X_{j_1} e^\alpha e_t$ and $h_2 = X_{i_2} e^\beta e_u - X_{j_2} e^\beta e_v$ with $|\alpha|, |\beta| = p - 1$ and with $i_1 > j_1$, $i_2 > j_2$ be two elements in H. It is $\text{in}(h_1) = X_{i_1} e^\alpha e_s$ and $\text{in}(h_2) = X_{i_2} e^\beta e_t$. We have to compute a standard expression for the S-pair $S(h_1, h_2)$ provided the initial terms involve the same basis elements, that is $e^\alpha e_s = e^\beta e_u$. If $i_1 = i_2$, then $S(h_1, h_2) = X_{j_1} e^\alpha e_t - X_{j_2} e^\beta e_v$ is a standard expression by the chosen term order. We may now assume that $i_1 \neq i_2$.

First we consider the case $s = u$. It follows that $e^\alpha = e^\beta$. The elements $g_1 = X_{i_1} e_s - X_{j_1} e_t$ and $g_2 = X_{i_2} e_u - X_{j_2} e_v$ belong to G. Since G is a Gröbner basis for $\Omega_1(I)$, we have a standard expression $S(g_1, g_2) = \sum_{k=1}^{l} h_k g_k$ with $g_k \in G$. With respect to the chosen term order for F the presentation $S(g_1, g_2) = \sum_{k=1}^{l} h_k (e^\alpha g_k)$ is standard and $e^\alpha g_k \in H$ for all k.

Let now $s \neq u$. Then e_s divides e^β and e_u divides e^α. Let $e^\gamma = e^\alpha / e_u$. We have

$$S(h_1, h_2) = X_{j_1} X_{i_2} e^\gamma e_u e_t - X_{i_1} X_{j_2} e^\gamma e_s e_v = X_{j_1} h_3 - X_{j_2} h_4 \tag{5}$$

where $h_3 = X_{i_2}(e^\gamma e_t) e_u - X_{j_2}(e^\gamma e_t) e_v$ and $h_4 = X_{i_1}(e^\gamma e_v) e_s - X_{j_1}(e^\gamma e_v) e_t$. The elements h_3 and h_4 belong to H. Since $\text{in}(X_{j_1} h_3) = X_{j_1} X_{i_2} e^\gamma e_t e_u$ and $\text{in}(X_{j_2} h_4) = X_{j_2} X_{i_1} e^\gamma e_v e_s$, the presentation in (5) is standard. This concludes the proof. □

4.5. Further applications

As direct consequences of Theorem 4.2.1 we get some known facts about standard graded Koszul algebras. Let $d \geq 1$ be an integer. In Chapter 1 we have defined the d^{th} Veronese subring $A^{(d)}$ of A and the tensor product of two standard graded K-algebras A and B which has the structure of a standard bigraded algebra, $A \otimes_K B = \bigoplus_{i,j \geq 0} A_i \otimes_K B_j$. The Segre product of A and B, denoted with $A * B$, is the $(1,1)$-diagonal of $A \otimes_K B$. We recover some well-known results (see Theorem 1.3.7).

Corollary 4.5.1. *Tensor products, Segre products and Veronese subrings of Koszul algebras are Koszul.*

Proof. Let $F_\bullet$ and $G_\bullet$ be the minimal graded free resolution of K over A and B respectively. Then the tensor product $G_\bullet \otimes_K F_\bullet$ forms the minimal graded free resolution of K over $A \otimes_K B$. Thus, if A and B are Koszul, $A \otimes_K B$ is Koszul. Now the Segre product $A * B$ is the $(1,1)$-diagonal $A \otimes_K B$ which is Koszul by Theorem 4.2.1.

Let A be a positively graded Koszul algebra and consider A as a standard bigraded algebra where all generators have degree $(1,0)$. Then $A^{(d)}$ is a diagonal of A and by Theorem 4.2.1 $A^{(d)}$ is Koszul. □

Let A be an arbitrary standard graded algebra and $M \in \mathcal{M}_{\mathbb{Z}}(A)$. Recall from [**3**] that the rate of M is given by $\operatorname{rate}_A(M) = \sup\{\frac{t_i(M)}{i} \colon i \geq 0\}$. A similar definition can be found in [**9**] where Backelin proves that $A^{(d)}$ is Koszul for $d \gg 0$. Note that an A-module M is naturally an $A^{(d)}$-module. Aramova, Barcanescu and Herzog have proved in [**3**] that

$$\operatorname{rate}_{A^{(d)}}(M) \leq \lceil \operatorname{rate}_A(M)/d \rceil \tag{6}$$

for an arbitrary K-algebra A and all $d \geq c$ where c is a constant depending on A. Moreover, they showed that $c = 1$ if A is a polynomial ring. For this, they used that the relative Veronese modules $A^{(d,j)} = \bigoplus_{i \geq 0} A_{id+j}$ for $j = 0, \dots, d-1$ have linear A-resolutions. Since the relative Veronese modules coincide with sidediagonal modules, it follows from Corollary 4.2.3 that (6) is valid for $d \geq 1$ provided A is Koszul. We get similar upper bounds for the regularity over Koszul algebras.

Corollary 4.5.2. *Let A be Koszul and $M \in \mathcal{M}_{\mathbb{Z}}(A)$. Then*

$$\operatorname{reg}_{A^{(d)}}(M) \leq \lceil \operatorname{reg}_A(M)/d \rceil$$

for all $d \geq 1$. In particular, $\operatorname{reg}_{A^{(d)}}(M) \leq 1$ if $d \geq \operatorname{reg}_A(M)$.

Proof. Consider A as a bigraded algebra generated in degree $(1,0)$. Let Δ be the $(d,0)$-diagonal of A. Then $A^{(d)} = A_\Delta$ and, as an $A^{(d)}$-module, we have $M = \bigoplus_{c=0}^{d-1} M_\Delta^{(c,0)}$. By Theorem 4.2.6 the claim follows. □

Finally, we study the consequences of Theorem 4.2.1 for bigraded semigroup rings. The following definition is an analogon to Definition 1.8.1.

Definition 4.5.3. Let $\Lambda \subset \mathbb{N}^d$ be a finitely generated semigroup. We call Λ *standard bigraded* if

(a) Λ is the disjoint union $\bigcup_{i,j\geq 0} \Lambda_{(i,j)}$,
(b) $\Lambda_{(0,0)} = 0$, $\Lambda_{(i,j)} + \Lambda_{(k,l)} \subset \Lambda_{(i+k,j+l)}$ for all integers $i, j, k, l \geq 0$, and
(c) Λ is generated by elements of $\Lambda_{(1,0)}$ and $\Lambda_{(0,1)}$.

We call the elements of $\Lambda_{(i,j)}$ bihomogeneous of degree (i, j). Let Λ be a standard bigraded semigroup which is minimally generated by $\alpha_1, \ldots, \alpha_n \in \Lambda_{(1,0)}$ and $\beta_1, \ldots, \beta_m \in \Lambda_{(0,1)}$ and let $K[T_1, \ldots, T_d]$ denote the polynomial ring. To a semigroup element $\lambda = (a_1, \ldots, a_d) \in \Lambda$ we associate the monomial $T^\lambda = T_1^{a_1} T_2^{a_2} \cdots T_d^{a_d}$. Recall that the semigroup ring $K[\Lambda]$ is the K-algebra generated by the monomials $T^{\alpha_i}, T^{\beta_j}$ where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let $\varphi : S \to K[\Lambda]$ be the epimorphism with $\varphi(X_i) = T^{\alpha_i}$ and $\varphi(Y_j) = T^{\beta_j}$. Set $J = \ker(\varphi)$ to be the toric ideal of the semigroup ring $K[\Lambda]$. If Λ is bigraded, then $K[\Lambda] = S/J$ is a standard bigraded algebra.

Example 4.5.4. If $I \subset S_x$ is a monomial ideal generated in one degree, then the Rees algebras $R(I)$ is a standard bigraded semigroup ring.

Let Λ be a bigraded semigroup. In analogy to the definition for graded K-algebras we set

$$\Lambda_\Delta = \bigcup_{i\geq 0} \Lambda_{(ia,ib)} \quad \text{and} \quad \Lambda_{\tilde{\Delta}} = \bigcup_{i,j\geq 0} \Lambda_{(ia,jb)}$$

for the (a, b)-diagonal Δ and the (a, b)-Veronese set $\tilde{\Delta}$ respectively. Note that Λ_Δ is graded and partially ordered by the induced ordering. If $\lambda \in \Lambda_{(ia,ib)}$, then we use $(\Gamma_\lambda)_\Delta$ for the order complex of the induced open interval $(0, \lambda) \subset \Lambda_\Delta$. Similarly, we define $(\Gamma_\lambda)_{\tilde{\Delta}}$ for $\lambda \in \Lambda_{(ia,jb)}$. Finally, we reformulate our main result for semigroup rings. By Proposition 1.8.3 the Koszul property of a semigroup ring is equivalent to the Cohen-Macaulay property of the finite divisor intervals. Therefore, we obtain:

Corollary 4.5.5. *Let $\Lambda \subset \mathbb{N}^d$ be a bigraded semigroup, Δ a diagonal and $\tilde{\Delta}$ a Veronese set. If Γ_λ is Cohen-Macaulay for all $\lambda \in \Lambda$, then:*
(a) *$(\Gamma_\lambda)_\Delta$ is Cohen-Macaulay for all $\lambda \in \Lambda_\Delta$.*
(b) *$(\Gamma_\lambda)_{\tilde{\Delta}}$ is Cohen-Macaulay for all $\lambda \in \Lambda_{\tilde{\Delta}}$.*

Bibliography

[1] D. Anick: On the homology of associative algebras, *Trans. Amer. Math. Soc.* **296** (1986), 641-659.

[2] L. L. Avramov: Infinite free resolutions, *in Six lectures in commutative algebra (Bellaterra, 1996),* Progr. Math. **166**, *Birkhäuser Verlag* (1998), 1-118.

[3] A. Aramova, S. Barcanescu, J. Herzog: On the rate of relative Veronese submodules, *Rev. Roumaine Math. Pures Appl.* **40** (1995), no. 3-4, 243-251.

[4] A. Aramova, K. Crona, E. De Negri: Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions, *J. Pure Appl. Algebra* **150** (2000), no. 3, 215-235.

[5] A. Aramova, J. Herzog, T. Hibi: Shellability of semigroup rings, to appear in *European J. Combin.* (1998).

[6] A. Aramova, J. Herzog, T. Hibi: Finite lattices and lexicographic Gröbner bases, *European J. Combin.* **21** (2000), no. 4, 431-439.

[7] L. L. Avramov, D. Eisenbud: Regularity of modules over a Koszul algebra, *J. Algebra* **153** (1992), no. 1, 85-90.

[8] L. L. Avramov, I. Peeva: Finite regularity and Koszul algebras, *Amer. J. Math.* **123** (2001).

[9] J. Backelin: On the rates of growth of the homologies of Veronese subrings, *in Algebra, algebraic topology and their interactions (Stockholm, 1983),* Lecture Notes in Math., **1183**, *Springer, Berlin-New York* (1986), 79-100.

[10] J. Backelin, R. Fröberg: Koszul algebras, Veronese subrings and rings with linear resolutions, *Rev. Roumaine Math. Pures Appl.* **30** (1985), no. 2, 85-97.

[11] S. Blum: Natürliche Schälbarkeit homogen erzeugter Halbgruppenringe, *Diplomarbeit, University of Essen* (1999).

[12] S. Blum: Initially Koszul algebras, *Beiträge Algebra Geom.* **41** (2000), 455-467.

[13] S. Blum: Base-sortable matroids and Koszulness of semigroup rings, *Preprint* (2000).

[14] S. Blum: Subalgebras of bigraded Koszul algebras, *Preprint* (2000).

[15] W. Bruns, J. Herzog: Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics **39**, *Cambridge University Press, Cambridge* (1993).

[16] W. Bruns, J. Herzog, U. Vetter: Syzygies and walks, *in Commutative algebra (Trieste 1992), World Sci. Publishing, River Edge, NJ* (1994), 36-57.

[17] A. Conca: Universally Koszul algebras, *Math. Ann.* **317** (2000), no. 2, 329-346.

[18] A. Conca, E. De Negri: personal communication (2000).

[19] A. Conca, J. Herzog: personal communication (2001).

[20] A. Conca, M. E. Rossi, G. Valla: Gröbner flags and Gorenstein algebras, *Preprint* (1999).

[21] A. Conca, N. V. Trung, G. Valla: Koszul property for points in projective spaces, to appear in *Math. Scand.* (1998).

[22] A. Conca, J. Herzog, N.V. Trung, G. Valla: Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces, *Amer. J. Math.* **119** (1997), no. 4, 859-901.

[23] E. De Negri: Toric rings generated by special stable sets of monomials, *Math. Nachr.* **203** (1999), 31-45.

[24] J. A. Eagon, V. Reiner: Resolutions of Stanley-Reisner rings and Alexander duality, *J. Pure Appl. Algebra* **130** (1998), no. 3, 265-275.

[25] J. Edmonds: Submodular functions, matroids and certain polyhedra, *Proc. Int. Conf. on Combinatorics (Calgary), Gordon and Breach, New York,* (1970), 69-87.

[26] D. Eisenbud: Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics, **150**, *Springer-Verlag, New York* (1995).

[27] D. Eisenbud, A. Reeves, B. Totaro: Initial Ideals, Veronese subrings and rates of algebras, *Adv. Math.* **109** (1994) 168-187.

[28] S. Eliahou, M. Kervaire: Minimal resolutions of some monomial ideals, *J. Algebra* **129** (1990), no. 1, 1-25.

[29] R. Fröberg: Koszul algebras, *Advances in commutative ring theory (Fez, 1997),* Lecture Notes in Pure and Appl. Math. **205**, *Dekker, New York* (1999), 337-350.

[30] M. L. Green: Generic initial ideals, *in Six lectures in commutative algebra (Bellaterra, 1996),* Progr. Math. **166**, *Birkhäuser Verlag* (1998), 119-186.

[31] D. Grayson, M. Stillman: MACAULAY2, a software system for algebraic geometry and commutative algebra, *available at: http://www.math.uiuc.edu/Macaulay*2

[32] J. Herzog, T. Hibi, G. Restuccia: Strongly Koszul algebras, *Math. Scand.* **86** (2000), no. 2, 161-178.

[33] J. Herzog: Algebra retracts and Poincaré-series, *Manuscripta Math.* **21** (1977), no. 4, 307-314.

[34] J. Herzog, D. Popescu, N.T. Trung: Gröbner bases and regularity of Rees algebras, *Preprint* (2000).

[35] J. Herzog, V. Reiner, V. Welker: The Koszul property in affine semigroup rings, *Pacific J. Math.* **186** (1998), no. 1, 39-65.

[36] J. Herzog, A. Simis, W. V. Vasconcelos: Approximation complexes of blowing-up rings, *J. Algebra* **74** (1982), no. 2, 466-493.

[37] T. Hibi: Distributive lattices, affine semigroup rings and algebras with straightening laws, *in Commutative Algebra and Combinatorics (M. Naga and H. Matsumura, Eds.),* Advanced Studies in Pure Math., Vol. **11**, *North-Holland, Amsterdam,* (1987), 93-109.

[38] E. Kunz: Einführung in die algebraische Geometrie, *Vieweg Studium* Vol. **87**, *Vieweg Verlag* (1997).

[39] C. Löfwall: On the subalgebra generated by one-dimensional elements in the Yoneda Ext-algebra, *in Algebra, algebraic topology and their interactions (Stockholm, 1983),* Lecture Notes in Math. **1183**, *Springer, Berlin-New York,* (1986), 79-100.

[40] C. J. H. McDiarmid: Rado's theorem for polymatroids, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 263-281.

[41] J. G. Oxley: Matroid theory, *Oxford Science Publications, The Clarendon Press, Oxford University Press, New York,* (1992).

[42] H. Ohsugi, J. Herzog, T. Hibi: Combinatorial pure subrings, *Osaka J. Math.* **37** (2000), no. 3, 745-757.

[43] H. Ohsugi, T. Hibi: Compressed polytopes, initial ideals and complete multipartite graphs, *Illinois J. Math.* **44** (2000), no. 2, 391-406.

[44] S. Priddy: Koszul resolutions, *Trans. Amer. Math. Soc.* **152** (1970), 39-60.

[45] I. Peeva, V. Reiner, B. Sturmfels: How to shell a monoid, *Math. Ann.* **310** (1998), no. 2, 379-393.

[46] T. Römer: Homological properties of bigraded algebras, *Preprint* (2000).

[47] J.-E. Roos, B. Sturmfels: A toric ring with irrational Poincaré-Betti series, *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 2, 141-146.

[48] V. Reiner, V. Welker: Linear syzygies of Sanley-Reisner ideals, to appear in *Math. Scand.*

[49] B. Sturmfels: Gröbner Bases and Convex Polytopes, University Lecture Series Vol. **8**, *Am. Math. Soc., Providence RI* (1996).

[50] D. J. A. Welsh: Matroid theory, L. M. S. Monongraphs, no. **8**, *Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York* (1976).

[51] N. White: The basis monomial ring of a matroid, *Adv. Math.* **24** (1977), 292-297.

[52] N. White: A unique exchange property for bases, *Linear Algebra Appl.* **31** (1980), 81-91.

Index of Symbols