Stochastic Mortality Modelling and Management of Longevity Risk with Pricing and Reserving Applications to Annuity Products

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Preface

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List of Abbreviations

ADST	General German mortality table	36
AIC	Akaike information criterion	114
AnlV	German Ordinance on the Investment of Restricted Assets of Insurance Undertakings	5
ARIMA	autoregressive integrated moving average	43
BaFin	German Federal Financial Supervisory Authority	3
BAV	German Supervisory Office for the insurance industry	4
BIC	Bayes information criterion	59
BMF	German Federal Ministry of Finance	3
bps	basis points	210
CEIOPS	Committee of European Insurance and Occupational Pensions Supervisors	3
cf.	compare (imperative of the Latin verb *conferre*)	
CMIB	Continuous Mortality Investigation Bureau	51
DAV	German Actuarial Society	36
Destatis	Federal Statistical Office of Germany	16
ECB	European Central Bank	2
ECtHR	European Court of Human Rights	6
e.g.	for the sake of example (abbreviation of the Latin expression *exempli gratia*)	
EIOPA	European Insurance and Occupational Pensions Authority	3
GDV	German Insurance Association	2
GLM	generalised linear model	67
GMAB	Guaranteed Minimum Accumulation Benefit	208
GMIB	Guaranteed Minimum Income Benefit	183
GMDB	Guaranteed Minimum Death Benefit	208
HGB	German Commercial Code	8
i.e.	that is (to say) (abbreviation of the Latin expression *id est*)	
i.i.d.	independent and identically distributed	79
ITA	Institute for Financial Security and Planning	2
LIBOR	London Interbank Offered Rate	151

MaRisk	minimum supervisory requirements for risk management 5
MLE	maximum likelihood estimation 65
MSE	mean squared error 124
PAYG	pay-as-you-go 1
SCOR	Survivor Credit Offer Rate 80
UK	United Kingdom 16
UN	United Nations 23
US	United States of America 16
VA	variable annuity 7
VAG	German Insurance Supervision Act 5
VAM	value at maturity 218
VVG	German Insurance Contract Act 5
WSS	weighted sum of squares 40

Change is hard for life insurance companies because many policyholders have long-term relationships with them, and they don't like change.

Aaron Kloch, research manager, SNL Financial, Virginia

CHAPTER 1

General introduction

1.1 Some recent challenges to the national and international life insurance business

When taking a closer look at the gradual changes in the national and international insurance landscape for the past decades, both the academic's and the practitioner's attention has been drawn by several factors.

First, it became obvious that over the second half of the 20th century highly-developed societies had become technically older in demographic terms. In particular, mortality improvements have taken place more favourably than generally assumed. On the one hand, socio-economic and demographic developments, especially the downturn of the total fertility rates and the growing share of pensioners, increasingly endanger(ed) the financial viability of pay-as-you-go (PAYG) statutory pension systems. More precisely, the benefits to be expected by current premium payers will fail to sustain the labour life prosperity level of pensioners such that the new business for supplementary private funded pension provision has been growing for some time now. On the other hand, the increasing life expectancy adversely affects national pension scheme projections and the actuarial reserve formation of the underwriter's existing business. In this connection, one of the most important tasks is the precise construction of life tables or population projections and hence, a prediction of mortality trends. It requires the use of a forecasting method, which ideally satisfies a range of quantitative and qualitative criteria. Driven by steadily increasing actuarial needs, the survival modelling experienced a nearly 90-year development from early static law-based projection survival models over more advanced dynamic but deterministic models, yielding one-dimensional projections of parameters or markers used. Some more recent articles apply stochastic time series models allowing for interval forecasts and parameter uncertainty. Nevertheless, the appropriate choice of a sample / model, adapted to the respective application and the desired degree of complexity, continues to be a major challenge for forecasters and actuaries.

Secondly, the measures taken to liberate and deregulate national and international insurance markets encouraged a noticeable crowding-out competition. The abolition of the lengthy approval procedure by the (national) state supervision in 1994 provided assurers with extensive freedom in their product design. The opening up of the European insurance market caused a growing competition concerning prices (low premium margins), product design (increasing product / tariff variety, e.g. target group or flexible modular concepts, different guarantee undertakings) and policy conditions (blurred products limits, increasing degree of clauses). On the one hand, the quality in the range of services increased and the innovation pressure created a variety of life insurance solutions based on different allocation concepts. On the other hand, insurers competed at the expense of their capital base and considerable capital investments whereas consumers lost a high degree of clarity and transparency. Thus, from the customer's point of view, there is a need for a reference standard with a broad market acceptance[1] to enable a meaningful product comparison. Within the scope of a competent counselling the consumer's personal risk preferences should help to pick out suitable concepts by scrutinizing the risk-reward profiles or the simulated yield distributions.

Thirdly, recent financial crises such as the dot-com economic bubble in 2001/2002, the stock market crash followed by the September 11 attacks in 2001 and the Sub-prime financial crisis in 2008/2009 threatened the insurance business with a downstream persistent low-level interest rate phase[2] and a considerable reduction in the unit-linked new business[3]. The bear financial market forced the life assurance sector to realise capital losses and high write-downs on equity positions according to a strict lower-of-cost-or-market principle. Big players stabilised their investment result by means of diversified long-term vehicles with excellent credit ratings. But many small insurance companies with a more risky investment policy ran into trouble as a result of the economic

[1] Some German consultancy service providers and analysts like the Institute for Financial Security and Planning (ITA) or Morgen&Morgen already offer (online) based broker tools like ITA SELECT and certified standards like Volatium® that support the choice of suitable pension products.

[2] The European Central Bank (ECB) lowered the key interest rate in the period 2001–2003 from 4.75% to 2% and in 2008–2009 from 4.25% to solely 1%. A similar pattern could be observed for the ten-year government bonds interest rate.

[3] From 2008 to 2010 the share of new business for unit-linked pensions by members of the German Insurance Association (GDV) decreased by nearly 32%.

crises[4]. As a consequence, insurers reduced their profit sharing[5] and contribution refunds and / or increased expense ratios. Furthermore, new product generations transferred parts of the potential risks back to the policyholder, for example, through deductibles or unit-linked concepts. At the recommendation of the German Actuary Association and the German Federal Financial Supervisory Authority (BaFin), the German Federal Ministry of Finance (BMF) stepwise readjusted the guaranteed interest rate to ensure the long-term financial viability of the insurance benefits. The downturn of margins and new business as well as the accumulation of negative valuation reserves lead to a conceiveable challenging market environment on the suppliers side. The lessons learned from the aftermath of the crises strengthened the demand for a sustainable (national) solvency protection and a tightening of supervisory requirements concerning a proper financial supervision, an effective risk management and a more accurate and reliable financial reporting.

For this reason, and as a fourth factor, the European Insurance and Occupational Pensions Authority[6] EIOPA prepares a fundamental restructuring of the European insurance supervision. Therefore, it accompanies the introduction of a new EU-wide directive, namely the Solvency II project, which constitutes the most important regulation draft in recent decades. The design is based on a three pillar approach, in which, the first pillar includes quantitative accounting requirements, the second pillar describes the supervisory review process and the qualitative aspect of risk management whereas the third pillar imposes publication requirements. In order to protect the policyholders' interests, the EU insurance regulation prescribes solvency rules for capital adequacy and hence the amount of the required actuarial reserve. Moreover, stakeholders receive an answer to the question of how well insurance companies are prepared for stress (market) scenarios. The directive's legislative implementation is planned for the first quarter of 2013. As early as 2004 a redesign and modernisation of (national) supervision responsibilities came into effect based on the results of the so called Müller-report[7] and prescribed binding minimum standards

[4] According to the GDV, the average volume of stocks and investment shares of German insurers since 2000 was 23%. Nevertheless, the near-collapse of the Mannheimer Lebensversicherung AG in 2003 could serve as an example for irresponsible speculation in equity and fund shares which was far above market average. Along the lines of Schiller and Weber (2009) the balance sheets exhibited an increasing appetite for shares and equity funds from 13% to 44% proportional to the total assets for the period 1995 to 2002.

[5] The GDV discloses a net investment return of 4.68% for 2002 (year 2008: 3.54%).

[6] The European Insurance and Occupational Pensions Authority (EIOPA) replaced the previously responsible European financial regulatory institution named Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) within a reorganisation of supervisory authorities in 2011.

[7] In 1994 the EU-parliament engaged a workgroup of representatives of different European

for solvency protection. The quantification of the minimum solvency requirement should be calculated deterministically from the premiums and losses. However, it has not considered the companies' individual strategic business or risk management measures nor asset strategies, in particular, non-actuarial risks so far. Early on the readjustment process, the EU commission detected this methodical weakness and launched a conceptional phase for an extensive EU-wide harmonisation of solvency systems.

The fact that, since to date, small to medium sized insurance companies like the Mannheimer Lebensversicherung AG have not performed (any) adequate and reasonable fine-tuning of insurance business and investments induced the European Commission to establish a (value-based) corporate management. The approach is based on an adequate risk model as an integral part of the first pillar requirements. Instead of using a simple and conservative standard model, underwriters are free to develop and obtain accreditation for an individual but cost-intensive internal risk model. The inclusion of a proper risk management has far-reaching potential consequences: The enhanced costs of risk capital especially for risky market sectors may result in increasing prices and cash flow fluctuations for smaller insurance companies. As a result, the products or guarantees could be outsourced to a few specialised providers[8] which, in turn, practise an active hedging. Efforts concerning a core business commitment in low-risk segments as well as the usage of diversification effects are also expected to increase. In the near future, the offering of modest unit-linked solutions, either with an outsourced guarantee or an underwriting from a non-European country with more liberal accounting rules, will therefore help to equally meet the expectations of shareholders and consumers. For this and other reasons, a combined and crisis-proof modelling of actuarial parameters marks a factor of success for the identification, analysis and assessment of biometric, market, credit and operational risks.

The fifth factor once again emphasises that, also at a German national level and an early stage, several regulatory and legal changes were necessary and have been passed. Essentially, these changes concern far-reaching requirements for the appropriateness and documentation of the insurer's internal risk management. Besides preventive regulation, efforts have been made concerning

supervisory authorities, which was headed by Dr. Helmut Müller as the former vice president of the German Supervisory Office for the insurance industry (BAV). The report – finished in 1997 – compared solvency supervision among the EU countries and included amendments and additions to the proven former solvency systems.

8 The same applies for long-term guarantees within pension products. For example, in 2007, the Zurich Deutscher Herold Lebensversicherung AG equipped their unit-linked pension with a premium refund guarantee exclusively provided by the DWS Investment GmbH.

a protective regulation, too. Moreover, the case of the German bankrupt Mannheimer Lebensversicherung AG in 2002/2003 attracted high media attention and became the first time intervention of the rescue company Protektor Lebensversicherungs-AG. Since all efforts concerning a voluntary merger, an acquisition or an injection of fresh capital failed, Protektor was originated in late 2002 by the GDV and authorised by the BaFin. From the date of the transfer of the insurance portfolio, the rescue company was responsible for the recapitalization of the investment portfolio and the resale / transfer of running insurance contracts. The accrued unrealised capital losses were and will be stepwise paid off by the Mannheimer Holding. In the wake of the rescue, the amendment of the German Insurance Supervision Act (VAG) in 2004 passed the establishment of a guarantee fund managed by Protektor[9] to pre-finance the accumulated liabilities and handle the run-off for portfolios of insolvent life insurance companies. However, the simultaneous collapse of two or more big players poses, notwithstanding extra contributions payable, a non-negligible viability risk.

Since 1999, the BaFin issued several circulars to concretise the prevailing German Ordinance on the Investment of Restricted Assets of Insurance Undertakings (AnlV) and the VAG part regulating the investment principles for tied assets, in particular shares and other equity. Thereby, the investment should follow general criteria like profitableness, security, liquidity, investment mix and diversification. In addition, the anticipation and preparation of appropriate (qualitative) risk management and extended disclosure requirements – revised in the 2008 VAG amendment – was meant to strengthen the position of the German insurance industry within the European competition. In order to ensure the binding interpretation, the BaFin published administrative regulation sketches called minimum supervisory requirements for risk management (MaRisk) since the year 2005. Furthermore, the profound amendment of the German Insurance Contract Act (VVG) in January 2008 pursues the idea of comprehensive consumer protection. It regulates qualitative requirements concerning stakeholder information, consulting and documentation as well as preliminary insurance coverage and exposes insurers to a certain pressure to react. Nevertheless, there is no doubt that in periods of economic turbulences

9 The VAG contains regulations on the guarantee fund, managed by Protektor, and provides an obligatory membership for all life insurance companies running their business in Germany and voluntary membership for German pension funds. The necessary amount to be deposited – market value as of 31 December 2010 was 716 million euros – is financed from the annual contributions (up to a maximum of 0.2‰ (on the whole 1‰) of the net technical provisions) of the members, which are at the same time the shareholders.

and for consumer protection reasons, importance is attached to the monitoring[10] of the insurers solvency.

As a sixth factor, the latest generation of unit-linked life insurance policies – supplemented by different guarantee components – became popular in continental European since the late nineties and, meanwhile, can be found in the product portfolios of nearly all major insurance companies. In times of volatile financial markets, consumers prefer products yielding rich guarantees, but also an adequate return on their risk or savings premiums[11]. On a national level, the sales slump of pure unit-linked contracts stands in contrast to a lucrative new business of those contracts offered with innovative guarantee concepts. The insurance landscape already offers an unmanageable range of product variants which, nevertheless, place high demands on their providers. Besides market-wide challenges like the reduction of the guaranteed interest rate[12] and the introduction of unisex tariffs[13] in 2012, the valuation[14] according to the introduction of Solvency II in 2013 and a continuing period of low interest rates, the underwriting of "innovative insurance products" includes further challenges.

For example, the product design of unit-linked products has already a major impact on the resulting risk management and pricing. On the one hand, the guarantee fulfilment itself restricts common premium flexibilities such that the guarantee amount or fee may be adjusted due to altering market situations. On the other hand, the design needs to be as cost-effective as possible selectively achieved by a tailored[15] range of funds and / or anchors[16] in the general terms and conditions. Furthermore, non-hedgeable financial and non-financial risks

[10] The adjustments and changes concerning domestic tax and welfare state reforms, which became necessary to accompany the realignment of the insurance supervision, will be discussed in Chapter 6 in the second part of the thesis.

[11] The market research institute YouGovPsychonomics AG and The Royal Bank of Scotland plc published a conjoint analysis in 2009 in which 1.000 representative German customers and 190 insurance brokers took part in an online questionnaire about the most important aspects affecting their choice of voluntary old-age provision products of the next generation. Two thirds of the respondents preferred a high guaranteed living benefit, 60% favour a flexible product design and a further 40%, though, expect high potential returns.

[12] As a result of the persistent period of low interest rates, the BMF has decided to lower the guaranteed interest from 2.25% to 1.75% on the first of January 2012.

[13] In the first quarter of 2011 the European Court of Human Rights (ECtHR) stipulated the application of unisex tariffs and benefits in terms of the equal treatment of men and women.

[14] The Solvency II directive intends a market consistent valuation of any guaranteed benefits or options offered within in the insurance wrapper.

[15] A small range of preferably index-tracking or volatility-control funds can noticeably reduce the hedge risk.

[16] Most providers of unit-linked products arrange exit clauses that allow for a guarantee

like policy behaviour or longevity[17] must be considered in the price-setting of products. Although, from the consumer's point of view, the modular concept with separately charged guarantee costs is highly transparent, whereas the assurer sells a complex product which affords a financial valuation and hedging of embedded guarantees.

Seventhly and finally, the securing of the guarantees included in modern capital market-oriented products differs considerably with various product variants. For instance, insurance concepts investing in a guarantee fund promise a certain minimal yield or intermediate fund peak. Thus, the guarantee is secured within the fund allocation at the expense of a lower yield compared to a direct index-fund investment. On the contrary, (dynamic) hybrid products utilise a conventional premium reserve stock (also available with an additional guarantee fund stock) whereas so called variable annuity (VA) offer guaranteed benefits which are exclusively detached from the fund investment. Therefore, the significance of a company's internal profitability calculations and business cases needs to be supported by the application of a stochastic / scenario-based simulation of maturity benefits.

The product calculation of unit-linked products including certain guaranteed benefits requires a certain degree of personnel expertise due to the implementation of allocation algorithms and pricing / hedging according to recognised financial principles for incomplete markets. Unit-linked pension contracts with embedded (annuity) options or guaranteed minimum benefits demand an actively managed (dynamic) hedging by means of derivative financial instruments since the options moneyness depends on the fund and the market interest rate development. The financial distress[18] of the British assurer Equitable Life Assurance Society in 2000 – at least the world's oldest life insurance company – showed that a mispricing due to a lack of knowledge in appropriate pricing and reserving methods and inadequacies of mortality projections can have severe impacts on the solvency of a life insurance company.

The rather conservative German accounting principles[19] prescribe a valuation

adjustment or a compulsory shift of funds in determined market or mortality scenarios.

[17] According to Blake et al. (2006c) longevity describes the risk that, on average, people live longer than anticipated.

[18] In the seventies and eighties Equitable Life offered a great number of deferred annuity contracts with an embedded option to convert the accumulated funds at retirement at a guaranteed fixed rate. Due to reduced market interest rates in the nineties and unanticipated mortality improvements especially for the retirement ages these guarantee undertakings became very valuable. As a consequence, Equitable Life had to allocate additional reserves since the company renounced an appropriate pricing and option hedging until that time. Therefore, the insurers solvency was threatened and new business had to be closed.

[19] In this context, the accounting principles according to § 11 and § 65 of the VAG as well as

at guaranteed interest rates such that a guarantee obligation affords additional reserves. As a consequence, hedging becomes capital-intensive and has been unattractive for domestic insurance companies[20] so far. As an alternative, insurance companies either agreed in expensive reinsurance contracts or engaged experienced and financially strong investment banks with the guarantee provision. However, even the AXA group – one of the leading life insurance companies and financial service providers in Germany – experienced financial problems with the hedging of its third layer VAs product during the Sub-prime financial crisis. As a result, the Twinstar invest product line was closed for new business in 2009 and redesigned until early 2010. A majority of the other providers either withdraw from the unit-linked market or raised their guarantee fees and included exit clauses respectively. Nevertheless, one of the capital market-oriented product's major advantages over traditional life insurance products is that in case of a rapid market recovery embedded guarantees in new tariffs could be increased without any delay.

1.2 Motivation and contribution overview

The present thesis deals with the quintessences and scientific issues of the the factors of Section 1.1 in various degrees of detail. It mainly focusses at contributing to the research field of private pension provision related to the challenging tasks of recent market and mortality developments. Thereby, the following research questions are intended to be answered: Which mortality developments have been responsible for the increase in the expected lifetime of industrialised, developed nations and what are the implications of longevity on actuarial, demographic and socio-political applications? Institutions may questionnaire which evolutionary relevant mortality models are suited for a precise description / projection of the current / future mortality pattern. Can well-known results from other fields of applied mathematics like financial mathematics or time series analysis be suitable in this respect? Furthermore, what are appropriate comparison criteria for a qualitative and quantitative measurement of the goodness of fit for certain model classes? When it comes to the pricing and reserving of pension contracts, its providers seek knowledge about the risk potential of (long-term) mortality trends and (short-term) population fluctuations in combination with a random capital market development. How

§ 341f of the German Commercial Code (HGB) prescribe a setting up of an actuarial reserve in accordance with principles of proper accounting and appropriate actuarial assumptions.

[20] The locational advantage was exploited by international insurance groups with subsidiaries in Ireland, Luxembourg or Liechtenstein which follow a more liberal accounting legislation.

has this to be interpreted / implemented in the light of a regulation intending a market consistent valuation under an EU-wide supervision. And of course, driven by the mentioned growing demand due to the popularity of modern unit-linked pension products with innovative guarantee concepts, how can different approaches for the product design, profitability and guarantee charge calculation be established?

1.3 Structure of the thesis

The thesis can be divided into two main parts followed by an appendix. **Part I** is focused upon the mathematical description and projection of the mortality of homogeneous populations or insurance cohorts. Besides a survey of the most important representatives we provide a comprehensive analysis and comparison of stochastic and deterministic mortality forecasting models. In particular, the first part is organized as follows: The introducing **Chapter 2** studies the most noticeable patterns in population ageing and recent mortality improvements since the late 19th century illustrated by German mortality data. Due to medical advances and improved life standards, the expected individual lifetimes have steadily increased which has induced a noticeable shift in the national ageing structure with corresponding consequences for the compulsory pension schemes and pension providers.

In **Chapters 3** and **4** we present a chronological survey of different mortality models. Thereby, we assert no claim of completeness, but rather list the, from our point of view, most landmarking approaches. With regard to deterministic mortality modelling, which is treated in **Chapter 3**, we foremost describe approaches concerning the projection of life tables via reduction factors as well as parametric and non-parametric graduation methods[21]. We analyse how these classical approaches can help to explain certain trends in mortality evolution. Moreover, we take a critical look whether deterministic models fulfil requirements of modern risk management. Most of the model frameworks are additionally illustrated by German life table data from the Human Mortality Database (2009). The subsequent **Chapter 4** addresses the issue of stochastic mortality modelling. First, we deduce a range of fundamental criteria to assess a model's appropriateness due to a preselected forecasting purpose. Thereafter, we give intuition for the concept of stochastic mortality based on a demographic visualisation of certain trends and effects inherent in German mortality data.

[21] Renshaw and Haberman (1996) define graduation as the set of principles and methods by which the observed (or crude) probabilities are fitted to provide a smooth basis for actuarial applications.

For this reason, we review both discrete and continuous time approaches based on time series, short-rate and market models already, beforehand, used in econometrics and finance.

Chapter 5 can be considered as a self-contained applied excursus on the stochastic projection of German population mortality. The model of choice is given by the Lee-Carter method and some of the most important modifications and multi-factor extensions. The forecasting methodologies constitute "distribution-free" approaches sampling the bivariate central death rate for the full age range. The parsimonious Lee-Carter model enjoys widespread scientific and practical popularity due to the straightforward fitting procedure by means of standard least squares or likelihood methods and practical use for long-term forecasts given a historic stable age-specific mortality evolution. Nevertheless, the Lee-Carter approach also suffers from drawbacks concerning precision, forecast reliability and flexibility. A number of modifications and extensions purposed the improvement of the fitting quality for historical data by including additional factors or effects, the robustness of the parameter estimates and the estimation procedure or enrich the comprehensiveness of the bivariate correlation structure. The estimation / projection results are subjected to an examination of a criteria catalogue including basic qualitative and a range of quantitative criteria like residual analysis, parsimony, variance explanation, mean squared error and fan chart comparison.

Chapter 6 forms a connection to **Part II**. We describe the German pension system and clarify the need for supplementary private retirement provision. We refrain, however, from an overly detailed discussion of additional provision forms of the first and second retirement saving layer, i.e. Rürup and Riester pensions as well as occupational pension schemes. Instead, we turn our attention to the third layer, especially endowment and pension insurance solutions. From the class of traditional pension insurance solutions we analyse a deferred life annuity which has been a popular retirement product for decades. The combination of a conservative and secure investment, mainly bond funds and fixed interest securities, together with a guaranteed minimum yield augmented with non-guaranteed bonus shares represents an advisable supplementary income for retirement ages. In contrast, a unit-linked pension insurance of the latest generation, e.g. a VA, offers higher potential returns but also contains elevated risk resulting from an investment in equity and strategy funds. Depending on the consumer's opportunity and risk preferences, the basic framework can for this reason be enriched by various additional guarantee components or riders which are explicitly priced as a percentage charge of the net asset value and therefore exhaust a part of the return. It is meanwhile broadly understood that actuarial applications in science and practice require a risk-adequate mortality modelling. We therefore analyse a full stochastic model approach for both the deferred conventional and the deferred unit-linked product as an application example of old-age provision.

Chapter 7 analyses the combined effects of stochastic mortality and interest rates on different pricing methods and risk capital allocation. In particular, we analyse the systematic risk which is inherent in a portfolio of deferred life annuities and therefore take into account stochastic mortality as well as stochastic interest rates. Both models are calibrated to German mortality data as well as the current Euro area yield curve. We use Monte Carlo simulations to approximate the variance of the discounted cash flow and its decomposition into a pooling and a non-pooling risk part. Furthermore, the principle of zero expected utility and the quantile principle are used to consider pricing effects. They are required since the chosen setting defines an incomplete market due to the non-tradable nature of mortality risk. The estimated risk premiums are benchmarked to the standard equivalence premium. Finally, we focus on solvency requirements which are based on the investment decisions and the associated shortfall probability of the annuity provider.

In **Chapter 8** we take a closer look on deferred VAs under stochastic mortality and investment risk. VAs describe unit-linked contracts commonly equipped with additional guaranteed living and death benefits which are priced as an annual fixed percentage charge of the net asset value decoupled from management and mortality loadings. Thus, an analysis of the fair guarantee fee in a complete market provides an option price and benchmark relative to common charged insurance market fees. We assume a full stochastic annuity contract with premiums invested in a mutual equity fund. In case of death within the deferment period the insured's dependants can enter a call option on the greater of fund value or premium paid until the time of death. At retirement the insured can either enter a lump-sum option or receive a minimum guaranteed pension in case of retirement. We show the existence and uniqueness of a fair percentage charge in accordance with the equivalence of benefits and the absence of arbitrage. In an illustrative part we consider different profitability measures from the customer's perspective (rate of return, pension amount at maturity and options moneyness) for different guarantee features offered to enhance the guaranteed payoff amount at retirement. From an insurer's perspective, we perform a real-world sensitivity analysis of the fair charge and option prices for a comprehensive selection of contract, financial and mortality process parameters as well as different guarantee features. Furthermore, several risk measures (expected rate of return , conditional tail expectation, inflation quantile) are considered.

The appendix contains calculations and detailed proofs for the applied models and parameters of investigation. More precisely, in **Appendix A** we list parameter estimation results and some statistical summaries for the Excursion Chapter 5 on the Lee-Carter model and modifications / extensions. **Appendix B** is related to Chapter 7 and, inter alia, contains calculations for the Hull-White interest rate model and the stochastic mortality rate processes. Furthermore, we carry out a detailed moment calculation for the discounted portfolio values and a sensitivity analysis for the present portfolio benefit variance. Similarly, the **Appendix C** comprises calculations for the Brownian Gompertz-Makeham mortality rate process used in Chapter 8. Furthermore, the existence and uniqueness of a fair percentage guarantee charge are proven.

Part I

Illustrated Review of Mortality Modelling: From the Beginnings to Modern Approaches

One of the largest sources of risk faced by life companies and pension funds is longevity risk: the risk that members of some reference population might live longer, on average, than anticipated.

David P. Blake, professor of pension economics and director of the Pensions Institute, Cass Business School

CHAPTER 2

An Introductory Overview of German Mortality Trends and Patterns since the late 19th Century

In the course of the 20th century a considerable demographic change in populations of sophisticated industrial nations[22] has taken place. Due to medical advances and improved life standards[23] the expected individual lifetimes increased which induced a change in the national ageing structure. An analysis of the German historical life expectancy from the Human Mortality Database (2009) unfolds that since the beginning of the 20th century the expected lifetime at birth has undergone a 32-years improvement for males and even an improvement of 34 years for females. For the so called retirement age group we record a similar drastic trend. More precisely, the residual lifetime of a 65-year old male pensioner has risen by 5 years and even by 7 years for a senior woman since the middle of the 20th century. Oeppen and Vaupel (2002) even find out that since 1840 the worldwide female life expectancy increased almost linearly when considering the respective national peak values. Olshansky et al. (2005) hold the opposite opinion and conclude a downturn for increasing life expectancy which emphasises that there exists high uncertainty regarding the future mortality development. However, the increasing longevity of future pensioners threatens the viability of PAYG pension schemes of social security systems. But also pension funds or annuity providers need to allocate increased risk adequate reserves due to elongated future liabilities. It is thus particularly

[22] For instance, Macdonald et al. (1998) perform a comprehensive international comparison of different trends and changes in population mortality.

[23] Gallop (2007) discusses key drivers for the decrease in mortality rates in the UK in the 20th century. These are primarily changes due to smoking, diet, medical advances, infectious diseases and social class affiliation (health care, cardiovascular diseases, housing conditions, occupation).

important to use appropriate mortality projections in order to avoid an underestimation of deferred life-long liabilities. Rüttermann (1999) shows that implicit safety margins in German annuity market tables were partly outrun by recent improvements in mortality especially those of elderly women. But also against the background of the forthcoming introduction of an EU-wide insurance supervision in form of Solvency II, pension funds and providers are afraid of additional capital charges to cover the longevity risk[24] of their insured parties. A vast illustration for Belgian mortality data can be found in Pitacco et al. (2008). Benjamin and Soliman (1993) and Renshaw et al. (1996) analyse changes in the period specific mortality rates in the United Kingdom (UK) and the United States of America (US) mortality data. The comprehensive paper of Macdonald et al. (1998) presents a survey of mortality data from the United Nations for the years 1970 to 1990. In 2006 the Federal Statistical Office of Germany (Destatis) presented the model calculations needed for the implementation of its cohort life table for generations born between 1871 and 2004 and enriched the report with various explanations. The press report released three years later by the Federal Statistical Office of Germany (2009) describes the results of the 12th coordinated German population projection[25].

It was Hippokrates of Kos[26] who originated the phrase "declare the past, diagnose the present, foretell the future" in one of his writings. Applied to the prediction of future mortality forecasters therefore need to analyse past mortality trends properly to obtain a meaningful starting point for the projection of future mortality. For this reason, the major trends in mortality evolution are briefly summarised and illustrated by German male mortality data[27] from the Human Mortality Database (2009) and the German Federal Statistical Office (2010). Before the foundation of the Federal Republic of Germany mortality data was taken from the overall German life tables 1876, 1906 and 1933. Along the lines of the procedure adopted in Babel et al. (2008)

[24] Longevity risk denotes the uncertainty in future mortality trends and thus the human life span. More specifically, mortality data has shown systematic deviations from the previously forecasted mortality rates of older age groups, i.e. future mortality and life expectancy improves more favourable than expected before. In contrast, mortality risk denotes the risk stemming from mortality rates which are systematically higher than expected.

[25] The projection study describes effects of the nowadays demographic trends on the future population based on different assumptions on birth rates, life expectancy and migration. Some of the main findings of the study were the change in the age structure especially the reduction and ageing in the working-age structure and thus an increasing share of (oldest) seniors.

[26] Ancient Greek physician (about 460-377 B.C.); quotation from *Epidemics*, Bk. I, Sect. XI.

[27] For reasons of simplification we assume that period life tables approximately describe the mortality at the rounded centre of their census period, e.g. the life table 1901/1910 describes year 1905.

only West German life tables for later periods 1950, 1971, 1992 were used since East German population had only little effect on overall life expectancy. After 2001 the combined current life table 2005 is considered as an explicit separation between East and West Germany did no longer take place. The period tables provide a basis to describe the temporal development of so called functions[28] such as central death rate, annual death probability or life expectancy.

Trend 1: Shifted force of mortality curve with accidental hump

The annual death probabilities[29] or respectively their logit-transforms[30] (Subfigure (2.1.1)) show a typical "bath tube" pattern. The curve shape starts with high rates around the pre-natal ages, minimal probabilities at childhood ages (years 10-15) followed by higher accidental mortality at young adulthood (years 20-25), increasing mortality at adulthood and retirement ages with nearly constant rate of increase. The so called "accident hump" at adolescence stands for higher probability and increasing volatility due to accidental deaths and lethal injuries caused by augmented risk-taking behaviour as well as increased suicide rates. Subfigure (2.1.1) shows that this specific shape particularly developed since World War II. For males aged 100 and older the rate of increase slows down approaching a rather flat shape since the cohort mainly consists of healthy, sprightly pensioners due to age selection. This effect of late-life deceleration was analysed by Gavrilov and Gavrilova (2001). Especially the noticeable oldest age improvement arouses discussion since demographic evidence for those ages is quite sparse and affected by high random fluctuations. Within the periodic specific "mortality profiles"[31] the overall mortality declined by-and-by whereas the rate of decline has not slowed down such that an end to this development is not in sight.

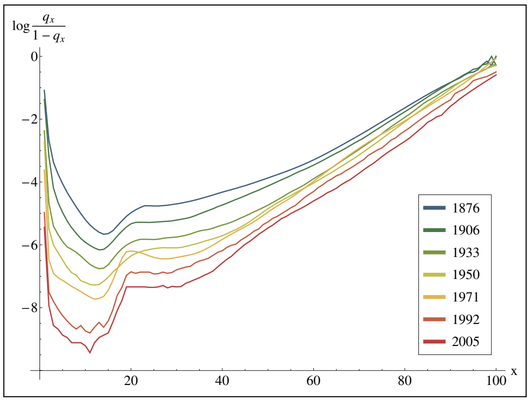
Trend 2: Rectangularisation and expansion

Under closer inspection of the period specific survival probabilities in Subfigure (2.1.2), the shape of more recent period life tables has become more rectangular in contrast to the diagonal curve for periods until the end of the 19th century. This phenomenon is a typical result of a simultaneous decline in mortality for

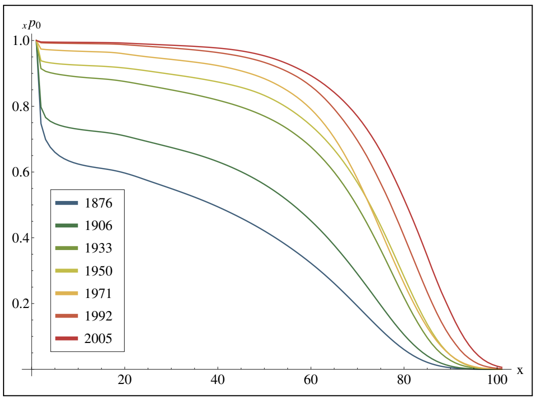
[28] See Section 3.1 for a precise definition of life table functions.
[29] The annual death probability at age x denotes the probability that an individual of exact age x will die within the same year. The definition and estimation of frequency measures is carried out in Section 3.1.1.
[30] The logarithmic scale illustrates even very small changes in mortality especially for infancy and thus is usually favoured for illustration.
[31] The term mortality profile refers to the graphical illustration of the yearly evolution of the mortality rate over time (or alternatively the central death rate and the annual death probability respectively) of a single age group conditional on survival until that age.

a broader range of adult age groups. A typical appearance for this era equals a nearly uniform distribution of the probability mass over the whole time axis. This phenomenon of rectangularisation or compression of mortality and its degree of severity respectively can, for example, be measured by the so called interquartile range (see Table 5.9) which was analysed by Wilmoth and Horiuchi (1999). The authors analyse the different behaviour of the interquartile range and thus the variability of US, Swedish and Japanese life table ages at death. From the Human Mortality Database (2009) mortality data it follows that the German female age at death variability fell from 67 years in 1876 to 13 years in 2005, the male interquartile indicator range from 64 to 17. Furthermore, the survival probabilities steadily tend to progress towards a biological maximum age limit $\omega > 100$ whose existence is still debated in demographic literature[32]. Both phenomenons directly favour the formation of longevity.

Illustration of Mortality Trend 1 and 2



(2.1.1) Force of mortality curve



(2.1.2) Rectangularisation and expansion

Figure 2.1: Temporal development of the shifted force of mortality curve and an illustration of the rectangularisation and expansion phenomenon based on German male period life table data from 1871 to 2006.

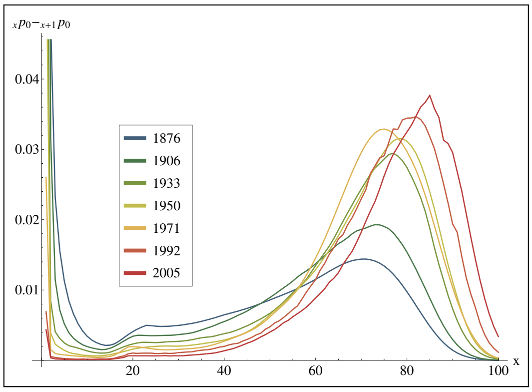
Trend 3: Increasing concentration of deaths around the mode

Subfigure (2.2.1) depicts the so called "curve of deaths" as the difference quotient of survival probabilities of a newborn individual for successive durations (i.e. $_xp_0 - {}_{x+1}p_0$ for ages $x \geq 0$). Due to extensive decrease of infant mortality (e.g. favoured by the application of vaccination or antibiotics) and old-age mortality (e.g. favoured by a decline in the severity of chronic diseases and its complications) based on improvements in public health and alimentation the function has evolved into an unimodal curve where the age at death increased and concentrated on a smaller age interval. Approximately 81%

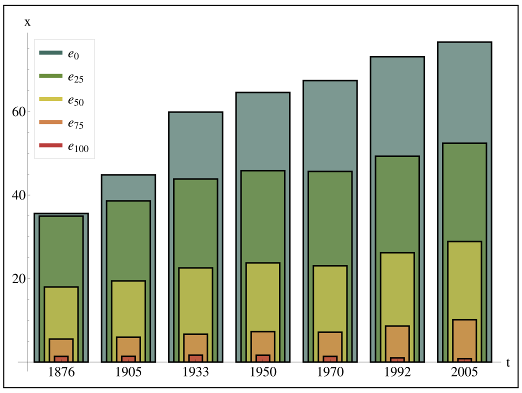
32 See, for example, Wachter and Finch (1997), Olshansky et al. (2001) or Waite (2005).

of the interquartile range decline already fall within the period from 1876 until 1933. The augmented probability mass aggregation in the range of the modal value equals a sharper decline of the survival curve in Subfigure (2.1.2). Therefore we observe an increase in the "most probable" age at death.

Illustration of Mortality Trend 3 and 4



(2.2.1) Curve of deaths



(2.2.2) Period life expectancy

Figure 2.2: Temporal development of the increasing concentration of deaths and an illustration of the period life expectancy based on German male period life table data from 1871 to 2006.

Trend 4: Increasing period life expectancy and ageing in the population pyramid

Subfigure (2.2.2) illustrates that life expectancy at birth (denoted by the life table function e_0) has more than doubled since 1870. The speed of the slope decreases when considering life expectancies e_{25}, e_{50}, e_{75} and e_{100} of different age groups whereas last mentioned even shows a slight reduction over time t due to insufficient empirical data on the expected age at death for oldest ages. The increasing life expectancies since the fifties, especially those for older age groups, caused a considerable increase of the share of pensioners in the total population. More specifically, based on the population tables of the Destatis we found out that since 2003 the proportion of people under 40 years decreased by an annual average of 0.5% (and a standard deviation of 0.7%). In contrast, the share of old and oldest age groups (>60 years) rose on average by 1% annually (with a standard deviation of also 1%) and constitutes 26% of the total population in 2009. Furthermore, we observe a decrease in the overall population level since 2003 due to a slight decline in the German birth rate[33] and a negative net migration. The demographic change illustrated in the ageing

[33] According to records of the Destatis the average number of children women had given birth to fell from 1.45 children in 1990 to 1.36 in 2009.

pyramid chart in Figure 2.3 intensifies among other things the pressure on the social security system planning.

Illustration of Mortality Trend 4 (continued)

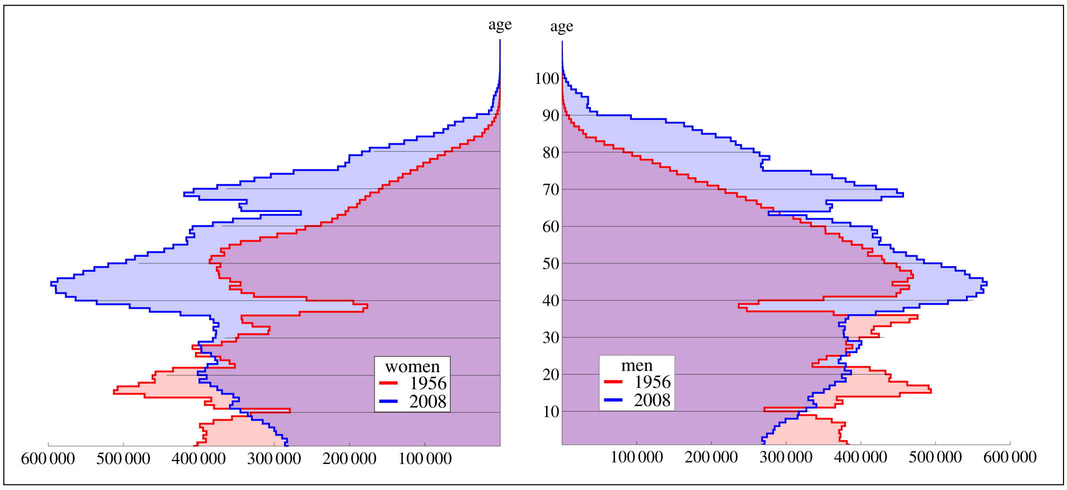
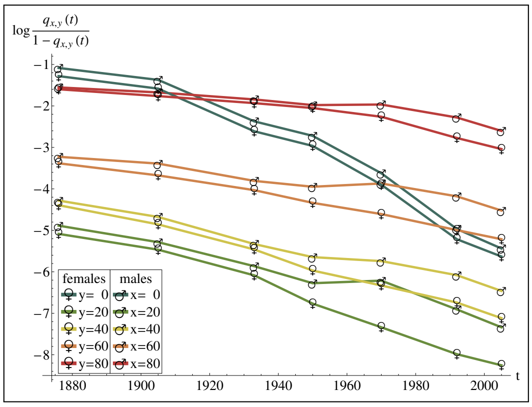


Figure 2.3: Gender-specific population pyramids based on West German annual population numbers for periods 1956 to 2008. The data originates from the Human Mortality Database (2009).

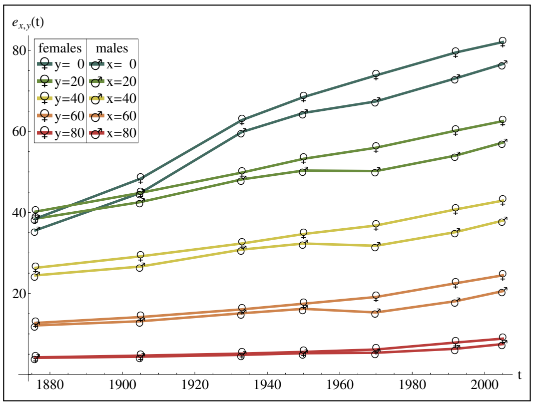
Trend 5: Heterogeneity in gender

Subfigure (2.4.1) shows that the newborn's mortality improved substantially for both females and males where especially the periods 1900-1930 and 1950-1990 recorded the strongest growth. For the remaining female age groups 20 to 80 mortality decreased almost uniformly whereas male mortality improvement stagnated or even inverted with the outbreak and progression of World War II. This effect becomes obvious for young adults which were involved in acts of war and generations born in the meanwhile. Since about 1970 logit rates are falling again. The changes in the death probabilities have a direct impact on the expected residual lifetimes for different ages. Demographic research shows that, on average, females tend to live longer than males. The amount of this gap becomes smaller with increasing gender-specific ages x_{female} and x_{male} respectively. In opposition, Subfigure (2.4.2) illustrates that life expectancy continuously increased especially for female individuals. Hereby, the amount and rate of improvement decreases for higher ages. A distinction by heterogeneity for different time periods can be extended to other cohort classifications, e.g. the socio-economic group an individual belongs to. Richards et al. (2007) observed that members of better classes underlie a more distinct improvement of mortality due to smoking behaviour, selection effects, nutrition, environmental conditions or lifestyle differences.

Illustration of Mortality Trend 5



(2.4.1) Logit death probabilities



(2.4.2) Life expectancy

Figure 2.4: Temporal development of the heterogeneity-in-gender trend based on German male period life table data from 1871 to 2006.

Life expectancy would grow by leaps and bounds if green vegetables smelled as good as bacon.

DOUG LARSON, US columnist and cartoonist

CHAPTER 3

Deterministic Mortality Modelling: Dynamic Life Tables and Graduation Methods

The chapter gives a classified survey of early dynamic actuarial methods used to describe mortality patterns. According to Booth (2006) there exist three different approaches for (demographic) forecasts: extrapolation based on statistical models, expectation based on models including underlying biomedical processes or expert opinions and explanation by means of causal models involving econometric variables[34]. In the following survey, only the former method will be considered while for the remaining methodologies we refer to the surveys of Wong-Fupuy and Haberman (2004), Booth (2006) and Antolin (2007). Extrapolation frameworks should provide beneficiaries of either official projections[35] or projections for the life and pension insurance business[36] with decision variables together with convenient forecasting opportunities and uncertainty measures. Besides approaches dealing with the projection of life tables via deterministic reduction factors in order to capture mortality trends we focus on parametric and non-parametric graduation. Parametric models represent mathematical formulas which became popular amongst demographers and biologists, e.g. the Gompertz mortality law (1825) and various extensions on it. The purpose of non-parametric methods is mainly to provide smooth patterns of crude mortality data.

[34] Econometric variables refer to exogenous variables such as socio-economic, governmental or health factors.

[35] Official population projections which are investigated regionally (statistical offices of the German states), nationally (Destatis) or internationally (United Nations (UN) Population Division) provide agencies with a starting point for economic, fiscal and social policy household planning.

[36] The private-sector life and pension insurance business applies projected life tables primarily in connection with pricing and reserving.

We analyse how classical approaches can help to understand certain trends in mortality evolution. Moreover, we take a critical look whether deterministic models fulfil requirements of modern risk management. Most of the model frameworks are illustrated by German life table data from the Human Mortality Database (2009). In Section 3.1 we give an overview over discrete-time mortality models like static and dynamic life tables as well as related life table functions. Continuous-time models divided into parametric and non-parametric graduation are handled in the subsequent Section 3.2. In particular, the concept of projection via reduction factors and mortality laws as well as indirect forecasting methods are highlighted and illustrated. Section 3.3 concludes with a critical discussion about the limits in the application of deterministic mortality frameworks.

3.1 Discrete-time frameworks

The idea behind the introduction of discrete-time mortality parameters lies in the nature of demographic population censuses and mortality registries which are based and published for integer age groups.

3.1.1 Definition of age-discrete probabilities

If not mentioned otherwise, we assume that mortality is time-invariant such that historical developments continue in the future. At first, we introduce some general functions describing frequency measures for the mortality events survival and death of an insurance portfolio or a population. Let $l_0, l_1, ..., l_\omega$ denote the sequenced numbers of age-specific survivors based on a fictitious cohort of individuals with l_0 newborns. Furthermore, we define $\tau_x^{(i)} = \tau_x$ as the random independent identically distributed residual lifetime of individual $i \in \{1,2,...,l_0\}$ aged x such that the observed cohort is assumed to be homogeneous. For natural numbers $x,t \geq 0$ the number of individuals that died during the interval $[x,x+t]$ is calculated as ${}_td_x = l_x - l_{x+t}$. Thus, the t-annual survival probability conditional on survival to age x equals the number of survivors reaching age $x+t$ related to the number of individuals having survived at least x years, i.e.

$$ {}_tp_x = P(\tau_x > t) = P(\tau_0 > x+t \,|\, \tau_0 > x) = \frac{l_{x+t}}{l_x} \tag{3.1} $$

Accordingly, the t-annual death probability is given as ${}_tq_x = 1 - {}_tp_x$. From Definition (3.1) we see that ${}_tp_x$ can be interpreted as the tail cumulative distribution function of τ_x. In particular, for $t = 1$ we obtain the annual probabilities

$$q_x = 1 - p_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}. \tag{3.2}$$

which represent common life table entries and are therefore called life table functions.

3.1.2 Static life tables

The earliest contributions dealing with life tables as a discrete version of the distribution of the individual[37] residual lifetime τ can be traced back until the mid of the 16th century[38]. Nowadays, the concept of life tables has become a fundamental procedure in demography to describe the mortality experience of a reference population which is embraced to be stationary in terms of a constant cohort size for each year of birth. Due to the estimation method of annual death probabilities (or equivalently other life table functions) two different types of single-decrement life tables are applied.

The plain type of a life table is called a cohort or generation life table. It contains observable long time data of a certain cohort equally aged at recording inception and thus describes specific characteristics of one single underlying cohort. The annual death probabilities are based on experienced survivor numbers of sequenced observation years. While this kind of observation takes durations up to $\omega + 1$ years (ω denotes a biological maximum age such that $l_\omega = 0$) in form of a longitudinal section, there is little supply of consistent and reliable data since in Germany proper mortality documentation did not start until the end of the 19th century[39]. The second table form is called a period or a current life table and records mortality data of a complete "in-force-population" of relevant ages over several adjacent years (usually one to three years[40]) in form of a cross-section[41]. Therefore we obtain a snap-shot of mortality throughout the observation period. Typically, life tables are complete in the sense that

[37] In the following, we assume a homogeneous population or portfolio of life insurance policies, i.e. identically distributed lifetimes τ.

[38] Graunt (1662) analysed the mortality rolls of London. Halley (1693) constructed a life table for the city of Breslau for the time period 1687-1691. Milne (1815) presented, from an actuarial point of view, the first correct construction of mortality tables of two English parishes during 1779-1787. The first German general life table was set up in the German Reich during 1871 and 1881.

[39] The Federal Statistical Office of Germany (2006) has published a generation life table for cohorts born between 1871 and 2004.

[40] Before 1930, the cross-sectional censuses took up to 10 years due to problems concerning the census and projection methods.

[41] See Pitacco et al. (2008) for further explanation.

the table entries correspond to one-year probabilities. Abridged life tables are based on different aggregated age groups.

Moreover, national mortality data is completed by age-specific death rates namely the central death rate m_x between exact age x and $x+1$. For positive, integer time periods u it holds

$$_u m_x = \frac{_u d_x}{_u L_x} = \sum_{i=0}^{u-1} \frac{d_{x+i}}{L_{x+i}} = \sum_{i=0}^{u-1} m_{x+i} \quad \text{with} \quad _u L_x := \int_0^u l_{x+t} dt = \sum_{i=0}^{u-1} L_{x+i} \tag{3.3}$$

where $_u L_x$ denotes the number of person years[42] lived between ages x and $x+u$. Its evaluation is based on life table data and requires distinct algebraic approximate assumptions based on different hypotheses on the number of survivors l_{x+t} for non-integer ages $x+t$. Figure 3.1 shows that the resulting functional forms are differentiable on the open interval $0 < t < i$ for each $i = 1, \ldots, u$ but discontinuous at the endpoints.

Linear Interpolation

W.l.o.g. we assume that $u = 1$. A popular and intuitive hypothesis assumes that l_{x+t} is treated as a linear function between consecutive cohort ages x and $x+1$, i.e.

$$l_{x+t} = (1-t)l_x + t l_{x+1} = l_x - t d_x, \tag{3.4}$$

such that deaths are uniformly distributed and occur on average half the interval, i.e. $_t q_x = t q_x$. Under (3.4) the number of person-years[43] reduces to

$$L_x = \int_0^1 l_{x+t} dt = \int_0^1 l_x - t\, d_x\, dt = l_x - \frac{1}{2}\left(l_x - l_{x+1}\right) = \frac{1}{2}\left(l_x + l_{x+1}\right)$$

and the central death rate as the average death rate over year x becomes

$$m_x = \frac{d_x}{\frac{1}{2}\left(l_x + l_{x+1}\right)} = \frac{l_x\left(1-p_x\right)}{l_x\left(\frac{1}{2}\left(1+p_x\right)\right)} = \frac{q_x}{\frac{1}{2}\left(2-q_x\right)} = \frac{q_x}{1-\frac{1}{2}q_x}. \tag{3.5}$$

Accordingly, the death probabilities are converted as follows

$$q_x = \frac{m_x}{1+\frac{1}{2}m_x}.$$

[42] L_{x+u} is also called exposure-to-risk or exposure and refers to the number of individuals adjusted for the duration being alive.
[43] The chosen definition ignores migration effects. According to Brouhns et al. (2002b) approximation (3.4) is still close enough for practical uses.

Exponential Interpolation

Another approach assumes an exponential decreasing relation of the number of survivors or a linear relation of the logarithmised number of survivals

$$\begin{aligned}\log\left(l_{x+t}\right) &= (1-t)\log\left(l_x\right) + t\log\left(l_{x+1}\right)\\ &= \log\left((l_x)^{1-t}(l_{x+1})^t\right) = \log\left(l_x\left(\frac{l_{x+1}}{l_x}\right)^t\right) = \log\left(l_x\left(p_x\right)^t\right), \end{aligned} \quad (3.6)$$

For death probabilities over periods less than one year ${}_tq_x = 1-(1-q_x)^t$ we obtain the number of person years as

$$L_x = \int_0^1 l_x^{1-t} l_{x+1}^t dt = l_x \int_0^1 \left(\frac{l_{x+1}}{l_x}\right)^t dt = l_x\left(\frac{p_x - 1}{\ln(p_x)}\right) = -\frac{l_x q_x}{\ln\left(p_x\right)}.$$

Under assumption (3.6) the relation between death and mortality rates becomes

$$m_x = -\log\left(p_x\right) \quad \text{as well as} \quad q_x = 1 - p_x = 1 - \exp(-m_x), \quad (3.7)$$

i.e. the central death rates are independent of t and constant over $[x,x+1]$.

Harmonic Interpolation

For non-integer survivor numbers presume a hyperbolic[44] decreasing shape

$$\frac{1}{l_{x+t}} = (1-t)\frac{1}{l_x} + t\frac{1}{l_{x+1}} \quad \text{or rather} \quad l_{x+t} = \frac{l_x\, l_{x+1}}{l_{x+1} + t\, d_x} \quad (3.8)$$

which is equivalent to a linear interpolation of the reciprocals of the survivor numbers. As a consequence, we obtain a linear functional relation

$${}_{1-t}q_{x+t} = -tq_x + q_x = (1-t)q_x \quad \text{for } (0 \le t \le 1).$$

For

$$L_x = \int_0^1 \frac{l_x\, l_{x+1}}{l_{x+1} + t\, d_x} dt = \frac{l_x\, l_{x+1}}{d_x} \ln\left(\frac{l_{x+1} + d_x}{l_{x+1}}\right) = -\frac{l_{x+1}}{q_x}\ln\left(p_x\right)$$

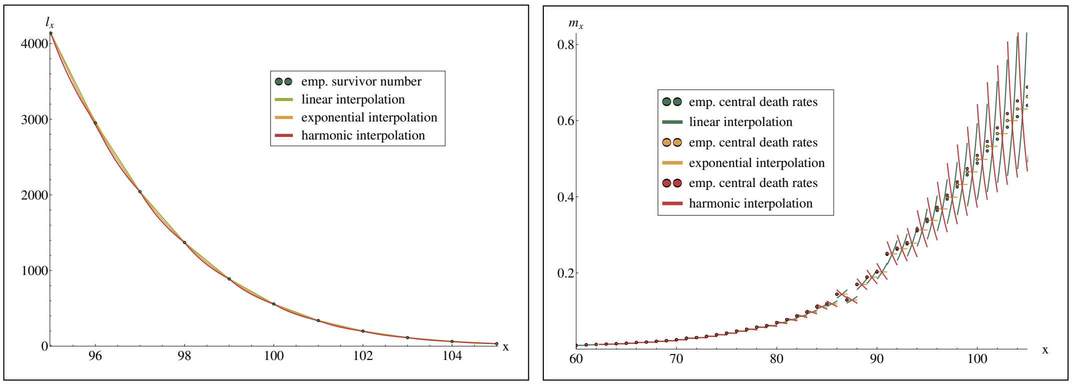
the central death rate becomes

$$m_x = -\frac{(q_x)^2}{p_x \ln\left(p_x\right)}.$$

44 The hypothesis is also called Balducci assumption since Balducci (1917) already used hyperbolic distributions for life table models.

Figure 3.1 illustrates the shapes for different hypotheses on the survivor distribution. Any mortality assumption produces a smooth survivor curve and different curves are hardly distinguishable (see Subfigure (3.1.1)). In Figure (3.1.2) the discontinuous interval endpoints become noticeable[45] for the sharp increase of central death rates at very old ages $x > 85$. In a complete life table

Linear, exponential and harmonic interpolation of life table functions



(3.1.1) Survivor number interpolation **(3.1.2)** Central death rate interpolation

Figure 3.1: Linear, exponential and harmonic interpolation of life table functions based on the German male period life table of 2006 from the Human Mortality Database (2009).

with yearly observable functions m_x or q_x any further function (see Subsec. 3.1.3) can be calculated. For period mortality observations the whole (synthetic) survivor sequence has to be recursively derived from the estimated frequency of deaths

$$l_1 = l_0(1 - q_0), l_2 = l_1(1 - q_1), ..., l_{x+1} = l_x(1 - q_x), ..., l_\omega = l_{\omega-1}(1 - q_{\omega-1}).$$

In case of a longitudinal observation of the survivor sequence $l_0, l_1, \dots, l_\omega$, the frequency measures can be calculated as $d_x = l_x - l_{x+1}$, $q_x = \frac{d_x}{l_x}$ and $p_x = 1 - q_x$. A collection of period life tables enables a comparison of mortality trends over time since each table corresponds to the mortality experience of a certain time period. Thus, they are suitable for an application in life insurance. Due to incomplete mortality data cohort tables consequently need to be reconstructed by stringing together series of past period tables or projected future period tables to so called aggregated life tables[46]. However most actuarial calculation concentrates on cohort tables since the evolution of one single age group is of

[45] London (1997) notes that exponential and hyperbolic interpolation seem inconvenient for life table calculations.
[46] The construction of aggregated life tables needs certain smoothing and projection procedures.

interest whenever measuring life expectancy or analysing certain age, period and cohort effects. Cohort tables are therefore commonly used for an application in private pension schemes and annuity insurance (see also Subsection (3.1.5)). In general, life tables that are based on census data and published by statistical offices serve as an import for population forecasts or medical analysis[47].

3.1.3 Related life table functions

The expected total number of years lived by a reference cohort, starting with l_0 newborn individuals, after reaching year x is denoted as

$$T_x := \sum_{i=0}^{\omega-x-1} L_{x+i} = {}_{\omega-x}L_x$$

and necessary to compute the expectation of the future lifetime as a measure for the mortality level of a population. For an individual aged x the average future lifetime equals

$$\mathring{e}_x = \frac{T_x}{l_x}, \tag{3.9}$$

such that the expected age at death amounts to $x + \mathring{e}_x$. For a life table environment under assumption (3.4) expression (3.9) reduces to the so called complete expectation of life[48] in terms of

$$\mathring{e}_x = \frac{T_x}{l_x} \overset{\sim}{=} \frac{\sum_{i=1}^{\omega-x} l_{x+i} + \frac{1}{2}l_x}{l_x} = \sum_{i=1}^{\omega-x} {}_ip_x + \tfrac{1}{2} =: e_x + \tfrac{1}{2}.$$

The functions L_x, T_x and $\mathring{e}_x$ denote duration measures since they are determined in person-years. In case of exponential and hyperbolic interpolation of non-integer age survivor numbers, no convenient analytic formulas for T_x or $\mathring{e}$ are available.

Actuarial calculations involve other location measures like the Lexis point or the median age at death which allow for a better understanding of characteristic patterns for the curve of deaths in Figure (2.2.1). Additionally, demographers may inspire certain variation measures like the standard deviation, the covariance, the entropy or the interquartile range to get comprehensive information about the distribution of the remaining lifetime τ_x (see Table 5.9).

47 For example, the expected lifetime within a country provides a development indicator for international comparisons.

48 Here, e_x denotes the curtate expectation of life $E\big[\lfloor\tau_x\rfloor\big] = \sum_{i=0}^{\omega-x} k\, {}_ip_x\, q_{x+i} = \sum_{i=1}^{\omega-x} {}_ip_x$ describing the number of complete integer years lived after age x. The second equation follows by summation by parts.

3.1.4 Dynamic life tables

According to Barrieu et al. (2010) standard life tables only give a restrictive view in the way that they fade out potential future mortality evolution. If future mortality evolution should be involved, e.g. for long-term insurance products like life annuities or pension plans, the insurer can make use of projected cohort tables. In contrast to static period tables, projected life table functions $\phi(x,t)$ are subjected to model as well as parameter risk[49]. The extrapolation task consists of an out-of-sample projection of future mortality. For a past and possibly fragmented observation period $[t_{\text{past}}, t_{\text{pres}}[$ (t_{pres} current calendar year), a dynamic mortality model provides estimates for the real-valued matrix $(\phi_{x,t})_{x=0,\dots,\omega-1;\, t=t_{\text{pres}}+1,\dots,t_{\text{fut}}}$ for a projection horizon $]t_{\text{pres}}, t_{\text{fut}}]$. The entries $\phi_{x,t}$ equal the life table function to be projected for an individual aged x at time t. Row vectors are derived from period life tables for a set of calendar year observations $\{t_1, \dots, t_k\}$ (with $t_1 \geq t_{\text{past}}$, $t_k \leq t_{\text{pres}}$) and different ages $\{x_1, \dots, x_l\}$ (with $x_1 \geq 0$, $x_l \leq \omega$). Alternatively, diagonal matrix vectors can originate from cohort tables for a set of different ages of birth $\{y_1, \dots, y_m\}$ (with $y_1 \geq t_{\text{past}} - \omega$, $y_m < t_{\text{pres}}$). The question which observation window duration suits best for a concrete projection application is an actuarial design[50] one. In general, graduation / projection procedures are subdivided into horizontal (period-specific), vertical (age-specific) or diagonal (cohort-specific)) estimation methods for the underlying life table function sequence. Therefore, forecast methods can, inter alia, be differentiated due to their use of data. Some dynamic models were actually introduced in the life insurance and pension business to project the mortality of pensioners.

Projection by Extrapolation

Extrapolation can be used straightforward to project life table functions assuming persistence in the experienced mortality data. This approach is equivalent to a direct horizontal trend estimation and allows for an age-independent pattern of two-factor life table functions like the survival function[51] $p_x(t)$ or the average future lifetime $\mathring{e}_x(t)$. Which graduation / extrapolation function is chosen depends, amongst other things, on the forecast purpose (e.g. age- or sex-specific), projection horizon, data disposability and, to a certain degree, on "expert opin-

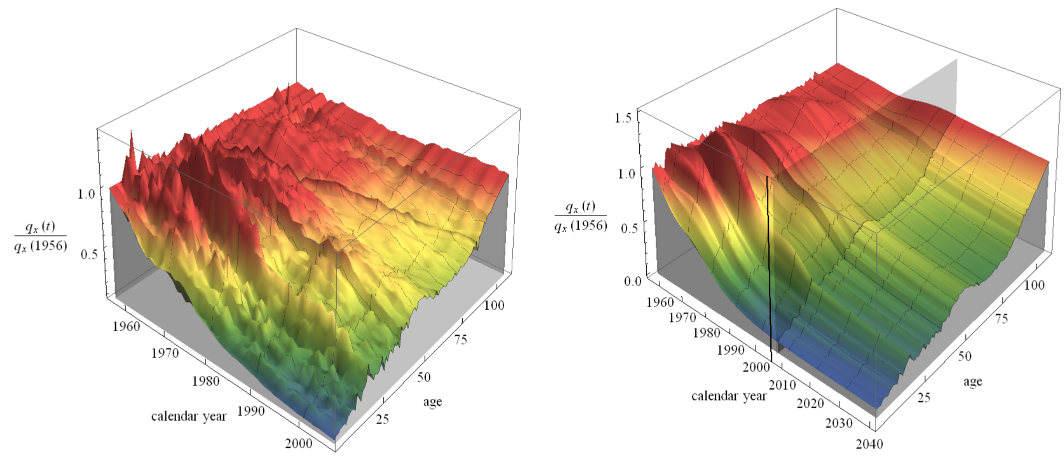
49 Parameter risk denotes the risk due to inaccurate estimation of certain model parameters. In turn, model risk arises from an inappropriate model choice.

50 For a discussion on the optimal calibration period see Pitacco et al. (2008).

51 The survival function denotes the discrete- or continuous-time functional representation for the survival probabilities. In other words, it defines the (cumulative) tail distribution function of the random residual lifetime. The corresponding definitions are given in Subsections 3.1.1 and 3.2.1.

ions" concerning the smoothness and the shape. Potential shortcomings like the high model risk and the large amount of age-specific parameters to be estimated have to be handled with caution since recent mortality changes themselves are subjected to random biological, medical and environmental driving factors. Inconsistencies in extrapolated mortality profiles make a further adjustment necessary. Without claiming completeness, Table 3.1 gives a brief methodological overview. We applied the extrapolation via exponential transformation to spline smoothed German period life table data from the Human Mortality Database (2009) and give an illustration in Figure 3.2. It is noticeable that

Exponential extrapolation of the death probability reduction factors



(3.2.1) Raw improvement rates

(3.2.2) Exponential extrapolation of spline smoothed improvements rates

Figure 3.2: Exponential extrapolation of the (spline smoothed) death probability reduction factors for German period life tables from the Human Mortality Database (2009) for periods 1956-2006.

compared to the initial frequency $q_x(1956)$, the probabilities for subsequent periods are generally falling for all ages but each to a different extent. Due to the fact that life table functions for ages 100 and older are updated annually by simple regression, the corresponding forecasted area has a smooth surface too. We even observe a rise of the period death probabilities to the historic mortality level $q_x(1956)$ due to an increase in the early nineties and stagnation afterwards. The remainder ages feature an excessive strong decline for infancy and retirement age and slower improvement rates for death probabilities of the age group 45-55. The projected rates continue this downward trend with negative exponential shape.

Projection of life table functions by extrapolation

Extrapolation via reduction factors

- the probabilities of death are proposed to follow a decreasing trend over time
- for the reduction factors $0 < r(x,t_{\text{fut}} - t_{\text{pres}}) = r(t_{\text{fut}} - t_{\text{pres}}) \leq 1$ the mortality trend is assumed to be age-invariant (which is reasonable for a limited set of age groups $x_1, x_2, ..., x_k$)
- the annual death probabilities of future calendar years $t_{\text{fut}} > t_{\text{pres}}$ are obtained according to

$$q_x(t_{\text{fut}}) = q_x(t_{\text{pres}}) \cdot r(x,t_{\text{fut}} - t_{\text{pres}}) \text{ for ages } x = 0,...,\omega - 1$$

- the concept of improvement scales was used for mortality forecasts in the UK and US

Extrapolation via exponential transformation

- the annual death probabilities are assumed to decline exponentially over time
- the exponential transform $\frac{q_x(t_{\text{fut}})}{q_x(t_{\text{pres}})} = e^{-\delta(x,t_{\text{fut}}-t_{\text{pres}})}$ describes a special case of reduction factor $r_{exp}(x,t_{\text{fut}} - t_{\text{pres}}) \leq 1$ for $t_{\text{fut}} > t_{\text{pres}}$ and $\delta(\,.\,,.\,) \geq 0$
- the extrapolation procedure then follows $q_x(t_{\text{fut}}) = q_x(t_{\text{pres}}) \cdot r_{exp}(x,t_{\text{fut}} - t_{\text{pres}})$
- for an expected asymptotic mortality $\Psi(x)q_x(t_{\text{pres}})$ the exponential formula can be extended by means of

$$q_x(t_{\text{fut}}) = q_x(t_{\text{pres}}) \cdot (\Psi(x) + (1 - \Psi(x)) \cdot r_{exp}(x,t_{\text{fut}} - t_{\text{pres}}))$$

  which was applied by the CMIB (1990) to allow for mortality improvements in the CMI standard tables of mortality based on the experiences for periods 1979 to 1982
- the formula $q_x(t_{\text{fut}}) = a(x) + b(x) \cdot c(x)^{t_{\text{fut}}}$ was proposed by the Institute of Actuaries in London in 1924 with age-parameters $a(x)$, $b(x)$, $c(x)$ estimated via observed mortality profile values

Extrapolation via non-exponential transformation

- in case of a presumed non-exponential behaviour for death probabilities, the following representations constitute a selection of potential extrapolation methods:
  - a rational function: $q_x(t_{\text{fut}}) = a(x) + \frac{b(x)}{t_{\text{fut}}}$
  - a polynomial function $q_x(t_{\text{fut}}) = a_0(x) + a_1(x)\,t_{\text{fut}} + ... + a_n(x)\,t_{\text{fut}}^n$ $(n \in \mathbb{N})$
  - a polynomial representation for the logit transform of the so called mortality odds $\ln\left(\frac{q_x(t_{\text{fut}})}{1-q_x(t_{\text{fut}})}\right) = a_0(x) + a_1(x)\,t_{\text{fut}} + ... + a_n(x)\,t_{\text{fut}}^n$ $(n \in \mathbb{N})$
- polynomials of high order describe the underlying data with high accuracy but tend to mismatch the expected trend for extrapolation and therefore possibly show implausible forecasts

Table 3.1: Projection of life table functions via reduction factors and (non-) exponential transformation.

Projection by Parameter Estimation

For vertical / diagonal estimation of life table functions the projection procedure works indirectly since each period / cohort table is described via parameter vectors which themselves have to be fitted in a horizontal way in order to extrapolate future life table function parameters. On the one hand these techniques lower the number of degrees of freedom but bring along possible interdependencies of parameters and high model risk due to the assumption of perfect correlation among the projections in age direction. For the vertical approaches each observed period life table yields an estimated parameter vector. Several indirect projection methods of the vertical / diagonal type can be found in Tables 3.2 and 3.3. An illustration by means of so called mortality law based vertical projection is given in Figure 3.7 and 3.8 of Subsection 3.2.2. The method of diagonal estimation is illustrated in Figure 3.3 using the Brass relational model presented in Table 3.2 for German generation table data from the German Federal Statistical Office (2010) with standard table "1871". In this context, the cohort survival functions of decennial birth years (1870-2040) correspond to the vertical mesh lines in the surface. Due to the linearity assumption which prohibits variations in the level and age patterns the projection surface turns out to be smooth and regular. Several extensions of the relational model approach were proposed including additional parameters to capture deviations from linearity of the logit values.

3.1.5 Life tables for actuarial practice

The calculation of smoothed projected life tables[52] in a dynamic context is of major importance for actuarial tasks like pricing and reserving of insurance contracts especially for those paying living benefits as they underlie mortality improvements due to long-term duration[53]. For practical issues assurers utilize deterministic applied market life tables of the generation type with implicit ad-hoc loadings capturing fluctuation and risk of chance since population and cohort mortality commonly diverge due to selection effects. In addition, certain trends in population mortality are indirectly calculated for annuity portfolios and social insurance programs. To a large extent, these selective market tables are based on mortality data from insurance portfolios. For example, German

[52] The concept of projected tables includes an adjustment of life tables according to anticipated future mortality trends. For an overview of different projection approaches see e.g. Pitacco et al. (2008).

[53] The underestimation of mortality improvements results in unexpected losses for annuity providers. Therefore a dynamic forecast sensitive to time changes in mortality pattern is a basic precondition.

Projection of life table functions by parameter estimation

Vertical mortality law-based projection

The most important representatives of period-specific force of mortality projection methods are:

- the Makeham mortality law version $m_x(t) = a(t) + b(t) \cdot c(t)^x$ introduced by Blaschke (1903)
- the Gompertz law version $m_x(t) = b(t) \cdot c^x$ used by Wetterstrand (1981)
- a generalised linear model as a multivariate approach proposed by Renshaw and Haberman (1996) which considers both the age variation in mortality and periodical changes

$$m_x(t) = \exp\Big(\sum_{i=0}^{s} a_i L_i\,(x')\Big) \exp\Big(\sum_{j=1}^{r} b_j\, t'^j\Big) \quad \text{with} \quad x' = \frac{2(x - x_{\min})}{x_{\max} - x_{\min}},\ t' = \frac{2(t - t_{\min})}{t_{\max} - t_{\min}}$$

with Legendre polynomials $L_i(x')$ of degree i generated by

$$L_0(x') = 1\,, \quad L_1(x') = 1\,, \quad \ldots, \quad L_{i+1}(x') = \tfrac{2i+1}{i+1}\, x\, L_i(x') - \tfrac{i}{i+1}\, L_{i-1}(x');$$

the first exponent describes a graduation formula for age effects distorted by the second calendar-year-adjustment term

Diagonal mortality law-based projection

Cohort-specific graduation methods depend on the year of birth $t - x$ (i.e. on the age x at calendar year t) and are, for example, given by

- the Makeham-based model with central death rate $m_x(t) = a(t-x) + b(t-x) \cdot c(t-x)^x$ applied by Davidson and Reid (1927) where $c(t - x) \equiv c$ and $a(t - x)$, $b(t - x)$ are estimated via cohort graduation for all years of birth $t - x$
- Kermack et al. (1934) suggest an age-cohort model $m_x(t) = a(x) \cdot b(t - x)$ where factor $a(x)$ describes age effects and $b(t - x)$ expresses cohort effects
- Tabeau et al. (2001) takes up an extension in form of an age-period-cohort model with $m_x(t) = a \cdot b(x) \cdot c(t) \cdot d(t - x)$ and parameter constraints $\sum_x \ln(a(x)) = 0$, $\sum_t \ln(b(t)) = 0$ and $\sum_{t-x} \ln(d(t - x)) = 0$
- Di Palo (2005) uses a dynamic version of the Weibull mortality law $m_x(t - x) = \frac{a(t-x)x^{c(t-x)-1}}{b(t-x)^{c(t-x)}}$

Limiting / optimal life tables

- Bourgeois-Pichat (1952) assumes a hypothetical (biological) limit in mortality decline due to humans constitution
- various verifiable trend interpolations based on current and "optimal" tables as well as expert opinions underline this assumption
- life table functions are projected with reference to that limit
- for exponential interpolation of the annual death probabilities q_x it holds $q_x(t) = q_x^{\text{limit}} + \big(q_x(t_{\text{pres}}) - q_x^{\text{limit}}\big) \cdot r_{exp}(x, t - t_{\text{pres}})$ with r_{exp} defined as in Table 3.1

Table 3.2: Projection of life table functions using law-based models and limiting life tables.

Projection of life table functions by reference to model life tables and by application of the Brass relational model and the Petrioli-Berti resistance transform

Model life tables
• demographers relate a set of different model tables with corresponding representative values in form of dynamic markers (e.g. life expectancy at birth) • each marker completely determines the table specific mortality profile with reference to an initial standard table with year of birth z_0 • annual life table functions are calculated by means of estimated marker parameters • due to limited vital statistics the United Nations presented model life tables for underdeveloped nations in 1956 describing age and gender patterns of mortality given by the marker "expected life at birth"

Brass relational model of mortality
• Brass (1974) analyses the time-dependent logit-transform $\Delta(x,t-x) = \frac{1}{2}\ln\left(\frac{1-{}_xp_0(t-x)}{{}_xp_0(t-x)}\right)$ of the survival function ${}_xp_0(t-x)$ for cohort data of successive birth years $z_1,...,z_n$ • under a linear assumption for logit values of sequenced years of birth z_k,z_{k+1} it holds $\Delta(x,z_{k+1}) = a_k + b_k \cdot \Delta(x,z_k) + \varepsilon_x$ for $k = 1,..,n-1$ • the procedure consists of an estimation of the level of mortality a_k, the shape-parameter b_k and the error term ε_x and an application on the inverse logit function to yield the cohort survival function ${}_xp_0(t-x) = (1+\exp(2\Delta(x,t-x)))^{-1}$

Petrioli-Berti resistance transform
• Petrioli and Berti (1979) address the resistance to death function defined as the ratio $$\Delta(x,\omega) = \frac{{}_xp_0\, x}{(1-{}_xp_0)\,(\omega-x)} = \frac{\frac{l_x}{l_0\,(\omega-x)}}{\frac{(l_0-l_x)}{l_0\, x}}$$ representing the average number of deaths between age x and the maximum age ω divided by the average number of deaths prior to x relative to the initial cohort size l_0 • the projection is preceded by a calibration of $\Delta(x,t) = a(t)x^{b(t)}(\omega-x)^{c(t)}$ including the estimation of the period parameters $a(t)$, $b(t)$, $c(t)$ • a conversion of the function $\Delta(x,\omega)$ yields the formula for the survival function ${}_xp_0 = \frac{\Delta(x,\omega)\,(\omega-x)}{x+(\omega-x)\,\Delta(x,\omega)}$ • the method was applied to project the Italian population mortality and adopted by the Italian Association of Insurers for projecting market tables

Table 3.3: Projection of life table functions by reference to model life tables, the Brass relational model and the Petrioli-Berti resistance function.

Diagonal projection via the Brass relational model

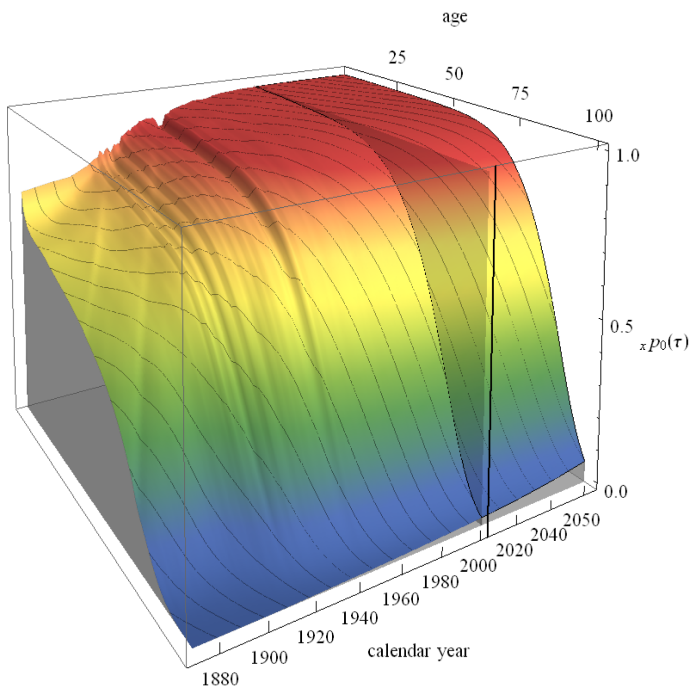


Figure 3.3: Diagonal projection using the Brass relational model based on German generation table data from 1871-2004 published by the German Federal Statistical Office (2010).

insurers make use of the DAV 2004R life table issued by the German Actuarial Society (DAV) for policies with endowment benefits and the DAV 2008T table for life insurance contracts. These life tables include an estimation of future mortality evolution by means of corresponding additional charges. The public retirement system relies on the General German mortality tables (ADSTs) (ADST 1986/88) issued by the Destatis. The so called Heubeck tables designed by the Heubeck AG are based on data from the social security administration system and serve as a basis for actuarial calculation of corporate pension plans from the private business sector.

3.2 Continuous-time frameworks

Despite the fact that mortality data exists in discrete, for instance, annual form continuous-time mortality modelling presents a natural extension since, in reality, mortality evolves continuously. Thus, the integer life table function values are extended for non-negative real ages x and time periods t.

3.2.1 Definition of age-continuous probabilities

The multi-annual survival probabilities are defined as

$$ {}_tp_x = P(\tau_x > t) = P(\tau_0 > x+t \,|\, \tau_0 > x) = \frac{P(\tau_0 > x+t)}{P(\tau_0 > x)} = \frac{{}_{x+t}p_0}{{}_xp_0} \tag{3.10} $$

describing the conditional tail distribution of a newborn's remaining lifetime τ_0 with survival function ${}_xp_0$. In other words, the remaining lifetime $\tau_x \in]0, \omega]$ of an individual aged x has a distribution function ${}_tq_x = 1 - {}_tp_x$ and density function $\frac{d}{dt}{}_tq_x = -\frac{d}{dt}{}_tp_x$ in form of an instantaneous probability measure of death at time $t \geq 0$. The relative instantaneous rate of death, i.e. the death probability for infinitesimal small time periods divided by the tail distribution function, is called (deterministic) force of mortality or instantaneous mortality rate and is defined as

$$ \mu_{x+t} = -\frac{\frac{d}{dt}{}_tp_x}{{}_tp_x} = -\frac{d}{dt}\ln({}_tp_x) \ (t \geq 0). \tag{3.11} $$

Equation (3.11) can be obtained as follows

$$ \begin{aligned} \mu_{x+t} &= \lim_{u \searrow 0} \frac{P\left(\tau_{x+t} \leq u\right)}{u} = \lim_{u \searrow 0} \frac{P\left(t < \tau_x \leq t+u \,|\, \tau_x > t\right)}{u} \\ &= \lim_{u \searrow 0} \frac{\mathbb{P}\left(t < \tau_x \leq t+u\right)}{u \cdot P\left(\tau_x > t\right)} = \lim_{u \searrow 0} \frac{P\left(\tau_x \leq t+u\right) - P\left(\tau_x \leq t\right)}{u \cdot P\left(\tau_x > t\right)} \\ &= \lim_{u \searrow 0} \frac{{}_tp_x - {}_{t+u}p_x}{u \cdot {}_tp_x} = \frac{-\frac{d}{dt}{}_tp_x}{{}_tp_x} = -\frac{d}{dt}\ln({}_tp_x) \end{aligned} $$

where $-\frac{d}{dt}\left({}_tp_x\right)$ denotes the unconditional density function of death at exact age $x+t$. Furthermore, it holds

$$ \mu_{x+t} = \frac{-\frac{d}{dt}{}_tp_x}{{}_tp_x} \quad \Leftrightarrow -\frac{d}{dt}\left({}_tp_x\right) = \frac{d}{dt}\left({}_tq_x\right) = {}_tp_x \cdot \mu_{x+t} $$

such that ${}_tp_x \cdot \mu_{x+t}$ forms the conditional probability density function of τ_x at time t given survival until $x+t$. Therefore we obtain

$$ P\left(\tau_x \leq u\right) = \int_0^u {}_tp_x \mu_{x+t} dt = \int_0^u -\frac{d}{dt}\left({}_tp_x\right) dt = 1 - {}_up_x = {}_uq_x, \tag{3.12} $$

i.e. the weighted arithmetic mean of the force of mortality over the interval [0,u] equals the probability that an individual alive between ages x and $x+u$ dies before attaining the exact age $x+u$. Conversely, the survival function for years $t \geq 0$ can be derived from Expression (3.11) by knowledge of the force of mortality

$$_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right).$$

Traditional static mortality models assume that $\mu_x(t) = \mu_x = \lim_{u \to 0}\, {}_um_x$ for all $t \geq 0$ and $x \leq \omega$, i.e. future mortality rates and survival probabilities at different ages are known and time-invariant. Forecasting the force of mortality seems to be more intuitive in a continuous-time framework on the one hand but demands a careful model choice / approximation[54] on the other hand. Life table functions are assumed to change continuously in the following, i.e.

$${}_ud_x = \int_0^u l_{x+t}\mu_{x+t}dt\ , \quad {}_uL_x = \int_0^u l_{x+t}\, dt.$$

Using integration by parts we obtain an expected lifetime

$$\mathring{e}_x = E\left[\tau_x\right] = \int_0^{\omega-x} t\, {}_tp_x\, \mu_{x+t}dt = \int_0^{\omega-x} t\left(-\frac{d}{dt}\, {}_tp_x\right) dt \tag{3.13}$$

$$= \left[-t\, {}_tp_x\right]_0^{\omega-x} + \int_0^{\omega-x} {}_tp_x dt = \int_0^{\omega-x} {}_tp_x dt. \tag{3.14}$$

The graduation or adjustment of crude mortality data describes a method which creates a smooth continuous shape together with a reasonable age-pattern. The graduation results serve as a starting point for subsequent forecasting applications. Actuarial and demographic literature provides various approaches which can be classified due to their functionality regarding the adjustment procedure, namely a distinction between parametric and non-parametric graduation.

3.2.2 Parametric graduation using mortality law models

In Section 3.1.2 we have learned that life tables represent a demographic method to model discrete-time table functions. For the continuous-time mathematical representation in form of a cross-sectional mortality snapshot a large number of approaches are available. The underlying objective is to obtain an analytic graduation of empirical observations with of a smooth one-factor parametrization function or so called mortality law for the mortality rate, the survival probability or any transform of age-specific probabilities by means of a small

[54] In contrast to realised death probabilities the instantaneous rate of mortality is not observable from population statistics and thus has to be estimated / approximated.

parameter set. The estimated set of parameter is each extrapolated and used to predict function values of the mortality law. In Tables 3.4 - 3.6 we chronologically present some non-polynomial parametrisation functions modelling the exponential shape of the force of mortality. In addition, there are further parametric formulas for annual death probabilities, mortality odds or "cause of death" functions available. However, their illustration is out of the scope of this thesis[55].

Historical force of mortality parametrisations

Gompertz (1825)

- widely used among biologists and demographers
- mortality rates are assumed to increase exponentially with age (especially at adult and retirement ages)
- driven by two parameters $b > 0$ (scaling of the adult mortality level), $c > 0$ (accelerating the shape)

$$\mu_{x+t} = b\, e^{c(x+t)}\,, \quad {}_tp_x = \exp\left(-\frac{b}{c}\, e^{cx}\left(e^{ct} - 1\right)\right) \quad (x > 0)$$

- causes a systematic underestimation at young adult ages and an overestimation at oldest ages

Makeham (1867)

- extension of the Gompertz model with additional constant term a describing ("background") non-senescent mortality independent of age (e.g. accidents)
- mortality rates assumed to increase exponentially with age x
- driven by three parameters $a \geq 0$, $b, c > 0$

$$\mu_{x+t} = a + b\, e^{c(x+t)}\,, \quad {}_tp_x = \exp\left(-at - \frac{b}{c}\, e^{cx}\left(e^{ct} - 1\right)\right) \quad (x > 0)$$

- improvement compared to the Gompertz case for ages $x > 30$ but still overestimation at oldest ages

Thiele (1871)

- the force of mortality equals a function with seven parameters $a, b, c, d, f, g, h > 0$

$$\mu_{x+t} = ae^{-b(x+t)} + c\, e^{-d((x+t)-f)^2} + ge^{h(x+t)} \quad (x > 0)$$

- no explicit survival function formula; for $a = b = d = 0$ we obtain the Makeham law
- the first term describes younger age patterns, the second one describes the "accident hump" at young-adult ages and the third term specifies senescent mortality at older ages
- the formula describes the whole lifespan

Table 3.4: Historical force of mortality parametrisations.

55 See Tabeau et al. (2001) for a detailed schedule of different parametric formulas.

Historical force of mortality parametrisations (continued)

Perks (1932)
• logistic model driven by five parameters $a \geq 0$ $b, c, d, f > 0$ such that $$\mu_{x+t} = \frac{a + b\, e^{c(x+t)}}{1 + d\, e^{c(x+t)} + f\, e^{-c(x+t)}}$$ $${}_tp_x = \exp\left(-at - \left(\frac{b - ad}{cd}\right) \log\left(\frac{1 + d\, e^{c(x+t)}}{1 + d\, e^{cx}}\right)\right) \text{ (for } f = 0 \text{ and } x > 0)$$
• extension of the Gompertz model, which equals the Makeham law for $d = f = 0$
• the denominator flattens out the exponential increase of the numerator for highest ages (finite positive asymptotic mortality)
• the special case $f = 0$ was used to graduate the UK immediate annuitants $a(55)$ mortality table

Weibull (1951)
• originally used for reliability modelling of technical systems in engineering
• driven by the two parameters $k, n > 0$ whereas k describes the scale and n the shape of the function $$\mu_{x+t} = k(x+t)^n\,, \quad {}_tp_x = \exp\left(-\frac{k}{n+1}\left((x+t)^{n+1} - x^{n+1}\right)\right) \quad (x > 0)$$

Beard (1959)
• driven by three parameters $b, c, d > 0$ $$\mu_{x+t} = \frac{b\, e^{c(x+t)}}{1 + d\, e^{c(x+t)}}\,, \quad {}_tp_x = \exp\left(-\frac{b}{cd} \log\left(\frac{1 + d\, e^{c(x+t)}}{1 + d\, e^{cx}}\right)\right) \quad (x > 0)$$
• extension of the Gompertz model ($d = 0$)
• special case of the Perks model for $a = f = 0$

Table 3.5: Historical force of mortality parametrisations (continued).

Chapter 2 has shown that for the mortality patterns of recent decades the exponential rate of increase declined for highest ages. Thus the classical exponential mortality laws from Tables 3.4 and 3.5 tend to fail the representation of increase mortality of old ages[56]. The logistic model variants in Tables 3.5 and 3.6 promise remedy since they consider the saturation effect of highest ages.

In order to illustrate the parametric graduation we fit the mortality rate profile of German male mortality data for the period 2006 provided by the Human Mortality Database (2009) by means of parametrisations from Tables 3.4 to 3.6. More precisely, we minimized the weighted sum of squares (WSS) for the

[56] See Pitacco (2004b) for a more comprehensive treatment on old age mortality under parametric models.

Recent force of mortality parametrisations

Siler (1979)

- the force of mortality equals the sum of three terms determining age independent, infancy and old age mortality
- driven by the five parameters $a, b, c, d, f > 0$ such that

$$\mu_{x+t} = a + b\,e^{-c(x+t)} + d\,e^{f(x+t)}$$
$$_tp_x = \exp\left(-at - \frac{b}{c}\left(e^{cx} - e^{-c(x+t)}\right) - \frac{d}{f}\left(e^{c(x+t)} - e^{-cx}\right)\right) \quad (x > 0)$$

- extension of the Makeham model by adding a component $b\,e^{-c(x+t)}$ (b determines the level and c the rate of decline) describing the high death risk in the first life year which rapidly declines for proceeding infancy years

Forfar et al. (1988)

- the force of mortality equals the general Gompertz-Makeham graduation formula $GM(r,s)$ divided into a polynomial term with r-parameters $(\alpha_0,\ldots,\alpha_{r-1})$ and an exponential term with s-parameters $(\alpha_r,\ldots,\alpha_{r+s-1})$

$$\mu_{x+t} = \sum_{i=0}^{r-1} \alpha_i (x+t)^i + \exp\left(\sum_{j=r}^{r+s-1} \alpha_j (x+t)^{j-r}\right)$$

- the Gompertz-Makeham law equals the $GM(1,2)$-formula
- for $r = 0$ the central death rate equals the exponential term, for $s = 0$ only the polynomial term is considered
- the CMIB (1990) used $GM(0,2)$, $GM(2,2)$ and $GM(1,3)$ versions for graduation purposes

Thatcher (1999)

- the force of mortality is given by a logistic function of age x based on the Perks formula
- driven by parameters $a, b, c > 0$ and "background" mortality level parameter $d \geq 0$

$$\mu_{x+t} = \frac{ab\,e^{c(x+t)}}{1 + b\,e^{c(x+t)}} + d\,, \quad {}_tp_x = \exp\left(-dt - \frac{a}{c}\log\left(\frac{1 + be^{c(x+t)}}{1 + be^{cx}}\right)\right) \quad (x > 0)$$

- for small $b\,e^{c(x+t)}$ (usually fulfilled for ages $x \leq 70$ and small $c < 1$) the death rates approximately follow a Gompertz-Makeham law, for advanced ages the function shows a plateau shape
- for $a = 1$ the robust three-parameter model successively fits limiting age mortality since it levels off at $1 + d$

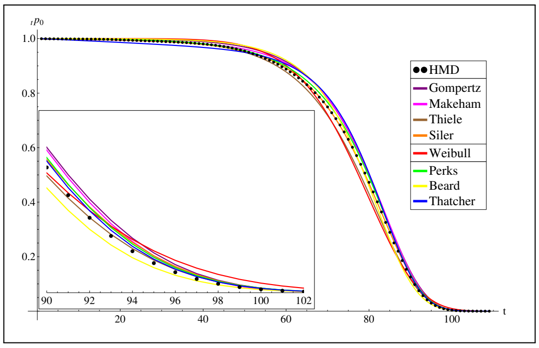
Table 3.6: Recent force of mortality parametrisations.

mortality rate residuals

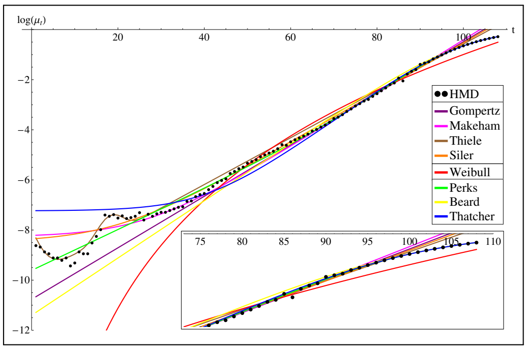
$$WSS(x) = \sum_{t=0}^{\omega-x} w_t \Delta_{x+t}^2 = \sum_{t=0}^{\omega} w_t(\mu_{x+t} - \hat{\mu}_{x+t})^2$$

for different sets of weights w_t ($\sum_{t=0}^{\omega-x} w_t = 1$) and crude central death rates $\hat{m}_{x+t}$ over the whole lifespan. Results are depicted in Subfigure (3.4.1) - (3.4.3). Subfigure (3.4.1) illustrates that all laws of the exponential model class

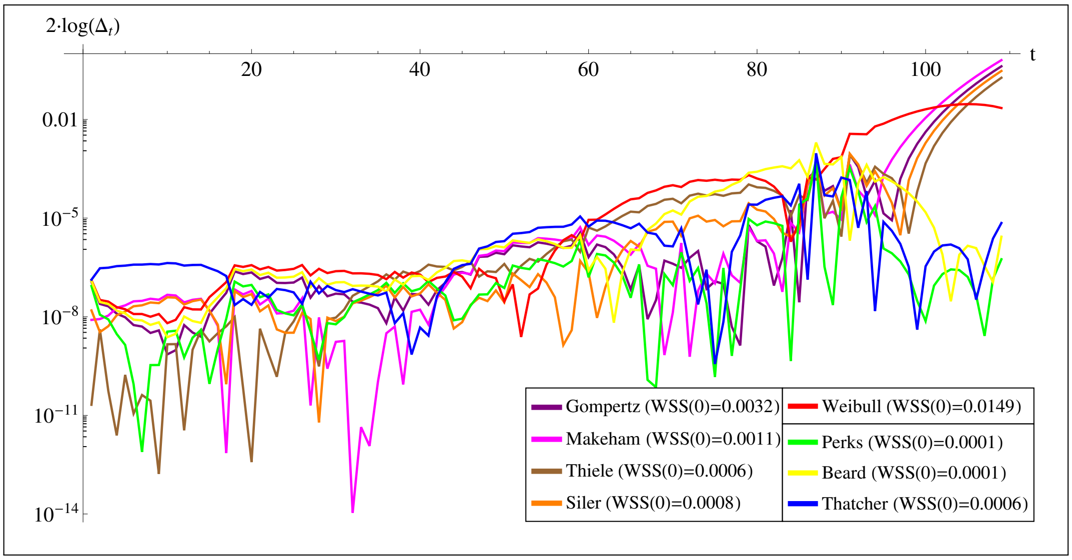
Mortality law parametrisations fitted to period life table data



(3.4.1) Fitted survival functions



(3.4.2) Fitted mortality curves



(3.4.3) Squared residuals of fitted survival curves (log-scales)

Figure 3.4: Mortality law parametrisations fitted to the 2006 German period life table data originated from the Human Mortality Database (2009).

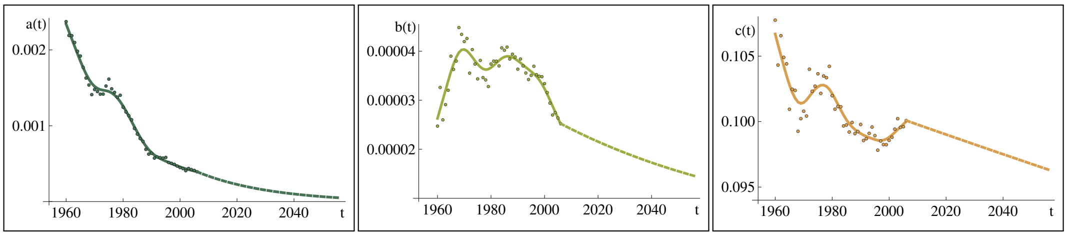
(Gompertz, Makeham, Thiele, Siler) fit the survival curve in a satisfactory way for adult ages except for the Thiele law. The Thiele parametrisation, in turn, successfully reproduces the accident hump effect (see Subfigure (3.4.2)). The only power function represented by the Weibull mortality law shows relatively poor fitting results over the whole age range. Additionally, Subfigure (3.4.3) attests good approximation results for a graduation with logistic models (Perks, Beard, Thatcher) in the saturation area of oldest age mortality. On the one hand, Beard and Perks slightly overestimate survival probabilities for ages 30-85 (respectively life expectancy) but, on the other hand, adequately fit advanced ages $x > 85$. Since the Beard model forms an extension of the Perks model its fitting results are slightly worse due to a loss of flexibility. The Perks model describes a fairly satisfactory fitting for the whole-life-span with minimal residuals Δ_t for young and oldest ages. The Thatcher law underestimates survival probabilities for young and old ages. All logistic models neither consider infant mortality nor the accident hump. Whereas the exponential class mostly overestimates the real curtate life expectancy $e_0 = 77$ of a newborn on average by 2%, the logistic class causes minor deviations up to 1.8%. We observe that none of the parameteric models is able to completely fit the whole age range of the period table. Since various laws describe certain age effects with different fitting quality Carriére (1992) uses a multi decrement mixture ${}_xp_0 = \sum_{k=1}^{m} \Psi_k \cdot {}_xp_0^{(k)}$ of parametric survival functions ${}_xp_0^{(1)}, {}_xp_0^{(2)}, \ldots, {}_xp_0^{(m)}$. The factor Ψ_k defines the probability that a newborn individual dies by cause $k = 1, \ldots, m$, e.g. neonatal, childhood / teenage and adult mortality. Therefore, the author constructs a static mortality model for the whole age range describing all stages[57] of a mortality profile.

If we are interested in the projection of parametric graduated mortality data we could, for example, follow the vertical law-based approach presented in Table 3.2. The procedure consists of an estimation of the period-specific law parameters based on past mortality experience. Thereafter, the different sets of model parameters are spline smoothed (alternatively, parametric graduation could also be considered) and projected by means of a (deterministic) function, e.g. the exponential function class, or a stochastic process, e.g. an autoregressive integrated moving average (ARIMA) process. A major drawback of this approach is the noticeable amount of subjective judgement concerning the choice of the smoothing parameter and projection function / process parameters, the lengthy data requirement and the disregarded correlation structure. Therefore the number of mortality law parameters plays a crucial role since estimates are commonly correlated (see e.g. the obvious negative correlation of the Gompertz-Makeham parameters $b(t)$ and $c(t)$ in Subfigures (3.5.2) and (3.5.3)) and parameter risk aggregates especially for long projection horizons. Figure 3.7 shows (deterministic) vertical law-based projections for the Gompertz-Makeham model according to Blaschke (1903) and the Perks law

[57] Inter alia, Carriére (1992) fits the US Decennial Life Table for the period 1979–1981 by a mixture of a Weibull, Inverse-Weibull and Gompertz law. The probabilities $\Psi_1, \ldots, \Psi_3$ relate to the probability of dying from childhood, teenage and adult causes respectively.

where parameters were estimated, spline-smoothed and exponentially projected based on German period life table data for calendar years 1956–2006.

Gompertz-Makeham survival function parameter estimates / extrapolation



(3.5.1) Parameter $a(t)$ **(3.5.2)** Parameter $b(t)$ **(3.5.3)** Parameter $c(t)$

Figure 3.5: Gompertz-Makeham survival function parameter estimates (dotted), spline functions (solid) and exponential extrapolation (dashed) for the law-based vertical method based on German life tables data from the Human Mortality Database (2009) for calendar years 1960–2006.

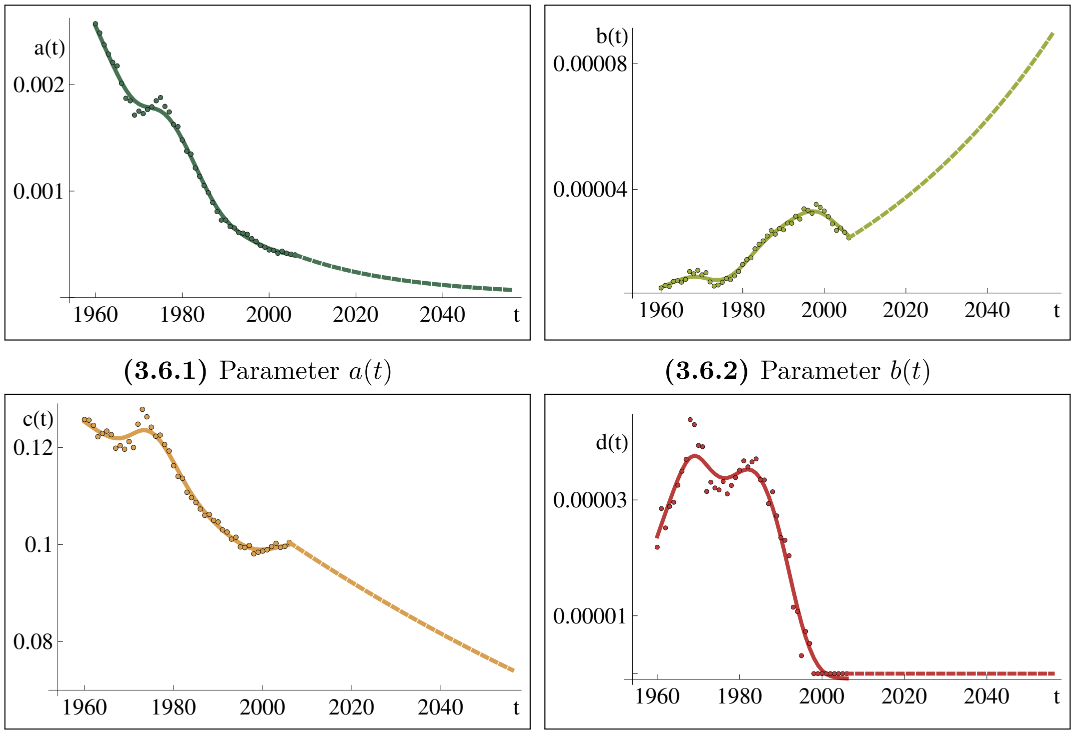
Whereas the Gompertz-Makeham law shows the typical estimation behaviour[58] already mentioned in Table 3.4, the Perks model reveals more flexibility and minimizes residuals especially for post-retirement ages at the price of high parameter risk due to a separate extrapolation[59] of four distinct parameter sets. Law-based approaches are partly controversially discussed in actuarial literature due to interdependencies of the estimated parameter sets. If correlation is not taken into account, which is obviously the case for a separate parameter extrapolation, forecast results may show implausible mortality patterns. Nevertheless, the law-based approach enables a strong reduction in the forecasting dimension combined with an “easy-to-interpret” representation. In regard to the fitting quality of the Gompertz-Makeham and the Perks model, the residual maps in Figure 3.8 underline the statements mentioned above. Both models fit older mortality profiles worse than more recent ones. The Gompertz-Makeham law graduation underestimates mortality for ages x younger than 35 and retirement ages between 65 and 80 years. On the other hand, it overestimates ages 40-60 and oldest ages $x > 80$. However, the Perks model shows a similar residual structure with lower absolute residual values and an additional underestimation of mortality at oldest ages $x > 80$.

For a more appropriate extrapolation method cross-correlated time series models have frequently been applied. In any case, according to Booth and

58 The Gompertz-Makeham law assumes a constant exponential rate of mortality increase. Empirical observations show a decline in the slope at oldest ages. Thus the Gompertz-Makeham model features noticeable deviations for ages $x > 100$.

59 We set the infancy scaling parameter f equal to zero in order to limit the parameter uncertainty and to obtain a closed-form for the survival function.

Perks survival function parameter estimates / extrapolation



(3.6.1) Parameter $a(t)$ **(3.6.2)** Parameter $b(t)$

(3.6.3) Parameter $c(t)$ **(3.6.4)** Parameter $d(t)$

Figure 3.6: Perks survival function parameter estimates (dotted), spline functions (solid) and exponential extrapolation (dashed) for the law-based vertical method based on German life table data from the Human Mortality Database (2009) for calendar years 1960–2006.

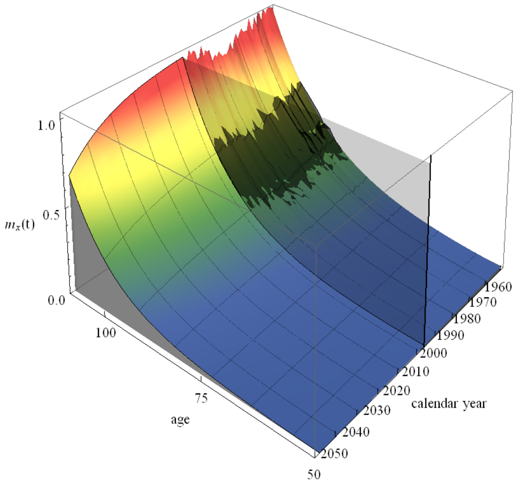
Tickle (2008) these methods contain a high amount of parameter as well as model risk such that the reliability of prediction intervals is limited to a certain degree. Furthermore, multivariate extrapolation may cause computational difficulties.

3.2.3 Non-parametric graduation

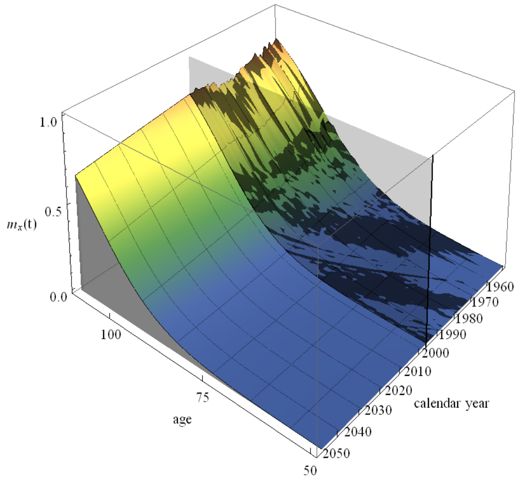
Along the lines of Debón et al. (2006) a non-parametric estimation of smooth initial mortality quantities excludes the risk of choosing inadequate functions or parameters. Non-parametric graduation or smoothing does not assume any underlying distribution but rather describes the shape of the regression function in a data-analytic way. In the following, we only refer to the graduation of crude central death rates m_x. Alternatively, we could also smooth any other life table function or a time series of period-specific parameter estimates for a parametric function[60]. The sequence $m_{x_1}, ..., m_{x_n}$ denotes the crude

[60] In Subsection 3.2.2 we exemplarily apply a vertical law-based extrapolation procedure using the Gompertz-Makeham and Perks mortality law.

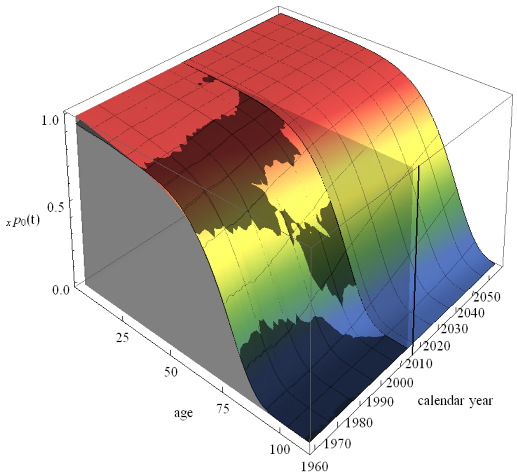
Fitted mortality law profiles and vertical law-based projections



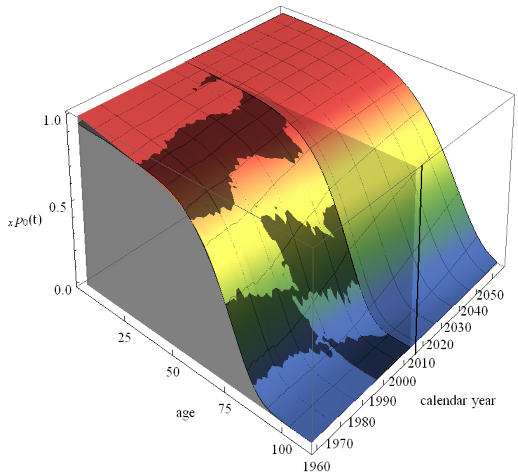
(3.7.1) Gompertz-Makeham mortality surface (exponential model)



(3.7.2) Perks mortality surface (logistic model)



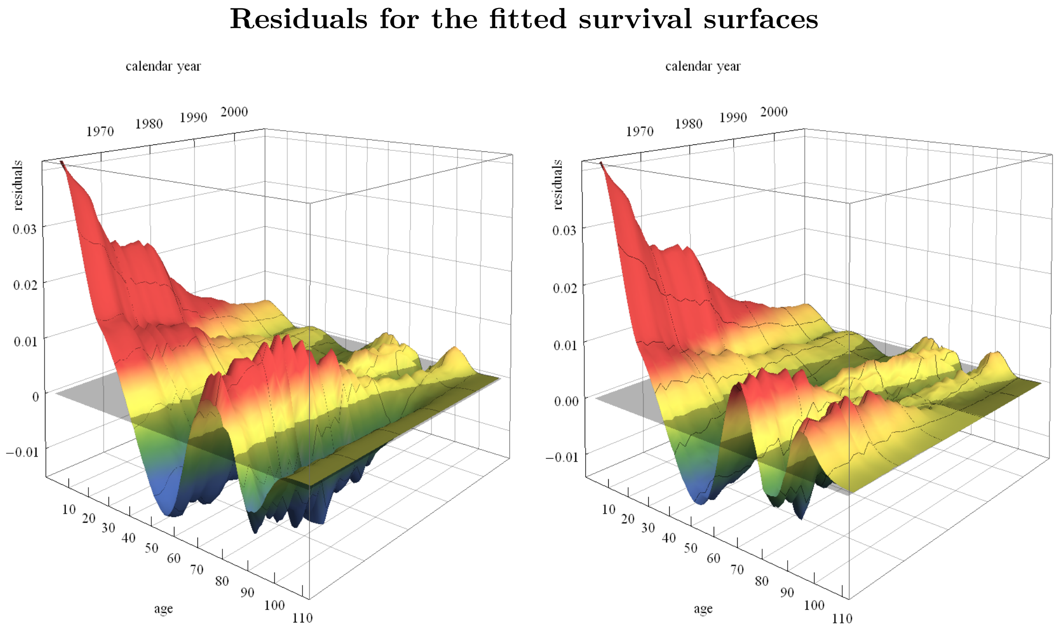
(3.7.3) Gompertz-Makeham survival surface (exponential model)



(3.7.4) Perks survival surface (logistic model)

Figure 3.7: Fitted mortality / survival surfaces together with vertical law-based projections for German life table data (dark regions) from the Human Mortality Database (2009) for periods 1960–2006.

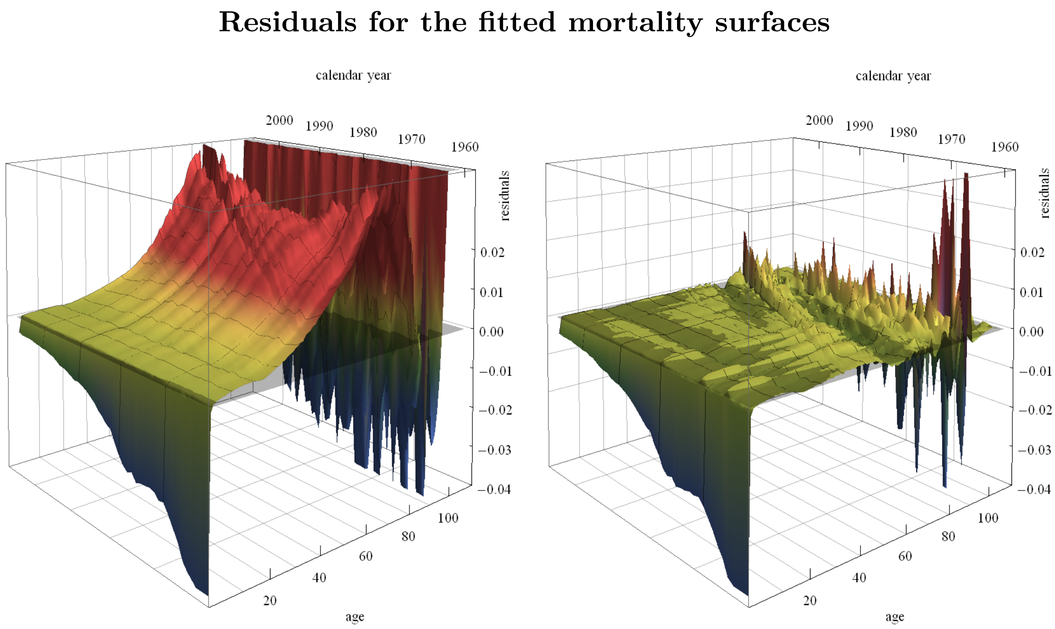
observed death rates for ages $x_1, ..., x_n$ from a connected series of numbers $\{1, 2, ..., n\} \in [0, \omega]$ where n denotes the cardinality of the age group set. A standard univariate non-parametric regression model describes the relation of a dependent variable m_{x_i} and the independent variable x_i via $m_{x_i} = \Psi(x_i) + \epsilon(x_i)$ with independent identically normally distributed error term $\epsilon(\,.\,)$ and unknown smoothing function Ψ. The main task is to locally approximate Ψ based on a ideally large sample with the objective of minimizing the sum of squared errors $\epsilon(x)$. The weighted mean estimator forms a smooth function with predictor x. For this purpose, four prominent smoothing methods (so called smoothers) are available to determine the local mean estimator of the regression function.



(3.8.1) Gompertz-Makeham (exponential model)

(3.8.2) Perks (logistic model)

Figure 3.8: Residuals for the fitted survival surfaces based on German life table data from the Human Mortality Database (2009) for periods 1960–2006.



(3.9.1) Gompertz-Makeham (exponential model)

(3.9.2) Perks (logistic model)

Figure 3.9: Residuals for the fitted mortality surfaces based on German life table data from the Human Mortality Database (2009) for periods 1960–2006.

More precisely, we distinguish between smoothing applying generalised moving average, kernels, regression polynomials or splines. A detailed list is given in Tables 3.7 - 3.9.

Actuaries could apply univariate non-parametric graduation as a supplementary and exploratory method to narrow down the range of potential parametric choices and thus to improve statistical performance. Some authors like Pagan and Ullah (1999) refer to semiparametric approaches which smooth estimates obtained by parametric graduation to improve the fitting quality. We apply this approach for the mortality law projections illustrated in Figure 3.6 and 3.9. The performance is frequently measured by the mean squared error, mean absolute error or mean absolute percentage error for the smoothed and raw data. This is carried out within the time series analysis in Section 5.4 of Chapter 5.

Non-parametric graduation (moving average and kernel smoothing)

Generalised moving weighted average smoothing

Moving average graduation describes the simplest smoothing method for univariate data. The local regression estimator is given, for example, by a weighted average of consecutive crude central death rates which forms a popular explorative approach to obtain an impression of underlying trends.

- based on a set of ages $\{x_1,x_2,...,x_n\}$ the k-point weighted moving average is calculated as

$$\hat{\Psi}(x_i) = \sum_{\nu=1}^{n} w(x_i,\nu) \cdot m_{x_\nu}$$

  where $N(x_i,k) := [x_{i-k},x_{i+k}]$ ($k \leq i$, $k \in \mathbb{N}$) describes the k-nearest neighbourhood of age x_i with range $2k+1$

- the weights are given as $w(x_i,\nu) = \begin{cases} \frac{1}{2k+1} & \text{if } -k \leq \nu - i \leq k \\ 0 & \text{otherwise} \end{cases}$ for all $\nu = 1,...,n$ and sum to one
- boundary ages can be estimated under application of special reflection boundary conditions

Kernel smoothing

Kernel smoothers represent a popular choice for the regression function estimator. The application solely requires the specification of a bounded, symmetric kernel function K in form of a probability density and the bandwidth b to control the smoothness of the estimator.

- possible regression estimator choices are the Nadaraya-Watson, Priestley-Chao or Gasser-Müller estimator using weight functions of Uniform, Normal, Gaussian, Triangular, Parzen or Epanechnikov type
- the kernel K assigns weight to each data point based on the distance and the bandwidth
- bandwidth selection due to subjective (e.g. rules of thumb or visual trial and error) or data-driven methods (e.g. cross-validation or local choice) implies the risk of under-/over-smoothing (trade-off between bias and variance of the estimator)
- small bandwidths reproduce the data, large b values result in an over-smoothed curve in form of the averaged response variables
- an increased bias for boundary ages requires the implementation of certain boundary kernels

Table 3.7: Univariate local moving weighted average smoothing and kernel smoothing.

Non-parametric graduation (polynomial regression smoothing)

Local polynomial weighted regression smoothing

The local polynomial weighted regression represents an generalisation of the local linear weighted regression and is based on the idea that the unknown smoothing function Ψ can be approximated locally by a Taylor polynomial.

- fitting takes place for a smooth curve on local neighbourhood subsets $N(x_i,k) := [x_{i-k},x_{i+k}]$ $(k \leq i \in \{1,...,n\})$ of the crude data by a local polynomial

$$g(a_0,a_1,...,a_r;x) := a_0 + a_1\,x + ... + a_r\,x^r$$

  with fixed degree r (i.e. $r = 1$ generates a linear and $r = 2$ a quadratic regression polynomial)

- methods includes boundary correction and addresses heteroscedastic data via different weights

$$w(x_i,\nu) = \frac{1}{b(x_i,k)} K\left(\frac{x_i - x_\nu}{b(x_i,k)}\right)$$

  for a kernel function K and bandwidth

$$b(x_i,k) = \max_{\substack{j,l \\ x_j,x_l \in N(x_i,k)}} \{|x_j - x_l|\}$$

- for every age x_i $(i = 1,...,n)$ the parameter estimates are solved by the weighted least square equation

$$\left(\hat{a}_0^{(i)},...,\hat{a}_r^{(i)}\right) = \operatorname*{argmin}_{a_0^{(i)},...,a_r^{(i)}} \sum_{\substack{\nu \\ x_\nu \in N(x_i,k)}} w(x_i,\nu)\left(m_{x_i} - g(a_0^{(i)},...,a_r^{(i)};x_\nu - x_i)\right)^2$$

  such that the local polynomial least squares estimator for x_i equals

$$\hat{\Psi}(x_i) = g(\hat{a}_0^{(i)},...,\hat{a}_r^{(i)};0) = \hat{a}_0^{(i)}$$

- the local polynomial weighted regression reveals certain advantages in comparison to the kernel regression such as the Nadaraya-Watson estimator

Table 3.8: Univariate local polynomial weighted regression smoothing.

Non-parametric graduation methods (spline smoothing)

Spline smoothing

- The Whitaker-Henderson method (see Macaulay (1931)) represents a discrete "spline approximation". The trade off between smoothness and goodness of fit is reached by minimizing the residual sum of squares and a roughness penalty

$$\sum_{i=1}^{n} (m_{x_i} - \hat{m}_{x_i})^2 + \lambda \sum_{i=1}^{n-k} \left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} m_{x_{i+k-j}} \right)^2$$

  for a smoothing preference parameter λ and the inclusion of the k-th forward difference of the crude observed values. For $\lambda = 0$ the data is interpolated, for large λ we obtain a global linear least-squares fitting.

- Regression splines $f(a_0,...,a_{r+m};x) = \sum_{j=0}^{r} a_j x^j + \sum_{l=1}^{m} a_{r+l} \left([x - x_l]^+\right)^r$ of order r represent a B-spline[61]curve composed of truncated polynomial B-spline base functions smoothly joined together and separated by so called (inner) spline knots $\xi_1,...,\xi_m$. An ordinary least-squares minimization subjected to $\sum_{l=1}^{m} a_{r+l}^2 \leq c$ for some $c > 0$ avoids an overfitting and defines the regression curve.

- The penalized likelihood approach (see Green and Silverman (1993)) describes the most widely investigated spline method minimizing the penalized sum of squares

$$(\hat{a}_0,...,\hat{a}_{d-1}) = \underset{a_0,...,a_{d-1}}{\operatorname{argmin}} \sum_{i=1}^{n} (m_{x_i} - f(a_0,...,a_{d-1};x_i))^2 + \lambda \int_{x_1}^{x_n} \left(\frac{d^k f(x)}{dx^k} \right)^2 dx$$

  over all smoothing splines $f(a_0,...,a_{d-1};x) := \sum_{i=0}^{d-1} a_i B_i(x)$ as a linear combination of $d = r + m + 1$ B-spline base functions of degree r. The choice of $k = 1$ ($k = 2$) evokes a quadratic (cubic) spline. The roughness of $f(a_0,...,a_{d-1};x_i)$ is defined as the integral of its squared k-times derivative and parameter λ penalizes the roughness of the spline. The spline value $\hat{\Psi}(x_i) = f(\hat{a}_0,...,\hat{a}_{d-1};x_i)$ yields the regression estimate for age x_i ($i = 1, ..., n$).

- Currie et al. (2004) apply a smoothing and projecting of mortality rates for two data sets provided by the Continuous Mortality Investigation Bureau (CMIB).

Table 3.9: Univariate local spline smoothing.

3.3 Limits of classical frameworks

Classical projection methods assume that mortality evolves time-invariant in the way that observations from the past will continue in the future. All kinds of projection methods dealing with extrapolation of life table functions in both parametric and non-parametric form are subjected to an considerable amount of model risk since past trends do not necessarily hold for the future such that forecasts can systematically deviate from actual mortality developments. Traditional models provide single numerical values in combination with high and low scenarios and ignore deviations around that estimates due to model

[61] The term B-spline is short for basis spline and denotes a parametric, cubic curve similar to Bèzier or Hermite curves.

and parameter risk and therefore disregard stochastic evolution or random trends of future mortality. Nevertheless, the combination of graduation and traditional extrapolation methods is still widely applied in actuarial practice. In particular, exponential mortality, limit life tables and law-based extrapolation mark popular techniques in that regard.

Demographic observations show that in the long run mortality evolution features long-term effects like rectangularisation, expansion or increasing life expectancy, but beyond that exhibits random short to mid-term effects which deserve a stochastic modelling. A detailed illustration of German mortality data reveals a general increase in life expectancy with random developments dependent on age, calendar year and birth cohort[62]. From this point of view, deterministic mortality functions seem to be feasible for the construction of (best estimate) life tables, smoothing or completion of raw data and scenario based forecasting but inadequate for the calculation of present benefit values and actuarial reserves. Moreover, a deterministic actuarial approach does not seem to be the model of choice for flexible forecasting and bears the systematic risk to misjudge mortality development especially when vertical estimation methods are applied.

[62] See Chapter 4 for a more detailed discussion on effects inherent in German mortality data.

We have two classes of forecasters: Those who don't know – and those who don't know they don't know.

JOHN K. GALBRAITH, Canadian-American economist and diplomat who lived from 1908 to 2006

CHAPTER 4

Stochastic Mortality Modelling – Time Series, Short-Rate and Market Methods

As already mentioned in Chapter 3, actuaries and demographers need to carefully identify past trends of mortality to predict its random future evolution. In addition to the traditional static approaches presented in Sections 3.1 and 3.2 we review stochastic models in both discrete and continuous time that are frequently applied in actuarial literature. Based on a brief illustration of certain effects inherent in German mortality data we give intuition for the concept of stochastic mortality and review the most important probabilistic process implementations. To a great extent the state-of-the-art concepts go back to already established short-rate and market models successively used in the field of financial mathematics due to a similar definition for the instantaneous interest, credit default and mortality rate. For further discussion on these similarities see Milevsky and Promislow (2001), Biffis (2003), Schrager (2006) or Cairns et al. (2006a, 2008).

The chapter is organised as follows: Section 4.1 discloses the major shortcomings of a time-invariant consideration of mortality and shows the volatility of mortality evolution caused by different age, period and cohort effects. Furthermore, the section contains a list of quantitative and qualitative criteria that measure a model's degree of appropriateness in the light of a chosen forecasting purpose. For this reason, discrete-time as well as continuous-time stochastic models are introduced in Sections 4.2 and 4.3 in form of time series, short-rate, forward and market models. In contrast to deterministic models which provide single point forecasts or trajectories respectively, their stochastic counterparts produce prediction intervals and therefore allow for a measurement of forecasting and, when appropriate, parameter uncertainty. However, approaches like the market model comply with a one-to-one transfer from well known interest rate frameworks and thus have to be checked for their suitability for mortality modelling. Some concluding remarks on Chapters 3 and 4 are given in Section 4.4.

4.1 Stochastic mortality modelling

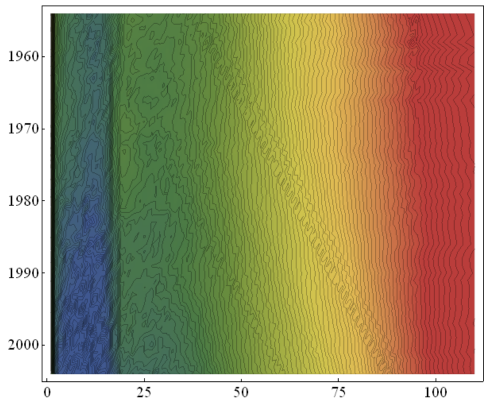
4.1.1 Motivation and empirical evidence

Several authors of recent actuarial and demographic literature make aware of the random nature of (long-term) improvements in mortality especially for the retirement age groups. For instance, Macdonald et al. (2003) observe a rapid improvement of mortality due to unknown driving factors. In Gallop (2007) smoking trends, increasing uncertainty at younger ages, medical advances, infectious diseases, obesity or social class formation are, apart from further factors, referenced as key forces for UK mortality evolution. Currie et al. (2004) state that prediction of future mortality is problematic due to random fluctuations as well as model and parameter risk. They disclose that for larger time horizons the width of the confidence intervals for the future development of mortality rates diverges and thus indicates growing uncertainty in the projection.

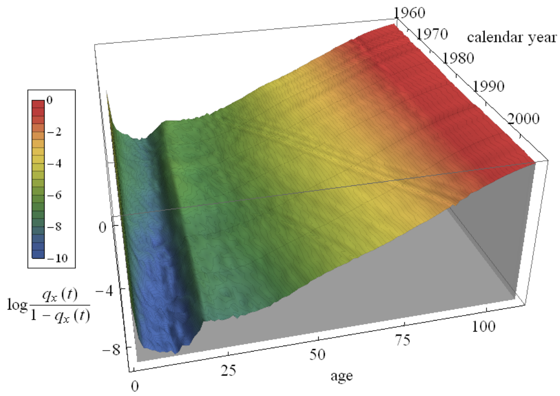
The different trends in Section 2 emphasise that mortality evolves in two dimensions namely age and time. The German logarithmic mortality odd surface in Figure 4.1 as a function of the annual death probabilities of an individual aged x and calendar year t shows random improvements in the mortality profiles for the periods 1960 to 2006 (Subfigure (4.1.1)) but also variable pattern improvements between different age groups x (Subfigure (4.1.2)). On the one hand, these two-dimensional improvements proceed in a continuous way due to long-term gradual developments like nutrition improvements or lifestyle changes. On the other hand, sudden mortality shocks caused by wars, pandemics but also medical advances or genetic discoveries occasionally appear.

A closer inspection of certain age group mortality curves in Subfigure (4.2.1) yields that basically mortality improved non-linearly for all birth cohorts since 1876 but every individual age group underwent its own random evolution which turned out seriously for young and only slight for highest ages (cf. Figure 3.2 in Chapter 3). Some age-specific improvement courses even show intersections which in turn speaks in favour for an age independent evolution. Furthermore, Subfigure (4.2.2) reveals that the most noticeable rates of change have appeared

Mortality surface for German life table data



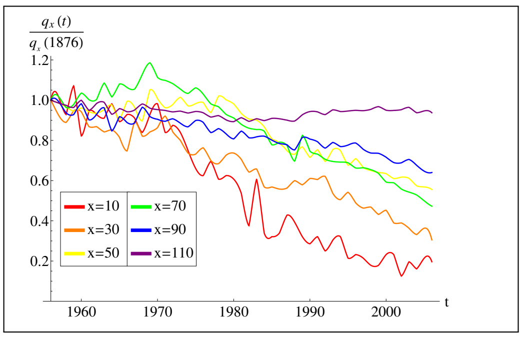
(4.1.1) Mortality surface (two-dimensional)



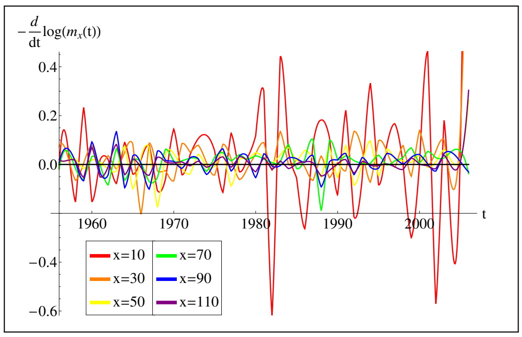
(4.1.2) Mortality space (three-dimensional)

Figure 4.1: Raw mortality surface based on German period life table data from the Human Mortality Database (2009).

Period and age effects for German life table data



(4.2.1) $q_x(t)$ relative to the 1876 cohort



(4.2.2) Logarithmised annual improvement rates of change

Figure 4.2: Observed period effects based on German male life tables for periods 1956-2006 from the Human Mortality Database (2009).

for old ages around the sixties and seventies and younger ages in the eighties. The trend appears similar for varying age groups but to some extent clearly differs in the corresponding shape. It seems that subsequent annual rates of change are negative autocorrelated in the way that positive and negative changes alternate due to raw mortality data used.

4.1.2 Impacts on the underwriting practice

The introduction of stochastic projection methods makes aware of systematic mortality risk when realised mortality rates differ from anticipated. This has been disregarded by traditional approaches so far. Pitacco (2004b) comments that deterministic projection leads to systematic underestimation of realised mortality improvements. In particular, for long time horizons the longevity risk forms an important risk factor. Underwriters as well as pension schemes lack effective risk management and hedging[63] tools and are therefore confronted with an unexpected elongated annuity payout period due to longer living retirees.

Besides demographic evidence stochastic modelling is justified as a result of the increasing competition through deregulation of the European insurance market and a growing product range including with-profit / unit-links variants with or without guaranteed options. Cairns et al. (2008) mention that actuarial applications like reserving (especially for those contracts with embedded guarantees that demand a conventional reserve), premium calculation or the pricing of mortality derivatives necessitates the application of stochastic simulation and option price theory.

Although stochastic mortality modelling has still not established that much in insurance practice[64], basic internal stochastic projection models serve for tasks like financial planning, the determination of risk capital as well as the validation against experience and pricing assumptions.

4.1.3 Qualitative selection criteria for stochastic mortality models

The question of the general adequacy of a (stochastic) mortality model can be answered by compliance with certain qualitative criteria. A verification of these criteria can either take place at the time of the model choice or immediately after the calibration of model parameters and extrapolation of life table functions respectively. Several authors like Luciano and Vigna (2006) or Cairns et al. (2009, 2008) agree in a set of characteristics a meaningful mortality model or alternatively its modelled values should optimally fulfil. In processing this criteria catalogue and further measures we get an evidence whether a model is more appropriate for an intended application or not. As expected there will be no single model dominating all other remaining models under all criteria. Moreover it can be assumed that some models perform better only under a selection of different criteria. An comprehensive verification of both qualitative and further common quantitative selection criteria for the well-known Lee-Carter model class is presented in Chapter 5. Qualitative

[63] In contrast to the risk of chance in mortality which manifests in deviations of expected and actual number of deaths the risk of change (e.g. longevity) cannot be diversified by an increasing number of cohort members. Thus, pricing and risk managing methods have to include adequate risk loadings and margins, respectively.

[64] The Society of Actuaries (2009) has commissioned a survey among US life insurers concerning their mortality risk management, inter alia, the extent of stochastic methods if applied.

selection criteria for mortality models are the following:

Transparency and simplicity

A mortality model which is intuitive and easy to understand minimizes the risk of an inappropriate implementation. Since an estimation of the degree of uncertainty is a subjective one, the demand for a transparent forecasting procedure is of a rather general nature.

Positive force of mortality

By definition, a crucial demand concerning the behaviour of the force of mortality is to claim positive rates which increase with age $x \in [0, \omega]$ for all future time points such that survival probabilities are well defined for all ages. Unfortunately, some stochastic short-rate processes exhibit negative mortality rates with positive probability.

Data Consistency

If a model prescribes certain independence or distributional assumptions which are not consistent with the structure of historical data its forecast performance needs to be mistrusted. Conversely, insufficient data leads to a certain degree of model risk, and, even in case of an appropriate model choice, still contains parameter risk. However, forecasted scenarios or uncertainty measures based on a chosen model should at least be compatible with trends or variation observed in mortality data[65]. The model should ideally be flexible enough to represent long-term trends as well as mid-term developments in mortality.

Biologically reasonable long-term dynamics

From a biological perspective, there are some requirements for a mortality model which are more or less controversially discussed in recent literature:

- Due to the process of ageing, mortality rates should rise with increasing age.
- Several authors argue whether there is a biological limiting age which serves as an upper bound for model considerations.
- Evolutional changes in mortality are linked to the incidence of random events, i.e. deterministic long-run mean-reversion of mortality processes might be unreasonable since there is no empiric evidence of a return to long-term average mortality. Luciano and Vigna (2006) therefore suggest to apply non mean reverting affine processes that better describe the rectangularization phenomenon.

[65] The appropriateness of forecasted scenarios or uncertainty measures can, for example, be verified by backtesting methods.

- There is an ongoing debate about the application of stochastic processes with either diffusion or jump noise term. For instance, Bauer et al. (2008a) use a Brownian motion due to distributional assumption. While short-rate approaches often feature (affine) diffusion processes driven by Brownian motions, some authors apply jump-diffusion or pure jump-processes. For biological reasons, especially jump models appear reasonable if random mortality shocks through pandemics, medical breakthroughs, natural disasters or wars are taken into account. Hereby, it is assumed that these events have a direct effect on life table functions or at least cause a subsequent impact on the overall mortality. Since diffusion processes reproduce a smooth mortality pattern, jump terms are foremost adequate to model short-term mortality shocks like epidemiological catastrophes or key medical breakthroughs. Furthermore, it is often observed that calibration leads to process diffusion parameters close to zero. For this reason somebody could argue whether to use a diffusion term at all.

Systematic developments and correlation structure

With the last 130 years (national) mortality data has shown systematic developments in form of long- and medium-term trends. More precisely, there is empirical evidence for age-, period- or cohort-specific[66] effects relating to the individual evolution in different (consecutive) age groups, calendar years or for cohorts with certain years of birth. Thus, a flexible mortality model should be sensible to such developments. In particular, improvements of mortality rates at different age groups have not necessarily been perfectly correlated among each other. A mortality model assuming a trivial correlation structure therefore conveys a illusory understanding of improvement.

Parameter interpretability

Underlying model parameters should ideally be interpretable variables (e.g. reduction factors, average mortality shape or speed of mortality decline) such that parameter forecasts can be back-tested by means of historical mortality data. At least, it should be possible to transform unobservable parameters into more concrete ones.

Robust model forecasts and parameter estimation

Parameter robustness means that a change in the age range or observation period should evoke only a rather slight adjustment for the particular parameters

[66] For example, Bauer et al. (2008a) suggest to model cohort effects and a constant general tendency of mortality evolution as diffusion coefficients of a multi-dimensional Brownian motion.

and thus provide a reasonable and consistent mortality forecast. A robustness towards a selection of different models out of an appropriate model class is also desirable in this regard.

Forecasting accuracy and levels of uncertainty

Besides the identification and the estimation of different sources of forecasting errors (e.g. model misspecification, parameter estimation errors, random variation, expert-related misjudgement), Booth (2006) declares the accuracy[67] of a model as a main objective for forecasting methods.

Analytic/numerical implementation

The underlying model should preferably offer a good trade-off between a straightforward analytic implementation or efficient numerical procedures on the one hand and flexibility in combination with goodness of fit on the other hand.

Parsimony

Within a defined class of (nested) mortality models modest representations in regard to their parameter cardinality are more preferable to complex ones. Parsimony can therefore be defined by the number of effective model parameters. For example, the Bayes information criterion (BIC) measures the additional likelihood value contributed by an additional model parameter. Thus, the model complexity is only raised in so far as the improvement in the data fitting is effective.

Parametrisation and sampling

Overparametrisation should in general be avoided, i.e. adding a further parameter needs to be justified by a remarkable performance augmentation. For example, extrapolation by means of time series requires a large number of parameters. In addition, a stochastic model should draw sample paths and allow for a calculation of prediction intervals to access the uncertainty in future mortality evolution. This is, for example, not the case for the spline graduation method by Currie and Durban (2002). The forecasting performance can, for example, be monitored by a back-testing of prediction interval charts similar to Dowd et al. (2010).

[67] The accuracy gives information how well the observed data is reproduced expressed by certain goodness of fit measures.

Parameter risk

Since data samples are incomplete the estimated parameters are subjected to parameter risk which should be considered by an adequate mortality model. Conversely, a model which disregards parameter uncertainty systematically underestimates uncertainty in future mortality.

The choice of a certain model variant out of a general model class may vary with different datasets or forecasting application such as

- the long-term projection of life table functions,
- the quantification of parameter and / or forecasting uncertainty,
- the valuation of insurance liabilities or
- the pricing of mortality-linked securities.

A brief verification of the qualitative selection criteria is executed in Chapter 5 based on the frequently used Lee-Carter model class. Of course, an actuaries personal belief how strong model assumptions or trends will turn out plays by no means an unimportant role. The bottom line is that no single model performs best for all of the criteria listened. A ranking of different model alternatives according to the degree of compliance with a single criterion (e.g. goodness of fit for historical data) might therefore give a biased impression and, at first glance, could be misleading.

4.2 Discrete-time frameworks

Discrete-time models are favourable in the way that they exactly fit the yearly raw mortality data, e.g. in form of the central death rates, by means of reliable statistical methods and also allow a stochastic projection into the future. In this connection, time series models introduced in Subsection 4.2.1 turn out to be the most promising approach. The implementation of so called market models (addressed in Subsection 4.2.3) is still in its infancy and assumes a liquid mortality (bond) market which currently seems far from being fulfilled.

4.2.1 Time series mortality projection models

Over the last two decades parametric time series modelling became a popular mortality extrapolation method. Depending on the number of random factors included, changes in mortality rates are classified into effects concerning certain age groups, calendar years and birth cohorts. Log-linear time series models are relatively parsimonious and parameter estimates are based on historical mortality trends only. A stochastic extrapolation of the time-dependent parameters based on subjective judgement and time series analysis allows for a probabilistic measurement of forecasting uncertainty by means of confidence and prediction intervals. Any prediction of the model implicitly assumes that past trends will continue in the future. Conversely, a prediction of future mortality changes due to endogenous social and biological factors is thereby not feasible. In contrast to law-based approaches the log-linear time series framework avoids a distributional assumption on parameter dependency. In Tables 4.1 - 4.3 we overview the most common time series approaches. Admittedly, no claim is made with regard to their completeness.

Two-factor Age-Period Models

Within the class of two-factor age-period models, the Lee and Carter model (1992) constitutes a pioneering approach and the most famous representative[68]. The graduation is based on an interaction of a single time-driven stochastic factor and a deterministic age function describing the averaged mortality curve. Most of the two-factor models assume perfect correlation for the mortality rates over the whole age range which is equivalent to independent identical normal distributed model residuals. Since raw mortality data is typically overdispersed[69] the rather strong distributional assumptions on the model residuals appear to be somewhat unrealistic. For this reason, several modifications relax this assumption to enlarge the model flexibility and improve the fitting quality.

Multi-factor Age-Period Models

Multi-factor age-period models based on maximum likelihood parameter estimation introduce an additional stochastic time factor to the Lee-Carter framework in order to model long-term longevity risk. As a result, the multi-factor variants successfully capture the imperfect correlation phenomenon across different ages. To be more precisely, one time factor captures the level and the other one the slope of period-related effects caused by industrialization, improvements in health care or climatic changes. While age-period models allow for imperfect correlation and partly produce smooth forecasts (at least for the logit-transform), they fail in reproducing the so called cohort effects.

[68] For a detailed description of the Lee-Carter model see Section 5.1.
[69] The residuals are called overdispersed if their observed variance is larger than the assumed variance, i.e. in case of an age-specific residual variance greater than one.

Age-Period-Cohort Models

Recently, several authors incorporated a cohort-related enhancement[70] into a two-factor age-period framework (see Table 4.3). Figure 4.3 reveals one feasible presentation[71] of cohort effects showing annual improvement rates for German raw central death rates. The contour map[72] shows an almost entirely general mortality improvement for the second half of the 20th century. In particular, juvenile and retirement ages experienced the strongest decrease in mortality. The red coloured areas denote a continuous improvement whereas the blue coloured cells indicate mortality deterioration compared to the previous calendar year. The lightness of the colour indicates the extent of the change in mortality.

Contour maps of the negative logarithmised central death improvement rates

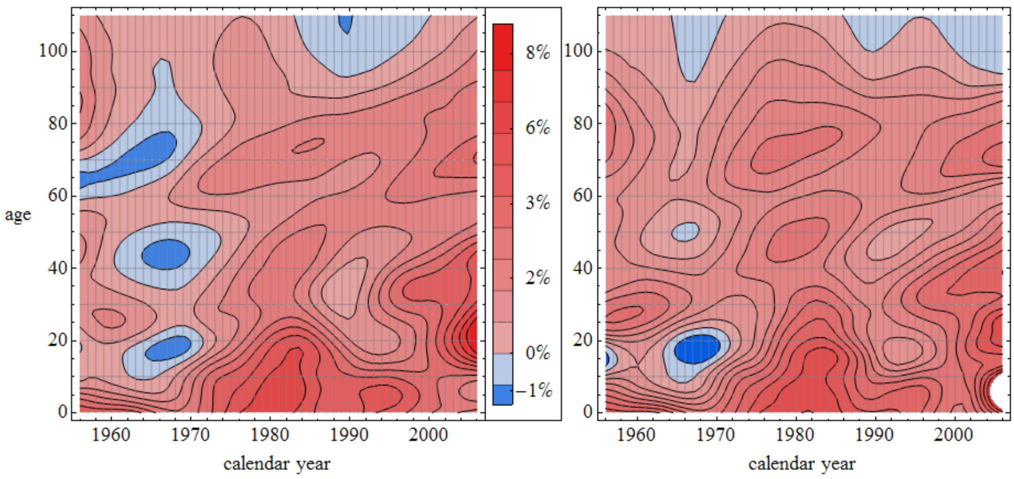


Figure 4.3: Shaded contour maps of the smoothed negative logarithmised central death improvement rates for German male (left) and female (right) mortality data from the Human Mortality Database (2009) for B-spline of the fourth order and a smoothing factor[73]of 0.02.

An analysis of the contour map attests visible cohort effects for male generations born around 1900 and 1930. Willets (2004) identifies certain driving factors for

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[70] Cohort effects result from impacts of wars or economic crises but also health reforms or generation-specific lifestyles and affect the mortality evolution of individual or a few consecutive birth cohorts.

[71] Cairns et al. (2009) use contour maps for an illustration of cohort effects in the historic mortality of England and Wales and the United States.

[72] Some authors also use the term "Lexis map" which was devised by Lexis (1875) for illustration. In fact, a Lexis diagram describes a chart methodology to analyse the mortality of a cohort in the age-period plane.

[73] A smoothing factor of 0.02 reveals that according to Table 3.9 only very small value is attached to the penalization of roughness and that the main focus is still on the interpolation.

UK cohort effects under a cause of death investigation. Richards et al. (2006) highlight the importance of cohort effects (concerning the year of birth) in addition to period effects (concerning the year of observation). According to Andreev and Vaupel (2005)[74] cohort and age-shifted effects in demographic data are characterised not so much by random temporal fluctuations as by systematic patterns and therefore should receive increasing interest in further research. The authors observed that the development of German central death rates varied with different pairs (x,t) (see contour map in Figure 4.3). More precisely, mortality rates increased gender-independent in the late fifties until the beginning of the seventies and declined especially for younger age groups (dark red coloured areas). Although period and cohort effects are hard to isolate, the "male map" shows diagonal improvement areas decreasing for cohorts aged 60 (40) in calendar year 1979 (1975). The female pattern does not exhibit comparable cohort effects. Furthermore, Figure 4.3 depicts typical period effects with increasing rates in the sixties and nineties (blue areas) as well as strong decrease between the late seventies, eighties and beyond year 2000 (red areas).

The CMIB (2007b) funded by the UK life insurance industry announced a "library" of future cumulative reduction factors based on sampling projections from the Lee-Carter and P-splines methodologies[75] to clarify actuaries the implications and appropriateness of stochastic approaches. Despite the last-mentioned approaches the application of time series models is still limited to demography and research in public health, biostatistics or epidemiology. In particular, age-period-cohort models are faced with difficulties in empirically estimating separate period and cohort effects based on aggregated mortality data (see Section 5.3).

4.2.2 Definitions for the forward mortality framework

Let τ_x denote the residual lifetime of an individual aged x at present time $t = 0$ represented by a non-negative random variable. Along the lines of Biffis et al. (2010), we define the stochastic force of mortality process on a filtered probability space $(\Omega, \mathbb{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\omega-x]}$ $(\mathcal{F}_0 = \{\emptyset, \Omega\})$. Furthermore, $\mathbb{F}$ is assumed to satisfy the usual conditions, i.e. right continuity and $(P, \mathcal{F})$-completeness, and to represent all available information for different times

[74] The authors present a comprehensive research of mortality patterns for several developed countries based on a smoothing technique using tensor-product splines.

[75] Along the lines of Andreev and Vaupel (2005) we apply penalized spline tensor product smoothing to graduate raw improvement rates as well as residuals (see Figure 4.3 and Subsection 5.4.2).

Two-factor age-period models

Dynamic distribution-free extensions of classical mortality laws are able to capture longevity trends and allow for uncertainty in future mortality rates.

- Lee and Carter (1992) model logarithmised time series of central death rates at different ages x and calendar years t as a stochastic log-bilinear process of the form

$$\ln\left(m_x(t)\right) = a(x) + b(x)k(t) + \varepsilon(x,t)$$

with non-parametric age-specific functions $a(x)$ and $b(x)$ expressing the averaged log-mortality age pattern and the level change at age x when $k(t)$ varies. The single time index $k(t)$ describes random period effects frequently extrapolated via an ARIMA processes. $\varepsilon(x,t)$ denotes a constant error term capturing residual age-time variation with zero mean and constant variance (for more details see Section 5.1). Parameters are estimated using singular value decomposition with a second stage re-estimation of $k(t)$ to ensure correspondence of the model and the raw number of deaths.

- Lee (2000) and Yang (2001) model annual death probabilities under a piecewise constant force of mortality (see (3.7)) and normal distributed number of death

$$d_x(t) \sim N(L_x(t)\cdot q_x(t), L_x(t)\cdot p_x(t)\cdot q_x(t)) \text{ such that } q_x(t) = q_x(t_0)\cdot \exp\left(Y(t)\right) \text{ for } t \geq t_0$$

for a deterministic expected probability $q_x(t_0)$ and a random reduction factor $Y(t) = X(t) - \frac{1}{2}\sigma_Y^2 + \sigma_Y\, Z_Y(t)$. The stochastic overall drift $X(t)$ follows a standard normal white noise process

$$X(t) = X(t-1) - \frac{1}{2}\sigma_X^2 + \sigma_X\, Z_X(t) \quad (X(t_0) = 0)$$

where Z_X and Z_Y are unit distributed and the term $\sigma_Y\, Z_Y(t)$ expresses year-specific environmental variations whereas the terms $-\frac{1}{2}\sigma_X^2$, $-\frac{1}{2}\sigma_Y^2$ ensure unit expectation of $\exp\left(Y(t)\right)$.

Table 4.1: Two-factor age-period models.

$t \in [0, \omega - x]$. More precisely, filtration $\mathbb{F}$ contains two strict subfiltrations signed $\mathbb{H}$ and $\mathbb{M}$. $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,\omega-x]}$ with σ-algebras $\mathcal{H}_t = \sigma\left(\left\{\mathbb{1}_{\{\tau_x \leq s\}} : 0 \leq s \leq t\right\}\right)$ denotes the minimal filtration needed to generate τ_x as a $\mathbb{F}$-stopping time for the first jump of the counting process $(\mathbb{1}_{\{\tau_x > t\}})_{t\geq 0}$ (i.e. information whether death occurs or not). Whereas the subfiltration $\mathbb{M} = (\mathcal{M}_t)_{t\in[0,\omega-x]}$ describes information $\mathcal{M}_t = \sigma\left(\{\mu_{x+s} : 0 \leq s \leq t\}\right)$ concerning the random evolution of the mortality state variables until time t. Furthermore, along the lines of Miltersen and Persson (2006) we assume that the filtrations $\mathcal{H}_t$ and $\mathcal{M}_t$ represent the minimal σ-algebras generated by the corresponding state variables.

In analogy to (3.10) the t-year survival probability under the real-world measure P for an individual aged x at time 0 equals

$$ {}_tp_x(0) := P\left(\tau_x > t \middle| \mathcal{H}_0 \vee \mathcal{M}_0\right) = E_P\left[\mathbb{1}_{\{\tau_x > t\}} \middle| \mathcal{H}_0 \vee \mathcal{M}_0\right]. \tag{4.1}$$

Two-factor age-period models (Lee-Carter extensions)

There exist various extensions for the Lee-Carter model improving different aspects like forecasting accuracy, smoothness, parsimony concerning the number of parameters or robustness: • Wilmoth (1993) fits the Lee-Carter model using an iterative procedure weighting the least squares matrix equation with weights corresponding to the raw number of deaths. Furthermore, maximum likelihood estimation (MLE) is applied based on a Poisson distributed number of deaths. Re-estimation can therefore be omitted. The MLE allows non-additive heteroscedastic residuals and avoids the assumption of constant variance. • Lee and Miller (2001) modify the Lee-Carter procedure by means of an observation window restriction, adjustment of the period factor by matching the life expectancy and forecasting starting with the present observation rather than an estimated value. • Brouhns et al. (2002b) improve the estimation method by assuming a Poisson distribution for the random number of deaths. The parameters are estimated by maximizing the Poisson log-likelihood function. Mixed Poisson, binomial or negative binomial distribution constitute alternative number of death distributions. • Booth et al. (2002) reduce the forecasting error by adjusting the period factor to the age-specific number of deaths rather than the total number of deaths and face non linearity in the period factor by an appropriate choice of the observation period. • Czado et al. (2005) combine a Poisson log-bilinear regression model with a Bayesian estimation to forego the re-estimation and to impose parameter smoothness. • De Jong and Tickle (2006) combine B-spline smoothing and estimation by means of the Kalman filter to fit a generalised version of the Lee-Carter model. In particular, the extension needs less parameters due to the spline representation and an integrated estimation / forecasting procedure. • Hyndman and Ullah (2007) generalise the Lee-Carter model and combine prior non-parametric smoothing of the raw data with functional data analysis. In addition, their principal component decomposition is robust against outlier mortality values. • Delwarde et al. (2007) smooth the estimated age-specific sensitivity factors from the Lee-Carter and Poisson log-bilinear models via a penalized least squares and MLE procedure, respectively. The optimal smoothing parameter is obtained by cross validation techniques.

Table 4.2: Two-factor age-period models (Lee-Carter extensions).

Therefore the random variable

$$ {}_t\tilde{p}_x := P\left(\tau_x > t \,\middle|\, \mathcal{M}_t\right) = E_P\left[\mathbb{1}_{\{\tau_x > t\}} \,|\, \mathcal{M}_t\right] \tag{4.2}$$

is called the survivor function or survivor index and describes the survival probability conditioned on a realised mortality[76] path until t. In particular, Expression (4.1) equals the averaged value of the survivor function over all possible mortality scenarios, i.e. by the law of total expectation we obtain

$$ {}_tp_x(0) = E_P\left[E_P\left[\mathbb{1}_{\{\tau_x > t\}}\middle|\,\mathcal{M}_t\right]\middle|\,\mathcal{H}_0 \vee \mathcal{M}_0\right] = E_P\left[{}_t\tilde{p}_x\middle|\,\mathcal{M}_0\right].$$

[76] Notice that in case of deterministic mortality, we have ${}_t\tilde{p}_x = {}_tp_x(0)$ for all $t > 0$.

Multi-factor age-period models

- Renshaw and Haberman (2003) extend the Lee-Carter framework and introduce a double bilinear predictor

$$\ln\left(m_x(t)\right) = a(x) + b_1(x)k_1(t) + b_2(x)k_2(t) + \varepsilon(x,t)$$

with distinct correlated period effects $k_i(t)$ $(i = 1,2)$ in form of appropriate stochastic processes (e.g. bivariate random walks) and constraints

$$\sum_x b_i(x) = 1 \text{ and } \sum_t k_i(t) = 0 \quad (i = 1,2)$$

ensuring the uniqueness of the parameter estimates.

- Cairns et al. (2006b) present an age-period model for the logit transform of the annual death probability

$$\ln\left(\frac{q_x(t)}{1-q_x(t)}\right) = b_1(x)k_1(t) + b_2(x)k_2(t) + \varepsilon(x,t)$$

where $(k_1(t),k_2(t))$ follows a correlated bivariate random walk with drift. The time factor $k_1(t)$ describes the level/intercept and $k_2(t)$ the slope coefficient. The model allows for imperfect correlation of changes in the mortality "age profiles". Furthermore, the logistic transform of the annual death probability corresponds to a special stochastic version of the Perks law of mortality in Subsection 3.2.2 under adequate parameter choice.

Table 4.3: Multi-factor age-period models.

The $(t-s)$-year conditional survival probability of an individual aged $x+s$ at time $s \leq t$ is calculated as follows

$$\begin{aligned}
P\left(\tau_x > t \,\middle|\, \mathcal{M}_s\right) &= P\left(\tau_x > s \,\middle|\, \mathcal{M}_s\right) \cdot P\left(\tau_x > t \,\middle|\, \mathcal{H}_s \vee \mathcal{M}_s\right) \\
&= E_P\left[\mathbb{1}_{\{\tau_x > s\}} \,|\, \mathcal{M}_s\right] \cdot E_P\left[\mathbb{1}_{\{\tau_x > t\}} \,\middle|\, \mathcal{H}_s \vee \mathcal{M}_s\right] \\
&= {}_s\tilde{p}_x \cdot E_P\left[E_P\left[\mathbb{1}_{\{\tau_x > t\}} \,\middle|\, \mathcal{H}_s \vee \mathcal{M}_t\right] \,\middle|\, \mathcal{M}_s\right] \\
&= {}_s\tilde{p}_x \cdot E_P\left[\frac{E_P\left[\mathbb{1}_{\{\tau_x > s\}} \cdot \mathbb{1}_{\{\tau_x > t\}} \,\middle|\, \mathcal{M}_t\right]}{E_P\left[\mathbb{1}_{\{\tau_x > s\}} \,\middle|\, \mathcal{M}_t\right]} \,\middle|\, \mathcal{M}_s\right] \\
&= {}_s\tilde{p}_x \cdot E_P\left[\frac{{}_t\tilde{p}_x}{{}_s\tilde{p}_x} \,\middle|\, \mathcal{M}_s\right] =: {}_s\tilde{p}_x \cdot {}_{t-s}p_{x+s}(s) \qquad (4.3)
\end{aligned}$$

where (4.3) follows by reason of $\mathbb{1}_{\{\tau_x > s\}} \cdot \mathbb{1}_{\{\tau_x > t\}} = \mathbb{1}_{\{\tau_x > t\}}$ and

$$E_P\left[\mathbb{1}_{\{\tau_x > s\}} \,\middle|\, \mathcal{M}_s\right] = E_P\left[\mathbb{1}_{\{\tau_x > s\}} \,\middle|\, \mathcal{M}_t\right] \quad \text{for all } t \geq s.$$

Following Cairns et al. (2008) the $(u-t)$-year forward survival probability of an individual aged $x+t$ at time t conditional on information concerning (individual) survival and mortality evolution until time $s \leq t$ can be obtained

Multi-factor age-period-cohort models

- Currie and Durban (2002) as well as Currie et al. (2004) use the generalised linear model (GLM) framework to apply P-splines (as a combination of (basis) B-splines and penalties on the roughness of the estimated coefficients) in order to smooth a surface of crude central death rates on the (x,t)-plane by

$$\ln(m_x(t)) = \sum_{i,j} \alpha_{i\,j}\, B_{i\,j}(x,t)$$

  where $B_{i\,j}(x,t) = B_i(x) \cdot B_j(t)$ are pre-specified bivariate basis polynomials driven by parameters $\alpha_{i\,j}$. The model lacks to sample paths directly since fitted and forecasted mortality rates are estimated simultaneously. The CMIB (2006) used age- and year-of-birth penalized spline regression to obtain a smooth cohort forecast.

- Renshaw and Haberman (2006) extend their age-period-model as follows

$$\ln(m_x(t)) = a(x) + b_1(x)k(t) + b_2(x)l(t-x) + \varepsilon(x,t)$$

  where $l(t-x)$ represents the overall mortality for the cohort born in year $t-x$. In order to avoid arbitrary revaluation of the parameters the constraints

$$\sum_x b_1(x) = \sum_x b_2(x) = 1 \text{ and } \sum_{x,t} l(t-x) = \sum_t k(t) = 0$$

  are imposed and parameters $k(t)$ and $l(t-x)$ are forecasted via univariate autoregressive time series. The model lacks smoothness in the age and cohort dimension. The CMIB (2007a) found a satisfactory out-of-sample description of cohort effects in UK mortality using the Renshaw-Haberman mortality model. Nevertheless, Cairns et al. (2008) argue that the model exhibits drawbacks resulting in potential implausible forecasts.

- Currie (2006) simplifies the discrete age-period-cohort model by Renshaw and Haberman (2006) taking constant age-specific improvement rates $b_1(x) = b_2(x) = 1$ for all ages $x \geq 0$, i.e.

$$\ln(m_x(t)) = a(x) + k(t) + l(t-x) + \varepsilon(x,t).$$

- Cairns et al. (2006b) give an age-period-cohort model for the logit transform of the mortality odds

$$\ln\left(\frac{q_x(t)}{1-q_x(t)}\right) = b_1(x)k_1(t) + b_2(x)k_2(t) + b_3(x)l(t-x) + \varepsilon(x,t)$$

  together with three parametric forms due to different choices for $b_i(x)$ (i=1,...,3). The model assumes smoothness in the mortality surface in the age direction but not in the diagonal cohort direction.

Table 4.4: Multi-factor age-period-cohort models.

via

$$P\left(\tau_x > u \,\middle|\, \mathcal{H}_t \vee \mathcal{M}_s\right) = E_P\left[\mathbb{1}_{\{\tau_x > u\}} \,\middle|\, \mathcal{H}_t \vee \mathcal{M}_s\right] =: {}_{u-t}p_{x+t}(s). \tag{4.4}$$

The quantities ${}_{t-s}p_{x+s}(s)$ and ${}_{u-t}p_{x+t}(s)$ define spot and forward survival probabilities respectively in analogy to Definitions (4.1) and (4.2). Under an alternative risk-neutral probability measure Q the corresponding survival probabilities are defined as well and, in the following, marked with the superscript Q. In contrast to deterministic approaches, forecasts are no longer single numerical values but rather equal realisations of random variables. Along the lines of Cairns et al. (2008) we distinguish between four different (theoretical) stochastic mortality rate models with former application in the field of term structure modelling.

4.2.3 Market mortality models

Due to well known market models trading rates which are directly accessible (e.g. from swap or LIBOR curves) the mortality market model approach attempts to apply this traditional approach to a mortality market. At the moment it seems that a so called mortality market is far from being perfectly liquid, frictionless and arbitrage-free. For this reason, we refer to traded assets like index-linked zero-coupon bonds on a hypothetical market for zero-coupon longevity bonds traded for all ages x and terms to maturity T. In the following, we present a filtration framework that facilitates the derivation of longevity bond prices. Let the cash account at time t be defined as

$$\beta_{t,T} = \exp\left(\int_t^T r_\vartheta \, d\vartheta\right)$$

compounding the instantaneous risk-free rate of interest r_t. The σ-algebra $\mathcal{I}_t = \sigma\left(\{r_s : 0 \leq s \leq t\}\right)$ therefore describes all information concerning the spot interest rate evolution until time t. For convenience, assume that the mortality and interest rate dynamics are independent. In particular, the information $\mathcal{I}_t$ and the union $\mathcal{H}_t \vee \mathcal{M}_t$ are independent for all $t \geq 0$ such that an overall filtration can be composed as $\mathbb{F} = \mathbb{H} \vee \mathbb{I} \vee \mathbb{M}$ for a subfiltration $\mathbb{I} = \left(\mathcal{I}_t\right)_{t\geq 0}$. Therefore, it is possible to split mortality-indexed bonds prices into interest and mortality related expectation operands. Furthermore, let $D(t,T)$ denote the price at time t of a zero-coupon bond paying one monetary unit at time T and let $\mathit{\Pi}_t(x,T)$ denote the price at time t of the zero-coupon longevity bond (abbreviated with $(x{,}T)$-bond[77]) paying ${}_T\tilde{p}_x$ at T. If the market is arbitrage free, the first fundamental theorem of asset pricing yields that there exists a risk-neutral measure Q equivalent to the real world measure[78] P ensuring

$$
\begin{aligned}
D(t{,}T) &= E_Q\left[\frac{\beta_{0,t}}{\beta_{0,T}}\,\middle|\,\mathcal{I}_t\right] = E_Q\left[\frac{1}{\beta_{t,T}}\,\middle|\,\mathcal{I}_t\right], && (4.5)\\
M(x{,}t{,}T) &= E_Q\left[{}_T\tilde{p}_x\middle|\,\mathcal{M}_t\right] = E_Q\left[E_Q\left[\mathbb{1}_{\{\tau_x>T\}}\middle|\,\mathcal{M}_T\right]\middle|\,\mathcal{M}_t\right]\\
&= E_Q\left[\mathbb{1}_{\{\tau_x>T\}}\middle|\,\mathcal{M}_t\right] = {}_t\tilde{p}_x\cdot{}_{T-t}p^Q_{x+t}(t)\\
\mathit{\Pi}_t(x{,}T) &= E_Q\left[\frac{\beta_{0,t}\,{}_T\tilde{p}_x}{\beta_{0,T}}\,\middle|\,\mathcal{M}_t\,,\mathcal{I}_t\right] = E_Q\left[\frac{{}_T\tilde{p}_x}{\beta_{t,T}}\,\middle|\,\mathcal{M}_t\,,\mathcal{I}_t\right]\\
&= E_Q\left[\frac{1}{\beta_{t,T}}\,\middle|\,\mathcal{I}_t\right]\cdot E_Q\left[{}_T\tilde{p}_x\,\middle|\,\mathcal{M}_t\right] = D(t{,}T)\cdot M(x{,}t{,}T) && (4.6)
\end{aligned}
$$

such that the discounted prices $M(x{,}t{,}T) = \frac{\mathit{\Pi}_t(x,T)}{D(t,T)}$ form a martingale[79] under Q. In terms of forward survival probabilities (i.e. $t \le T_0 < T_1$) it holds

$$
\begin{aligned}
{}_{T_1-T_0}p^Q_{x+T_0}(t) &= \frac{{}_{T_1-t}p^Q_{x+t}(t)}{{}_{T_0-t}p^Q_{x+t}(t)} = \frac{M(x{,}t{,}T_1)}{M(x{,}t{,}T_0)} = \frac{\mathit{\Pi}_t(x{,}T_1)\,D(t{,}T_0)}{\mathit{\Pi}_t(x{,}T_0)\,D(t{,}T_1)} \quad \text{and}\\
{}_{T_1-T_0}p^Q_{x+T_0}(t) &= \frac{M(x{,}t{,}T_1)}{M(x{,}t{,}T_0)} = \frac{\mathit{\Pi}_t(x{,}T_1)}{{}_{T_0}\tilde{p}_x\,D(t{,}T_1)} \quad \text{for times } T_0 < t < T_1. && (4.7)
\end{aligned}
$$

The martingale property of $M(x{,}t{,}T)$ and Expression (4.7) ensure that for all $t < T$

$$
\begin{aligned}
E_Q\left[{}_{T-t}p^Q_{x+t}(t+1)\,\middle|\,\mathcal{M}_t\right] &= \frac{1}{{}_t\tilde{p}_x}E_Q\left[\frac{\mathit{\Pi}_{t+1}(x{,}T)}{D(t+1{,}T)}\,\middle|\,\mathcal{M}_t\right]\\
&= \frac{\mathit{\Pi}_t(x{,}T)}{{}_t\tilde{p}_x D(t{,}T)} = {}_{T-t}p^Q_{x+t}(t).
\end{aligned}
$$

An example of a discrete-time market mortality model approach is listed in Table 4.8.

4.3 Continuous-time frameworks

The intuition associated with the introduction of stochastic models describing the development of the instantaneous force of mortality or either the (real time) survival function is to analyse the exact evolution for non-integer intermediate

[77] The payout of the $(x{,}T)$-bond is linked to the mortality evolution until time T.

[78] Note that for the real world probability measure P and a non unique equivalent measure Q the expressions $E_P\left[\mathbb{1}_{\{\tau_x>s\}}\middle|\,\mathcal{M}_t\right]$ and $E_Q\left[\mathbb{1}_{\{\tau_x>t\}}\middle|\,\mathcal{M}_t\right]$ $(s \le t)$ are not necessarily equal.

[79] Equations (4.5) and (4.6) imply that the discounted prices $\frac{D(t,T)}{\beta_{0,t}}$ and $\frac{\mathit{\Pi}_t(x,T)}{\beta_{0,t}}$ themselves form Q-martingales.

time points and to apply methods from continuous-time stochastic calculus. Furthermore, continuous-time models can substantially support a better insight into actuarial tasks such as the pricing of mortality-linked derivatives or the premium and reserve calculation for (index-linked) life insurance policies. In contrast to Subsection (3.2), the (spot) force of mortality $\mu_{x+s}(s)$ describing the instantaneous death rate for an individual aged $x+s$ at time $s \geq 0$ evolves stochastically over time. For the true survival probabilities under the real world measure[80] P it holds

$$ {}_{t-s}p^{P}_{x+s}(s) = E_P\left[\frac{{}_t\tilde{p}_x}{{}_s\tilde{p}_x} \middle| \mathcal{M}_s \right] = E_P\left[\exp\left(-\int_s^t \mu_{x+\vartheta}(\vartheta)\, d\vartheta \right) \middle| \mathcal{M}_s \right]. \tag{4.8} $$

According to Cairns et al. (2008) there exist several term structure related modelling approaches describing the dynamics of the conditioned expected survivor function $M(x,s,t)$ under a risk-neutral measure Q. The martingale $M(x,s,t)$ forms a t-measurable expectation under Q about future mortality evolution as a spot price of mortality risk (see Subsection 4.3.2).

4.3.1 Short-rate mortality models

According to well established continuous one-factor short-rate models for the risk-free spot interest rate, e.g. the Vasićek, the Cox, Ingersoll and Ross or the Hull and White model, it is possible to directly transfer this framework to mortality modelling. Having recognised that the force of mortality and the spot interest rate have nearly identical representations, actuarial literature[81] started to adapt and use interest rate model insights and results. Along the lines of Dahl (2004) the instantaneous force of mortality can be defined as follows

$$ d\mu_{x+t}(t) = a(x,t) + b(x,t)^T dW(t) $$

where $W(t)$ denotes a standard d-dimensional Brownian motion under the risk-neutral measure Q. The drift process $a(x,t)$ and volatility vector $b(x,t)$ assign deterministic processes[82]. Since $b(x,t)$ and $W(t)$ are $(d \times 1)$ - vectors,

80 A subscripted P indicates that expectation is taken with respect to the real-world measure P. In the following, we seek for a risk neutral valuation / pricing approach to take into account for a market expectation of future mortality.

81 Several authors such as Milevsky and Promislow (2001), Dahl (2004) or Cairns et al. (2008) advise of this commonalities. However, there exist some important differences between mortality and interest rate modelling. For example, mortality rates are assumed to stay positive and show non mean reversion behaviour. The length of each sample path for spot interest rate models is limited by the maximum long-term zero bond maturities.

82 The one-row matrix $b(x,t)^T$ defines the transpose of the one-column matrix $b(x,t)$. In the following, the transpose of a matrix is designated by the superscript T.

we are able to model age-specific short-term evolution of mortality rates by d different driving factors. This effect of variation between different age groups can be observed for long-term empirical data (see Figure 4.3). The drift and the volatility function may additionally depend on underlying diffusion or jump processes adapted to the same mortality evolution $\mathcal{M}_t$. Under certain sufficient conditions on $a(x,t)$ and $b(x,t)$ we obtain a closed-form affine solution for the risk-neutral spot survival probability defined as

$$ {}_{t-s}p^{Q}_{x+s}(s) = \frac{M(x,s,t)}{M(x,s,s)} = E_Q\left[\exp\left(-\int_s^t \mu_{x+\vartheta}(\vartheta)\,d\vartheta\right)\middle|\,\mathcal{M}_s\right]. \tag{4.9} $$

Without postulating completeness, the most frequently used "short-rate mortality models" are listed chronologically in Table 4.5 and 4.6 due to their first application. Furthermore, Figure 4.4 illustrates that survival functions implied by an affine process with deterministic parameters fail to capture the complete rectangularisation effect for more recent period life tables. On average, the 2006 survival functions in Subfigures (4.4.1) and (4.4.3) underestimate raw survival probabilities for ages $x + t < 85$ and conversely overstate mortality data for ages $x + t \geq 85$. The extent of misjudgement turns out stronger for younger age profiles where the rectangularisation as well as the expansion phenomena is substantial. A comparison of different survival functions / profiles based on different calibration years (Subfigures (4.4.2) and (4.4.4)) reveals that phenomenons such as rectangularisation and expansion intensified within the last 30 years. The intersection point of the model and raw survival curve is still roughly at age 85. Here, again, probabilities ${}_tp_x$ for $x + t < 80$ are underestimated and vice versa overestimated for $x + t \geq 85$. Both, the square root diffusion and the compound Poisson jump process provide very similar calibration results even in connection with the lack of flexibility in the curve shape. The jump process shows less absolute deviations in the 2006 survival functions of different ages.

A pragmatic heuristic vertical extrapolation[83] for the short-rate framework is addressed in Figure 4.5. In this connection, the choice of the (optimal) observation window and especially its width has a substantial influence on the forecasting result[84]. The model and parameter risk can become considerably high due to a misspecification of the projection function and an inappropriate

[83] We extrapolate the period-specific sets of parameter estimates and thus the initial mortality rate curve $\mu_{65}(t_0)$ as well as the survival function ${}_tp_{65}(t_0)$ using a three parameter exponential function similar to the parametric graduation of Subsection 3.2.2.

[84] Booth et al. (2002) present systematic methods to detect the most appropriate observation window width and location.

choice of the observation window. In a first instance, we therefore smooth the period-specific estimated parameters by means of P-splines (see Section 3.2.3 for further details on non-parametric graduation) to reduce noise and increase the forecasting reliability. As already mentioned in Subsection 3.2.2 the forecasting error grows with the number of model parameters and extrapolation horizon. Subfigure (4.5.2) illustrates that future survival functions ${}_tp_{65}(2056)$ can show implausible behaviour (mortality deterioration) by crossing the current survival function ${}_tp_{65}(2006)$. This is a direct result of both an inappropriate extrapolation function and a disregarded parameter correlation structure and occurs for (narrow) observation windows starting in year 1987 and beyond.

The decision for or against the application of extrapolation via exponential reduction factors has to be carefully thought out since graduation demands a subjective expert judgement as well as an extensive time series analysis for the parameter estimates which is fraught with additional risk. Furthermore, the applicability of the resulting forecasts depends keenly on the quality of the estimates. Another extrapolation possibility lies in the application of an autoregressive stochastic process. On the one hand, it also bears susceptibility to inappropriate estimation and thus implausible forecasts. But, on the other hand, stochastic forecasting enables an analysis of parameter as well as forecasting uncertainty. A validation concerning the choice of a process variant and parameters can be double-checked using a simple backtest[85]. The model / parameter is applied to historical data to evaluate the ex-post forecasting performance.

4.3.2 Forward mortality model

Under a risk-neutral martingale measure Q the forward force of mortality surface is defined by

$$f^{\mu}_{x+t}(s) := -\frac{d}{dt}\ln\left(E_Q\left[\frac{{}_t\tilde{p}_x}{{}_s\tilde{p}_x}\middle|\mathcal{M}_s\right]\right) = -\frac{d}{dt}\ln\left({}_{t-s}p^Q_{x+s}(s)\right) \tag{4.10}$$

for $t \geq s \geq 0$. Here, the random variable $M(x,s,t)$ forms a martingale under Q and expresses market expectation concerning future survival (cf. Bauer and Ruß (2006)). Conversely, for the spot survival probabilities Equation (4.9) holds such that

$$ {}_{t-s}p^Q_{x+s}(s) = \exp\left(-\int_s^t f^{\mu}_{x+\vartheta}(s)d\vartheta\right). $$

85 Mortality backtesting is, inter alia, used in Dowd et al. (2010).

Short-rate mortality models

Stochastic Mortality Law Models

- Milevsky and Promislow (2001) use an age-invariant generalised mean reverting Gompertz process in terms of

$$\mu_{x+t}(t) = \mu_x(0)\exp\left(\gamma t + \sigma\, Y(t)\right) \quad \text{for} \quad dY(t) = -\theta Y(t)\, dt + dW^\mu(t)$$

  with a non mean reverting Ornstein-Uhlenbeck process Y and γ, σ, $\mu_x(0) > 0$, $\theta \geq 0$ as well as a deterministic starting distribution $Y(0) = 0$. The continuous-time model is applied for the valuation of the European option to annuitise embedded in VA contracts.

- Korn et al. (2006) suggest to use a continuous-time Gompertz model with

$$\mu_{x+t}(t) = \alpha(t)\exp\left(\beta(t)x\right)$$

  and parameter processes

$$d\alpha(t) = -\gamma\,\alpha(t)dt \quad \text{and} \quad d\beta(t) = \delta dt + \sigma dW^\mu(t)$$

  where $\beta(0) = \beta_0$, $\alpha(0) = \alpha_0$, $\delta, \gamma > 0$. The risk-neutral survival probabilities are used to price longevity bonds with coupon payments depending on the realised mortality of a certain German age cohort.

Affine Diffusion Models

Affine models allow for the calculation of closed-form solutions for the spot survival probabilities.

- Dahl (2004) constitutes mortality rates following an extended square root diffusion process with a stochastic differential equation

$$d\mu_{x+t}(t) = \left(\theta(x{,}t) - \gamma(x{,}t)\mu_{x+t}(t)\right)dt + \sigma(x{,}t)\sqrt{\mu_{x+t}(t)}dW^\mu(t).$$

  The mean reversion level is given by $\frac{\theta(x,t)}{\gamma(x,t)}$ and the deterministic positive bounded continuous functions $\theta(x{,}t)$, $\gamma(x{,}t)$ and $\sigma(x{,}t)$ ensure the existence of an affine mortality structure which serves for the pricing of mortality-linked insurance contracts and mortality derivatives.

- Schrager (2006) applies a Gaussian Makeham model

$$\mu_{x+t}(t) = Y_1(t) + Y_2(t)\, c^x \quad (Y_1, Y_2 > 0,\, c > 1)$$

  for the calculation of a market price of mortality risk and a guaranteed annuity (put) option. The model equals a special case of a general affine mortality model class with processes Y_1, Y_2 of Ornstein-Uhlenbeck type

$$dY_1(t) = \alpha_1\left(\theta - Y_1(t)\right)dt + \sigma_1 dW_1^\mu(t) \quad \text{and} \quad dY_2(t) = -\alpha_2 Y_2(t)dt + \sigma_2 dW_2^\mu(t).$$

Table 4.5: Short-rate mortality models.

Short-rate mortality models (continued)

Affine Jump Diffusion Models

- Biffis (2005) models the force of mortality by

$$\mu_{x+t}(t) = \alpha(t) + \beta(t)\,Y(t)$$

where the term $\alpha(t) > 0$ describes a best-estimate assumption at time t and $\beta(t) \cdot Y(t)$ the random deviations from α. The term Y, in turn, forms a jump-diffusion process with dynamics

$$dY(t) = \theta\,(\bar{\gamma}(t) - Y(t))\,dt + \sigma dW^{\mu}(t) - dJ(t),\ \ \theta, \bar{\gamma}(t), \sigma > 0$$

where θ denotes the speed of mean reversion, $\bar{\gamma}(\,.\,)$ a time-varying mean-reversion term and σ the instantaneous volatility. The stochastic process J is a compound Poisson jump process. The closed form survival probabilities capture the rectangularization and expansion properties well and enable the fair valuation of various life insurance contracts.

- Luciano and Vigna (2006) apply a non mean reverting Ornstein-Uhlenbeck and a square root diffusion process of the form

$$d\mu_{x+t}(t) = \gamma\mu_{x+t}(t) + \sigma dW^{\mu}(t)\ (-dJ(t))\ , \quad \gamma, \sigma > 0,$$
$$d\mu_{x+t}(t) = \gamma\mu_{x+t}(t) + \sigma\sqrt{\mu_{x+t}(t)}dW^{\mu}(t)\ (-dJ(t))\ , \quad \gamma, \sigma > 0$$

optionally with an additional jump term (given in brackets).

- Biffis and Millossovich (2006) propose a general two-factor model including the mortality evolution of different age cohorts to span a triangular Markov-random-field $\mu = \big(\mu_{x+t}(t)\big)_{\substack{0\le t\le \omega-x \\ x\le\omega}}$ with $\mu_{x+t}(t) = a(x,t) + b(x,t)Y(x,t)$ and matrix-valued functions $a(\,.\,)$, $b(\,.\,)$. The dynamics of the ω-dimensional diffusion process $Y(\,.\,,t)$ are given by

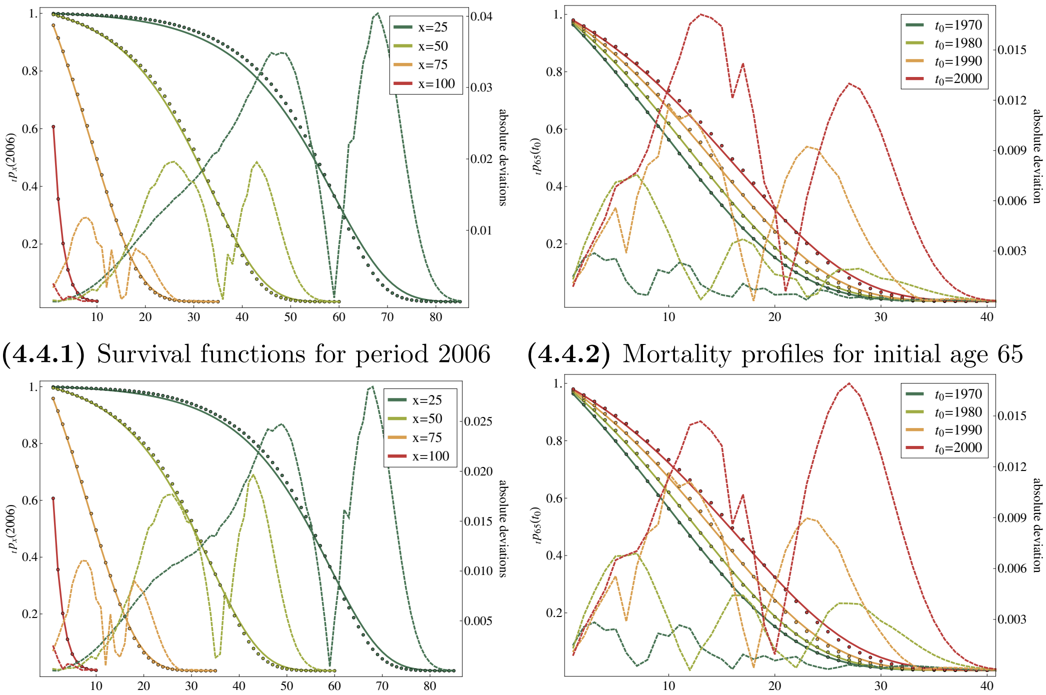
$$\frac{d}{dt}Y(x,t) = \gamma(x,t,Y(x,t))dt + \sigma(x,t,Y(x,t))\frac{d}{dt}W^{\mu_x}(t)$$

whereas W^{μ_x} denotes a $(\omega - x)$-dimensional Brownian motion and γ, σ are matrix-valued functions ensuring a unique solution. The mortality model is part of a valuation framework for longevity bonds and guaranteed annuity options.

- Bravo (2008) models the force of mortality by means of a non mean reverting square-root jump diffusion process according to Luciano and Vigna (2006) extended by a Poisson process with double asymmetric exponentially distributed jump size.

Table 4.6: Short-rate mortality models (continued).

Illustration of German period survival data and fitted survival curves



(4.4.1) Survival functions for period 2006 **(4.4.2)** Mortality profiles for initial age 65

(4.4.3) Survival functions for period 2006 **(4.4.4)** Mortality profiles for initial age 65

Figure 4.4: Illustration of German male period survival data (dotted) from the Human Mortality Database (2009) and fitted survival curves (solid) for a non mean reverting square root diffusion process (upper row) and a compound Poisson jump process (lower row). The absolute deviation between the model survival function and the raw survival probabilities is displayed by the dashed lines. Definitions and closed-form solutions for the survival functions of both mortality models are given in Chapter 7.

Therefore $f^{\mu}_{x+t}(s)$ can be interpreted as a best estimate of the future force of mortality for individuals aged $x+t$ at time t based on information $\mathcal{M}_s$ until time s[86]. Due to the law of iterated expectations the spot price of mortality risk $\{M(x,s,t)\}_{s\geq 0}$ forms a positive Q-martingale. Therefore Itô's formula yields a stochastic differential equation

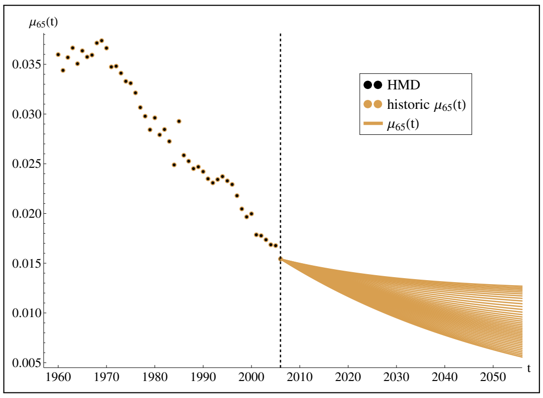
$$dM(x,s,t) = M(x,s,t)\, V(x,s,t)^T\, dW^{\mu}(s),$$

where $V(x,s,t)$ specifies a family of previsible vector processes. Furthermore, let the forward force of mortality follow the Q-dynamics

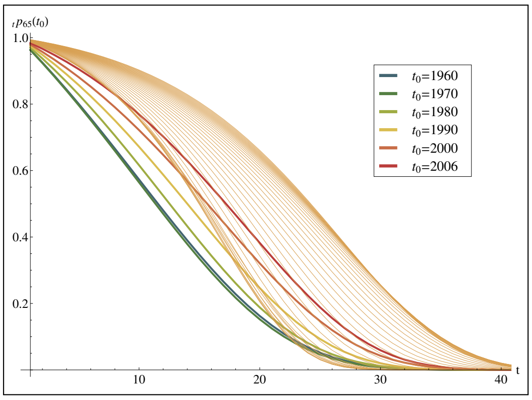
$$df^{\mu}_{x+t}(s) = \alpha(x,s,t)\, dt + \beta(x,s,t)^T\, dW^{\mu}(s),\ (f^{\mu}_{x+t}(0) > 0) \tag{4.11}$$

[86] For $t = s$ the forward force of mortality $f^{\mu}_{x+s}(s)$ equals the spot force of mortality $\mu_{x+s}(s)$. In case of $t > s$, the forward rate matches the (risk-neutral) market expectation of future spot rates.

Exponential projection of the short-rate process parameters



(4.5.1) Initial mortality rate



(4.5.2) Survival function

Figure 4.5: Exponential extrapolation (orange coloured) of the initial mortality rate (left) and survival function (right) of a non mean reverting square root diffusion mortality rate process for year 2056. The extrapolation was executed w.r.t. a male individual aged $x = 65$ based on observation windows $[1965 + k, 2006]$ of different length ($k = 0,\ldots,37$). Historical parameter values (survival curves) are estimated from period life table mortality data originated by the Human Mortality Database (2009).

with previsible processes $\alpha(x,s,t)$ and $\beta(x,s,t)$ and a Q-Brownian motion W^{μ}. Possible approaches describing the relationship between $V(x,s,t)$, $\alpha(x,s,t)$ and $\beta(x,s,t)$ are presented in Table 4.7.

4.3.3 Positive mortality model

A framework which ensures positive mortality rates (on the hole age range) follows an approach[87] in analogy to Flesaker and Hughston (1996). For a probability measure $\tilde{P}$ equivalent to Q and a given random process $\{A(x,s)\}_{s\geq 0}$ of $\mathcal{M}_s$-adapted strictly positive supermartingales[88] define the spot survival probability

$$ {}_{t-s}p^{Q}_{x+s}(s) = \frac{M(x,s,t)}{M(x,s,s)} = E_{\tilde{P}}\left[\left.\frac{A(x,t)}{A(x,s)}\right| \mathcal{M}_s\right] \tag{4.12}$$

ensuring that $0 < {}_{t-s}p^{Q}_{x+s}(s) \leq 1$ with ${}_{t-s}p^{Q}_{x+s}(s)$ decreasing in $t > s$. Therefore, the force of mortality is, by definition, positive. In analogy to Cairns (2004), the resulting dynamics of $A(x,s)$ under $\tilde{P}$ exhibit drift $-\mu_{x+s}(s)\, A(x,s)$. For a setup of the process $A(x,s)$ see Table 4.7. Currently, however, no explicit positive mortality model variants have been provided by actuarial research.

[87] The Flesaker and Hughston approach (1996) is used for interest rate derivative pricing such as caps or swaptions.

[88] A stochastic process $\{A(x,s)\}$ is called a supermartingale with respect to a filtration $\mathbb{M} = (\mathcal{M}_t)_{t\in[0,\omega-x]}$ if it holds $0 < E\left[A(x,t)|\, \mathcal{M}_s\right] \leq A(x,s)$ for all $s,t \geq 0$ with $s \leq t$.

4.3.4 Market mortality models

In accordance with Subsection 4.2.3 the whole term-structure of mortality is modelled under utilisation of continuous-time stochastic calculus. In contrast to the spot rate models (Subsection 4.3.1) and forward rate model (Subsection 4.3.2) the market approach satisfies no-arbitrage conditions by directly modelling certain forward survival probabilities or annuity prices calibrated to (potentially) observable market prices. Table 4.9 lists different drafts of mortality market models. Forward and market models allow for analytic tractability. However, they require full market information about spot SCOR rates which themselves correspond to a theoretical construct at the present moment. In addition, the forward maturity is limited by maximum market SCOR maturity.

4.4 Conclusion

Chapter 3 and 4 present a survey and an exemplary presentation of mortality modelling and projection methodologies in both discrete and continuous time with parameter estimation based on historical German period life table data from the Human Mortality Database (2009). Whether actuaries carry out calculations concerning the pricing or reserving of pension products or demographers carry out population forecasts, the first choice in projecting the age-pattern of mortality[90] has commonly been a graduation-extrapolation mixture. In particular, age-specific extrapolation via reduction factors or polynomials and period- / cohort-specific mortality laws or transforms are used frequently in insurance practice. The former horizontal methods present a straightforward forecast but show high parameter dimensionality and may cause implausible forecasting results. Vertical / diagonal parametric approaches like mortality laws or transforms of life table functions reduce the degrees of freedom but, at the same time, introduce forecasting difficulties due to parameter dependencies. In contrast, non-parametric graduation methods create a smooth age-pattern but require subjective judgement from the forecaster concerning the degree of smoothness. This circumstance implies the risk to smooth short- to mid-term trends to extensively. Projection methods are based on the assumption that mortality evolves time-invariant such that observations from the past will

[89] According to the principle of equivalence the fair annuity rate value equals the single premium payment and expected forward annuity contract benefits.

[90] The age-pattern of mortality denotes a reporting-date-related cross-section of the mortality of an single insurance cohort or a whole population.

Forward and positive mortality models

Forward Mortality Models

- Cairns et al. (2006a) apply a Heath-Jarrow-Morton related framework and set the conditions

$$\alpha(x,s,t) = -V(x,s,t)^T\,\beta(x,s,t) \text{ (drift condition) where } \beta(x,s,t) = \frac{\partial}{\partial t}V(x,s,t)$$

ensuring an arbitrage free market. However, the determination of an adequate form of $V(x,s,t)$ is left for further research.

- Miltersen and Persson (2006) consider the case of correlated mortality and interest rates in a Heath-Jarrow-Morton framework. The volatility β has therefore to be estimated from historical forward force of mortality data or liquidly traded mortality derivatives whereas, at present, only the former data source seems reliable. The authors use the dynamics of the stochastic forward rate to model the price process of a pure endowment insurance based on a market term structure of pure endowment contracts.

- Zhu and Bauer (2011) demonstrate the opportunities of the forward mortality factor model class with forward force of mortality dynamics

$$df^{\mu}_{x+t}(s) = \left(\left(\tfrac{\partial}{\partial t} - \tfrac{\partial}{\partial x}\right) f^{\mu}_{x+t}(s) + \alpha(x,s,t)\right)\,dt + \beta(x,s,t)^T\,dW^{\mu}(s),\ (f^{\mu}_{x+t}(0) > 0)$$

with a multidimensional Q-Brownian motion W^{μ} and common Heath-Jarrow-Morton drift restriction for $\alpha(x,s,t)$. The volatility structure is captured by a matrix-valued function derived from a principal component analysis and parameters are subsequently (re)calibrated based on MLE.

Positive Mortality Model

Cairns et al. (2006a) consider the rational lognormal model according to Flesaker and Hughston (1996) incorporating positive mortality rates and define a state-price deflator

$$A(x,s) = \int_s^{\infty} N(x,s,\vartheta)d\vartheta,$$

where $\{N(x,s,t)\}$ is a family of positive martingales under a measure $\tilde{P}$ equivalent to Q. The family $\{A(x,s)\}$, in turn, is a strictly positive supermartingale and the spot survival probabilities are given by the integral representation

$${}_{t-s}p^{Q}_{x+s}(s) = E_{\tilde{P}}\left[\frac{A(x,t)}{A(x,s)}\middle|\,\mathcal{M}_s\right] = \frac{\int_t^{\infty} N(x,s,\vartheta)d\vartheta}{\int_s^{\infty} N(x,s,\vartheta)d\vartheta} \quad \text{for } 0 \le s \le t < \infty.$$

Just like the forward mortality framework the modelling of the dynamic of the $N(x,s,.)$ is left for further research. In this connection, a common financial market approach aims at a calibration of initial (theoretical) values for $N(x,0,.)$ to the observed mortality term structure. A differentiation with respect to the survival probability yields

$$N(x,0,t) = -\frac{\partial}{\partial t}\,{}_tp^{Q}_{x}(0) = f^{\mu}_{x+t}(0)\cdot {}_tp^{Q}_{x}(0).$$

Therefore, the implied survival probabilities ${}_tp^{Q}_{x}(0)$ which are, according to (4.9), themselves derived from market quotes for zero-coupon (longevity) bonds, constitute the basis for the calibration.

Table 4.7: Forward and positive mortality models.

Discrete-time mortality market models

- Olivier and Jeffery (2004) and Smith (2005) formulate a one-factor forward-rate model for future one-year forward survival probabilities driven by a gamma distributed deterioration factor

$$p^Q_{x+T}(t+1) = p^Q_{x+T}(t)^{a(x,t+1,T)\, G_{t+1}}$$

where $(G_i)_{i\in\mathbb{N}}$ is a sequence of Gamma independent and identically distributed (i.i.d.) random variables with unit mean and variance $\delta > 0$. The bias corrector a denotes a $\mathcal{M}_t$-measurable function ensuring the martingale property. The Gamma distribution ensures closed form solutions and reasonable projected probabilities with domain equal to the unit interval.

- Cairns (2006) extend the flexibility in the volatility term structure of the spot survival probabilities and allow for (non-trivial) correlated changes in the mortality rates. More precisely, the approach uses a two-dimensional array of dependent Gamma distributed variables $\tilde{G}_i(x,T) \sim \Gamma(\delta(x,t+1,T),\delta(x,t+1,T))$ for each age x and each maturity T generated using multivariate Gaussian copulas. The normalising constants $a(x,t+1,T)$ are derived from

$$a(x,t+1,T) = -\frac{\delta(x,t+1,T)\left(p^Q_{x+T}(t)^{-1/a(x,t+1,T)} - 1\right)}{\log\left(p^Q_{x+T}(t)\right)}.$$

Table 4.8: Discrete-time mortality market models.

continue in the future. However, a detailed illustration of German mortality data reveals a general increase in life expectancy (especially during the course of the 20th century) with significant random structural changes depending on age, calendar year and birth cohort. Therefore, a deterministic approach includes the risk to misjudge mortality developments[91].

During the last two decades, modern stochastic forecasting concepts were introduced and consider age-specific variation around an uncertain direction of future trends (due to parameter uncertainty). In this connection, the two-factor time series model of Lee and Carter (1992) presents a significant actuarial milestone since it provides variability measures for future projections using the bootstrap method by Efron (1979) based on a single mortality data sample and unknown distribution. The method shows robustness concerning log-linear trends of the central death rate and can basically be applied for endowment or term life insurance calculation as long as no distinct period or cohort effects are present. Nevertheless, due to strong (distributional) model assumptions forecasting results have to be interpreted accordingly[92]. In addition, sparse empirical data for highest ages leads to implausible parameter / projected

[91] Almost all mortality projections carried out in the second half of the 20th century underestimated the increase in life expectancy.

[92] A detailed discussion of the Lee-Carter model with graduation / projection implementation based on German mortality data is presented in Chapter 5.

Continuous-time mortality market models

- Cairns et al. (2006a) introduce the Survivor Credit Offer Rate (SCOR) market model as an analogue to the LIBOR market model by Miltersen et al. (1997) and Brace et al. (1997) for pricing interest rate derivatives. More precisely, each SCOR-implied forward rate (only spot SCORs are market observable) is modelled under its forward arbitrage-free measure, i.e. the model is given by a vector of forward SCOR dynamics for different forward rates of certain durations and maturities. The fair forward SCOR based on zero-coupon survivor bond prices is given as

$$L(x,s,t,t+1) = \frac{\Pi_s(x,t) - \Pi_s(x,t+1)}{\Pi_s(x,t+1)}.$$

  Hence, a forward SCOR equals the ratio of a portfolio of longevity bonds positions relative to the price of the longer one which suffers as a numéraire under a measure $P_t \approx Q$. Therefore, $L(x,s,t,t+1)$ forms a martingale with dynamics

$$dL(x,s,t,t+1) = L(x,s,t,t+1)\big(\sigma^r_{\text{SCOR}}(x,s,t)^T dW^{t+1,r} + \sigma^\mu_{\text{SCOR}}(x,s,t)^T dW^{t+1,\mu}(t)\big)$$

  for independent Brownian motions $W^{t+1,r}$ and $W^{t+1,\mu}$ and volatility functions σ^r_{SCOR}, σ^μ_{SCOR}. For deterministic volatility functions the forward SCOR $L(x,s,t,t+1)$ is log-normally distributed under P_t, such that the well-known Black-Formula can be applied for pricing issues.

- Cairns et al. (2006a) additionally develop a SCOR extension eliminating some drawbacks of their first approach and present the Perks-SCOR market model as a specific variant assuming a stochastic Perks law for mortality modelling.

- Cairns et al. (2008) introduce an annuity market model describing the dynamics of the price process of the so called forward annuity rates for (deferred) single-premium life annuity contracts in analogy to the swap market model introduced by Jamshidian (1996). Under contract-specific assumptions (e.g. for mortality-linked securities and annuity futures) the fair value[89]of an immediate annuity rate equals

$$F(x,s,t) = \frac{D(s,t)\,M(x,s,t)}{\sum_{u=t+1}^{\infty} D(s,u)\,M(x,s,u)} =: \frac{\Pi_s(x,t)}{X(s)} > 0$$

  and forms a martingale under an equivalent pricing measure $P_X \approx P$. Itô's formula yields a stochastic differential equation

$$dF(x,s,t) = F(x,s,t)\big(\sigma^r_{\text{FR}}(x,s,t)^T dW^r(t) + \sigma^\mu_{\text{FR}}(x,s,t)^T dW^\mu(t)\big)$$

  with independent standard Brownian motions W^r and W^μ under P_X and appropriate previsible interest rate and mortality rate volatility functions $\sigma^r_{\text{FR}}(x,s,t)$ and $\sigma^\mu_{\text{FR}}(x,s,t)$. If the latter are deterministic in s,t with $s < t \leq u$ the forward annuity rate $F(x,t,u)$ is log-normally distributed under P_X. Cairns et al. (2008) extended the approach to annuity contracts with minimum guarantee options.

Table 4.9: Continuous-time mortality market models.

values and thus additional risk resulting in wider prediction intervals. Several modifications and extensions of the Lee-Carter framework relaxed the model assumptions and enhanced parameter estimation efficiency. If projection aims at reproducing an exact image of cohort- and period-specific phenomenons, multi-factor time series models have proved to be particularly suitable. Additionally, period and/or cohort factors improve the fitting performance and describe longevity trends in an appropriate probabilistic way. Therefore, age-period-cohort models are principally applicable to long term forecasts and pensions pricing. Once again, the estimation bias and confidence interval appearance strongly depends on the availability of lengthy series of historical (cohort) data.

Lee-Carter is not Black-Scholes for longevity but a real step forward!

GUIDO GRÜTZNER, head of Life & Modelling, Secquaero Advisors AG, Switzerland

CHAPTER 5

Excursus on Modelling and Forecasting Mortality using the Lee-Carter Model and Multi-factor Extensions

This chapter is an adapted version of Steuten (2012).

5.1 Lee-Carter age-period model

The extrapolative method proposed by Lee and Carter (1992) constitutes a milestone of the development of stochastic mortality projection models. The matrix of the logarithm of the central death rate is modelled assuming a log-linear long term trend in the single time factor for different age groups. Due to the comparably low number of parameters, which are easy to interpret and feature a trivial correlation structure, the method is suitable for various applications: It serves for an appropriate description of discrete life table data and shows robustness concerning log-linear trends in the central death rate. Furthermore, the method can principally be applied for endowment or term life insurance calculation as long as no distinct period or cohort effects are observable from mortality data. However, the strong (distributional) model assumptions restrict the significance of forecasting results and thus prevent an application for long-term projections commonly utilised in pension modelling[93]. It is often used because of its straightforward applicability[94] on the one hand.

[93] For example, Lee-Carter population forecasts served as the basis of population forecasts for the US Social Security System. See Lee (2000) for a further treatment.

[94] It has to be admitted that there are no analytical formulas for any of the other life table functions available. The latter are complex non-linear functions of the Lee-Carter parameters.

On the other hand, generating bootstrapped parameter series and projecting of the time factor using standard Box-Jenkins techniques is possible. The model (hereafter referred to as LC92) describes the time-dependent evolution of the central death rate (3.3) for different age groups by means of a small number of parameters which can be easily interpreted.

In particular, the bilinear behaviour results from a distinction into the sum of an average log-mortality rate over time $a(x)$ and an improvement rate $b(x)$ sensitive to changes in the period effect $k(t)$ (indexing the level of uncertainty). Thus, we obtain an expression

$$\text{LC92} : \ln\left(m_x(t)\right) = a(x) + b(x)k(t) + \varepsilon_{\text{LC92}}(x,t). \tag{5.1}$$

The residual parameter $\varepsilon_{\text{LC92}}(x,t) \sim N\left(0, \sigma^2_{\text{LC92}}\right)$ covers age-specific random disturbances not captured by the model. The variance $\sigma^2_{\text{LC92}} > 0$ is assumed to be constant thus residuals are homoskedastic. This distributional assumption implies that temporal variation in the mortality profile are ignored and the rate of change $\frac{d\ln(m_x(t))}{dt}$ stays constant respectively. However, the constraints prevent so called mortality crossovers, i.e. an overlap of different mortality profiles, or other biologically implausible patterns.

Lee and Carter (1992) introduce two constraints controlling a translation ($\sum_x b(x) = 1$) and a scaling ($\sum_t k(t) = 0$) to ensure that $a(x)$ becomes the age-specific average shape of the log mortality rate over time and thus to get a unique least-squares solution

$$\left(\hat{a}(x),\hat{b}(x),\hat{k}(t)\right) = \underset{a(x),b(x),k(t)}{\text{argmin}} \sum_{x,t} \left(\ln\left(m_x(t)\right) - a(x) - b(x)k(t)\right)^2 \tag{5.2}$$

in case of overparametrisation. Since the parameters on the right side of Equation (5.2) are not observable, the forecasting procedure is divided. Firstly, parameters $\hat{a}(x)$, $\hat{b}(x)$ and $\hat{k}(t)$ are estimated by principal component analysis as an application of the singular value decomposition[95]. Afterwards, the parameters $\hat{k}(t)$ are iteratively reestimated in a separate step such that estimated and observed total number of deaths coincide

$$\sum_x d_x(t) \overset{!}{=} \sum_x L_x(t)\,\hat{m}_x(t) = \sum_x L_x(t)\,\exp\left(\hat{a}(x) + \hat{b}(x)\cdot\hat{k}^{\text{reest}}(t)\right) \quad \forall t \geq 0$$

for the (raw) number of deaths $d_x(t)$ and the number of person years $L_x(t)$.

[95] More precisely, the maximum likelihood estimators are determined in terms of the singular value decomposition of the (centred) log-mortality rate matrix. According to Girosi and King (2007) the Lee-Carter model forms a special case of a principal component method with a (first) single component $k(t)$ fitting mortality rates under a loss of information.

Figure 5.1 shows parameter estimates fitted to 1956-2006 German mortality data from the Human Mortality Database (2009). Thereafter, the new values $\hat{k}^{\text{reest}}$ are modelled as an ARIMA process preferably as a random walk according to the Box and Jenkins time series approach (1970) and extrapolated to obtain mortality forecasts (see Figure (5.1.3)). The extrapolation presupposes that observed historical trends persist in the future. However, this also means that (short-term) deterioration of mortality evolution due to structural changes (shocks) are not considered by the model.

5.2 Age-period(-cohort) modifications and extensions of the Lee-Carter model

In the following, we present three different Poisson log-bilinear modifications / extensions of the age-period model by Lee and Carter. Therefore, we focus on different approaches by Brouhns et al. (2002a), Renshaw and Haberman (2003) as well as Renshaw and Haberman (2006) since they share some characteristics:

- The models provide analytical graduation formulas to fit the age-period surface of the central death rate $m_x(t)$.
- The number of deaths $d_x(t)$ is modelled as an independent response variable following a Poisson distribution $\Pi(L_x(t) \cdot \hat{m}_x(t))$ such that the expected number of deaths equals $L_x(t) \cdot \hat{m}_x(t)$. This approach allows for a Poisson likelihood maximization.
- The period / cohort effects can under certain conditions be extrapolated via ARIMA(0,1,0) processes, i.e. simple random walks with drift.
- The standardised deviance residuals are appropriate in the context of (Poisson) bilinear regression to monitor the models' fitting quality towards crude mortality rates.

In order to avoid the singular value decomposition of the Lee-Carter approach we implement an iterative Newton-Raphson algorithm according to Brouhns et al. (2002b) to estimate the parameters. Since $d_x(t) \sim \Pi(\hat{d}_x(t))$ the density function for the number of deaths has the form

$$f(k,\hat{d}_x(t)) = \frac{\left(\hat{d}_x(t)\right)^k}{k!} \exp\left(-\hat{d}_x(t)\right) \quad \text{for integers } k \geq 0.$$

Thus, we obtain a log-likelihood function

$$\begin{aligned}
\ln(\mathcal{L}) &= \ln\left(\prod_{x,t} \frac{\left(\hat{d}_x(t)\right)^{d_x(t)} \exp\left(-\hat{d}_x(t)\right)}{d_x(t)!}\right) \\
&= \sum_{x,t} \ln\left(\frac{\left(\hat{d}_x(t)\right)^{d_x(t)} \exp\left(-\hat{d}_x(t)\right)}{d_x(t)!}\right) \\
&= \sum_{x,t} d_x(t) \ln\left(\hat{d}_x(t)\right) - \hat{d}_x(t) - \ln\left(d_x(t)!\right) \\
&= \sum_{x,t} d_x(t) \ln\left(\hat{m}_x(t)\right) - L_x(t)\hat{m}_x(t) + \underbrace{d_x(t) \ln\left(L_x(t)\right) - \ln\left(d_x(t)!\right)}_{\text{const.}} \qquad (5.3)
\end{aligned}$$

for estimated death rates $\hat{m}_x(t)$. A maximization of $\ln(\mathcal{L})$ is realised by an iterative stepwise update of the maximum likelihood estimates series

$$\hat{\xi}^{(\nu+1)} = \hat{\xi}^{(\nu)} - \frac{\partial \ln(\mathcal{L}(\hat{\xi}^{(\nu)}))/\partial \xi}{\partial^2 \ln(\mathcal{L}(\hat{\xi}))^{(\nu)}/\partial \xi^2}$$

already introduced by Goodman (1979) to estimate log-linear models with bilinear terms.

The "second-stage" re-estimation for the Poisson log-bilinear model can be omitted since the number of deaths is directly matched within the log-likelihood estimation. If the algorithm converges to the maximum likelihood both the observed and the estimated number of deaths coincide. The extrapolation task is solved using standard Box-Jenkins methods. A replication of the original data with Poisson noise in the number of death realizations allows for a distinction into a sample error in the estimated parameters (see for example Subfigures (5.2.1) - (5.2.3)) and a forecasting error related to the period-(cohort-)specific factors (Subfigure (5.2.4)). Firstly, this is done by estimating the model parameters for every bootstrapped replicate. Secondly, period-(cohort-)factors are extrapolated by means of the estimated ARIMA parameters. On the whole, we obtain insights concerning the variability of the central death rate. We fitted central mortality rates for life tables from the Human Mortality Database (2009) for the periods $t \in \{1960,...,2006 = t_0\}$ and an age range $x \in \{0,...,109\}$. The extrapolation was exemplarily executed for an 65-year old male with a time horizon of 50 years, i.e. until year 2056. We analyse the sampling and forecast errors for the parameters based on $5 \cdot 10^3$ bootstrapped samples.

5.2.1 Age-period model by Brouhns, Denuit and Vermunt (2002)

Brouhns et al. (2002a) model the incisive Gaussian LC92 error structure as a heteroscedastic Poisson distribution and introduce a variant (hereafter referred to as BDV02) with a single estimation stage using loglikelihood maximization. The bilinear structure of the LC92 model remains, i.e. we have

$$\text{BDV02}: \quad \ln\left(m_x(t)\right) = a(x) + b(x)\,k(t) + \varepsilon_{\text{BDV02}}(x,t).$$

The parameter updating scheme of the iterative procedure is depicted in Table 5.1. Let $\hat{d}_x(t)^{(\nu)} = L_x(t) \cdot \exp\left(\hat{a}(x)^{(\nu)} + \hat{b}(x)^{(\nu)} \cdot \hat{k}(t)^{(\nu)}\right)$ denote the estimated number of deaths after ν parameter updates. The updating circle is repeated

Iteration procedure for the BDV02 model

Set the start values as $\hat{a}(x)^{(0)} = 0$, $\hat{b}(x)^{(0)} = 1$, $\hat{k}(t)^{(0)} = 0$ and the stop criterion to $limit = 10^{-6}$. The Parameter updates after iteration step ν are obtained as:

1. update $\hat{a}(x)^{(\nu+1)} = \hat{a}(x)^{(\nu)} + \dfrac{\sum_t \left(d_x(t) - \hat{d}_x(t)^{(\nu)}\right)}{\sum_t \hat{d}_x(t)^{(\nu)}}$,
2. fix $\hat{b}(x)^{(\nu+1)} = \hat{b}(x)^{(\nu)}$, $\hat{k}(t)^{(\nu+1)} = \hat{k}(t)^{(\nu)}$,
3. update $\hat{k}(t)^{(\nu+2)} = \hat{k}(t)^{(\nu+1)} + \dfrac{\sum_x \left(d_x(t) - \hat{d}_x(t)^{(\nu+1)}\right) \hat{b}(x)^{(\nu+1)}}{\sum_x \hat{d}_x(t)^{(\nu+1)} \left(\hat{b}(x)^{(\nu+1)}\right)^2}$,
4. fix $\hat{a}(x)^{(\nu+2)} = \hat{a}(x)^{(\nu+1)}$, $\hat{b}(x)^{(\nu+2)} = \hat{b}(x)^{(\nu+1)}$, $\hat{k}(t)^{(\nu+2)} = \hat{k}(t)^{(\nu+1)}$,
5. update $\hat{b}(x)^{(\nu+3)} = \hat{b}(x)^{(\nu+2)} + \dfrac{\sum_t \left(d_x(t) - \hat{d}(t)^{(\nu+2)}\right) \hat{k}(t)^{(\nu+2)}}{\sum_t \hat{d}_x(t)^{(\nu+2)} \left(\hat{k}(t)^{(\nu+2)}\right)^2}$,
6. fix $\hat{a}(x)^{(\nu+3)} = \hat{a}(x)^{(\nu+2)}$, $\hat{k}(t)^{(\nu+3)} = \hat{k}(t)^{(\nu+2)}$
7. repeat 1.–6. while $\left|\ln(\mathcal{L}(\hat{\xi}^{(\nu+1)})) - \ln(\mathcal{L}(\hat{\xi}^{(\nu)}))\right| > limit$.

Table 5.1: Iteration procedure for the BDV02 age-period model.

until the increase in the log-likelihood function (5.3) becomes negligible small and converges respectively. In order to fulfil the LC92 model constraints

$$\hat{a}(x) = \tfrac{1}{\#t} \sum_t \ln\left(\hat{m}_x(t)\right), \quad \sum_x b(x) = 1, \quad \sum_t k(t) = 0$$

needed to ensure an unique likelihood maximum we apply a parameter transformation

$$\hat{a}(x) \to \hat{a}(x) + \hat{b}(x) \cdot \tfrac{1}{\#t} \sum_t \hat{k}(t),$$

$$\hat{k}(t) \to \left(\hat{k}(t) - \tfrac{1}{\#t}\sum_t \hat{k}(t)\right)\sum_x \hat{b}(x),$$
$$\hat{b}(x) \to \frac{\hat{b}(x)}{\sum_x \hat{b}(x)}$$

where $\#t$ denotes the cardinality of the observation window. The value for $\hat{d}_x(t)$ is not affected by this transformation since

$$\hat{a}(x)+\hat{b}(x)\cdot\tfrac{1}{\#t}\sum_t \hat{k}(t)+\frac{\hat{b}(x)}{\sum_x \hat{b}(x)}\left(\hat{k}(t) - \tfrac{1}{\#t}\sum_t \hat{k}(t)\right)\sum_x \hat{b}(x) = \hat{a}(x)+\hat{b}(x)\cdot\hat{k}(t).$$

Mortality rates are extrapolated for future times $t > t_0$ by means of the extrapolated logarithmised mortality reduction factors $\hat{b}(x)\left(k(t) - \hat{k}(t_0)\right)$, i.e. based on bootstrapped forecasts $\{k(t)\}_{t=2007,\ldots,2056}$ and the latest available central death rate $\hat{m}_x(t_0)$ we obtain

$$m_x(t) = \hat{m}_x(t_0)\exp\left(\hat{b}(x)\left(k(t) - \hat{k}(t_0)\right)\right).$$

The parametric bootstrap procedure provides an extrapolation path for every simulated realisation $d_x(t) \sim \mathit{\Pi}(\hat{d}_x^{\text{LC92}}(t))$ were $\hat{d}_x^{\text{LC92}}(t)$ denotes the fitted number of deaths from the LC92 model. This also implies re-estimation of each parameter $\hat{a}(x)$, $\hat{b}(x)$ and $\hat{k}(t)$ as well as the drift and diffusion parameters of the random walks.

5.2.2 Age-period model with age-specific enhancement by Renshaw and Haberman (2003)

The extension by Renshaw and Haberman (2003) (hereafter referred to by the abbreviation RH03) introduces an additional age-specific term to ensure equality between age-specific actual and expected total number of deaths. Due to the additional period factor the rather strong assumption of perfect correlation of changes across different age groups can be relaxed. The logarithmised central death rates are calculated as

$$\text{RH03}: \quad \ln\left(m_x(t)\right) = a(x) + b_1(x)\,k_1(t) + b_2(x)\,k_2(t) + \varepsilon_{\text{RH03}}(x,t).$$

Thus, the introduction of a double bilinear predictor structure aims at capturing age-specific systematic trends in mortality patterns. As mentioned above, the Poisson response modelling can be extended to a multi-factor framework. The updating scheme has therefore to be reformulated according to Table 5.2. The univariate time series $\{k_i(t)\}_{t=1965,\ldots,2006}$ $(i = 1,2)$ enables a projection based

Iteration procedure for the RH03 model

Set the start values as $\hat{a}(x)^{(0)} = 0$, $\hat{b}(x)^{(0)} = 1$, $\hat{k}(t)^{(0)} = 0$ and the stop criterion to $limit = 10^{-4}$. The parameter updates after iteration step ν (i=1,2) are obtained as:

1. update $\hat{a}(x)^{(\nu+1)} = \hat{a}(x)^{(\nu)} + \dfrac{\sum_t \left(d_x(t) - \hat{d}_x(t)^{(\nu)}\right)}{\sum_t \hat{d}_x(t)^{(\nu)}}$,
2. fix $\hat{b}_i(x)^{(\nu+1)} = \hat{b}_i(x)^{(\nu)}$, $\hat{k}_i(t)^{(\nu+1)} = \hat{k}_i(t)^{(\nu)}$,
3. update $\hat{k}_1(t)^{(\nu+2)} = \hat{k}_1(t)^{(\nu+1)} + \dfrac{\sum_x \left(d_x(t) - \hat{d}_x(t)^{(\nu+1)}\right) \hat{b}_1(x)^{(\nu+1)}}{\sum_x \hat{d}_x(t)^{(\nu+1)} \left(\hat{b}_1(x)^{(\nu+1)}\right)^2}$,
4. fix $\hat{a}(x)^{(\nu+2)} = \hat{a}(x)^{(\nu+1)}$, $\hat{b}_i(x)^{(\nu+2)} = \hat{b}_i(x)^{(\nu+1)}$, $\hat{k}_2(t)^{(\nu+2)} = \hat{k}_2(t)^{(\nu+1)}$,
5. update $\hat{b}_1(x)^{(\nu+3)} = \hat{b}_1(x)^{(\nu+2)} + \dfrac{\sum_t \left(d_x(t) - \hat{d}(t)^{(\nu+2)}\right) \hat{k}_1(t)^{(\nu+2)}}{\sum_t \hat{d}_x(t)^{(\nu+2)} \left(\hat{k}_1(t)^{(\nu+2)}\right)^2}$,
6. fix $\hat{a}(x)^{(\nu+3)} = \hat{a}(x)^{(\nu+2)}$, $\hat{b}_2(x)^{(\nu+3)} = \hat{b}_2(x)^{(\nu+2)}$, $\hat{k}_i(t)^{(\nu+3)} = \hat{k}_i(t)^{(\nu+2)}$,
7. update $\hat{k}_2(t)^{(\nu+4)} = \hat{k}_2(t)^{(\nu+3)} + \dfrac{\sum_x \left(d_x(t) - \hat{d}_x(t)^{(\nu+3)}\right) \hat{b}_2(x)^{(\nu+3)}}{\sum_x \hat{d}_x(t)^{(\nu+3)} \left(\hat{b}_2(x)^{(\nu+3)}\right)^2}$,
8. fix $\hat{a}(x)^{(\nu+4)} = \hat{a}(x)^{(\nu+3)}$, $\hat{b}_i(x)^{(\nu+4)} = \hat{b}_i(x)^{(\nu+3)}$, $\hat{k}_1(t)^{(\nu+4)} = \hat{k}_1(t)^{(\nu+3)}$,
9. update $\hat{b}_2(x)^{(\nu+5)} = \hat{b}_2(x)^{(\nu+4)} + \dfrac{\sum_t \left(d_x(t) - \hat{d}(t)^{(\nu+4)}\right) \hat{k}_2(t)^{(\nu+4)}}{\sum_t \hat{d}_x(t)^{(\nu+4)} \left(\hat{k}_2(t)^{(\nu+4)}\right)^2}$,
10. fix $\hat{a}(x)^{(\nu+5)} = \hat{a}(x)^{(\nu+4)}$, $\hat{b}_1(x)^{(\nu+5)} = \hat{b}_1(x)^{(\nu+4)}$, $\hat{k}_i(t)^{(\nu+5)} = \hat{k}_i(t)^{(\nu+4)}$,
11. repeat 1.−10. while $\left|\ln(\mathcal{L}(\hat{\xi}^{(\nu+1)})) - \ln(\mathcal{L}(\hat{\xi}^{(\nu)}))\right| > limit$.

Table 5.2: Iteration procedure for the RH03 age-period model.

on the present observation year t_0 by

$$m_x(t) = \hat{m}_x(t_0) \exp\left(\hat{b}_1(x)\left(k_1(t) - \hat{k}_1(t_0)\right) + \hat{b}_2(x)\left(k_1(t) - \hat{k}_2(t_0)\right)\right).$$

The functionality of the bootstrap algorithm is the same as for the BDV02 model.

5.2.3 Age-period-cohort model by Renshaw and Haberman (2006)

The LC92 and BDV02 age-period models were supplemented by the cohort-based extension of Renshaw and Haberman (2006) (referred to as RH06). Consequently, the additional bilinear term explains cohort effects for different years of birth $t - x$. Mortality rates are modelled via

$$\text{RH06}: \quad \ln\left(m_x(t)\right) = a(x) + b_1(x)\,k(t) + b_2(x)\,l(t-x) + \varepsilon_{\text{RH06}}(x,t)$$

and we use the following iterative procedure (Table 5.3) with projected values

$$m_x(t) = \hat{m}_x(t_0)\exp\left(\hat{b}_1(x)\left(k(t)-\hat{k}(t_0)\right)+\hat{b}_2(x)\left(l(t-x)-\hat{l}(t_0-x)\right)\right).$$

Here as well, the bootstrap procedure equals the one applied for the BDV02 model.

Iteration procedure for the RH06 model

Set the start values as $\hat{a}(x)^{(0)} = 0$, $\hat{b}(x)^{(0)} = 1$, $\hat{k}(t)^{(0)} = 0$ and the stop criterion to $limit = 10^{-4}$. The parameter updates after iteration step ν (i=1,2) are obtained as:

1. update $\hat{a}(x)^{(\nu+1)} = \hat{a}(x)^{(\nu)} + \dfrac{\sum_t \left(d_x(t)-\hat{d}_x(t)^{(\nu)}\right)}{\sum_t \hat{d}_x(t)^{(\nu)}}$,
2. fix $\hat{b}_1(x)^{(\nu+1)} = \hat{b}_1(x)^{(\nu)}$, $\hat{k}_1(t)^{(\nu+1)} = \hat{k}_1(t)^{(\nu)}$, $\hat{b}_2(x)^{(\nu+1)} = \hat{b}_2(x)^{(\nu)}$, $\hat{l}(t-x)^{(\nu+1)} = \hat{l}(t-x)^{(\nu)}$
3. update $\hat{k}_1(t)^{(\nu+2)} = \hat{k}_1(t)^{(\nu+1)} + \dfrac{\sum_x \left(d_x(t)-\hat{d}_x(t)^{(\nu+1)}\right)\hat{b}_1(x)^{(\nu+1)}}{\sum_x \hat{d}_x(t)^{(\nu+1)}\left(\hat{b}_1(x)^{(\nu+1)}\right)^2}$,
4. fix $\hat{a}(x)^{(\nu+2)} = \hat{a}(x)^{(\nu+1)}$, $\hat{b}_i(x)^{(\nu+2)} = \hat{b}_i(x)^{(\nu+1)}$, $\hat{l}(t-x)^{(\nu+2)} = \hat{l}(t-x)^{(\nu+1)}$,
5. update $\hat{b}_1(x)^{(\nu+3)} = \hat{b}_1(x)^{(\nu+2)} + \dfrac{\sum_t \left(d_x(t)-\hat{d}(t)^{(\nu+2)}\right)\hat{k}_1(t)^{(\nu+2)}}{\sum_t \hat{d}_x(t)^{(\nu+2)}\left(\hat{k}_1(t)^{(\nu+2)}\right)^2}$,
6. fix $\hat{a}(x)^{(\nu+3)} = \hat{a}(x)^{(\nu+2)}$, $\hat{b}_2(x)^{(\nu+3)} = \hat{b}_2(x)^{(\nu+2)}$, $\hat{k}_1(t)^{(\nu+3)} = \hat{k}_1(t)^{(\nu+2)}$, $\hat{l}(t-x)^{(\nu+3)} = \hat{l}(t-x)^{(\nu+2)}$,
7. update $\hat{l}(t-x)^{(\nu+4)} = \hat{l}(t-x)^{(\nu+3)} + \dfrac{\sum_{\substack{t,x\\t-x}} \left(d_x(t)-\hat{d}_x(t)^{(\nu+3)}\right)\hat{b}_2(x)^{(\nu+3)}}{\sum_{\substack{t,x\\t-x}} \hat{d}_x(t)^{(\nu+3)}\left(\hat{b}_2(x)^{(\nu+3)}\right)^2}$,
8. fix $\hat{a}(x)^{(\nu+4)} = \hat{a}(x)^{(\nu+3)}$, $\hat{b}_i(x)^{(\nu+4)} = \hat{b}_i(x)^{(\nu+3)}$,$\hat{k}_1(t)^{(\nu+4)} = \hat{k}_1(t)^{(\nu+3)}$,
9. update $\hat{b}_2(x)^{(\nu+5)} = \hat{b}_2(x)^{(\nu+4)} + \dfrac{\sum_t \left(d_x(t)-\hat{d}(t)^{(\nu+4)}\right)\hat{l}(t-x)^{(\nu+4)}}{\sum_t \hat{d}_x(t)^{(\nu+4)}\left(\hat{l}(t-x)^{(\nu+4)}\right)^2}$,
10. fix $\hat{a}(x)^{(\nu+5)} = \hat{a}(x)^{(\nu+4)}$, $\hat{b}_1(x)^{(\nu+5)} = \hat{b}_1(x)^{(\nu+4)}$, $\hat{k}_1(t)^{(\nu+5)} = \hat{k}_1(t)^{(\nu+4)}$, $\hat{l}(t-x)^{(\nu+5)} = \hat{l}(t-x)^{(\nu+4)}$
11. repeat 1.−10. while $\left|\ln(\mathcal{L}(\hat{\xi}^{(\nu+1)})) - \ln(\mathcal{L}(\hat{\xi}^{(\nu)}))\right| > limit$.

Table 5.3: Iteration procedure for the RH06 age-period-cohort model.

5.3 Estimation results of the mortality models LC92–RH06

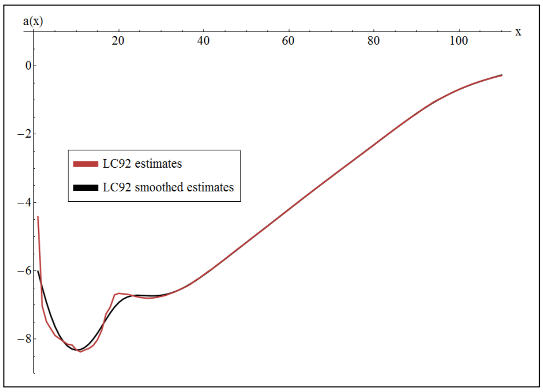
In order to avoid localised age- and period-specific anomalies we smoothed all historic parameter estimates with cubic P-spline interpolation[96] for the illustration and the goodness of fit analysis in Subsection 5.4. Notice that according to Brouhns et al. (2005) the computationally intensive models BDV02–RH06 allow for randomness due to a sampling fluctuation for each of the parameters as well as a forecast error for the period- and / or cohort factors by means of a parametric Poisson bootstrap for the number of deaths. Therefore, each of the parameters in the Tables A.1 - A.4 in Appendix A contains (smoothed) historic estimate values (marked with a hat) and, if available, average value (marked with a bar), standard error (marked with SE) and 0.05- as well as 0.95-quantiles (marked with $q_{0.05}$ and $q_{0.95}$). Additionally, the Figures 5.1 - 5.4 present parameter sampling errors given by the confidence intervals (marked with CI) and illustrate prediction intervals (marked with PI) for the simulated projected period and cohort factor(s) as well as the central death rates in order to measure the forecasting uncertainty.

LC92 parameters:

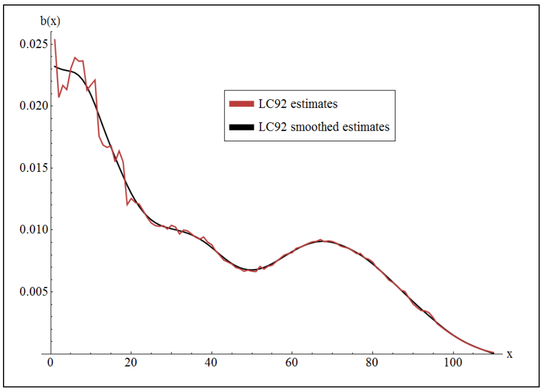
Parameter $a(x)$ (Subfigure (5.1.1)) describes the average shape of the logarithmised age-specific mortality profile similar to Subfigure (2.1.1) with increasing force of mortality for adult ages. The pattern for the “sensitivity parameter” $b(x)$ (Subfigure (5.1.2)) is positive, slightly undulated and monotonically decreasing. Since $b(x)$ expresses the sensitivity of central death rates to changes in the period factor $k(t)$, we can conclude that improvements at younger ages tend to be more rapidly and scattered. The time trend $k(t)$ shows nearly linear decreasing behaviour since 1970 starting with a value of 29 and ending at -61. This supports an extrapolation by means of a random walk with drift using data from 1970 and afterwards. The negative fitted values for $k(t)$ indicate that mortality improvements beyond year 1986 proceed more slowly compared to changes in the previous observation window [1960,1986]. This is especially true for infancy ages where parameter $b(x)$ shows its maximum. Both parameters $b(x)$ and $k(t)$ exhibit a rough shape in age and period direction respectively. This roughness is partly owed to the Likelihood-based methods.

[96] For an overview of smoothing and different choices for smoothing kernels see Subsection 3.2.3. However, we used tensor product penalized spline smoothing with parameter $\lambda = 0.02$ penalizing roughness and a spline function of fourth degree.

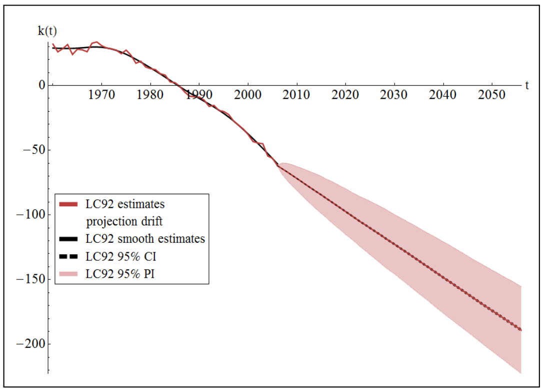
Parameter estimates, projections and forecasting error for the LC92 model



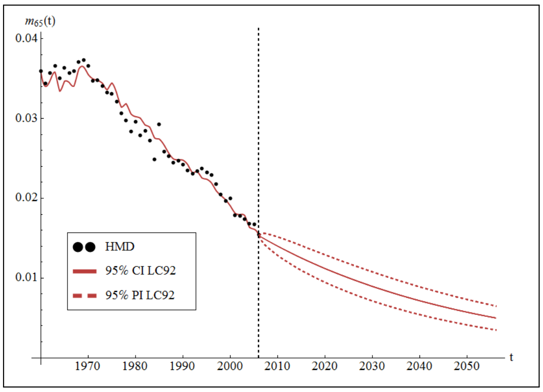
(5.1.1) Age-specific component $a(x)$



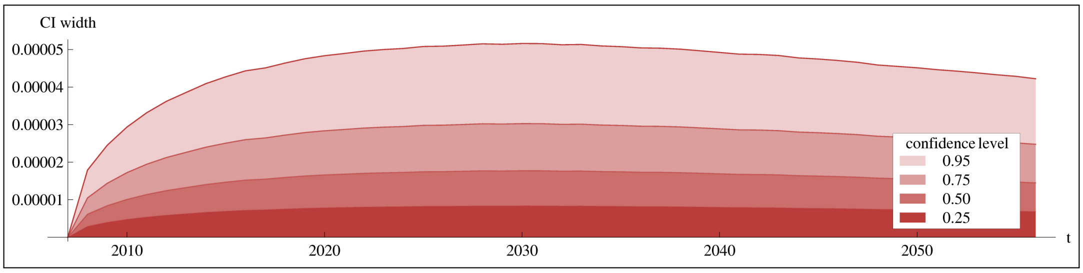
(5.1.2) Age-specific component $b(x)$



(5.1.3) Time-varying parameter $k(t)$



(5.1.4) HMD mortality rates and central death rates



(5.1.5) Confidence interval width for the projected central death rates

Figure 5.1: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the LC92 model. The projections are based on $5 \cdot 10^3$ iterations.

BDV02 parameters:

The age dependent Poisson estimates of model BDV02 show some extreme values[97] for parameters $a(x)$ and $b(x)$ for ages $x > 100$ due to inconsistent and sparse number of death data. In particular, we observe negative values for parameter $b(x)$ (see also Table A.2), i.e. an alternating mortality evolution with a supposed age specific mortality deterioration. The prediction intervals for the LC92 and BDV02 model are of nearly the same width such that the additional sampling risk is negligible here. Moreover, since the assumption of normality on the model residuals is relaxed, we observe asymmetric confidence bounds (see Subfigure (5.2.5)). The parameter sampling uncertainty is illustrated in each of the Subfigures (5.2.1) - (5.2.3) by means of the 0.95-prediction interval and 0.95-confidence interval of the bootstrapped historic and projected parameters. It should be noted that due to a lack of robustness the confidence interval can show systematic deviations from the solid historic estimates. This effect can also be observed for the quantile values of Table A.2.

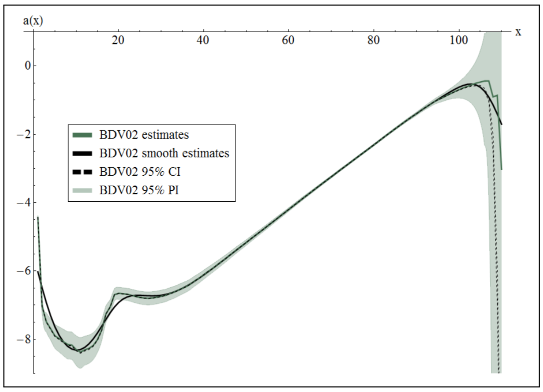
RH03 parameters:

At first, we have a look at the two distinct period factors allowing for more flexibility in the reproduction of period effects provided that the underlying data exhibits no regular pattern[98]. Factor $k_1(t)$ determines the level of the logarithmised mortality rate curve over time and shows general period-specific improvements for all ages. In particular, it decreases over the complete observation window with low forecasting error indicated by the prediction interval width for projected values. In contrast, period factor $k_2(t)$ forms the slope of the mortality profiles. $k_2(t)$ increases until year 1976 and decreases afterwards with the same slope according to the amount. The curve shape for $b_2(x)$ proceeds sinusoidal with ages 8, 17, 42 and 76 being sensitive to variations in the time index $k_2(t)$. This means that for the mentioned ages mortality improvements were more pronounced prior to the mid-seventies and rates of change became less thereafter. Moreover, the prediction interval is wider. Parameters $a(x)$ and $b_1(x)$ are similar to those of the LC92 and BDV02 approach. It is remarkable that the 0.95-prediction interval forecast of the values is considerably thinner than in the BDV02 model. This can be explained by the fact that the forecast errors of both time factors $k_1(t)$ and $k_2(t)$, taken together, do not reach the uncertainty of the BDV02 period factor. Compared to the BDV02 estimates, the central death rate estimates are better fitted to demographic data for the

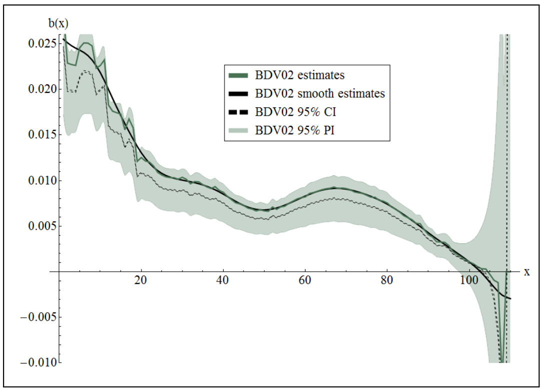
[97] This effect has also been observed for models RH03 and RH06.

[98] For a short outline on the identifiability problem of certain period- and cohort-related effects see Section 4.1.

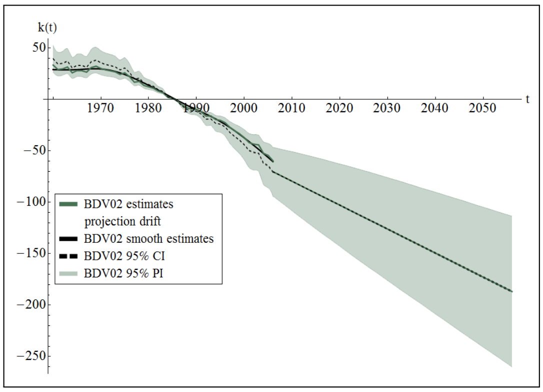
Parameter estimates, projections and forecasting error for the BDV02 model



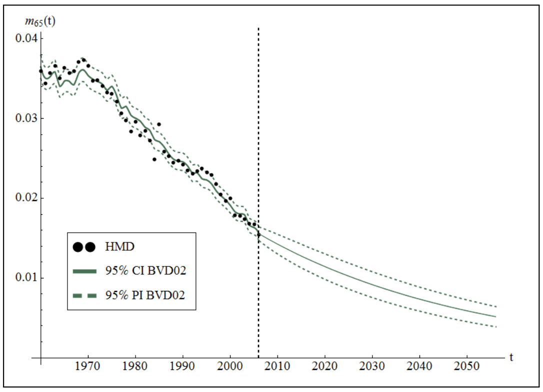
(5.2.1) Age-specific component $a(x)$



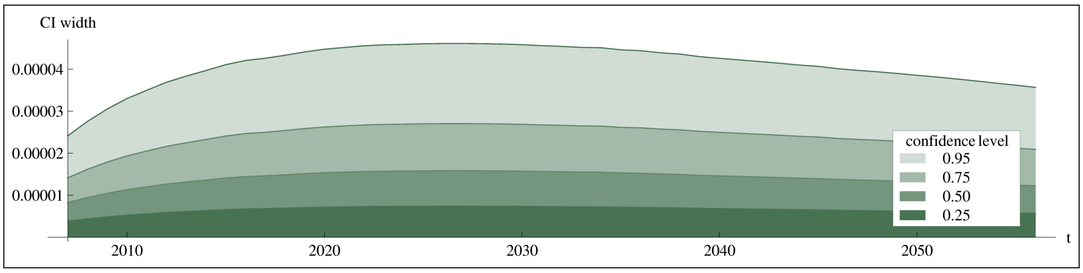
(5.2.2) Age-specific component $b(x)$



(5.2.3) Time-varying parameter $k(t)$



(5.2.4) HMD mortality rates and central death rates

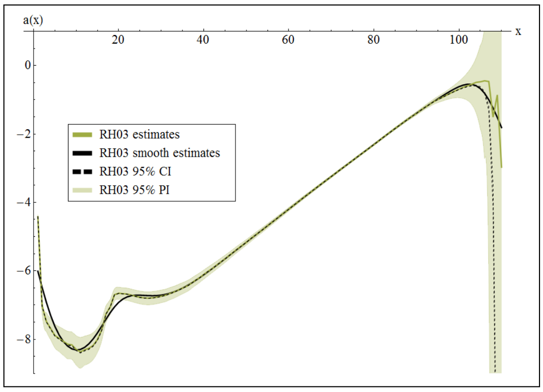


(5.2.5) Confidence interval width for the projected central death rates

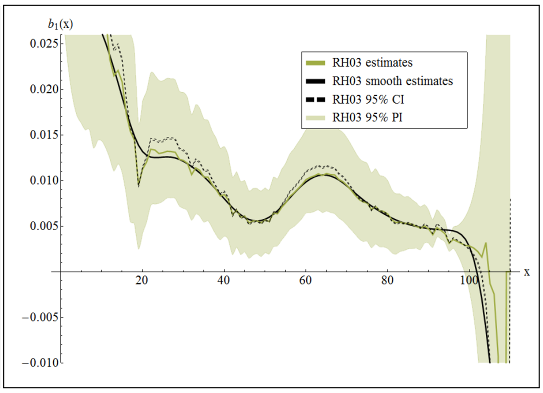
Figure 5.2: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the BDV02 model. The projections are based on $5 \cdot 10^3$ iterations.

decades 1960-1970 and 1975-1985 (see Figure 5.5).

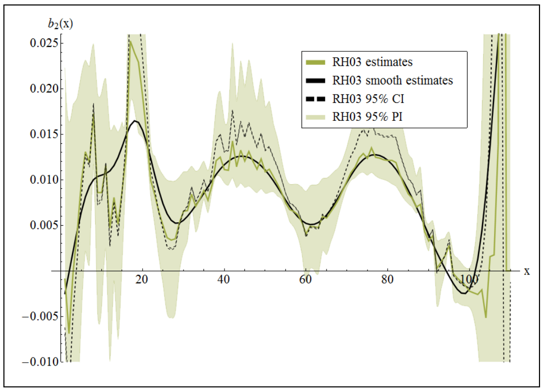
Parameter estimates, projections and forecasting error for the RH03 model



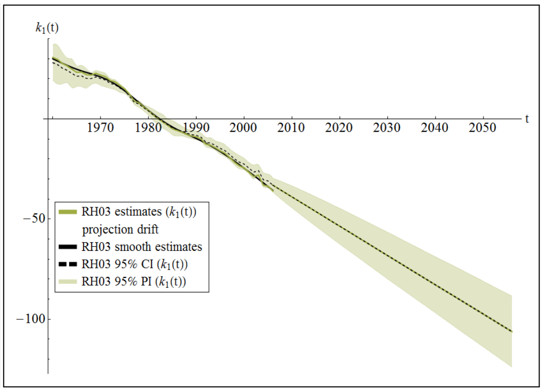
(5.3.1) Age-specific component $a(x)$



(5.3.2) Age-specific component $b_1(x)$



(5.3.3) Age-specific component $b_2(t)$

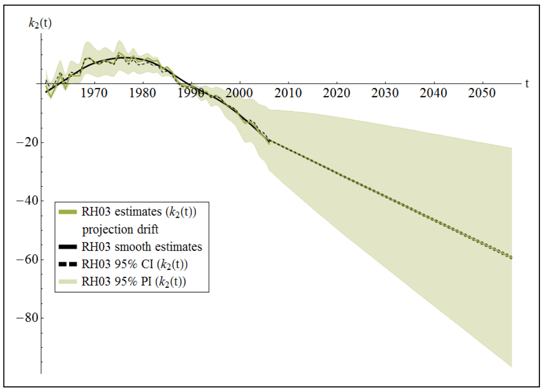


(5.3.4) Time-varying parameter $k_1(t)$

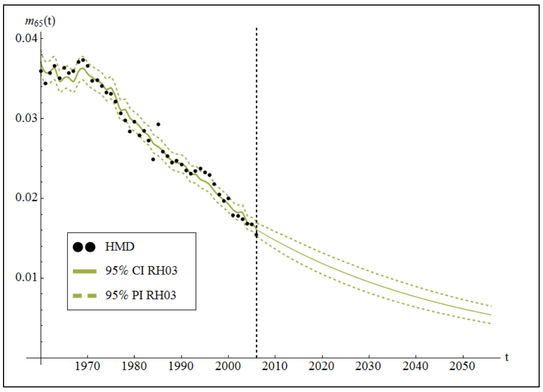
Figure 5.3: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the RH03 model. The projections are based on $5 \cdot 10^3$ iterations.

RH06 parameters:

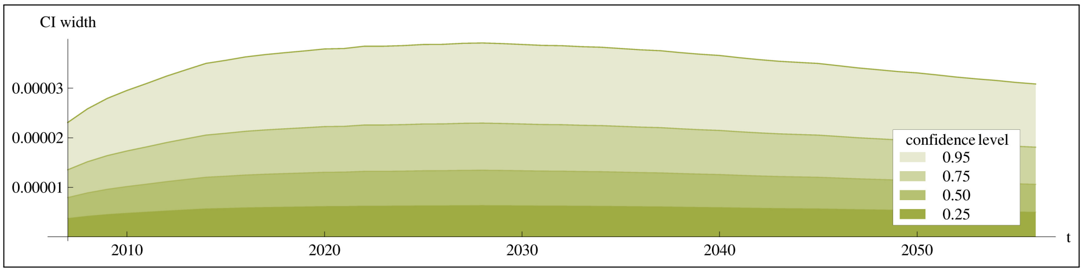
A glance at the parameter estimates $b_1(x)$ and $b_2(x)$ that control the age-specific period and cohort sensitivity reveals a similar but varying behaviour especially in the border age regions. The non-smoothness and bootstrap dispersion may indicate an overfitting of the mortality data and result in a lack of robustness concerning the choice of the graduation window or the initial number of deaths. The period factor $k(t)$ shows a change of inclination in the mid-seventies which was already observed for the RH03 parameter analysis. Thereafter, the slope proceeds almost linear such that a choice in favour for the random walk seems appropriate. The additional cohort factor $l(t-x)$ reveals a series of parameters with distinct variation in the slope for generations born at the end of World War I, the 1919 influenza and World War II. Willets (2004) mentions that these



(5.3.5) Time-varying parameter $k_2(t)$



(5.3.6) HMD mortality rates and central death rates



(5.3.7) Confidence interval width for the projected central death rates

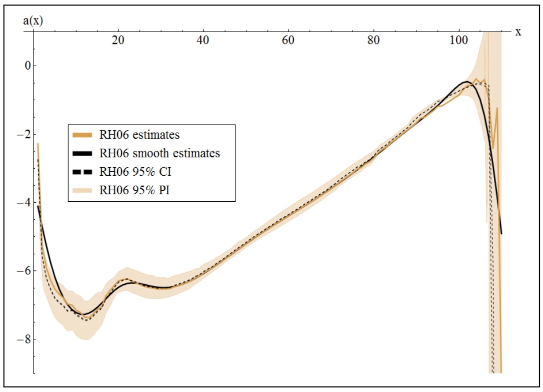
Figure 5.3: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the RH03 model. The projections are based on $5 \cdot 10^3$ iterations.

cohorts exhibit rapid mortality improvements relative to deteriorations in times of war or similar incidents. The prominent break in cohort year 1870 followed by a linear decreasing trend stands for a falling cohort mortality and a period effect independent[99] of $k(t)$. The bootstrap noise for the cohort factor up to calendar year 1870 is accompanied by the wide CIs in that region. For the historic central death rate estimates we record strong oscillations for the time period 1975 until 1985 including extreme outlier values. However, the goodness of fit for model RH03 is surpassed especially for periods since the late-seventies. Thus, the overall absolute deviation of model and historic central death rates is three times lower than in the default LC92 case and at least 2.4 times lower than the RH03 total absolute deviation (see Figure 5.5). The forecasting error measured by the confidence interval width (see Subfigure (5.4.7)) diverges funnel-shaped and has the largest value compared to the remaining time series models LC92–RH03.

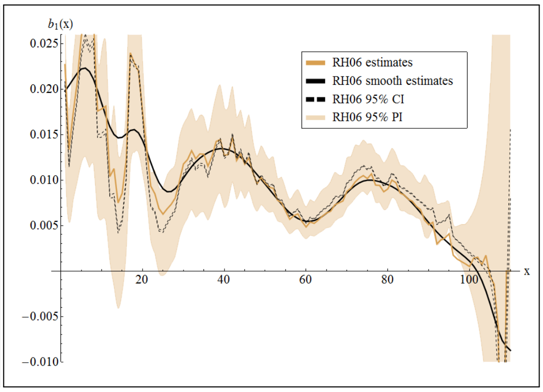
As already mentioned, the parameter estimates of the Lee-Carter extensions should not only undergo a reasonable smoothing procedure but rather be

[99] See Cairns et al. (2009) for a comprehensive presentation of the mentioned phenomenons illustrated by data from England and Wales as well as the US.

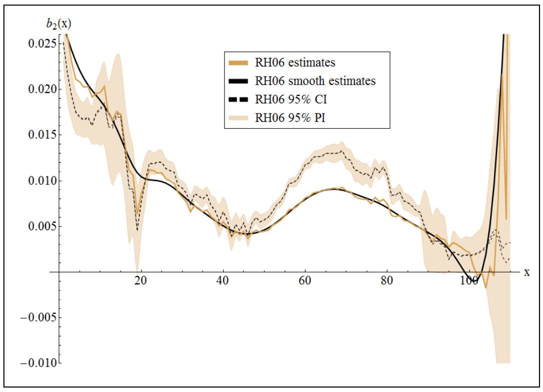
Parameter estimates, projections and forecasting error for the RH06 model



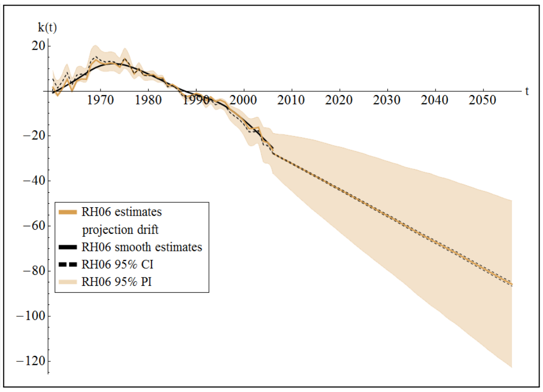
(5.4.1) age-specific component $a(x)$



(5.4.2) Age-specific component $b_1(x)$



(5.4.3) Age-specific component $b_2(t)$



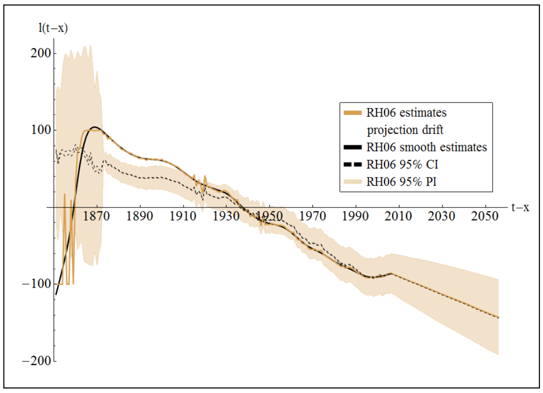
(5.4.4) Time-varying parameter $k(t)$

Figure 5.4: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the RH06 model. The projections are based on $5 \cdot 10^3$ iterations.

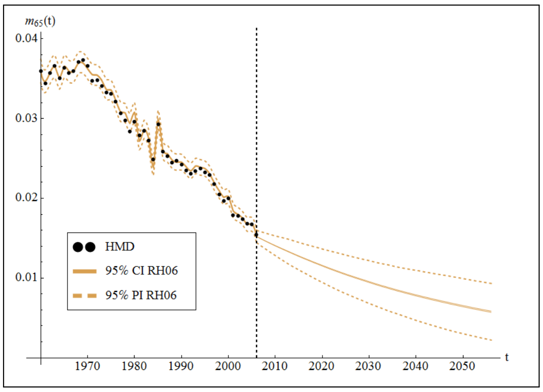
graduated by deterministic functions for age- and period-intervals where only sparse demographic data is available, i.e. for ages beyond 100 and cohorts born before 1870. Barrieu et al. (2010) suggest to smooth or adjust data up to age 90 and to extrapolate the raw data by a local parametric shape under certain fitting constraints. Nevertheless, any subjective graduation of the parameters has a hardly predictable impact on a subsequent mortality projection.

5.4 Comparison of the mortality models

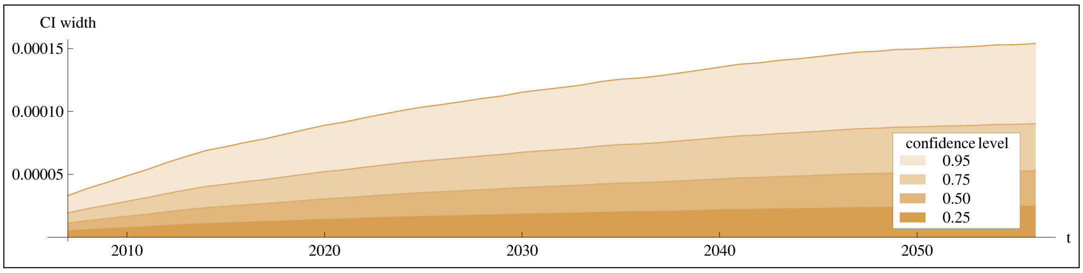
When it comes to a comparison of the different models it seems plausible to subject them to an initial verification of fifteen different qualitative benchmark criteria already discussed in Subsection 4.1.3. To a certain extent and for a part of the mortality models LC92–RH06, this has already been done in actuarial literature. e.g. see Cairns et al. (2009), Cairns et al. (2008) and Plat (2009).



(5.4.5) Cohort parameter $l(t-x)$



(5.4.6) HMD mortality rates and central death rates



(5.4.7) Confidence interval width for the projected central death rates

Figure 5.4: Parameter estimates/projections, 0.95-confidence intervals (CI, solid) and prediction intervals (PI, dashed) for the RH06 model. The projections are based on $5 \cdot 10^3$ iterations.

Absolute deviation between the raw mortality profile and the estimated central death rate function for the mortality models LC92–RH06

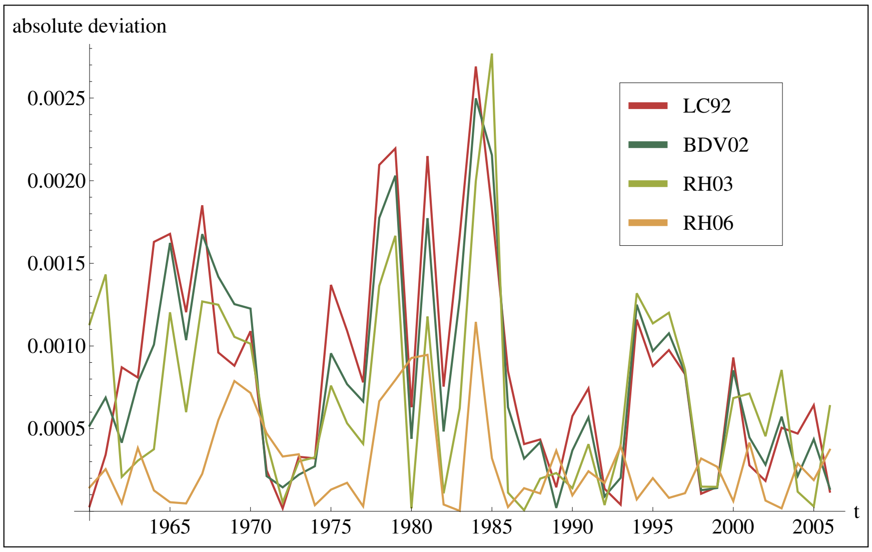


Figure 5.5: Absolute deviation between the raw mortality profile and the estimated central death rate function w.r.t. a male individual aged 65 years and observation years 1960-2006.

Moreover, in Subsections 5.4.2–5.4.6 we present a comprehensive treatment of quantitative criteria and check them against the results for the nested mortality models.

5.4.1 Comparison of qualitative model criteria

The choice in favour for or against a certain mortality model strongly depends on the prioritization in the underlying application predefined by the user. This means that factors like data availability, forecasting purpose and projection horizon may effect the selection process to different degrees. Besides more or less basic criteria like positive force of mortality, biological reasonable long-term behaviour or consistency with historical data, model representatives should ideally exhibit a number of additional features which are listened in Table 5.4. The fulfilment of several of these criteria (indicated with a question mark) can neither be verified nor negated since they are only partly fulfilled or a subjective assessment is required. A verification of the formal features provides the following results:

- Time series projection models provide simple, transparent and positive

Comparison of qualitative model criteria

Model selection criteria (in accordance with Section 4.1.3)	Lee-Carter variants and extensions			
	LC92	BDV02	RH03	RH06
transparency and simplicity	?	?	?	?
positive mortality rates	√	√	√	√
data consistency	?	?	√	√
biological reasonable long-term dynamics	√	√	√	√
consideration of (period- / cohort-)effects	×	×	√	√
non-trivial correlation structure	×	×	√	?
parameter interpretability	√	√	√	√
robust model forecasts and parameter estimation	√	√	×	×
calculation accuracy / speed of convergence	√	?	×	×
incorporation of parameter / model uncertainty	√	√	√	√
analytic / numerical implementation	√	√	√	√
parsimonious representation	√	√	?	?
path sampling and prediction interval calculation	√	√	√	√
full age range mapping	?	?	?	?
application for pricing and reserving	√	√	√	√

Table 5.4: A comparison of qualitative assessment criteria for the mortality models LC92–RH06. The checkmark stands for a compliance with the corresponding criterion, the cross correspondingly indicates a non-fulfilment. The question mark indicates criteria where (non-) fulfilment cannot be definitely answered. The red coloured characters originate from the results in Cairns et al. (2009), Cairns et al. (2008) and Plat (2009).

bivariate mortality rate functions with a rather intuitive parameter structure.

- Two-factor models (LC92, BDV02) are relatively parsimonious compared to the multi-parameter extensions (RH03, RH06). Nevertheless, the fitting quality is worse and a perfect correlation for changes in the underlying mortality rates at different ages has to be assumed.
- The programming of the Lee-Carter model is straightforward. For the extensions a more comprehensive programming is required. The inclusion of age-specific enhancements or cohort-factors is accompanied by robustness and convergence difficulties. Parameter estimates commonly lack smoothness in the age or year of birth dimension.
- In practice, the use of multi-factor extension forecasts is limited due to incomplete and minor quality cohort data.
- The sampling of mortality-linked cash-flows (under parameter and model uncertainty) yields probabilistic prediction interval forecasts. Thus, the pricing of, e.g. longevity derivatives, can be theoretically performed under a change of measure using a risk neutral version of the model.
- None of the models tested is superior under all listed features. In order to verify the suitability of a certain model further quantitative ranking criteria need to be involved.

5.4.2 Analysis of the estimation error and examination of the model assumptions

Based on the parameter estimation method there are different functional representations available to measure the residuals and thus to monitor the quality of the graduation. For the least-squares Lee-Carter estimation with a Gaussian error structure the raw or response residuals $d_x(t) - \hat{d}_x(t)$ with estimated number of deaths

$$\hat{d}_x(t) = L_x(t) \cdot \hat{m}_x(t) = L_x(t) \cdot \exp\left(\hat{a}(x) + \hat{b}(x)\hat{k}(t)\right)$$

are standardised to ensure an unit asymptotic variance. The function

$$R^{SP}(x,t) = \frac{d_x(t) - \hat{d}_x(t)}{\sqrt{\hat{d}_x(t)}} \tag{5.4}$$

is called the (standardised) Pearson residual for the value pair (x,t) and equals the raw residual divided (standardised) by its asymptotic standard deviation. Thus, the Pearson residual is approximately standard normally distributed due to the central limit theorem. Alternatively, the single value decomposition residual estimate from Definition (5.1) given by

$$\hat{\varepsilon}(x,t) = \log\left(\tfrac{m_x(t)}{\hat{m}_x(t)}\right) = \log\left(\tfrac{d_x(t)}{\hat{d}_x(t)}\right) \tag{5.5}$$

with standardised residual function

$$R^{SSVD}(x,t) = \frac{\hat{\varepsilon}(x,t)}{\sqrt{\frac{1}{\text{DF}}\sum_{x,t}\left(\hat{\varepsilon}(x,t)\right)^2}}$$

can be used. Hereby, DF determines the model specific degree of freedom in terms of the number of data cells minus the number of independent parameters. The denominator describes the unbiased standard deviation of the estimated error term. The second equality in (5.5) is due to the adjustment step $\hat{d}_x(t) = L_x(t) \cdot \hat{m}_x(t)$ from the Lee-carter estimation procedure. In case of a MLE approach with Poisson error structure, a further residual concept which is more likely normally distributed is favourable. The deviance residual[100] is given by

$$R^D(x,t) = \text{sign}\Big(d_x(t) - \hat{d}_x(t)\Big)\sqrt{dev(x,t)} \quad \text{with deviance}$$
$$dev(x,t) = 2\Big(d_x(t)\log\left(\tfrac{d_x(t)}{\hat{d}_x(t)}\right) - \Big(d_x(t) - \hat{d}_x(t)\Big)\Big)$$

and is approximately normally distributed. The standardised variant[101] has the form

$$R^{SD}(x,t) = \text{sign}\Big(d_x(t) - \hat{d}_x(t)\Big)\sqrt{\frac{dev(x,t)}{\frac{1}{\text{DF}}\sum_{x,t} dev(x,t)}} \tag{5.6}$$

which is therefore approximately standard normally distributed. Here again, the denominator equals the unbiased squared deviance of the Poisson likelihood. In the following, we apply standardised Pearson residuals (5.4) for the LC92 model and standardised deviance residuals (5.6) for the likelihood models BDV02–RH06.

For a Poisson distributed number of deaths $d_x(t)$ we have equality of mean and variance which is only rarely the case for demographic data. Therefore, we possibly fade out potential future variability when assuming a Poisson distribution. Since there always exist variation not explained by the model, mortality data is typically overdispersed[102]. More precisely, for the two-factor

[100] Deviance residuals are e.g. applied by Renshaw and Haberman (1996), Brouhns et al. (2005) and Cossette et al. (2007) to measure the goodness of fit.

[101] Pitacco et al. (2008) suggest to use standard deviance residuals for monitoring in case of Poisson, Binomial or Negative Binomial distributed number of deaths.

[102] Overdispersion can be identified for the mortality data of several countries. Cairns et al. (2009) explain this phenomenon as a result of estimated exposure data $L_x(t)$. Delwarde et al.

models LC92 and BDV02 the sample variance for the observed residuals are larger than the model predicts (see Table 5.5) such that we observe a violation of the standard normal distribution assumption. The goodness of fit measured by the unbiased sample variance of the standardised residuals improves significantly when additional period (RH03) or cohort factors (RH06) are introduced. Demographic and actuarial literature offers various visualisation methods for the residual analysis. The most common types are illustrated in the following by means of the observed residual values for the mortality models LC92–RH06.

Sample moments and coefficient of variation for the standardised residuals of the mortality models LC92–RH06

	LC92	BDV02	RH03	RH06
DF	4905	4905	4750	4640
$E\,[R(x,t)]$	0.0042	-0.0003	-0.0016	0.0026
$Var\,[R(x,t)]$	2.29	1.08	0.96	0.79
$CV\,[R(x,t)]$	361.11	-2973.33	-595.67	335.36

Table 5.5: Sample moments and coefficient of variation for the standardised residuals of the mortality models LC92 – RH06.

Residual contour maps

We assume the (standardised) residuals to be randomly sized over the age-period surface such that, vice versa, any structured or regular pattern indicates the presence of systematic mortality changes. In this case, the underlying model is unable to capture all age-, time- or cohort-specific trends appropriately. Shaded contour maps constitute a visual demographic method to measure the fitting precision and to illustrate systematic effects in the underlying German male life table data for periods 1956 to 2006 originated by the Human Mortality Database (2009). A clumping or accumulation of areas with negative (blue coloured) or positive (red coloured) residuals conflicts with assumptions for the residual distribution and indicates systematic false estimation of the number of deaths. Negative residuals show an overestimation of the raw number of deaths and, accordingly, positive residuals imply a previous underestimation of mortality.

LC92 map:

The maps in Figure 5.6 show period-specific (vertical clustered areas) and cohort-specific (diagonal clustered areas) effects since single-factor time series

(2006) justify the normal distribution violation with the lack of smoothness for the estimated age factors $b_i(x)$ $(i = 1, 2)$.

fail to account for changes in the mortality patterns across different age groups and years of birth. The smoothed[103] contour map reveals that the LC92 model underestimates infant and oldest age mortality until the mid-seventies and since the mid-nineties. Especially the strong improvements for the age groups 10-25 in the time interval from 1975 to 1985 is not accounted for. We assume two cohort effects[104] apparent for male generations born around 1903 and 1930 that evoke a clustering of positive residuals and are therefore not captured. We refer the reader to Subsection 5.4.2 for a more detailed description of the mentioned cohort effects. The noticeable systematic diagonal lines with alternating positive and negative residuals for generations born in 1911-1919 constitute a distinctive feature. It seems plausible that this sharp improvements are caused by a progressing adjustment of consecutive abbreviated period life tables[105] containing incomplete or unreliable mortality data during World War I (1914-1919) and the Spanish Flu in 1919.

Crude and smoothed standardised residuals for the LC92 model

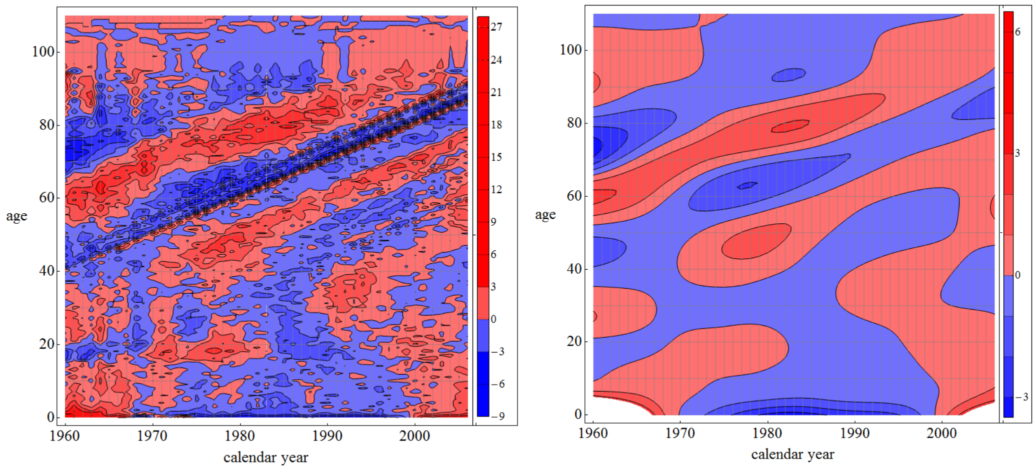


Figure 5.6: Crude (left) and smoothed (right) standardised Pearson residuals $R^{SP}(x,t)$ for the LC92 model.

[103] In the following, we applied tensor product penalized spline smoothing with penalty parameter $\lambda = 0.02$ is adopted to filter out random noise and outliers to get an impression for systematic effects.
[104] Cohort effects are determined by adjacent continuous diagonal bands of clustered positive residuals accompanied by one or two bands with negative residuals. The (enclosed) generation whose year of birth forms the origin of the positive band experienced stronger mortality improvements than the surrounding cohorts. Thus, the mortality only effects a single generation (diagonal direction) instead of an entire age group (horizontal direction).
[105] During the observation window, German population census solely took place in 1961, 1970 and 1987. The resulting complete life tables were updated by abbreviated life tables for interim periods. As time went by, this mortality projection became very unreliable especially for young and oldest age groups.

BDV02 map:

Compared to the LC92 model, the Poisson log-bilinear extension shows a slightly improved fitting with reduced absolute residual amounts. In particular, the mortality of juveniles is less strongly understated and the amounts of the residual in this region are lowered. However, the mortality for oldest ages $x > 90$ and between ages 30 and 50 is now overstated. The two diagonal bands are diminished whereas the smoothed band from 1930 exhibits two distinct elevations. Both the LC92 and the BDV02 model fail to reproduce the strong time-dependent changes in the raw central death rate data especially for childhood and adolescence.

Crude and smoothed standardised residuals for the BDV02 model

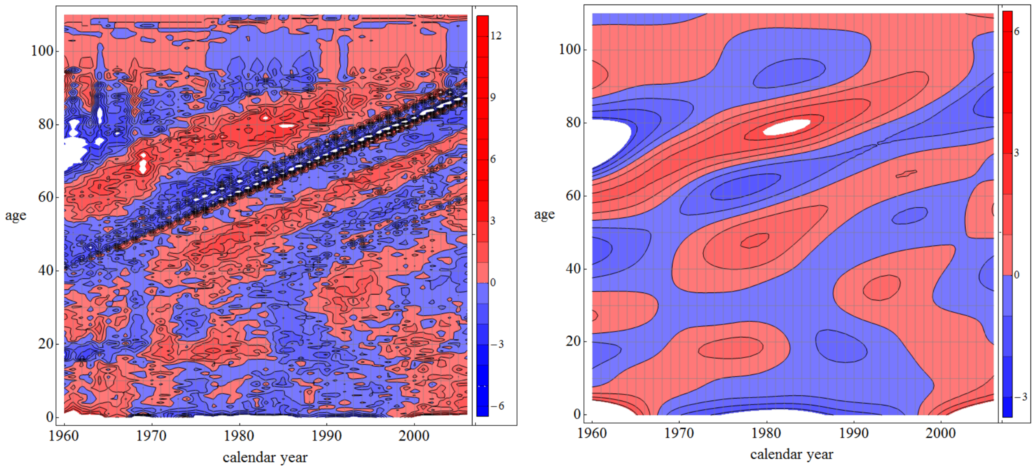


Figure 5.7: Crude (left) and smoothed (right) standardised deviance residuals $R^{SD}(x,t)$ for the BDV02 model.

RH03 map:

While there are distinct period effects, recognisable by vertical bands of clumped negative residuals for periods 1968-1970 (age groups 0-60) and 1985-1988 (age groups 0-50), the RH03 model partly weakens this areas (cf. Figure 5.8). The same holds for the whole childhood and adolescence age region. Furthermore, the model attests a strong change in mortality rates over time for the end of the age scale. i.e. age 105 and older. It should be emphasised that for this age group raw life table values are commonly extrapolated by means of non-linear regression due to a sparse database.

RH06 map:

By comparison, the contour map in Figure 5.9 for the RH06 model shows model-compliant random residuals, i.e. the systematic under- and overestima-

Crude and smoothed standardised residuals for the RH03 model

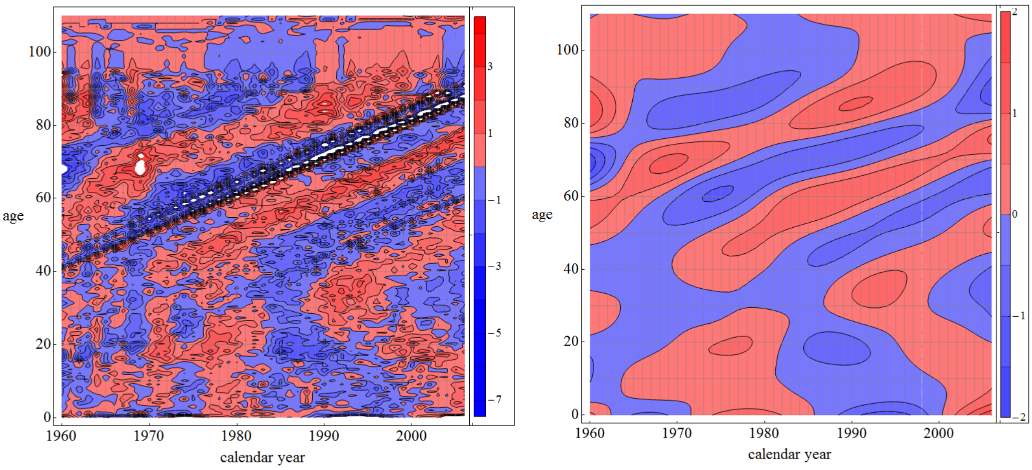


Figure 5.8: Crude (left) and smoothed (right) standardised deviance residuals $R^{SD}(x,t)$ for the RH03 model.

tion of mortality for certain age groups and calendar years has largely been eliminated[106]. This can partly be explained by the consideration of cohort effects and thus age-period correlation and partly results from the successful reproduction of pronounced period-dependent changes for juvenile mortality. In case of the raw contour map, we still observe a slight clumping in the rectangular areas spanned by the plane coordinates (1968,25) and (1970,50), (1978,95) and (1989,105) as well as (1984,20) and (1988,40). However, the smoothed map shows many low-valued random elevations which are paler coloured than those in the maps of models LC92–RH03. Vertical and diagonal bands are almost entirely caught by (additional) model factors.

Residual scatter plots

Residual scatter plots[107] denote the projection of the contour map onto the age or period axis as well as the year of birth diagonals. In this way, age-, period- and year-of-birth-related systematic trends which are not captured by the underlying model can be separately identified.

LC92 scatter plots:

The crude period scatter plot in Figure 5.10 shows that residuals vary randomly around zero but obviously with variance greater than one. The smoothed version in Figure 5.11 reveals oscillatory behaviour with systematic understated

[106] The highest age groups are generally problematic with regard to graduation.
[107] A scatter plot or scattergraph visualises a dependent variable plotted against different control parameters, e.g. age, calendar year or year of birth, inside a diagram. It reveals correlations as clusters or linear structures.

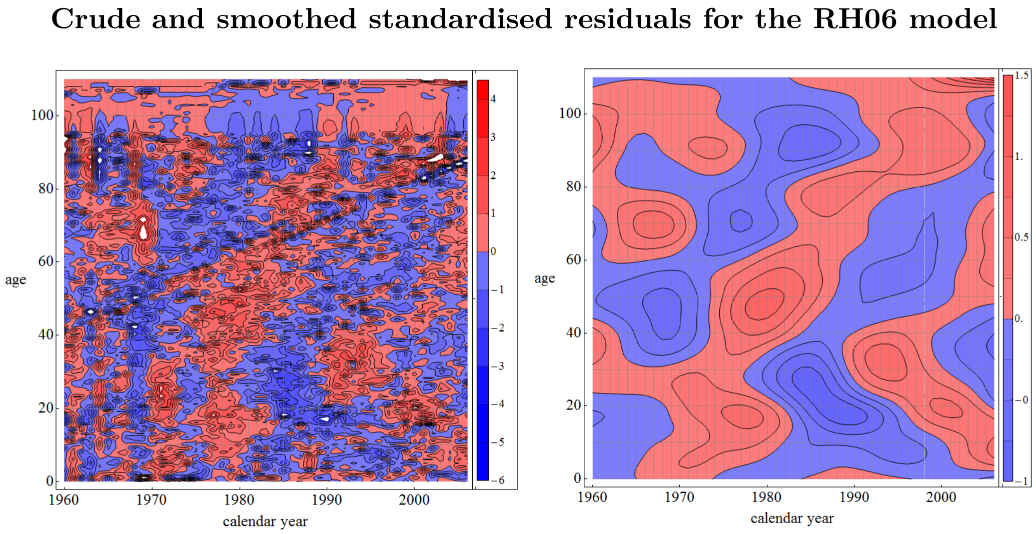


Figure 5.9: Crude (left) and smoothed (right) standardised deviance residuals $R^{SD}(x,t)$ for the RH06 model.

infancy mortality ($x < 5$) for the periods before year 1965 and after year 2000 and overstated old age mortality ($x \in [70,80]$) until the mid-sixties. These observations clearly indicate that residuals behave heterogeneous over time. The single-factor time series LC92 model does not allow for sufficient flexibility and therefore overestimates mortality for the age groups 0-5, 40-50 and 55-70. For ages 100 and older the raw death rates are given by regressed and extrapolated values from the Human Mortality Database (2009) such that age-specific patterns are barely noticeable in this region (cf. Figure 5.10). A comparison of different "year-of-birth bands" shows characteristic ripple effects since an explanatory variable / factor is missing. In particular, there are obvious cohort effects for generations born around 1903 and 1930 and discontinuities between adjacent years 1919-1921 (which are eliminated after smoothing in Figure 5.11). We can assume that the generation 1903 featured a stronger mortality improvement compared to the generations born during times of World War I and the Spanish Flue which are thus overstated by the LC92 model. A similar development could have been undergone by generations born around 1930 who have not been exposed to the aftermaths of the postwar periods and experienced stronger improvements in mortality compared to newborn generations involved in World War II (1939-1945). Furthermore, we observe weaker such phenomena for generations born in 1957 and during the Hong Kong Flu 1968-1969. A likely explanation for the temporary increase of male mortality in the fifties could be health impairments caused by World War II especially for young men. For this reason, a model extension including an additional cohort or at least period factor achieves better results.

Crude scatter plots of the standardised residuals for model LC92

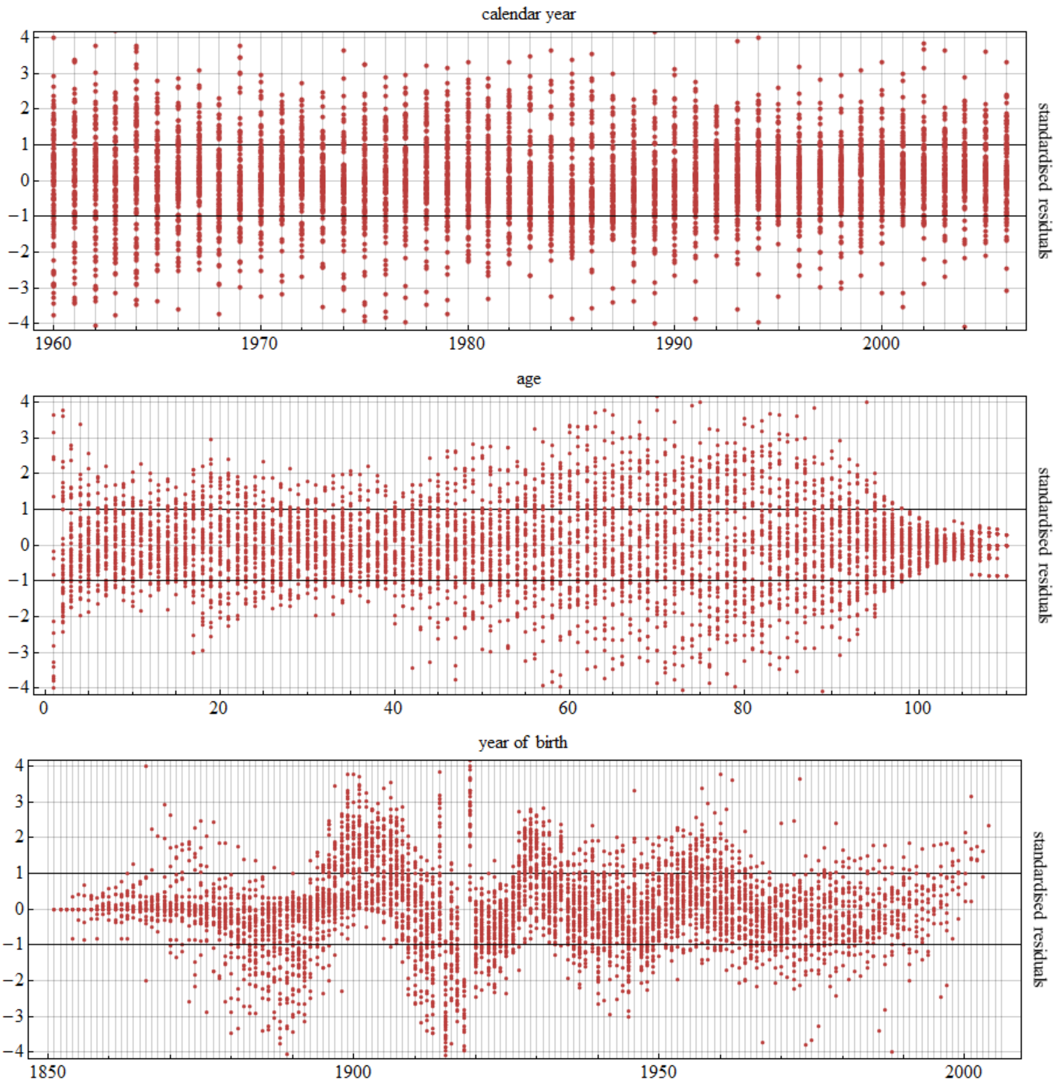


Figure 5.10: Crude scatter plots of the standardised Pearson residuals for the LC92 model.

BDV02 scatter plots:

In case of the BDV02 model, the systematic underestimation of infancy and old age mortality persists as well but is to a certain extent reduced. For a Poisson error assumption heterogeneity by age group also indicates overdispersion. The amount of false estimation has decreased in contrast to the LC92 model. In particular, the course of the accident hump is reproduced in an appropriate way (see Figure 5.13). The appearance of the year-of-birth scatter diagram in Figure 5.12 remains nearly unchanged except for slightly lower residuals that are visible in the smoothed variant in Figure 5.13. However, we must admit that the incomplete cohort mortality data for generations born before 1870 and after 1980 fails to provide enough meaningful quantitative information

Smoothed scatter plots of the standardised residuals for model LC92

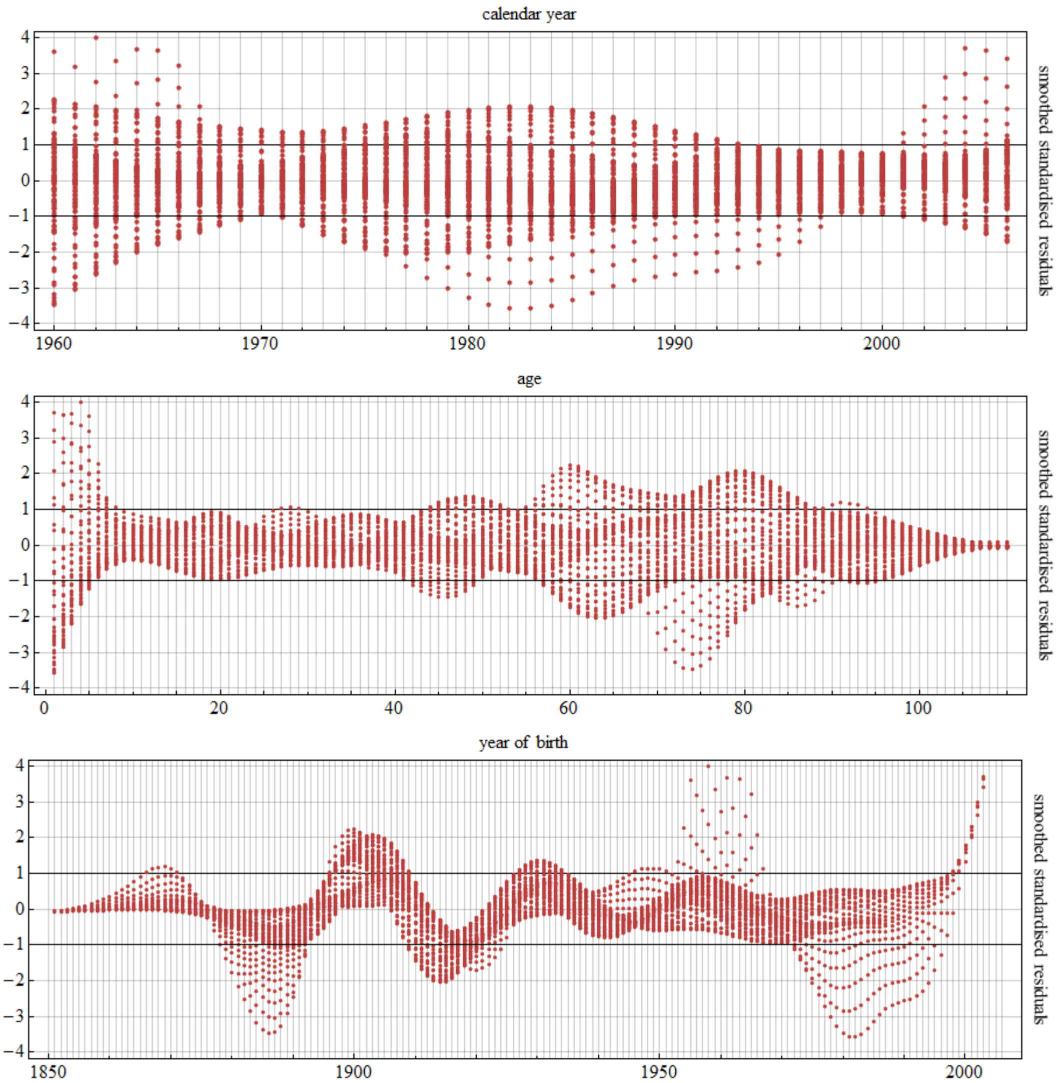


Figure 5.11: Smoothed scatter plots of the standardised Pearson residuals for the LC92 model.

such that in the following these ranges are excluded from the analysis (cf. the approach in Cairns et al. (2009)).

RH03 scatter plots:

The introduction of an additional period factor ensures that the standardised residuals are randomly distributed over the whole period observation window. More precisely, the continuing decrease in infant mortality and the formation of the distinct accident hump in the course of the proliferation of motor vehicles are sufficiently considered. For the age axis scatter plot, markable fitting improvements can already be registered. The amount of residual deviations is reduced especially for age ranges showing a systematic trend behaviour. Interestingly, parts of the alleged cohort effects are captured implicitly by the

Crude scatter plots of the standardised residuals for model BDV02

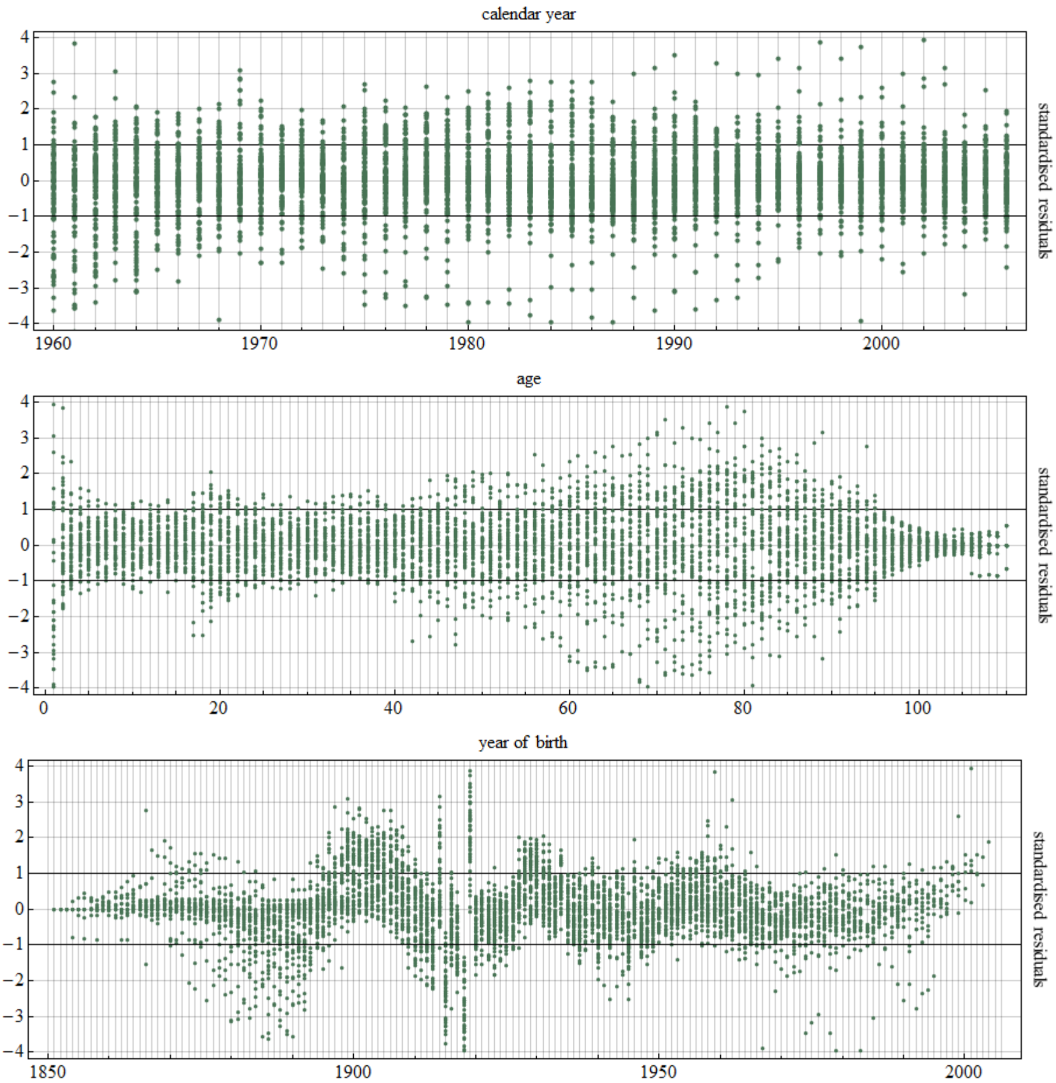


Figure 5.12: Crude scatter plots of the standardised deviance residuals for the BDV02 model.

introduced period factor $k_2(t)$. Nevertheless, the cohort effects in the interval 1900-1950 remain although to an alleviated extent.

RH06 scatter plots:

The RH06 model shows significant fitting improvements over all the other model variants. For all three main effects concerning age, calendar and cohort year we observe an uniformly scattered appearance such that ripple effects are almost completely removed. The results of the contour map and scatter plot analysis clearly endorse the use of the RH06 time series model.

Smoothed scatter plots of the standardised residuals for model BDV02

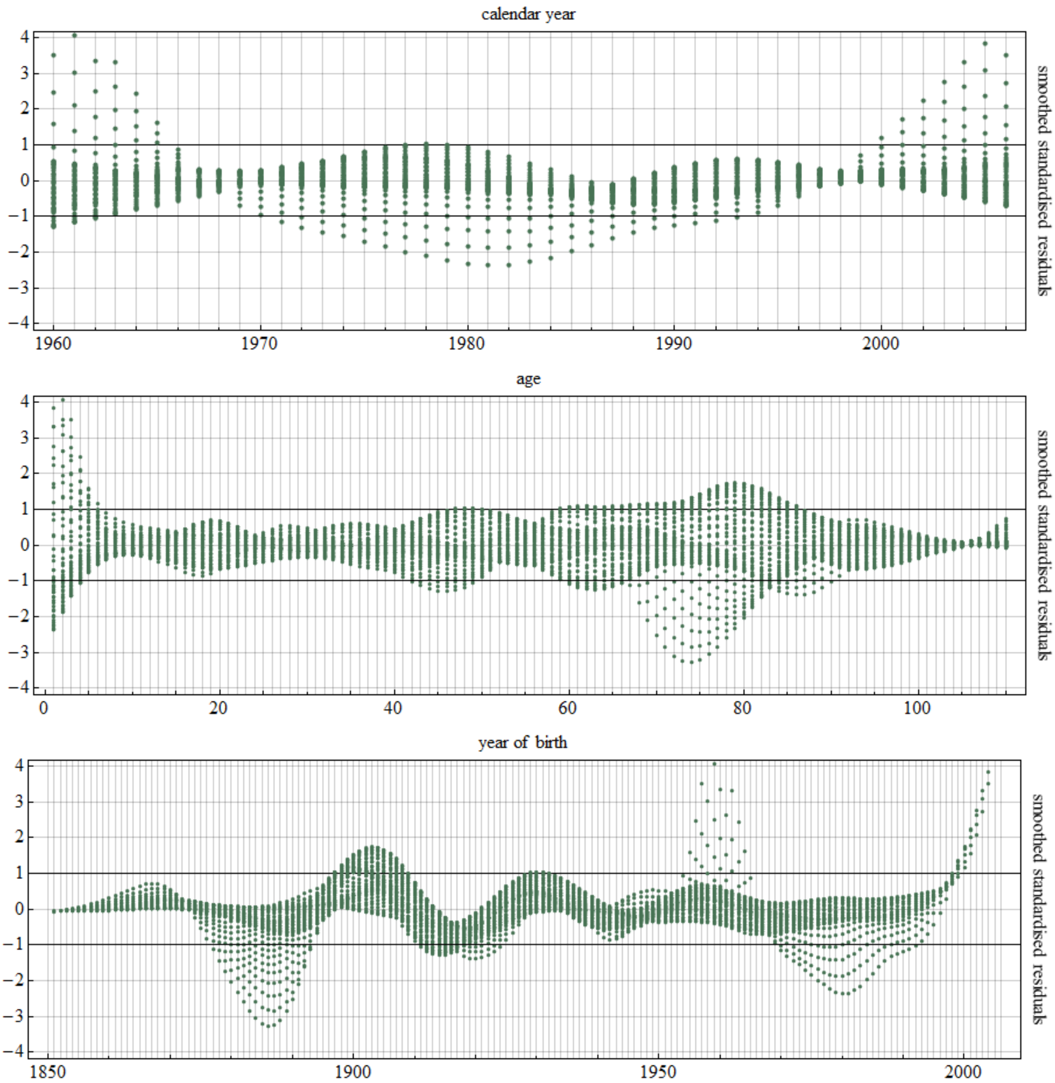


Figure 5.13: Smoothed scatter plots of the standardised deviance residuals for the BDV02 model.

Residual quantile plots

The assumption of a Poisson distributed number of deaths $d_x(t)$ implies that for large estimated values $\hat{d}_x(t)$ (which is fulfilled for mortality data) the Poisson distribution can be approximated by the Gaussian normal distribution. Thus, for appropriate estimates $\hat{b}_i(x)$, $\hat{k}_i(t)$ $(i = 1, 2)$, $\hat{l}(t - x)$ residuals are normally distributed as well. In particular, the standardised variant (5.6) is "standard normal" in age and calendar year direction. The Pearson residuals (5.4) of the LC92 model are by definition approximately standard normally distributed. For this reason, a quantile plot of normal quantiles against model quantiles constitutes a further valid and useful diagnostic method to verify the normal distribution assumption on the standardised residuals. If the points

Crude scatter plots of the standardised residuals for model RH03

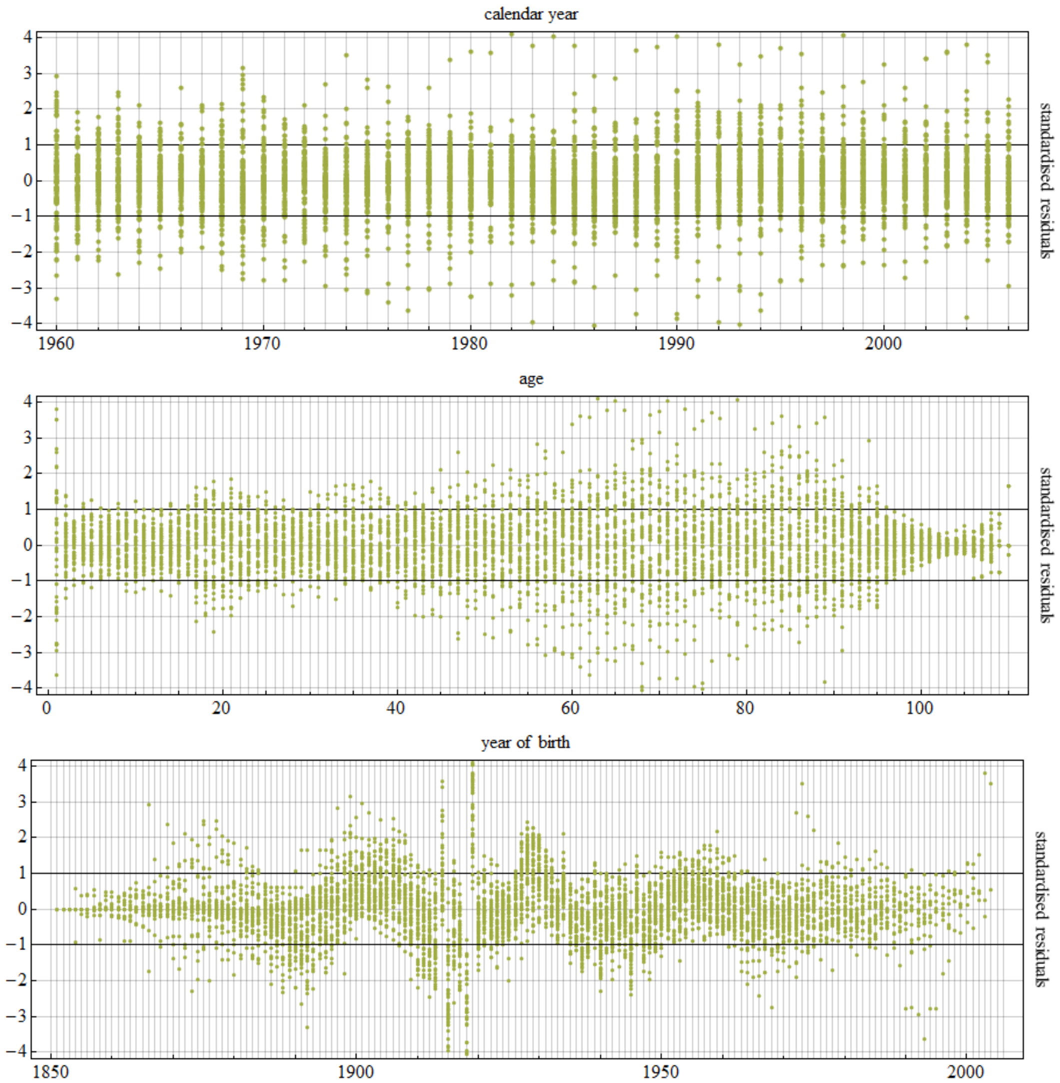


Figure 5.14: Crude scatter plots of the standardised deviance residuals for the RH03 model.

deviate systematically from the target straight (benchmark) line through the axes origin, there is statistical evidence against standard normally distributed residuals which is the case for the underlying mortality data (cf. Figure 5.18). We calculate "quantile-lines" for a 65-year old male and the whole period range based on 10 different Poisson bootstrap responses (coloured) and the historic estimation based on the raw numbers of deaths (black). A first look at the quantile plots indicates that all models clearly show systematic regions and extreme outliers at the outer edges. The strong deviations indicate poor adaptability whereas goodness of fit increases with the number of period and/or cohort factors applied. The mean of the bootstrap sampled residuals fluctuates around zero indicated by the downward exceedance for negative residuals and upward exceedance for positive residual values. This systematic course of the

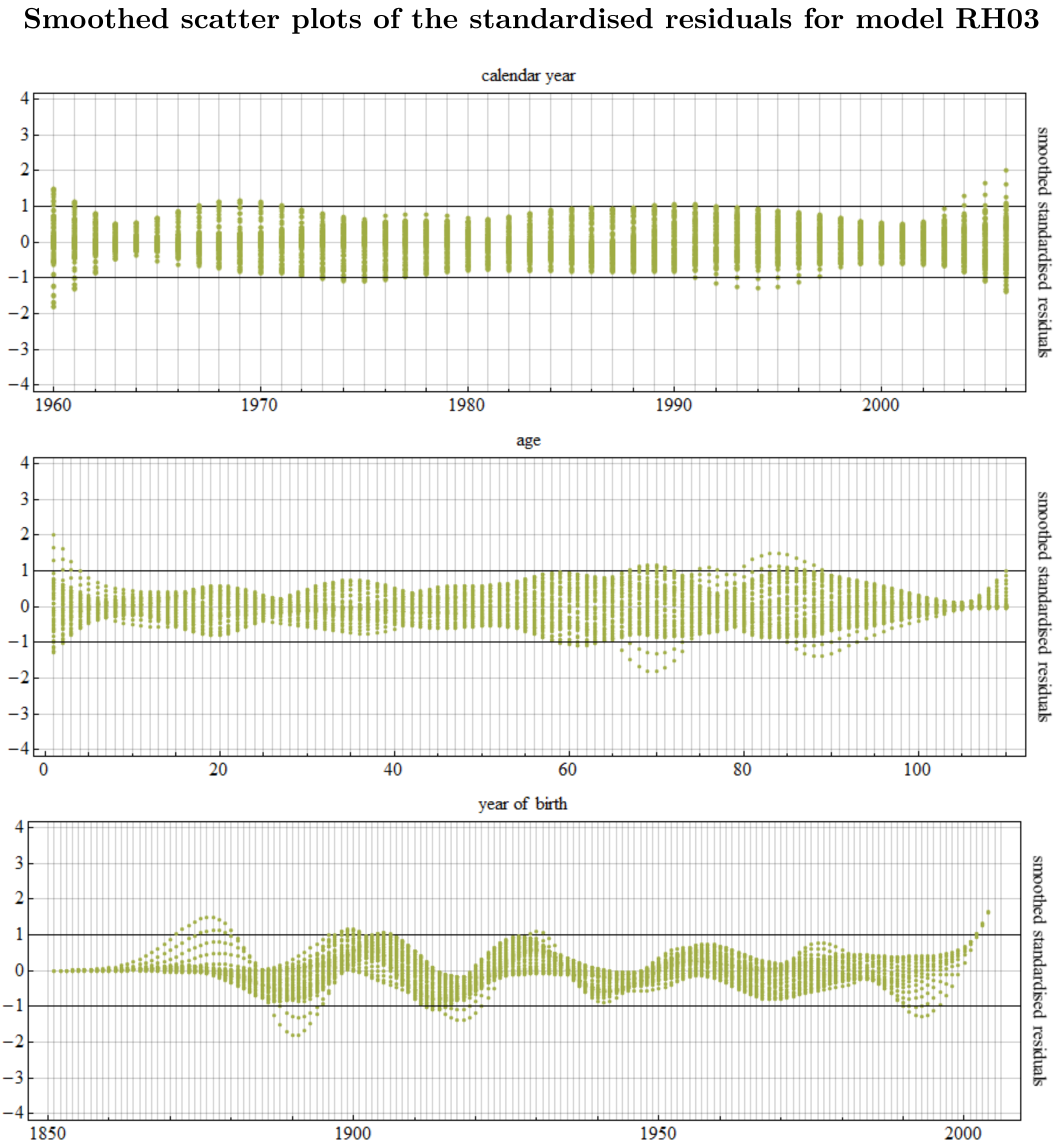


Figure 5.15: Smoothed scatter plots of the standardised deviance residuals for the RH03 model.

quantile-lines is typical for data with positive kurtosis which is equivalent to a relatively "peaked" distribution when compared to the Gaussian bell shape. Consequently, a large part of the variance results from more rare extreme peaks. In case of model RH06, the pattern of the quantile lines indicates an underlying peaked and left skewed distribution. Therefore, we can assume that the sample distribution has more pronounced tails. The quantile-lines based on the smoothed parameter values (drawn in black) show a comparatively moderate shape but still with deviations for quantiles in the boundary region. Therefore the quantile plot detects an overdispersed residual distribution. Possible reasons for the "peak behaviour" might be the already mentioned general poor fitting results for oldest ages due to sparse or projected data, a slow convergence speed and associated accuracy (especially for the models RH03 and RH06) and, in

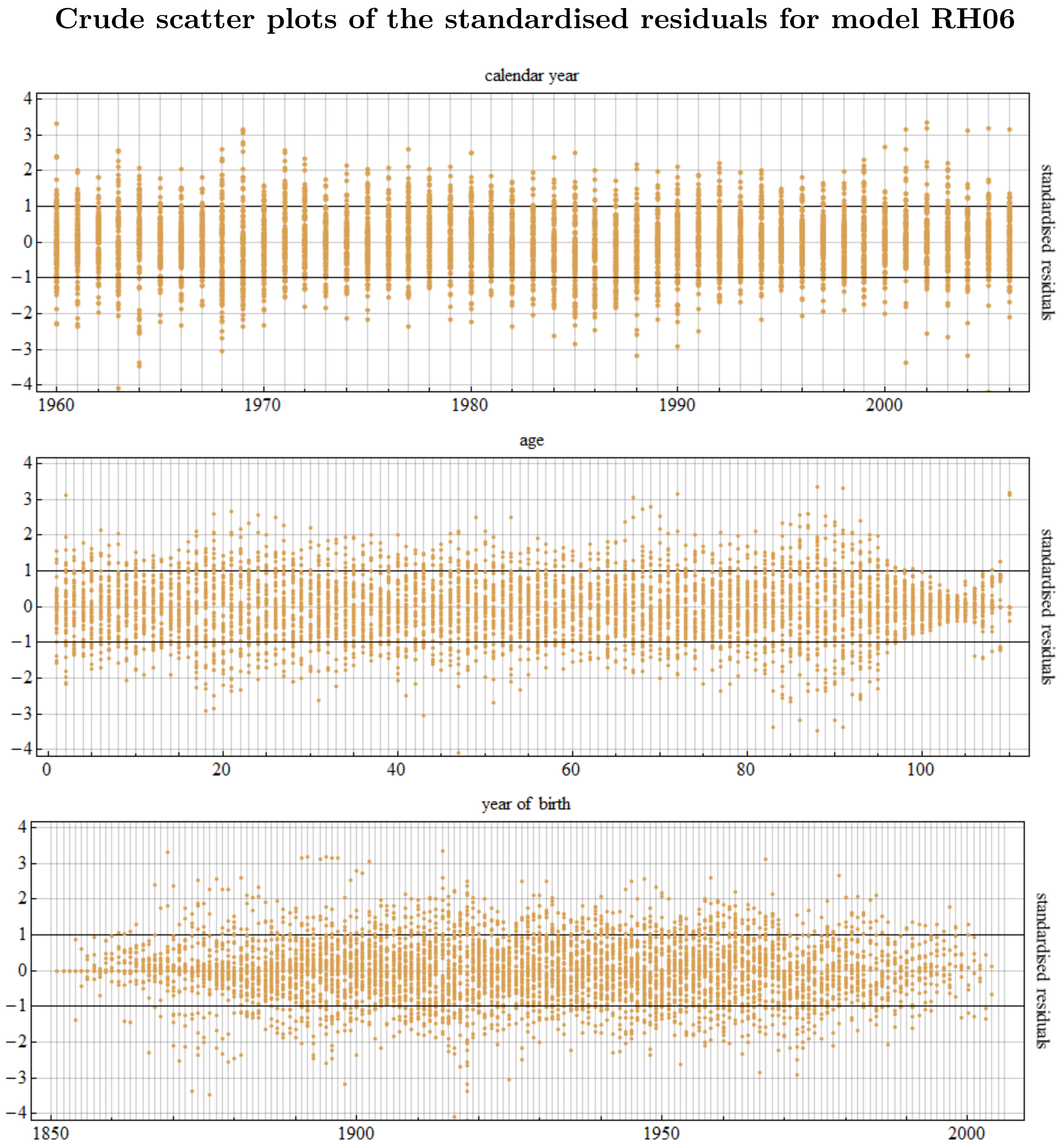


Figure 5.16: Crude scatter plots of the standardised deviance residuals for the RH06 model.

particular, a lack of model robustness due to a distorted number of deaths (cf. parameter Figures 5.1 - 5.4).

5.4.3 Analysis of the BIC

A further quantitative model criterion, often applied in the context of (stochastic) parametric mortality models, is represented by the BIC. It penalizes overparametrised models out of a certain class using MLE. Therefore, the BIC allows a comparison of nested models with different numbers of parameters since maximum likelihood values increase with the number of additional parameters.

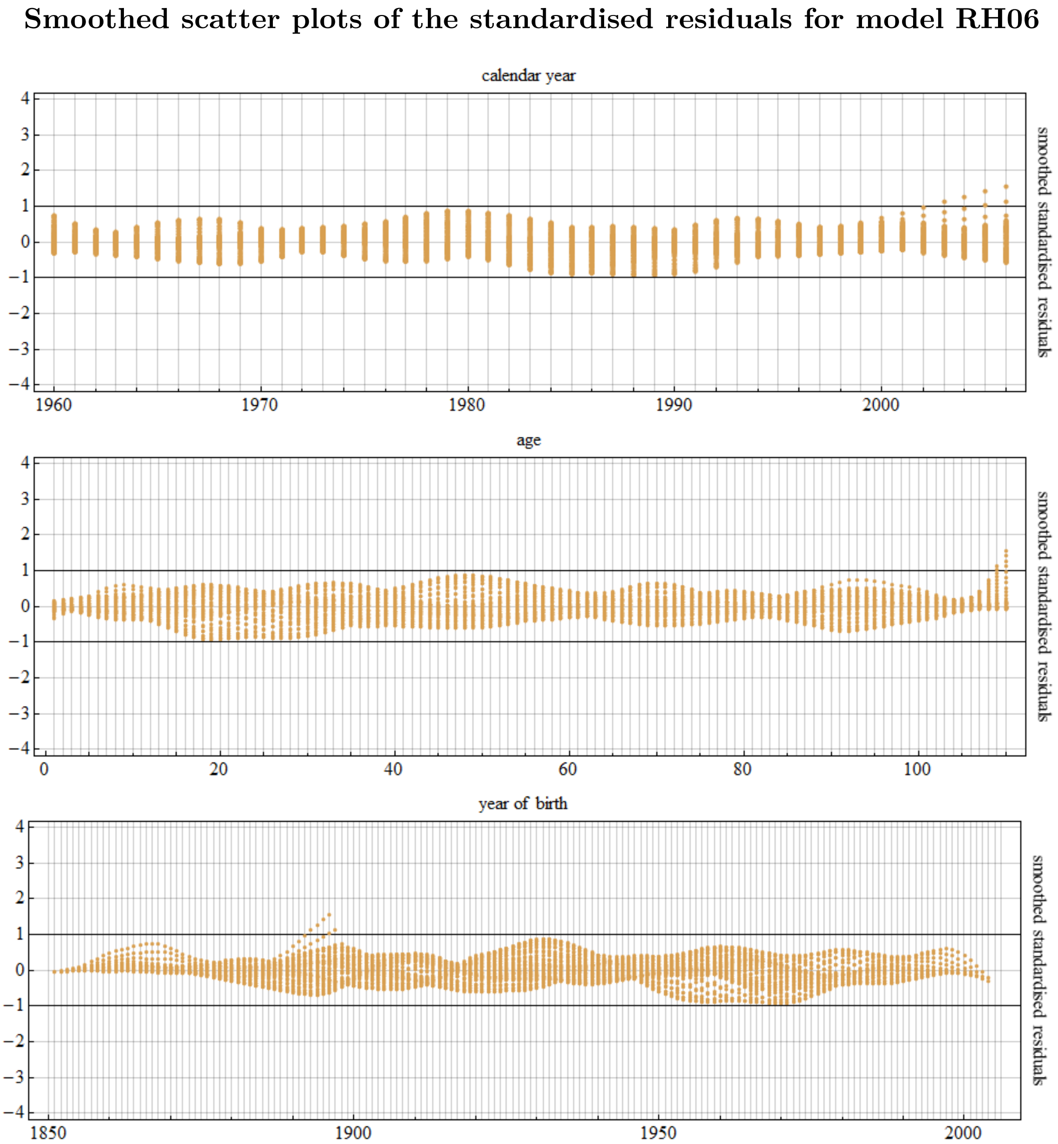


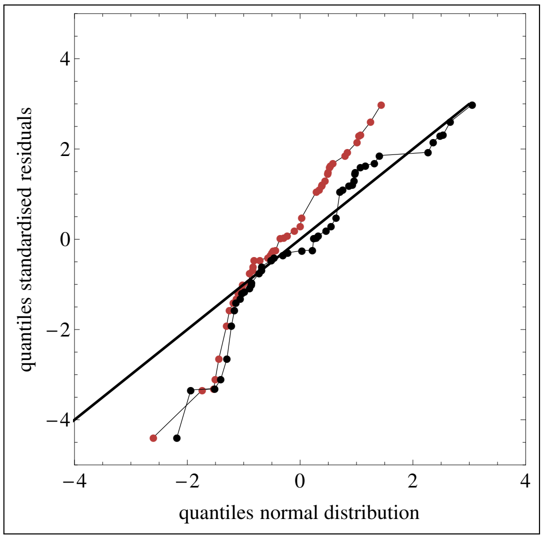
Figure 5.17: Smoothed scatter plots of the standardised deviance residuals for the RH06 model.

The BIC-function is defined as follows

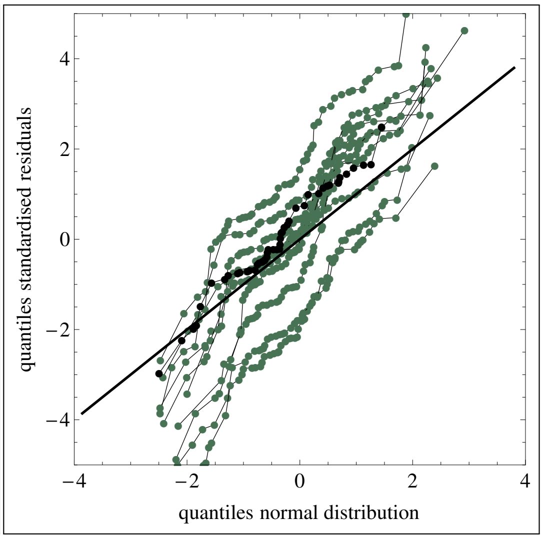
$$BIC(\hat{\xi},n,k) = -2\ln\left(\mathcal{L}\left(\hat{\xi}\right)\right) + k\ln(n)$$

where ξ represents a model parameter vector and $\hat{\xi}$ the corresponding maximum likelihood estimate. The penalization function consists of the number of observations n and the logarithmised number of free (effective) parameters k. Hereby, lower values for the BIC are preferable either evoked by a better fitting or parameter parsimony (results are shown in Table 5.6). Other related criteria that account for the gain in likelihood caused by extra parameters are the

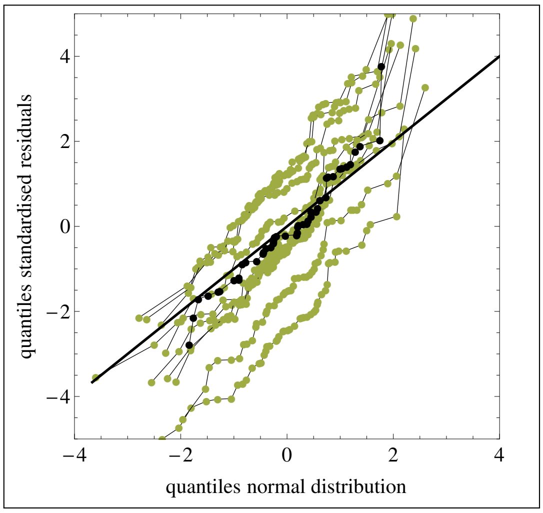
Quantile plots of the standardised residuals for the mortality models LC92–RH06



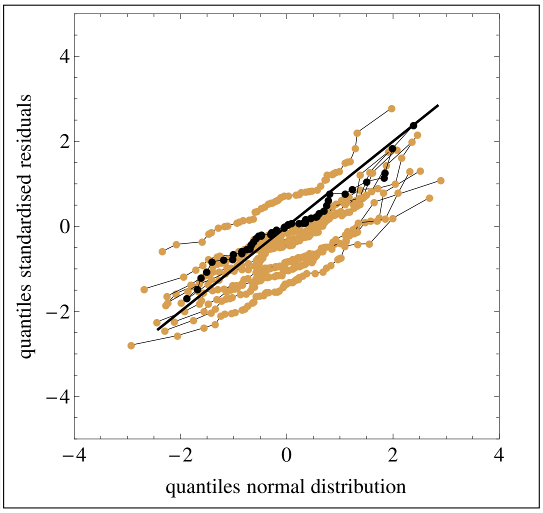
(5.18.1) LC92 mortality model



(5.18.2) BDV02 mortality model



(5.18.3) RH03 mortality model



(5.18.4) RH06 mortality model

Figure 5.18: Quantile plots of the standardised residuals for a 65-year old male and mortality models LC92 – RH06.

Akaike information criterion (AIC)

$$AIC(\hat{\xi},k) = -2\ln\left(\mathcal{L}\left(\hat{\xi}\right)\right) + 2k$$

and the likelihood-ratio statistic

$$L^2(\hat{\xi}) = 2\sum_{x,t} d_x(t)\,\ln\left(\frac{d_x(t)}{\hat{d}_x(t)}\right)$$

where smaller values are preferable, too. According to Brouhns et al. (2002a) $L^2(\hat{\xi})$ measures discrepancies between the fitted and the current number of deaths.

However, the BIC penalizes the model complexity level more severe than the AIC for a number of observations greater than 10. Both BIC and AIC attest that including additional period / cohort factors enhances fitting quality due to an improvement in the log-likelihood value albeit the change is marginal in the range of several basis points. Generally speaking, those extrapolation models with higher likelihood exhibit lower variance in the standardised residuals (cf. Table 5.5). The increase of model complexity in form of additional (free) parameters is therefore acceptable. Nevertheless, the likelihood-ratio statistic value $L^2(\hat{\xi})$ shows considerable improvements and a clear ranking result.

Maximum likelihood, BIC and AIC for the mortality models LC92–RH06

model	$\ln(\mathcal{L}(\hat{\xi}))$	number of observations n	number of free parameters k	$BIC(\hat{\xi},n,k)$ (rank)	$AIC(\hat{\xi},k)$ (rank)	$L^2(\hat{\xi})$ (rank)
BDV02	$-1.87668 \cdot 10^7$	5170	265	$3.7535866 \cdot 10^7$ (3)	$3.7534130 \cdot 10^7$ (3)	10470 (3)
RH03	$-1.87649 \cdot 10^7$	5170	420	$3.7533392 \cdot 10^7$ (2)	$3.7530640 \cdot 10^7$ (2)	6645 (2)
RH06	$-1.87629 \cdot 10^7$	5170	530	$3.7530324 \cdot 10^7$ (1)	$3.7526860 \cdot 10^7$ (1)	2764 (1)

Table 5.6: Model choice as a trade off between goodness of fit and parsimony for models BDV02 – RH06 in accordance to Cairns et al. (2009). The number of free parameters k equals the number of observations n minus the degree of freedom DF given in Table 5.5.

5.4.4 Analysis of the coefficient of determination

The R^2-square measure or coefficient of determination quantifies how much percentage of the (overall) variance can be explained by the underlying model. In particular, it measures the proportion of the variance of the logarithmised central death rate (at fixed age) accounted for by variation in the period factor $\hat{k}_i(t)$ $(i = 1, 2)$ and / or cohort factor $\hat{l}(t - x)$. Therefore, the unexplained variation in form of the age-specific variance of the raw residuals

$$Var_x\left[\hat{\varepsilon}(x,t)\right] = \tfrac{1}{\#t-1}\sum_t \left(\hat{\varepsilon}(x,t) - \tfrac{1}{\#t}\sum_t \hat{\varepsilon}(x,t)\right)^2$$

is related to the total variance of the logarithmised raw mortality rates

$$Var_x\left[\ln\left(m_x(t)\right)\right] = \tfrac{1}{\#t-1}\sum_t \left(\ln\left(\frac{m_x(t)}{\overline{m}_x}\right)\right)^2$$

with average age-specific central death rate $\overline{m}_x = \frac{1}{\#t} \sum_t m_x(t)$. Thus, the R^2 coefficient equals

$$0 \leq R^2(x) = 1 - \frac{Var_x\left[\hat{\varepsilon}(x,t)\right]}{Var_x\left[\ln\left(m_x(t)\right)\right]} \leq 1.$$

If R^2 lies near zero, central death rate estimates are given by the rather naive mean predictor $\overline{m}_x = \hat{a}(x)$ which does not account for variation at all. If R^2 is close to one, nearly all of the variation in the central death rates is explained. For a coefficient equal to one we observe a perfect prediction in the way that the estimate $\hat{m}_x(t)$ completely reflects reality for all observation periods. The coefficient of determination indicates the appropriateness of future projections, i.e. how likely they are predicted by the underlying model. Figure 5.19 exhibits

Coefficient of determination for the mortality models LC92–RH06

age x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
LC92	0.92	0.9	0.94	0.93	0.93	0.94	0.94	0.94	0.92	0.89	0.94	0.91	0.9	0.88	0.92	0.94	0.92	0.92	0.89	0.94	0.91	0.93
BDV02	0.92	0.9	0.95	0.93	0.94	0.95	0.94	0.94	0.93	0.9	0.94	0.92	0.91	0.89	0.93	0.95	0.92	0.92	0.89	0.94	0.92	0.94
RH03	0.98	0.98	0.98	0.97	0.96	0.97	0.97	0.96	0.96	0.93	0.96	0.94	0.94	0.92	0.95	0.95	0.92	0.93	0.91	0.95	0.92	0.94
RH06	1.	0.98	0.97	0.98	0.97	0.97	0.97	0.96	0.95	0.93	0.97	0.93	0.95	0.93	0.95	0.95	0.94	0.95	0.92	0.96	0.94	0.95

age x	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
LC92	0.94	0.93	0.93	0.94	0.93	0.92	0.95	0.95	0.94	0.95	0.95	0.94	0.94	0.95	0.94	0.95	0.96	0.95	0.95	0.93	0.94	0.93
BDV02	0.95	0.94	0.94	0.95	0.94	0.93	0.96	0.95	0.95	0.95	0.95	0.94	0.94	0.95	0.94	0.95	0.96	0.95	0.95	0.92	0.93	0.92
RH03	0.95	0.94	0.95	0.96	0.95	0.94	0.96	0.95	0.94	0.95	0.95	0.94	0.94	0.94	0.94	0.95	0.96	0.96	0.95	0.95	0.94	0.95
RH06	0.96	0.94	0.95	0.94	0.93	0.93	0.95	0.95	0.94	0.95	0.95	0.95	0.96	0.96	0.96	0.97	0.97	0.96	0.96	0.97	0.96	0.97

age x	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
LC92	0.93	0.94	0.91	0.93	0.94	0.94	0.94	0.96	0.94	0.96	0.96	0.96	0.96	0.97	0.96	0.95	0.96	0.96	0.97	0.97	0.97	0.98
BDV02	0.93	0.93	0.9	0.92	0.93	0.94	0.93	0.96	0.94	0.96	0.96	0.96	0.97	0.97	0.97	0.96	0.97	0.97	0.97	0.98	0.97	0.98
RH03	0.94	0.96	0.92	0.94	0.95	0.95	0.95	0.97	0.95	0.96	0.96	0.96	0.97	0.97	0.97	0.97	0.97	0.97	0.98	0.98	0.98	0.98
RH06	0.97	0.97	0.95	0.97	0.98	0.98	0.97	0.99	0.98	0.99	0.99	0.99	0.99	0.99	0.99	1.	1.	1.	1.	1.	1.	1.

age x	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87
LC92	0.98	0.97	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.97	0.98	0.97	0.97	0.97	0.97	0.96	0.96	0.96	0.97	0.95	0.95	0.93
BDV02	0.98	0.97	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.97	0.97	0.96	0.97	0.96	0.96	0.96	0.95	0.96	0.95	0.94	0.93
RH03	0.99	0.98	0.98	0.98	0.98	0.99	0.98	0.99	0.99	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.97	0.96	0.97	0.95	0.94	0.92
RH06	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	0.99	0.99	0.99	0.98	0.98	0.97

age x	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109
LC92	0.94	0.93	0.89	0.88	0.84	0.8	0.79	0.75	0.68	0.62	0.54	0.46	0.38	0.3	0.23	0.17	0.12	0.07	0.04	0.02	0.01	0.
BDV02	0.94	0.93	0.89	0.89	0.84	0.81	0.78	0.75	0.68	0.61	0.52	0.44	0.35	0.24	0.15	0.12	0.	0.	0.	0.	0.	0.
RH03	0.94	0.93	0.89	0.93	0.87	0.82	0.79	0.82	0.78	0.74	0.69	0.65	0.6	0.55	0.45	0.49	0.	0.	0.	0.	0.	0.
RH06	0.96	0.95	0.92	0.92	0.88	0.83	0.85	0.85	0.8	0.73	0.59	0.48	0.2	0.2	0.07	0.	0.	0.	0.	0.	0.	0.

Table 5.7: Coefficient of determination $R^2(x)$ for different ages and mortality models LC92–RH06.

correlations across different ages with increasing proportion of temporal variance until age 80. Moreover, we notice a consistently high coverage of at least 95% of the total variance for retirement age groups 60-80. Thereby, the ratio of explained variance of the multi-factor time series models RH03 and RH06 dominates the R^2 values reached by the LC92 or BDV02 model especially for infancy ($x < 10$) and adult ages 30-60. In particular, for the age-period-cohort variant we observe an overall explained variance[108] of 96% which is quite good in relation to the poor fitting results leading to R^2 values near zero for ages 100

[108] Lee and Carter (1992) propose an overall measure for the goodness of fit relating the sum of age-specific unexplained variances and the sum of total variance over all ages.

and older. Interestingly, solely the LC92 model obtains visible positive values in this age range with an overall determination of at least 93%. This effect can be explained by the fact that the more complex models feature unreasonable behaviour for the logarithmized central death rates towards the end of the age scale (cf. Subfigures (5.2.1), (5.3.1) and (5.4.1) in contrast to (5.1.1)) associated with a lack of proportion in the temporal development of the variance.

Coefficient of determination for the mortality models LC92–RH06

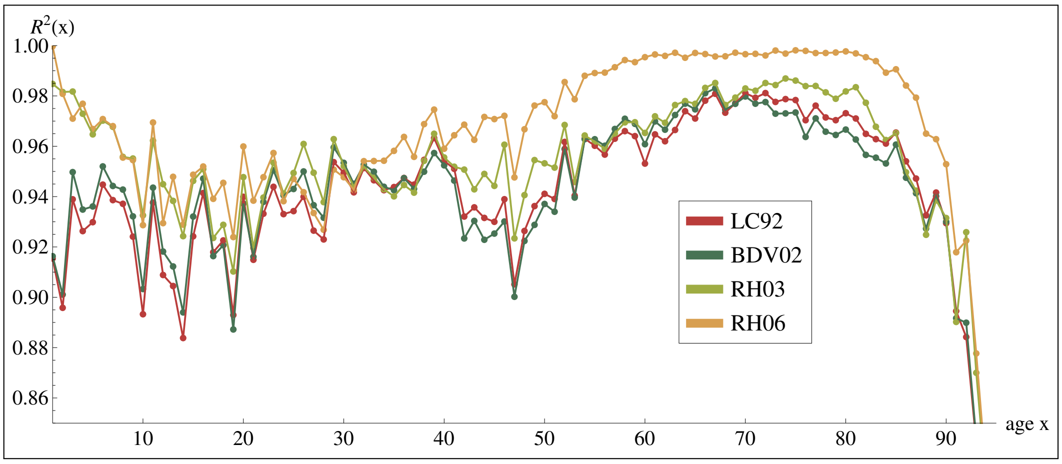


Figure 5.19: Coefficient of determination $R^2(x)$ for different ages and mortality models LC92–RH06.

5.4.5 Analysis of the mortality rate fan charts

In order to capture the uncertainty in forecasts of time series models for different mortality related functions such as survival function, life expectancy or present values of particular annuity contracts, the concept of fan charts[109] describes a promising way of illustration. Fan charts are suitable to illustrate stress-tests or to backtest forecast errors or to perform a (separate) parameter risk analysis. For a future period, values in the boundary regions of the chart indicate low probability for their incidence.

[109] The illustration via fan charts was taken up by King (2004), Blake et al. (2006a) or Dowd et al. (2007). The uncertainty in future mortality projections is visualised by means of prediction intervals of contrasting colours for different probabilities together with the central projection sample mean. The colour scheme is designed in such a manner that the darker the related prediction area the higher the likelihood for the outcome. The degree of projection risk and, if included, parameter risk is quantified by the prediction interval width. Obviously, the uncertainty increases with growing forecasting horizon, i.e. the fan legs drift further apart.

Projection of the central death rate

Using the Lee-Carter model as an example, Figure 5.20 shows central death rate estimates and prediction interval bounds (risk bounds) for different prediction probabilities together with a central projection. Since the standard deviation of raw central death rates strongly increases for retirement ages 60-95 and the reduction factor $\exp\left(\hat{b}(x)\left(k(t)-\hat{k}(t)\right)\right)$ is decreasing in x, the prediction interval width increases for older age profiles.

Lee-Carter fan charts of the central death rate

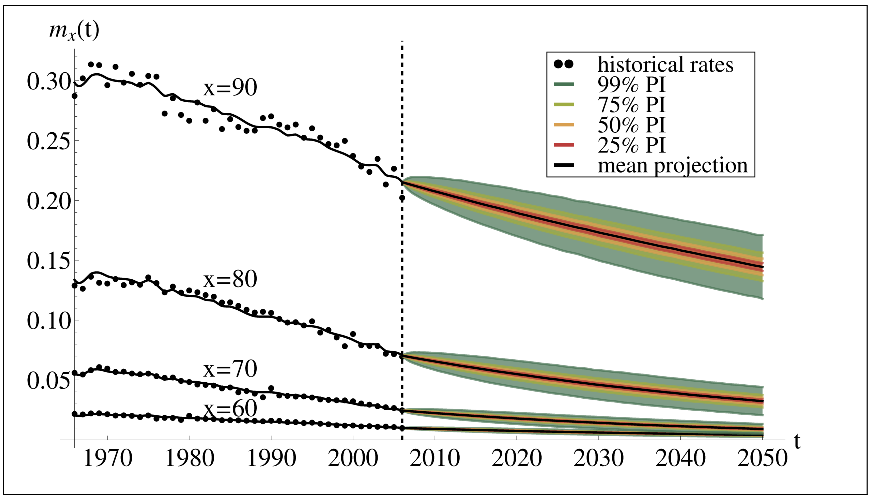


Figure 5.20: Lee-Carter fan charts of the central death rate based on $5 \cdot 10^4$ sample paths for different age groups for German male mortality data from the Human Mortality Database (2009).

Subfigure (5.21.1) additionally illustrates the impact of the model choice on the appearance of the fan chart. Hereby, the width of the PIs (of estimated values) for the mortality models LC92-RH03 is almost identical. Especially the charts of the LC92 and BDV02 model show equal prediction interval width since the number of parameters remains unchanged. Solely the cohort variant RH06 bears a substantial projection risk which may halfway be explained by the strong parameter uncertainty illustrated in Figure 5.4. Even more important than the prediction interval width is the direction of the central projection given by the bootstrap sample mean. Results show that, on average, the models LC92-RH03 forecast the same rate of mortality improvement whereas the RH06 sample mean proceeds above the former central projections. As already noted, the likelihood models BDV02–RH06 feature growing prediction interval width for increasing projection horizon.

Projection of the survival function

In principle, there exist two different calculation methods for the survival function depending on the use of (projected) central death rates. On the one hand, we apply the model simulated period survival function defined by

$$_{k}p_{65}^{\uparrow}(t) = \prod_{j=0}^{t-1} p_{65+j}(t) \overset{(\#)}{=} \prod_{j=0}^{k-1} \exp\left(-m_{65+j}(t)\right) = \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t)\right).$$

The curve originates from mortality experience of a single period t. On the other hand, the corresponding cohort survival function

$$\begin{aligned} _{k}p_{65}^{\nearrow}(t) &= \prod_{j=0}^{k-1} p_{65+j}(t+j) \overset{(\#)}{=} \prod_{j=0}^{k-1} \exp\left(-m_{65+j}(t+j)\right) \\ &= \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t+j)\right) \end{aligned}$$

is based on the realised mortality experience of a male cohort aged 65 in year t. Thus, the cohort variant requires comprehensive mortality projections of a whole generation and therefore includes noticeable model as well as forecasting risk. The superscript notations $\uparrow$ and $\nearrow$ denote calculation in vertical (period-specific) and diagonal (cohort-specific) direction. The second equality (#) is based on the common assumption[110] of constant central death rates for non integer ages and periods, i.e.

$$m_{x+\Delta x}(t+\Delta t) = m_x(s) \quad \text{for} \quad \Delta x, \Delta t \in [0,1[, \tag{5.7}$$

such that the annual survival probabilities of an individual aged x at time t is $p_x(t) = \exp\left(-m_x(t)\right)$. Subfigure (5.21.2) clarifies the difference between the period and cohort survival function concept. While the period specific survival probabilities $p_{65+k}(2006)$ are known (thick solid lines) the corresponding cohort probabilities $p_{65+k}(2006+k)$ (coloured fans) have to be projected and therefore contain a considerable amount of forecasting uncertainty. Obviously, the period survival function also bears risk as soon as we refer to future reference periods.

We observe that due to the use of forecasted central death rates the period survival function values proceed below the cohort life expectancies. A resampling of the graduated central death rates yields cohort survival function fan charts that contain additional uncertainty. Furthermore, it is noticeable that all cohort survival function fan charts show similar patterns. The small prediction interval

[110] See Subsection 3.1.2 for more details.

width for the RH03 model might be caused by an inappropriate choice[111] for the ARIMA process parameters of the second period factor. Here again, the RH06 fan bears the most forecast uncertainty. This has to be implicitly considered when using the age-period-cohort extension for actuarial calculations.

Projecting the life expectancy

The (future) period life expectancy (cf. Equation 3.14 of Section 3.1.3) of an individual aged x at time t under assumption (5.7) is calculated as

$$\begin{aligned}\mathring{e}^{\uparrow}_{65}(t) &= \int_{0}^{110-65} {}_{s}p^{\uparrow}_{65}(t)ds = \sum_{k=0}^{110-65} {}_{k}p^{\uparrow}_{65}(t)\left(-\frac{q_{65+k}(t)}{\ln(p_{65+k}(t))}\right) \\ &= \sum_{k=0}^{45} \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t)\right)\cdot\frac{q_{65+k}(t)}{m_{65+k}(t)}. \end{aligned} \tag{5.8}$$

Thus, the 65-year-old individual in t dies in year $t+\mathring{e}^{\uparrow}_{65}(t)$ at age $x+\mathring{e}^{\uparrow}_{65}(t)$ using the period life table in t. For the cohort life expectancy we allow for an evolution of central death rates over time and obtain

$$\begin{aligned}\mathring{e}^{\nearrow}_{65}(t) &= \int_{0}^{110-65} {}_{s}p^{\nearrow}_{65}(t)ds = \sum_{k=0}^{110-65} {}_{k}p^{\nearrow}_{65}(t)\left(-\frac{q_{65+k}(t+k)}{\ln(p_{65+k}(t+k))}\right) \\ &= \sum_{k=0}^{45} \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t+j)\right)\cdot\frac{q_{65+k}(t+k)}{m_{65+k}(t+k)}. \end{aligned} \tag{5.9}$$

It is likely that similar to the course of the period and cohort survival function in Subfigure (5.21.2) the period life expectancy values proceed below the cohort life expectancies as well. Moreover, the width of the prediction intervals increases the further the fixed calendar year t lies in the future. For the sampling of the complete life expectancy we only make use of the period approach (5.8) since central death rates are solely projected up to year 2056 for reasons of computational effort. Consequently, the evaluation of Expression (5.9) would only allow for a series of cohort life expectancies for years $t \in [2006{,}2011]$ since the index of the summands is limited to 2011+45=2056.

The historic German male life expectancy increased almost linearly since the late-sixties which coincides with demographic observations made by the German Federal Statistical Office (2010) and in Subsection 2. The projection of the life expectancy in Subfigure (5.21.3) illustrates uncertainty in longevity and potential tendency and direction of mortality development for the underlying

[111] Subfigure (5.3.5) illustrates that the course of this factor is less linear but more quadratic such that a random walk with drift might be unsuitable in this context.

population. The fan charts for the different analysed forecasting models are directly related to the shape of the corresponding survival function. All models satisfactorily capture historic life expectancy and the parameter misspecification is moderate. We observe only slight model-specific differences in the fan charts for the models LC92, BDV02 and RH03 concerning the width and the forecasting tendency. Thus the BDV02 and RH03 model only include few parameter uncertainty. As expected, the noticeable amount of projection risk for the RH06 model also appears for the life expectancy fan chart. All charts have in common that the slope does not continue the linear trend of the past but is alleviated with a slight concave curve shape.

Projection of immediate annuity present values

We calculate the projected values of an immediate starting life annuity paying an advanced annual pension of one monetary unit for a male individual aged $x = 65$ years in year t. For this purpose, we use deterministic[112] discount rates based on the (normal) euro area government bond yield curve[113] $(Y_k)_{k\in[0,44]}$. The period annuity price is derived by the formula

$$\ddot{a}^{\uparrow}_{65}(t) = \sum_{k=0}^{110-65-1} \frac{{}_kp^{\uparrow}_{65}(t)}{(1+Y_k)^k} = 1 + \sum_{k=1}^{44} (1+Y_k)^{-k} \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t)\right).$$

Similarly, the cohort annuity present value under assumption (5.7) equals

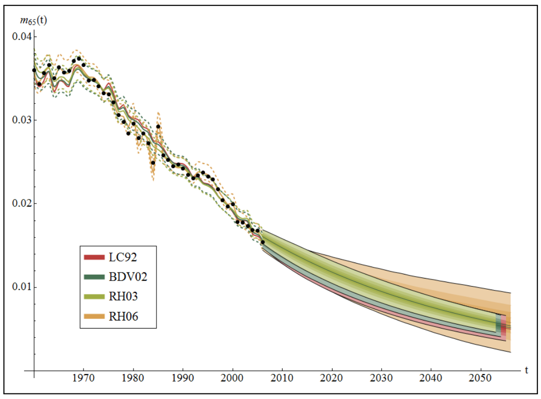
$$\ddot{a}^{\nearrow}_{65}(t) = \sum_{k=0}^{110-65-1} \frac{{}_kp^{\nearrow}_{65}(t+k)}{(1+Y_k)^k} = 1 + \sum_{k=1}^{44} (1+Y_k)^{-k} \exp\left(-\sum_{j=0}^{k-1} m_{65+j}(t+j)\right).$$

By analogy with the projection of the life expectancy function, we only present the period annuity present value due to the fixed forecasting horizon. A growing average life expectancy has a considerable impact on life annuity values and thus the reserving of an annuity provider. The latter is well-advised to use the forecasting-extensive cohort annuity values in order to account for a temporal variation in the central death rates. The projected values in Subfigure (5.21.4) show a nearly identical course compared to Subfigure (5.21.3). This is not surprising at all since both the expected lifetime and the present annuity value mainly compose from survival function values of more recent future horizons at

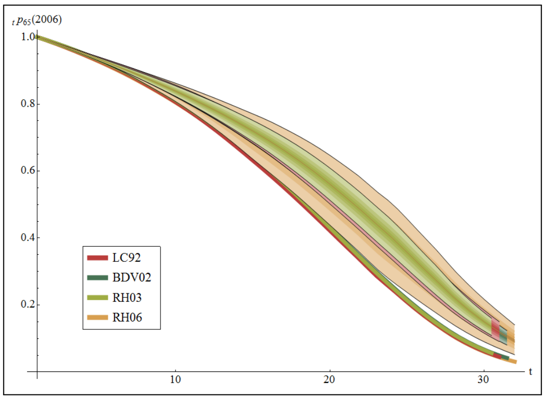
[112] Note that even if the assumption of deterministic interest rates over the long contract duration might not be too realistic our primary objective lies in the illustration of uncertainty concerning the mortality projection.
[113] The euro area government bond yield curve for maturities up to 30 years is provided by the European Central Bank (2010). We use quotes with reference date 08/10/10 and applied logarithmic extrapolation for (integer) maturities $k \in]30,44]$.

which the fan charts proceed almost uniformly. The mentioned "saturation" effect of the life expectancy charts is slightly more pronounced for the projection of the present annuity values.

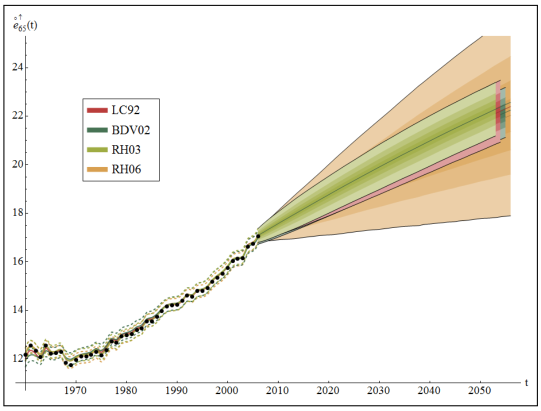
Fan chart analysis of related functions for models LC92-RH06



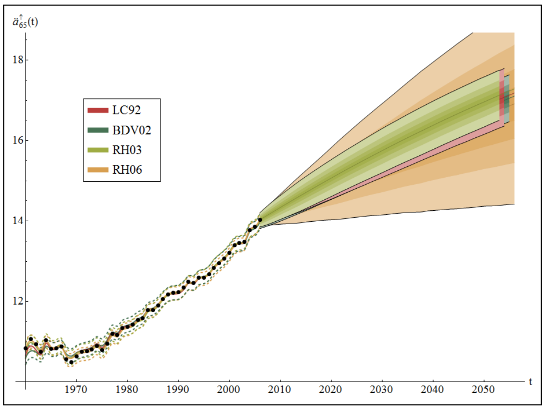
(5.21.1) Central death rate projection



(5.21.2) Cohort and period survival function projection



(5.21.3) Period life expectancy projection



(5.21.4) Period life annuity projection

Figure 5.21: Historic parameter estimation results with 0.95-prediction intervals for selected related functions for mortality models LC92-RH06 and an individual aged 65. The illustration of the projection risk is presented by means of fan charts (with prediction probabilities (0.25,0.5,0.7,0.95)) based on $5 \cdot 10^3$ projections. The period survival functions are illustrated by thick solid lines. The mortality data is taken from the Human Mortality Database (2009).

5.4.6 Further statistical goodness of fit measures

In addition, we present further measures and markers that are suited to quantify fitting capability and to obtain a description of the probability distribution of the residual lifetime.

Mean squared error

The central death rate mean squared error (MSE) measures the squared deviations of the raw and the estimated logarithmised central death rates for the calendar year t as follows

$$MSE(\omega, t) = \frac{1}{\omega} \sum_{x=0}^{\omega} \left(\ln\left(m_x(t)\right) - \ln\left(\hat{m}_x(t)\right)\right)^2 .$$

The measure equals the second sample moment of the estimated death rates at time t containing both the variance and the squared ordinary estimation bias. The method is applicable for both the Lee-Carter single value decomposition and MLE methods. A small MSE value indicates low values for graduation bias and variance of the estimated central death rates. Figure 5.22 illustrates the noticeable high squared errors (especially for periods 1960-1980) for the model extensions BDV02-RH06 on the logarithmic scale when all ages are included (i.e. $\omega = 109$). This is a result of the poor parameter fitting of the average mortality pattern $\hat{a}(x)$ for ages 100 and older (cf. Subfigures (5.1.1) - (5.4.1)). Admittedly, these errors become negligible for a limiting age $\omega = 100$. Figure 5.22 also shows that the underestimation of mortality patterns for ages beyond 100 years slightly improves for more recent observation periods. Table 5.8 indicates that the LC92 model features estimates that are robust in the age range to be fitted. By comparison, we observe the lowest time average MSE value ($6.5 \cdot 10^{-3}$) as long as the whole age range is considered for analysis of the graduation error. An exclusion of the oldest ages with weak empirical evidence supports in turn the multi-factor extensions (with a time average MSE level of $4.4 \cdot 10^{-3}$ for the RH03 and $3.8 \cdot 10^{-3}$ for the RH06 mortality model).

Time average mean squared errors for the mortality models LC92–RH06

	Time average MSE level			
age limit	LC92	BDV02	RH03	RH06
$\omega = 109$	$6.5 \cdot 10^{-3}$	$8.1 \cdot 10^{-2}$	$1.1 \cdot 10^{-1}$	$5.4 \cdot 10^{-1}$
$\omega = 100$	$6.9 \cdot 10^{-3}$	$6.6 \cdot 10^{-3}$	$4.4 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$

Table 5.8: Time average mean squared errors for the period life tables 1960 to 2006 and mortality models LC92–RH06.

Location and variability measures

The goodness or appropriateness of a parametric model can furthermore be assessed by a comparison of meaningful location and variability measures. In Table 5.9 we give a short overview following Pitacco et al. (2008). However, we

Mean squared error for different periods and Mortality models LC92–RH06

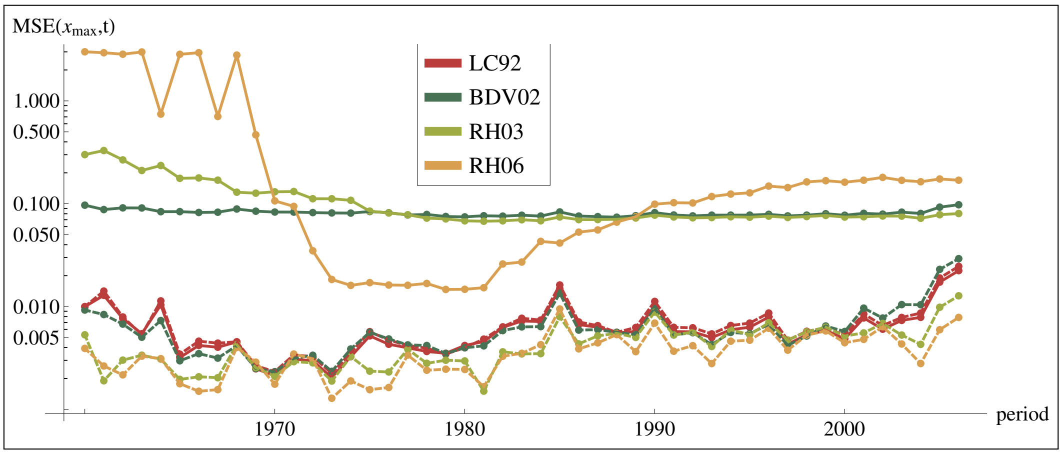


Figure 5.22: Mean squared error $MSE(t)$ for different periods and mortality models. The solid lines indicate the case $\omega = 109$, the dashed lines the truncated case $\omega = 100$.

abstain from a concrete derivation of the presented location and variability measures for the probability distribution of the random residual lifetime. Wilmoth and Horiuchi (1999) suggest to solely use the interquartile measure because all other location and variability measures show strong positive correlation.

Location and variability measures for the residual lifetime

Measure	Definition	Description / application
Expected residual lifetime	$\mathring{\boldsymbol{e}}_{\boldsymbol{x}} = E[\tau_x] = \int_0^{\omega-x} {}_tp_x \, dt = \frac{T_x}{l_x}$	An increase in the expected residual lifetime or complete expectation of life (and therefore the most probable age of death) serves as an indicator for the level of health in different populations or the health development of a representative individual.
Mode	$\boldsymbol{Mod}\,[\boldsymbol{\tau_x}]$	The mode for the curve of deaths (value at which the probability mass function reaches its maximum) is called Lexis point. The location of the Lexis point at old ages provides information about the degree of the expansion phenomenon, i.e. the mode moves towards oldest ages.
Median	$P\,(\tau_x > \boldsymbol{Med}\,[\boldsymbol{\tau_x}]\,) = 0.5$	The median age at death or median future lifetime is based on the mortality data only and equals the 0.5-quantile. For statistical applications, the median is commonly benchmarked with official forecasts.
Standard deviation	$\boldsymbol{Std}[\boldsymbol{\tau_x}] = \sqrt{2 \int_0^{\omega-x} t \cdot {}_tp_x \, dt - (\mathring{e}_x)^2}$	The standard deviation of the complete future lifetime measures the extent of dispersion around the expected residual lifetime. If the variation in the age at death decreases, then the standard deviation decreases as well.
Coefficient of variation	$\boldsymbol{COV}[\boldsymbol{\tau_x}] = \frac{\sqrt{Var[\tau_x]}}{E[\tau_x]}$	The coefficient of variation denotes the unitless ratio of the standard deviation and the mean of the residual lifetime distribution. An increasing COV indicates higher dispersion around the expected age at death. It provides an appropriate measure to compare the degree of variation in multivariate survival data.
Entropy	$\boldsymbol{H}[\boldsymbol{\tau_x}] = -\frac{1}{\mathring{e}_x} \int_0^{\omega-x} {}_tp_x \, \ln({}_tp_x) \, dt$	The entropy of the survival function measures the extent of the rectangularization phenomenon which is associated with an increasing concentration of deaths around the mode of the curve of deaths. A decreasing entropy towards zero indicates a concave and a more pronounced rectangular shape respectively.
Maximum downward slope	$\boldsymbol{MDS}[\boldsymbol{\tau_x}] = \max_x \left\{ -\frac{d}{dt} {}_tp_x \right\}$	The slope at the age of fastest decline measures the degree of concentration in the curve of deaths and equals its ordinate value at the mode. Conversely, the steeper the slope of the survival curve the lower the variability in the age at death distribution.
Endurance	$P\,(\tau_x > \boldsymbol{\xi_{10}}[\boldsymbol{\tau_x}]\,) = 0{,}90$	The Endurance defines the age barrier (0.1-quantile $\xi_{10}[\tau_x]$) which is reached in 90% of the cases of death.
Interquartile range	$\boldsymbol{IQR}\,[\boldsymbol{\tau_x}] = \xi_{75}[\tau_x] - \xi_{25}[\tau_x]$	The interquartile range indicates the distance between the lower ($\xi_{25}[\tau_x]$) and the upper ($\xi_{75}[\tau_x]$) quartile for the residual lifetime distribution. A decreasing interquartile range reveals that the age at death distribution becomes more concentrated. Statistically spoken, the individual lifetimes of the middle 50% of the population lie within the interquartile age interval.

Table 5.9: Location (upper row block) and variability measures (lower row block) for the random residual lifetime.

5.5 Conclusion and critical appraisal of time series mortality models

In this chapter, we have undertaken an excursus concerning the modelling and forecasting of mortality using the classical Lee-Carter mortality model and (multi-factor) modifications / extensions fitted to the German male life table data for periods 1956–2006 originated by the Human Mortality Database (2009). The Lee-Carter mortality model combines a parsimonious and intuitive interpretation of the underlying age- and period-parameters with robust projected age-specific patterns. However, it reveals several drawbacks preventing an application for long-term mortality forecasting, for instance, pension modelling. Since the LC92 model assumes independence between the residuals and the age-/period-parameter estimates, it therefore introduces model risk in terms of understated residual variance and thus comparatively narrow confidence bounds since parameter uncertainty is not explicitly accounted for. For reasons of manageability, the error term is further normally distributed with constant variance for all ages and calendar years, i.e. the model error is homoskedastic[114]. The assumption of a normally distributed error term causes a symmetric interval forecast with possibly unrealistic (long-term) behaviour since the rate of decline for certain age patterns stays constant over time. Furthermore, the fitting procedure based on least-squares minimization is twofold such that the time dependent parameters need to be re-adjusted in a second step. Thus, the bias is inherited from the first term estimates and incoherence could arise.

Several log-bilinear extensions based on MLE produce relief in the way that the parameters vary by distribution. For example, the Poisson regression model used by Brouhns et al. (2002b) treats the observed number of deaths as realisations of an univariate Poisson distribution with mean equal to the expected number of deaths under the LC92 model[115]. Thus, a second stage adjustment of the time-dependent factors becomes obsolete. Forecasts are sampled by parametric bootstrapping and therefore introduce additional parameter estimation uncertainty. Nevertheless, the overdispersion problem remains and is expressed by excessively narrow confidence intervals. More precisely, the

[114] On the basis of increasing dispersion of the logarithmic central death rates from young to oldest ages due to a smaller number of deaths this assumption seems to be somehow unrealistic. Actuarial literature tackled this discrepancy in different ways. For example, the Lee-Carter extension of Koissi and Shapiro (2006) treats parameters as random fuzzy numbers with distributions obtained by a Bayesian technique.

[115] Renshaw and Haberman (2008) additionally present a joint Poisson-Gamma and a negative Binomial extension in order to model the second moment of the number of deaths distribution. Thus they allow for unconstrained dispersion.

Poisson error distribution assumption of equal mean and variance conveys an illusory degree of precision in the way that future dispersion in the central death rate is understated. Subfigures (5.1.5) and (5.2.5) illustrate these characteristics for the 95% confidence (CI) intervals in case of the LC92 and BDV02 model. Over time, a number of modifications of the original Lee-Carter framework (cf. Table 4.2) have been proposed to improve the fitting quality or robustness. Nonetheless, all single-factor age-period models have in common that certain systematic period effects and trends inherent in historic mortality data are fitted unsatisfactorily.

For this reason, upgrades to multi-factor age-period models (see Table 4.3) have been introduced and promised fitting improvements by a description of additional deterministic period patterns not captured by the LC92 period factor. We focussed on two extensions introduced by Renshaw and Haberman (2003, 2006) to perform a quantitative analysis of graduation as well as projection results. The two factor age-period model RH03 showed higher flexibility due to the additional individual bilinear term. Hereby, the second period factor explained a major part of the variance in mortality rates for young and oldest age groups and therefore improved the fitting quality compared to the single-factor LC92 and BDV02 models. Nevertheless, certain characteristics for cohorts with different year of birth remained unexplained.

In contrast, the RH06 model adds a year-of-birth dependent factor to capture systematic cohort effects. An analysis of the standardised residuals showed that mortality trends evoked by age, period or cohort effects are largely captured. The year-of-birth dependence of the residuals vanished almost completely. Admittedly, the complex multi-factor model variants (RH03, RH06) exhibit drawbacks concerning numerical instability and a lack of robustness for the parameter fitting although they provide a good fit for historic death rates or related functions (see Figure 5.21). In this connection, Renshaw and Haberman (2006), CMIB (2007b) or Cairns et al. (2009) mention that due to the linear dependence (year of birth equals the current calendar year minus the cohort age) the modelling might be problematic in the way that resulting parameter sets are not unique. The iterative scheme (5.3) shows very slow convergence caused by a flat likelihood function and / or multiple maxima such that computing time increases considerably. Furthermore, the estimation algorithm lacks robustness in the initial parameter values and the observation window. Especially when the sample size is sparse, e.g. for estimated number of person years (3.3) at oldest ages, we obtained forecasts which are (biologically) implausible. However, caution is advised concerning the projection of the additional cohort effect which involves a high degree of parameter risk. Basically, log-linear model variants with more than two non-linear stochastic factors (e.g. the Cairns-Blake-Dowd model with a cohort effect presented in Cairns et al. (2009))) are also conceivable but further increase estimation and forecasting uncertainty and struggle with the same drawbacks mentioned above.

We have tested the selection of models LC92–RH06 against basic qualitative and a range[116] of quantitative criteria like residual analysis, parsimony, variance explanation, mean squared error and fan chart comparison. As already mentioned, the residual inspection attests the RH06 extension the best graduation performance. The ranking regarding the parsimony remains unchanged when considering the proportion of the explained temporal variance. Furthermore, a review of the fan charts for projected central death rates and related functions shows that, on the one hand, the parameter uncertainty of the models BDV02 and RH03 is negligible in contrast to the LC92 type[117]. On the other hand, the RH06 model which we preferred for fitting quality reasons reveals considerable parameter as well as forecasting risk[118] for estimated and projected life table functions. What is more, markers such as the expected residual lifetime or a life annuity present value measure the mortality development and provide a decision support for state authorities or annuity providers. A (longevity) fan chart comparison reveals homogeneous concave shape with notable differences among the mortality models. Robustness and numerical shortcomings for the multi-factor representatives are revealed by a comparison of the mean squared error.

Finally, it should be noticed that a decision for or against a certain extrapolation model must always comply with the requirements and data quality / availability of the selected application. The stochastic models under consideration are only able to quantify model risk and forecasting uncertainty. Other elements of uncertainty such as process, basis or judgement risk are neglected. The single-factor bilinear models are robust but limited in their fitting ability due to the number of parameters. For the multi-factor extensions, precisely the opposite is the case.

[116] Any rank order based on a single criterion only might be ambiguous and, for example, should not be limited to fitting performance of historic data only.
[117] We observed similar results concerning direction and width of the fan charts for these extrapolation models.
[118] The forecast error is assessed, for example, by the (mean) absolute deviation or the mean squared error. The width and the form of the prediction interval allows for a assessment of the forecasting risk.

Part II

Application of Stochastic Mortality Models to Life Insurance Products

People may live as much retired from the world as they like, but sooner or later they find themselves debtor or creditor to some one.

Johann W. von Goethe, German playwright, poet, novelist and dramatist who lived from 1749 to 1832

CHAPTER 6

The German pension system and the need for private retirement provision

Due to the ongoing pension debates since the nineties – treating an expected demographic adverse development on the long-term financial viability of the German pension system – the topic of retirement provision has gained increasing importance. When talking about retirement provision, we refer to all provision arrangements during the period of employment that aim at maintaining the livelihood for a subsequent retirement desirably without any forfeit of welfare. Within the framework of Reich Chancellor Bismarck's social legislation at the end of the 19th century, the statutory pension system (for the time being capital covered) has been introduced. Due to the economic aftermaths of the Great Depression (hyperinflation), World War I/II and the currency reform, the contribution reserves have been nearly devalued or exhausted by substantially reduced contributions on the one hand and considerable disability and widow's benefit payouts on the other hand. Right before the parliamentary elections in 1957, Konrad Adenauer, the Federal Chancellor being in office at that time, came to a decision with serious consequences: he enforced a pension reform introducing a PAYG system despite the concerns expressed by his former German Minister of Economic Affairs Ludwig Erhard. Among various (long overdue) reform adjustments, this reallocation-financed system endures down to the present day. The great reform adopted an indexation[119] of the statutory pension to the general development of the employer's gross wages[120] to ensure

[119] Since the introduction of the German PAYG pension system in 1957, pensions had been adjusted for forty-nine times (until 2010) according to the gross growth of wages. Admittedly, due to a growing low-wage sector income-related pension increase was on average less than one percent since the beginning of the new millennium.

[120] Because age groups with low birth rates born in the seventies began their entry into the workforce and life expectancy steadily increased, the 1992 pension reform implemented the indexation according to the net wage development.

participation of retirees in the increasing level of welfare of contributors. The pension was no longer conceived as a livelihood subsidy but rather as a wage substitution.

With the legal inclusion of the company pension scheme in 1974 under the so called Employers' Retirement Benefits Act, retirement provision offered by three different carriers was rounded out. In accordance with the World Bank model, the concept of different supporting legs was constructed as a three-pillar concept. The first pillar denotes the statutory public pension scheme as an integral part of the basic social security system. In 2008, 85% of the income of persons aged 65 and older[121] was made up by the statutory pension. Its sustainability is based on a PAYG system, i.e. under the assumption of a stable population structure a major part[122] of the pension payments to current benefit recipients is funded by the contributions of the current employees which in turn acquire a pension benefit entitlement. It was already in 1975 when the Federal Chancellor Adenauer was thoroughly mistaken as he defended his reform of a dynamic pension plan with the rather flippant explanation that "People do always have children.". However, demographic development after that time has upset reform plans because population sizes have been declining since the nineties which will lead to a future mismatch in the proportion of retirees and contributors. One of the core sentences associated with German pension policy was stated by the former Federal Minister for Labour and Social Affairs Norbert Blüm who announced before the elections in 1986 that "Because one thing is certain: The legal pension.". There have been only a few statements in pension politics that aroused more controversial discussion. From today's point of view, the only thing that is for sure concerning the pension is insecurity about the fact that the paid mandatory contributions cover future old-age livelihood. Additionally, several circumstances[123] have threatened and threaten the long-term financial viability of the public PAYG pension system. If one is to believe the predictions of the Commission for Sustainability in Financing the Social Security Systems (Rürup Comission) the proportion of retirees and contributors will continuously rise due to structural changes among the age distribution of the German population. A higher life expectancy together with

[121] The data was obtained from the supplementary report of the German Federal Ministry of Health and Social Security (2008).

[122] For the year 2007 the German Federal Ministry of Health and Social Security (2008) recorded a total income of about 238 billion euros of which 73% were employees contributions, 26% were federal subsidies and a further 1% came from other financial resources.

[123] The pension settlement within the reunification of Germany, high tax-funded state subsidies, the growing low-wage sector in the course of the economic crisis, the high rate of unemployment (in particular in the new federal states) and a significant proportion of partial/early retirement have had a negative effect on the funding of statutory pensions.

a decreasing birth rate lead to a pronounced ageing of the population with serious impacts on the pension planning. More precisely, the proportion of the income bracket in the age range 20-64 to pensioners 65+ is expected to shrink from 3.25 to 1 in 2003 to a ratio of 2.2 to 1 for the year 2030. While in 2003 a group of 100 active contributors of the statutory public pension scheme financed the pensions to approximately 48 retirees, population forecasts yield a value of 68 for the year 2030 at the expense of the contributors. As a consequence, legal modifications concerning the contributions, the wage adoption, the working life as well as the amounts of the reduced earning capacity pension and the survivor's benefits were and are inevitable to secure the future viability of the PAYG social system. Following the reference scenario of the Commission for Sustainability in Financing the Social Security Systems (Rürup Comission), the pension reform means that the standard replacement ratio[124] shrinks from 48% in 2003 to 40% in the year 2030 whereas at the same time contributions increase from 19.5% to 22% of the gross income. Against the background of a temporary pay freeze[125] of indexation of statutory pensions and legal changes within the pension reform[126], a supplementary funded private preservation in order to maintain the pre-retirement standard of living seems more reasonable than ever. The statutory pension will then solely suffer as a livelihood security.

In contrast, the second pillar is given by the tax-privileged voluntary occupational pension based on a funded provision. Wage earners acquire a legitimate payment claim[127] against their employers in case of the insured events retirement, disability or death during the accumulation phase. Claims are either employer-financed, employee-financed (in form of a pay conversion) or a combination of both. For this purpose, a total of five permissible realisation methods are available yielding economic incentives for both insurance takers and providers. On the one hand, the employer directly grants defined benefits through (reinsured) benefit funds or company-based pension schemes which

[124] The standard replacement ratio describes the relation of the acquired gross pension and the current average gross income just before retirement. It is thus a measure for the retirement income gap. On principle, consumer groups assume that a pension amounting to 80% of the last earned net wage is sufficient to maintain the pre-retirement standard of living.

[125] Effectively, a zero pension rate adjustment means a reduction when considering a positive inflation rate.

[126] The so called bill of pension sustainability takes account of the pronounced ageing of the population. In particular, the bill included the introduction of a so called sustainability factor measuring the ratio of contributors to retirees and an increased statutory retirement age. The sustainability factor attenuates the pension increase and thus should achieve a long-term stability in the contribution rate.

[127] The achievement of a claim and its legal vesting period depends, inter alia, on the length of the employment, the underlying wage agreement and the offered realisation method.

is most frequently used at the moment. On the other hand, the company indirectly engages an external provider to provide occupational retirement provision for its employees by means of a direct insurance, a staff pension insurance or a pension fund. Up to now, little use is being made of occupational pension schemes since solely 5% of old-age income result from this kind of provision form. The topic of company pension schemes shall not be discussed any further subsequently.

The third and final pillar describes the voluntary private provision which is funded by the contributor and at the same time beneficiary, e.g. in form of equity fund saving plans (non-state-subsidised), government certified Riester and Rürup pension policies (state-subsidised demand oriented basic provision), life insurance policies (non-state-subsidised) and private home pension schemes (Wohn-Riester is state-subsidised). The savers accumulate capital and participate from distributed profits. The resulting entitlement is paid out as a lifelong pension or a (partial) lump-sum payment. In accordance with the German Federal Ministry of Health and Social Security (2008) the present pension income from private provision products lies around 10% such that there is plenty room for a re-attachment of weight from the first to the private funded pillar. Nevertheless, the largest part of the retirement earnings will also be hold by the public pension scheme in the (near) future.

With the introduction of the Retirement Income Act in 2005 and the revision of the taxation of pension provisions and retirement earnings involved, the three-pillar system was supplemented by a three-tier model. The components in form of three provision layers are basic provision (deferred taxation), supplementary provision (deferred taxation) and investment products (taxed in advance) distinguished according to the fiscal handling. Since the statutory main pillar is about to collapse in the foreseeable future and therefore threatens the stability of the entire pillar system, a layered arrangement with an equally distributed provision model seems more stabilizing in this regard. Nowadays, a provision willing person can select out of a wide range of retirement products yielding a life-long payment and thus protection against outliving his/her accumulated assets. However, on the supplier side the insurance sector is faced with the challenging task to account for future changes in the demand and requirements concerning funded provision products. In particular, due to the continuing demographic development, products like single-premium immediate annuities for generations 50+ and (unit-linked) deferred annuities with inherent life phase concept[128] could become a sought-after provision form. Thus, a dynamic

[128] Within the life phase approach for private pension insurance the accumulation and the

product design constitutes a significant success indicator. It has to ensure that highly complex insurance products become both simpler (transparency) and more consumer-friendly (cost efficiency, flexibility) to achieve a long-term customer retention in markets that are becoming increasingly saturated. Furthermore, the post-Global-Financial-Crisis era caused a demand for security and long-term guarantees.

Traditional life insurance solutions like endowment life policies and private (deferred) pension policies underlie an investment in a conventional premium reserve stock and therefore a conservative allocation almost away from equities. Based on the report of the GDV for the year 2010, the life or pension insurance is still Germany's most favoured retirement product with an overall annual premium sum of 1.51 billion euros which equals a share in the new business of 24.6%. Hereby, especially the new business for single-premium policies with a comparatively small premium sum of 330 million euros recorded a percentage increase of 64% compared to business year 2009. The high ratio of a secure investment (contractually guaranteed actuarial interest rate and annuity factor) is bought by a low expiry yield[129] and thus implicite costs within the insurance wrapper.

Alternatively, the customer can choose in favour for a modern "state-of-the-art" policy with (permanent) high (self-determined) fund exposure in form of a capital market based contract enriched with selectable guaranteed benefits. Common products are unit-linked endowment and pension policies with "guarantee funds" (often endorsed with a peak lock-in mechanism), (dynamic) hybrid products or so called variable annuities. It is especially the latter product variant which offers the highest (guaranteed) pension rates and, at the same time, nearly 100% participation in the underlying fund portfolio. Moreover, unit-linked annuities offer the benefit of a deferred taxation of the pension's profit share at a reduced retirement tax rate in comparison to a direct fund investment[130] burdened with a withholding tax. In times of decreasing guaranteed interest rates[131] and an enhanced need for private provision, flexible

decumulation phase are no longer fixed but are fully flexible designed and can be adapted to the policyholder's individual needs and living conditions. Inter alia, this comprises the inclusion/exclusion as well as the increase/reduction of benefits from certain contractual parts or allied perils and the implementation of different (partial) lump sum payments or retirements.

129 In 2010 the GDV announced an average regular net return of 4.13% for the life insurance business.

130 Kling et al. (2005) show that with a correspondingly long contract duration the expiry yield of a unit-linked pension dominates the one of a mutual fund savings plan.

131 Since its peak in 2000, the German guaranteed interest rate for insurance companies has been stepwise reduced from 4% to 1.75% for the forthcoming year 2012 by decision of the

and cost transparent unit-linked concepts – albeit niche products so far – may become a serious alternative to conventional life and pension insurance products. The proportion of unit-linked pension contracts to the overall new business rose steadily from 11.7% in 2005 to 23.7% in 2008 and declined afterwards due to the global economic recession to 16% in 2010. However, the "brave new pension world" involves several new challenging actuarial tasks. Besides the risk concerning decreasing (market) interest rates and continuing mortality improvements the unit-linked concept introduces further market risks (concerning the fund price, fund volatility and currency) as well as financially rational policyholder behaviour which tremendously complicates risk management and rises solvency capital requirements. The need for an adequate hedging of the stated guarantees introduces further risks like operational, credit and basis risks.

In the following two chapters we take a detailed account of common insurance solutions of the conventional as well as the unit-linked form. More precisely, two short-rate mortality models out of the stochastic framework introduced in Section 4.3 of the first part of the thesis are each calibrated to the German population mortality and used to model both the individual assured's survival and the general (insurance) cohort mortality. In combination with a full stochastic financial market model, this necessitates the use of pricing and reserving methods applicable in incomplete markets[132]. We are paying particular attention to these issues in Chapter 7 which treats the analysis of a traditional deferred life annuity from the perspective of the underwriter. In the subsequent Chapter 8 we take a closer look on a deferred variable annuity contract with guaranteed living and death benefits. On the one hand, we also tackle pricing issues, notably the estimation of a fair guarantee percentage charge, and a sensitivity analysis concerning process parameter misspecification. On the other hand, we apply a profitability analysis from the point of view of the policyholder including the simulation of the net rate of return on the premiums, the moneyness of stated guarantees and the distribution of the total contract value at maturity.

Federal Finance Ministry.

[132] For instance, the topic of an incomplete insurance market is treated in Milevsky et al. (2006) who describe how the law of large numbers breaks down when pricing is performed under stochastic instead of deterministic mortality rates. The authors use a pre-specified instantaneous Sharpe ratio to price mortality contingent claims. An alternative indifference pricing approach is analysed in Ludkovski and Young (2008). Olivieri and Pitacco (2003, 2008) regard the adequate insurer's capital allocation and illustrate the impacts of uncertainty in the level of future mortality. They consider solvency requirements for life annuity portfolios and funded pension plans. Additional references are given in Section 7.1 of Chapter 7.

Buy an annuity cheap, and make your life interesting to yourself and everybody else that watches the speculation.

CHARLES DICKENS, English novelist and writer who lived from 1812 to 1870

CHAPTER 7

Deferred Life Annuities - On the Combined Effects of Stochastic Mortality and Interest Rates

This chapter is based on Mahayni and Steuten (2013). The publisher's version is available at link.springer.com.

7.1 Introduction

Due to the growing importance of the private pension scheme, various annuity insurance contracts and their analysis receive more and more attention. Besides unit-linked annuities, the conventional annuity and pension insurance registered the largest increase[133] of German new business in 2009. The accumulation and decumulation period of the products depend on the random residual lifetime of the insured such that the demographic risk in the sense of mortality and longevity risk plays a crucial role in the pricing and risk management process.

A life annuity secures lifelong annual payments to the insured. In financial terms, the payoff to the insurance cohort can be interpreted as a portfolio of zero-coupon bonds with stochastic maturities and face values. Therefore, the insurance company is exposed to longevity risk and interest rate risk. This is particularly true in the case of deferred annuities where pension payments are postponed. Traditional pricing and valuation models which are based on a conservative time-invariant[134] mortality assumption and a deterministic flat

[133] The redeemed new annuity and pension insurance business contributes regular premiums amounting to 1.3 billion euros with an overall share of 31%. The business in force at the end of 2009 consists of 18.3 million contracts (equals a share of 20%), see German Insurance Association (GDV).

[134] See Chapter 3 for an overview over traditional mortality modelling.

yield curve do not adequately model the risk factors. In a deterministic model setup, the insurance company is solely exposed to a perfectly diversifiable risk. Because of the law of large numbers, the random number of bonds can be replaced by deterministic survival probabilities. Admittedly, besides a diversification or pooling risk which stems from an insufficient large insurance cohort, there is also a non-pooling risk part which cannot be diversified[135] since it affects each policyholder's mortality in the same way. In addition, the benefits and contributions are considerably exposed to interest rate risk as a result of long terms to maturity. Therefore, it is neither convenient to assume deterministic interest rates nor deterministic mortality rates. Even in the (realistic) case that the interest rate and the mortality risk are independent, the former risk is only (perfectly) hedgeable if one abstracts from stochastic mortality rates. As a consequence, the pricing as well as any solvency control must take into account the combined effects of stochastic mortality and interest rates.

Our contributions to the existing literature on the effects of stochastic mortality and interest rates are as follows: We calibrate state of the art models to current interest and mortality data. The results of the stochastic models are not considered by their own but are benchmarked to conveniently chosen deterministic counterpart models. This allows us to quantify the effects of different degrees of randomness. Basically, our analysis refers to four risk scenarios: interest rates and mortality rates are both stochastic, only one of the two is stochastic and the (benchmark) case where both are deterministic. Due to the calibration to the same financial market and demographic data, the expected discounted portfolio cash-flow of both, the premium income and the pension benefit, is the same within all models/risk scenarios. However, the risk profile which is given in terms of higher moments varies. For the specification of the (stochastic) mortality rate, we consider a pure diffusion model as well as a compound Poisson jump model. It turns out that the results are rather similar for both setups. For financial modelling we apply a Gaussian Heath, Jarrow and Morton framework. For the sake of simplicity, we use a one-factor Hull-White interest rate process. The term structure of interest rates is calibrated for different assumptions on the degree of randomness, i.e. we calibrate the term structure for varying spot rate volatilities. We approximate the higher moments of the discounted cash flows by means of Monte Carlo simulations. One emphasis is on the variance and its decomposition into a

[135] Note that at present, the mortality derivative market is almost illiquid and reinsurance solutions are cost-intensive and tailor-made. Thus, we do not consider the non-pooling risk to be a traded risk.

pooling and a non-pooling part. We also consider different pricing principles that have already been adapted successfully in incomplete markets[136]. The resulting premiums are benchmarked to the equivalence premium in terms of the (discounted) expected cash-flow. For example, we consider an utility based (indifference) approach along the lines of Pelsser (2005) to obtain a market-consistent valuation of insurance benefits and a quantile premium principle which ensures a given value-at-risk condition posed on the underwriter's loss at the contract maturity. Both principles require assumptions on the insurer's investment/hedging strategy. We compare the "no hedging" case with a static hedge consisting of an adequate number of zero bonds with different maturities. We also consider the effects of forward starting strategies which avoid a pre-financing of the future premiums. It turns out that the full stochastic scenario demands a substantial risk premium. However, even in the case of stochastic mortality, the "as good as possible hedge" with zero bonds reduces the required risk premiums to a large extent. Finally, we also take solvency requirements based on the shortfall probability of the annuity provider into account. In addition, we consider the expected shortfall to measure the extent of a default. Again, it turns out that, compared to the (deterministic) benchmark model setup, the consideration of the combined effects of stochastic mortality and interest rates has a substantial impact on the solvency requirements.

There are different strands of related literature. Without postulating completeness, we mention the following. A simple and illustrative example how the law of large numbers breaks down when switching from deterministic to stochastic mortality rates is given in Milevsky et al. (2006). The impact of longevity risk is studied in Blake and Burrows (2001), Olivieri (2001, 2007), Di Lorenzo and Sibillo (2002), Coppola et al. (2003) or Pitacco (2004a). They all conclude that mortality data shows a decreasing trend with different random developments for different ages, periods or gender (cf. Chapter 2). Some methods to model stochastic mortality rates are, for example, given in Biffis (2005), Schrager (2006), Dahl (2004), Pitacco (2004b), Bayraktar et al. (2009), Dahl and Møller (2006) and Korn et al. (2006)[137]. A subdivision of the variance of the present benefit value was first introduced in Parker (1997) who divides the total variance of an endowment life insurance portfolio into a sum

[136] Up to now, insurance risk is neither fully hedgeable nor sufficiently traded. In contrast to the interest rate derivatives market, which constitutes the largest derivatives market in the world, the trading of so called mortality derivatives is still in its infancy.

[137] Biffis (2005) and Schrager (2006) choose an affine mortality process, Korn et al. (2006) use a stochastic version of the Perks-/Gompertz-Mortality law and Dahl and Møller (2006) a square root diffusion process, whereas Bayraktar et al. (2009) analyse a general diffusion process.

of two components either conditioned on investment or insurance risk. The systematic risk component which stems from permanent mortality trends is, inter alia, considered in Olivieri (2001), Coppola et al. (2000, 2002) or Pitacco (2004a). The authors use a finite range of future mortality scenarios instead of stochastic realisations and differentiate between pooling and non-pooling variance parts. In contrast, Hári et al. (2008) and De Waegenaere et al. (2010) consider pension annuities in a generalised Lee-Carter setting. Christiansen and Helwich (2008) use a stochastic Gompertz-model in the case of both, pure endowment and temporary life insurance. Dahl et al. (2008) rely on a Cox-Ingersoll-Ross square-root diffusion mortality model and simulate the risk per policy of an endowment life insurance under different risk-minimizing strategies. For an application of the principle of zero expected utility to the pricing of life annuities we refer to the work of Pelsser (2005), Hainant and Devolder (2008) and Ludkovski and Young (2008). In particular, Pelsser (2005) postulates requirements on (meaningful) utility functions which are satisfied by functions with double hyperbolic absolute risk aversion. Concerning the relevance of the quantile principle we refer to Olivieri (2001, 2007) and the literature given herein. Solvency aspects are, for example, discussed in Bauer and Weber (2008) and Hári et al. (2008) who calculate the value-at-risk and the expected shortfall for immediate starting life annuities under different static hedging scenarios.

The outline of the chapter is given as follows. Section 7.2 presents the contract definition, introduces the (stochastic) discounted portfolio value and gives the model setup for the combined mortality and interest rate model. Section 7.3 describes the calibration procedure and illustrates the calibration results. In Section 7.4, we approximate the variance of the discounted portfolio value by means of Monte Carlo simulations. We also illustrate the pooling and the non-pooling risk parts w.r.t. different interest rate volatilities, cohort sizes and deferment periods. Section 7.5 concerns the pricing. We give a brief review of the indifference/zero expected utility and quantile pricing approaches. We approximate and compare the resulting risk premiums for different risk scenarios and investment / hedging assumptions. The associated shortfall probabilities and the expected shortfall of the insurance (business) loss at maturity are considered in Section 7.6. In addition, we simulate the solvency margins w.r.t. exogenously postulated (maximum) shortfall probabilities. Section 7.7 concludes the chapter.

7.2 Contract and model specification

Throughout the following, we consider a portfolio of N_0 identical forward starting life annuities with annual pension payments c starting at $T > 0$ as long as the insured is living but at most until the contract expiration $\bar{T}$. We set $\bar{T} = \omega - x$ where x denotes the current age of the insured and ω can be interpreted as a biological age limit. The periodic premiums π are paid in advance on a constant regular basis, i.e. at times $t = 0,...,T-1$. All annuity contracts under consideration are initiated at $t = 0$. Conditioned on the mortality law, the residual lifetime $\tau_x^{(n)}$ ($n = 1,...,N_0$) of the number of survivors are i.i.d.. The payment stream[138] of a deferred life annuity with deferment period of length T is depicted in Figure 7.1. Essentially, two different scenarios are possible. If the insured n dies before T, i.e. the event $\{\tau_x^{(n)} < T\}$, we assume that there are no further premium payments and no delayed annuity payments. If the insured survives T, i.e. $\{\tau_x^{(n)} \geq T\}$, he receives an annually payment c starting at T until his death. All premium payments contributed

Payment stream of a deferred life annuity

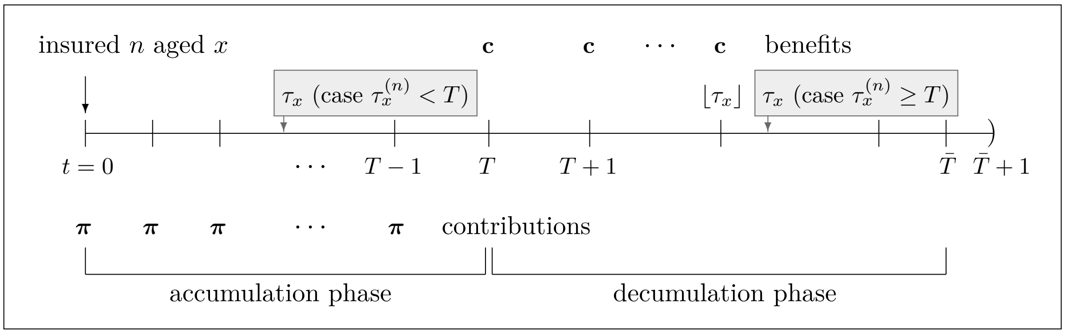


Figure 7.1: Payment stream of a T-deferred life annuity with periodic premium π and pension payment c. The figure was first published in Mahayni and Steuten (2013).

by the in-force insurance portfolio are conditioned on the event that the insured is still living during the deferment period, i.e. $\pi \, \mathbb{1}_{\{\tau_x^{(n)} > t\}}$ is the premium paid by the insured $n \in \{1,..., N_0\}$ at time $t \in \{0,1,...,T-1\}$. Summing up, the portfolio premium income at year t is $\pi \cdot N_t$ where N_t denotes the size of the insurance cohort at time t, i.e.

$$N_t := \sum_{n=1}^{N_0} \mathbb{1}_{\{\tau_x^{(n)} > t\}}.$$

[138] The accumulation phase $[0,T[$ represents some kind of savings plan, whereas the subsequent decumulation phase $[T,\bar{T}]$ forms an immediate life annuity.

On the benefits side, payments to the annuitants are conditioned on the event that the insured is still living after retirement. Thus, the portfolio payoff immediately after year $T+t$ is $c \cdot N_{T+t}$.

In the following, the stochastic process β denotes the bank account, i.e.

$$\beta_{t,\bar{t}} := \exp\Big(I(t,\bar{t})\Big) \text{ where } I(t,\bar{t}) := \int_t^{\bar{t}} r_u \, du \quad (t \le \bar{t} \le \omega - x) \tag{7.1}$$

and $r = (r_t)_{0 \le t \le \bar{T}}$ is the process of the continuously compounded spot interest rate. For $t \in [0,T]$, the discounted [139] portfolio premium income value X_t^{Π} and the discounted portfolio benefit value Z_t^{Π} are given by

$$X_t^{\Pi} = \pi \sum_{i=0}^{T-1} \frac{N_i}{\beta_{t,i}} \quad \text{and} \quad Z_t^{\Pi} = c \sum_{i=T}^{\bar{T}-1} \frac{N_i}{\beta_{t,i}}. \tag{7.2}$$

As a consequence of the periodic premium payments, the insurer faces also the risk of unanticipated deaths of the insureds during the accumulation phase. In addition, notice that both the discounted income X and the discounted benefit value Z take into account two random components: the number of survivors N and the (stochastic) bank account β. Thus, X and Z (the underwriting loss $Z - X$ respectively) are convenient to analyse the effects of stochastic mortality and stochastic interest rates on the risk profile of the annuity provider.

All stochastic processes under consideration are defined on a filtered probability space $(\Omega,\mathbb{F},P)$ with filtration (information structure) $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\bar{T}]}$ $(\mathcal{F}_0 = \{\emptyset,\Omega\})$ satisfying the usual conditions, i.e. right continuity and $(P,\mathcal{F})$-completeness. $\mathcal{F}_t$ represents all available information at time $t \in [0,\bar{T}]$. In particular, $\mathcal{F}$ contains the (sub-)information about the interest rates $\mathcal{I}$, the number of survivors $\mathcal{H}$ as well as the mortality rates $\mathcal{M}$. To be more precise, $\mathcal{I}_t = \sigma\left(\{r_s \,:\, 0 \le s \le t\}\right)$ summarizes the information about the interest rate evolution up to time t. The σ-algebra $\mathcal{H}_t = \sigma\Big(\{\mathbb{1}_{\{\tau_x > s\}} \,:\, 0 \le s \le t\}\Big)$ contains information concerning the residual lifetime indicator and thus describes the evolution of the cohort size whereas $\mathcal{M}_t = \sigma\left(\{\mu_{x+s}(s) \,:\, 0 \le s \le t\}\right)$ describes the evolution of the mortality rate up to time t. The instantaneous death probability at t is denoted by $\mu_{x+t}(t)$ and is the same for all annuitants who are still living at t (and aged $x+t$), i.e. for $\Delta t > 0$ small enough it holds

$$P\Big(\tau_x^{(n)} \le t + \Delta t \mid \mathcal{F}_t\Big) \simeq \mu_{x+t}(t) \cdot \Delta t \quad \text{for } n = 1, \ldots, N_0.$$

[139] To be more precise, the discounted portfolio value is expressed by taking the stochastic bank account as the numéraire.

Conditioned on the mortality information until time t, the residual lifetimes $\tau_x^{(n)}$ are i.i.d. , i.e. for $0 \leq s \leq t$ it holds

$$P\left(\tau_x^{(n)} > t \mid \mathcal{H}_s \vee \mathcal{M}_t\right) = E\left[\mathbb{1}_{\{\tau_x^{(n)} > t\}} \mid \mathcal{H}_s \vee \mathcal{M}_t\right] = \exp\left(-\int_s^t \mu_{x+u}(u)\, du\right) \tag{7.3}$$

for $n = 1,...,N_0$. It is assumed that the interest rates and the mortality rates are independent, i.e. $\mathcal{I}$ and $\mathcal{M}$ are independent. In addition, the "vital" status of an annuitant (the cohort size respectively) has no impact on the financial market, i.e. $\mathcal{I}$ is even independent of the union of $\mathcal{M}$ and $\mathcal{H}$.

Remark 7.2.1. Throughout the chapter, we consider a combined model framework which is specified by an interest rate and a mortality rate model. In general, there are two models needed, one for the pricing and one for the risk management purpose. While the pricing model is based on the so called pricing measure, the risk management model is formulated under the real world measure. In practice, the pricing measure is typically obtained implicitly by calibrating the underlying asset pricing processes to liquid products quoted in the market. In the context of mortality risk, the estimation of a pricing measure is difficult if not impossible[140]. Even in the case that one appraises the mortality derivative market as liquid (which we do not), there is still some basis risk stemming from the differences in the underlying cohorts left. We consider the mortality risk as a non-traded and, because of the randomness of the mortality rates, a non-diversifiable risk. The mortality rate model is formulated under the real world measure which is also immanent in the model calibration to observed mortality probabilities. In contrast, the interest rate market is assumed to be complete and we model the interest rate dynamics under the martingale/pricing measure. Due to the incompleteness arising from the mortality side, the risk management of annuities should take into account the risk premium included in the interest rate derivative products. However, the estimation of the relevant risk premiums is beyond the scope of this chapter. We set it equal to zero such that the measures coincide. To simplify the exposition, we state the interest rate dynamics under one measure, only.

[140] For a discussion on the topic of fair valuation of insurance liabilities and the market price of mortality risk we refer to Biffis et al. (2010).

7.2.1 Mortality model

For the dynamics of the mortality rate, we restrict ourselves to the class of affine processes. This assumption implies closed-form solutions for the survival probabilities which are used to calibrate the models to the period life table data originated by the Human Mortality Database (2009). We consider two versions of non mean reverting processes, namely Model (I) and Model (II). Model (I) is a Poisson jump process and Model (II) is a square-root diffusion process. Both models are based on the initial mortality $\mu_x(0)$ which is of Gompertz-Makeham type, i.e. $\mu_x(0) = \alpha + \beta\gamma^x$ for $\alpha \geq 0$, $\beta > 0$, $\gamma > 1$.

Definition 7.2.2 (Model (I)). Model (I) is defined by a non mean reverting Poisson jump process with dynamics

$$d\mu_{x+t}(t) = \kappa_1 \mu_{x+t}(t)dt + dJ_t, \quad (\kappa_1 > 0). \tag{7.4}$$

$J_t = \sum_{i=1}^{M(t)} \varepsilon_i$ is a compound Poisson jump process[141]. M denotes a Poisson counting process with jump arrival intensity λ. The jump sizes ε_i are independent and identically double asymmetric exponentially distributed with density

$$f(z) = \frac{\varphi_1}{\eta_1} e^{-\frac{z}{\eta_1}} \mathbb{1}_{\{z \geq 0\}} + \frac{\varphi_2}{\eta_2} e^{\frac{z}{\eta_2}} \mathbb{1}_{\{z<0\}}.$$

φ_1 and φ_2 are the probabilities for bidirectional jumps, i.e. $\varphi_1, \varphi_2 \geq 0$ and $\varphi_1 + \varphi_2 = 1$. $\eta_1 > 0$ denotes the average jump size of a positive jump and $\eta_2 > 0$ denotes the average (absolute) size of a negative jump, i.e. $E[\varepsilon_i] = \varphi_1\eta_1 - \varphi_2\eta_2$. The process M and the random variables ε_i $(i \geq 1)$ are assumed to be independent such that $E[J_t] = E[M_t]E[\varepsilon_i]$ where $E[M_t] = \lambda t$.

Negative jumps in J (with average size η_2) can be interpreted in terms of mortality improvements like medical breakthroughs or healthier life styles. The result is a shift in the modal value of the curve of deaths. Positive jumps (with average size η_1) can be motivated by a biological age limit, natural catastrophes, wars or pandemics like the bird flu. Unfortunately, processes which are specified along the lines of Equation (7.4) imply negative mortality rates with positive probability. However, the probability is very small in the case of the relevant process parameters and is thus considered as negligible.

Definition 7.2.3 (Model (II)). Model (II) is defined by the dynamics of a non mean reverting square-root diffusion process (of the Cox-Ingersoll-Ross

141 Kou (2002) introduces a compound Poisson jump-diffusion process with double asymmetric jump size distribution in the context of option pricing. Bravo (2008) takes this approach to model mortality rates under bidirectional shocks.

type), i.e.

$$d\mu_{x+t}(t) = \kappa_2 \mu_{x+t}(t)dt + \sigma^{\mu}\sqrt{\mu_{x+t}(t)}\, dW_t^{\mu}, \quad (\kappa_2 > 0, \sigma^{\mu} \geq 0). \tag{7.5}$$

In particular, Model (II) ensures non-negativity.

7.2.2 Interest rate model

For the financial setting we define $D(t,\bar{t})$ as the value at time t of one monetary unit paid at time $\bar{t} \geq t$, i.e. $D(t,\bar{t})$ denotes the time t price of a zero coupon bond with maturity $\bar{t}$. Recall that $r = (r_t)_{0 \leq t \leq \bar{T}}$ denotes the continuously compounded spot interest rate process. We assume a complete and arbitrage-free financial market model under interest rate risk where the dynamics of the zero coupon bonds $D(.,\bar{t})$ with maturity $\bar{t} \in [0,\bar{T}]$ are lognormal, i.e. the interest rate dynamics follow a Gaussian Heath, Jarrow and Morton model (1992), i.e.

$$\frac{dD(t,\bar{t})}{D(t,\bar{t})} = r_t dt + \sigma_{\bar{t}}(t)dW_t, \quad D(t,t) = 1. \tag{7.6}$$

W denotes a one-dimensional Brownian Motion independent of W^{μ}. The volatility of the zero coupon bond $\sigma_{\bar{t}}$ is a deterministic, time-dependent function with $\sigma_{\bar{t}}(\bar{t}) = 0$ and satisfies the usual regularity conditions. The zero bond price process is $\mathcal{I}$-adapted where $\mathcal{I}_t$ can be interpreted as the filtration generated by W, i.e. $\mathcal{I}_t = \sigma(\{W_s : s \leq t\})$. It holds

$$D(t,\bar{t}) = E\left[\exp\left(-I(t,\bar{t})\right)\middle|\mathcal{I}_t\right] = E\left[\frac{1}{\beta_{t,\bar{t}}}\middle|\mathcal{I}_t\right]. \tag{7.7}$$

A convenient possibility to analyse and illustrate the effects of the interest rate volatility is given by the (one-factor) Hull and White model (1990) where the interest rate dynamics follow

$$d\,r_t \;=\; (\theta(t) - a\,r_t)\,dt + \sigma_{spot}\,d\,W_t. \tag{7.8}$$

The parameter a denotes the speed of mean reversion, $\frac{\theta(t)}{a}$ equals the mean reversion level and σ_{spot} describes the spot rate volatility. The time-dependent drift θ allows to calibrate the model to the initial interest rate curve which is currently observed at the market. The simple model choice is convenient for our purpose. However, the model is not able to capture all possible movements of the term structure.

7.3 Calibration

7.3.1 Calibration of the mortality Model (I) and (II)

In order to calibrate Model (I) and Model (II) to mortality data, it is convenient to consider the (observable) survival probabilities instead of the mortality rates. Let ${}_{T-t}p_{x+t}(t)$ denote the probability at time t that an individual who is $(x+t)$-aged is still living at T, i.e.

$$ {}_{T-t}p_{x+t}(t) := E\left[\mathbb{1}_{\{\tau_x > T\}}\middle|\mathcal{F}_t\right] = E\left[\exp\left(-\int_t^T \mu_{x+s}(s)\,ds\right)\middle|\mathcal{M}_t\right]. \tag{7.9}$$

W.r.t. the above models, the survival probabilities are given by the closed-form solution of an ordinary Riccati differential equation.

LEMMA 7.3.1. *If the mortality dynamics are given as in*

(i) Equation (7.4), then it holds

$$ {}_{T-t}p^{(I)}_{x+t}(t) = \exp\left(A_1(T-t) + B_1(T-t)\,\mu_{x+t}(t)\right), \tag{7.10}$$

where

$$\begin{aligned} A_1(T-t) &= \frac{\lambda\varphi_1}{\eta_1 - \kappa_1}\left(-\kappa_1(T-t) + \ln\left[1 - \eta_1\, B_1(T-t)\right]\right) \\ &\quad + \frac{\lambda\varphi_2}{\eta_2 + \kappa_1}\left(\kappa_1(T-t) - \ln\left[1 + \eta_2\, B_1(T-t)\right]\right) - \lambda(T-t), \end{aligned}$$

and

$$ B_1(T-t) = \frac{1}{\kappa_1}\left(1 - e^{\kappa_1(T-t)}\right)$$

for $\eta_1, \eta_2 > 0$, $\lambda \geq 0$ *defined for* $-\frac{1}{\eta_2} < B_1(T-t) < \frac{1}{\eta_1}$.

(ii) Equation (7.5), then it holds

$$ {}_{T-t}p^{(II)}_{x+t}(t) = \exp\left(A_2(T-t) + B_2(T-t)\,\mu_{x+t}(t)\right), \tag{7.11}$$

where

$$ A_2(T-t) = 0 \quad \text{and} \quad B_2(T-t) = \frac{1 - e^{b^\mu(T-t)}}{c^\mu + d^\mu\, e^{b^\mu(T-t)}}$$

for

$$b^{\mu} = -\sqrt{\kappa_2^2 + 2\left(\sigma^{\mu}\right)^2},\ c^{\mu} = \frac{b^{\mu} + \kappa_2}{2},\ d^{\mu} = \frac{b^{\mu} - \kappa_2}{2}$$

and $b^{\mu}, c^{\mu}, d^{\mu} < 0, \eta - c^{\mu} - (d^{\mu} + \eta)e^{b^{\mu}(T-t)} > 0.$

Proof. A proof of the results (i) and (ii) is given in the Appendix B.2 , Subsections B.2.1 and B.2.2. □

Model (I) and Model (II) are now calibrated to the survival function ${}_tp_{45}(0)$ of the 2006 German period life table originated by the Human Mortality Database (2009). We use a least-squares minimization w.r.t. the difference between the model probabilities in terms of Equation (7.10) (Equation (7.11)) and the probabilities observed in the data. The resulting parameter constellations are summarised in Subtable (7.1.1) (Subtable (7.1.2)). For example, $\lambda = 0.1$ means that, on average, there is a jump every ten years with expected jump size $E[\varepsilon] = 0.1 \cdot 9 \cdot 10^{-5} - 0.9 \cdot 3 \cdot 10^{-5} = -1.8 \cdot 10^{-5}$. The average jump sizes η_1 and η_2 are in line with the related literature.[142] This holds also for the very low estimated volatility σ^{μ} (cf. for example Luciano and Vigna (2006) or Dahl and Møller (2006)). As illustrated in Figure 7.2, both models slightly underestimate the multi-annual survival probabilities for durations between 20 and 40 years. On the contrary, they tend to overestimate the curve for survival periods less than 20 and between 40 and 60 years. The model specific expected residual lifetimes of a 45-year-old person equal 34.30 years for Model (I) and 34.28 years for Model (II). In both cases, the demographic life expectancy of 33.29 years is exceeded by around one single year. The deviations, especially in the "rectangularization area" at retirement age, can be reduced by the usage of time and age dependent parameters. However, a more accurate description of the data means cutbacks concerning an analytic formula for the survival function.

In the following, we talk about deterministic mortality whenever random mortality evolution is neglected and only uncertainty concerning the residual lifetime of the insureds is regarded.

Remark 7.3.2. Analogous to Definition (3.11) the deterministic counterpart

[142] For example, Bravo (2008) estimates the values $\eta_1 = 2.8 \cdot 10^{-2}$ and $\eta_2 = 9 \cdot 10^{-5}$ for the survival function of an 65-year-old insured and the 2004 Portuguese projected life table. Luciano and Vigna (2006) state that negative jumps are adequate to describe mortality variations. They estimate an average size of $\eta = 3 \cdot 10^{-5}$ for the calendar year 1945 and a retiree aged 65.

Benchmark parameters for the mortality models

α	β	γ	κ_1	λ	η_1	η_2	φ_1	φ_2
$5.89 \cdot 10^{-4}$	$4.46 \cdot 10^{-4}$	1.0319	0.0957	0.1	$9 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	0.1	0.9

(7.1.1): Mortality Model (I)

α	β	γ	κ_2	σ^μ
$5.89 \cdot 10^{-4}$	$4.46 \cdot 10^{-4}$	1.0319	0.0946	10^{-4}

(7.1.2): Mortality Model (II)

Table 7.1: Benchmark parameters for the mortality Models (I) and (II) based on survival data from the Human Mortality Database (2009).

Fitting results for the mortality models

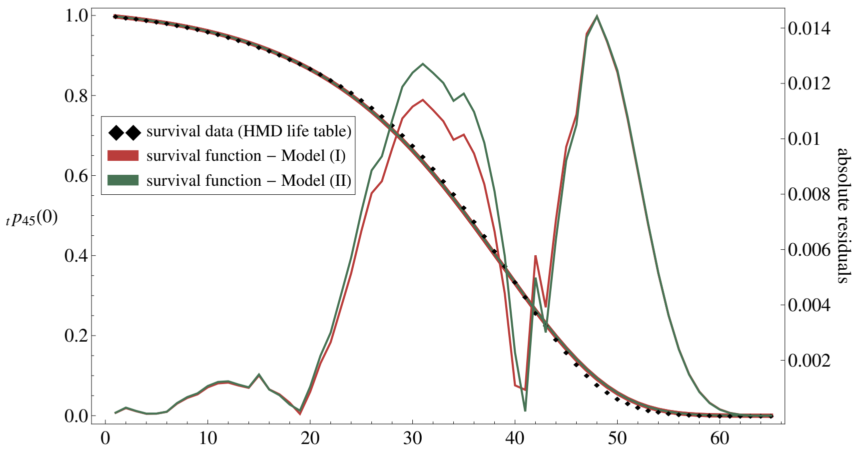


Figure 7.2: Survival functions and residuals for the mortality Models (I) and (II) based on survival data from the Human Mortality Database (2009). The figure was first published in Mahayni and Steuten (2013).

of Model (I) is defined by

$$
\begin{aligned}
\mu_{x+t} &= \frac{-\frac{d}{dt}{}_tp_x(0)}{{}_tp_x(0)} = -\frac{d}{dt}\ln\left({}_tp_x(0)\right) = -\frac{d}{dt}\ln\left(\exp\left(A_1(t) + \mu_x(0)B_1(t)\right)\right) \\
&= -\frac{d}{dt}A_1(t) - \mu_x(0)\frac{d}{dt}B_1(t) \\
&= -\lambda\left(\frac{\varphi_1}{1 - \frac{\eta_1}{\kappa_1}\left(1 - e^{\kappa_1 t}\right)} + \frac{\varphi_2}{1 + \frac{\eta_2}{\kappa_1}\left(1 - e^{\kappa_1 t}\right)} - 1\right) + \mu_x(0)\, e^{\kappa_1 t}
\end{aligned}
$$

where the force of mortality or hazard function μ_{x+t} denotes the instantaneous mortality rate at age $x + t$. The deterministic counterpart of Model (II) is

$$
\mu_{x+t} = -\frac{d}{dt}A_2(t) - \mu_x(0)\frac{d}{dt}B_2(t) = \frac{4\mu_x(0)(b^\mu)^2\, e^{b^\mu t}}{((\kappa_2 + b^\mu)(b^\mu - \kappa_2)e^{b^\mu t})^2}.
$$

The deterministic counterpart formulas are based on the closed form solutions of Lemma 7.3.1 and standard differential calculus.

7.3.2 Calibration of the interest rate model

The assumption that the interest rate dynamics are given by Equation (7.8) together with the bond pricing Equation (7.7) determines the diffusion coefficient $\sigma_{\bar{t}}(t)$ of the bond dynamics (7.6), i.e. $\sigma_{\bar{t}}(t) = \frac{\sigma_{spot}}{a}\left(1 - e^{-a(\bar{t}-t)}\right)$. The Hull-White zero coupon bond price at time t with maturity $\bar{t}$ is given by

$$D(t,\bar{t}) = \exp\left(A(t,\bar{t}) - B(t,\bar{t})r_t\right) \text{ where} \tag{7.12}$$

$$A(t,\bar{t}) = \ln\left(\frac{D(0,\bar{t})}{D(0,t)}\right) + B(t,\bar{t})\, f(0,t) - \frac{\sigma_{spot}^2}{4a}\left(B(t,\bar{t})\right)^2\left(1 - e^{-2at}\right), \tag{7.13}$$

$$B(t,\bar{t}) = \frac{1}{a}\left(1 - e^{-a(\bar{t}-t)}\right). \tag{7.14}$$

$f(0,t) = -\frac{\partial \ln(D(0,t))}{\partial t}$ denotes the current instantaneous forward rate prevailing at t. Notice that Equations (7.12)–(7.14) define the zero bond price implied by the Hull White model. The model parameters are chosen such that the model prices given by Equation (7.12) coincide with the ones observed at the market, i.e. $D(0,\bar{t}) = D^M(0,\bar{t})$. The superscript M indicates the reference to market prices (rates respectively). Notice that $D(0,\bar{t}) = D^M(0,\bar{t})$ is easily achieved by plugging the market prices D^M and the forward rates f^M into the right hand side of Equation (7.13). Hereby, fitting the initial term structure is possible for arbitrary parameters a and σ_{spot}. The time-dependent (interest rate) model parameter $\theta(t)$ is then given as in Table 7.3. We obtain $D^M(0,\bar{t})$ as follows. We use financial market data in the form of current Euro London Interbank Offered Rate (LIBOR) and swap rates to calculate the short-term LIBOR and bootstrapped swap discount factors. The joined zero coupon or yield curve $D^M(0,\bar{t})$ is interpolated and extrapolated logarithmically for maturities $\bar{t} > 30$. Then, the forward rate function $f^M(0,\bar{t}) = -\frac{\partial \ln(D^M(0,\bar{t}))}{\partial \bar{t}}$ is calculated. The resulting term structure is presented in Figure 7.3.

The calibration procedure is completed by an estimation of the speed of mean reversion parameter a and the spot rate volatility σ_{spot} using implied at-the-money[143] European forward swaption volatilities. More precisely, we minimize the associated sum of squared errors w.r.t. the at-the-money swaption straddle prices. The market prices are given by the Black formula and the model prices by the Hull-White formula. The formula are, for example, given in Brigo and Mercurio (2007). The calibration results are illustrated in Figure 7.4. For the sake of completeness, the market straddle prices (implied volatilities

[143] A swaption is called at-the-money if its strike equals the forward par swap rates.

respectively) and the calibration errors are given in Table 7.2. Similar to the

Swaption straddle implied volatility quotes and calibration results

		t	e	n	o	r					
		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
e	1Y	60.5 (64.%)	44.3 (39.%)	38.3 (14.%)	34.1 (0.%)	31.3 (-10.%)	29.2 (-17.%)	27.6 (-22.%)	26.5 (-26.%)	25.7 (-29.%)	24.7 (-31.%)
x	2Y	43.5 (8.%)	34.2 (5.%)	30.8 (-5.%)	28.6 (-13.%)	27.1 (-19.%)	25.9 (-24.%)	25.0 (-27.%)	24.3 (-30.%)	23.7 (-32.%)	23.3 (-34.%)
p	3Y	32.4 (-7.%)	27.4 (-7.%)	25.6 (-13.%)	24.3 (-18.%)	23.4 (-22.%)	22.8 (-26.%)	22.3 (-29.%)	21.8 (-30.%)	21.4 (-32.%)	21.2 (-34.%)
i	4Y	26.2 (-13.%)	23.5 (-14.%)	22.4 (-18.%)	21.5 (-21.%)	21.0 (-24.%)	20.5 (-26.%)	20.2 (-28.%)	19.9 (-30.%)	19.7 (-32.%)	19.6 (-33.%)
r	5Y	22.6 (-16.%)	20.8 (-16.%)	20.2 (-20.%)	19.8 (-23.%)	19.4 (-25.%)	19.1 (-27.%)	18.9 (-28.%)	18.7 (-30.%)	18.6 (-31.%)	18.5 (-32.%)
y	7Y	18.4 (-17.%)	17.8 (-18.%)	17.6 (-20.%)	17.4 (-22.%)	17.3 (-24.%)	17.1 (-25.%)	17.1 (-27.%)	17.1 (-28.%)	17.2 (-30.%)	17.3 (-31.%)
	10Y	15.7 (-13.%)	15.4 (-13.%)	15.6 (-16.%)	15.7 (-18.%)	15.9 (-21.%)	16.0 (-23.%)	16.2 (-25.%)	16.4 (-27.%)	16.7 (-29.%)	16.9 (-31.%)

Table 7.2: Euro at-the-money swaption straddle implied volatility quotes taken from Thomson Reuters (2010) with date 14/06/10. The percentage differences of the model and the market swaption volatilities are given in brackets.

results of Park (2004) we observe quite stable volatility estimators. In contrast, the speed of mean reversion parameter is sensitive to changes in the market data as well as in the start values for the root search algorithm. In order to control the long run mean and the negative spot rate behaviour, we restrict the optimization by means of reasonable constraints. Inter alia, we want to ensure that the forward rate and expected interest rate curve exhibit similar long term shapes which is illustrated in Figure 7.5. The mean reversion level $\frac{\theta(t)}{a}$ ensures that the model implied volatility declines with maturity. In practice, however, swaption volatility surfaces often show a characteristic hump between short and medium maturity terms leading to characteristic deviations[144] around this region. The estimated model parameters are summarised in Table 7.3.

Market instantaneous forward rates and discount factors

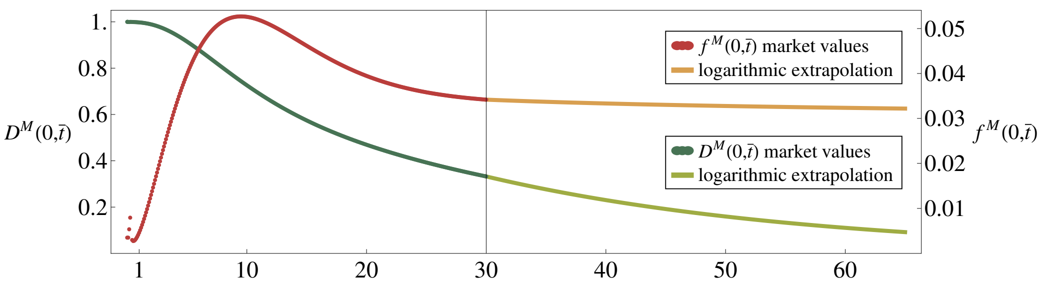
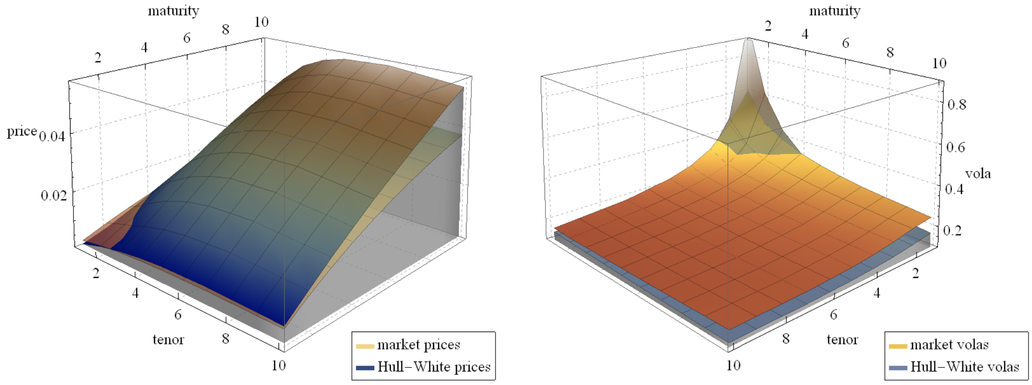


Figure 7.3: Yield curve and term structure of the market discount factors (valuation date 16/06/10) stripped from market LIBOR and Euro swap rates from the CLP Structured Finance (2010) and the Financial Times (2010). The figure was first published in Mahayni and Steuten (2013).

It is worth mentioning that, in the case of a sensitivity analysis in Section 7.4 we vary with respect to the spot rate volatility σ_{spot}, the adjustment in $\theta(t)$

[144] Relief is provided by introducing a time-varying volatility $\sigma_{spot}(t)$ to get a better trade-off between application and model stationarity.

Swaption price surface and implied volatility surface



(7.4.1) Swaption prices **(7.4.2)** Implied volatilities

Figure 7.4: At-the-money European swaption price surface and implied volatility surface for different swap expiries and tenors. The data is taken from Reuters / ICAP: VCAP1 on 16/06/10. The figure was first published in Mahayni and Steuten (2013).

Long run behaviour of the spot rate process

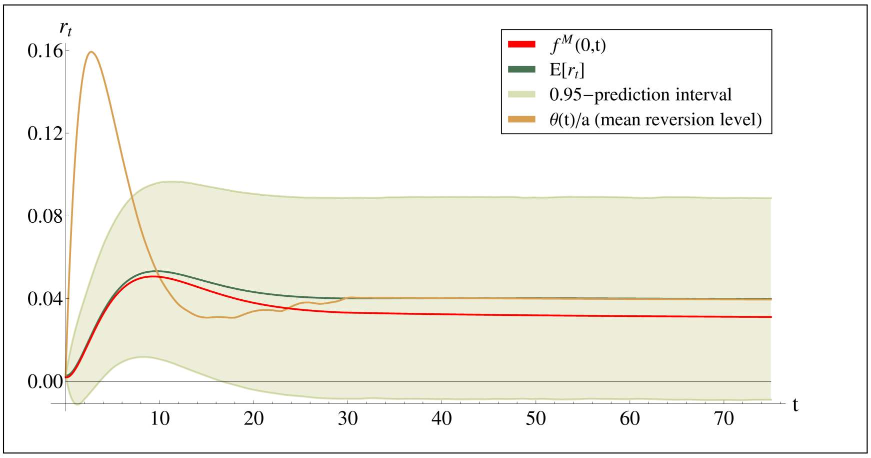


Figure 7.5: Long run behaviour of the spot rate process (7.8) using the (estimated) parameters given in Table 7.3. The figure was first published in Mahayni and Steuten (2013).

Benchmark parameters for the Hull-White model

$r_0^M = f^M(0,0)$	a	σ_{spot}	$\theta(t)$
0.0028	0.0709	0.0094	$\frac{\partial f^M(0,t)}{\partial t} + a\, f^M(0,t) + \frac{\sigma_{spot}^2}{2a}\left(1 - e^{-2at}\right)$

Table 7.3: Benchmark parameters for the Hull-White interest rate process.

still ensures consistency with the current interest rate structure. However, the implied swaption volatilities may be different. $\sigma_{spot} \to 0$ is interpreted as the deterministic counterpart of the model given by Equation (7.8), i.e. $r_t = f^M(0,t)$ for all $t \in [0,\bar{T}]$. For simplicity, we use the notation $\sigma_{spot} = 0$ instead of $\sigma_{spot} \to 0$.

7.4 Variance analysis

7.4.1 Distribution of the portfolio values

We consider a cohort which is currently aged $x = 45$, a deferment period of $T = 20$ years and an contract expiration after $\bar{T} = 75$ years. The advanced regular annuity payments amount to $c = 1$. The periodic premiums π are given by the principle of equivalence which implies that at time $t = 0$ the discounted benefits and liabilities coincide in expectation, i.e. $E[Z_0^\Pi] = E[X_0^\Pi]$. For $t \in \{0, \dots, T-1\}$, the expected discounted portfolio values are given by

$$E[X_t^\Pi|\mathcal{F}_t] = E\left[\pi \sum_{i=0}^{T-1} \frac{N_i}{\beta_{t,i}} \middle| \mathcal{F}_t\right] = \pi \sum_{i=0}^{t} N_i\, \beta_{i,t} + \pi\, N_t \sum_{i=t+1}^{T-1} D(t,i)\ {}_{i-t}p_{x+t}(t), \tag{7.15}$$

$$E[Z_t^\Pi|\mathcal{F}_t] = E\left[c \sum_{i=T}^{\bar{T}-1} \frac{N_i}{\beta_{t,i}} \middle| \mathcal{F}_t\right] = cN_t \sum_{i=T}^{\bar{T}-1} D(t,i)\ {}_{i-t}p_{x+t}(t). \tag{7.16}$$

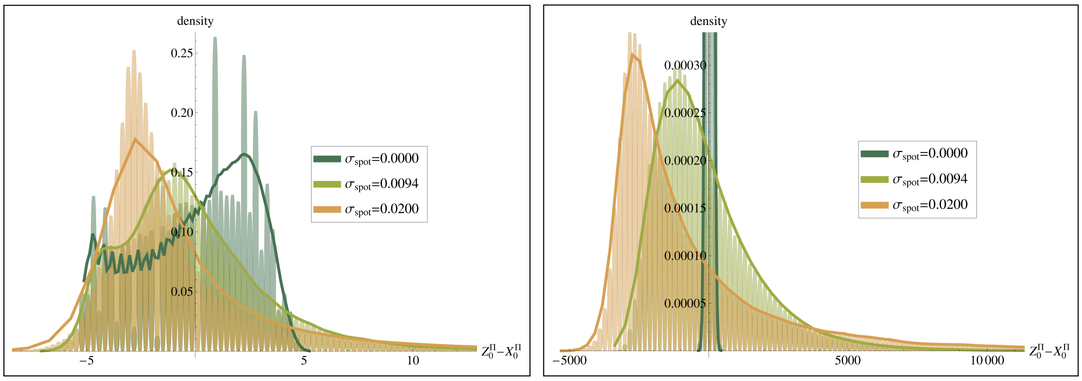
In particular, for $t = 0$ the expected discounted portfolio values are specified in terms of the current zero bond prices $D(0,\bar{t})$ ($\bar{t} = 1,\dots,\bar{T}-1$). Further moments of the present values depend on the assumptions which are posed on the zero bond volatilities (the spot rate volatilities respectively). Since all mortality and interest rate models are calibrated to the same data, the periodic premium π varies only slightly due to the calibration errors immanent w.r.t. the mortality data. The mortality Model (I) gives $\pi = 0.349$ and Mortality Model (II) implies $\pi = 0.348$.

First, we consider the effect of different interest rate volatilities σ_{spot} on the distribution of the present portfolio premium income value X_0^Π, the present portfolio benefit value Z_0^Π and the present portfolio underwriting loss $Z_0^\Pi - X_0^\Pi$. Figure 7.6 illustrates the densities which are approximated by Monte Carlo simulations. The arithmetic means, sample variances[145] and coefficients of

[145] Notice that in the following the expectation is given in analytic closed-form, cf. Equations (7.15) and (7.16). The variance has a semi-analytic solution since the calculation of higher moments of the residual lifetime requires Monte Carlo approximation.

variation (which are marked with a hat) are summarised in Table 7.4. Due to the model calibration, the expected discounted values are, except for the estimation inaccuracies, unaffected by changes in the spot rate volatility. Intuitively, it is clear that a "full stochastic model setup" exhibits much more deviation than the deterministic interest rate case ($\sigma_{spot} = 0$). From Subfigure (7.6.1) we observe that the distribution of the single policy loss is left skewed for the deterministic counterpart model ($\sigma_{spot} = 0$) but it is right skewed if the spot rate is stochastic. However, this effect is explained by a small cohort size (see Subfigure (7.6.2)). Central limit arguments justify that the risk profile can be measured in terms of the variance (per policy) if the cohort size is sufficiently large.

Densities of the discounted portfolio loss value



(7.6.1) Cohort size $N_0 = 1$ **(7.6.2)** Cohort size $N_0 = 1000$

Figure 7.6: Probability density of the discounted portfolio loss value $Z_0^\Pi - X_0^\Pi$ (mortality Model (I)). The histograms are based on $2 \cdot 10^4$ drawn samples. The solid lines represent the Gaussian kernel density estimators. The figure was first published in Mahayni and Steuten (2013).

Moments of the portfolio values and the portfolio loss

σ_{spot}	mortality Model (I)							
	$\hat{E}[X_0^\Pi]$	$\hat{V}ar[X_0^\Pi]$	$\hat{CV}[X_0^\Pi]$	$\hat{E}[Z_0^\Pi]$	$\hat{V}ar[Z_0^\Pi]$	$\hat{CV}[Z_0^\Pi]$	$\hat{E}[Z_0^\Pi - X_0^\Pi]$	$\hat{V}ar[Z_0^\Pi - X_0^\Pi]$
0.0000	4.94 (4937)	0.45 (452)	0.14 (0.00)	4.96 (4973)	8.11 (9421)	0.57 (0.02)	0.03 (36)	6.68 (7863)
0.0094	4.98 (4973)	0.70 (244463)	0.17 (0.10)	4.85 (4835)	14.86 $(4.97 \cdot 10^6)$	0.79 (0.46)	-0.13 (-137)	11.93 $(3.52 \cdot 10^6)$
0.0200	4.97 (4971)	1.69 $(1.19 \cdot 10^6)$	0.26 (0.22)	4.78 (4838)	53.23 $(3.53 \cdot 10^7)$	1.53 (1.23)	-0.19 $(-133.)$	44.03 $(2.76 \cdot 10^7)$

σ_{spot}	mortality Model (II)							
	$\hat{E}[X_0^\Pi]$	$\hat{V}ar[X_0^\Pi]$	$\hat{CV}[X_0^\Pi]$	$\hat{E}[Z_0^\Pi]$	$\hat{V}ar[Z_0^\Pi]$	$\hat{CV}[Z_0^\Pi]$	$\hat{E}[Z_0^\Pi - X_0^\Pi]$	$\hat{V}ar[Z_0^\Pi - X_0^\Pi]$
0.0000	4.93 (4931)	0.45 (452)	0.14 (0.00)	4.97 (4967)	8.07 (9817)	0.57 (0.02)	0.04 (36)	6.65 (8232)
0.0094	4.97 (4968)	0.70 (245903)	0.17 (0.10)	4.85 (4831)	15.06 $(4.96 \cdot 10^6)$	0.80 (0.46)	-0.12 (-137)	12.08 $(3.50 \cdot 10^6)$
0.0200	4.97 (4969)	1.71 $(1.19 \cdot 10^6)$	0.26 (0.22)	4.84 (4821)	69.05 $(3.42 \cdot 10^7)$	1.72 (1.21)	-0.12 (-148)	59.07 $(2.65 \cdot 10^7)$

Table 7.4: Moments of the portfolio values for different spot rate volatilities w.r.t. an individual aged $x = 45$, a deferment period $T = 20$, a contract duration $\bar{T} = 75$ and a cohort size $N_0 = 1$. The values implied by a cohort size of $N_0 = 1000$ are given in brackets. The results are based on 10^5 simulation runs.

7.4.2 Variance decomposition

The law of large numbers is the core of the insurance principle. In an ideal world where the insurance company faces independent and homogeneous policies, the distribution of the liabilities per policy becomes more stable if the insurance cohort increases. Without the introduction of stochastic mortality rates, the insureds outliving their expected lifetime are matched by the ones who do not live as long as expected. In this case, the variance per policy vanishes for a sufficiently large cohort size. In contrast, the risk pooling breaks down if one takes into account stochastic mortality. Assuming that the law of the residual lifetime is stochastic implies a dependence structure. While it is still assumed that the survival of one insured does not influence the survival probability of another insured, a change of the mortality law effects the survival probabilities of the whole insurance cohort at the same time. A variance decomposition is a convenient way to quantify this effect. The pooling part is described by the expected conditional variance, i.e. the variance which is associated with a particular mortality law. The remaining variance part is then caused by the dispersion (variance) of the expected values due to the different (possible) mortality laws. This gives the non-pooling risk. To be more precise, the above is achieved by conditioning on the mortality information which prevails at the terminal date $\bar{T}$. It includes the mortality laws.

In general, the variance decomposition can be based on different information sets and/or random variables. Recall that $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\bar{T}]}$ is the overall information structure. The information $\mathcal{F}_t$ contains the information about the cohort sizes, the mortality rates and the interest rates up to time t. The subfiltrations $\mathcal{I}_t$ ($\mathcal{M}_t$) refer to the interest rate (mortality rate) information until time t, only. After $\bar{T}$ years, when all annuities cease to exist, $\mathcal{M}_{\bar{T}}$ contains the (whole) information about the mortality rate evolution but not the numbers of cohort sizes N_t for $t = 0, \dots \bar{T}$ (or the interest rate information). Thus, in the case of deterministic interest rates, the variance decomposition from the conditioning on $\mathcal{M}_{\bar{T}}$ allows to split the overall variance into a pooling (p) and a non-pooling (np) part which is due to stochastic mortality. However, in the case of stochastic interest rates, the interpretation of the variance decomposition is more demanding. There is also a non-pooling risk part stemming from the interest rate dynamics. Since the interest rate model is complete, i.e. the interest risk is a traded risk, we adjust for this effect by conditioning on the combined information consisting of the mortality and interest rate information $\mathcal{G}_{\bar{T}} := \mathcal{I}_{\bar{T}} \vee \mathcal{M}_{\bar{T}}$. For $t \in \{0, \dots, \bar{T}-1\}$ and a portfolio value $PV_t \in \{X_t^{\Pi}, Z_t^{\Pi}, Z_t^{\Pi} - X_t^{\Pi}\}$, the variance decomposition is

$$Var\left[PV_t|\,\mathcal{F}_t\right] = E\left[Var\left[\,PV_t|\,\mathcal{G}_{\bar{T}}\right]|\,\mathcal{F}_t\right] + Var\left[\,E\left[PV_t|\,\mathcal{G}_{\bar{T}}\right]|\,\mathcal{F}_t\right]. \tag{7.17}$$

Recall that a pooling risk is defined by the part of the per policy variance which vanishes if the cohort size becomes large, i.e. if for some σ-algebra $\mathcal{A}$ it holds

$$\lim_{N_t\to\infty} E\left[Var\left[\left.\frac{PV_t}{N_t}\right|\mathcal{A}\right]\middle|\,\mathcal{F}_t\right] = 0.$$

In the case of stochastic mortality and interest rates, the above is true if (and only if) we use $\mathcal{A} = \mathcal{G}_{\bar{T}}$, i.e. if we condition on both the mortality and the interest rate information. We use the notation Var_t^{p} to denote the first summand of the right hand side of Equation (7.17) and Var_t^{np} for the second one. Notice that for $t = 0$ we can omit $\mathcal{F}_0$ such that

$$Var_0^{\mathrm{p}} = E\left[Var\left[\left.\frac{PV_0}{N_0}\right|\mathcal{G}_{\bar{T}}\right]\right],\; Var_0^{\mathrm{np}} = Var\left[E\left[\left.\frac{PV_0}{N_0}\right|\mathcal{G}_{\bar{T}}\right]\right]. \tag{7.18}$$

Remark 7.4.1 (Numerical determination of pooling and non-pooling variance). The variance parts are approximated by nested Monte Carlo simulations. The first and second moments are approximated by the average values of the simulation. The conditioning on the (sub-) information about the mortality and the interest rates implies that we need to simulate the interest and the mortality rate path in an outer Monte Carlo simulation. For each simulated mortality and interest rate path, we also need an inner simulation. The interest rate path immediately gives the discount factor which is needed. The present value is also based on the number of livings at each subsequent annuity (reference) date. These numbers are simulated according to the mortality laws (mortality rate path) implied by the outer simulation, cf. Equation (7.3).

In Table 7.5 we compare the variance (per policy) implied by the mortality Model (I) and (II) and their deterministic counterpart models. Thus, one can argue that we compare the reality in the sense of the mortality Model (I) and (II) with the simplified deterministic setup. Obviously, the pooling risk decreases in the cohort size while the non-pooling variance per policy increases. Observe that a cohort size of $N_0 = 2 \cdot 10^4$ is large enough to pool the risk. Here, the remaining variance is exclusively given in terms of the non-pooling variance. The risk is substantial. According to Table 7.5, Model (I) gives a per policy variance of 14.13. A similar result is true for Model (II). However, the largest part of the non-pooling risk stems from the interest rate side. Compared to the stochastic mortality setup, the deterministic mortality counterpart still implies a non-pooling variance of 13.53, i.e. 96% of the non-pooling risk is due to the stochastic interest rates. The interest rate effect is further emphasised in Table

Influence of the cohort size on the variance (per policy)

N_0	mortality Model (I)			mortality Model (II)		
	$Var\left[\frac{Z_0^{\Pi}-X_0^{\Pi}}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]	$Var\left[\frac{Z_0^{\Pi}-X_0^{\Pi}}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]
1	27.24 (27.24)	48.13 (50.34)	51.87 (49.66)	27.21 (27.21)	47.86 (50.41)	52.14 (49.59)
10	15.44 (14.90)	8.49 (9.20)	91.51 (90.80)	15.49 (14.86)	8.41 (9.23)	91.59 (90.77)
10^2	14.26 (13.66)	0.92 (1.00)	99.08 (99.00)	14.32 (13.63)	0.91 (1.01)	99.09 (98.99)
10^3	14.14 (13.54)	0.09 (0.10)	99.91 (99.90)	14.20 (13.51)	0.09 (0.10)	99.91 (99.90)
10^4	14.13 (13.53)	0.01 (0.01)	99.99 (99.99)	14.19 (13.49)	0.01 (0.01)	99.99 (99.99)
10^5	14.13 (13.53)	0.00 (0.00)	100.00 (100.00)	14.19 (13.49)	0.00 (0.00)	100.00 (100.00)
10^6	14.13 (13.53)	0.00 (0.00)	100.00 (100.00)	14.19 (13.49)	0.00 (0.00)	100.00 (100.00)

Table 7.5: Influence of the cohort size on the variance (per policy) and variance decomposition w.r.t. an individual aged $x = 45$ and a deferment period of $T = 20$ years. The simulations are based on $5 \cdot 10^5$ iterations. The values implied by the deterministic counterpart mortality models are given in brackets.

7.6 which gives the non-pooling risk for varying spot rate volatilities. Here, the cohort size $N_0 = 2 \cdot 10^4$ is large enough such that the pooling variance part is close to zero. Model (I) and (II) are combined with stochastic interest rate models which differ[146] w.r.t. their spot rate volatility σ_{spot}. The results are very similar in the case of mortality Model (I) and (II). In both cases, the non-pooling variance per policy increases exponentially with increasing interest rate volatility. We further study the effect of postponing the deferment period T and thus a reduction of the decumulation phase (cf. Table 7.7). The non-pooling variance per policy therefore decreases almost linearly with increasing deferment period. The following illustrations 7.8 and 7.9 address the issue of dividing the non-pooling variance per policy into components regarding both the systematic interest and mortality rate risk.

7.4.3 Detailed variance decomposition

This subsection concerns the question how to divide the non-pooling variance further with respect to the interest and the mortality risk. Following the approach of Christiansen and Helwich (2008), we consider a causation-oriented variance decomposition of the portfolio value PV_t into

$$PV_t = E\left[PV_t|\, \mathcal{G}_{\bar{T}}\right] + \left(PV_t - E\left[PV_t|\, \mathcal{G}_{\bar{T}}\right]\right) \tag{7.19}$$

[146] Recall that, according to Table 7.6, only $\sigma_{spot} = 0.0094$ is consistent with the swaption straddle volatilities. However, for varying σ_{spot}, we nevertheless calibrate to the same initial term structure of interest, i.e. we use the function $\theta(t)$ as in Table 7.3. Thus, we still have the same expected values, independent of the choice of σ_{spot}.

Influence of the interest rate volatility on the variance (per policy)

$\boldsymbol{\sigma_{spot}}$	mortality Model (I)			mortality Model (II)		
	$Var\left[\frac{Z_0^\Pi - X_0^\Pi}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]	$Var\left[\frac{Z_0^\Pi - X_0^\Pi}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]
0.0000	0.38 (0.00)	0.11 (100.00)	99.89 (0.00)	0.44 (0.00)	0.09 (100.00)	99.91 (0.00)
0.0050	3.70 (3.27)	0.01 (0.01)	99.99 (99.99)	3.76 (3.26)	0.01 (0.01)	99.99 (99.99)
0.0094	14.13 (13.53)	0.00 (0.01)	100.00 (99.99)	14.19 (13.49)	0.00 (0.01)	100.00 (99.99)
0.0150	52.77 (51.50)	0.00 (0.00)	100.00 (100.00)	52.83 (51.37)	0.00 (0.00)	100.00 (100.00)
0.0200	166.60 (163.13)	0.00 (0.00)	100.00 (100.00)	166.65 (162.70)	0.00 (0.00)	100.00 (100.00)

Table 7.6: Influence of the interest rate volatility on the variance (per policy) and variance decomposition w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$ years and cohort size $N_0 = 20000$. The simulations are based on $5 \cdot 10^5$ iterations. The values implied by the deterministic counterpart mortality models are given in brackets.

Influence of the deferment period on the variance (per policy)

$\boldsymbol{T}$	mortality Model (I)			mortality Model (II)		
	$Var\left[\frac{Z_0^\Pi - X_0^\Pi}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]	$Var\left[\frac{Z_0^\Pi - X_0^\Pi}{N_0}\right]$	Var_0^{p} [%]	Var_0^{np} [%]
15	19.09 (18.27)	0.00 (0.00)	100.00 (100.00)	19.18 (18.23)	0.00 (0.00)	100.00 (100.00)
20	14.13 (13.53)	0.00 (0.01)	100.00 (99.99)	14.19 (13.49)	0.00 (0.01)	100.00 (99.99)
25	8.62 (8.21)	0.01 (0.01)	99.99 (99.99)	8.65 (8.18)	0.01 (0.01)	99.99 (99.99)
30	4.16 (3.93)	0.01 (0.01)	99.99 (99.99)	4.18 (3.91)	0.01 (0.01)	99.99 (99.99)

Table 7.7: Influence of the deferment period on the variance (per policy) and variance decomposition w.r.t. an individual aged $x = 45$ and cohort size $N_0 = 20000$. The simulations are based on $5 \cdot 10^5$ iterations. The values implied by the deterministic counterpart mortality models are given in brackets.

where the first addend refers to the interest risk and systematic mortality risk and the second addend determines the unsystematic mortality risk. Taking the expectation on both sides of Equation (7.19) conditioned on information $\mathcal{G}_{\bar{T}}$ yields

$$E\left[PV_t|\,\mathcal{G}_{\bar{T}}\right] = E\left[PV_t|\,\mathcal{M}_{\bar{T}}\right] + \left(E\left[PV_t|\,\mathcal{G}_{\bar{T}}\right] - E\left[PV_t|\,\mathcal{M}_{\bar{T}}\right]\right). \tag{7.20}$$

By means of this separation we isolate the systematic mortality risk and the interest rate risk in the bracket term. Taking the variance on both sides of (7.19) and using Expression (7.20) for the first addend we obtain a variance decomposition

$$\begin{aligned} Var\left[PV_t|\,\mathcal{F}_t\right] &= Var\left[E\left[PV_t|\,\mathcal{M}_{\bar{T}}\right]|\,\mathcal{F}_t\right] + E\left[Var\left[E\left[PV_t|\,\mathcal{G}_{\bar{T}}\right]|\,\mathcal{M}_{\bar{T}}\right]|\,\mathcal{F}_t\right] \\ &\quad + E\left[Var\left[PV_t|\,\mathcal{G}_{\bar{T}}\right]|\,\mathcal{F}_t\right] =: Var_t^{\mathrm{smr}} + Var_t^{\mathrm{sir}} + Var_t^{\mathrm{umr}}. \end{aligned}$$

In doing so, we separate the risk components for systematic mortality risk Var^{sm}, interest risk Var^{sir} and unsystematic mortality risk Var^{umr}. In this context the first two terms describe non-pooling risk whereas the third part equals the pooling variance part Var^{p}. Results for different cohort sizes and spot rate volatilities are listened in Tables 7.8 and 7.9. In the case of approximative diversification ($N_0 = 2 \cdot 10^4$) the absolute values for both systematic mortality (smr) and interest rate risk (sir) per policy exhibit absolute positive limits 0.38 and 13.75 for Model (I) as well as 0.44 and 13.75 for Model (II). Therefore, the systematic mortality risk part constitutes only 3% of the systematic financial risk which is subjected to considerably stronger fluctuations during the contract duration. This fact has already been mentioned in the former Subsection 7.4.2. From Table 7.9 we observe that for increasing interest rate diffusion systematic mortality and interest risk develop symmetrically inclined in relation to each other. Thus, the systematic deviations in the mortality rate evolution have a visible partial impact for volatilities $\sigma_{spot} < 0.0050$, i.e. for "calm" interest rate markets with particularly low volatility. Otherwise, the financial market risk constitutes the determining factor. The goodness of the approximation is measured by the standard errors implied in the variance approximation which are at least less than $2 \cdot 10^{-6}$ and therefore negligible small.

Influence of the cohort size on the variance (per policy) parts

	mortality Model (I)				mortality Model (II)			
N_0	$Var[Z_0^{II} - X_0^{II}]$	Var_0^{umr} [%]	Var_0^{sir} [%]	Var_0^{smr} [%]	$Var[Z_0^{II} - X_0^{II}]$	Var_0^{umr} [%]	Var_0^{sir} [%]	Var_0^{smr} [%]
1	27.24 (27.24)	48.13 (50.34)	50.48 (49.66)	1.40 (0.00)	27.21 (27.21)	47.86 (50.41)	50.52 (49.59)	1.62 (0.00)
10	15.44 (14.90)	8.49 (9.20)	89.04 (90.80)	2.47 (0.00)	15.49 (14.86)	8.41 (9.23)	88.75 (90.77)	2.85 (0.00)
10^2	14.26 (13.66)	0.92 (1.00)	96.41 (99.00)	2.67 (0.00)	14.32 (13.63)	0.91 (1.01)	96.01 (98.99)	3.08 (0.00)
10^3	14.14 (13.54)	0.09 (0.10)	97.21 (99.90)	2.69 (0.00)	14.20 (13.51)	0.09 (0.10)	96.80 (99.90)	3.10 (0.00)
10^4	14.13 (13.53)	0.01 (0.01)	97.29 (99.99)	2.70 (0.00)	14.19 (13.49)	0.01 (0.01)	96.88 (99.99)	3.11 (0.00)
10^5	14.13 (13.53)	0.00 (0.00)	97.30 (100.00)	2.70 (0.00)	14.19 (13.49)	0.00 (0.00)	96.89 (100.00)	3.11 (0.00)
10^6	14.13 (13.53)	0.00 (0.00)	97.30 (100.00)	2.70 (0.00)	14.19 (13.49)	0.00 (0.00)	96.89 (100.00)	3.11 (0.00)

Table 7.8: Influence of the cohort size on the different (un)systematic variance (per policy) parts w.r.t. an individual aged $x = 45$, a deferment period $T = 20$. The simulations are based on $5 \cdot 10^5$ iterations. The values implied by the deterministic counterpart mortality models are given in brackets.

Influence of the spot rate volatility on the variance (per policy) parts

	mortality Model (I)				mortality Model (II)			
σ_{spot}	$Var[Z_0^{II} - X_0^{II}]$	Var_0^{umr} [%]	Var_0^{sir} [%]	Var_0^{smr} [%]	$Var[Z_0^{II} - X_0^{II}]$	Var_0^{umr} [%]	Var_0^{sir} [%]	Var_0^{smr} [%]
0.0000	0.38 (0.00)	0.11 (100.00)	0.00 (0.00)	99.89 (0.00)	0.44 (0.00)	0.09 (100.00)	0.00 (0.00)	99.91 (0.00)
0.0050	3.70 (3.27)	0.01 (0.01)	89.69 (99.99)	10.30 (0.00)	3.76 (3.26)	0.01 (0.01)	88.26 (99.99)	11.73 (0.00)
0.0094	14.13 (13.53)	0.00 (0.01)	97.30 (99.99)	2.70 (0.00)	14.19 (13.49)	0.00 (0.01)	96.89 (99.99)	3.11 (0.00)
0.0150	52.77 (51.50)	0.00 (0.00)	99.28 (100.00)	0.72 (0.00)	52.83 (51.37)	0.00 (0.00)	99.16 (100.00)	0.83 (0.00)
0.0200	166.60 (163.13)	0.00 (0.00)	99.77 (100.00)	0.23 (0.00)	166.65 (162.70)	0.00 (0.00)	99.73 (100.00)	0.26 (0.00)

Table 7.9: Influence of the spot rate volatility on the different (un)systematic variance (per policy) parts w.r.t. an individual aged $x = 45$, a deferment period $T = 20$ and cohort size $N_0 = 20000$. The simulations are based on $5 \cdot 10^5$ iterations. The values implied by the deterministic counterpart mortality models are given in brackets.

7.5 Pricing effects

7.5.1 Pricing principles

Now, we consider three different pricing principles to determine the periodic premium π. Namely, the principle of equivalence or net premium (NP) principle, the zero expected utility or indifference principle (IP) and the quantile principle (QP).

The principle of equivalence

According to the principle of equivalence, the periodic premiums are determined such that the present value of the expected discounted benefits and the present value of the expected discounted contributions coincide, i.e.

$$\mathrm{NP} = \left\{\pi \in \mathbb{R}_+ \middle| E\left[Z_0^{\Pi}\right] = E\left[X_0^{\Pi}(\pi)\right]\right\}.$$

Notice that the principle of equivalence is consistent with the deterministic mortality counterpart models, only. In this case, the mortality risk is a pooling risk and the remaining interest rate risk is a traded risk. In general, this premium principle serves only as a benchmark for a premium principle which also takes into account for non-pooling and non-tradeable risk. The principle of equivalence defines a lower bound for the premium. In other words, any meaningful pricing principle includes a premium loading on the equivalence premium.

The expected-utility principle

One possibility to address the pricing of risks which can neither be diversified nor traded is based on (expected) utility. In the above context, a suitable approach is to compare the expected utility in the case of underwriting and not underwriting an annuity cohort. The indifference premium is indirectly defined by the condition

$$\mathrm{IP} = \inf\left\{\pi \in \mathbb{R}_+ \mid E\left[u\left(-L_{\bar{T}}(\pi)\right)\right] = E\left[u\left(S_0\,\beta_{0,\bar{T}}\right)\right]\right\} \tag{7.21}$$

where $L_{\bar{T}}(\pi)$ denotes the insurance loss at time $\bar{T}$, u denotes a non-decreasing, concave utility function[147] describing the insurers risk aversion behaviour and S_0

[147] A common choice for the utility function is exponential utility due to its simplicity. The indifference pricing approach under exponential utility was already applied by Ludkovski and Young (2008) and Hainant and Devolder (2008) for pure endowment and (temporary) life annuity contracts.

denotes some starting capital. Consequently, the indifference premium defines an individual periodic premium at which the insurer is indifferent between selling or not selling a police.

Definition 7.5.1 (Indifference pricing under DHARA utility). Let K^* denote the terminal wealth of the annuity provider. Along the lines of Pelsser (2005), we assume that the insurers final utility is characterised by a double hyperbolic absolute risk aversion (DHARA) function [148] with

$$u(K^*) = \begin{cases} -\frac{\rho_2}{\rho_1 - 1}\left(1 + \frac{K^*}{\rho_2}\right)^{1-\rho_1} & \text{for } K^* \geq 0 \\ -\frac{\rho_2}{\rho_1 + 1}\left(1 - \frac{K^*}{\rho_2}\right)^{1+\rho_1} - \frac{2\rho_2}{\rho_1^2 - 1} & \text{for } K^* < 0 \end{cases} \qquad \text{and } \rho_1, \rho_2 > 0.$$

In particular, the Arrow-Pratt measure of absolute risk aversion[149] A is given by

$$A(K^*) := -\frac{u''(K^*)}{u'(K^*)} = \left\{ \begin{array}{ll} \frac{\rho_1}{\rho_2 + K^*} & \text{for } K^* \geq 0 \\ \frac{\rho_1}{\rho_2 - K^*} & \text{for } K^* < 0 \end{array} \right\} > 0.$$

There are a two parameters ρ_1 and ρ_2 denoting adjustments of the utility function in the ordinate and abscissa respectively.

As an alternative to the DHARA utility function actuarial literature commonly applies a utility function out of the constant absolute risk aversion (CARA) class although it might be not very realistic for a market-consistent valuation of insurance liabilities.

Definition 7.5.2 (Indifference pricing under CARA utility). The insurers final utility is characterised by the exponential utility function

$$u(K^*) = -\frac{1}{\rho}\exp\left(-\rho K^*\right)$$

which has a simple functional form as it depends on the Arrow-Pratt coefficient of the absolute risk aversion $A(K^*) = \rho$ only. Thus, the value of the insurance contract is independent of the initial wealth of the pension provider and the effect of risk aversion on the annuity price can be isolated. Under the assumption

[148] According to Pelsser (2005), a market-consistent valuation of insurance liabilities demands a utility function class fulfilling certain requirements:

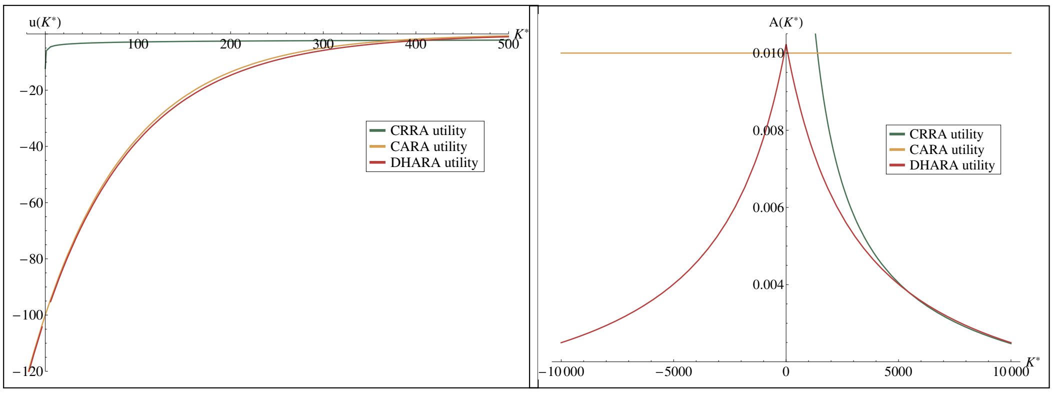
- The probability of negative wealth respectively positive loss at maturity is small but positive. Therefore, we require $0 < \lim_{K^* \to \pm 0} A(K^*) = \frac{\rho_1}{\rho_2} < \infty$.
- The insurer aims at avoiding underwriting loss such that risk aversion increases finitely for terminal wealth $K^* < 0$ approaching zero.
- The risk aversion decreases asymptotically for $K^* > 0$ with $\lim_{K^* \to \infty} \inf A(K^*) = 0$.

[149] The Arrow-Pratt measure is named after the economists Pratt (1964) and Arrow (1965). For a terminal wealth K^* the partial derivatives $u'(K^*)$ and $u''(K^*)$ denote the marginal utility and the curvature of the utility function respectively.

that $S_0 = 0$, i.e. no additional solvency loading to buffer a shortfall in the liability payment, the premium principle (7.21) is identical to the principle of zero utility.

An illustration of the constant relative risk aversion (CRRA), the CARA and DHARA utility functions is given in Figure 7.7. The CARA and CRRA utility frameworks are less suitable since they fail to meet all requirements for a market-consistent valuation of insurance liabilities. Basically, a DHARA utility function combines the characteristics of negative wealth from the CARA functions (cf. Subfigure (7.7.1)) together with decreasing absolute risk aversion behaviour for a positive wealth (cf. Subfigure (7.7.2)) from the CRRA class.

CARA, CRRA and DHARA utility functions and associate absolute risk aversion



(7.7.1) Utility functions **(7.7.2)** Absolute risk aversion

Figure 7.7: CARA, CRRA and DHARA utility functions with associated absolute risk aversion. The figure was first published in Mahayni and Steuten (2013).

The quantile premium principle

The third principle, the quantile principle, is closely linked to the topic of solvency[150] which is discussed in Section 7.6. Formally, the quantile premium (QP) satisfies the condition that the probability of an underwriting loss (gain) is less than ε (greater than $1-\varepsilon$), i.e.

$$\mathrm{QP} = \inf\{\pi \in \mathbb{R}_+ \,|\, P\left(L_{\bar{T}}(\pi) \leq 0\right) \geq 1 - \epsilon\}.$$

For example, we consider quantiles of order $1-\varepsilon = 0.95$ and $1-\varepsilon = 0.99$.

[150] Ballotta et al. (2006) state that the quantile approach was considered by the European Commission to be incorporated for the Solvency II agenda. A discussion of solvency aspects is given in Section 7.6.

7.5.2 Different hedging scenarios

Before we can give some illustration, we need an assumption on the investment policy. In a first instance, we propose two benchmark hedging scenarios in which the annuity provider follows a trading strategy in zero bonds given by a $\mathcal{F}$-adapted process $\phi = \left(\phi_t^{(i)}\right)_{\substack{i=T,\ldots,\bar{T}-1 \\ t\in\Phi}}$. The set Φ denotes discrete trading times at which $\phi_t^{(i)}$ zero bonds $D(t,i)$ are acquired.

Investment in the stochastic bank account

The first strategy simply states that the annuity provider holds only cash positions and refrains from hedging his liabilities, i.e. for a discrete trading strategy $\phi_0^{(i)} = 0$ with payoff dates $i = T,\ldots,\bar{T}-1$ the terminal accumulated costs of the (initial) hedge at contract inception is

$$C_{\bar{T}}^{ini}(\phi) := \left(\sum_{i=T}^{\bar{T}-1} \phi_0^{(i)}\, D(0,i)\right)\beta_{0,\bar{T}} = 0.$$

To be more precisely, its initial capital and premium income are invested or rolled over according to the (stochastic) bank account (7.1). A straightforward consequence of leaving the future pension payments unhedged lies in a high uncertainty about the total duplication costs at maturity $\bar{T}$

$$C_{\bar{T}}^{dup}(\phi) := \sum_{i=T}^{\bar{T}-1} \left(cN_i - \phi_0^{(i)}\right)\beta_{i,\bar{T}} = c\sum_{i=T}^{\bar{T}-1} N_i\, \beta_{i,\bar{T}}.$$

For this reason, we call the resulting loss a roll over loss L^{RO}, i.e.

$$L_{\bar{T}}^{RO} := C_{\bar{T}}^{dup}(\phi) - S_0\, \beta_{0,\bar{T}} - \pi \sum_{i=0}^{T-1} N_i\, \beta_{i,\bar{T}} \tag{7.22}$$

where S_0 denotes some starting capital. In this section, we set $S_0 = 0$.

Investment in a static zero bond hedge strategy

In contrast, we consider an annuity provider who relies on a "as good as possible" hedge in zero bonds according to the expected number of retirees during the decumulation phase. At contract inception the annuity provider takes the positions $\phi_0^{(i)} = cN_0 \cdot {}_ip_x(0)$ in the zero bonds $D(0,i)$ with maturities $i = T,\ldots,\bar{T}-1$. The time-$\bar{T}$ value of the (initial) hedging strategy is therefore

$$C_{\bar{T}}^{ini}(\phi) = \left(\sum_{i=T}^{\bar{T}-1} \phi_0^{(i)}\, D(0,i)\right)\beta_{0,\bar{T}} = \left(\sum_{i=T}^{\bar{T}-1} cN_0\, {}_ip_x(0)\, D(0,i)\right)\beta_{0,\bar{T}}.$$

The tracking error at time i is $cN_i - \phi_0^{(i)}$ and consequently the accumulated duplication costs amount to

$$C_{\bar{T}}^{dup}(\phi) = \sum_{i=T}^{\bar{T}-1} \left(cN_i - \phi_0^{(i)} \right) \beta_{i,\bar{T}}.$$

Thus, the (terminal) loss $L_{\bar{T}}^{ZB}$ is

$$L_{\bar{T}}^{ZB} := C_{\bar{T}}^{dup}(\phi) + C_{\bar{T}}^{ini}(\phi) - S_0 \beta_{0,\bar{T}} - \pi \sum_{i=0}^{T-1} N_i \, \beta_{i,\bar{T}}. \tag{7.23}$$

While the benchmark case (7.22) refers to no-hedging, Equation (7.23) gives the loss associated with a static risk-minimizing[151] hedging strategy where the zero bond positions $\phi_0^{(i)} = cN_0 \, {}_i p_x(0)$ minimize the variance of $L_{\bar{T}}^{ZB}(\phi_0)$.

Remark 7.5.3. The above defined zero bond hedge is only a benchmark example which allows us to give some meaningful illustrations. Using the variance of the terminal loss as an optimality criterion is convenient in view of the problem formulation of this section, i.e. the pricing of non-pooling risk. From the perspective of the risk management, the strategies under consideration are not satisfying. For example, one would like to consider more sophisticated optimality criteria, allow for dynamic hedging and focus on intermediate losses, too. However, the main focus is not on an optimal risk management policy but on realistic hedging strategies and their outcomes which are convenient benchmarks for the pricing and the solvency requirements. On the one hand, "no-hedging" is the scenario which gives the worst case strategy in terms of fluctuations in the terminal loss. On the other hand, one may argue that the static variant of a variance minimizing zero bond hedge generates hedging outcomes which tend to be better than the ones achieved in practice. In practice, the hedging relies on a bucket structure, i.e. not all maturities (zero bonds respectively) are used. A reduction of the number of hedging instruments/maturities is therefore suboptimal. In addition, there is model risk, i.e. in reality the hedging strategies are based on model assumptions which deviate from the true dynamics. It is also worth to emphasise that the provider does not borrow money for his hedging purpose. However, the static zero bond hedge ZB in combination with a periodic premium principle implies

[151] For continuous-time dynamic risk minimizing hedging strategies, we refer to Dahl (2004) and Dahl and Møller (2006). The authors consider dynamic risk minimizing hedging strategies under the so called minimal martingale measure introduced by Schweizer (2001). In the following, the term risk-minimizing is equivalent to a minimization of the insurers hedge costs. In this connection, Chen (2008) analyses the net loss of an insurance company under discrete-time risk-minimization.

that the (uncertain) future premium income has to be pre-financed in order to initiate the hedge at contract inception.

Motivated by the above remark, we add a static and a dynamic forward starting strategy which avoid the pre-financing of the future premium income. Both strategies are initiated at the last premium date $T-1$, i.e. at the time when all premiums are collected by the provider and there exists certainty about the number of retirees.

Investment in a forward starting static zero bond hedge stra-tegy

First, consider the forward starting version of the static zero bond hedge ZB which we call for short $fwd\ ZB$. The premiums π are rolled over according to the stochastic bank account (7.1) until the time $T-1$. At $T-1$, the annuity provider takes the positions $\phi^{(i)}_{T-1} = cN_{T-1} \cdot {}_{i-T+1}p_{x+T-1}(T-1)$ in the zero bonds[152] $D(T-1,i)$ with maturities $i = T,...,\bar{T}-1$. The time-$\bar{T}$ value of the hedging strategy is

$$\begin{aligned} C^{ini}_{\bar{T}}(\phi) &= \left(\sum_{i=T}^{\bar{T}-1} \phi^{(i)}_{T-1} D(T-1,i) \right) \beta_{T-1,\bar{T}} \\ &= \left(\sum_{i=T}^{\bar{T}-1} cN_{T-1} \cdot {}_{i-T+1}p_{x+T-1}(T-1) \cdot D(T-1,i) \right) \beta_{T-1,\bar{T}}. \end{aligned} \tag{7.24}$$

The resulting tracking error at time i is $cN_i - \phi^{(i)}_{T-1}$ and the accumulated duplication costs for the forward static zero bond hedge aggregate to

$$C^{dup}_{\bar{T}}(\phi) = \sum_{i=T}^{\bar{T}-1} \left(cN_i - \phi^{(i)}_{T-1} \right) \beta_{i,\bar{T}}.$$

Thus, the (terminal) loss $L^{fwd\ ZB}_{\bar{T}}$ is

$$L^{fwd\ ZB}_{\bar{T}} := C^{dup}_{\bar{T}}(\phi) + C^{ini}_{\bar{T}}(\phi) - S_0 \beta_{0,\bar{T}} - \pi \sum_{i=0}^{T-1} N_i\, \beta_{i,\bar{T}}. \tag{7.25}$$

Investment in a forward starting dynamic zero bond hedge strat-egy

The (discrete time) dynamic forward starting strategy ($fwd\ ZB^d$) is based on an annual reallocation of the hedging positions, i.e. at time i ($i = T-1, \ldots \bar{T}-2$) the number of bonds with maturities $j = i+1 \ldots \bar{T}-1$ is $\phi^{(j)}_i = cN_i \cdot$

[152] Note that the amount of hedge positions is ex ante unknown and in case of a stochastic mortality evolution might noticeably change over the accumulation phase.

${}_{j-i}p_{x+i}(i)$. Therefore, we obtain the accumulated (initial) hedge costs (7.24) and accumulated duplication costs

$$C_{\bar{T}}^{dup}(\phi) = \sum_{i=T}^{\bar{T}-1} \left(cN_i - \phi_{i-1}^{(i)} \right) \beta_{i,\bar{T}}.$$

The discrete-time update of the hedge positions evokes additional accumulated rebalancing costs

$$C_{\bar{T}}^{reb}(\phi) = \sum_{i=T}^{\bar{T}-2} \left(\sum_{j=i+1}^{\bar{T}-1} \left(\phi_i^{(j)} - \phi_{i-1}^{(j)} \right) D(i,j) \right) \beta_{i,\bar{T}}.$$

Altogether, the terminal loss is composed of

$$L_{\bar{T}}^{fwd\, ZB^d} := C_{\bar{T}}^{dup}(\phi) + C_{\bar{T}}^{ini}(\phi) + C_{\bar{T}}^{reb}(\phi) - S_0 \beta_{0,\bar{T}} - \pi \sum_{i=0}^{T-1} N_i \, \beta_{i,\bar{T}}. \qquad (7.26)$$

It is worth to emphasise that we focus on the loss in terms of money paid at $\bar{T}$ and not on the discounted loss in terms of money paid at contract inception. In particular, the above dynamic hedge does not necessarily imply a lower variance of the terminal loss $L_{\bar{T}}$ than the static counterpart version.

7.5.3 Illustration of the hedge costs and the duplication error

The application of a discrete-time hedge strategy (whether in a static or a dynamic form) together with an incomplete annuity market introduces hedge costs (initial hedge and rebalancing costs) on the one hand and duplication costs on the other hand. It becomes almost impossible to completely duplicate future liabilities and, at the same time, to stay self-financed although the financial market is assumed to be complete. According to the Definitions (7.22)–(7.26), the loss at maturity equals the aggregation of the initial hedge costs (originated at the begin ($t = 0$) or at the end ($t = T - 1$) of the accumulation phase) plus the rebalancing costs (for the dynamic variant) and duplication costs during the decumulation phase minus the accumulated premium income and solvency margin. Table 7.10 clarifies the allocation of the total liabilities into different hedge component parts for the scenarios introduced in Subsection 7.5.2.

First of all, notice that under deterministic interest rates the expected total hedge costs and corresponding variances of the benchmark strategies RO and ZB coincide. The moments for the forward starting hedge scenarios are slightly reduced due a time-decreasing survival function and a comparatively small variation in the number of survivors. In other words, if the hedging decision is

Moments of the hedge and the duplication costs

hedging decision	parameter choice: mortality rate	parameter choice: interest rate	initial hedge and rebalancing costs: $\hat{E}[C_T^{ini} + C_T^{reb}]$	initial hedge and rebalancing costs: $\hat{V}ar[C_T^{ini} + C_T^{reb}]$	duplication costs: $\hat{E}[C_T^{dup}]$	duplication costs: $\hat{V}ar[C_T^{dup}]$
RO	deterministic	deterministic	0	0	65380	$1.46 \cdot 10^6$
		stochastic	0	0	126311	$1.16 \cdot 10^{10}$
	stochastic	deterministic	0	0	65881	$1.75 \cdot 10^6$
		stochastic	0	0	127534	$1.28 \cdot 10^{10}$
ZB	deterministic	deterministic	65487	0	10	$1.40 \cdot 10^6$
		stochastic	166878	205294	11	$8.09 \cdot 10^6$
	stochastic	deterministic	65487	0	517	$1.75 \cdot 10^6$
		stochastic	168353	210929	847	$9.96 \cdot 10^6$
$fwd\,ZB$	deterministic	deterministic	64498	573341	-3	844109
		stochastic	133602	$1.61 \cdot 10^{10}$	14	$4.31 \cdot 10^6$
	stochastic	deterministic	64976	853111	8	862303
		stochastic	134199	$1.70 \cdot 10^{10}$	24	$4.39 \cdot 10^6$
$fwd\,ZB^d$	deterministic	deterministic	64485	$1.25 \cdot 10^6$	0	10725
		stochastic	134909	$1.68 \cdot 10^{10}$	1	64924
	stochastic	deterministic	64994	$1.56 \cdot 10^6$	-12	10796
		stochastic	136066	$1.75 \cdot 10^{10}$	-21	66662

Table 7.10: Expected value and variance of the hedge and duplication costs for the different hedging scenarios and mortality Model (II). Calculations are based on $2 \cdot 10^4$ sample paths drawn w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$.

postponed to the end of the accumulation phase in $T - 1$, the insurer stays on the safe side, i.e. benefits are superhedged (non-positive duplication costs) in the mean (see also Figure 7.8 and 7.9). However, these values noticeably change if the investment is based on a random interest rate development. Although the expected costs (especially the hedge and rebalancing costs) for the static zero bond benchmark lie above the expected values of the remaining strategies, the variances of the initial hedge and rebalancing costs are considerably smaller. The major influence comes from the (stochastic) bank account as a part of the hedge costs[153]. In particular, the (total) expected hedge costs are two-and-a-half times increased compared to the deterministic counterpart.

7.5.4 Illustration of the terminal loss distribution

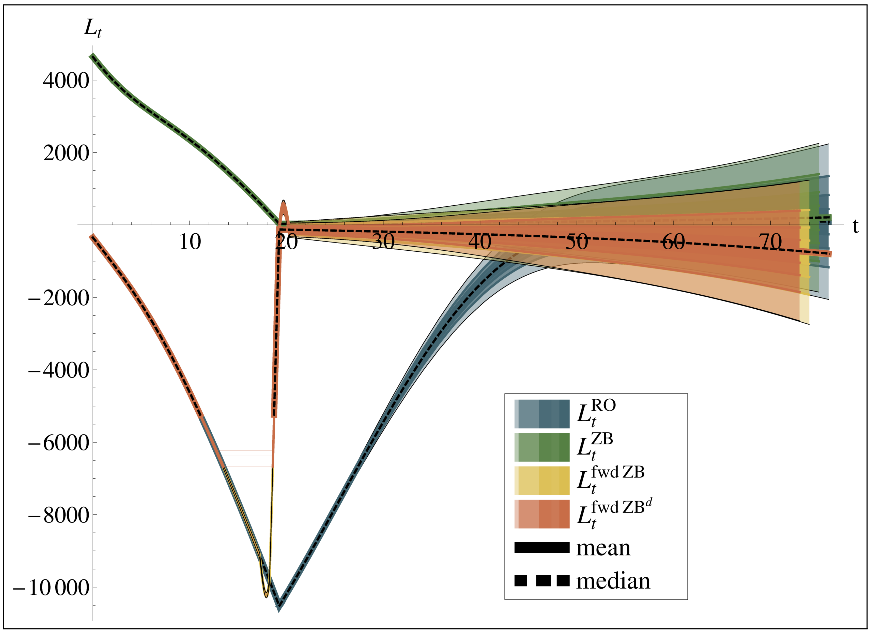
The following illustrations are based on the calibration results given in Section 7.3 where both interest rate and mortality model are considered in their

[153] For example, a mortality improvement accompanied with an underestimation of the survival function evokes a subhedge in the mean of the duplication costs for the ZB scenario (845) and a superhedge for the $fwd\,ZB$ variant (-21).

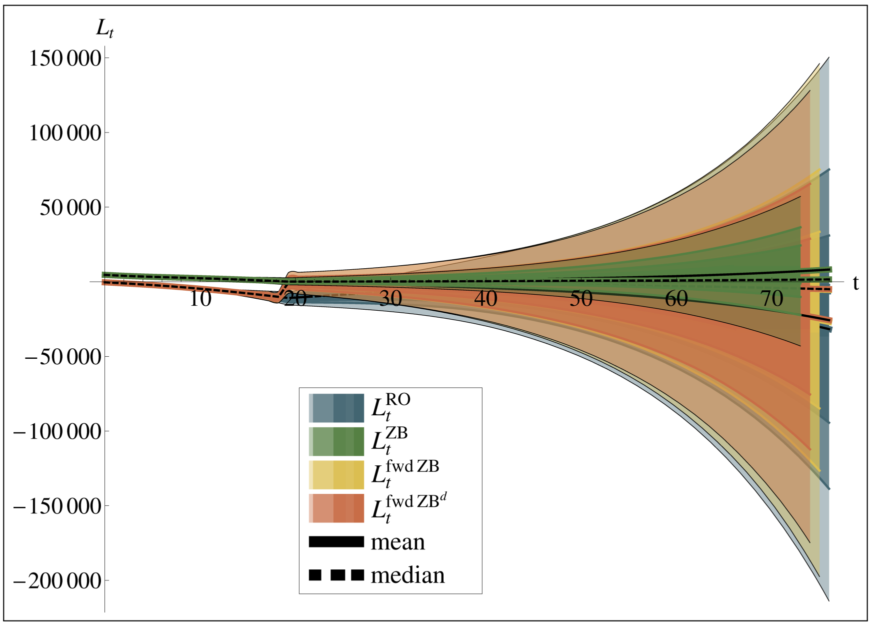
stochastic and deterministic version. Figure 7.8 shows the fan charts[154] for the portfolio loss process equipped with a start capital $S_0 = 0$ and $S_0 = 500$. Observe that analogue to the previous discussion the consideration of stochastic mortality and interest rates has a strong effect on the width of the prediction intervals and location measures such as mean and median. Figure 7.9 illustrates the densities of the portfolio terminal loss $L_{\bar{T}}$ corresponding to the benchmark hedging scenarios (Subfigure (7.9.1)) and the forward starting strategies in contrast to the ZB-Hedge (Subfigure (7.9.2)). In addition, Table 7.11 contains the quantile values for all hedging scenarios. In the deterministic setup the terminal portfolio loss distribution is approximately symmetric for the “immediately starting” hedge scenarios (see Table 7.11 and Figure 7.8). This is due to the fact that under deterministic mortality and interest rates the accumulated hedge costs at $\bar{T}$ for the forward starting strategies are more favourable than for the static zero bond scenario (cf. Table 7.10) such that the insurer stays on the safe side with high probability (see Subfigure (7.8.1)). Whereas in the full stochastic variant without any start capital S_0 the roll-over and the forward starting strategies are left skewed and the benchmark zero bond strategy is slightly right skewed (cf. Figure 7.8.2). The main reason for this phenomenon is the log-normal distribution[155] of the stochastic bank account (7.1) and positive skewness respectively. Thus, the terminal loss for the static zero bond strategy ZB is likely advantaged by the high accumulated initial hedge costs. In contrast, the forward starting hedge strategies react to “high-interest-scenarios” with a terminal profit while for more rarely “low-interest-rates” paths it is precisely the other way round. As a result, the 0.95-quantile values of Table 7.11 are comparatively high when estimated for the zero bond hedging. Accordingly, the probability densities in Figure 7.9 show the above-mentioned skewness behaviour. Subfigures (7.8.4) and (7.9.4) reveal that the strong variation in the static zero bond terminal loss can be counterbalanced for an additional capital $S_0 > 500$ such that its distribution also becomes right-skewed.

[154] The terminal portfolio loss fan chart constitutes an appropriate instrument to illustrate the uncertainty in the projections of the future cash-flow. For this purpose, the central projections for the median and mean are enriched with a set of prediction interval bounds.
[155] The low speed of mean reversion a compared to the volatility σ_{spot} causes large variation in the accumulated hedge costs of all zero bond strategies (see Table 7.10) which has a substantial impact on the loss at maturity values.

Prediction interval charts for the portfolio loss process

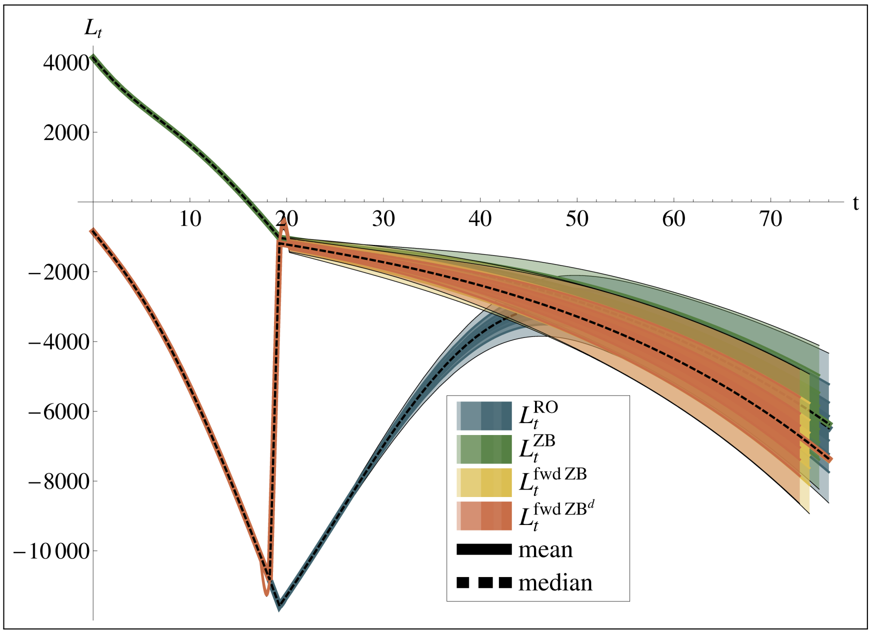


(7.8.1) Deterministic parameters ($S_0 = 0$)

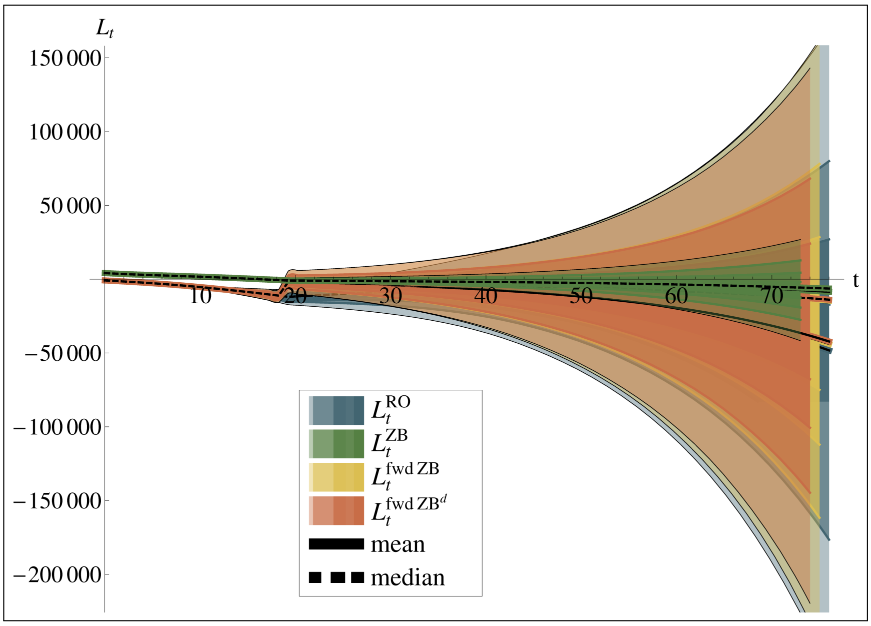


(7.8.2) Stochastic parameters ($S_0 = 0$)

Figure 7.8: Prediction intervals (5%, 25%, 50%, 75% and 95% intervals are marked in different shades) and location measures mean (solid) and median (dashed) for the loss processes under different hedging scenarios and mortality Model (II). Calculations are based on $2 \cdot 10^4$ sample paths drawn w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$.



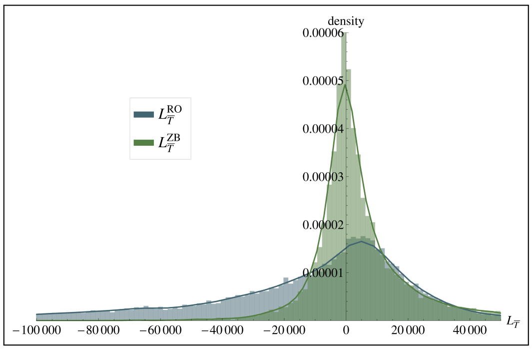
(7.8.3) Deterministic parameters ($S_0 = 500$)



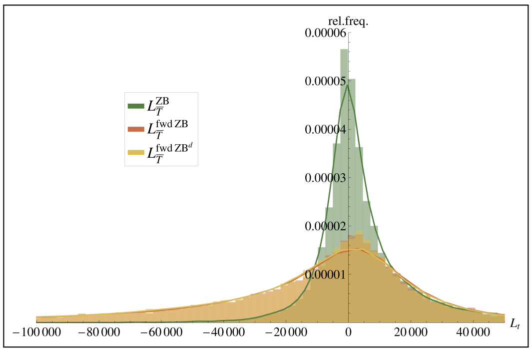
(7.8.4) Stochastic parameters ($S_0 = 500$)

Figure 7.8: Prediction intervals (5%, 25%, 50%, 75% and 95% intervals are marked in different shades) and location measures mean (solid) and median (dashed) for the loss processes under different hedging scenarios and mortality Model (II). Calculations are based on $2 \cdot 10^4$ sample paths drawn w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$.

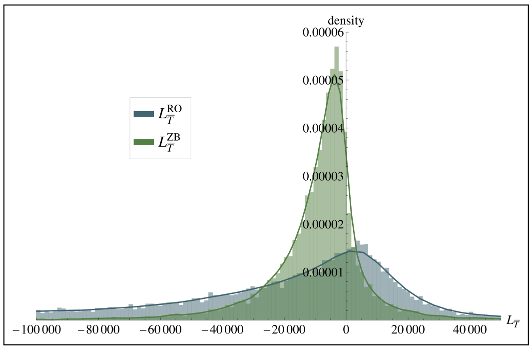
Probability densities of the terminal portfolio loss



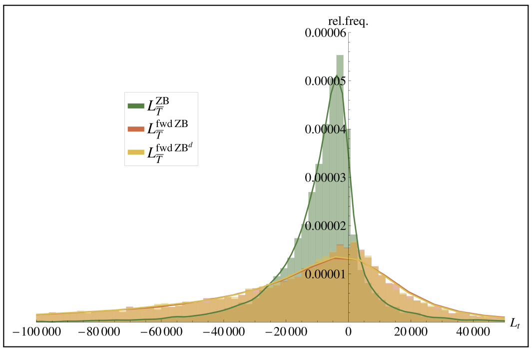
(7.9.1) Benchmark strategies ($S_0 = 0$)



(7.9.2) Forward starting strategies ($S_0 = 0$)



(7.9.3) Benchmark strategies ($S_0 = 500$)



(7.9.4) Forward starting strategies ($S_0 = 500$)

Figure 7.9: Probability density of the terminal portfolio loss under mortality Model (I) w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$. The histograms are based on $2 \cdot 10^4$ drawn samples. The solid lines represent the Gaussian kernel density estimators.

Quantile values for the terminal portfolio loss

hedging scenario	parameter choice		Model (I)					Model (II)				
	mortality rate	interest rate	$Q_{0.05}$	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	$Q_{0.95}$	$Q_{0.05}$	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	$Q_{0.95}$
RO	deterministic	deterministic	-2218	-907	-4	896	2215	-2308	-974	-82	809	2129
		stochastic	-181031	-42246	-6923	8783	33778	-177213	-42952	-7042	8813	33569
	stochastic	deterministic	-1805	-382	611	1574	2976	-1960	-494	518	1531	2957
		stochastic	-186741	-42310	-6529	9345	34148	-198381	-41419	-10421	9994	36590
ZB	deterministic	deterministic	-2180	-923	-5	912	2203	-2368	-1049	-120	796	2144
		stochastic	-18484	-4637	116	8249	45140	-18503	-4644	130	8153	45591
	stochastic	deterministic	-1813	-378	574	1538	2970	-1990	-509	497	1513	2988
		stochastic	-16619	-3993	683	9364	47811	-17648	-4327	317	6736	38720
fwd ZB	deterministic	deterministic	-3190	-1893	-1010	-112	1153	-3308	-2023	-1106	-182	1102
		stochastic	-162979	-36829	-6489	8890	40964	-161813	-37253	-5922	8833	41350
	stochastic	deterministic	-2872	-1418	-432	533	1913	-2991	-1536	-532	498	1921
		stochastic	-162204	-36355	-5730	9318	42047	-183613	-49175	-12853	8475	37590
fwd ZB^d	deterministic	deterministic	-3161	-1883	-1006	-107	1198	-2818	-1597	-814	38	923
		stochastic	-164770	-37214	-6148	9125	40611	-164868	-37260	-6118	8757	40558
	stochastic	deterministic	-2813	-1390	-440	524	1907	-2925	-1488	-485	511	1893
		stochastic	-163360	-37332	-5627	9476	41686	-161124	-35755	-5523	9635	43398

Table 7.11: Quantile values Q for the terminal portfolio loss. Calculations are based on $2 \cdot 10^4$ sample paths drawn w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$.

7.5.5 Pricing for different hedging scenarios

When considering the impact on the periodic premiums, Table 7.12 summarises the estimation results according to the three premium principles (NP), (IP) and (QP) as well as the percentage premium loadings of (IP) and (QP) additionally charged on the net premium (NP). First of all notice that under the full deterministic model approach the premiums are the same for the RO and ZB hedging/investment alternatives – except for simulation inaccuracies – since the total hedge costs coincide (cf. Table 7.10 and Subfigure (7.8.1)). In this regard, the forward strategies are preferable in the sense that duplication costs are further reduced due to a deferred hedging decision. Independent of the hedging scenario, random mortality or interest rates demand additional premium loadings. Interestingly, a consideration of a random mortality development only evokes an increase of the percentage loading in the range of only 1%-3% while in connection with stochastic spot interest rates this effect is much more pronounced. However, along the lines of the results due to the variance decompositions, the impact of stochastic mortality is visibly smaller than the one implied by random interest rates. With regard to the underlying hedging/investment scenario, the zero bond investments under hedge scenario ZB reduce the premium loading in addition to the equivalence premium to a large extent. For example, while the premium loading implied by the indifference price (IP) under the DHARA utility[156] is 257% in the case of RO (and the stochastic variant of mortality Model (II)), it is only 35% if the annuity provider invests in an adequate number of zero bonds at the inception of the contract. By comparison, the forward starting strategies which are identical to the RO scenario during the accumulation phase imply a premium loading of 294% in the case of the static hedge $fwd\ ZB$ and 234% for the dynamic version $fwd\ ZB^d$. The corresponding charges for the 0.99-quantile prices (QP) amount to 107% (RO), 25% (ZB), 99% ($fwd\,ZB$) as well as 98% ($fwd\,ZB^d$). Thus, it appears that the risk premium requirements vary according to the underlying pricing principle.

7.6 Solvency effects

Currently, the topic of solvency requirements receives growing attention both from a practical and a theoretical point of view. The EU Commission's

[156] Table 7.12 shows that indifference premiums under DHARA utility are slightly higher than under CARA utility since losses are stronger penalized using a DHARA utility function and the applied parameters.

Periodic premiums for different hedging scenarios

hedging scenario	parameter choice: mortality	parameter choice: interest rate	Model (I) (NP)	(IP) CARA		(IP) DHARA		(QP) $\varepsilon = 0.05$		(QP) $\varepsilon = 0.01$	
RO	deterministic	deterministic	0.349	0.361	+4.%	0.366	+5.%	0.359	+3.%	0.363	+4.%
		stochastic	0.349	1.191	+242.%	1.221	+250.%	0.566	+62.%	0.719	+106.%
	stochastic	deterministic	0.349	0.368	+6.%	0.373	+7.%	0.362	+4.%	0.366	+5.%
		stochastic	0.349	1.260	+261.%	1.370	+293.%	0.571	+64.%	0.732	+110.%
ZB	deterministic	deterministic	0.349	0.360	+3.%	0.365	+5.%	0.360	+3.%	0.364	+4.%
		stochastic	0.349	0.479	+37.%	0.480	+38.%	0.412	+18.%	0.435	+25.%
	stochastic	deterministic	0.349	0.366	+5.%	0.372	+7.%	0.363	+4.%	0.367	+5.%
		stochastic	0.349	0.488	+40.%	0.492	+41.%	0.413	+18.%	0.437	+25.%
$fwd\,ZB^d$	deterministic	deterministic	0.349	0.358	+3.%	0.362	+4.%	0.355	+2.%	0.358	+3.%
		stochastic	0.349	1.032	+196.%	1.061	+204.%	0.546	+57.%	0.673	+93.%
	stochastic	deterministic	0.349	0.369	+6.%	0.373	+7.%	0.358	+3.%	0.362	+4.%
		stochastic	0.349	1.320	+279.%	1.345	+286.%	0.554	+59.%	0.690	+98.%
$fwd\,ZB$	deterministic	deterministic	0.349	0.358	+3.%	0.363	+4.%	0.354	+1.%	0.358	+3.%
		stochastic	0.349	1.040	+198.%	1.072	+207.%	0.549	+57.%	0.681	+95.%
	stochastic	deterministic	0.349	0.363	+4.%	0.368	+6.%	0.358	+3.%	0.362	+4.%
		stochastic	0.349	1.081	+210.%	1.113	+219.%	0.550	+58.%	0.689	+98.%

hedging scenario	parameter choice: mortality	parameter choice: interest rate	Model (II) (NP)	(IP) CARA		(IP) DHARA		(QP) $\varepsilon = 0.05$		(QP) $\varepsilon = 0.01$	
RO	deterministic	deterministic	0.348	0.360	+3.%	0.365	+5.%	0.358	+3.%	0.362	+4.%
		stochastic	0.348	1.144	+228.%	1.190	+242.%	0.569	+63.%	0.714	+105.%
	stochastic	deterministic	0.348	0.369	+6.%	0.373	+7.%	0.362	+4.%	0.366	+5.%
		stochastic	0.348	1.214	+249.%	1.243	+257.%	0.573	+65.%	0.721	+107.%
ZB	deterministic	deterministic	0.348	0.365	+5.%	0.369	+6.%	0.359	+3.%	0.363	+4.%
		stochastic	0.348	0.446	+28.%	0.447	+28.%	0.410	+18.%	0.434	+25.%
	stochastic	deterministic	0.348	0.364	+5.%	0.368	+6.%	0.363	+4.%	0.367	+5.%
		stochastic	0.348	0.470	+35.%	0.471	+35.%	0.413	+19.%	0.436	+25.%
$fwd\,ZB$	deterministic	deterministic	0.348	0.358	+3.%	0.363	+4.%	0.354	+2.%	0.357	+3.%
		stochastic	0.348	0.995	+186.%	1.053	+202.%	0.545	+57.%	0.671	+93.%
	stochastic	deterministic	0.348	0.363	+4.%	0.367	+5.%	0.357	+3.%	0.362	+4.%
		stochastic	0.348	1.340	+285.%	1.373	+294.%	0.547	+57.%	0.693	+99.%
$fwd\,ZB^d$	deterministic	deterministic	0.348	0.360	+3.%	0.364	+5.%	0.354	+2.%	0.357	+3.%
		stochastic	0.348	1.091	+213.%	1.113	+220.%	0.549	+58.%	0.685	+97.%
	stochastic	deterministic	0.348	0.362	+4.%	0.366	+5.%	0.357	+3.%	0.362	+4.%
		stochastic	0.348	1.134	+226.%	1.163	+234.%	0.554	+59.%	0.688	+98.%

Table 7.12: Periodic premiums for different hedging scenarios w.r.t. an individual aged $x = 45$, a deferment period of $T = 20$, a cohort size $N_0 = 1000$ and a start capital $S_0 = 0$. The DHARA parameters $\rho_1 = 33$ and $\rho_2 = 3225$ are fitted to an exponential utility function with risk aversion $\rho = 0.01$ (see Figure 7.7). The simulations are based on $2 \cdot 10^4$ iterations.

European directive Solvency II includes a standardisation and reformation of the national insurance supervisions. Thereby, the introduction of solvency rules concerning a (minimum) regulatory capital[157] plays a major part within the quantitative pillar and is planned for the first quarter of 2013. Additionally, actuarial literature supports and promotes the Solvency II project progress by developing risk-sensitive methods to measure insurance liabilities and to construct biometric valuation bases or solvency tests (see e.g. Hári et al. (2008),

[157] According to Olivieri and Pitacco (2008) the 99% confidence level margin can be seen as the amount of required capital (in addition to the fair premium) consistent with a solvency rule for internal models with a time horizon matching the contract duration.

Christiansen (2010) or Hayes (2010)).

7.6.1 Definition of solvency risk measures

The first to consider is how to define the term solvency as well as solvency requirements. For example, solvency requirements depend on the concrete asset definition, time horizon under consideration and borrowing constraints. With respect to a portfolio of annuity contracts, it is also important to differentiate between a run-off and a going concern approach. In the following, we consider a weak form of solvency (ruin respectively) which is based on a run-off approach. While the run-off approach does not allow for new contracts in addition to the ones of the cohort at inception, this is the case in the going concern approach. On the one hand, this simplifies the analysis. On the other hand, allowing new contracts (their premiums respectively) to subsidise old contracts is a very critical point, anyway. In addition, we do not focus on a ruin event prior to the date $\bar{T} = \omega - x$ where our cohort ceases to exist. This is equivalent to the (unrealistic) assumption that it is possible to borrow unlimited prior to the terminal date.[158] In reality, there is a stronger form of solvency requirement necessary which checks the cash flow emergence yearly or monthly and thus records a loss as soon as it occurs. The term shortfall is to be interpreted in a technical sense since only the maturity date is regarded and any fees and costs within annuity products are ignored. In particular, a stronger version of solvency affords a definition of a bankruptcy level which must not be violated together with a pricing rule for future liabilities. The task of a capital adjustment consists in the determination of a solvency margin S_0 such that (exogenous) solvency requirements are fulfilled. We therefore follow an indirect approach by considering the shortfall probability, i.e. the probability that the terminal loss value $L_{\bar{T}}$ is positive.

Definition 7.6.1 (Solvency Requirements - Weak Form). Let ϵ denote a predefined confidence level, then the initial solvency requirements are defined by the rule

$$S_0^* := \inf\{S_0 \in \mathbb{R}_+ \mid P\left(L_{\bar{T}} > 0\right) \leq \epsilon\} \tag{7.27}$$

where $L_{\bar{T}}$ is the underwriters loss at time $\bar{T}$. The solvency capital S_0^* is therefore defined by the minimal amount which is needed to limit the (terminal) shortfall and stay on the safe side with probability $1 - \epsilon$. In particular, we refer to the

[158] Among other authors, Bauer and Weber (2008) stress that it might be unrealistic to finance intermediate shortfalls against a savings account rather than money at the market interest rate.

probability $P(L_{\bar{T}} > 0)$ as a weak form of shortfall probability as only the loss at maturity is considered.

To avoid the problem of specifying a pricing rule which is, in an incomplete market, not necessarily independent of the investment decisions, we refer to the weak form given above. Due to longevity risk, annual losses typically come along with an overall loss at maturity as we will see subsequently. In addition to the solvency capital S_0^* which is implied if the periodic premiums are calculated according to the equivalence premium (NP), we measure the extent of the shortfall by the conditional expected shortfall given default.

Definition 7.6.2 (Expected Shortfall Given Default). The expected shortfall is defined as the expected terminal portfolio value conditioned on the occurrence of a shortfall. In case of a continuous net loss distribution, we have

$$E[L_{\bar{T}} \mid L_{\bar{T}} > 0] = \frac{E\left[L_{\bar{T}} \mathbb{1}_{\{L_{\bar{T}} > 0\}}\right]}{P(L_{\bar{T}} > 0)}. \tag{7.28}$$

The actuarial task lies in the minimization of the expected loss under a budget constraint depending on the investment decision.

In contrast to the shortfall probability, the expected shortfall measure constitutes a coherent risk measure and reflects both the probability and the severity of a potential default. Note that due to the Bayes theorem, the conditional expectation depends, inter alia, on the shortfall probability.

7.6.2 Estimation of solvency margins

In general, there is no analytic loss distribution available such that we have to estimate (7.27) and (7.28) by Monte-Carlo simulation. For mortality Model (II) the Monte Carlo standard error for the shortfall probability is at least smaller than $1.1 \cdot 10^{-4}$. The sample mean for 1000 simulated conditional expected shortfall values[159] amounts to 17578 and has standard deviation of 6.44. Therefore, the number of simulations seems appropriate. As before, the illustrations refer to the calibration results of Section 7.3. Table 7.13 gives the resulting solvency capital S_0^*. Again, the results are similar for both mortality models. Along the lines of Table 7.4, the expected discounted premiums for Model (II) are equal to 4931. Observe that, in the full stochastic setup and no-hedging (*RO*) strategy, 73% (126%) of the expected discounted premiums

[159] The determination of the Monte Carlo error is based on the full stochastic *RO*-scenario using $2 \cdot 10^4$ iterations.

Start capital / solvency margins for different hedging scenarios

hedging decision	parameter choice: mortality rate	parameter choice: interest rate	Model (I) premium $\pi = 0.349$, $\epsilon = 0.05$	Model (I) premium $\pi = 0.349$, $\epsilon = 0.01$	Model (II) premium $\pi = 0.348$, $\epsilon = 0.05$	Model (II) premium $\pi = 0.348$, $\epsilon = 0.01$
RO	deterministic	deterministic	135	189	135	188
		stochastic	3448	6432	3507	6153
	stochastic	deterministic	181	240	185	248
		stochastic	3513	6521	3583	6201
ZB	deterministic	deterministic	134	191	136	191
		stochastic	711	936	704	955
	stochastic	deterministic	181	243	187	249
		stochastic	733	958	714	961
$fwd\,ZB$	deterministic	deterministic	73	126	73	126
		stochastic	3171	5590	3169	5567
	stochastic	deterministic	116	179	122	188
		stochastic	3308	5880	3224	5886
$fwd\,ZB^d$	deterministic	deterministic	70	123	72	125
		stochastic	3214	5670	3308	5718
	stochastic	deterministic	117	177	121	187
		stochastic	3235	5775	3253	5755

Table 7.13: Start capital / solvency margins for different hedging scenarios w.r.t. an individual aged $x = 45$, a deferment period $T = 20$ and a cohort size $N_0 = 1000$. The simulations are based on $2 \cdot 10^4$ iterations.

are needed as solvency capital if the maximum tolerable shortfall probability is limited to 5% (1%). Thus, no-hedging is prohibitive if one takes into account both stochastic interest rates and stochastic mortality rates. Although the static hedge in zero bonds reduces theses numbers to 15% (20% respectively), the fraction of the expected discounted premium which is needed to provide an adequate solvency condition can still be considered as (too) high. The solvency capital requirements for the forward hedging strategies lie in between the margins for the benchmark cases RO and ZB. On the one hand, the forward hedging causes comparatively low duplication costs (implying a superhedge). On the other hand, a steady mortality improvement might evoke high initial hedge costs which form the major part of the total hedge costs. In combination with random interest rates the forward scenarios are therefore subjected to strong variations similar to the no-hedging scenario (cf. Subfigures (7.8.2) and (7.9.2)).

7.6.3 Illustration of the shortfall probability and amount

The above mentioned effects are additionally emphasised by Table 7.14 which summarises the shortfall probability for different solvency margins and hedge scenarios. In analogy to the observations made in Subsection 7.5.4, the interaction of stochastic mortality and / or interest rates causes a high shortfall probability such that values above 0.4 are not surprising in the absence of additional solvency capital. However, in the case of the static ZB-hedging, the shortfall probability is even higher for stochastic interest rates and low additional[160] solvency capital $S_0^* < 183$ (mortality Model (II)). On closer inspection, the hedge costs of the static zero bond strategy are strongly driven by the interest rate evolution[161] throughout the long-term contract duration. For a high interest rate level, the resulting high total hedge costs can even cause a severe loss at maturity (cf. Table 7.11) such that for additional solvency margins the shortfall probabilities are partially counterbalanced. In case of the forward zero bond scenarios, we observe comparatively low shortfall probabilities for deterministic actuarial parameters according to the hedge intention. However, the introduction of stochastic mortality and interest rates evokes values slightly beneath those of the RO-strategy.

Shortfall probabilities for different hedging scenarios

hedging	parameter choice		solvency margin S_0 – Model (I)						solvency margin S_0 – Model (II)					
decision	mortality rate	interest rate	0	100	250	500	1000	2000	0	100	250	500	1000	2000
RO	deterministic	deterministic	0.50	0.11	0.00	0.00	0.00	0.00	0.50	0.11	0.00	0.00	0.00	0.00
		stochastic	0.40	0.38	0.35	0.30	0.23	0.12	0.41	0.39	0.35	0.31	0.22	0.12
	stochastic	deterministic	0.66	0.24	0.01	0.00	0.00	0.00	0.66	0.24	0.01	0.00	0.00	0.00
		stochastic	0.41	0.39	0.36	0.31	0.23	0.12	0.42	0.40	0.36	0.31	0.23	0.13
ZB	deterministic	deterministic	0.50	0.11	0.00	0.00	0.00	0.00	0.49	0.11	0.00	0.00	0.00	0.00
		stochastic	0.51	0.42	0.30	0.13	0.01	0.00	0.51	0.43	0.30	0.13	0.01	0.00
	stochastic	deterministic	0.66	0.23	0.01	0.00	0.00	0.00	0.65	0.24	0.01	0.00	0.00	0.00
		stochastic	0.54	0.45	0.33	0.15	0.01	0.00	0.53	0.45	0.32	0.14	0.01	0.00
$fwd\,ZB$	deterministic	deterministic	0.22	0.02	0.00	0.00	0.00	0.00	0.23	0.02	0.00	0.00	0.00	0.00
		stochastic	0.40	0.38	0.35	0.30	0.21	0.11	0.41	0.39	0.35	0.30	0.22	0.11
	stochastic	deterministic	0.38	0.07	0.00	0.00	0.00	0.00	0.38	0.08	0.00	0.00	0.00	0.00
		stochastic	0.41	0.39	0.35	0.30	0.22	0.11	0.41	0.38	0.35	0.30	0.22	0.11
$fwd\,ZB^d$	deterministic	deterministic	0.22	0.02	0.00	0.00	0.00	0.00	0.23	0.02	0.00	0.00	0.00	0.00
		stochastic	0.40	0.38	0.35	0.30	0.22	0.11	0.40	0.38	0.35	0.30	0.22	0.11
	stochastic	deterministic	0.38	0.07	0.00	0.00	0.00	0.00	0.39	0.08	0.00	0.00	0.00	0.00
		stochastic	0.41	0.39	0.35	0.30	0.22	0.12	0.41	0.39	0.36	0.31	0.23	0.12

Table 7.14: Shortfall probabilities for different hedging scenarios w.r.t. an individual aged $x = 45$, a deferment period $T = 20$ and a cohort size $N_0 = 1000$. The simulations are based on $2 \cdot 10^4$ iterations.

Additionally, we consider the extent of a shortfall. The conditional expected shortfalls are listed in Table 7.15. Under deterministic interest rates the forward strategies are advantageous with regard to the minimization of the duplication costs. Therefore, we obtain a similar pattern to the survival probability values. In case of uncertainty in the bank account, the variation in the hedge and rebalancing costs is noticeably with direct consequences for the shortfall amounts. For example, observe that for mortality Model (II) the average

[160] In other words, a start capital of $S_0 = 183$ leads to the same shortfall probability of 38% for the RO and ZB scenario and equals a percentage charge of 53% on top of the total premium income.
[161] Whereas, the duplication costs are significantly lower affected.

Conditional expected shortfall amount for different hedging scenarios

hedging	parameter choice		solvency margin S_0 – Model (I)						solvency margin S_0 – Model (II)					
decision	mortality rate	interest rate	0	100	250	500	1000	2000	0	100	250	500	1000	2000
RO	deterministic	deterministic	1067.	651.	306.	0.	0.	0.	1066.	650.	251.	0.	0.	0.
		stochastic	17222.	16804.	16188.	15186.	13266.	11074.	17242.	16699.	16068.	15023.	13331.	11074.
	stochastic	deterministic	1407.	838.	510.	0.	0.	0.	1442.	879.	534.	0.	0.	0.
		stochastic	17368.	16870.	16224.	15217.	13612.	11106.	17280.	16809.	16086.	15101.	13461.	11134.
ZB	deterministic	deterministic	1080.	651.	344.	0.	0.	0.	1079.	665.	327.	0.	0.	0.
		stochastic	18711.	17304.	15607.	13083.	9056.	0.	18726.	17436.	15684.	13308.	9441.	0.
	stochastic	deterministic	1383.	860.	528.	0.	0.	0.	1436.	893.	480.	0.	0.	0.
		stochastic	18797.	17464.	15691.	13320.	9671.	0.	18734.	17482.	15706.	13722.	11972.	0.
$fwd\,ZB$	deterministic	deterministic	754.	481.	119.	0.	0.	0.	762.	473.	251.	0.	0.	0.
		stochastic	20263.	19791.	18982.	18086.	16309.	13614.	20118.	19664.	18952.	17885.	16110.	13622.
	stochastic	deterministic	995.	650.	371.	0.	0.	0.	1053.	678.	390.	0.	0.	0.
		stochastic	20604.	20082.	19283.	18148.	16466.	13906.	20625.	20102.	19219.	18092.	16112.	13666.
$fwd\,ZB^d$	deterministic	deterministic	772.	479.	0.	0.	0.	0.	760.	494.	380.	0.	0.	0.
		stochastic	20363.	19785.	19108.	17832.	15871.	13059.	20539.	19988.	19159.	17929.	16013.	13431.
	stochastic	deterministic	1002.	653.	455.	0.	0.	0.	1033.	667.	654.	0.	0.	0.
		stochastic	20668.	20102.	19434.	18227.	16327.	13871.	21454.	20938.	20307.	19103.	17120.	14456.

Table 7.15: Conditional expected shortfall amount for different hedging scenarios w.r.t. an individual aged $x = 45$, a deferment period $T = 20$ and a cohort size $N_0 = 1000$. The simulations are based on $2 \cdot 10^4$ iterations.

shortfall size for a solvency capital of 1000 (approximately 20% of the expected total premium income) and a static zero bond hedge is still 13461 (which equals 273% of the expected total premium income). Combined with high shortfall probabilities in case of insufficient initial solvency capital, the static zero bond benchmark returns higher expected losses too. The corresponding values for the forward starting strategies are even higher than in the RO-scenario whereas the severity of a shortfall can be further reduced by means of a relatively cost-effective dynamic rebalancing. To sum up, the outcomesalso emphasise the magnitude of the impact of stochastic interest rates when compared to the impact of stochastic mortality rates. However, the combined effects of stochastic mortality and interest rates heavily affect the solvency requirements.

7.7 Conclusion

In this chapter, we have shown that the valuation, risk management and solvency assessment of a life annuity depend on the assumptions which are posed on the underlying mortality and interest rate dynamics. By means of different (stochastic and deterministic) mortality models which are fitted to the same population period life table and interest rate models calibrated to the current market term structure we focussed on the question how important it is to take into account for (random) changes in the mortality. We used Monte Carlo simulations to approximate the variance of the discounted cash flow and its decomposition into a pooling and a non-pooling risk part. We also considered pricing effects using the principle of zero expected utility and the quantile principle. The estimated risk premiums for a selection of different hedge / investment strategies were benchmarked to the equivalence premium. Finally, we focussed on the definition of solvency requirements and meaningful shortfall measures. The associated shortfall probability and conditional expected shortfall of the annuity provider are sensitive to changes in the underlying hedge strategy and degree of uncertainty with regard to the underlying mortality and interest model. In summary, the results emphasise that the impact of stochastic mortality is low if compared to the impact of stochastic interest rates. To some extent, one can argue that the major risk which is due to stochastic mortality stems from its interaction with stochastic interest rates. However, against the background of a risk-adequate assessment this should have a lasting effect on the insurers pricing and reserving.

But overall, these new annuity products are not bad, [...] If they would get their total expenses down a percent, they would be very good.

David B. Jacobs, founder and president of Pathfinder Financial Services, LLC, Hawaii

CHAPTER 8

Deferred Variable Annuities with Guaranteed Minimum Death and Income Benefits under Stochastic Mortality and Investment Risk

This chapter is an adapted version of Steuten (2011).

8.1 Introduction

Over the course of the last decades, the social pension schemes of many industrialised countries were seriously affected by an continuing longevity trend regarding the pensioners residual lifetimes and sustaining decreasing total fertility rates among females of childbearing age. As a consequence, PAYG pension schemes are expected to be unable to completely protect an ageing society against outliving its assets. Governments therefore subsidise private pension coverage and renew the incentive scheme through a reform of the current social security systems. The stimulation of the demand for private attractive capital funded life annuities should counterbalance the so called "annuity-puzzle" phenomenon[162]. In case of the German insurance market, the pension reform of the statutory pension insurance scheme in 2001, which lowered the replacement ratio to 64%, forced current contributors to consider additional occupational pension programs as well. Inter alia, equity-linked policies – about one third disposed in a Riester version (as an integral part of the 2nd pillar representing old-age retirement provision) – are state-subsidised

[162] On the one hand, private pension coverage seems meaningful from an economic view. But on the other hand, it can be observed that the annuity market is still narrow in relation to the total number of pensioners and thus potential consumers.

as long as the insurer guarantees a deferred life annuity calculated on at least a money-back basis, i.e. the insured receives the aggregated premiums.

In this context, a specific line of unit-linked annuity products called variable annuities deserves a closer inspection. Variable annuity[163] products (in the following abbreviated by VAs) fill the gap left by (domestic) demand between traditional life annuity contracts with low but guaranteed rate of return (currently 2.25%) and pure unit-linked products without any guaranteed retirement income. From the policyholder's point of view, VAs are characterised by a combination of high guaranteed payments together with potential returns of unit-linked products. Besides a successful sales launch in the US (1980), Japan (1999), the UK (2005) or Canada (2007), VAs are also offered in Continental Europe since 2005[164]. More precisely, variable annuities describe tax-privileged deferred life annuities with equity-linked savings phase containing at least one option on a minimal insurance benefit which is separately charged as a percentage of the fund value. A wide range of options is offered concerning enhanced financial flexibility and/or enhanced downside, longevity and dependants protection. Commonly, policyholders are allowed to compose their individual option package and even change it within the duration of the contract against additional fees. The mutual fund or the mix of funds can also be shifted and switched during the contract duration. Besides equity-linked funds, the capital can be invested for example in fund of funds, retail funds or money market funds. Due to the risky investment and the selling of an option with long-dated maturity, it seems necessary that VA providers enhance internal risk control mechanisms, apply dynamic hedging strategies and reduce their reporting timescales.

The chapter is motivated by addressing the cost transparency[165] aspect of VAs which has been paid little attention in recent academic literature so far. Charges relating to fund, mortality and expenses, administration or

[163] In contrast to dynamic hybrid products in which capital market guarantees are outsourced into a capital preservation fund or conventional premium reserve stock, variable annuities allow for an almost complete investment in equity funds and are (dynamically) hedged by means of derivative financial instruments. Variable annuities were first introduced in the United States in 1952 without any additional guarantee rider. In 1980 the first contract with guaranteed minimum death benefit was sold. Contracts with income guaranteed benefits followed in 1996.

[164] According to the German Insurance Association (GDV), the market share of new business for unit-linked pension insurance increased from 15.9% in 2006 over 23.7% in 2008 to 18.5% in 2009. Moreover, 4.88 million unit-linked endowment policies and 8.46 million unit-linked annuity policies were underwritten by GDV members at the end of the year 2009. These sales volumes made up 5.3% and 9.3% of the total life assurance business.

[165] Besides a transparent rider charge reporting, the VA product concept renders transparency in terms of the fund value proposition and amount of the benefits.

guarantees are separately accounted for such that capital investment and guarantee management fees are charged separately. For a VA contract calculated with common market charges a direct comparison to a fund savings plan is almost impossible. Even in the case of a traditional deferred annuity, a comparison is meaningful only to a limited extent since the guarantee charges can only be estimated by making assumptions due to the investment and surplus distribution policy of the insurance company. Nevertheless, the question whether a stated periodic charge is a fair charge has to be answered. In practice, quoted rider charges reduce the contract gross rate of return considerably such that their amount forms a relevant sales pitch. Insurance buyers commonly endorse guarantees but are willing to pay for them insofar as the rate of returns exceeds at least tax and inflation. Since insurance demand is out of the scope of this chapter, we concentrate on a fair charge estimation which is, from an insurer's perspective, equivalent to a minimal charge needed to cover the expected guarantee payout.

W.l.o.g., we assume a VA contract with premiums invested in a single mutual fund. The equity share of the fund is assumed to be equal to 100%. In case of death within the deferment period, the insurer guarantees the higher of the fund value and the premiums paid to the insureds heirs. If the policyholder dies within the decumulation phase solely the fund value is paid to the dependants. At the agreed retirement T the insured can enter a lump-sum option either by taking the fund value as a lump sum payment, converting the portfolio into a life annuity calculated with respect to the prevailling annuity market conditions[166] or benefiting from a life annuity paying a prespecified guaranteed periodic amount. Moreover, the retiree can request to defer the lump-sum decision within a so called "annuity prolongation phase". The guaranteed annuitisation option is also known as Guaranteed Minimum Income Benefit (GMIB). Furthermore, the GMIB option is equipped with different enhancements features which determine the total amount of the guarantee base. In the context of GMIBs typical features are the return of the gross premiums paid optionally accumulated with a guaranteed rate (roll-up), the maximum of the sum of premiums and historic fund value peaks (ratchet) and the greater of roll-up or ratchet. Obviously, by taking a short position in these long-dated options the insurer is exposed to risk sources like equity, interest rate as well as mortality risk. For this reason, we decided in favour for a full stochastic model framework. In particular, the fund prices are driven by a geometric Brownian motion, the spot interest rates follow a one-factor Hull-White process and the mortality rate evolution is described

[166] The accumulated fund assets is annuitised according to the mortality and interest conditions valid at that time.

by a non mean reverting Ornstein-Uhlenbeck process in slight modification to Ballotta and Haberman (2006). Furthermore, we assume a complete financial market model which allows for correlated bond and fund prices.

The fair premium principle applied by Nielsen and Sandmann (2002) serves as a pricing basis and combines two different risk diversification principles. On the one hand, we have the no-arbitrage pricing principle leading to diversification by duplication and, on the other hand, the equivalence premium principle covering diversification through balancing within the insurance portfolio. The principle by Nielsen and Sandmann is based on the analysis of an equity-linked life insurance contract using no-arbitrage pricing considered in Brennan and Schwartz (1976). An application to the guaranteed benefits of a VA contract yields equality between the accumulated expected present charge income and the present value difference of the accumulated expected discounted benefit values of the VA contract and a comparable unit-linked endowment contract without any guarantee. For the sake of completeness, the proof of the existence and the uniqueness of a fair guarantee charge is given in analogy to Nielsen and Sandmann (2002). Considering a contract with regular premiums, the guaranteed minimum death benefit option and the retirement option in the so called prolongation phase are closely related to an arithmetic Asian options. For this reason, we calculate expected values by applying Monte Carlo techniques. Our contribution to the existing literature on pricing and the analysis of the return/risk profile of VAs are as follows: In a first step we take the view of an insurance buyer and carry out an assessment concerning the contribution return and the pension amount at maturity (distribution). In particular, the impact of different rider features on the options moneyness and payoff value are discussed. From the perspective of an insurance seller, we estimate the risk-neutral prices of living and death benefit guarantees under fair charges for a comprehensive constellation of contract, financial and mortality parameters and different guarantee features. In a risk-neutral context charges are often mispriced particularly in combination with different guarantee enhancement options.

In the research field of unit-linked insurance various contract variants and pricing approaches have been proposed. Without postulating completeness, we mention the following: Bacinello and Ortu (1994) extend the pioneering work of Brennan and Schwartz (1976) to simulate single up-front and periodic premiums of an unit-linked and whole life insurance contract. The authors assume an endogenously minimum guarantee which is a function of the total premiums paid. Nielsen and Sandmann (1995) present numerical solutions for the price of an unit-linked endowment insurance including an Asian call-option due to periodic premium payments. The authors estimate a fair premium under random fund value and interest rate evolution with different scenarios for the initial term structure. In Nielsen and Sandmann (2002) the existence of a fair percentage charge for a unit-linked term and pension insurance is deduced and price bounds are derived. Schrager and Pelsser (2004) price an unit-linked endowment contract which guarantees a refund of the accumulated premiums at maturity. The authors calculate price bounds under stochastic interest rates and take the position of exogenously given charges and deterministic survival data from a life table. The article of Ho et al. (2006) investigates the impact of correlated equity prices and interest rates on the cost of guarantee of a single upfront premium VA with GMIB rider. Besides lapsation and hedging issues the authors simulate fair at-the-money guarantee values using the extended linear path methodology for a range of correlations. Estimates are based on an extension of the discrete two factor arbitrage-free interest model introduced in Ho and Lee (2004) and on the exclusion of mortality risk. Fair VA guarantee percentage charges are determined in Bauer et al. (2008b) based on a fund price driven by a geometric Brownian motion and deterministic assumptions on the interest and the mortality rates. Therefore, Monte Carlo simulation provides numerical values in case of deterministic surrender probabilities and a multidimensional finite mesh discretisation yields values under rational policyholder behaviour. The more recent literature concerning VAs with living benefits considers the following: the work of Jiang and Chang (2010) applies Monte Carlo simulation to derive charges for an annuity contract with upfront premium and guaranteed accumulation and death benefit riders. Interest rates are driven by a square diffusion process and the fund prices are modelled using a geometric Brownian motion. However, mortality risk is neglected, too. The charge is subjected to a sensitivity analysis concerning interest rate and contract parameters. A valuation of a pure GMIB option with a greater of roll-up or ratchet clause is undertaken by Marshall et al. (2010) who extend the model of Bauer et al. (2008b). More precisely, the paper treats the Monte Carlo estimation of no-arbitrage prices for a term certain annuity financed by a single upfront premium in a combined financial market, i.e. a fund price driven by a geometric Brownian motion and a correlated Hull-White interest rate process. Mortality risk is completely excluded and charges are illustrated for a comprehensive choice of financial and contract parameters.

The remainder of the chapter is organised as follows. Section 8.2 describes the assumptions made with respect to the analysed contract. In Section 8.3 a stochastic complete and arbitrage-free financial market model and a stochastic mortality model are introduced. Section 8.4 deals with the existence and uniqueness of a fair charge for variable annuities with bundled GMDB and GMIB rider assuming independence of financial risk and mortality risk as well as market neutrality towards mortality risk. In Section 8.5, numerical results are provided on the one hand for fair charges and option prices, and on the other hand for the rate of return and the pension amount at maturity. Section 8.6 concludes the chapter.

8.2 Contract specification

8.2.1 Definition of the (mutual) fund value

The analysis is restricted to the case of constant premium payments, i.e. either a single up-front premium or constant periodic premiums. Let $t_N = T$ denote the deferment period of the contract and $t_0 = 0$ the inception date. In the case of a regular premium payment the insured aged x at time $t_0 = 0$ pays a gross premium $\pi^G_{t_i} = \pi \cdot \mathbb{1}_{\underline{T}}(t_i)$ at each time $t_i \in \underline{T} := \{t_0, t_1, ..., t_{N-1} \in \mathbb{N} \,|\, t_0 < t_1 < ... < t_{N-1} < t_N\}$ given that the policyholder survived until $t_i > 0$. When considering single up-front premium contracts, the insurer solely receives $\pi^G_{t_i} = \pi \cdot \mathbb{1}_{\{t_0\}}(t_i)$ at inception date t_0. According to Schrager and Pelsser (2004) we assume that there exists certainty about premium payments such that policy surrender and lapsation are excluded a priori. We do neither consider an exemption from payment nor the involvement of additional riders. Therefore, the insurer has to hold hedge positions for the whole insurance portfolio which leads to higher charges than under surrender or lapse assumptions. The fixed costs[167] ϕ are deducted from $\pi^G_{t_i}$ to obtain the net premium $\pi^N_{t_i}$. After an infinitesimal small time period the asset-based fee[168] as a fixed percentage φ of the fund value is deducted[169]. The resulting investment premium $\pi^I_{t_i}$ is immediately invested into a single (third party) mutual fund.

We assume there exists no credit risk, no customer withdrawal (reducing the

[167] The fixed cost building block as a percentage of the gross premium is evoked by acquisition and selling expenses, contract and investment fees.

[168] Fund value related costs are usually denoted as mortality and expense (M&E) fees containing distribution, regular administration costs as well as the coverage of the standard death and annuitisation risk. For any optional guarantee rider and corresponding benefit base beyond the standard insurance protection, however, additional rider fees have to be charged. According to Nielsen and Sandmann (2002) $1 - \varphi$ is called the investment share.

[169] This approach is contrary to Schrager and Pelsser (2004) who charge the first asset-based fee at payment date t_1. Note that the amount of charges, each time t_i withdrawn by the insurer, is therefore exposed to equity risk in form of price fluctuations

guarantee amount) during the deferment period and that the fund is completely invested in shares and hence, as opposed to so called guarantee funds (frequently offered in cooperating with life insurers), the mutual fund does not provide any downside protection. Of course, the number of shares in a mutual fund purchased at time t_i depends on the prevailing market value S_{t_i} of the fund assets and is given by $\pi^I_{t_i} \cdot S_{t_i}$.

Definition 8.2.1. The market value of the insureds portfolio at time $t > 0$ equals

$$A(t,\pi,\varphi) := \sum_{i=0}^{M(x,t)} \pi^I_{t_i} \cdot \frac{S_t}{S_{t_i}} = \sum_{i=0}^{M(x,t)} (1-\varphi)^{M(x,t)+1-i}\, \pi^N_{t_i} \cdot \frac{S_t}{S_{t_i}} \qquad (8.1)$$

with start value $A(0,\pi,\varphi) = 0$ and summation limit

$$M(x,t) := n^*(\min\{\tau_x,t\}) \quad (n^*(t) := \max\{j \in \underline{T} \mid t_j < t\}).$$

According to Definition 3.1, τ_x denotes the random residual lifetime of an insured aged x at time $t_0 = 0$. The investment premium

$$\pi^I_{t_i} := (1-\varphi)\left(\pi^G_{t_i} - \phi\right) - \varphi \cdot A(t_i,\pi,\varphi) = (1-\varphi)\pi^N_{t_i} - \varphi \cdot A(t_i,\pi,\varphi)$$

thus becomes path dependent.

Notice that a proof of the second Equation in (8.1) follows the argumentation of Schrager and Pelsser (2004) and, for the sake of completeness, is given in Appendix C.2.1. Although there is only one investment made at the inception date t_0 within single premium contracts, the withdrawals of the charge are assumed to proceed periodically as well. For the sake of simplicity we decided against a consideration of annual fees like acquisition, commission and selling expenses or third-party management fees[170]. The basic unit-linked contract can be extended by adding guarantees providing security for surviving dependants and/or against times of economic downfalls or bad fund performance at maturity. These options are called GMDB and GMIB respectively. Hereafter, they are each described in detail.

[170] Throughout the chapter we act on the assumption of a single fixed reference fund such that investment choices for sub-accounts do not take place and therefore the investment expenses can be neglected.

8.2.2 The Guaranteed Minimum Death Benefit (GMDB) rider

Typically, the majority of the contracts with guaranteed income benefits include a guaranteed minimal death benefit such that in the case of death at time t within the deferment period $]\,t_0,T\,[$ the insureds dependants receive the gross premiums paid $g_D(t)$ until time t plus a bonus amounting to the repayment level ξ times the greater of the sum of premiums paid and current fund value $A(t,\pi,\varphi)$. In general, the GMDB option expires at retirement T.

Definition 8.2.2. Following the notation of Nielsen and Sandmann (2002) the total guaranteed death benefit $G_D(t)$ equals

$$\begin{aligned} G_D(t) &:= \max\left\{\xi\, A(t,\pi,\varphi) + (1-\xi)g_D(t),g_D(t)\right\} \\ &= g_D(t) + \xi\left[A(t,\pi,\varphi) - g_D(t)\right]^+ \\ &= \sum_{k=0}^{i-1} \pi_{t_k}^G + \xi\left[\sum_{k=0}^{i-1}(1-\varphi)^{i-k}\,\pi_{t_k}^N \cdot \frac{S_t}{S_{t_k}} - \sum_{k=0}^{i-1}\pi_{t_k}^G\right]^+ \end{aligned} \tag{8.2}$$

if the insured dies in year t_{i-1}, i.e. at any real-valued time $t \in]t_{i-1},t_i]$. The guarantee can be interpreted as a portfolio of European call options.

For a repayment level ξ equal to one the death payment equals the maximum of the benefit base $g_D(t)$ and portfolio value $A(t,\pi,\varphi)$ paid upon the death of the policyholder. This variant is called the return of premium death benefit and forms the basic death benefit frequently included in VAs with GMIB rider. For $\xi = 0$ the payment equals the guaranteed amount $g_D(t)$ in form of a simple premium refund commonly offered in conventional pension contracts. In the following, we restrict ourselves to the case $\xi = 1$, i.e. the full amount of the bonus payment. A reduced level $\xi \in [0,1[$ solely leads to a decreased option price and thus a decreased rider fee to be charged.

8.2.3 The Guaranteed Minimum Income Benefit (GMIB) rider

On expiry of the preset deferment or waiting period $[0,T]$ (typically at least 10 years), the insured has the contractual right to decide between four different options. Hereby, any pension drawdown within a so called early retirement phase will be disregarded.

(a) First of all, the insured can allow the GMIB option to lapse and make the pension provider pay out the market value of his investment portfolio as a lump sum (either as a cash payment or as fund shares) given by $A(T,\pi,\varphi)$. Once this opportunity has been chosen, the contract between the annuity provider and the insured ends immediately at maturity T.

(b) Secondly, the insured can choose to convert the market value of his portfolio into an immediately starting life annuity calculated with the actuarial basis in force, i.e. the pension payment is not guaranteed. The estimation of the fair value of the prevailing "market annuity factor" $\ddot{a}_{x+T}(T)$ and its reciprocal, the payout rate $r_{x+T}(T)$ at time T, takes place under the risk neutral measure[171] P^*. Hence, the fair annuity value given the information[172] $\mathcal{F}_T$ is defined as

$$\ddot{a}_{x+T}(T) := E_{P^*}\left[\sum_{j=N}^{\omega-x} \left(\beta_{T,t_j}\right)^{-1} 1_{\{\tau_x > t_j\}} \middle| \mathcal{F}_T\right] \tag{8.3}$$

where τ_x denotes the random residual lifetime of a cohort-representative policyholder aged x entering the contract at time $t_0 = 0$. β_{T,t_j} describes the stochastic bank account at time T with maturity t_j. A definition of the money market account follows in Subsection 8.3.2. In particular, the annuity factor $\ddot{a}_{x+T}(T)$ equals the expected discounted value at time T of an immediately starting life annuity with advanced annual payments of one monetary unit throughout the remaining lifetime of a person aged $x+T$ at time T. Vice versa, the market annuity payout rate $r_{x+T}(T)$ specifies the annual amount paid out by an immediately starting life annuity, negotiated at time T for a single premium of one monetary unit. Later on, the market factor $\ddot{a}_{x+T}(T)$ – which is a function of both interest and mortality rates – will be discussed more detailed.[173] Under the assumption of the market annuity factor being a fair annuity factor, the option to convert the investment portfolio just equals the market value of the investment portfolio, since

$$A(T,\pi,\varphi) \cdot \underbrace{r_{x+T}(T) \cdot \ddot{a}_{x+T}(T)}_{=1} = A(T,\pi,\varphi),$$

where $A(T,\pi,\varphi) \cdot r_{x+T}(T)$ corresponds to the market pension from the

[171] Let P^* denote the joint probability measure of the financial market measure and the mortality measure. Due to a complete financial market the marginal distribution concerning the financial market is given by the unique risk-neutral measure P^*. Mortality risk is quantified via the real-world marginal measure P due to the (current) absence of a market price of mortality risk.

[172] A complete filtration setup for the underlying asset as well as mortality model is explained in Section 8.3.

[173] For example Boyle and Hardy (2003), Pelsser (2003) or Ballotta and Haberman (2006) analyse a contract containing an embedded put option on the annuity factor $\ddot{a}_{x+T}$, a so-called guaranteed annuity option. These minimum return guarantees were commonly issued within pension policies of British insurance companies in the seventies and eighties.

investment portfolio[174].

(c) Third, the insured can choose an immediately starting life annuity paying out a fixed annual pension. The fixed amount is assumed to be pre-specified at contract inception date $t_0 = 0$. The expected discounted value (market price) $\ddot{a}_{x+T}(T)$ at time T of a life annuity with guaranteed benefit base $L(T)$ annuitised at a fixed guaranteed rate $r^G_{x+T}(T)$ is given by

$$g_I(T,x) := L(T) \cdot r^G_{x+T}(T) \cdot \ddot{a}_{x+T}(T) \tag{8.4}$$

where the product $L(T) \cdot r^G_{x+T}(T)$ equals the fixed annual pension payment. The representation for the guaranteed periodic annuity amount as a certain lump sum $L(T)$ times a certain guaranteed rate $r^G_{x+T}(T)$ was selected to allow for GMIBs calculated with different guaranteed interest yields $\delta \geq 0$. More precisely, given a certain predetermined[175] rate δ and charge φ, the factors $L(T,\delta,\varphi)$ and $r^G_{x+T}(T,\delta)$ are called GMIB payout amount and rate respectively. Usually, the guaranteed GMIB payout $L(T,\delta,\varphi)$ is provided as one of the guarantee features listened in Table 8.4 where variable indication was omitted for the sake of clarity. The fixed payout rate is defined via

$$r^G_{x+T}(T) := \left(\sum_{j=N}^{\omega-x} e^{-\delta(t_j - T)} \cdot {}_{t_j - T}\bar{p}_{x+T} \right)^{-1}, \tag{8.5}$$

where ω denotes the maximum attainable age from a conservative life table at time $t_0 = 0$ and ${}_s\bar{p}_x$ the corresponding s-year survival probability in line with a prudent product design. Normally, the insurer selects a life table which ensures that $\frac{r^G_{x+T}(T)}{r_{x+T}(T)} < 1$ under current mortality expectation. With deciding for opportunity (b) as well as (c) no further dependants protection[176] elements are assumed to be included in the contract.

(d) Finally, the insured can decide to enter an unit-linked prolongation phase $\varDelta_S$, which usually lasts 7-10 years. At the beginning of the extension in $t_N = T$ the option to hold the policy for one more year and to extend the fund investment is available. One year later, the opportunities (a)-(c) as well as the prolongation (d) can be selected. In general, at each time $t_j \in \overline{T} := \{t_N, t_{N+1}, \ldots, t_{N+m-1} \in \varDelta_S \,|\, t_N < t_{N+1} < \ldots < t_{N+m-1}\}$ the opportunities (a)-(d) can be chosen and at time t_{N+m} solely the choices

[174] Up to this point we are still in the case of an unit-linked pension contract.
[175] The guaranteed rate δ usually exceeds the prevailing risk-free (market) interest rate.
[176] In practice, current VA product generations include a minimum duration of guaranteed annuity payments independent of the insureds survival during the pension phase. Furthermore, the insured can write a conventional supplementary insurance providing living benefits confined to the widow / widower.

(a)-(c) are available[177]. If the policyholder exercises the option to prolongate the contract at any time t_j, we assume the insurer to withdraw the amount of $\varphi A(t_j,\pi,\varphi)$ from the insureds investment portfolio. Depending on the different enhancement types from Table 8.1 the guarantee is, if necessary, upgraded in case of an exercise of the prolongation option. In the case of death within the annuity prolongation phase the dependants receive the insureds portfolio value as a lump sum payment, i.e. no downside protection elements are included.

Variable annuity guarantee features

Feature	Definition of the benefit base	Description
Return of premium	$L^{\text{RP}} = \sum_{i=0}^{N-1} \pi^G_{t_i}$	The standard return of premium option offers a benefit basis equal the total gross premiums paid (corresponds to a guaranteed minimum yield of 0%).
Roll-up	$L^{\text{RO}} = \sum_{i=0}^{N-1} \pi^G_{t_i} \cdot e^{\delta(T-t_i)}$	The (annual) roll-up benefit equals the respective premium amount accumulated at a fixed roll-up rate δ. This basis represents the standard GMIB market feature with a minimal annual increase commonly in the range of 3-6%.
Ratchet	$L^{\text{RA}} = \max_{i=1,\ldots,N} \{L^{\text{RP}}, A(t_i,\pi,\varphi)\}$	The (annual) ratchet benefit equals the maximum of the guarantee amount and the highest historic fund value at different exercise dates, i.e. a lookback-option with annual peak level lock-in.
Greater of ratchet or roll-up	$L^{\text{RORA}} = \max_{i=1,\ldots,N} \{L^{\text{RO}}, A(t_i,\pi,\varphi)\}$	The enhancement is calculated as the greater amount under roll-up or ratchet feature.

Table 8.1: Common guarantee features offered within VA contracts.

Throughout the following we assume that at retirement date T the insured behaves financially rational such that the GMIB option is exercised if it is in the money, i.e. it exhibits positive intrinsic value. Assuming this kind of behaviour tends to result in higher fair charges than those demanded at the market[178]. With the objective of showing the existence of a fair guarantee charge, option (d) will for the time being ignored.

Definition 8.2.3. Conditional upon the survival of the insured until maturity

[177] For fiscal reasons, the time t_{N+m} is limited from above by the tax-related maximum retirement age derived from the applied life table.
[178] A policyholder could for example choose the lump sum payment even if the guarantee is in the money due to a personal financial shortage or a deteriorated health condition.

T the total contract value equals

$$\begin{aligned} G_I(T,x) &:= \max\left\{A(T,\pi,\varphi), g_I(T,x)\right\} = g_I(T,x) + \left[A(T,\pi,\varphi) - g_I(T,x)\right]^+ \\ &= L(T)\cdot r^G_{x+T}(T)\cdot \ddot{a}_{x+T}(T) \\ &\quad + \left[\sum_{i=0}^{N-1}(1-\varphi)^{N-i}\pi^N_{t_i}\frac{S_T}{S_{t_i}} - L(T)\cdot r^G_{x+T}(T)\cdot \ddot{a}_{x+T}(T)\right]^+, \end{aligned} \quad (8.6)$$

i.e. the guarantee $g_I(T,x)$ plus the value of an European exchange option including the right to trade / surrender the terminal fund value in exchange for the market value of a life annuity with a guaranteed pension payment.

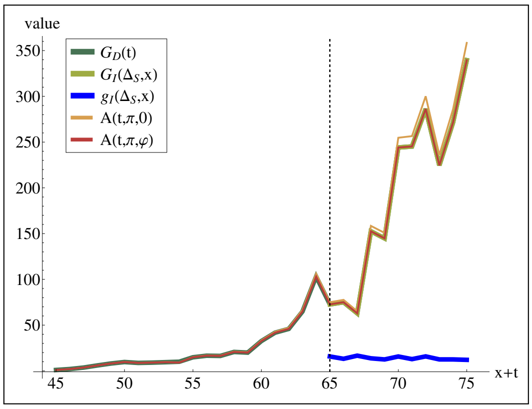
A conclusion on the intrinsic value at the end of the deferment period thus depends not only on the fund performance but also on the evolution of the annuity factors. In Figures 8.1 and 8.2 we illustrate real-world sample paths of the guarantees (8.2) and (8.6) for a deferment period of length $T = 20$, a prolongation phase of $\Delta_S = 10$ years and the different pension benefit features listened in Table 8.1. Since rider charges increase with the richness of the benefit base $L(T)$ we assume a common market charge sequence $\{\varphi^{\text{RP}},\varphi^{\text{RO}},\varphi^{\text{RA}},\varphi^{\text{RORA}}\} = \{25, 40, 60, 85\}$ for the GMDB and GMIB bundle measured in basis points. In each case the rider fee for the guaranteed refund of insurance premiums in case of death amounts to 10 basis points. In scenario A the guarantee base increases due to a good fund performance. Hence, the ratchet feature leads to the highest option strike level $g_I(\Delta_S,x)$ (Subfigure (8.1.3)). For the RORA case (Subfigure (8.1.4)) the higher cost deduction caused by high fund returns decreases the option value accordingly. Conversely, in scenario B the guarantees are in the money (especially the GMDB guarantee rider) and claim payouts are significant especially for ages $x \leq 70$. Due to the underperforming mutual fund, the roll-up feature with a guaranteed interest δ (Subfigure (8.2.2)) constitutes the highest downside protection for the customer. For the ratchet benefit (Subfigure (8.2.3)) the GMIB rider is merely "out of the money".

8.3 Model specification

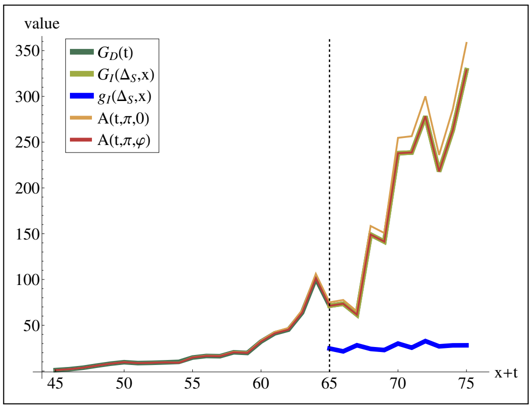
8.3.1 Combined filtration setup

In the following, we introduce stochastic processes for the fund prices, spot interest rates, bond prices and mortality rates. All processes are defined on a filtered probability space $(\Omega,\mathbb{F},P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual

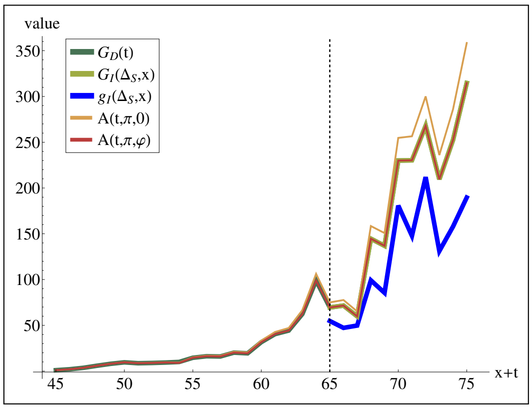
Guarantee value and fund value paths for different features - scenario A



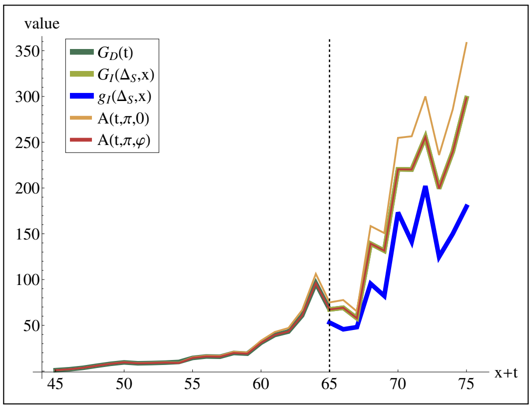
(8.1.1) $L(T) = L^{\mathrm{RP}}$ $(\varphi^{\mathrm{RP}} = 25\,bps)$



(8.1.2) $L(T) = L^{\mathrm{RO}}$ $(\varphi^{\mathrm{RO}} = 40\,bps)$



(8.1.3) $L(T) = L^{\mathrm{RA}}$ $(\varphi^{\mathrm{RA}} = 60\,bps)$



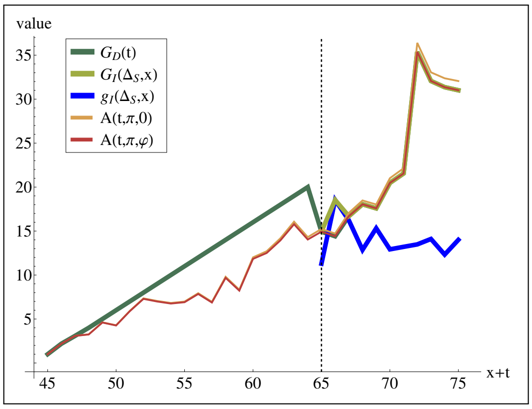
(8.1.4) $L(T) = L^{\mathrm{RORA}}$ $(\varphi^{\mathrm{RORA}} = 85\,bps)$

Figure 8.1: Real-world guarantee values for different features in case of a good fund performance (scenario A) w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$, a prolongation phase $\Delta_S = 10$ years and a guarantee rate $\delta = 0.04$. The process parameters are taken from Tables 8.2-8.4.

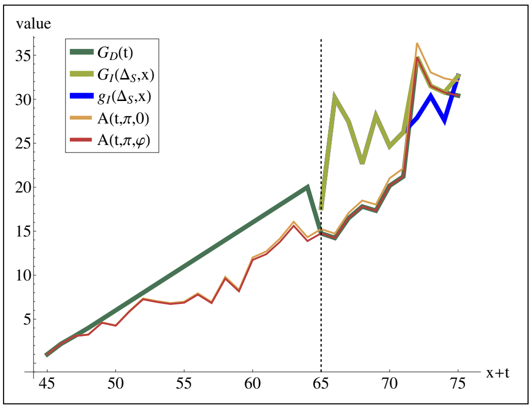
conditions and representing all available information at different times $t \geq 0$. The filtration $\mathbb{F}$ is composed as an aggregation of two strict subfiltrations signed $\mathbb{G}$ and $\mathbb{H}$. Subfiltration $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,\omega-x]}$ with $\mathcal{H}_t = \sigma\left(\{\mathbb{1}_{\{\tau_x \leq s\}} : 0 \leq s \leq t\}\right)$ describes the risk of random fluctuations of the residual lifetime and corresponds to the minimal filtration containing information whether death occurs or not. Subfiltration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,\omega-x]}$ contains information in form of a minimal σ-algebra $\mathcal{G}_t = \sigma\left(\{\mu_{x+s}, r_s, S_s : 0 \leq s \leq t\}\right)$ concerning the evolution of interest rate, fund price and mortality rate state variables until time t. $\mathcal{G}_t = \mathcal{I}_t \vee \mathcal{M}_t$ equals a disjoint union of filtrations $\mathcal{I}_t = \sigma\left(\{r_s, S_s : 0 \leq s \leq t\}\right)$ and $\mathcal{M}_t = \sigma\left(\{\mu_{x+s}(s) : 0 \leq s \leq t\}\right)$ for all $t \in [0, \omega - x]$, i.e. the financial risk can be decoupled from mortality risk via independence assumption.[179] In

[179] The independence assumption of mortality and financial risk is quite common in actuarial literature, see e.g. Bacinello and Ortu (1994), Nielsen and Sandmann (1996), Milevsky and Promislow (2001) or Dahl and Møller (2006).

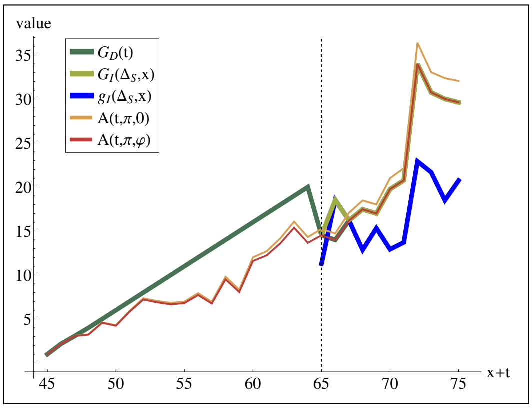
Guarantee value and fund value paths for different features - scenario B



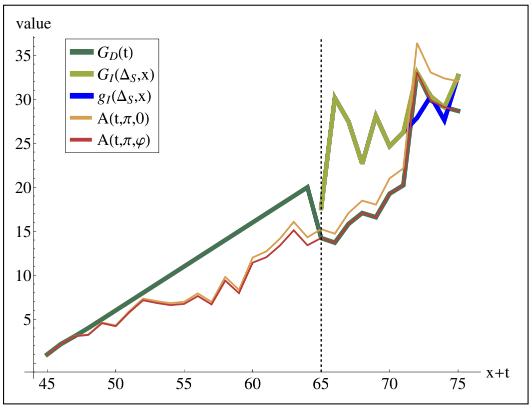
(8.2.1) $L(T) = L^{\text{RP}}$ ($\varphi^{\text{RP}} = 25\,bps$)



(8.2.2) $L(T) = L^{\text{RO}}$ ($\varphi^{\text{RO}} = 40\,bps$)



(8.2.3) $L(T) = L^{\text{RA}}$ ($\varphi^{\text{RA}} = 60\,bps$)



(8.2.4) $L(T) = L^{\text{RORA}}$ ($\varphi^{\text{RORA}} = 85\,bps$)

Figure 8.2: Real-world guarantee values for different features in case of a bad fund performance (scenario B) w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$, a prolongation phase $\Delta_S = 10$ years and a guarantee rate $\delta = 0.04$. The process parameters are taken from Tables 8.2-8.4.

particular, filtration $\mathcal{I}_t$ is independent of the union $\mathcal{H}_t \vee \mathcal{M}_t$ for times $t \geq 0$.

8.3.2 Interest rate model

For the financial setting (cf. Subsection 7.2.2) we define $\beta_{t,\bar{t}}$ as the stochastic bank account at time t with maturity $\bar{t}$ of one monetary unit invested at time t ($0 \leq t \leq \bar{t} \leq \omega - x$) given by

$$\beta_{t,\bar{t}} = \exp\left(\int_t^{\bar{t}} r_u\, du\right) \quad \text{following the dynamics} \quad d\beta_{t,\bar{t}} = \beta_{t,\bar{t}}\, r_t dt\,,$$

where $r = (r_t)_{0 \leq t \leq \omega - x}$ denotes the continuously compounded spot interest rate. Let $D(t,\bar{t})$ denote the time t price of a zero coupon bond with maturity $\bar{t}$ defined by

$$D(t,\bar{t}) = E_P \left[\exp\left(- \int_t^{\bar{t}} r_u \, du \right) \middle| \mathcal{I}_t \right] = E_P \left[\left(\beta_{t,\bar{t}} \right)^{-1} \middle| \mathcal{I}_t \right]. \qquad (8.7)$$

In the following, we assume a complete and arbitrage-free financial market model under interest rate risk where the dynamic of the zero coupon bonds is lognormal. Thus, the interest rate dynamics follow a Gaussian Heath, Jarrow and Morton model (1992), i.e.

$$dD(t,\bar{t}) = D(t,\bar{t}) \left(r_t dt + \sigma_{\bar{t}}(t) dW_t^r \right), \quad D(t,t) = 1, \qquad (8.8)$$

where W^r denotes a one-dimensional $\mathcal{I}_t$-adapted Brownian motion with respect to the real world measure P, and $\sigma_{\bar{t}}$ satisfies the usual regularity conditions. In particular, the volatility of the zero coupon bond $\sigma_{\bar{t}}(t)$ is a time-dependent function[180] with $\sigma_{\bar{t}}(\bar{t}) = 0$. For illustrative purposes, we apply a (one-factor) Hull and White model (1990) where

$$d\, r_t \;=\; (\theta(t) - a\, r_t)\, dt + \sigma_{spot}\, d\, W_t^r, \quad r_0 > 0 \qquad (8.9)$$

for speed of mean reversion a, long term reversion level $\theta(t)/a$ and spot rate volatility σ_{spot}. In contrast to Subsection 7.3.2 the parameters a and σ_{spot} are exogenous and not calibrated to market swaption quotes. The closed form solution for the zero coupon bond price is given by

$$D(t,\bar{t}) = \exp\left(A(t,\bar{t}) - B(t,\bar{t}) r_t \right)$$

where

$$A(t,\bar{t}) = \ln\left(\tfrac{D^M(0,\bar{t})}{D^M(0,t)} \right) + B(t,\bar{t})\, f(0,t) - \tfrac{\sigma_{spot}^2}{4a} \left(B(t,\bar{t}) \right)^2 \left(1 - e^{-2at} \right),$$
$$B(t,\bar{t}) = \tfrac{1}{a} \left(1 - e^{-a(\bar{t}-t)} \right).$$

The quantity $D^M(0,\bar{t})$ denotes the market discount factor[181] and market instantaneous forward rates are derived from $f^M(0,\bar{t}) = -\frac{\partial \ln(D^M(0,\bar{t}))}{\partial \bar{t}}$. The benchmark parameter constellation is given in Table 8.2. The uncertainty of the market annuity factor $\ddot{a}_{x+T}(T)$ at retirement T as an expected discounted value additionally exposes the contract value to interest rate risk.

[180] In particular, it holds $\sigma_{\bar{t}}(t) = \frac{\sigma_{spot}}{a}(1 - e^{-a(\bar{t}-t)})$ for the Hull-White interest rate model. The time-dependent model parameter $\theta(\,.\,)$ is used to calibrate the model to the observed market zero bond prices $D^M(0,\bar{t})$.

[181] For reasons of simplification and ease of use, we assume initial market prices $D^M(0,\bar{t}) = e^{-r_0^M \bar{t}}$ resulting from a flat initial term structure at r_0^M continuously compounded.

8.3.3 Mutual fund model

A common choice for the price process of a single mutual (reference) fund or stock $\{S_t : t \geq 0\}$ is given by the geometric Brownian motion[182] along the lines of Black and Scholes (1973), i.e.

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S, \quad S_0 > 0 \tag{8.10}$$

for an instantaneous rate of return μ_S, volatility σ_S and a one-dimensional $\mathcal{I}_t$-adapted Brownian motion with respect to P satisfying the usual regularity conditions. Therefore, Itô's lemma yields real world fund prices

$$S_t = S_0 \exp\left(\left(\mu_S - \frac{\sigma_S^2}{2}\right)\, t + \sigma_S W_t^S\right)$$

such that future fund prices are lognormally distributed with $S_t \sim LN(S_0 e^{\mu_S t}, S_0^2 e^{2\mu_S t}(e^{\sigma_S^2 t} - 1))$. As equations (8.2) and (8.6) indicate, the contract value at time $t \in \,]\,0, T\,]$ is strongly affected by the performance of the mutual fund. In the following, we concentrate on a single reference fund S quoted in national currency (see Table (8.3) for the parameters used in the numerical analysis under Section 8.5) such that the insured does not need to select out of a number of funds. Thus, the so called right to switch or shift between different offered mutual funds during the accumulation period should no longer be regarded.

8.3.4 Complete and arbitrage-free financial market model

In the following, we define a combined equity and interest rate market model. Therefore the financial market consists of two traded assets, a fund and a bond. Under the financial market measure P^* the risk-neutral fund and bond prices[183] follow

$$\begin{aligned} dS_t &= S_t\left(r_t dt + \sigma_S dW_t^*\right) \\ dD(t,\bar{t}) &= D(t,\bar{t})\left(r_t dt + \sigma_{\bar{t}}(t) dW_t^{1*}\right) \end{aligned}$$

where W^* and W^{1*} denote Brownian motions with respect to P^*. We assume the fund and bond returns to be correlated with cross-variation $d\left\langle W^*, W^{1*}\right\rangle_t = \rho dt$

[182] For the sake of simplicity, we choose a geometric Brownian motion although it might not be suitable to model stock prices for a long term equity-linked life insurance contract. Furthermore, we excluded stochastic (fund) volatility. For further research on stochastic asset volatility in the context of variable annuities see Benhamou and Gauthier (2009).
[183] The Girsanov Theorem ensures the existence of $dW_t^* = dW_t^S + \left(\frac{\mu_S - r_t}{\sigma_S}\right)dt$ where $\frac{\mu_S - r_t}{\sigma_S}$ denotes the classical form for the market price of risk.

and correlation coefficient $\rho \in [-1,1]$ respectively. Furthermore, the discounted price process $\{S_t\}_{0 \le t \le u}$ of the mutual fund as well as the discounted price process $\{D(t,u)\}_{0 \le t \le u}$ (normalised under the money market account) of all unit face value zero coupon bonds with maturity $u \in \mathbb{R}^+$ are martingales under P^*, i.e.

$$\begin{aligned} S_t &= E_{P^*}\left[\left(\beta_{t,\bar{t}}\right)^{-1} S_{\bar{t}} \middle| \mathcal{I}_t\right] \quad \forall\, \bar{t} \ge t, \\ D(t,u) &= E_{P^*}\left[\left(\beta_{t,\bar{t}}\right)^{-1} D(\bar{t},u) \middle| \mathcal{I}_t\right] \quad \forall\, \bar{t} \in [t,u], \forall u \ge \bar{t}. \end{aligned}$$

The risk-neutral martingale measure is the probability measure employing the risk-free money-market account as the numeráire. Given an unique risk-neutral measure, the pricing of financial instruments is applied by path dependent computation of the expected discounted future cash flows. As a next step, consider a Brownian motion W^{2*} independent from W^{1*} with $d\left\langle W^{1*}, W^{2*}\right\rangle_t = 0$. In order to decorrelate the interest rate and fund asset risk we express W^* via the Cholesky decomposition

$$dW_t^* = \rho\, dW_t^{1*} + \sqrt{1-\rho^2}\, dW_t^{2*},$$

such that

$$dS_t = S_t\left(r_t dt + \sigma_S\left(\rho\, dW_t^{1*} + \sqrt{1-\rho^2}\, dW_t^{2*}\right)\right), \tag{8.11}$$

$$dD(t,\bar{t}) = D(t,\bar{t})\left(r_t dt + \sigma_{\bar{t}}(t) dW_t^{1*}\right). \tag{8.12}$$

Along the lines of Jamshidian (1991) or Geman et al. (1995) we change the numeráire from the money-market account towards zero coupon bond prices with maturity u ($t \le \bar{t} \le u$) which leads to a u-forward risk-adjusted measure P^u determined by the Radon-Nikodym derivative

$$\left.\frac{dP^u}{dP^*}\right|_{\mathcal{I}_t} = \exp\left(\int_0^t \sigma_u(s) dW_s^{1*} - \frac{1}{2}\int_0^t \left(\sigma_u(t)\right)^2 ds\right).$$

Under this measure, expected cash flows of the financial instruments discounted with the bond numeráire are martingales[184] under P^u such that

$$S_t = D(t,u)\, E_{P^u}\left[\frac{S_u}{D(u,u)} \middle| \mathcal{I}_t\right] \quad \forall\, u \ge t,$$

[184] Nielsen and Sandmann (1996) give more details and necessary regularity conditions with respect to the financial market.

$$D(t,T) = D(t,u)\ E_{P^u}\left[\frac{D(u,T)}{D(u,u)}\middle|\mathcal{I}_t\right] \quad \forall\, u \in [t,T]$$

with normalised price dynamics

$$\begin{aligned} d\left(\frac{S(t)}{D(t,u)}\right) &= \frac{S(t)}{D(t,u)}\left((\rho\sigma_S - \sigma_t(u))\, dW_t^{1*,u} + \sqrt{1-\rho^2}\,\sigma_S\, dW_t^{2*,u}\right) \\ d\left(\frac{D(t,\bar{t})}{D(t,u)}\right) &= \frac{D(t,\bar{t})}{D(t,u)}\left(\sigma_t(\bar{t}) - \sigma_t(u)\right) dW_t^{1*,u} \end{aligned}$$

for Brownian motions $dW_t^{1*,u} = (dW_t^{1*} - \sigma_t(u))\, dt,\ dW_t^{2*,u} = dW_t^{2*}$ under P^u. Hence, the expected (forward) accumulated fund shares until time u ($T \geq u \geq t_i \geq 0;\ i = 0,...,n^*(u)$) accrued against a unit premium under the forward measure P^u can be written as

$$\begin{aligned} E_{P^u}\left[\sum_{i=0}^{n^*(u)} \frac{S_u}{S_{t_i}}\middle|\mathcal{I}_0\right] &= \sum_{i=0}^{n^*(u)} E_{P^u}\left[\frac{S_u}{S_{t_i}}\middle|\mathcal{I}_0\right] = \frac{1}{D(0,u)} \sum_{i=0}^{n^*(u)} E_{P^*}\left[(\beta_{0,u})^{-1}\frac{S_u}{S_{t_i}}\middle|\mathcal{I}_0\right] \\ &= \frac{1}{D(0,u)} \sum_{i=0}^{n^*(u)} E_{P^*}\left[\frac{(\beta_{0,t_i})^{-1}}{S_{t_i}} E_{P^*}\left[(\beta_{t_i,u})^{-1} S_u\middle|\mathcal{I}_{t_i}\right]\middle|\mathcal{I}_0\right] \\ &= \frac{1}{D(0,u)} \sum_{i=0}^{n^*(u)} E_{P^*}\left[(\beta_{0,t_i})^{-1}\frac{S_{t_i}}{S_{t_i}}\middle|\mathcal{I}_0\right] = \sum_{i=0}^{n^*(u)} \frac{D(0,t_i)}{D(0,u)}. \end{aligned} \tag{8.13}$$

Therefore, application of the forward measure allows a path-independent calculation of the accumulated fund shares, which is not the case for the equivalent martingale measure P^*.

Parameters for the Hull-White model

r_0^M	a	$\theta(t)$	σ_{spot}
0.035	0.25	$ar_0^M + \frac{\sigma_{spot}^2}{2a}\left(1 - e^{-2at}\right)$	0.015

Table 8.2: Parameters for the Hull-White interest rate process. For the sake of simplicity, we assume a flat (initial) term structure[185] r_0^M.

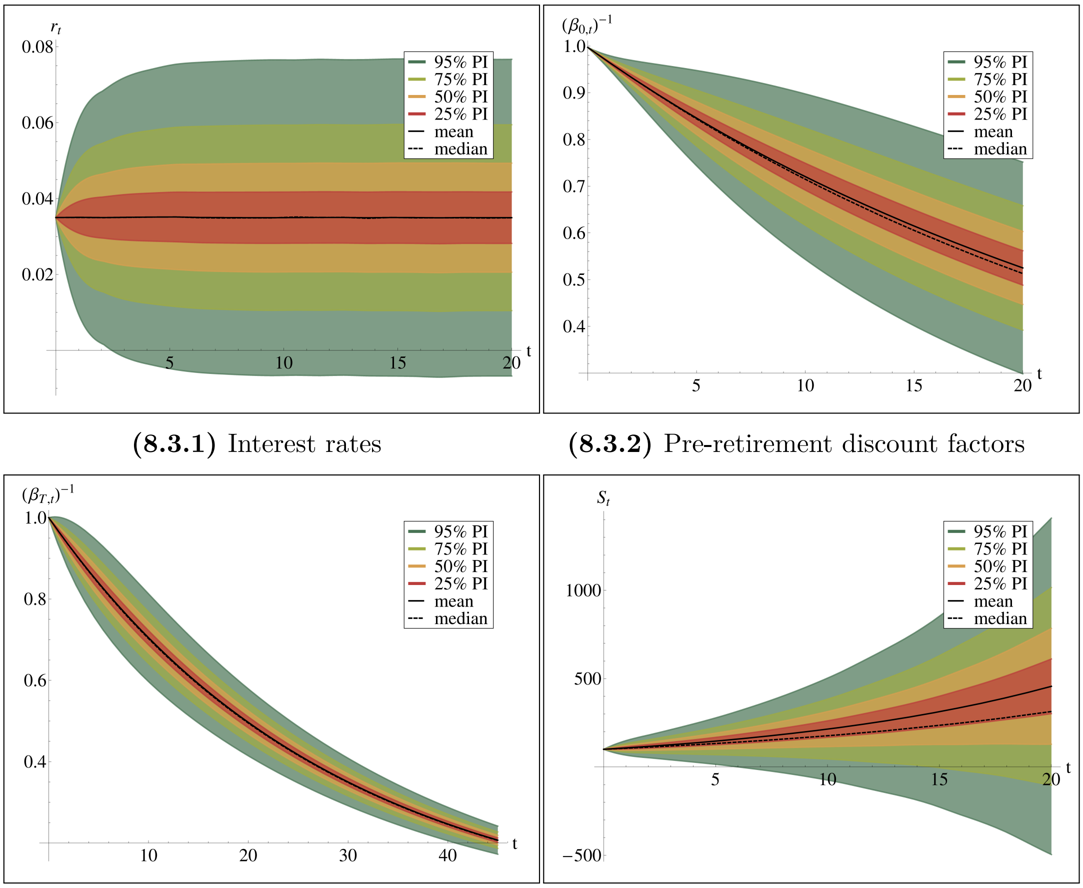
Parameters for the fund value process

S_0	μ_S	σ_S	ρ
100	0.08	0.2	-0.1

Table 8.3: Parameters for the mutual fund value process.

[185] A calibration of the Hull and White model on the basis of the market yield curve and swaption straddle implied volatilities was already executed in Subsection 7.3.2.

Real-world process prediction



(8.3.1) Interest rates

(8.3.2) Pre-retirement discount factors

(8.3.3) Post-retirement discount factors

(8.3.4) Mutual fund prices

Figure 8.3: Real-world process prediction interval fan charts and location measures mean (solid) and median (dashed) for $5 \cdot 10^4$ sample paths. The sampling is based on the parameters listed in Tables 8.2-8.4.

8.3.5 Mortality model

The motivation and necessity of modelling mortality stochastically was given in Section 4.1 of Chapter 4. The stochastic mortality risk model presented by Ballotta and Haberman (2006) allows for time and age dependence of the mortality rate and furthermore for integrated modelling of unsystematic and systematic mortality risk. Unsystematic mortality risk describes random fluctuations around a trend and can be diversified by enlarging the cohort size using the law of large numbers. Ballotta and Haberman (2006) combine a traditional mortality law, i.e. a parametric age-dependent function of the mortality rate, with a stochastic exponential reduction factor. A reduction factor determines the rate of change in mortality rates over time.

We assume that the initial force of mortality $\mu_x(0)$ is given by a standard

Gompertz-Makeham mortality law of the form

$$\mu_{x+t}(0) = \alpha_0 + \beta_0\, \gamma_0^{x+t} \quad \alpha_0 \geq 0;\, \beta_0, \gamma_0 > 0.$$

The initial mortality law $\mu_{x+t}(0)$ equals the mortality rate of a person aged $x+t$ years at time 0. Combining the age-dependent mortality law with an age-period exponential reduction factor yields

$$\mu_{x+t}(t) = \mu_{x+t}(0) \exp\left((\alpha + \beta(x+t))t + \gamma Y_t\right) \quad \alpha \geq 0, \beta, \gamma > 0 \tag{8.14}$$

$(Y_t)_{t\geq 0}$ forms an Ornstein-Uhlenbeck process with a long-run mean reversion[186] level that equals zero and a speed factor κ defined by the differential equation

$$dY_t = -\kappa Y_t\, dt + dW_t^{\mu}, \quad Y_0 = 0, \kappa > 0$$

with a solution given by

$$Y_t = \int_0^t e^{-\kappa u} dW_u^{\mu} \;\text{ such that }\; Y_t \sim N\Big(0, \tfrac{1-e^{-2\kappa t}}{2\kappa}\Big)\; \forall\, t > 0.$$

W^{μ} denotes a standard Wiener process under the physical measure P assumed to be independent of random motions in the financial market. The process (8.14) is similar to the non mean reverting Brownian Gompertz[187] process introduced in Milevsky and Promislow (2001) and the logarithmised mortality rate at time $t > 0$ is normally distributed[188] with mean $(\alpha + \beta(x+t))t + \ln(\mu_{x+t}(0))$ and variance $\frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa t})$.

LEMMA 8.3.1. *The mortality rate (8.14) follows a diffusion process of the form*

$$d\mu_{x+t}(t) = \zeta(\mu_{x+t}(t), t)\, \mu_{x+t}(t)\, dt + \gamma\, \mu_{x+t}(t)\, dW_t^{\mu}, \quad \mu_{x+t}(0) > 0 \tag{8.15}$$

with positive drift $\zeta(\mu_{x+t}(t), t)$ defined by the equation

[186] Ballotta and Haberman (2006) put the mean reversion property of mortality improvements up for discussion. Some authors like Luciano and Vigna (2006) conclude that the use of (positive) mean reverting affine processes which are successively used in finance implicates some undesirable characteristics providing a non-adequate image of mortality evolutional phenomenons like rectangularization or (high-)age expansion.

[187] In contrast, the process introduced by Milevsky and Promislow (2001) does not capture deterministic time dependence such that all period effects should be captured by the volatility parameter γ. Stochastic reduction factor models were also used by Dahl and Møller (2006) and Biffis et al. (2010).

[188] A proof of the normal distribution for Y_t and $\log(\mu_{x+t}(t))$ is given in Appendix C.1.

$$\zeta(\mu_{x+t}(t),t) = \alpha + \beta(x+2t) + \tfrac{1}{2}\gamma^2 + \kappa \ln\left(\mu_{x+t}(0)\right) + \kappa\left(\alpha + \beta(x+t)\right)t \\ - \kappa \ln\left(\mu_{x+t}(t)\right).$$

Proof. See Appendix C.1.2. □

The tail distribution of the remaining lifetime τ_x of a person aged x at time $t_0 = 0$ yields conditional survival probabilities equal to

$$\begin{aligned} {}_tp_x(0) &= P\left(\tau_x > t \middle| \mathcal{H}_0 \vee \mathcal{M}_0\right) = E_P\left[\mathbb{1}_{\{\tau_x > t\}} \middle| \mathcal{H}_0 \vee \mathcal{M}_0\right] \\ &= E_P[{}_t\tilde{p}_x | \mathcal{M}_0] = E_P\left[\exp\left(-\int_0^t \mu_{x+u}(u)du\right) \middle| \mathcal{M}_0\right], \\ {}_{T-t}p_{x+t}(t) &= P\left(\tau_x > T \middle| \mathcal{H}_t \vee \mathcal{M}_t\right) = E_P\left[\mathbb{1}_{\{\tau_x > T\}} \middle| \mathcal{H}_t \vee \mathcal{M}_t\right] \\ &= E_P\left[\frac{{}_T\tilde{p}_x}{{}_t\tilde{p}_x} \middle| \mathcal{M}_t\right] = E_P\left[\exp\left(-\int_t^T \mu_{x+u}(u)du\right) \middle| \mathcal{M}_t\right], \end{aligned} \tag{8.16}$$

where ${}_t\tilde{p}_x = E_P\left[\mathbb{1}_{\{\tau_x > t\}} \,|\, \mathcal{M}_t\right]$ is called the survivor function or survivor index and describes the survival probability conditioned on a realised mortality path until t. Since the integral (infinite sum) of lognormal mortality rates is not necessary lognormally distributed, the survival function ${}_{T-t}p_{x+t}(t)$ has no closed-form solution and thus has to be approximated by means of Monte Carlo simulation. Within the mortality model (8.14) parameter α, if chosen deterministically, accounts for changes in the mortality pattern over different time periods whereas the parameter β is responsible for age specific changes in the mortality intensity. Since systematic mortality risk describes period specific systematic deviations from the mortality expectation, Ballotta and Haberman (2006) introduce systematic parameter uncertainty by modelling $\alpha = \tilde{\alpha}$ as a random variable when sampling post-retirement survival probabilities. More precisely, in order to generate a single mortality rate path, $\tilde{\alpha}$ is taken from a certain discrete or continuous probability distribution[189]. The path drawn from this $\tilde{\alpha}$ yields one possible mortality evolution out of a family of future projected mortality rates. By taking the risk-neutral expectation we implicitely consider a different survival distribution and thus a longevity loading factor inducing higher charge values. Vice versa, for a fixed α, model (8.14) solely allows

[189] Ballotta and Haberman (2006) propose $\tilde{\alpha}$ to be uniform distributed. However, since longevity constitutes the predominant mortality trend we use an abscissa-reflected general Gamma $\Gamma(6{,}3 \cdot 10^{-3}{,}0.22)$ distribution which is left-skewed (i.e. longevity sensitive) and similar to the uniform approach concerning the moments.

for unsystematic mortality risk. In Figure 8.4 we illustrate post-retirement multi-annual survival probabilities for a fixed α (Subfigure (8.4.1)) and an uniformly distributed $\tilde{\alpha}$ introducing systematic mortality deviations (Subfigure (8.4.2)).

Benchmark parameters for the Brownian Gompertz-Makeham model

x	ω	α_0	β_0	γ_0	α	β	γ	κ
45	110	0	$1.29 \cdot 10^{-4}$	1.0829	-0.1051	$1.25 \cdot 10^{-3}$	0.1	0.5

Table 8.4: Parameters for the non mean reverting Brownian Gompertz-Makeham mortality model. The Gompertz-Makeham parameters (α_0, β_0, γ_0) were fitted to central death rates of the 2006 German male period life table from the Human Mortality Database (2009). Parameters γ and κ were taken from Ballotta and Haberman (2006), α and β were fitted to the post-retirement survival curve of the 2006 period life table since these parameters represent the deterministic part of the mortality model.

8.4 Pricing of variable annuities with bundled GMDB and GMIB rider

When considering VA contracts there arise some elementary differences to traditional (unit-linked) deferred annuity policies. This has implications both on the choice of the pricing measure and definition of fair contract parameters.

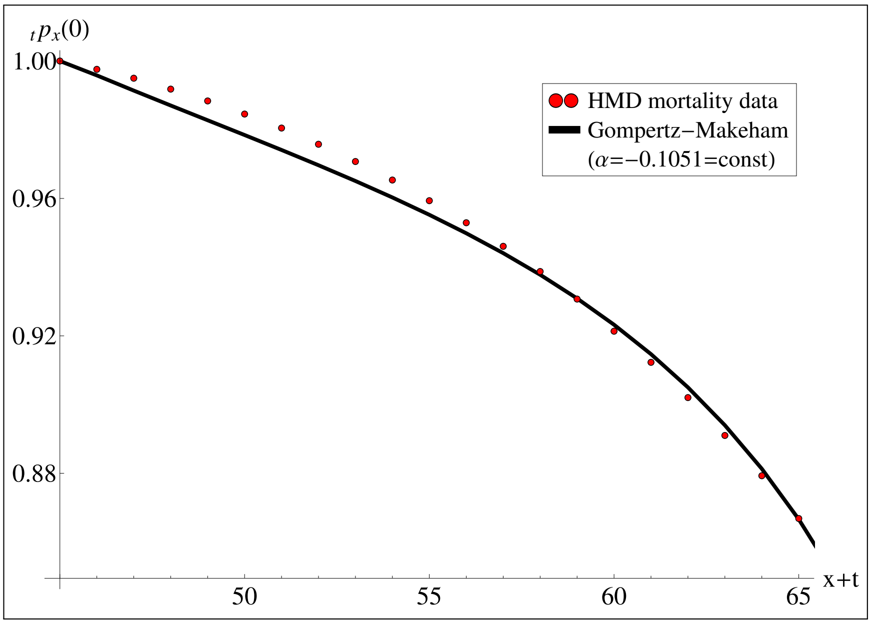
8.4.1 Derivation of the market annuity factor

As an integral part of the income benefit guarantee (8.4) the market annuity factor (8.3) has to be reviewed.

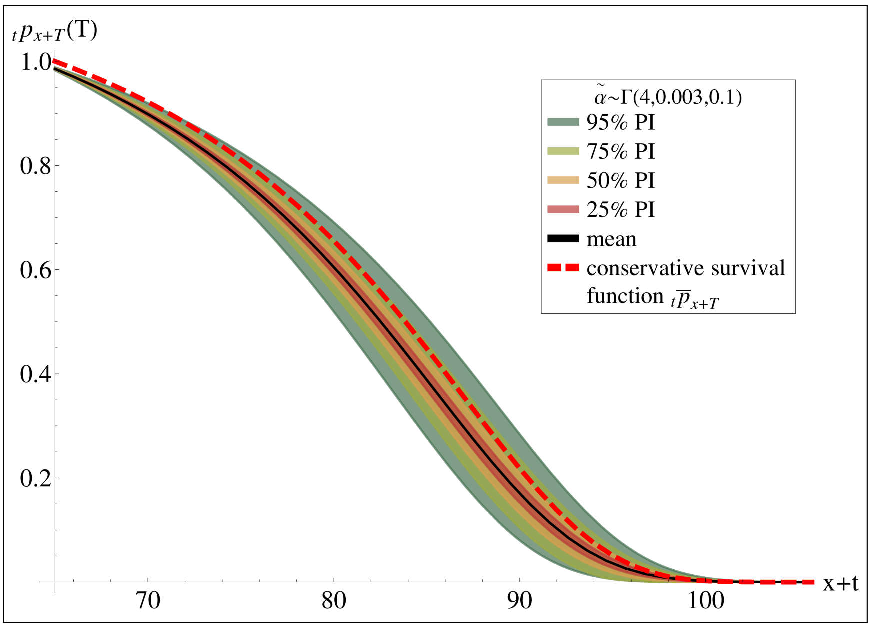
Remark 8.4.1. The independence assumption of financial and mortality risk together with the Definition (8.16) yields

$$\begin{aligned}
\ddot{a}_{x+T}(T) &= E_{P^*}\left[\sum_{j=N}^{\omega-x}\left(\beta_{T,t_j}\right)^{-1} 1_{\{\tau_x>t_j\}}\middle|\mathcal{F}_T\right] \\
&= E_{P^*}\left[\sum_{j=N}^{\omega-x}\left(\beta_{T,t_j}\right)^{-1}\middle|\mathcal{I}_T\right] E_P\left[1_{\{\tau_x>t_j\}}\middle|\mathcal{H}_T\vee\mathcal{M}_T\right] \\
&= {}_T\tilde{p}_x\sum_{j=N}^{\omega-x} D(T,t_j)\, E_P\left[\exp\left(-\int_T^{t_j}\mu_{x+u}(u)du\right)\middle|\mathcal{M}_T\right] \\
&= {}_T\tilde{p}_x\sum_{j=N}^{\omega-x} D(T,t_j)\, {}_{t_j-T}p_{x+T}(T). \qquad (8.17)
\end{aligned}$$

Pre-retirement and post-retirement survival functions



(8.4.1) Pre-retirement survival function (cf. Eq. (8.5))



(8.4.2) Post-retirement survival function (cf. Eq. (8.17))

Figure 8.4: Survival functions without and with systematic mortality risk based on life table data from the Human Mortality Database (2009). The random post-retirement survival probabilities are illustrated by means of prediction intervals against the conservative survival function used to calculate the guaranteed payout rate (8.5).

For the calculation of fair percentage charges and contract values we assume survival probabilities to coincide under the real-world and risk-neutral probability measure[190] (i.e. the market price of mortality risk is zero) to calculate a fair charge according to the fair premium principle which is equal to the assumption of the market being risk neutral with respect to systematic and unsystematic mortality risk.

8.4.2 Existence and uniqueness of a fair percentage charge

From a life insurers point of view, the traditional role of a service provider covering biometric risk is substituted by the role of a seller of complex financial and mortality risk bearing options. Traditional reserving schemes had to be replaced by hedging strategies[191] in case were an actuarial reserve is completely hold in fund shares. When considering the differences compared to traditional annuity pricing, the common equivalence premium principle has to be reviewed. Therefore, the fair premium principle applied by Nielsen and Sandmann (2002) combines risk-neutral valuation and diversification arguments given by the principle of equivalence since the contract payoff is driven by both financial and mortality risk. An application of the fair premium principle ensures the existence and uniqueness of a fair premium in the context of traditional equity-linked endowment insurance contracts. Nielsen and Sandmann assume the market to be risk-neutral towards systematic mortality risk, i.e. any insurance risk can theoretically be fully diversified. However, this approach presupposes that a mortality derivative market[192] is complete such that there exists a dynamic self-financing replication portfolio. More precisely, the portfolio contains liquidly traded securities which perfectly match the payoff of the underlying VA contract. According to the no-arbitrage principle the option price given as the expected discounted contract payoff under the unique martingale measure and the initial

[190] This approach was, for example, also considered by Milevsky and Promislow (2001), Schrager and Pelsser (2004) or Ballotta and Haberman (2006). Dahl (2004) and Ballotta and Haberman (2006) assume an incomplete market for mortality-linked securities. This assumption is justified by the (current) absence of market prices of mortality risk. Hence, there is no adequate numeráire available that allows to derive an unique artificial probability measure, see e.g. Dahl (2004), Lin and Cox (2005) and Biffis et al. (2010).

[191] Dynamic discrete hedging techniques are applied to secure the insurers solvency. The effectiveness primarily depends on the choice of a robust and reliable underlying model. In any case, the hedging of embedded annuity options is particularly difficult due to high transaction costs, long-maturities and limited liquidity.

[192] At the moment, a mortality derivative market is far from being complete since there are no actively traded financial instruments suitable for a hedge of systematic mortality risk especially longevity risk. Several bilateral tailor-made reinsurance solutions like the longevity bond emitted in December 2010 by the Swiss Re reinsurance company represent first efforts in longevity securitisation.

value of the hedge portfolio coincide.

We apply the fair premium principle to the guaranteed benefit of a VA contract defined in Section 8.2 with an exogenously given premium. The result is an implicit equation for the fair percentage guarantee charge which can be solved numerically. Nevertheless, the general technique and line of argumentation of Nielsen and Sandmann (2002) can be retained. In order to price the guarantee riders the VA contract is assumed to be compared to a pure unit-linked endowment insurance contract without any guarantees.

REMARK 8.4.2. For a given guarantee charge φ and a net premium $\pi^N > 0$ the risk-neutral prices $C_D(x,t)$ and $C_I(x,t,T)$ at time t for the guaranteed death and living benefits equal the discounted expected values of the random future option payoffs under the forward risk adjusted measure. In particular, using Equations (8.2) and (8.6) the prices are given as follows

$$
\begin{aligned}
C_D(x,t) &= \int_t^T E_{P^*}\left[(\beta_{t,u})^{-1} \cdot \left(g_D(u) + [A(u,\pi,\varphi) - g_D(u)]^+ \right) \mathbb{1}_{\{\tau_x = u\}} \middle| \mathcal{F}_t \right] du \\
&= \int_t^T D(t,u)\, E_{P^u}\left[\left(g_D(u) + [A(u,\pi,\varphi) - g_D(u)]^+ \right) \mathbb{1}_{\{\tau_x = u\}} \middle| \mathcal{F}_t \right] du \\
&= \int_t^T D(t,u)\, {}_t\tilde{p}_x \cdot \tfrac{d}{du} \left({}_{u-t}q_{x+t}(t) \right) \left(g_D(u) + E_{P^u}\left[[A(u,\pi,\varphi) - g_D(u)]^+ \middle| \mathcal{F}_t \right] \right) du,
\end{aligned}
\tag{8.18}
$$

$$
\begin{aligned}
C_I(x,t,T) &= E_{P^*}\left[(\beta_{t,T})^{-1} \cdot \left(g_I(T,x) + [A(T,\pi,\varphi) - g_I(T,x)]^+ \right) \mathbb{1}_{\{\tau_x > T\}} \middle| \mathcal{F}_t \right] \\
&= D(t,T)\, E_{P^T}\left[\left(g_I(T,x) + [A(T,\pi,\varphi) - g_I(T,x)]^+ \right) \mathbb{1}_{\{\tau_x > T\}} \middle| \mathcal{F}_t \right].
\end{aligned}
\tag{8.19}
$$

The following Lemma provides the different parts of a conditional equation for the fair percentage guarantee charge.

LEMMA 8.4.3. *Let η denote the repayment level of the put option payoffs (8.2) and (8.6). The expected discounted present value of the total rider fee (RF) income charged from the insureds fund value is calculated as*

$$
\begin{aligned}
EDPV^{RF}(\varphi) &= E_{P^*}\left[\sum_{i=0}^{N-1} \frac{\varphi A(t_i,\pi,\varphi)}{\beta_{0,t_i}} \middle| \mathcal{F}_0 \right] = \sum_{i=0}^{N-1} \varphi\, E_{P^*}\left[\frac{A(t_i,\pi,\varphi)}{\beta_{0,t_i}} \middle| \mathcal{F}_0 \right] \\
&= \sum_{i=0}^{N-1} \varphi \sum_{j=0}^{i-1} (1-\varphi)^{i-j} \pi^N_{t_j}\, E_{P^*}\left[(\beta_{0,t_i})^{-1} \frac{S_{t_i}}{S_{t_j}} \middle| \mathcal{I}_0 \right] E_P\left[\mathbb{1}_{\{\tau_x > t_i\}} \middle| \mathcal{H}_{t_j} \vee \mathcal{M}_{t_j} \right] \\
&= \sum_{i=0}^{N-1} \varphi \sum_{j=0}^{i-1} (1-\varphi)^{i-j} \pi^N_{t_j}\, D(0,t_i)\, E_{P^{t_i}}\left[\frac{S_{t_i}}{S_{t_j}} \middle| \mathcal{I}_0 \right] {}_{t_j}\tilde{p}_x \cdot {}_{t_i - t_j}p_{x+t_j}(t_j)
\end{aligned}
$$

$$= \sum_{i=0}^{N-1} \varphi \sum_{j=0}^{i-1} (1-\varphi)^{i-j} \pi_{t_j}^N \, D(0,t_j) \, {}_{t_j}\tilde{p}_x \cdot {}_{t_i - t_j} p_{x+t_j}(t_j).$$

As a benefit in return, the insurer guarantees a death benefit with expected discounted present value

$$\begin{aligned} EDPV_\eta^{GMDB}(\varphi) &= E_{P^*}\left[\int_0^T (\beta_{0,u})^{-1} \left(\eta \, G_D(u) - \sum_{i=0}^{n^*(u)} \pi_{t_i}^N \frac{S_u}{S_{t_i}} \right) \mathbb{1}_{\{\tau_x = u\}} \, du \middle| \mathcal{F}_0 \right] \\ &= \int_0^T D(0,u) \left(E_{P^u}\left[\eta \, G_D(u) \mathbb{1}_{\{\tau_x = u\}} \middle| \mathcal{F}_0 \right] \right. \\ &\quad \left. - \sum_{i=0}^{n^*(u)} \pi_{t_i}^N E_{P^u}\left[\frac{S_u}{S_{t_i}} \middle| \mathcal{I}_0 \right] \tfrac{d}{du}({}_u q_x(0)) \right) du \\ &= \eta \, C_D(x,0) - \int_0^T \sum_{i=0}^{n^*(u)} \pi_{t_i}^N \, D(0,t_i) \, \tfrac{d}{du}({}_u q_x(0)) \, du, \end{aligned} \tag{8.20}$$

i.e. the expected discounted payoff of a VA with GMDB during the savings phase minus the expected discounted payoff of an unit-linked life endowment insurance. Furthermore, we obtain the expected discounted present value of the guaranteed living benefits via

$$\begin{aligned} & EDPV_\eta^{GMIB}(\varphi) = E_{P^*}\left[(\beta_{0,T})^{-1} \left(\eta \, G_I(T,x) - \sum_{i=0}^{N-1} \pi_{t_i}^N \frac{S_T}{S_{t_i}} \right) \mathbb{1}_{\{\tau_x > T\}} \middle| \mathcal{F}_0 \right] \\ &= D(0,T) \left(E_{P^T}\left[\eta \, G_I(T,x) \mathbb{1}_{\{\tau_x > T\}} \middle| \mathcal{F}_0 \right] - \sum_{i=0}^{N-1} \pi_{t_i}^N \, E_{P^T}\left[\frac{S_T}{S_{t_i}} \middle| \mathcal{I}_0 \right] {}_T p_x(0) \right) \\ &= \eta \, C_I(x,0,T) - \sum_{i=0}^{N-1} \pi_{t_i}^N \, D(0,t_i) \, {}_T p_x(0) \end{aligned} \tag{8.21}$$

which equals the benefit related difference of the expected discounted payoff of a variable annuity contract with GMIB feature and the expected discounted payoff of an unit-linked assurance on survival to a stipulated age.

In the following Definition, we consider the question how to specify a fair contract, i.e. how to specify a fair percentage guarantee charge φ^* for a given periodic net premium $\pi^N > 0$.

Definition 8.4.4 (Fair Percentage Charge). Consider a variable annuity contract with a bundled GMDB and GMIB rider as specified in Section 8.2, i.e. the repayment level η equals one. For each net premium $\pi^N > 0$ the percentage guarantee charge φ^* is called a fair charge, if it forms a solution to

$$EDPV^{RF}(\varphi^*) = EDPV^{GMDB}_{\eta=1}(\varphi^*) + EDPV^{GMIB}_{\eta=1}(\varphi^*), \tag{8.22}$$

i.e. if the expected discounted present value of the total charge income coincides with the difference between the expected discounted present benefit of the variable annuity and a unit-linked endowment contract.

The object of the following investigation is the proof of the existence of a fair charge satisfying Definition 8.4.4. The subsequent proposition presents properties of the percentage charge in the case of extreme contract specifications without any guarantees.

PROPOSITION 8.4.5. *Consider a net premium $\pi^N > 0$ and a variable annuity contract without any guaranteed payments, i.e. $g_D = g_I = 0$.*

(a) Assume that, the repayment level η is equal to one and in the case of death at time $t \in \,]\,0,T\,]$ during the deferment period the insurer pays out the market value of the insureds investment portfolio $A(t,\pi,\varphi)$. Furthermore, it is assumed that, if the insured survives until retirement T, the market value of $A(T,\pi,\varphi)$ is paid out. Then the fixed percentage charge φ is a fair charge if and only if $\varphi = 0$.

(b) Assume that the above contract representation is modified in a sense of reduced repayments, i.e. we have payouts $\eta\, A(t,\alpha,P)$ and $\eta\, A(T,\alpha,P)$ respectively for $\eta \in \,]\,0,1\,[$. Then there exists a unique fair charge φ^* and it holds $\varphi^* < 0$.

Proof. See Appendix C.2.2. □

REMARK 8.4.6. With the objective of deriving a unique fair charge φ^* a restriction with respect to the total guaranteed amount for all $t \in \,]\,0,T\,]$ has to be formulated. Assume that the expected present discounted value of the payoff in case of a guarantee charge φ equal to one[193] is bounded from above by the expected present discounted payoff value of a unit-linked endowment insurance contract without any guarantee. Otherwise, a comparison of the VA and unit-linked endowment contract makes no sense in the way that a fair charge cannot be uniquely determined. The relationship can be expressed by

[193] A guarantee charge equal to one ensures that the full premium amount is available to hedge the guarantee. A fund investment does not take place.

the following inequality

$$\int_0^T D(0,u)\, E_{P^u}\left[g_D(u)\,\mathbb{1}_{\{\tau_x=u\}}\middle|\,\mathcal{F}_{t_0}\right] du \,+\, D(0,T)\, E_{P^T}\left[g_I(T,x)\,\mathbb{1}_{\{\tau_x>T\}}\middle|\,\mathcal{F}_{t_0}\right]$$

$$< \int_0^T \sum_{i=0}^{n^*(u)} \pi_{t_i}^N\, D(0,t_i)\, \tfrac{d}{du}\left({}_u q_x(0)\right) du \,+\, \sum_{i=0}^{N-1} \pi_{t_i}^N\, D(0,t_i)\, {}_T p_x(0)$$

which is equivalent to

$$EDPV_{\eta=1}^{GMDB}(1) + EDPV_{\eta=1}^{GMIB}(1) < 0 = EDPV^{RF}(1). \tag{8.23}$$

Theorem 8.4.7 (Existence and Uniqueness of a Fair Percentage Charge). *Suppose a variable annuity contract as specified in Section 8.2, a net premium $\pi^N > 0$ and guarantees g_D and g_I satisfying assumption (8.23). Let the value of the mutual fund S_t be continuously distributed for any time t. Then there exist a unique fair percentage charge φ^* and it holds $\varphi^* \in \,]\,0,1\,[$.*

Proof. See Appendix C.2.3. □

8.5 Numerical analysis

8.5.1 Variable annuities with a bundled GMDB and GMIB rider offered in the German insurance market

In September 2008 the German government introduced a bill to change the Insurance Supervision Act which aimed at a loosening of actuarial reserve accounting by means of close-to-market accounting rules. However, due to growing concern after the financial crisis as from 2007 this draft was postponed indefinitely only three months later. Until now, regulation changes that promote a VA business from a reserving point of view are still in progress. The common reserving using a fixed guaranteed interest rate makes the offering of VA products for German insurance companies without foreign subsidiary companies nearly impossible under economic considerations. In particular, even if a company is perfectly secured by a dynamic hedge portfolio[194], it

[194] Graf et al. (2006) analyse the concept and the effects of a dynamic delta(-rho)-hedging exemplified by a standard Guaranteed Minimum Accumulation Benefit (GMAB) unit-linked contract. Typically, market Guaranteed Minimum Death Benefits (GMDBs) are delta-hedged by exchange-traded index options / futures with weekly rebalancing. Contracts with a GMIB rider are dynamically secured by a delta-rho-hedge using equity index / bond futures as well as interest rate futures / swaps.

still has to establish additional actuarial reserves for strongly uncertain future (guarantee) liabilities. Hence, foreign assurers which are not under the control of the national state supervision represented by the BaFin benefit from this competitive disadvantage. Despite or even because of this regulation rule the German insurance market was one of the first European countries in which VAs were offered.

The currently traded VA products of the third pillar with GMIB and GMDB riders during the savings phase are the AXA TwinStar Invest (since April 2006 with sales stop in April 2009 and relaunch for the new business in January 2010) and the Swiss Life Champion (since October 2008). The policies are processed by AXA Life Europe Limited located in Ireland and Swiss Life Products S.A. located in Luxembourg respectively. As of November 2010, the annual M&E-charge for the TwinStar Invest product variant "Chance / Index" amounted to 80 basis points. The cost of the bundled[195] GMDB and GMIB rider calculated with a RO benefit base[196] lay between 75 and 190 basis points[197] of the fund value. In the 2009 "Active" variant Swiss Life customers paid 10 basis points for the GMDB amounting to the return of premiums paid in case of death during the savings phase. For the GMIB rider 100-250 basis points were charged depending on the selected fund composition. Both products offer individual fund selection out of a predefined portfolio and the option of dynamic benefit adjustments between 1 and 10 percent.

8.5.2 Calibration and parameter values

Schrager and Pelsser (2004) derive bounds for the valuation of (8.18) and (8.19) since no closed form for the option prices exist. Nevertheless, we can estimate a fair charge using nested Monte-Carlo simulation under the risk-neutral measure. Similar to Vellekoop et al. (2006) we initially simulate $2 \cdot 10^4$ sample paths for discretised versions of the processes (8.9), (8.10) and (8.14) using the parameters listed in Tables 8.2-8.4. Thereafter, we calculate the discounted guarantee payoffs and the discounted present value of the total rider fee. The differences of the path dependent benefits of a VA and an unit-linked life endowment insurance contract on the one hand and the rider income on the other hand

[195] In the following, we use the term bundled GMDB and GMIB rider in the sense that the guaranteed death benefits equal the refund of premiums paid and solely living benefit enhancement features are varied.
[196] At the time of the TwinStar launch in 2007 AXA granted a roll-up rate of 3-5% depending on the respective contract duration and age at inception.
[197] The exact guarantee charge values depend on the contract duration, method of premium payment, fund investment and the policyholders age at inception.

are gradually summed up. The root of the sum of differences yields the fair charge estimate. Its determination is carried out by using a simple numerical static line search method.

In the following, we assume a 45-year-old male signing a regular premium unit-linked annuity insurance contract with a guaranteed living benefit only at maturity $T = 20$. The contract is optionally provided with a guaranteed death benefit during the accumulation period with full repayment level $\xi = 1$. For reasons of simplification, we set $\Delta_S = 0$ and the net premium amount π^N equal to one, i.e. the selection phase and fixed costs are excluded. We assume that during the savings phase the customer neither withdrawals from the fund value, neither undertakes additional payments nor negotiates a premature redemption. Furthermore, besides equity returns or guaranteed interest on the living benefits, the policyholder is not allocated to any profit participation. We neglect the common process management for the last years prior to maturity, at which the fund shares are successively shifted into a money market fund or comparable retail funds with low volatility. The fixed payout rate $r^G_{x+T}(T)$ is calculated using a conservative survival function containing a safety margin of 3% on the parameter α and a guaranteed discount rate δ, if not mentioned otherwise, amounting to 4%. The market annuity factor $\ddot{a}_{x+T}(T)$ is simulated using Equation (8.17). The Monte Carlo standard errors for the fair percentage charge φ^* are at least smaller than $3.1 \cdot 10^{-5}$ for $2 \cdot 10^3$ simulated charges which are themselves expected values of $2 \cdot 10^4$ individual simulations. The simulation errors for the corresponding option prices $C_D(x,0)$ and $C_I(x,0,T)$ are at least less than $3.2 \cdot 10^{-4}$ and $3.7 \cdot 10^{-3}$ respectively. Therefore, the sample path number seems appropriate.

8.5.3 Analysis of the fair charge and option price

Estimation of fair percentage guarantee charges for different guarantee features

Similar to the results of Bauer et al. (2008b), Table 8.5 illustrates that GMDB options are in general overpriced[198] (see e.g. Milevsky and Posner (2001)) with market charges of 15-35 basis points (bps). Sometimes, the standard return of premium death benefit is included without taking an explicit additional charge for it which is the case for the TwinStar Invest policy. On the contrary, single GMIB options (charges generally range from 15 to 75 bps) are underpriced partly

[198] Common GMDB guarantees exhibit longer durations up until the tax-related maximum retirement age. The rider equipment varies from the paid gross premiums (RP) to an annual ratchet base (RA) and a 5%-6% annual roll-up base (RO).

due to assumptions on the insureds irrational behaviour concerning surrender and lump sum payment even if the option is valuable for the policyholder. More precisely, the product pricing based on a financially rational customer behaviour causes charges that exceed the market prices several times over especially for those features with a common roll-up benefit base. It can be assumed that in practice an insurer utilizes cross-subsidies in bundling products with different riders which are less correlated among each other.

Sun (2006) points out that a bundling of the GMDB and GMIB rider (both provided with a RORA benefit base) barely offers cost reduction or natural hedging potential since both riders are nearly perfectly positively correlated with regard to the option's moneyness (cf. Figures 8.1 and 8.2). Depending on the scope of guarantee benefits the bundled charge deduction may use up any further cost reduction potential. In other words, the joint fund value is reduced more severe which consequently leads to an increase in the overall option price. For the RP variant we observe a moderate cost decrement of 6% in the bundled rider fee compared to the (direct) sum of fees which is similar to the estimation results in Sun (2006). In the following, we analyse the impact of variations

Fair percentage charges for different guarantee features

rider composition	guarantee feature			
	RP	RO	RA	RORA
GMDB (RP)	3	—	—	—
GMIB	32	174	78	201
GMDB (RP) + GMIB	33	171	79	196

Table 8.5: Fair percentage charges (in bps) for different guarantee features w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$ and guarantee rate $\delta = 0.04$. The charge estimates are based on $2 \cdot 10^4$ iterations.

in input factors like contract, financial market and mortality parameters on the amount of the fair charge for a VA contract with a GMDB rider during the savings phase and a GMIB rider valid at retirement. In other words, we investigate relationships between the different input parameters and the fair charges / option prices.

Sensitivity analysis of variations in the contract parameters

Intuitively, Table 8.6 indicates that higher guarantee rates δ increase the fair percentage charge substantially since the option strike $g_I(T,x)$ is subjected to a strong increase. This is due to a growth in both the fixed payout rate and the roll-up benefit base. As the underlying annuity option becomes more valuable we observe a strong increase in the fair charge. An upward movement of the market yield r_0^M causes a decline in the post-retirement discount factors and hence also in the market annuity factor $\ddot{a}_{x+T}(T)$. Thus, the option price and fair charge decrease. This effect is intensified by the assumed negative market correlation between the equity fund and interest rates which further lowers the option payouts and thus fair charges. Since Euro area yield curves recently experienced rather downturn movements, the hedging costs[199] for the VA guarantees generally increased. Notice that occasionally customary market roll-up rates lie above the initial (market) term structure r_0^M valid at that time. In order to provide a counterbalance, the insurer determines a fixed payout rate $r^G_{x+T}(T)$ below the anticipated market annuity payout rate $r_{x+T}(T)$ at retirement. As a consequence, the retirement option (8.6) may be still in the money even if the guaranteed benefit base exceeds the market value of the investment portfolio. Nevertheless, in case of low interest rates together with high guaranteed roll-up rates and deferment periods $T > 10$ fair guarantee charges according to Theorem 8.4.7 do not exist. Basically, a raise in the accumulation period T implicates lower charges due to an increase in the option strike. Therefore, selling contracts with long option maturity or a prolongation phase forms a contractual way to ensure a competitive option fee enabled by a reduced option price.

A look at the risk neutral option prices in Figure 8.5 reveals two distinct effects, namely the "parameter effect" and the "charge effect" which in turn is indirectly induced by the first one. In particular, a strong increase in the fair percentage charge lowers the fund value considerably and therefore can compensate the impact caused by a change of an underlying parameter. Subfigure (8.5.1) shows that although $C_D(x{,}0)$ is parameter independent from δ the increasing rider-fees evoke a decrease in the payout value especially for a low market yield and a short-term deferment period. Just like the values for $C_I(x{,}0{,}T)$, an increasing market yield causes a reduction in the option price which is much more pronounced in the living benefit case. A comparison of the differently coloured price surfaces yields that a postponement of the retirement date evokes a decrease in the risk-neutral prices primarily caused by a decreasing bond price combined with increasing guaranteed benefit bases.

Sensitivity analysis of variations in the financial market parameters

Depending on the correlation of the Brownian motions driving the zero bond and equity fund prices, changes in the fund volatility σ_S and interest rate volatility σ_{spot} have multiple effects on the cost of guarantee (Table 8.7) as well as option

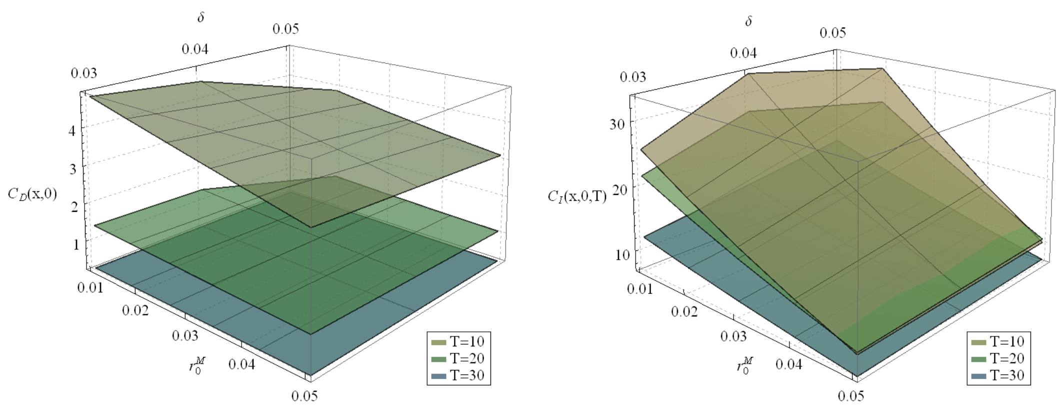
[199] Interest rate risk can for example be rho-hedged by swaptions or interest futures.

Sensitivity of the guarantee charge to a variation in the contract parameters

initial yield r_0^M	T=10			T=20			T=30		
	guarantee rate δ			guarantee rate δ			guarantee rate δ		
	0.03	0.04	0.05	0.03	0.04	0.05	0.03	0.04	0.05
0.01	996	2137	6742	555	1385	n.a.	345	873	n.a.
0.02	416	949	1961	245	534	1278	164	332	812
0.03	172	414	896	118	249	530	81	161	322
0.04	72	186	420	57	118	246	40	77	156
0.05	30	80	185	27	58	123	19	39	78

Table 8.6: Sensitivity of the bundle guarantee charge to a variation in the contract parameters (in bps) w.r.t. an individual aged $x = 45$ and a guarantee level $L(T) = L^{\text{RO}}$. The charge estimates are based on $2 \cdot 10^4$ iterations. For table entries marked with "n.a." a fair guarantee according to Theorem 8.4.7 does not exist.

Sensitivity of the option prices to a variation in the contract parameters



(8.5.1) $C_D(x,0)$ **(8.5.2)** $C_I(x,0,T)$

Figure 8.5: Sensitivity of the spline-smoothed option prices to a variation in the contract parameters w.r.t. an individual aged $x = 45$, a guarantee level $L(T) = L^{\text{RO}}$ and a smoothing parameter of 0.02. The fair charges are taken from Table 8.6. The charge estimates are based on $2 \cdot 10^4$ iterations.

prices (Figure 8.7). Generally speaking, increasing fund volatility on the one hand evokes increasing annuity (call) option prices[200] $C_D(x,0)$ and $C_I(x,0,T)$ resulting in higher rider fees. On the other hand, an increase in the spot rate volatility causes a wider range of outcomes for the annuity factor $\ddot{a}_{x+T}(T)$ due to greater dispersion in the post-retirement discount factors and larger fluctuation in the drift of the mutual fund process such that fair estimates also rise though to a comparatively less extent. Under the assumption of correlation between the development of the investment portfolio and interest rates we observe various partial effects which drive the charge and prices respectively in

[200] A higher variability in the fund price volatility σ_S induces higher probabilities of larger fund values.

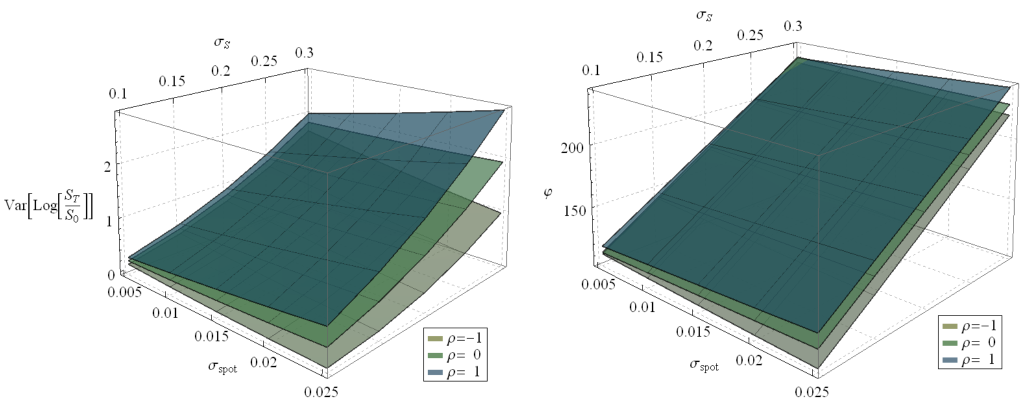
different directions and thus can counterbalance or even overlie each other.

Sensitivity of the guarantee charge to a variation in the financial parameters

spot rate volatility σ_{spot}	$\rho=-1$					$\rho=0$					$\rho=1$				
	fund volatility σ_S					fund volatility σ_S					fund volatility σ_S				
	0.10	0.15	0.20	0.25	0.30	0.10	0.15	0.20	0.25	0.30	0.10	0.15	0.20	0.25	0.30
0.005	113	140	171	199	226	113	144	172	202	232	119	145	175	205	233
0.010	110	139	167	198	224	116	144	171	201	230	122	150	177	208	233
0.015	109	136	164	195	225	114	145	173	202	228	123	154	177	203	238
0.020	108	132	161	194	222	116	145	172	203	223	124	153	183	212	242
0.025	109	129	160	193	219	118	147	172	208	228	129	160	184	213	241

Table 8.7: Sensitivity of the bundle guarantee charge to a variation in the financial parameters (in bps) w.r.t. an individual aged $x=45$, a contract maturity $T=20$ and a guarantee level $L(T)=L^{\mathrm{RO}}$. The charge estimates are based on $2\cdot 10^4$ iterations.

Sensitivity of the logarithmised fund return and the fair charge to a variation in the financial parameters



(8.6.1) Variance of the logarithmised fund return

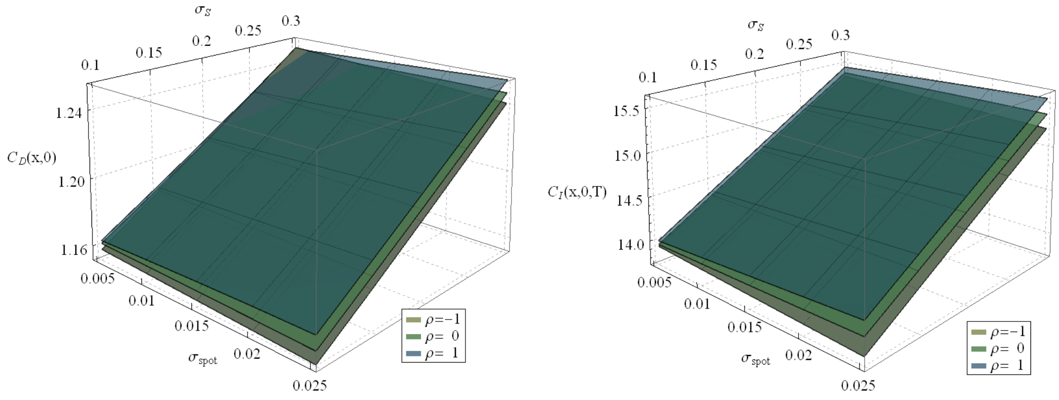
(8.6.2) Fair charge estimates

Figure 8.6: Sensitivity of the logarithmised fund return at maturity and the spline-smoothed fair charge estimates to a variation in the financial parameters w.r.t. an individual aged $x=45$, a contract maturity $T=20$, a guarantee level $L(T)=L^{\mathrm{RO}}$ and a smoothing parameter of 0.02. The fair charges are taken from Table 8.7. The charge estimates are based on $2\cdot 10^4$ iterations.

In terms of correlation only, Marshall et al. (2010) quote that the GMIB option price constitutes an increasing[201] function of the correlation coefficient. Further notice that according to Brigo and Mercurio (2007), the variance of the logarithmised fund returns at maturity, which is a key determinant for the (GMIB) charge and option price estimates, also enhances with increasing correlation coefficient. This phenomenon is illustrated in Figure 8.6. Despite from simulation inaccuracies, the fair charge estimates show an analogous

[201] Marshall et al. (2010) also assume a fund volatility of $\sigma^S=0.2$ and an interest rate volatility of $\sigma_{spot}=0.015$.

Sensitivity of the option prices to a variation in the financial parameters



(8.7.1) $C_D(x,0)$ **(8.7.2)** $C_I(x,0,T)$

Figure 8.7: Sensitivity of the spline-smoothed option prices to a variation in the financial parameters w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$, a guarantee level $L(T) = L^{\text{RO}}$ and a smoothing parameter of 0.02. The fair charges are taken from Table 8.7. The charge estimates are based on $2 \cdot 10^4$ iterations.

behaviour. The mentioned effect of changes in the annuity factor is almost negligible for the default financial parameters listed in Tables 8.2 and 8.3. In particular, for negative correlation the annuity provider expects equity and zero bond prices to develop oppositionally. Therefore, if market interest rates rise, the fund performance as well as charge and option price decrease. Although the dispersion of the annuity market factor increases, insurance market annuities may be even more attractive for the insured (see Subfigure (8.7.2)). In case of uncorrelated fund and bond prices, increasing spot rate volatility causes higher volatility in the annuity factor as well as the drift of the fund prices. The resulting overall change can hardly be explained such that fee and option price estimates slightly increase for low fund fluctuation but slightly decrease for high values of σ_S. For perfectly correlated price processes decreasing market rates result in an enhancement of the fund returns[202] and the annuity factor. From a policyholders point of view the annuity contract becomes more valuable. The Brownian motion parts of Equations (8.11) and (8.12) evolve unidirectional and thus charges and option prices are increasing in spot rate volatility as well as fund volatility.

Sensitivity analysis of variations in the mortality parameters

A change of parameters that affect the exponential mortality reduction factor have a direct impact on the market annuity factor $\ddot{a}_{x+T}(T)$ and thus influence

[202] It should be noted that in certain cases an option to switch the fund selection (due to bad performance) can become highly valuable such that the annuity provider has to restrict his offered fund selection to reduce volatility. This can be achieved by a limitation of fund choices within the product design.

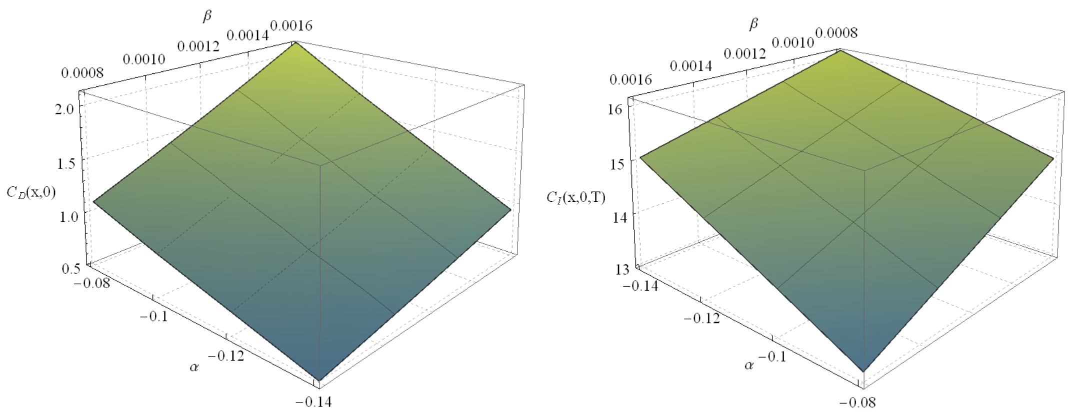
the fair charge and the guarantee payoff values. In the following, we carry out a separate sensitivity analysis for parameters driving the deterministic and the stochastic part of the mortality rate process (8.15). The variation of the "deterministic" parameter values for α and β is chosen in such a way that the conditional expected age at death of an individual aged $x = 45$ at inception lies within an age period of 82 years minus 5 plus 10-11 years[203]. The more

Sensitivity of the guarantee charge to a variation in the "deterministic" mortality parameters

age specific rate of change β	period specific growth rate α			
	-0.14	-0.12	-0.10	-0.08
0.0008	213	201	189	183
0.0010	203	189	181	175
0.0012	191	180	169	163
0.0014	185	173	166	156
0.0016	170	165	158	148

Table 8.8: Sensitivity of the bundle guarantee charge to a variation in the "deterministic" mortality parameters (in bps) w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$ and a guarantee level $L(T) = L^{\text{RO}}$. The charge estimates are based on $2 \cdot 10^4$ iterations.

Sensitivity of the option prices to a variation in the "deterministic" mortality parameters



(8.8.1) $C_D(x{,}0)$ **(8.8.2)** $C_I(x{,}0{,}T)$

Figure 8.8: Sensitivity of the spline-smoothed option prices (8.18) and (8.19) to a variation in the "deterministic" mortality parameters w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$, a guarantee level $L(T) = L^{\text{RO}}$ and a smoothing parameter of 0.02. The fair charges are taken from Table 8.8. The charge estimates are based on $2 \cdot 10^4$ iterations.

negative the logarithmic rate of period change level α, i.e. anticipations about future mortality patterns, the stronger the post- and pre-retirement survival

[203] The age period is chosen asymmetric towards older ages to consider the general trend of increasing life expectancy.

probabilities improve due to a downward trend of mortality rates. As a result, the guarantee $g_I(T,x)$ increases since the product $r^G_{x+T}(T) \cdot \ddot{a}_{x+T}(T)$ increases[204]. Consequently, the fair charges in Table 8.8 increase and the discounted expected GMIB payoff increases (see Figure 8.8) whereas the GMDB payoff falls in value. When considering the age-related logarithmic rate of change, we observe, conversely, that a raise in the age specific parameter β lowers the estimated rider charge since survival probabilities experience a downturn due to an increasing exponential reduction factor. In this case, the guarantee $g_I(T,x)$ falls and ensures that the GMDB option price increases and that the GMIB value decreases. A variation in the parameters κ and γ driving the stochastic reduction factor Y yields the results presented in Table 8.9 and Figure 8.9. According to Ballotta and Haberman (2006), we observe that the variation in charges and option prices is only marginal if not negligible.

Sensitivity of the guarantee charge to a variation in the "stochastic" mortality parameters

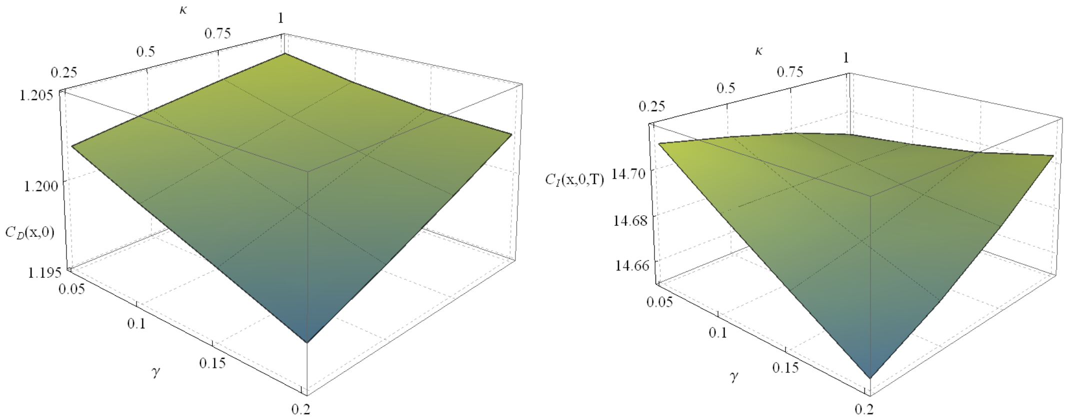
speed of mean reversion κ	reduction factor diffusion γ			
	0.05	0.10	0.15	0.20
0.25	173	170	170	173
0.50	174	172	174	170
0.75	174	172	174	173
1.00	175	175	172	169

Table 8.9: Sensitivity of the bundle guarantee charge to a variation in the "stochastic" mortality parameters (in bps) w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$ and a guarantee level $L(T) = L^{\mathrm{RO}}$. The charge estimates are based on $2 \cdot 10^4$ iterations.

Positive changes in the speed of convergence κ towards the long run mean zero increase the survival probabilities and therefore slightly enhance the fair charges. We observe that the stronger the diffusion of the Brownian motion W^μ the more sensitive the mean reversion effect of κ turns out. For very small convergence speed κ the process Y behaves almost non mean reverting and stronger diffusion γ lowers the survival probabilities and raises the death probabilities. Apart from simulation inaccuracies, this leads to decreasing fair charges and a decreasing option price $G_I(x,0,T)$ as well as an increasing value for $G_D(x,0)$. Conversely, for a strong mean reversion effect survival probabilities do not decrease to the same extent due to a deterioration in diffusion. Consequently, the guarantee charges slightly decrease, the living benefits slightly increase and the death

[204] Note that the value of the guaranteed payout rate has a strong impact on the income benefits and therefore has to be carefully determined by the insurance company. In case of the AXA Twinstar Invest policy, the calculation of the fixed payout rate $r^G_{x+T}(T)$ is based on the selection life table DAV 2004R plus a safety margin of 10%.

Sensitivity of the option prices to a variation in the "stochastic" mortality parameters



(8.9.1) $C_D(x,0)$ **(8.9.2)** $C_I(x,0,T)$

Figure 8.9: Sensitivity of the spline-smoothed option prices (8.18) and (8.19) to a variation in the "stochastic" mortality parameters w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$, a guarantee level $L(T) = L^{\text{RO}}$ and a smoothing parameter of 0.02. The fair charges are taken from Table 8.9. The charge estimates are based on $2 \cdot 10^4$ iterations.

benefits increase although to a lesser extent.

8.5.4 Analysis of the rate of return and the pension amount at maturity

Analysis of the rate of return and key risk-return figures

From a customers point of view, the desire for embedded guarantees is closely related to a lower chance of outperformance compared to a contract without additional guarantee riders. We illustrate this effect by simulating the discrete annualised net yield to maturity[205] (referred to as y_a^N) under the physical measure conditioned on the insured's survival of the deferment period. Furthermore, we determine key risk-return figures. In Subtable (8.10.1) we show the expected (discrete) rate of return on total premiums paid denoted as $E\left[y_a^N\right]$, the conditional tail expectation[206] at level 0.1 given as $E\left[y_a^N\middle|\ \text{"10\%"}\right]$ and the

[205] The discrete annualised net yield to maturity y_a^N denotes the annual discrete compounding rate which ensures that the total gross premiums paid – each compounded at y_a^N – equates to the total contract value available at retirement date. For every sample path $k \in \mathbb{N}$ the pathwise simulated yield $y_{a,k}^N$ solves the equation

$$y_{a,k}^N = \left\{ y_a \in \mathbb{R} \,\middle|\, \text{VAM} = \sum_{i=0}^{N-1} \pi^N (1+y_a)^{T-t_i} = \pi^N \frac{((1+y_a)^T - 1)(1+y_a)}{y_a} \right\}$$

where the value at maturity (VAM) equals $A(T,\pi,\varphi)$ in case of the single GMDB rider and $G_I(T,x)$ for the GMIB or the bundled GMDB and GMIB rider.

[206] The conditional tail expectation at level 0.1 describes the expectation of the 10% worst rate of return real world scenarios.

inflation quantile $P\left(y_a^N < 0.02\right)$ as a shortfall measure considering scenarios with yields below the inflation rate[207] using the fair guarantee charges estimated in Table 8.5.

Notice that in case of the return of premium feature (RP) the expected rate of return for the different rider compositions have almost the same amount despite the heterogeneous fee structure listed in Table 8.5. As expected, the discrete rate of return estimates lie above the risk-neutral instantaneous expected fund performance $\mu_S - \frac{1}{2}\sigma_S^2$ of 6% per annum. The choice of the benefit base has only a minor effect on the amount of the expected return. However, relevant differences can be found in the clustering of the bar charts in Subfigures (8.10.2)-(8.10.4) and the inflation shortfall measure. By means of the conditional tail expectation values we observe that in case of guaranteed income benefits the rate of return can also become negative[208]. This means that in contrast to a VA with a GMAB the guarantee (8.4) does not necessarily reach the total (gross) premiums paid. In fact, the GMIB rider provides a minimal predetermined annual pension income granted by a guaranteed annuity factor and benefit base at retirement. The quantile bar chart in Subfigure (8.10.2) illustrates that every fifth VA contract with GMDB generates a yield which is only less or equal to the rate of inflation. For the GMIB (Subfigure (8.10.3)) and bundled GMDB and GMIB (Subfigure (8.10.4)) rider with RP guarantee feature the probability of suffering a real loss is even higher since the rate of return is close to zero whenever the option is in the money. In addition, the RO and thus the RORA feature completely exclude returns below the inflation rate due to the high guarantee benefit bases. In combination with rather high fair charges this ensures that for the roll-up clause nearly three out of five policies earn a return between 2% and 6%. In contrast, the distribution of the yield for the ratchet feature RA is more regular distributed in the range of yields less than 8% and the probability for an outperformance (greater than or equal 8%) is 40%. Actually, the demanding hedging of the (ratchet) lookback option becomes relatively cheap since the in-the-moneyness probability is low (see also Subsection 8.5.4) and assumptions concerning customer lapse behavior are commonly taken into account. Since we excluded acquisition, selling and administrative costs (AXA TwinStar deducts 9.5% and Swiss Life Champion deducts 11.77% of the total premium sum) for insurance coverage as well as fund management fees (commonly 1.5% total expense ratio), the actual effective

[207] A yield less than the inflation rate is synonymous to a real loss. In the following we assume an inflation rate of 2%.

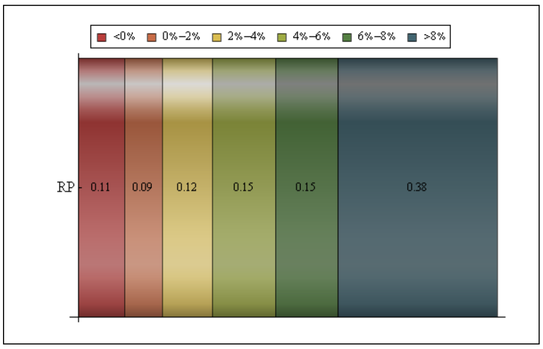
[208] Note that the low-return ten percent of GMIB or bundled GMDB and GMIB contracts exhibit an average negative yield to maturity of -0.11%. This tail expectation should become more adverse if fixed costs ϕ are included.

reduction in yield should be in the range of TwinStar values[209] estimated in Ortmann (2007, 2010). To sum up, in terms of a risk-return profile, a VA with GMIB obtains yields above the rate of return for conventional pension policies but also exhibits a significant probability of solely reaching the guarantee (depending on the underlying rider equipment). Furthermore, a proper analysis of the net return on the savings premiums or key risk-return figures constitutes an opportunity to compare the guarantee costs for insurance products with different guarantee concepts.

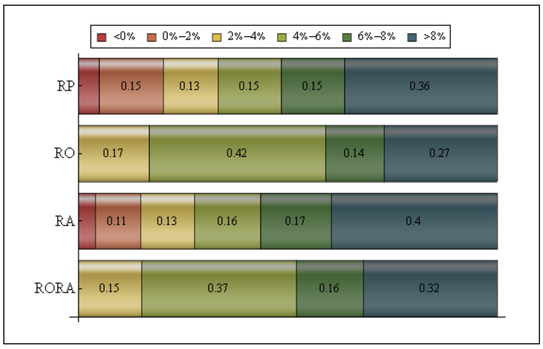
Risk-return figures and rate of return quantile bar charts for different guarantee riders

rider composition	risk-return measure	guarantee feature			
		RP	RO	RA	RORA
GMDB (RP)	$E\left[y_a^N\right]$	0.065	—	—	—
	$E\left[y_a^N \middle| \text{“10\%”}\right]$	-0.026	—	—	—
	$P\left(y_a^N < 0.02\right)$	0.193	—	—	—
GMIB	$E\left[y_a^N\right]$	0.066	0.066	0.070	0.070
	$E\left[y_a^N \middle| \text{“10\%”}\right]$	-0.001	0.034	0.001	0.034
	$P\left(y_a^N < 0.02\right)$	0.205	0.000	0.151	0.000
GMDB (RP) + GMIB	$E\left[y_a^N\right]$	0.065	0.066	0.070	0.070
	$E\left[y_a^N \middle| \text{“10\%”}\right]$	-0.001	0.034	0.001	0.034
	$P\left(y_a^N < 0.02\right)$	0.206	0.000	0.152	0.000

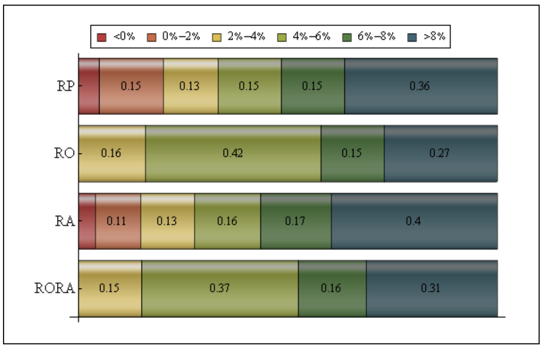
(8.10.1): Risk-return figures



(8.10.2): Quantile bar chart GMDB rider



(8.10.3): Quantile bar chart GMIB rider



(8.10.4): Quantile bar chart bundled GMDB and GMIB rider

Table 8.10: Risk-return figures and rate of return quantile bar charts for different guarantee riders w.r.t. an individual aged $x = 45$ and a contract maturity $T = 20$. The simulations are based on 10^5 iterations.

[209] Ortmann (2007, 2010) estimates a reduction in yield due to total effective costs (contract, investment, guarantee) for the AXA Twinstar Invest variant of 3.63% (3.65%) which lowers the predicted gross rate of return on the premiums paid from 8% per annum (see Table 8.3) to a rate of return of 5.87% (5.85%) as well as an effective yield (expected net return on the savings premiums) of only 4.37% (4.35%). This rate of return is below the real expected return of historic broad-market stock indices. The model calculations are based on a 35 year-old insured paying a monthly premium of $\pi^G = 100$ over a period of $T = 30$ years.

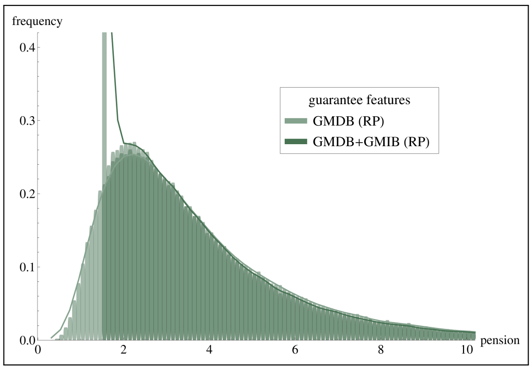
Analysis of the pension amount at maturity

At maturity, a financially rational policyholder of a VA contract with at least a GMIB rider decides in favour for an annual guaranteed pension $L(T) \cdot r^G_{x+T}(T)$ if and only if it exceeds the market value of the investment portfolio $A(T,\pi,\varphi)$ annuitised by the market payout rate $r_{x+T}(T)$ valid at that time. In other words, the insured exercises the annuitisation option if the fixed annual pension is higher than the prevailing market pension. In contrast to a VA contract with GMAB rider, we occasionally observe scenarios (especially for the return of premium (RP) enhancement feature) in which the policyholder allows the option (8.6) to lapse although the guaranteed benefit base $L(T)$ already exceeds the value of the investment portfolio $A(T,\pi,\varphi)$. Due to a comparison of the pension amounts, the insured may decide in favour for the market pension. Figure 8.10 illustrates that based on the insured's survival the GMIB rider yields by definition a minimum guaranteed pension amount at maturity T. Subfigures (8.10.1) and (8.10.3) show that in case of the RP and RA enhancements the policyholder receives at least an annual deferred life annuity of 1.58 against a previously paid regular premium of 1. This minimum pension is reached approximately in 12% (RP) and 7% (RA) of cases. For the RO and RORA scenario[210] the minimum pension amounts to 2.47 but the probability of exercising the exchange option is even 45% (RO) and 37% (RORA). For VAs without any guaranteed living benefits the pension amount at maturity is to a larger extent subjected to previous fund value performance. For this reason, the guaranteed pensions 1.58 (2.47) are fallen short of by more than 11% (33%). Nevertheless, a total loss of the survival benefit is excluded entirely for the parameter constellation of the selected financial market model.

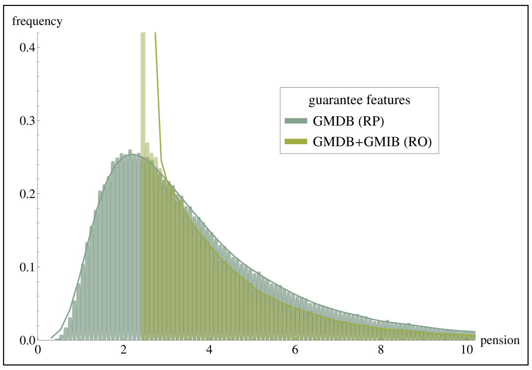
In addition, Table 8.11 shows the real-world "in-the-moneyness" probabilities for different rider and enhancement feature combinations. Since in the event of a claim approximately 67% of the GMDB contracts with premium refund are valuable for the annuity provider, the rider is often already included in GMIB annuities without triggering any additional guarantee charge. In this connection, a rational policy behaviour takes place, if at all, only by surrendering a contract which is, from the customer's point of view, far out of the money during the deferment period. For the single GMIB put option we observe that the more comprehensive the benefit base $L(T)$ the rarer the option becomes valuable. While for the premium refund enhancement 87% of the simulated fund values outperform the benefit base, only every seventh contract of the RORA type

[210] The density shapes under the RO (Subfigure (8.10.2)) and RORA feature (Subfigure (8.10.4)) are quite similar since option payoffs approximately coincide in 50% of the cases.

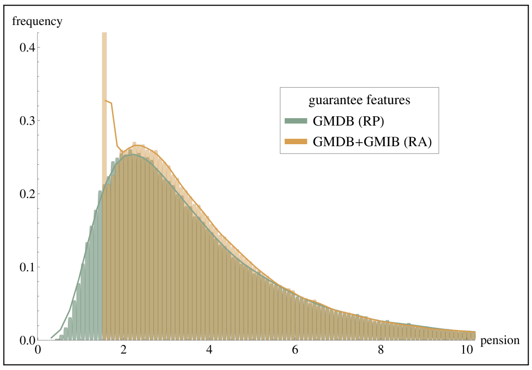
Frequency distributions of the pension amount at maturity for different guarantee features



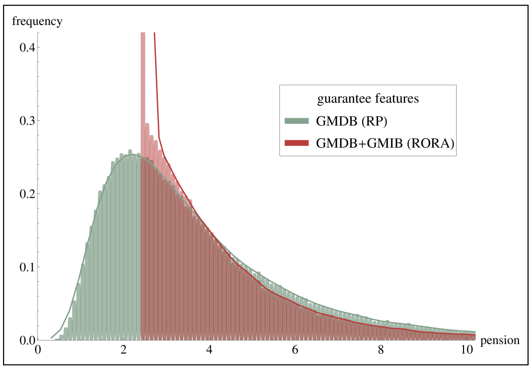
(8.10.1) $L(T) = L^{\mathrm{RP}}$



(8.10.2) $L(T) = L^{\mathrm{RO}}$



(8.10.3) $L(T) = L^{\mathrm{RA}}$



(8.10.4) $L(T) = L^{\mathrm{RORA}}$

Figure 8.10: Probability densities of the pension amount at maturity for different guarantee features w.r.t. an individual aged $x = 45$, a contract maturity $T = 20$ and charges taken of from Table 8.5. The histograms are based on 10^5 drawn samples. The solid lines represent the Gaussian kernel density estimators.

does so although the corresponding fair charge is more than six times higher (see Table 8.5). Furthermore, the increase in the option's moneyness compared to the GMDB and GMIB bundle is only marginal since the withdrawal of the total guarantee charge ensures that the option is slightly earlier in the money.

Summing up, we conclude that GMIB products offer an attractive guaranteed pension together with an expected rate of return which is still competitive. At the same time, however, it should be noted that the probability of reaching the minimum income is quite high especially for those enhancement features widely sold in the market. In practice, the supply of additional features like ratchets (RA) in GMIB contracts serves as a method to control or affect the customer's behaviour concerning surrender and option exercise. Whenever the put option moves considerably out of the money, the enhancement of the peak value guarantee reduces the risk of selective lapsation, i.e it prevents the insured from surrendering in times of strong fund price increase. For instance, we calculate that a financially rational insured, who has selected the bundled

GMDB and GMIB variant with ratchet clause, should decide in nearly 70% of cases in favour for the guarantee.

In-the-moneyness probabilities

rider composition	guarantee feature			
	RP	RO	RA	RORA
GMDB (RP)	0.67	—	—	—
GMIB	0.87	0.54	0.21	0.14
GMDB (RP) + GMIB	0.88	0.55	0.21	0.15

Table 8.11: In-the-moneyness probabilities at the exercise date of the put options (8.2) and (8.6) for different guarantee features w.r.t. an individual aged $x = 45$ and a contract maturity $T = 20$. The probabilities are based on $5 \cdot 10^4$ iterations.

8.6 Conclusion

In this chapter, we have analysed a modern, flexible unit-linked pension product providing customers with a high predetermined fund participation and optional riders guaranteeing minimal death and living benefits. Over the last decades, these so-called variable annuities have gained more and more attraction in the life insurance sectors of highly-developed service societies. This is attributed to the merits of a successful combination of downside protection as offered in conventional pension insurance with potential returns from unit-linked products. Insurance takers are free to select out of a range of investment funds and guarantee riders to combine guaranteed death, living and withdrawal benefits. We focussed on a regular-premium variant – which is typical for the German VA market – with guaranteed minimum income benefit rider. Most of the policies are optionally provided with a guaranteed minimum death benefit during the savings phase which was marginally discussed in academic literature so far. For the sake of simplification, we only considered risk stemming from changes in the mortality and financial market evolution. We questioned whether some features or different benefit bases are offered for contract design or consumers' benefit. However, a bundled sale in favour for an incentive has a strong impact on the financially rational option exercise and the contract profitability. We applied a risk neutral pricing approach to value the underlying embedded options based on the approach by Nielsen and Sandmann (2002). More precisely, we estimated a fair guarantee charge which is sufficient to cover additional beneficial guarantees when compared to a pure unit-linked endowment contract. By means of Monte Carlo simulation we presented charge estimates based on common financial market parameters with GMIB and/or

GMDB rider calibrated to mortality data from the Human Mortality Database (2009).

We also performed a sensitivity analysis concerning parameter misspecification for the contract, financial and mortality framework. It became evident that fair charge estimates strongly differ from usual market fees and a variation in the contract specific parameters leads to substantial changes in the fair percentage guarantee charge. Hereby, a well-balanced choice of the guaranteed roll-up rate in the presence of the prevailing market yield curve describes a key determinant for the amount of the fee and thus its market competitiveness. The influence of financial market parameters – in particular interest rate and fund volatility – is no less important for the charge estimation especially against the quite realistic assumption of correlated zero bond and fund prices. A change in the parameters driving the deterministic part of the mortality rate process has similar distinct effects and therefore should not be disregarded when practising aggregated long-term portfolio risk management. Furthermore, note that the choice of the mortality process parameters also affects the determination of the conservative survival function used to calculate the guaranteed payout rate.

In addition, we analysed the rate of return distribution and key risk-return figures as well as the pension amount at maturity under the real-world measure. By means of quantile bar charts we showed that the simulated rates of return are differently distributed among the individual yield classes depending on the selected rider and benefit base. The policyholder principally earns a guaranteed pension bought through a lower chance of outperformance. Since a GMIB rider guarantees a predetermined lifelong pension payment, it does not necessarily prevent the policyholder from a yield to maturity below the inflation rate. A choice in favour for the so-called roll-up enhancement feature, which is commonly selected as a protection against inflation, can counteract bearish market scenarios. Conversely, a consumer could be interested in a lock-in of interim fund price gains which is ensured by the ratchet clause. The effects of both enhancement features (individually or in combination) on the rate of return and pension amount were discussed in detail. From the perspective of an annuity provider, the supplementary offer of certain enhancement features provides opportunities to control the policyholder's lapse behaviour and thus saving potential for the quantification of necessary hedge positions. We showed that features which increase the benefit base at maturity also increase the option's intrinsic value and thus its moneyness. Admittedly, with an enhanced benefit base the probability that the annuity buyer or dependants solely receive the guaranteed benefit also increases.

Summary and Outlook

The main focus of this thesis is on several aspects of mortality modelling and life insurance mathematics. In particular, we discussed significant demographic changes in mortality and, in this connection, presented a wealth of appropriate graduation / forecasting techniques. Based on the choice of a concrete model, we also dealed with tasks concerning the pricing, management of (longevity) risk and risk capital allocation of private funded pension products. Special attention was given to the interaction of (systematic) mortality and financial market risk. Therefore, the first part of the thesis gave an illustrated overview of mortality modelling from the beginnings to recent modern approaches. The second part applied selected stochastic mortality models to life annuity products in order to examine pricing or reserving issues in incomplete markets. The results can be summarised as follows:

Summary Part I:

The introductory Chapter 1 of the thesis treated some recent challenges to the national and international life insurance business. In particular, the modelling and management of longevity risk motivates a contribution in terms of pricing and reserving applied to annuity products. The chapter was rounded off by a description of the remaining structure. Chapter 2 illustrated the different demographic changes which have taken place more or less significant in most industrialised nations in the course of the 20th century. Since in the first half of the century primarily young ages have experienced an improvement in mortality due to medical advances (e.g. a reduction of infectious diseases), this has been the case for old ages due to improved life standards (accompanied by an apparent decline in chronic diseases) in the second half. The overall mortality has steadily declined with different force of improvement and significant random structural changes depending on age, calendar year and birth cohort. In particular, the oldest age groups have experienced a clear shift (expansion phenomenon) and shape modification (rectangularization effect) of their period specific survival functions. As a consequence, the expected individual lifetime increased which induced a dramatic "upwards" shift in the national age structure with immediate consequences for the public pension and health care systems, life insurance companies and annuity providers as well as life settlement investors.

A detailed chronological overview of the most important representatives of mortality (forecasting) models in actuarial mathematics together with details concerning respective fields of application were developed in Chapter 3. In this context, mortality was assumed to evolve time-invariant such that observations from the recent to medium-term past will continue in the future. Some of the presented methods were additionally illustrated based on historical German period life table data from the Human Mortality Database (2009). On the one hand, we have surveyed discrete-time dynamic life table projection methods using either a horizontal extrapolation of the projection object itself or a vertical / diagonal extrapolation of related functional parameters. As a result, the methods are subjected to a high parameter dimensionality and may cause implausible forecasts. On the other hand, we dealed with continuous-time graduation procedures in parametric as well as non-parametric form. Parametric approaches like mortality laws or transforms of life table functions reduce the degrees of freedom but, at the same time, introduce forecasting difficulties due to parameter dependencies. Non-parametric graduation techniques create a smooth age-pattern but require subjective judgement from the forecaster concerning the degree of smoothness containing the risk to flatten short- and mid-term trends too much. Therefore, the method of choice in projecting the age-pattern of mortality has commonly been a consecutive combination of graduation and extrapolation. Nevertheless, a deterministic approach includes the risk to systematically misjudge mortality developments[211] thus is subjected to limits concerning its applicability.

In the subsequent Chapter 4 we provided a description of modern stochastic forecasting methods which gained in importance during the last two decades. The reasons for this are obvious: They allow for random short-term variations (e.g. triggered by either natural catastrophes and epidemics or sudden medical breakthroughs) and long-term mortality trends (e.g. the continuing development of longevity). In this context, the well-known discrete-time model of Lee and Carter (1992) describes a distribution-free approach which provides a mean estimate together with a measure of (forecasting) uncertainty. Several modifications and extensions of the Lee-Carter framework relaxed the rather strong (distributional) model assumptions and enhanced parameter estimation efficiency. Additional period and / or cohort factors improve the fitting

[211] Almost all mortality projections carried out in the second half of the 20th century underestimated the increase in life expectancy.

performance and describe systematic mortality trends more detailed. Finally, we reviewed a continuous-time mortality term structure approach recently discussed in actuarial literature[212]. The approach uses stochastic processes already applied in no-arbitrage pricing frameworks for interest rate derivatives due to a similar definition of the spot interest and mortality rate. In analogy to the interest rate definitions, the underlying explanatory variables are either represented by the short, forward or market mortality rate.

In Chapter 5, we undertook a separate, illustrated excursion on the parametric graduation and forecasting of mortality data using the classical Lee-Carter mortality model and modifications as well as (multi-factor) extensions. More precisely, we focussed on a Poisson regression modification treated by Brouhns et al. (2002b) and two extensions introduced by Renshaw and Haberman (2003, 2006). We have tested different models against a range of basic qualitative and quantitative criteria like residual analysis, parsimony, variance explanation, mean squared error and fan chart comparison. To sum up, a decision for or against a certain model variant must always comply with the requirements and data quality of the selected application. The bilinear models were robust but limited in the fitting ability due to a small number of parameters. For the complex multi-factor extensions, precisely the opposite was the case.

Outlook Part I:

A decision which model suits best in case of a concrete application may be deduced by verification of several suitable qualitative and quantitative criteria. The fulfilment of qualitative aspects must therefore be considered more or less as a basic prerequisite. Nevertheless, a comparison, which qualitative criteria are necessary and which quantitative criteria are convincing, deserves a closer examination.

In Chapter 4 we introduced four theoretical concepts following Cairns et al. (2008) which originate from financial market models. For this reason, the short-rate, market, forward mortality and positive mortality models need to be examined carefully for their suitability for mortality modelling. At this time, the application is very limited insofar as the estimation of risk-neutral spot and forward survival probabilities presupposes a unique market price for systematic mortality risk only. Furthermore, the concrete implementation of market models which is closely related to the pricing of mortality-linked securities is hindered so far in that a mortality derivative or life market is far

212 For an overview of risk-neutral frameworks for pricing and hedging mortality risk we refer to Cairns et al. (2006a) or Blake et al. (2006b).

from being perfectly liquid, transparent, arbitrage-free or frictionless contrary to the (no-arbitrage) pricing assumptions.

Summary Part II:

The German state pension problem and the need for supplementary private funded retirement provision was presented in Chapter 6. More precisely, we briefly described the development from the capital covered statutory pension system introduced during Bismarck's social legislation at the end of the 19th century to the pay-as-you-go pension system under the government of Chancellor Adenauer in the middle of the 20th century. On the institutional side, the following dramatic continuing age shift in the population structure necessitated legal modifications concerning future contributions or benefits. On the contributors side, the reduced state pension benefits have to be compensated by private provision forms from savings models from all three pillars / layers of retirement saving. Besides voluntary occupational pensions, an important provision element is given by life insurance products of the third layer. The benefits of deferred life annuities either with conventional or unit-linked savings phase are, besides risk stemming from the financial market, subjected to the risk of longevity which was the major object of research in the subsequent chapters.

Chapter 7 treated the valuation, risk management and solvency assessment of the traditional life annuity business which strongly depend on the assumptions that are posed on the underlying mortality and interest rate dynamics. By means of different (stochastic and deterministic) mortality models which are fitted to the same mortality data and interest rate models calibrated to the market term structure we focussed on the question how important it is to take into account for (random) changes in the mortality. We used Monte Carlo simulations to approximate the variance of the discounted cash-flow and its decomposition into a pooling and a non-pooling risk part. We also considered effects of pricing in incomplete markets using the principle of zero expected utility and the quantile principle. The estimated risk premiums were benchmarked to the equivalence premium for a selection of different hedge / investment strategies. Finally, we focussed on the definition of solvency requirements and meaningful shortfall measures. The associated shortfall probability and conditional expected shortfall of the annuity provider are sensitive to changes in the underlying hedge strategy and degree of uncertainty with regard to the underlying mortality models and interest model. In summary, the results emphasise that the impact of stochastic mortality is low if compared to the impact of stochastic interest rates. To some extent, one can argue that the major risk which is due to stochastic mortality stems from its interaction with stochastic interest rates. However, against the background of a risk-adequate assessment this should have a long-lasting effect on the insurers pricing and reserving.

In Chapter 8, we analysed a modern, flexible unit-linked life insurance contract. More precisely, we dealed with a deferred variable annuity providing customers with a high predetermined fund participation and optional riders guaranteeing minimal death and living benefits. We focussed on a regular-premium variant with a minimum income benefit rider guarantee at retirement optionally provided with a guaranteed minimum death benefit during the savings phase. For the sake of simplification, we only considered risk stemming from the underlying mortality and financial market. We applied Monte Carlo simulation to estimate a fair guarantee charge which is sufficient to cover the additional beneficial guarantees when the variable annuity contract is compared to a pure unit-linked endowment contract. The charge estimates were based on common financial market parameters and mortality parameters calibrated to mortality data from the Human Mortality Database (2009). It became evident that fair charge estimates clearly differ from usual market fees and a variation of the contract specific parameters leads to substantial changes in the fair percentage guarantee charge. In particular, we performed a sensitivity analysis of the guarantee charges and option prices concerning parameter misspecification for the contract, financial and mortality framework. We also analysed the rate of return distribution as well as the pension amount at maturity. Depending on the selected rider and benefit base, the policyholder principally earns a comparatively high guaranteed pension. But high guarantees are bought through a smaller chance of attractive rates of return. In this context, the roll-up enhancement feature, which is commonly selected as a protection against inflation, and the ratchet clause ensuring an automatic lock-in of interim fund price gains are popular variants. Since those features increase the option's intrinsic value and thus its moneyness, the probability of solely reaching the guaranteed benefit also increases with different consequences for both the annuity writer and the policyholder.

Outlook Part II:

The requirements to be met by modern risk management are multifaceted. Firstly, the combined management of longevity and financial market risks plays a key role in safeguarding the assurer's solvency situation. We emphasised that life insurers and pension providers are exposed to significant systematic mortality risk which has a strong influence on the pricing and risk capital assessment.

A comprehensive look at the value process of the written insurance business helped us to determine an adequate (risk) premium and solvency margin using simple bond hedging / investment scenarios to partly catch the financial risk. Secondly, there is still a need for further research regarding effective dynamic hedging (strategies) using mortality derivatives[213], bulk annuity transfers for closed pension funds, swap-like reinsurance or natural hedging approaches[214].

Especially the mortality-linked financial securities have aroused increasing interest in actuarial literature and practice since they theoretically enable an almost perfect hedge against longevity or catastrophic risk. Typical representatives of this new asset class of derivatives are survivor bonds[215] as well as swaps[216], annuity futures and further (exotic) mortality options. On the one hand, derivative issuers like pension schemes are provided with an instrument to hedge their longevity exposure. On the other hand, buyers like hedge funds or other financial institutions could be interested in these derivatives because of liquidity and low betas relative to their existing investment portfolio. However, the practical experience has shown that investors and issuers had scarcely any appetite for longevity risk. Hardly assessable factors[217] like the high degree of basis and credit risk (with standardised hedge instruments), the lack of bond concepts with sufficient duration and standard valuation methodologies (concerning the risk premium in incomplete markets where no-arbitrage pricing is not applicable) hindered the market development so far. The main objective of further research is therefore to thoroughly examine pricing issues and hedging performance. Nevertheless, recent actuarial literature, e.g. Loeys et al. (2007), Blake et al. (2008) and Bertocchi et al. (2010), agrees that, if certain entry barriers can be overcome, a mortality market will soon evolve with an increasing number of longevity protection willing sellers, advanced acknowledgement and technical potential.

A further interesting point is to extend the model framework in favour for a

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213 Mortality derivatives are correlated with a mortality index or number of survivors of a reference cohort. For a detailed description see for example Blake and Burrows (2001), Lin and Cox (2005), Dowd et al. (2006) or Blake et al. (2006b).

214 Natural hedging can be done “across time” as well as “across different lines of business”. See Pitacco et al. (2008) for information on the topic of natural hedging.

215 In 2004 the European Investment Bank (EIB) aimed to launch a longevity bond with a maturity of 25 years. However, the bond was withdrawn for redesign (term and model inaccuracies) and the low demand from investors out of the pension insurance sector. Recently in 2010, the Swiss Re launched a series of longevity risk bonds via an off-balance sheet investment vehicle named Kortis with maturity in 2017.

216 Swiss Re issued the first longevity swap in 2007 with the UK life assurer Friends’ Provident which is valued at 1.7 billion £. Nevertheless, the mortality swap corresponded to an reinsurance contract and was not traded on a life market.

217 See Cairns et al. (2006a) for a detailed description on risks within pricing issues.

more complex but more customary contract. Deferred life annuities commonly include embedded conventional death benefits[218], variable periodic premium payments or a lump-sum option at maturity. Additionally, it appears useful to refine of the model by correlated[219] future mortality and interest rates under the pricing measure and transaction costs for rebalancing the liability portfolio.

Driven by recent market crises in the last decade and near collapses of a few variable annuity provider, which relied on self-insurance only, the actuarial literature agrees that a meaningful risk management demands the introduction of a comprehensive hedge program. Therefore, the insurance company needs to hold an actively managed hedge portfolio sensitive to changes in the actuarial and financial factors and irrational policyholder behaviour concerning lapse, investment and option exercise. Hedging, in turn, introduces further sources of uncertainty (liquidity, credit and operational risk) and is rarely perfect due to limitations concerning the long maturity and the liquidity of adequate derivatives. For this reason, we leave the analysis of parameter uncertainty in the curse of dynamic hedging effectiveness for further research.

We have shown that variable annuities contain quite a few different embedded guarantees or flexibilities available to the customer. For example, at maturity, the insured can exercise a premature retirement or a prolongation option within certain limits to outlast periods with weak fund performance. A verification whether the prolongation option is actually meaningful from a financial point of view or originates from product selling reasons is left for further research. In particular, the offering of so called guarantee riders, which admittedly enhance the guaranteed living or death benefit base but also enhance the complexity since look-back and barrier options are involved, has to be examined. The question must be asked whether such flexibilities enrich an option or simply serve as a method to control or affect the customers behaviour concerning surrender and option exercise. Several authors consider the so called option to lapse[220] the contract during the accumulation phase. In addition, the customer

[218] A standard feature is given by a term insurance amounting to a certain percentage ratio of the overall premium sum or a premium refund.
[219] Miltersen and Persson (2006) use the Heath-Jarrow-Morton framework to model the correlated development of future term structures of forward force of mortality rates and to price mortality derivatives.
[220] Milevsky and Salisbury (2001) determine a suitable asset-based charge that funds a variable annuity guarantee under rational surrender behaviour and a frictional deferred surrender charge. Mudavanhu and Zhou (2002) value the GMDBs and the option to lapse during periods of weak fund performance by means of a deterministic piecewise step surrender charge function. In contrast, Kolkiewicz and Tan (2006) price the lapse option assuming a lapsation behaviour depending on economic factors such as the equity volatility.

usually receives several investment options or rights either to switch[221] and / or shift funds out of the individual basket or to pick out of a range of different (actively managed) strategy depots. To sum up, the impact of behavioural assumptions (surrender, selection phase, withdrawal) or contract restrictions (fund and guarantee management) on pricing and hedging has to be investigated more precisely.

[221] See Mahayni and Schneider (2012), Mahayni and Schoenmakers (2011) or Zieling (2010) for a more detailed treatment of the option-to-switch topic.

Zusammenfassung (Summary in German)

Die vorliegende Arbeit setzt sich mit der Untersuchung verschiedener Facetten der Sterblichkeitsmodellierung und Mathematik von Lebensversicherungen auseinander. Insbesondere werden nachhaltige Veränderungen innerhalb der Bevölkerungssterblichkeit und zu deren Beschreibung eine Vielzahl unterschiedlicher Ausgleichs- und Extrapolationsverfahren vorgestellt. Anhand ausgewählter Sterblichkeitsmodelle werden aktuarielle Themen wie Prämienkalkulation, Management biometrischer Risiken und die Bemessung von ausreichendem Risikokapital für private Rentenversicherungen behandelt. Ein besonderes Augenmerk liegt dabei auf der Untersuchung des Zusammenwirkens von Sterblichkeits- und Finanzmarktrisiko. Aus diesem Grund enthält der erste Teil der Arbeit einen illustrierten Überblick über verschiedene Varianten von Sterblichkeitsmodellen angefangen bei ersten parametrischen Modellen bis hin zu moderneren Zeitreihenansätzen. Der zweite Teil hingegen beschäftigt sich mit der Anwendung ausgewählter stochastischer Modellierungsformen zwecks Bestimmung von Risikoprämie und -kapital für private Rentenversicherungen in einem unvollständigem Marktumfeld. Die Ergebnisse lassen sich wie folgt zusammenfassen:

Zusammenfassung Teil I:

Das einleitende Kapitel 1 behandelt unterschiedliche Entwicklungen, die das Lebensversicherungsgeschäft vor große Herausforderungen stellten und auch weiterhin stellen werden. Insbesondere die Modellierung und der Umgang mit Langlebigkeitsrisiken motiviert zu einem Forschungsbeitrag in Form einer Untersuchung der Auswirkungen auf die Kalkulation und Reservierung beliebter Altersversorgungsprodukte. Auf die Darstellung der Motivation und Fragestellung folgt eine Beschreibung der übrigen Kapitelinhalte. Kapitel 2 zeigt die verschiedenen Entwicklungen der Bevölkerungssterblichkeit hochindustrialisierter Länder im Verlauf des 20. Jahrhunderts auf. Aufgrund des anhaltenden medizinischen Fortschrittes (z.B. bei der Bekämpfung von Infektionskrankheiten) in der ersten Hälfte des Jahrhunderts haben vor allem junge Altersgruppen eine deutliche Verbesserung der Sterblichkeitsverhältnisse erfahren. In der zweiten Hälfte waren vor allem die sogenannten Rentnerjahrgänge durch eine spürbare Verbesserung der Lebensbedingungen (einhergehend mit einem Rückgang chronischer Erkrankungen) begünstigt. Es läßt sich beobachten, dass die Gesamtsterblichkeit einer nahezu stetigen Abnahme mit unterschiedlich starken und zufälligen zwischenzeitlichen Ausprägungen abhängig von Alter, Beobachtungsjahr und Geburtenjahrgang unterlag. Insbesondere für die älteren Jahrgänge läßt sich eine Verschiebung und Verformung ("Rechteckbildung") der periodenspezifischen Überlebensfunktion feststellen. Infolgedessen stieg die durchschnittliche Lebenserwartung merklich an und verursachte unter anderem Finanzierungsengpässe bei Leistungsversprechen ausgesprochen durch den gesetzlichen Rentenversicherungsträger, Lebensversicherungsunternehmen sowie Zweitmarktanbieter.

Im darauffolgenden Kapitel 3 wurden die aus aktuarieller Sicht wichtigsten Vertreter von Sterblichkeitsvorhersagemodellen chronologisch aufgearbeitet und deren Anwendungsfeldern, sofern bekannt, beschrieben. In diesem Zusammenhang setzt man für die Projektion voraus, dass sich kurz- bis mittelfristig beobachtete vergangene Sterblichkeitstrends auch in Zukunft fortsetzen werden. Einige der dargestellten Methoden wurden zusätzlich anhand deutscher Sterblichkeitsdaten der Human Mortality Database (2009) illustriert.

Zum einen wurde ein Überblick über diskrete Methoden zur Projektion dynamischer Sterbetafeln gegeben. Dabei bedient man sich entweder einer horizontalen Fortschreibung der beobachteten Größe oder einer vertikalen bzw. diagonalen Extrapolation der zu Grunde liegenden Parameter. Dadurch benötigt man allerdings für die Fortschreibung eine hohe Anzahl an Parametern verbunden mit dem Risiko unplausibler Extrapolationsergebnisse. Zum anderen wurden stetige Ausgleichs- bzw. Glättungsmechanismen in parametrischer und nicht-parametrischer Form betrachtet. Parametrische Ansätze wie Sterbegesetze oder Transformierte von Sterbetafelfunktionen eignen sich zwar zur Reduktion der Freiheitsgrade, erhöhen aber auch gleichzeitig den Extrapolationsaufwand aufgrund möglicher korrelierter Modellparameter. Nicht-parametrische Ausgleichsverfahren erzeugen glatte altersspezifische Verläufe der Modellgröße, deren Erscheinungsbild jedoch maßgeblich durch das subjektive Einschätzen des Anwenders bestimmt wird. Dabei ist auch unter Anwendung bestimmter Näherungsregeln nicht auszuschliessen, dass kurz- bis mittelfristige Trends zu stark ausgeglättet werden. Aus diesem Grund nutzt man üblicherweise eine Kombination aus einem Ausgleich roher altersspezifischer Sterbedaten und einer anschliessenden parametrischer Projektion. Nichtsdestotrotz beinhaltet ein deterministischer Ansatz das Risiko einer systematischen Fehleinschätzung[222] der Sterblichkeitsentwicklung und somit eine Einschränkung der Anwendbarkeit.

Kapitel 4 enthält eine Beschreibung moderner stochastischer Vorhersagemodelle, die für die Sterblichkeitsforschung während der letzten 20 Jahre an Bedeutung zunahmen, da unter anderem die Möglichkeit zur Modellierung zufallsbedingter Schockszenarien (beispielsweise ausgelöst durch Naturkatastrophen, Epidemien oder aber einen medizinischen Durchbruch) und Langzeiteffekte (etwa der anhaltende Trend zur Langlebigkeit) besteht. In diesem Zusammenhang bietet das wohlbekannte zeitdiskrete Modell nach Lee und Carter (1992) einen verteilungsfreien Ansatz zur Ermittlung zukünftiger Projektionswerte mitsamt den zugehörigen Konfidenzintervallen. Einige Modifizierungsvarianten des Lee-Carter Modells lockern die teils starken (Verteilungs-)Annahmen und erhöhen die Effizienz der Parameterschätzung. Zusätzliche zeitabhängige und / oder Geburtsjahr-abhängige Faktoren verschiedener Modellerweiterungen sind darüber hinaus in der Lage, das Anpassungsvermögen zu verbessern und somit systematisch bedingte Sterblichkeitstrends genauer abbilden zu können. Zu guter Letzt wurden zeitstetige Modellvarianten zur Beschreibung der Sterblichkeitsstruktur besprochen, die in diesem Zusammenhang erst kürzlich durch versicherungsmathematische Literatur[223] aufgegriffen wurden. Da Zins- und Sterblichkeitsrate gewisse konzeptionelle Ähnlichkeiten aufweisen, verwendet man für die Ansätze stochastische Prozesse, die bereits für arbitragefreie Modelle zur Bewertung von Zinsderivaten verwendet werden. Analog zur Definition der Zinsrate ist die zu Grunde liegende erklärende Sterblichkeitsrate dabei entweder als Momentanrate, Forward-Rate oder Marktrate definiert.

In Kapitel 5 wurde ein gesonderter, illustrierter Exkurs zur parametrischen Be- bzw. Fortschreibung von Sterbedaten unter Zuhilfenahme des Lee-Carter Modells beziehungsweise einige der wichtigsten Modellverfeinerungen und -erweiterungen durchgeführt. Dies sind im Einzelnen der auf Poisson-Regression beruhende Ansatz nach Brouhns u.a. (2002) und die von Renshaw und Haberman (2003) bzw. Renshaw und Haberman (2006) dargestellten Mehrfaktormodelle. Die getesteten Modellvarianten wurden einer ausführlichen Analyse qualitativer und quantitativer Gütekriterien wie Residuenanalyse oder der Untersuchung von Freiheitsgraden, Bestimmtheitsmaß, mittlerer quadratischer Abweichung und Vorhersageintervall-Charts unterzogen. Zusammenfassend läßt sich sagen, dass die Wahl für oder wider eine bestimmte Modellvariante in erheblichem

[222] Nahezu alle in der zweiten Hälfte des 20. Jahrhunderts durchgeführten Sterblichkeitsvorausberechnungen unterschätzten den tatsächlichen Anstieg der Lebenserwartung.
[223] Ein Überblick über risikoneutrale Ansätze zur Bewertung und Absicherung des Sterblichkeitsrisikos wird unter anderem in Cairns u.a. (2006) oder Blake u.a. (2006) gegeben.

Maße davon abhängt, welche konkreten Anforderungen an die Vorhersage bestehen und welche Datengüte beziehungsweise -umfang der Anwendung zu Grunde liegt. Bilineare Modelle wie das Lee-Carter Modell sind zwar robust aber begrenzt in ihrem Anpassungsvermögen aufgrund der geringen Anzahl an Parametern. Für die Mehrfaktor-Erweiterungen gilt hingegen das genaue Gegenteil.

Ausblick Teil I:

Eine Einschätzung darüber, welches Modell sich für ein bestimmtes Anwendungsgebiet am besten eignet, läßt sich durch eine Überprüfung geeigneter qualitativer und quantitativer Kriterien gewinnen. Dabei ist die Erfüllung eines Großteils der qualitativen Aspekte in den meisten Anwendungsfällen als Grundvoraussetzung anzusehen. Nichtsdestotrotz bedarf es einer genauen Überprüfung, welche qualitativen Kriterien zwingend notwendig und welche quantitativen Kriterien aussagekräftig sind.

In Kapitel 4 wurden vier unterschiedliche Konzepte analog zu Cairns u.a. (2008) vorgestellt, deren Ursprung in der Modellierung der Zinsstruktur von Finanzmärkten liegt. Aus diesem Grund sollten die erklärenden Variablen Momentanrate, Forwardrate oder Marktrate kritisch auf ihre Eignung zur Modellierung der Sterblichkeit hin untersucht werden. Derzeit findet die Anwendung lediglich auf Ebene aktuarieller Forschung statt, da die Bestimmung risikoneutraler Momentan- und Forward-Überlebenswahrscheinlichkeiten einen eindeutigen Marktpreis für systematisches Sterblichkeitsrisiko voraussetzt. Die Entwicklung eines liquiden, transparenten, arbitragefreien oder reibungslosen Marktes für Sterblichkeitsderivate befindet sich allerdings noch in den frühesten Kinderschuhen und verhindert somit bisher auch die Anwendung von Marktmodellen und die finanzmathematische Kalkulation sterblichkeitsindexierter Finanzinstrumente.

Zusammenfassung Teil II:

Das Kapitel 6 stellt die Finanzierungsproblematik des deutschen Rentenversicherungssystems und die daraus bedingte erhöhte Nachfrage nach privater Altersversorgung dar. Genauer gesagt wird die Entwicklung von der Einführung des kapitalgedeckten gesetzlichen Rentensystems im Zuge der Bismarckschen Sozialgesetze gegen Ende des 19.Jahrhunderts bis hin zur umlagefinanzierten Versorgung unter Kanzler Konrad Adenauer seit Mitte des 20.Jahrhunderts kurz dargestellt. Auf institutioneller Seite beinhaltet die anhaltende, dramatische Verschiebung der Altersstruktur innerhalb der Bevölkerung ein Erfordernis hinsichtlich gesetzlicher Anpassungen zukünftiger Beiträge bzw. Leistungen.

Für den Beitragszahler hingegen bedeutet dies, dass die sinkende Leistung aus der gesetzlichen Rente über ein Zusatzeinkommen aus den verbleibenden Absicherungsformen unterschiedlicher Schichten der Altersvorsorge aufgebaut werden muss. Diesen Zweck erfüllen, neben betrieblicher Altersversorgung, Lebensversicherungsprodukte der sogenannten dritten Schicht und bieten somit ein wichtiges zusätzliches Standbein. Das lebenslängliche Alterseinkommen aufgeschobener Leibrenten wahlweise mit konventioneller oder fondsgebundener Ansparphase unterliegt jedoch in starkem Maße dem Risiko allgemeiner Langlebigkeit – einem Thema, dem sich die folgenden Kapitel der Arbeit widmen.

In Kapitel 7 wurden die Bewertung, das Risikomanagement und die Risikokapitalermittlung für das traditionelle Lebensversicherungsgeschäft betrachtet. Diese hängen maßgeblich von den Annahmen hinsichtlich der Entwicklung der Sterblichkeit und des Marktzinses ab. Mit Hilfe verschiedener (stochastischer und deterministischer) Modelle, die jeweils an aktuelle Sterblichkeitsdaten und die aktuelle Zinsstrukturkurve kalibriert wurden, zeigt das Kapitel, welchen Einfluss zufallsbedingte Sterblichkeits- und Kapitalmarktentwicklungen auf die anfangs geschilderten Versicherungsprozesse haben. Anhand von Monte Carlo simulierten Varianzwerten für den diskontierten Geschäftsverlust des Versicherers fand eine Aufteilung in einen diversifizierbaren und nicht-diversifizierbaren Bestandteil statt. Des Weiteren wurde die Kalkulation von Risikobeiträgen anhand des Nullnutzen- und Quantilprinzips durchgeführt. Die simulierten Prämienaufschläge wurden für eine Auswahl verschiedener Zero-Bond Hedge-/Investitionsstrategien in Relation zur Äquivalenzprämie gesetzt. Abschließend wurden geeignete Solvenzkriterien und Solvenzmaße zur Bewertung der Zahlungsfähigkeit des Versicherers definiert. Die damit verbundene Ausfallwahrscheinlichkeit und die bedingte Ausfallhöhe des Cash-Flow Prozesses sind aus Sicht des Versicherers stark abhängig von der zu Grunde liegenden Hedgestrategie sowie dem Unsicherheitsgrad bezüglich des Sterblichkeits- und Zinsmodells. In der Zusammenfassung der Ergebnisse wurde darauf eingegangen, dass die Höhe des Sterblichkeitsrisikos im Vergleich zum Kapitalmarktrisiko eher gering erscheint und das Potential größtenteils aus beider Zusammenspiel resultiert. Dennoch ist der Einfluss systematischer Sterblichkeitstrends auf die Kalkulation und die Reservierung als sehr nachhaltig anzusehen.

Das folgende Kapitel 8 beschäftigte sich mit der Analyse einer modernen und flexiblen Variante fondsgebundener Rentenversicherungsprodukte, den sogenannten aufgeschobenen Variablen Annuitäten. Diese Form der Leibrente bietet Verbrauchern eine hohe Aktienmarktpartizipation und gleichzeitig optionale Garantien hinsichtlich Hinterbliebenenschutz und Ablaufleistung. Gegenstand der Untersuchung ist eine Variante gegen periodischen Beitrag mit garantierter Altersrente zum Rentenbeginn und minimaler Todesfallleistung in der Aufschubzeit. Aus Gründen der Vereinfachung wurden lediglich Sterblichkeits- und Finanzmarktrisiken betrachtet. Durch den Vergleich der Monte Carlo simulierten Leistungsbarwerte einer Variablen Annuität mit zusätzlicher Garantiekomponente und einer fondsgebundenen Kapitallebensversicherung ohne garantierte Leistungen konnten faire Garantiegebühren ermittelt werden. Die Simulation der erforderlichen Garantiegebühren basierte dabei auf üblichen Annahmen hinsichtlich der Finanzmarktparameter. Die Sterblichkeitsparameter wurden an aktuelle Sterbedaten der Human Mortality Database (2009) kalibriert. Es zeigte sich, dass die Schätzer für die faire Garantiegebühr teils deutlich von den marktüblichen Gebühren abweichen und vor allem eine Änderung der Vertragsparameter (wie risikoloser Marktzins, Vertragslaufzeit oder Garantieverzinsung) erheblichen Einfluss auf die Höhe der erforderlichen Gebühr haben. Insbesondere wurde eine umfangreiche Sensitivitätsanalyse der Garantiekosten und -optionspreise hinsichtlich einer Veränderung der Inputfaktoren aus dem Versicherungsmantel, Finanzmodell und Sterblichkeitsmodell durchgeführt. Des Weiteren sind auch die (Ablauf-)Renditeverteilung sowie die Altersrente zum Rentenbeginn einer Analyse unterzogen worden. Der Versicherungsnehmer erhält abhängig von der gewählten Garantieart bzw. -höhe zwar eine vergleichsweise hohe garantierte Altersrente, die allerdings durch eine entsprechend geringere Chance auf attraktive Ablaufrenditen erkauft wird. In diesem Zusammenhang stellen die verzinsliche Ansammlung der eingezahlten Beiträge als Schutz gegen Inflationsrisiken und die Absicherung zwischenzeitlich erreichter Fondshöchststände beliebte Garantieerhöhungsvarianten dar. Die Wahrscheinlichkeit, im Leistungszeitpunkt lediglich den Garantiebetrag zu erreichen, steigt durch den Einschluss bestimmter Erhöhungsvarianten, da der innere Wert der Option und somit ihre Werthaltigkeit gesteigert werden. Dies hat unterschiedliche Konsequenzen für den Rentenanbieter und für den Policeninhaber.

Ausblick Teil II:

Die Anforderungen an ein modernes Risikomanagement können mitunter sehr vielschichtig sein. Zum einen spielt die Interaktion von Langlebigkeits- und Finanzmarktrisiken eine tragende Rolle bei der Bemessung eines angemessenen Solvenzkapitals eines Lebensversicherers. Es wurde herausgestellt, dass Lebensversicherer und Rentenanbieter in hohem Maße dem Risiko systematischer Sterblichkeitsentwicklungen ausgesetzt sind mit entsprechenden Auswirkungen auf die Beitragskalkulation sowie Deckungskapitalbereitstellung. Eine umfassende Untersuchung des Cash-Flow Prozesses liefert dabei Aussagen über die erforderlichen Risikobeiträge und Solvenzmittel unter Einsatz verschiedener Hedgestrategien zur Eindämmung finanzieller Risiken. Zum anderen ist weitere Forschungsarbeit hinsichtlich einer effektiven, dynamischen Absicherung anhand sogenannter Sterblichkeitsderivate[224], der vollständigen oder partiellen Übertragung geschlossener Pensionspläne an sogenannte Bulk Annuity Provider, Swap ähnlicher Rückversicherungslösungen oder natürlichen Hedging-Ansätzen[225] erforderlich.

Besonders die Verbriefung biometrischer Risiken weckt zunehmendes Interesse in Forschung und Versicherungspraxis gleichermaßen, da dieser Risikotransfer zumindest theoretisch ein vollständige Absicherung gegen Langlebigkeits- oder Katastrophenrisiken erlaubt. Typische Vertreter dieser neuen Anlageklasse sind Langlebigkeitsanleihen wie Longevity oder Survivor Bonds[226] sowie Longevity Swaps[227], Annuity Futures und weitere (exotische) sterblichkeitsindexierte Optionen. Auf der einen Seite erhalten die Emitenten solcher Papiere einen effektiven Schutz gegen das Langlebigkeitsrisiko ihres Versichertenbestandes. Auf der anderen Seite bietet der Kauf der Derivate Hedgefonds oder anderen Finanzinstituten Liquidität und die Aussicht auf ein Investment-Portfolio mit geringem Betafaktor. Jedoch zeigen Erfahrungen aus der Praxis, dass mögliche Investoren und Emitenten bisher kaum an einer Verbriefung der Langlebigkeit interessiert waren. Mögliche Gründe für diese schleppende Entwicklung eines "Sterblichkeitsmarktes" liegen in den nur schwer einschätzbaren Risikofaktoren[228] wie beispielsweise hohem Basis- und Ausfallrisiko (für standardisierte Hedgeinstrumente), fehlender Bonds mit ausreichender Laufzeit und Bewertungsstandards (hinsichtlich der Berechnung von Risikoprämien in unvollständigen Märkten). Aus diesem Grund muss das Hauptaugenmerk zukünftiger

[224] Sterblichkeitsderivate sind an bestimmte Sterblichkeitsindizes oder die Anzahl der Überlebenden einer Referenzkohorte gekoppelt. Eine detaillierte Beschreibung wird beispielsweise in Blake und Burrows (2001), Lin und Cox (2005), Dowd u.a. (2006) oder Blake u.a. (2006) ausgeführt.

[225] Ein natürlicher Hedge von biometrischem Risiko kann ausgleichend über die Zeit oder über verschiedene Lebenssparten wirken. Für weitere Ausführungen zu diesem Thema siehe Pitacco u.a. (2008).

[226] Die Europäische Investitionsbank (EIB) plante 2004 die Auflage eines Longevity Bonds mit einem Anlagehorizont von 25 Jahren. Allerdings wurde der Bond aufgrund von Designproblemen und dadurch mangelnder Nachfrage vom Markt genommen. Die Swiss Re legte 2010 eine Reihe von Longevity Bonds mit Laufzeitende in 2017 über ein ausserbilanzielles, strukturiertes Anlagevehikel names Kortis auf.

[227] Die Swiss Re emitierte 2007 den ersten Longevity Swap zusammen mit dem britischen Lebensversicherer Friend's Provident über ein Volumen von 1,7 Milliarden Pfund. Allerdings handelt es sich hierbei um einen maßgeschneiderten Rückversicherungsvertrag und keinen gehandelten Standardkontrakt.

[228] Siehe Cairns u.a. (2006) für eine detaillierte Beschreibung von Risiken verbunden mit der Bewertung von Langlebigkeit.

Forschungsanstrengungen auf der sorgfältigen Durchdringung der Kalkulations- und Hedgefragestellungen liegen. Nichtsdestotrotz ist sich die derzeitige aktuarielle Literatur, wie zum Beispiel Loeys u.a. (2007), Blake u.a. (2008) und Bertocchi u.a. (2009), einig darüber, dass sich ein Sterblichkeitsmarkt mit ansteigender Zahl an Marktteilnehmern, Methodenwissen und technischen Möglichkeiten entwickeln kann, sofern bestimmte Markteintrittshürden erfolgreich passiert wurden.

Ein weiteres interessantes Forschungsvorgehen liegt in der Ausweitung des Modellrahmens auf einen komplexeren aber gleichzeitig praxisnäheren Versicherungsvertragstyp. Aufgeschobene Leibrenten beinhalten üblicherweise einen konventionellen integrierten Risikoschutz[229], variable periodische Prämien oder ein Kapitalwahlrecht zum Ablauf der Aufschubzeit. Zusätzlich erscheint es sinnvoll, eine Verfeinerung des Modells hinsichtlich der Korrelation[230] von zukünftigen Sterberaten und Zinsraten unter dem Bewertungsmaß und der Einbeziehung von Transaktionskosten für die Hedgestrategien vorzunehmen.

Aufgrund jüngster (Finanz-)Marktkrisen des letzten Jahrzehnts und der Beinahe-Zusammenbruch einiger Anbieter von Variablen Annuitäten ohne echte Risikozession vertritt die derzeitige Literatur die Meinung, dass ein sinnvolles Risikomanagement die Einführung eines umfänglichen Hedgeprogrammes erfordert. Demnach sollte das Versicherungsunternehmen ein aktiv gemanagtes Hedge-Portfolio aufsetzen, welches hinsichtlich der Allokation sensibel auf finanzielle oder biometrische Änderungen sowie Verbraucherverhalten (Storno, Kapitalanlage oder Optionsausübung) reagiert. Ein umfassendes Hedging birgt allerdings auch zusätzliche Risiken (wie etwa Liquiditäts-, Ausfall- oder operationelle Risiken) mit sich und ist aufgrund der Beschränkungen hinsichtlich der Laufzeit und Liquidität der Hedgeinstrumente als unvollständig zu betrachten. Aus diesem Grund ist die Analyse der Parameterunsicherheit hinsichtlich der Effektivität von dynamischen Hedgestrategien eine interessante Forschungsfrage.

Es wurde gezeigt, dass Variablen Annuitäten für Versicherungsnehmer eine Reihe verschiedener zusätzlicher Garantien oder Flexibilitäten bereithalten. So hat dieser etwa zum Ende der Aufschubzeit die Möglichkeit, durch einen vorzeitigen Abruf flexibel zu verrenten oder durch eine Verlängerung der Ansparphase eine Aktienbaisse "auszusitzen". Die Frage, ob eine Verlängerungsoption aus

[229] Als garantierten Todesfallschutz findet man häufig einen festen Prozentsatz der Gesamtbruttobeitragssumme oder eine Rückgewähr bereits gezahlter Bruttobeiträge.

[230] Miltersen und Persson (2006) verwenden den Heath-Jarrow-Morton Ansatz unter korrelierter Entwicklung der Zinsstrukturkurve und der Forward Sterberaten zur Kalkulation von Sterblichkeitsderivaten.

finanzökonomischer Sicht auch sinnvoll erscheint oder nur vertriebliche Relevanz besitzt ist noch zu beantworten. Insbesondere dem Angebot sogenannter Zusatzoptionen, die zwar die garantierte Erlebensfall- oder Todesfallleistung aber auch die Komplexität über Look-back- und Barrier-Optionen erhöhen, sollte weitere Aufmerksamkeit geschenkt werden. Hierbei muss man die Frage stellen, ob solche Flexibilitäten den Optionswert aus Sicht des Kunden nachhaltig bereichern oder als Instrument zur Steuerung beziehungsweise Beeinflussung des Kundenverhaltens hinsichtlich Rückkauf und Optionsausübung dienen. Einige Autoren betrachten die Option, den Vertrag je nach Werthaltigkeit während der Aufschubzeit zu stornieren[231]. Zusätzlich erhält der Versicherungsnehmer weitere Anlageoptionen oder -rechte über Fondsswitches[232] oder -shifts innerhalb seines Fondsbaskets oder über eine Auswahl vordefinierter (aktiv gemanagter) Strategiedepotlösungen. Zusammenfassend läßt sich festhalten, dass zum Einfluss von Annahmen an das Kundenverhalten (bezüglich Storno, Verlängerung, oder vorzeitigen Konsum) oder vertraglichen Beschränkungen (für das Fonds- und Garantiemanagement) auf die Kalkulation und das Hedging weitere Studien durchgeführt werden sollten.

[231] Milevsky und Salisbury (2001) bestimmen eine geeignete fondsbasierte Garantiegebühr, indem sie von rationalem Kundenverhalten und einer bedingten aufgeschobenen Stornogebühr für die sogenannte “Real Option to Lapse” ausgehen. Mudavanhu und Zhou (2002) bewerten die Stornooption innerhalb von Vertragsperioden mit schwacher Aktienperformance anhand einer deterministischen Stufenfunktion für die Stornogebühr. Im Gegensatz dazu berechnen Kolkiewicz und Tan (2006) die Option unter Annahme kapitalmarktabhängiger Ausführung etwa basierend auf der Aktienvolatilität.

[232] Mahayni und Schneider (2010), Mahayni und Schoenmakers (2010) oder Zieling (2010) behandeln das Thema der Switchoption eingehender.

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APPENDIX to Chapter 5

A.1 Parameter and symbol definition for the estimation results

Smoothed historic estimate values:

$\hat{a}(x)$ age-specific estimates determining the empirical average shape over time of the age profile in age group

$\hat{b}(x),\ b_i(x)$ $(i = 1, 2)$ age-specific estimates capturing the age-specific improvement rate sensitive to changes the general level of mortality

$\hat{k}(t),\ k_i(t)$ $(i = 1, 2)$ time-varying estimates representing the variation in the general level of mortality

$\hat{l}(t-x)$ estimates representing additional cohort effects in the age-time surface

Measures of central tendency and dispersion:

$\bar{a}(x),\ \bar{b}(x),\ \bar{b}_i(x)$ $(i = 1, 2)$, $\bar{k}(x),\ \bar{k}_i(x)$ $(i = 1, 2)$, $\bar{l}(t-x)$ denote the raw corresponding expected values

$SE_{\bar{a}},\ SE_{\bar{b}},\ SE_{\bar{b}_i}$ $(i = 1, 2)$, $SE_{\bar{k}},\ SE_{\bar{k}_i}$ $(i = 1, 2)$, $SE_{\bar{l}}$ denote the raw standard error of the mean

$q_{0.05},\ q_{0.95}$ represent the 0.05- as well as 0.95-quantiles

A.2 Estimation results for the age-period model by Lee and Carter

age x	$\hat{a}(x)$	$\hat{b}(x)$	age x	$\hat{a}(x)$	$\hat{b}(x)$	year t	$\hat{k}(t)$
0	-6.0138	0.0232	55	-4.5866	0.0075	1960	28.93
1	-6.4846	0.0230	56	-4.4896	0.0077	1961	28.78
2	-6.9227	0.0229	57	-4.3924	0.0079	1962	28.68
3	-7.3068	0.0229	58	-4.2952	0.0081	1963	28.66
4	-7.6269	0.0228	59	-4.1980	0.0083	1964	28.70
5	-7.8809	0.0227	60	-4.1008	0.0084	1965	28.86
6	-8.0719	0.0225	61	-4.0039	0.0086	1966	29.13
7	-8.2052	0.0221	62	-3.9072	0.0088	1967	29.43
8	-8.2858	0.0216	63	-3.8109	0.0089	1968	29.69
9	-8.3178	0.0209	64	-3.7150	0.0090	1969	29.75
10	-8.3026	0.0201	65	-3.6195	0.0090	1970	29.52
11	-8.2417	0.0192	66	-3.5245	0.0091	1971	28.99
12	-8.1377	0.0183	67	-3.4300	0.0091	1972	28.17
13	-7.9948	0.0175	68	-3.3358	0.0091	1973	27.08
14	-7.8199	0.0167	69	-3.2419	0.0090	1974	25.72
15	-7.6238	0.0159	70	-3.1482	0.0089	1975	24.12
16	-7.4210	0.0151	71	-3.0547	0.0088	1976	22.27
17	-7.2285	0.0143	72	-2.9614	0.0087	1977	20.24
18	-7.0599	0.0136	73	-2.8682	0.0085	1978	18.10
19	-6.9254	0.0130	74	-2.7751	0.0084	1979	15.88
20	-6.8279	0.0124	75	-2.6818	0.0082	1980	13.60
21	-6.7647	0.0119	76	-2.5885	0.0080	1981	11.28
22	-6.7300	0.0115	77	-2.4951	0.0078	1982	8.89
23	-6.7165	0.0111	78	-2.4014	0.0075	1983	6.45
24	-6.7164	0.0108	79	-2.3074	0.0073	1984	3.96
25	-6.7225	0.0106	80	-2.2133	0.0070	1985	1.47
26	-6.7290	0.0104	81	-2.1192	0.0067	1986	-0.99
27	-6.7314	0.0103	82	-2.0254	0.0064	1987	-3.39
28	-6.7269	0.0102	83	-1.9321	0.0061	1988	-5.70
29	-6.7137	0.0101	84	-1.8394	0.0058	1989	-7.89
30	-6.6910	0.0100	85	-1.7475	0.0055	1990	-10.03
31	-6.6586	0.0099	86	-1.6567	0.0052	1991	-12.18
32	-6.6168	0.0098	87	-1.5670	0.0049	1992	-14.37
33	-6.5662	0.0097	88	-1.4789	0.0045	1993	-16.63
34	-6.5075	0.0096	89	-1.3924	0.0042	1994	-19.02
35	-6.4413	0.0095	90	-1.3080	0.0039	1995	-21.59
36	-6.3685	0.0093	91	-1.2260	0.0036	1996	-24.37
37	-6.2898	0.0091	92	-1.1471	0.0034	1997	-27.38
38	-6.2064	0.0088	93	-1.0714	0.0031	1998	-30.59
39	-6.1193	0.0086	94	-0.9993	0.0028	1999	-33.97
40	-6.0293	0.0083	95	-0.9307	0.0025	2000	-37.49
41	-5.9370	0.0080	96	-0.8654	0.0022	2001	-41.11
42	-5.8431	0.0078	97	-0.8034	0.0020	2002	-44.82
43	-5.7479	0.0075	98	-0.7444	0.0017	2003	-48.63
44	-5.6518	0.0073	99	-0.6886	0.0015	2004	-52.56
45	-5.5550	0.0071	100	-0.6358	0.0013	2005	-56.53
46	-5.4580	0.0070	101	-0.5860	0.0011	2006	-60.54
47	-5.3609	0.0069	102	-0.5391	0.0010		
48	-5.2639	0.0068	103	-0.4949	0.0008		
49	-5.1670	0.0068	104	-0.4531	0.0006		
50	-5.0702	0.0068	105	-0.4135	0.0005		
51	-4.9735	0.0069	106	-0.3755	0.0004		
52	-4.8770	0.0070	107	-0.3389	0.0003		
53	-4.7803	0.0071	108	-0.3029	0.0002		
54	-4.6836	0.0073	109	-0.2673	0.0000		

Table A.1: Parameter estimates for the LC92 model based on German male period life tables 1956-2006 from the Human Mortality Database (2009).

A.3 Estimation results for the age-period model by Brouhns, Denuit and Vermunt

age x	$\hat{a}(x)$	$\overline{a}(x)$	$SE_{\overline{a}}$	$q^a_{0.05}$	$q^a_{0.95}$	age x	$\hat{b}(x)$	$\overline{b}(x)$	$SE_{\overline{b}}$	$q^b_{0.05}$	$q^b_{0.95}$
0	-6.02	-4.43	0.0006	-4.4858	-4.3822	0	0.0255	0.0245	0.0001	0.0199	0.0284
1	-6.49	-7.04	0.0023	-7.2364	-6.8651	1	0.0251	0.0198	0.	0.0165	0.0228
2	-6.93	-7.5	0.0029	-7.7471	-7.2806	2	0.0248	0.0198	0.	0.0165	0.0227
3	-7.31	-7.71	0.0032	-7.9805	-7.4631	3	0.0245	0.0197	0.	0.0165	0.0226
4	-7.63	-7.91	0.0036	-8.2247	-7.6422	4	0.0243	0.0213	0.	0.0179	0.0243
5	-7.88	-8.	0.0038	-8.3317	-7.7201	5	0.024	0.022	0.	0.0184	0.025
6	-8.07	-8.09	0.004	-8.4421	-7.7965	6	0.0237	0.0219	0.	0.0184	0.0249
7	-8.21	-8.17	0.0041	-8.5312	-7.8677	7	0.0232	0.0216	0.	0.0182	0.0245
8	-8.29	-8.19	0.0041	-8.5478	-7.8839	8	0.0226	0.0195	0.	0.0163	0.022
9	-8.32	-8.33	0.0044	-8.7213	-7.9996	9	0.0218	0.0199	0.	0.0168	0.0226
10	-8.3	-8.4	0.0046	-8.8023	-8.0605	10	0.021	0.0204	0.	0.0172	0.0232
11	-8.24	-8.34	0.0044	-8.7194	-8.0118	11	0.02	0.016	0.	0.0134	0.0182
12	-8.14	-8.29	0.0043	-8.6615	-7.9753	12	0.019	0.0155	0.	0.013	0.0176
13	-7.99	-8.2	0.0041	-8.5564	-7.8985	13	0.018	0.0152	0.	0.0127	0.0172
14	-7.82	-8.03	0.0037	-8.3476	-7.7496	14	0.0171	0.0151	0.	0.0126	0.0172
15	-7.62	-7.75	0.0032	-8.0179	-7.5068	15	0.0162	0.0136	0.	0.0114	0.0156
16	-7.42	-7.26	0.0025	-7.4749	-7.0711	16	0.0154	0.0145	0.	0.0121	0.0167
17	-7.22	-7.06	0.0022	-7.2466	-6.8836	17	0.0145	0.0137	0.	0.0114	0.0158
18	-7.05	-6.7	0.0019	-6.8541	-6.5539	18	0.0138	0.0105	0.	0.0087	0.0121
19	-6.92	-6.66	0.0018	-6.8105	-6.5155	19	0.013	0.0109	0.	0.009	0.0125
20	-6.82	-6.67	0.0018	-6.8259	-6.5294	20	0.0124	0.0106	0.	0.0087	0.0122
21	-6.76	-6.69	0.0018	-6.8402	-6.5407	21	0.0119	0.0105	0.	0.0087	0.0121
22	-6.73	-6.72	0.0019	-6.8753	-6.5716	22	0.0115	0.0101	0.	0.0083	0.0116
23	-6.71	-6.75	0.0019	-6.91	-6.6014	23	0.0111	0.0095	0.	0.0078	0.0109
24	-6.71	-6.78	0.0019	-6.942	-6.6287	24	0.0108	0.0092	0.	0.0076	0.0106
25	-6.72	-6.8	0.002	-6.9604	-6.6446	25	0.0106	0.009	0.	0.0075	0.0104
26	-6.73	-6.8	0.002	-6.9696	-6.6514	26	0.0104	0.009	0.	0.0074	0.0103
27	-6.73	-6.79	0.0019	-6.9578	-6.6415	27	0.0102	0.009	0.	0.0074	0.0103
28	-6.72	-6.77	0.0019	-6.9374	-6.6242	28	0.0101	0.0088	0.	0.0073	0.0101
29	-6.71	-6.75	0.0019	-6.9079	-6.5976	29	0.01	0.009	0.	0.0074	0.0103
30	-6.69	-6.72	0.0019	-6.8762	-6.5713	30	0.01	0.0088	0.	0.0073	0.0101
31	-6.66	-6.67	0.0018	-6.8237	-6.527	31	0.0099	0.0083	0.	0.0069	0.0096
32	-6.61	-6.63	0.0018	-6.7772	-6.4856	32	0.0098	0.0086	0.	0.0071	0.0099
33	-6.56	-6.57	0.0017	-6.7153	-6.4326	33	0.0097	0.0086	0.	0.0071	0.0099
34	-6.51	-6.51	0.0017	-6.6509	-6.3766	34	0.0095	0.0083	0.	0.0068	0.0096
35	-6.44	-6.45	0.0016	-6.5906	-6.324	35	0.0094	0.0081	0.	0.0067	0.0093
36	-6.37	-6.38	0.0016	-6.5101	-6.2535	36	0.0092	0.0079	0.	0.0065	0.0091
37	-6.29	-6.28	0.0015	-6.4079	-6.162	37	0.009	0.0081	0.	0.0066	0.0093
38	-6.2	-6.21	0.0015	-6.328	-6.0913	38	0.0088	0.0077	0.	0.0064	0.0089
39	-6.12	-6.11	0.0014	-6.2256	-5.9998	39	0.0085	0.0075	0.	0.0062	0.0087
40	-6.03	-6.02	0.0013	-6.1285	-5.9134	40	0.0082	0.0072	0.	0.0059	0.0083
41	-5.94	-5.94	0.0013	-6.0438	-5.8366	41	0.008	0.0068	0.	0.0056	0.0079
42	-5.84	-5.84	0.0012	-5.9395	-5.742	42	0.0077	0.0066	0.	0.0054	0.0076
43	-5.75	-5.74	0.0012	-5.8388	-5.6503	43	0.0075	0.0065	0.	0.0053	0.0074
44	-5.65	-5.66	0.0011	-5.7474	-5.567	44	0.0073	0.0063	0.	0.0051	0.0072
45	-5.55	-5.55	0.0011	-5.6366	-5.4653	45	0.0071	0.0061	0.	0.005	0.007
46	-5.46	-5.45	0.001	-5.5372	-5.3736	46	0.0069	0.006	0.	0.0049	0.0069
47	-5.36	-5.35	0.001	-5.4335	-5.2776	47	0.0068	0.0058	0.	0.0047	0.0067
48	-5.26	-5.26	0.0009	-5.3392	-5.1896	48	0.0068	0.0059	0.	0.0048	0.0068
49	-5.17	-5.17	0.0009	-5.2416	-5.0985	49	0.0068	0.0058	0.	0.0047	0.0067
50	-5.07	-5.06	0.0008	-5.1336	-4.9974	50	0.0068	0.0058	0.	0.0047	0.0066
51	-4.97	-4.97	0.0008	-5.0376	-4.9072	51	0.0069	0.0062	0.	0.005	0.0071
52	-4.88	-4.88	0.0008	-4.9406	-4.8156	52	0.007	0.0059	0.	0.0048	0.0069
53	-4.78	-4.79	0.0007	-4.8468	-4.7269	53	0.0071	0.0062	0.	0.005	0.0071
54	-4.68	-4.68	0.0007	-4.7415	-4.6271	54	0.0073	0.0062	0.	0.005	0.0072
55	-4.59	-4.59	0.0007	-4.6434	-4.5338	55	0.0075	0.0065	0.	0.0053	0.0075
56	-4.49	-4.49	0.0006	-4.5442	-4.4392	56	0.0077	0.0067	0.	0.0054	0.0078
57	-4.39	-4.39	0.0006	-4.4413	-4.3409	57	0.0079	0.0069	0.	0.0056	0.008
58	-4.29	-4.3	0.0006	-4.3475	-4.2508	58	0.0081	0.0071	0.	0.0057	0.0082
59	-4.2	-4.2	0.0006	-4.2455	-4.153	59	0.0083	0.0071	0.	0.0057	0.0083
60	-4.1	-4.1	0.0005	-4.1448	-4.0559	60	0.0085	0.0075	0.	0.006	0.0086
61	-4.	-4.	0.0005	-4.0468	-3.9614	61	0.0087	0.0075	0.	0.0061	0.0088
62	-3.91	-3.9	0.0005	-3.9456	-3.8635	62	0.0088	0.0077	0.	0.0062	0.0089
63	-3.81	-3.81	0.0005	-3.8486	-3.7695	63	0.0089	0.0078	0.	0.0063	0.0091
64	-3.71	-3.72	0.0005	-3.7559	-3.6794	64	0.009	0.0079	0.	0.0063	0.0092
65	-3.62	-3.62	0.0005	-3.6533	-3.5797	65	0.0091	0.0079	0.	0.0064	0.0092
66	-3.52	-3.52	0.0004	-3.5578	-3.4866	66	0.0091	0.0081	0.	0.0065	0.0094
67	-3.43	-3.43	0.0004	-3.4652	-3.3961	67	0.0091	0.0079	0.	0.0063	0.0092
68	-3.34	-3.33	0.0004	-3.3686	-3.3015	68	0.0091	0.008	0.	0.0064	0.0093
69	-3.24	-3.24	0.0004	-3.2777	-3.2124	69	0.0091	0.0079	0.	0.0063	0.0092
70	-3.15	-3.15	0.0004	-3.1797	-3.1162	70	0.009	0.0078	0.	0.0062	0.0091
71	-3.05	-3.05	0.0004	-3.0832	-3.0212	71	0.0089	0.0076	0.	0.006	0.0089
72	-2.96	-2.96	0.0004	-2.9917	-2.9309	72	0.0087	0.0075	0.	0.006	0.0088
73	-2.87	-2.87	0.0004	-2.8989	-2.8393	73	0.0086	0.0075	0.	0.0059	0.0087
74	-2.77	-2.77	0.0004	-2.8045	-2.7459	74	0.0084	0.0073	0.	0.0058	0.0085
75	-2.68	-2.68	0.0004	-2.7083	-2.6506	75	0.0083	0.0071	0.	0.0056	0.0083
76	-2.59	-2.59	0.0004	-2.6182	-2.5609	76	0.0081	0.0071	0.	0.0056	0.0083
77	-2.49	-2.5	0.0004	-2.5255	-2.4685	77	0.0079	0.0068	0.	0.0054	0.008

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78	-2.4	-2.4	0.0004	-2.4337	-2.3768	78	0.0076	0.0067	0.	0.0053	0.0079
79	-2.31	-2.31	0.0004	-2.3408	-2.2838	79	0.0074	0.0065	0.	0.0052	0.0077
80	-2.21	-2.22	0.0004	-2.2442	-2.187	80	0.0071	0.0062	0.	0.0049	0.0072
81	-2.12	-2.12	0.0004	-2.1452	-2.0874	81	0.0069	0.006	0.	0.0047	0.007
82	-2.03	-2.02	0.0004	-2.0529	-1.9941	82	0.0066	0.0057	0.	0.0045	0.0067
83	-1.93	-1.93	0.0004	-1.9626	-1.9022	83	0.0063	0.0054	0.	0.0042	0.0063
84	-1.84	-1.84	0.0004	-1.8698	-1.8075	84	0.006	0.0052	0.	0.004	0.0061
85	-1.75	-1.75	0.0004	-1.7813	-1.7165	85	0.0056	0.005	0.	0.0038	0.0059
86	-1.66	-1.66	0.0004	-1.6906	-1.6226	86	0.0053	0.0046	0.	0.0036	0.0055
87	-1.57	-1.57	0.0004	-1.6041	-1.532	87	0.005	0.0046	0.	0.0035	0.0055
88	-1.48	-1.48	0.0005	-1.5193	-1.442	88	0.0046	0.004	0.	0.003	0.0049
89	-1.39	-1.4	0.0005	-1.4445	-1.3604	89	0.0043	0.0036	0.	0.0026	0.0043
90	-1.31	-1.31	0.0006	-1.3546	-1.263	90	0.0039	0.0033	0.	0.0024	0.0041
91	-1.23	-1.22	0.0006	-1.2728	-1.1714	91	0.0036	0.0029	0.	0.002	0.0037
92	-1.14	-1.14	0.0007	-1.2029	-1.0885	92	0.0033	0.0029	0.	0.0019	0.0037
93	-1.06	-1.06	0.0008	-1.1269	-0.9973	93	0.003	0.0028	0.	0.0018	0.0037
94	-0.98	-0.99	0.0009	-1.0721	-0.921	94	0.0027	0.0025	0.	0.0015	0.0035
95	-0.9	-0.94	0.0011	-1.0306	-0.8521	95	0.0024	0.002	0.	0.0009	0.003
96	-0.83	-0.87	0.0013	-0.9838	-0.7711	96	0.0021	0.0017	0.	0.0006	0.0029
97	-0.75	-0.81	0.0016	-0.9475	-0.6889	97	0.0017	0.0014	0.	0.0001	0.0028
98	-0.68	-0.76	0.002	-0.9209	-0.6043	98	0.0014	0.0012	0.	-0.0003	0.0028
99	-0.62	-0.7	0.0024	-0.91	-0.5148	99	0.0011	0.001	0.	-0.0009	0.0029
100	-0.57	-0.66	0.0031	-0.9233	-0.4198	100	0.0007	0.0007	0.	-0.0016	0.0031
101	-0.54	-0.61	0.004	-0.9576	-0.3131	101	0.0003	0.0005	0.	-0.0024	0.0034
102	-0.53	-0.58	0.0054	-1.048	-0.1957	102	-0.0001	0.0001	0.	-0.0038	0.0039
103	-0.55	-0.57	0.0074	-1.2411	-0.0551	103	-0.0006	-0.0003	0.0001	-0.0063	0.0047
104	-0.62	-0.59	0.0107	-1.599	0.0903	104	-0.0011	-0.0011	0.0001	-0.0104	0.005
105	-0.74	-0.69	0.0168	-2.1608	0.247	105	-0.0017	-0.0029	0.0002	-0.0154	0.006
106	-0.91	-0.98	0.03	-4.338	0.4439	106	-0.0022	-0.0067	0.0003	-0.0387	0.007
107	-1.14	-2.24	0.1126	-7.5754	0.6187	107	-0.0026	-0.0141	0.0005	-0.0679	0.0072
108	-1.41	-6.48	0.2436	-41.3164	0.7236	108	-0.0028	-0.0186	0.0008	-0.09	0.0454
109	-1.7	-14.22	0.3386	-41.2898	0.8905	109	-0.0029	0.1612	0.0036	-0.0065	0.3755

year t	$\hat{k}(t)$	$\overline{k}(t)$	$SE_{\overline{k}}$	$q^k_{0.05}$	$q^k_{0.95}$	**year t**	$\hat{k}(t)$	$\overline{k}(t)$	$SE_{\overline{k}}$	$q^k_{0.05}$	$q^k_{0.95}$
1960	30.89	39.45	0.1341	32.831	48.4001	1983	5.36	7.66	0.0269	6.3221	9.4768
1961	30.41	34.33	0.1184	28.4931	42.2779	1984	2.93	2.33	0.0094	1.837	3.0035
1962	29.97	35.05	0.1196	29.1415	43.0455	1985	0.51	1.04	0.0046	0.7976	1.3862
1963	29.61	37.21	0.1261	30.9871	45.6265	1986	-1.87	-2.02	0.0058	-2.4018	-1.7224
1964	29.33	30.5	0.1054	25.311	37.5829	1987	-4.2	-6.49	0.0207	-7.8338	-5.4605
1965	29.18	33.05	0.1126	27.4869	40.5801	1988	-6.43	-9.83	0.032	-11.9317	-8.2527
1966	29.16	32.94	0.1125	27.3848	40.4693	1989	-8.57	-11.17	0.0372	-13.6247	-9.3314
1967	29.2	31.24	0.1069	25.9679	38.3978	1990	-10.67	-11.65	0.0391	-14.2439	-9.7138
1968	29.23	36.65	0.1237	30.5425	44.8929	1991	-12.76	-14.41	0.0485	-17.6371	-12.0132
1969	29.13	38.12	0.1288	31.7598	46.6966	1992	-14.89	-19.27	0.0647	-23.5749	-16.0781
1970	28.8	35.52	0.1206	29.577	43.5798	1993	-17.08	-19.01	0.0644	-23.2922	-15.8374
1971	28.2	33.86	0.1152	28.1816	41.5602	1994	-19.39	-23.24	0.0788	-28.5048	-19.3575
1972	27.33	32.8	0.1116	27.2918	40.2478	1995	-21.86	-24.17	0.082	-29.6503	-20.1292
1973	26.19	31.55	0.1073	26.2634	38.7161	1996	-24.52	-26.91	0.0916	-33.0308	-22.3871
1974	24.78	28.6	0.0976	23.7925	35.1279	1997	-27.38	-32.28	0.1101	-39.6542	-26.8613
1975	23.14	30.49	0.1033	25.3981	37.3824	1998	-30.41	-36.17	0.1237	-44.4617	-30.0666
1976	21.25	26.03	0.0887	21.6606	31.9594	1999	-33.59	-39.6	0.1359	-48.7228	-32.8929
1977	19.19	19.71	0.0681	16.3591	24.2986	2000	-36.9	-44.16	0.1514	-54.3017	-36.6792
1978	17.03	20.64	0.0708	17.1538	25.3873	2001	-40.3	-49.75	0.1709	-61.2005	-41.3176
1979	14.79	16.04	0.0557	13.2997	19.7903	2002	-43.77	-51.58	0.1775	-63.4862	-42.8213
1980	12.49	14.37	0.0501	11.8958	17.7556	2003	-47.33	-52.43	0.1809	-64.5891	-43.5009
1981	10.16	12.74	0.0441	10.5651	15.7061	2004	-50.99	-62.03	0.2134	-76.3555	-51.4974
1982	7.78	9.43	0.0333	7.7826	11.6879	2005	-54.7	-64.49	0.2225	-79.4428	-53.5061
						2006	-58.43	-70.71	0.244	-87.099	-58.6734

Table A.2: Parameter estimates, simulation mean, standard error (SE) and quantiles ($q_{0.05}$, $q_{0.95}$) for the BDV02 model based on German male period life tables 1956-2006 from the Human Mortality Database (2009).

A.4 Estimation results for the age-period model with age-specific enhancement by Renshaw and Haberman

age x	$\hat{a}(x)$	$\overline{a}(x)$	$SE_{\overline{a}}$	$q^a_{0.05}$	$q^a_{0.95}$	age x	$\hat{a}(x)$	$\overline{a}(x)$	$SE_{\overline{a}}$	$q^a_{0.05}$	$q^a_{0.95}$
0	-6.01	-4.42	0.0011	-4.4717	-4.3679	55	-4.5858	-4.59	0.0012	-4.6452	-4.5328
1	-6.48	-7.04	0.0039	-7.2357	-6.8577	56	-4.4887	-4.49	0.0011	-4.546	-4.4382
2	-6.92	-7.49	0.005	-7.7466	-7.2729	57	-4.3915	-4.39	0.0011	-4.443	-4.3399
3	-7.3	-7.7	0.0055	-7.9832	-7.4556	58	-4.2942	-4.3	0.001	-4.3489	-4.2498
4	-7.62	-7.91	0.0062	-8.2228	-7.6325	59	-4.197	-4.2	0.001	-4.2469	-4.152
5	-7.88	-8.	0.0065	-8.3366	-7.7119	60	-4.0999	-4.1	0.001	-4.1461	-4.055
6	-8.07	-8.09	0.0069	-8.4396	-7.7879	61	-4.0029	-4.	0.0009	-4.048	-3.9604
7	-8.2	-8.17	0.0072	-8.5443	-7.8611	62	-3.9063	-3.9	0.0009	-3.9468	-3.8626
8	-8.28	-8.19	0.0071	-8.5599	-7.8764	63	-3.81	-3.81	0.0008	-3.8498	-3.7686
9	-8.31	-8.32	0.0077	-8.728	-7.9926	64	-3.7142	-3.72	0.0008	-3.7569	-3.6785
10	-8.3	-8.4	0.008	-8.8132	-8.0531	65	-3.6188	-3.62	0.0008	-3.6545	-3.5789
11	-8.24	-8.33	0.0076	-8.7289	-8.0061	66	-3.5239	-3.52	0.0008	-3.5589	-3.4857
12	-8.13	-8.29	0.0074	-8.6729	-7.9701	67	-3.4294	-3.43	0.0007	-3.4663	-3.3954
13	-7.99	-8.2	0.0071	-8.5626	-7.8948	68	-3.3353	-3.33	0.0007	-3.3698	-3.3009
14	-7.82	-8.03	0.0064	-8.3584	-7.7442	69	-3.2415	-3.24	0.0007	-3.279	-3.212
15	-7.62	-7.75	0.0055	-8.0309	-7.5006	70	-3.1478	-3.15	0.0007	-3.1811	-3.1159
16	-7.42	-7.27	0.0044	-7.4898	-7.0704	71	-3.0544	-3.05	0.0007	-3.0847	-3.0211
17	-7.23	-7.06	0.0039	-7.2594	-6.8835	72	-2.9611	-2.96	0.0007	-2.9934	-2.9311
18	-7.06	-6.7	0.0032	-6.8653	-6.5556	73	-2.8679	-2.87	0.0006	-2.9007	-2.8395
19	-6.92	-6.66	0.0032	-6.8189	-6.5154	74	-2.7746	-2.78	0.0006	-2.8061	-2.7459
20	-6.82	-6.67	0.0032	-6.8329	-6.5276	75	-2.6814	-2.68	0.0006	-2.7102	-2.6509
21	-6.76	-6.68	0.0032	-6.846	-6.5382	76	-2.588	-2.59	0.0006	-2.6197	-2.5609
22	-6.73	-6.72	0.0033	-6.8815	-6.5696	77	-2.4945	-2.5	0.0006	-2.527	-2.4686
23	-6.71	-6.75	0.0033	-6.9155	-6.5989	78	-2.4008	-2.41	0.0006	-2.4351	-2.3768
24	-6.71	-6.78	0.0034	-6.9468	-6.6253	79	-2.3069	-2.31	0.0006	-2.3422	-2.2838
25	-6.72	-6.8	0.0034	-6.9667	-6.6422	80	-2.213	-2.22	0.0006	-2.2457	-2.187
26	-6.73	-6.8	0.0034	-6.9757	-6.6492	81	-2.1192	-2.12	0.0006	-2.1467	-2.0874
27	-6.73	-6.79	0.0034	-6.9627	-6.6386	82	-2.0258	-2.02	0.0006	-2.0543	-1.994
28	-6.72	-6.77	0.0034	-6.9431	-6.622	83	-1.9331	-1.93	0.0006	-1.9639	-1.902
29	-6.71	-6.75	0.0033	-6.9128	-6.5958	84	-1.8411	-1.84	0.0007	-1.871	-1.8072
30	-6.69	-6.72	0.0033	-6.882	-6.5695	85	-1.7501	-1.75	0.0007	-1.7826	-1.7161
31	-6.66	-6.67	0.0032	-6.8296	-6.5238	86	-1.66	-1.66	0.0007	-1.6919	-1.6222
32	-6.61	-6.63	0.0031	-6.7812	-6.4828	87	-1.571	-1.57	0.0008	-1.6057	-1.5316
33	-6.56	-6.57	0.003	-6.7197	-6.4296	88	-1.483	-1.48	0.0008	-1.5206	-1.4413
34	-6.51	-6.51	0.0029	-6.6565	-6.3745	89	-1.3959	-1.4	0.0009	-1.4458	-1.3595
35	-6.44	-6.45	0.0029	-6.5958	-6.321	90	-1.3095	-1.31	0.001	-1.3562	-1.2621
36	-6.37	-6.38	0.0028	-6.5155	-6.2503	91	-1.2237	-1.22	0.0011	-1.2736	-1.1694
37	-6.29	-6.28	0.0026	-6.4126	-6.1608	92	-1.1386	-1.14	0.0012	-1.204	-1.0865
38	-6.2	-6.21	0.0025	-6.3335	-6.0903	93	-1.0538	-1.06	0.0014	-1.1287	-0.9952
39	-6.12	-6.11	0.0024	-6.2302	-5.9991	94	-0.9693	-0.99	0.0016	-1.0752	-0.9194
40	-6.03	-6.02	0.0023	-6.1328	-5.9117	95	-0.8851	-0.94	0.0019	-1.0327	-0.8477
41	-5.94	-5.94	0.0022	-6.0492	-5.8359	96	-0.8017	-0.87	0.0023	-0.9869	-0.7664
42	-5.84	-5.84	0.0021	-5.9442	-5.7412	97	-0.7206	-0.81	0.0028	-0.95	-0.684
43	-5.75	-5.74	0.002	-5.8437	-5.6502	98	-0.6449	-0.75	0.0034	-0.9266	-0.5978
44	-5.65	-5.66	0.0019	-5.7518	-5.5663	99	-0.5792	-0.7	0.0043	-0.9202	-0.5067
45	-5.55	-5.55	0.0018	-5.6411	-5.4649	100	-0.5302	-0.65	0.0055	-0.9403	-0.4086
46	-5.46	-5.45	0.0018	-5.5414	-5.3732	101	-0.5072	-0.61	0.0073	-0.9908	-0.3002
47	-5.36	-5.35	0.0017	-5.4372	-5.2769	102	-0.5213	-0.59	0.0099	-1.0998	-0.1749
48	-5.26	-5.26	0.0016	-5.3432	-5.1895	103	-0.5855	-0.58	0.0143	-1.3522	-0.0354
49	-5.17	-5.17	0.0015	-5.2448	-5.0979	104	-0.7133	-0.63	0.0217	-1.9187	0.1304
50	-5.07	-5.06	0.0015	-5.1367	-4.9968	105	-0.9162	-0.78	0.0344	-2.6324	0.259
51	-4.97	-4.97	0.0014	-5.0404	-4.9065	106	-1.201	-1.24	0.0647	-4.6769	0.4628
52	-4.88	-4.88	0.0013	-4.9432	-4.815	107	-1.5647	-4.23	0.5018	-9.9448	0.6218
53	-4.78	-4.79	0.0013	-4.849	-4.7262	108	-1.9895	-14.07	1.0494	-100.266	0.6868
54	-4.68	-4.68	0.0012	-4.7438	-4.6262	109	-2.4564	-43.92	1.6913	-100.245	0.7034

age x	$\hat{b}_1(x)$	$\overline{b}_1(x)$	$SE_{\overline{b}_1}$	$q^{b_1}_{0.05}$	$q^{b_1}_{0.95}$	age x	$\hat{b}_2(x)$	$\overline{b}_2(x)$	$SE_{\overline{b}_2}$	$q^{b_2}_{0.05}$	$q^{b_2}_{0.95}$
0	0.03	0.041	0.0003	0.0346	0.0563	0	-0.0033	-0.01	0.0005	-0.0368	0.0058
1	0.03	0.0362	0.0003	0.0291	0.0506	1	-0.001	-0.01	0.0005	-0.0458	-0.0014
2	0.03	0.0339	0.0003	0.0261	0.0486	2	0.0014	0.	0.0004	-0.029	0.0051
3	0.03	0.0324	0.0003	0.0244	0.0472	3	0.0038	0.	0.0003	-0.0172	0.0092
4	0.03	0.0332	0.0003	0.0243	0.0489	4	0.0059	0.01	0.0002	-0.005	0.015
5	0.03	0.0331	0.0003	0.024	0.0491	5	0.0076	0.01	0.0002	0.0027	0.0186
6	0.03	0.0334	0.0003	0.0241	0.0494	6	0.0088	0.01	0.0002	0.0012	0.0174
7	0.03	0.0313	0.0003	0.0224	0.0469	7	0.0096	0.02	0.0001	0.0138	0.0236
8	0.03	0.0306	0.0003	0.0222	0.0452	8	0.01	0.01	0.0002	-0.005	0.0133
9	0.03	0.0313	0.0003	0.0228	0.0467	9	0.0102	0.01	0.0002	-0.0042	0.0137
10	0.03	0.0311	0.0003	0.0223	0.0468	10	0.0103	0.01	0.0002	0.0034	0.017
11	0.02	0.0263	0.0002	0.0191	0.039	11	0.0107	0.	0.0002	-0.0102	0.0086
12	0.02	0.0242	0.0002	0.0174	0.0362	12	0.0113	0.01	0.0001	-0.0009	0.0118
13	0.02	0.0248	0.0002	0.018	0.0367	13	0.0123	0.	0.0002	-0.0081	0.0092
14	0.02	0.0232	0.0002	0.0167	0.0345	14	0.0135	0.01	0.0001	0.0016	0.0128
15	0.02	0.0189	0.0002	0.0135	0.0284	15	0.0149	0.01	0.0001	0.0118	0.0177
16	0.02	0.0157	0.0002	0.0106	0.0245	16	0.016	0.03	0.0002	0.0257	0.045
17	0.01	0.0147	0.0002	0.01	0.0229	17	0.0165	0.03	0.0002	0.0242	0.0421

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18	0.01	0.0093	0.0001	0.0059	0.015	18	0.0164	0.03	0.0002	0.0228	0.0437
19	0.01	0.0114	0.0001	0.0078	0.0176	19	0.0155	0.02	0.0002	0.0199	0.0337
20	0.01	0.0126	0.0001	0.009	0.0189	20	0.0141	0.02	0.0001	0.0156	0.0233
21	0.01	0.0145	0.0001	0.0109	0.0212	21	0.0123	0.01	0.0001	0.0088	0.0133
22	0.01	0.0144	0.0001	0.0109	0.021	22	0.0105	0.01	0.0001	0.0059	0.0114
23	0.01	0.0142	0.0001	0.0108	0.0205	23	0.0087	0.01	0.0001	0.0021	0.0092
24	0.01	0.0143	0.0001	0.0111	0.0205	24	0.0072	0.	0.0001	-0.0011	0.0077
25	0.01	0.0146	0.0001	0.0114	0.0209	25	0.0061	0.	0.0001	-0.0045	0.0062
26	0.01	0.0146	0.0001	0.0114	0.0208	26	0.0054	0.	0.0001	-0.005	0.0059
27	0.01	0.0145	0.0001	0.0113	0.0207	27	0.0051	0.	0.0001	-0.0045	0.0062
28	0.01	0.0134	0.0001	0.0103	0.0194	28	0.0051	0.01	0.0001	0.0004	0.0079
29	0.01	0.0133	0.0001	0.0101	0.0192	29	0.0054	0.01	0.0001	0.003	0.0092
30	0.01	0.0128	0.0001	0.0097	0.0186	30	0.0059	0.01	0.0001	0.0041	0.0096
31	0.01	0.0114	0.0001	0.0085	0.0167	31	0.0064	0.01	0.	0.0073	0.011
32	0.01	0.0123	0.0001	0.0093	0.0179	32	0.0071	0.01	0.0001	0.005	0.0099
33	0.01	0.0119	0.0001	0.009	0.0174	33	0.0078	0.01	0.	0.0065	0.0105
34	0.01	0.011	0.0001	0.0082	0.0162	34	0.0085	0.01	0.	0.0081	0.0116
35	0.01	0.0111	0.0001	0.0083	0.0161	35	0.0092	0.01	0.	0.0069	0.0104
36	0.01	0.0101	0.0001	0.0075	0.0149	36	0.0099	0.01	0.	0.0092	0.013
37	0.01	0.0093	0.0001	0.0067	0.014	37	0.0106	0.01	0.0001	0.0124	0.0186
38	0.01	0.0085	0.0001	0.006	0.0129	38	0.0112	0.01	0.0001	0.0133	0.0205
39	0.01	0.0088	0.0001	0.0064	0.0131	39	0.0117	0.01	0.0001	0.0114	0.017
40	0.01	0.008	0.0001	0.0058	0.0121	40	0.0122	0.01	0.0001	0.0117	0.0177
41	0.01	0.0061	0.0001	0.004	0.0096	41	0.0125	0.02	0.0001	0.0149	0.0265
42	0.01	0.0068	0.0001	0.0048	0.0104	42	0.0127	0.01	0.0001	0.0122	0.0195
43	0.01	0.0058	0.0001	0.0039	0.0092	43	0.0127	0.02	0.0001	0.0139	0.0244
44	0.01	0.006	0.0001	0.0041	0.0094	44	0.0127	0.01	0.0001	0.0128	0.0213
45	0.01	0.0051	0.0001	0.0033	0.0082	45	0.0126	0.02	0.0001	0.0137	0.0245
46	0.01	0.0054	0.0001	0.0036	0.0085	46	0.0124	0.01	0.0001	0.0128	0.0217
47	0.01	0.0055	0.0001	0.0038	0.0086	47	0.0121	0.01	0.0001	0.012	0.0197
48	0.01	0.0052	0.0001	0.0035	0.0082	48	0.0117	0.02	0.0001	0.013	0.0224
49	0.01	0.0056	0.0001	0.0039	0.0087	49	0.0112	0.01	0.0001	0.0118	0.019
50	0.01	0.0054	0.0001	0.0037	0.0084	50	0.0106	0.01	0.0001	0.012	0.0197
51	0.01	0.0065	0.0001	0.0047	0.0099	51	0.01	0.01	0.0001	0.011	0.017
52	0.01	0.0065	0.0001	0.0047	0.0097	52	0.0093	0.01	0.0001	0.01	0.0152
53	0.01	0.0073	0.0001	0.0054	0.0108	53	0.0085	0.01	0.	0.0087	0.0125
54	0.01	0.0079	0.0001	0.0061	0.0115	54	0.0078	0.01	0.	0.0071	0.0098
55	0.01	0.0087	0.0001	0.0068	0.0126	55	0.0071	0.01	0.	0.0059	0.0088
56	0.01	0.0092	0.0001	0.0072	0.0132	56	0.0064	0.01	0.	0.0053	0.0086
57	0.01	0.0097	0.0001	0.0077	0.0139	57	0.0059	0.01	0.0001	0.0043	0.0083
58	0.01	0.0103	0.0001	0.0082	0.0146	58	0.0054	0.01	0.0001	0.0026	0.0075
59	0.01	0.011	0.0001	0.0089	0.0154	59	0.0051	0.	0.0001	-0.0008	0.0063
60	0.01	0.0112	0.0001	0.009	0.0158	60	0.005	0.	0.0001	0.001	0.0072
61	0.01	0.0113	0.0001	0.0092	0.016	61	0.005	0.	0.0001	0.0007	0.0072
62	0.01	0.0116	0.0001	0.0094	0.0163	62	0.0052	0.	0.0001	0.0003	0.0072
63	0.01	0.0113	0.0001	0.0092	0.016	63	0.0055	0.01	0.0001	0.0031	0.0084
64	0.01	0.0116	0.0001	0.0094	0.0164	64	0.006	0.01	0.0001	0.0026	0.0083
65	0.01	0.0114	0.0001	0.0092	0.0162	65	0.0066	0.01	0.0001	0.0039	0.009
66	0.01	0.0113	0.0001	0.0091	0.0161	66	0.0073	0.01	0.0001	0.0051	0.0096
67	0.01	0.0107	0.0001	0.0085	0.0153	67	0.0081	0.01	0.	0.0068	0.0104
68	0.01	0.0104	0.0001	0.0082	0.015	68	0.0089	0.01	0.	0.0082	0.0115
69	0.01	0.0099	0.0001	0.0077	0.0143	69	0.0097	0.01	0.	0.0095	0.0132
70	0.01	0.0092	0.0001	0.0071	0.0135	70	0.0105	0.01	0.	0.0108	0.0155
71	0.01	0.0086	0.0001	0.0065	0.0127	71	0.0112	0.01	0.0001	0.0117	0.0173
72	0.01	0.008	0.0001	0.0059	0.0119	72	0.0118	0.01	0.0001	0.0131	0.02
73	0.01	0.0077	0.0001	0.0056	0.0115	73	0.0123	0.02	0.0001	0.0137	0.0212
74	0.01	0.0076	0.0001	0.0056	0.0113	74	0.0127	0.01	0.0001	0.0131	0.0201
75	0.01	0.0067	0.0001	0.0047	0.0102	75	0.0128	0.02	0.0001	0.0148	0.0233
76	0.01	0.0071	0.0001	0.0052	0.0108	76	0.0129	0.01	0.0001	0.0133	0.0206
77	0.01	0.0066	0.0001	0.0048	0.01	77	0.0128	0.01	0.0001	0.0134	0.0209
78	0.01	0.0065	0.0001	0.0047	0.0099	78	0.0126	0.01	0.0001	0.0132	0.0205
79	0.01	0.0062	0.0001	0.0045	0.0095	79	0.0123	0.01	0.0001	0.0132	0.0207
80	0.01	0.0055	0.0001	0.0039	0.0084	80	0.0118	0.01	0.0001	0.0133	0.0213
81	0.01	0.0052	0.0001	0.0036	0.008	81	0.0113	0.01	0.0001	0.013	0.021
82	0.01	0.0053	0.0001	0.0039	0.0081	82	0.0106	0.01	0.0001	0.0115	0.018
83	0.01	0.0052	0.0001	0.0038	0.0078	83	0.0098	0.01	0.0001	0.0102	0.0157
84	0.01	0.0053	0.	0.004	0.0079	84	0.0089	0.01	0.	0.0089	0.0135
85	0.01	0.0052	0.	0.0039	0.0076	85	0.008	0.01	0.	0.0084	0.0128
86	0.	0.005	0.	0.0039	0.0073	86	0.0071	0.01	0.	0.0072	0.0107
87	0.	0.0047	0.	0.0036	0.0069	87	0.006	0.01	0.	0.0077	0.0119
88	0.	0.0052	0.	0.0042	0.0073	88	0.005	0.	0.	0.0039	0.0058
89	0.	0.0049	0.	0.004	0.0067	89	0.004	0.	0.	0.0025	0.0046
90	0.	0.0042	0.	0.0033	0.0057	90	0.0029	0.	0.	0.0037	0.0052
91	0.	0.0053	0.	0.0045	0.0069	91	0.0019	0.	0.0001	-0.0039	0.0015
92	0.	0.0049	0.	0.0041	0.0063	92	0.001	0.	0.	-0.002	0.0021
93	0.	0.0044	0.	0.0037	0.0056	93	0.0001	0.	0.	0.0001	0.0028
94	0.	0.0031	0.	0.0025	0.0038	94	-0.0007	0.	0.	0.0027	0.004
95	0.	0.0037	0.	0.0025	0.0045	95	-0.0015	0.	0.	-0.0031	0.0008
96	0.	0.0035	0.	0.0018	0.0047	96	-0.0022	0.	0.	-0.0037	0.0002
97	0.	0.0032	0.	0.0008	0.0049	97	-0.0026	0.	0.	-0.0041	-0.0003
98	0.	0.0028	0.0001	-0.0004	0.0053	98	-0.0026	0.	0.	-0.0042	-0.0006
99	0.	0.0024	0.0001	-0.0022	0.0057	99	-0.0021	0.	0.	-0.0035	-0.0003
100	0.	0.0019	0.0001	-0.0048	0.0063	100	-0.0009	0.	0.0001	-0.0031	0.0015
101	0.	0.001	0.0002	-0.0088	0.0074	101	0.0013	0.	0.0001	-0.003	0.0068
102	0.	-0.0004	0.0003	-0.0142	0.0087	102	0.0045	0.	0.0003	-0.0035	0.0119
103	-0.01	-0.0036	0.0005	-0.0322	0.01	103	0.0088	0.01	0.0007	-0.0043	0.0401

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104	-0.01	-0.01	0.0009	-0.0712	0.0115	104	0.0142	0.02	0.0013	-0.0043	0.0934
105	-0.01	-0.0222	0.0014	-0.0925	0.0146	105	0.0202	0.03	0.0023	-0.0076	0.166
106	-0.02	-0.0464	0.002	-0.1264	0.0143	106	0.0263	0.05	0.0031	-0.0236	0.2356
107	-0.02	-0.0644	0.0028	-0.2024	0.0144	107	0.0314	0.02	0.0035	-0.0932	0.1789
108	-0.03	-0.0619	0.0037	-0.3194	0.0252	108	0.0343	-0.06	0.0042	-0.3742	0.0284
109	-0.03	0.0066	0.0012	0.	0.0217	109	0.0365	0.	0.0009	-0.0259	0.

year t	$\hat{k}_1(t)$	$\overline{k}_1(t)$	$SE_{\overline{k}_1}$	$q^{k_1}_{0.05}$	$q^{k_1}_{0.95}$	**year t**	$\hat{k}_2(t)$	$\overline{k}_2(t)$	$SE_{\overline{k}_2}$	$q^{k_2}_{0.05}$	$q^{k_2}_{0.95}$
1960	29.6	28.1	0.1501	20.1593	32.1077	1960	-2.3	1.37	0.0623	-1.188	3.224
1961	28.5	27.12	0.1682	18.4475	31.6335	1961	-1.32	-1.83	0.0495	-3.9411	-0.4741
1962	27.42	25.35	0.1367	18.1245	29.0132	1962	-0.31	1.	0.0552	-1.2421	2.6322
1963	26.38	24.27	0.1066	18.3827	27.1461	1963	0.72	4.16	0.0693	1.315	6.227
1964	25.41	22.71	0.1255	16.0936	26.1166	1964	1.75	0.33	0.0472	-1.6322	1.723
1965	24.53	21.32	0.0899	16.3041	23.7646	1965	2.81	4.07	0.0632	1.4255	5.9423
1966	23.75	21.48	0.0928	16.3373	24.0163	1966	3.86	3.78	0.0617	1.255	5.6289
1967	23.04	20.26	0.086	15.4719	22.5997	1967	4.88	3.76	0.059	1.3003	5.5122
1968	22.36	20.15	0.0535	16.8619	21.6521	1968	5.82	8.46	0.0887	3.9943	11.0823
1969	21.66	21.01	0.0558	17.5795	22.5183	1969	6.62	8.78	0.0923	4.137	11.5163
1970	20.85	20.14	0.0592	16.5629	21.7506	1970	7.26	7.49	0.082	3.4561	9.9571
1971	19.88	19.41	0.0594	15.8537	21.0213	1971	7.76	6.86	0.0765	3.1304	9.1723
1972	18.73	17.69	0.044	14.9608	18.8706	1972	8.15	7.84	0.0801	3.7433	10.1759
1973	17.38	17.1	0.0436	14.4014	18.2736	1973	8.46	7.4	0.0765	3.5182	9.6455
1974	15.83	15.43	0.0388	13.0274	16.4838	1974	8.7	6.73	0.0697	3.193	8.7759
1975	14.1	13.96	0.0256	12.9499	15.0681	1975	8.87	9.79	0.0881	4.9762	12.2432
1976	12.21	11.51	0.0228	10.8152	12.53	1976	8.93	8.64	0.0764	4.425	10.7327
1977	10.24	8.29	0.0199	7.6496	9.2132	1977	8.89	6.8	0.0591	3.5043	8.4079
1978	8.25	7.41	0.0327	6.3193	8.7306	1978	8.76	8.35	0.0687	4.4166	10.1335
1979	6.27	5.52	0.0285	4.554	6.6864	1979	8.52	6.58	0.0542	3.4756	7.9971
1980	4.33	3.91	0.0382	2.657	5.3862	1980	8.18	6.8	0.0539	3.652	8.1737
1981	2.46	1.93	0.0518	0.3363	4.0057	1981	7.72	7.36	0.0554	4.0311	8.6597
1982	0.68	0.22	0.0544	-1.4568	2.3826	1982	7.12	6.43	0.0473	3.5618	7.5113
1983	-0.98	-1.26	0.0614	-3.0796	1.2966	1983	6.37	6.42	0.0456	3.6222	7.4395
1984	-2.51	-2.92	0.0523	-4.5123	-0.6421	1984	5.47	4.04	0.0277	2.3327	4.598
1985	-3.92	-4.46	0.0634	-6.2724	-1.6227	1985	4.47	4.43	0.0292	2.6269	4.9998
1986	-5.2	-5.38	0.0569	-6.9722	-2.7183	1986	3.4	3.04	0.0194	1.904	3.3498
1987	-6.39	-6.13	0.0422	-7.3211	-4.0259	1987	2.29	0.61	0.0102	0.3972	1.0081
1988	-7.51	-7.14	0.0368	-8.1468	-5.1891	1988	1.21	-0.83	0.0169	-1.2815	-0.1595
1989	-8.6	-7.87	0.037	-8.8488	-5.8547	1989	0.21	-1.09	0.0208	-1.7147	-0.2546
1990	-9.7	-8.22	0.0381	-9.2579	-6.1321	1990	-0.72	-1.12	0.022	-1.7998	-0.2271
1991	-10.85	-9.2	0.036	-10.1787	-7.1643	1991	-1.6	-2.15	0.0298	-3.0455	-0.8365
1992	-12.07	-11.66	0.0413	-12.7475	-9.2459	1992	-2.43	-3.39	0.0414	-4.6219	-1.4821
1993	-13.36	-12.07	0.046	-13.3641	-9.4455	1993	-3.25	-2.87	0.04	-4.0929	-1.1055
1994	-14.73	-13.37	0.0413	-14.4978	-10.8748	1994	-4.08	-4.6	0.0528	-6.189	-2.0697
1995	-16.2	-14.75	0.0519	-16.1964	-11.6929	1995	-4.97	-4.13	0.0525	-5.7147	-1.7385
1996	-17.76	-15.9	0.0521	-17.3587	-12.7929	1996	-5.93	-4.99	0.0599	-6.7696	-2.1821
1997	-19.4	-17.91	0.0505	-19.3263	-14.8215	1997	-7.	-6.86	0.0755	-9.1626	-3.161
1998	-21.1	-20.03	0.056	-21.6415	-16.6013	1998	-8.16	-7.7	0.0848	-10.2954	-3.534
1999	-22.84	-21.67	0.0582	-23.3829	-18.099	1999	-9.41	-8.6	0.0946	-11.511	-3.962
2000	-24.61	-22.6	0.0518	-24.0153	-19.4101	2000	-10.74	-10.73	0.109	-13.9413	-5.1377
2001	-26.4	-24.64	0.0518	-25.9449	-21.4778	2001	-12.12	-12.67	0.1249	-16.3231	-6.1609
2002	-28.21	-26.61	0.0627	-28.3703	-22.7657	2002	-13.54	-12.34	0.1257	-16.0743	-5.8908
2003	-30.05	-25.88	0.0547	-27.2945	-22.6365	2003	-15.01	-13.36	0.1333	-17.2913	-6.4494
2004	-31.93	-30.43	0.0639	-32.0228	-26.4915	2004	-16.54	-15.94	0.1524	-20.3004	-7.846
2005	-33.82	-31.04	0.0621	-32.5608	-27.3471	2005	-18.09	-16.94	0.1621	-21.6314	-8.3594
2006	-35.72	-33.15	0.0633	-34.5588	-29.4538	2006	-19.65	-19.16	0.1777	-24.1673	-9.5983

Table A.3: Parameter estimates, simulation mean, standard error (SE) and quantiles ($q_{0.05}$, $q_{0.95}$) for the RH03 model based on German male period life tables 1956-2006 from the Human Mortality Database (2009).

A.5 Estimation results for the age-period-cohort model by Renshaw and Haberman

age x	$\hat{a}(x)$	$\overline{a}(x)$	$SE_{\overline{a}}$	$q^a_{0.05}$	$q^a_{0.95}$	age x	$\hat{a}(x)$	$\overline{a}(x)$	$SE_{\overline{a}}$	$q^a_{0.05}$	$q^a_{0.95}$
0	-4.1	-2.95	0.0467	-3.5284	-2.5812	55	-4.6965	-4.73	0.0171	-4.9385	-4.5971
1	-4.7	-5.77	0.0442	-6.2965	-5.4111	56	-4.6172	-4.65	0.0168	-4.8474	-4.5098
2	-5.25	-6.39	0.0435	-6.9279	-6.0212	57	-4.5388	-4.56	0.0169	-4.7615	-4.4206
3	-5.75	-6.72	0.0429	-7.2665	-6.3439	58	-4.4611	-4.5	0.0181	-4.7086	-4.35
4	-6.17	-6.98	0.0449	-7.5649	-6.5807	59	-4.3839	-4.43	0.0195	-4.6535	-4.2738
5	-6.5	-7.11	0.0456	-7.7108	-6.7046	60	-4.3069	-4.35	0.0193	-4.5651	-4.1924
6	-6.77	-7.21	0.0463	-7.8271	-6.8014	61	-4.2299	-4.28	0.0202	-4.5056	-4.1208
7	-6.97	-7.36	0.0461	-7.9812	-6.9432	62	-4.1526	-4.19	0.0198	-4.4157	-4.0384
8	-7.11	-7.34	0.046	-7.9567	-6.9388	63	-4.0748	-4.11	0.0193	-4.3262	-3.9592
9	-7.21	-7.48	0.0483	-8.1387	-7.0329	64	-3.9963	-4.04	0.0198	-4.2619	-3.8884
10	-7.25	-7.55	0.0499	-8.2286	-7.0949	65	-3.9168	-3.95	0.0192	-4.1647	-3.8034
11	-7.26	-7.62	0.0449	-8.2378	-7.2136	66	-3.8361	-3.86	0.0187	-4.0811	-3.7263
12	-7.22	-7.6	0.0459	-8.218	-7.2059	67	-3.754	-3.79	0.0193	-4.025	-3.6466
13	-7.15	-7.49	0.0462	-8.1104	-7.1069	68	-3.6704	-3.71	0.0199	-3.9741	-3.57
14	-7.05	-7.36	0.0449	-7.9466	-6.9796	69	-3.5851	-3.61	0.018	-3.8534	-3.4768
15	-6.93	-7.24	0.0388	-7.7472	-6.8922	70	-3.4984	-3.52	0.0184	-3.7861	-3.3883
16	-6.8	-6.92	0.0337	-7.3369	-6.6283	71	-3.4104	-3.4	0.0164	-3.6456	-3.2827
17	-6.67	-6.75	0.0298	-7.1275	-6.4764	72	-3.3217	-3.31	0.0173	-3.5804	-3.1952
18	-6.56	-6.54	0.0224	-6.8142	-6.3286	73	-3.2323	-3.22	0.0166	-3.4783	-3.1039
19	-6.46	-6.41	0.024	-6.722	-6.1824	74	-3.1422	-3.14	0.0164	-3.3744	-3.0223
20	-6.4	-6.37	0.0259	-6.7089	-6.1345	75	-3.0511	-3.04	0.0171	-3.2904	-2.9189
21	-6.36	-6.35	0.0283	-6.7166	-6.1106	76	-2.9587	-2.99	0.0166	-3.2054	-2.8697
22	-6.34	-6.41	0.0283	-6.7707	-6.1604	77	-2.8642	-2.88	0.016	-3.0857	-2.7696
23	-6.35	-6.45	0.0288	-6.8227	-6.2019	78	-2.7674	-2.81	0.0162	-3.0028	-2.7018
24	-6.37	-6.5	0.0288	-6.8724	-6.2492	79	-2.6683	-2.69	0.0137	-2.849	-2.5934
25	-6.39	-6.55	0.0281	-6.9143	-6.2992	80	-2.5674	-2.52	0.0105	-2.6399	-2.4436
26	-6.42	-6.58	0.0277	-6.94	-6.329	81	-2.466	-2.43	0.011	-2.557	-2.3502
27	-6.45	-6.58	0.0273	-6.9446	-6.3412	82	-2.3648	-2.35	0.0107	-2.4795	-2.2737
28	-6.47	-6.6	0.0249	-6.9369	-6.3749	83	-2.2641	-2.22	0.0088	-2.3305	-2.1598
29	-6.48	-6.59	0.0242	-6.9202	-6.3712	84	-2.1642	-2.13	0.0087	-2.235	-2.0653
30	-6.48	-6.59	0.023	-6.8971	-6.3717	85	-2.0651	-2.03	0.0084	-2.1333	-1.9655
31	-6.48	-6.56	0.0207	-6.848	-6.3652	86	-1.9669	-1.93	0.0085	-2.029	-1.8586
32	-6.46	-6.52	0.0223	-6.8204	-6.3112	87	-1.8693	-1.83	0.0082	-1.936	-1.7635
33	-6.43	-6.47	0.0215	-6.7656	-6.2738	88	-1.7722	-1.6	0.0115	-1.6872	-1.4678
34	-6.38	-6.43	0.0206	-6.7088	-6.236	89	-1.6753	-1.49	0.0103	-1.5773	-1.3769
35	-6.34	-6.38	0.021	-6.6631	-6.1867	90	-1.5781	-1.38	0.0077	-1.4462	-1.2838
36	-6.28	-6.32	0.0186	-6.5744	-6.1424	91	-1.4801	-1.3	0.0093	-1.3839	-1.1948
37	-6.21	-6.23	0.0172	-6.47	-6.068	92	-1.3795	-1.22	0.0079	-1.2933	-1.121
38	-6.14	-6.17	0.0149	-6.3759	-6.0205	93	-1.2751	-1.13	0.0072	-1.1983	-1.0365
39	-6.07	-6.08	0.016	-6.2984	-5.9253	94	-1.166	-1.03	0.0023	-1.06	-0.9961
40	-5.99	-6.	0.0142	-6.1938	-5.858	95	-1.0521	-0.99	0.0043	-1.0432	-0.9296
41	-5.91	-5.92	0.0111	-6.0801	-5.8044	96	-0.9336	-0.92	0.0037	-0.964	-0.8664
42	-5.82	-5.83	0.0129	-6.0121	-5.7055	97	-0.8133	-0.85	0.004	-0.9241	-0.8067
43	-5.73	-5.74	0.0115	-5.9044	-5.6253	98	-0.6971	-0.8	0.005	-0.8965	-0.7495
44	-5.65	-5.66	0.0128	-5.8419	-5.5426	99	-0.5949	-0.74	0.0071	-0.8854	-0.6738
45	-5.56	-5.55	0.0099	-5.6944	-5.4531	100	-0.5213	-0.69	0.01	-0.896	-0.5924
46	-5.47	-5.48	0.0123	-5.6465	-5.3614	101	-0.4964	-0.65	0.0139	-0.9425	-0.5264
47	-5.38	-5.39	0.0131	-5.5639	-5.2692	102	-0.5432	-0.61	0.0206	-1.0329	-0.4408
48	-5.29	-5.3	0.0119	-5.4603	-5.1884	103	-0.6865	-0.59	0.0302	-1.1849	-0.3411
49	-5.2	-5.21	0.012	-5.3721	-5.1028	104	-0.9514	-0.65	0.0672	-2.543	-0.2169
50	-5.11	-5.12	0.0122	-5.2785	-5.0073	105	-1.3563	-0.75	0.0988	-2.9294	-0.017
51	-5.03	-5.03	0.0127	-5.1995	-4.9243	106	-1.9108	-1.02	0.142	-3.3273	0.0786
52	-4.94	-4.96	0.0141	-5.1409	-4.8417	107	-2.6049	-1.68	0.3319	-9.2091	0.5001
53	-4.86	-4.88	0.0142	-5.0589	-4.7623	108	-3.4094	-42.49	14.7306	-413.35	0.8294
54	-4.78	-4.8	0.0153	-4.9882	-4.6747	109	-4.2915	-138.4	23.1162	-453.36	0.6931

age x	$\hat{b}_1(x)$	$\overline{b}_1(x)$	$SE_{\overline{b}_1}$	$q^{b_1}_{0.05}$	$q^{b_1}_{0.95}$	age x	$\hat{b}_2(x)$	$\overline{b}_2(x)$	$SE_{\overline{b}_2}$	$q^{b_2}_{0.05}$	$q^{b_2}_{0.95}$
0	0.02	0.0213	0.0005	0.0189	0.0269	0	0.0271	0.03	0.0002	0.0243	0.0297
1	0.02	0.0123	0.0004	0.0093	0.0172	1	0.0258	0.02	0.0003	0.0205	0.0268
2	0.02	0.0166	0.0005	0.0126	0.0238	2	0.0245	0.02	0.0002	0.0182	0.0236
3	0.02	0.0201	0.0006	0.0151	0.0295	3	0.0233	0.02	0.0002	0.0165	0.0211
4	0.02	0.0256	0.0008	0.019	0.0386	4	0.0223	0.02	0.0002	0.0158	0.0199
5	0.02	0.0278	0.0009	0.0203	0.043	5	0.0215	0.02	0.0002	0.0155	0.0194
6	0.02	0.0263	0.0008	0.0192	0.0405	6	0.0208	0.02	0.0002	0.0157	0.0199
7	0.02	0.0274	0.0009	0.0195	0.0428	7	0.0202	0.02	0.0001	0.015	0.0185
8	0.02	0.0163	0.0006	0.011	0.0262	8	0.0196	0.02	0.0002	0.0161	0.0212
9	0.02	0.0166	0.0006	0.0106	0.0284	9	0.0191	0.02	0.0003	0.0163	0.0218
10	0.02	0.0171	0.0007	0.011	0.0296	10	0.0185	0.02	0.0003	0.0168	0.0228
11	0.02	0.0093	0.0005	0.0061	0.0168	11	0.0177	0.02	0.0003	0.0144	0.02
12	0.01	0.0094	0.0007	0.0066	0.0169	12	0.0169	0.02	0.0005	0.0141	0.0196
13	0.01	0.0052	0.0006	0.0035	0.0104	13	0.016	0.02	0.0005	0.0153	0.0218
14	0.01	0.0065	0.0006	0.0045	0.0124	14	0.015	0.02	0.0004	0.0153	0.0208
15	0.01	0.0113	0.0006	0.0076	0.0185	15	0.0139	0.01	0.0004	0.0123	0.0155
16	0.02	0.0253	0.0009	0.0157	0.0385	16	0.0129	0.01	0.0005	0.0075	0.0094
17	0.02	0.0241	0.0007	0.0149	0.0357	17	0.0119	0.01	0.0003	0.0074	0.0089
18	0.01	0.0237	0.0007	0.0147	0.0347	18	0.0112	0.	0.0004	0.0024	0.005
19	0.01	0.018	0.0005	0.0112	0.0261	19	0.0108	0.01	0.0002	0.0066	0.0077
20	0.01	0.012	0.0003	0.0074	0.0169	20	0.0105	0.01	0.0002	0.0095	0.0113

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21	0.01	0.0078	0.0003	0.0061	0.0111	21	0.0104	0.01	0.0001	0.0113	0.0138
22	0.01	0.0073	0.0003	0.0054	0.0107	22	0.0103	0.01	0.0001	0.0111	0.0136
23	0.01	0.0053	0.0003	0.0034	0.008	23	0.0103	0.01	0.0001	0.0113	0.0139
24	0.01	0.0051	0.0003	0.0033	0.0076	24	0.0102	0.01	0.0001	0.0112	0.0139
25	0.01	0.0062	0.0004	0.0038	0.0098	25	0.01	0.01	0.0001	0.0107	0.0125
26	0.01	0.0069	0.0003	0.0045	0.0108	26	0.0097	0.01	0.0001	0.0105	0.0122
27	0.01	0.0076	0.0003	0.0051	0.0117	27	0.0094	0.01	0.0001	0.0104	0.0119
28	0.01	0.0101	0.0004	0.0073	0.015	28	0.009	0.01	0.0001	0.0087	0.0098
29	0.01	0.0116	0.0004	0.0086	0.0171	29	0.0087	0.01	0.	0.0084	0.0095
30	0.01	0.0124	0.0004	0.0095	0.018	30	0.0083	0.01	0.0001	0.0077	0.0089
31	0.01	0.0133	0.0004	0.0104	0.0191	31	0.0079	0.01	0.0001	0.0063	0.0077
32	0.01	0.0121	0.0004	0.0094	0.0173	32	0.0075	0.01	0.0001	0.0077	0.0089
33	0.01	0.0126	0.0004	0.0099	0.0179	33	0.0072	0.01	0.0001	0.0073	0.0086
34	0.01	0.0126	0.0004	0.01	0.0179	34	0.0068	0.01	0.0001	0.0069	0.0083
35	0.01	0.0112	0.0003	0.0089	0.0159	35	0.0065	0.01	0.0001	0.0075	0.0087
36	0.01	0.0128	0.0004	0.0103	0.0182	36	0.0062	0.01	0.0001	0.0061	0.0076
37	0.01	0.0145	0.0004	0.0119	0.0202	37	0.0059	0.01	0.0001	0.0052	0.0071
38	0.01	0.0154	0.0004	0.0125	0.0215	38	0.0056	0.01	0.0001	0.0037	0.0062
39	0.01	0.0133	0.0004	0.0107	0.0187	39	0.0053	0.01	0.0001	0.005	0.007
40	0.01	0.0137	0.0004	0.0113	0.0188	40	0.005	0.01	0.0001	0.0041	0.0061
41	0.01	0.0158	0.0004	0.0136	0.0212	41	0.0048	0.	0.0001	0.0022	0.0045
42	0.01	0.013	0.0003	0.0112	0.0174	42	0.0046	0.	0.0001	0.0038	0.0056
43	0.01	0.0138	0.0003	0.0122	0.0182	43	0.0045	0.	0.0001	0.0032	0.005
44	0.01	0.0122	0.0003	0.0109	0.0159	44	0.0044	0.01	0.0001	0.0044	0.0057
45	0.01	0.0135	0.0003	0.0119	0.0177	45	0.0044	0.	0.0001	0.0026	0.0044
46	0.01	0.0112	0.0002	0.0101	0.0144	46	0.0044	0.01	0.0001	0.0046	0.0056
47	0.01	0.0099	0.0002	0.009	0.0127	47	0.0045	0.01	0.	0.0053	0.0062
48	0.01	0.0108	0.0002	0.0098	0.014	48	0.0046	0.01	0.0001	0.0047	0.0057
49	0.01	0.0102	0.0002	0.0092	0.0132	49	0.0048	0.01	0.0001	0.005	0.0059
50	0.01	0.0096	0.0002	0.0086	0.0126	50	0.005	0.01	0.	0.0053	0.0062
51	0.01	0.0099	0.0002	0.0087	0.013	51	0.0053	0.01	0.0001	0.0058	0.0069
52	0.01	0.0081	0.0002	0.0073	0.0106	52	0.0056	0.01	0.	0.0068	0.0075
53	0.01	0.0081	0.0002	0.0071	0.0109	53	0.0059	0.01	0.	0.0071	0.008
54	0.01	0.0072	0.0002	0.0062	0.0097	54	0.0063	0.01	0.	0.0079	0.0087
55	0.01	0.0062	0.0001	0.0053	0.0086	55	0.0067	0.01	0.	0.0091	0.0099
56	0.01	0.0067	0.0002	0.0057	0.0094	56	0.007	0.01	0.	0.0091	0.01
57	0.01	0.0071	0.0002	0.006	0.0099	57	0.0074	0.01	0.	0.0094	0.0103
58	0.01	0.0063	0.0001	0.0052	0.0089	58	0.0077	0.01	0.	0.0102	0.0111
59	0.01	0.0056	0.0001	0.0046	0.008	59	0.0081	0.01	0.	0.0112	0.012
60	0.01	0.0062	0.0001	0.0051	0.0088	60	0.0084	0.01	0.	0.0113	0.0122
61	0.01	0.006	0.0001	0.005	0.0086	61	0.0086	0.01	0.	0.0119	0.0128
62	0.01	0.0066	0.0002	0.0054	0.0093	62	0.0088	0.01	0.	0.0119	0.0129
63	0.01	0.0071	0.0002	0.0058	0.0101	63	0.009	0.01	0.0001	0.0117	0.0129
64	0.01	0.0074	0.0002	0.0062	0.0104	64	0.0091	0.01	0.	0.012	0.0133
65	0.01	0.0078	0.0002	0.0065	0.011	65	0.0092	0.01	0.0001	0.0118	0.0133
66	0.01	0.0086	0.0002	0.0072	0.0121	66	0.0092	0.01	0.0001	0.0116	0.0133
67	0.01	0.0086	0.0002	0.0074	0.0118	67	0.0092	0.01	0.0001	0.0119	0.0133
68	0.01	0.0087	0.0002	0.0077	0.012	68	0.0091	0.01	0.0001	0.0121	0.0141
69	0.01	0.0101	0.0002	0.0089	0.0137	69	0.009	0.01	0.0001	0.011	0.0132
70	0.01	0.0103	0.0002	0.0093	0.0138	70	0.0089	0.01	0.0001	0.011	0.0136
71	0.01	0.0112	0.0002	0.0101	0.0149	71	0.0087	0.01	0.0001	0.0098	0.0125
72	0.01	0.0113	0.0002	0.0105	0.0146	72	0.0085	0.01	0.0001	0.0101	0.0128
73	0.01	0.0118	0.0002	0.0109	0.0152	73	0.0084	0.01	0.0001	0.0097	0.0125
74	0.01	0.0114	0.0002	0.0105	0.0147	74	0.0082	0.01	0.0001	0.0098	0.0121
75	0.01	0.0115	0.0002	0.0108	0.0144	75	0.008	0.01	0.0001	0.0098	0.0122
76	0.01	0.0106	0.0002	0.0097	0.0137	76	0.0079	0.01	0.0001	0.0104	0.0122
77	0.01	0.0103	0.0002	0.0097	0.0132	77	0.0076	0.01	0.	0.01	0.0116
78	0.01	0.0096	0.0002	0.0089	0.0125	78	0.0074	0.01	0.	0.0105	0.0118
79	0.01	0.0101	0.0002	0.0093	0.0131	79	0.0071	0.01	0.0001	0.0093	0.0109
80	0.01	0.0111	0.0002	0.0104	0.0141	80	0.0068	0.01	0.0001	0.0073	0.0088
81	0.01	0.0105	0.0002	0.0099	0.0133	81	0.0065	0.01	0.0001	0.0075	0.0087
82	0.01	0.0095	0.0002	0.0087	0.0121	82	0.0062	0.01	0.0001	0.0074	0.009
83	0.01	0.0094	0.0002	0.0086	0.012	83	0.0059	0.01	0.0001	0.0064	0.0081
84	0.01	0.009	0.0002	0.0082	0.0114	84	0.0056	0.01	0.0001	0.0062	0.0078
85	0.01	0.0087	0.0002	0.0079	0.0109	85	0.0054	0.01	0.0001	0.0059	0.0075
86	0.01	0.0081	0.0001	0.0074	0.01	86	0.0051	0.01	0.0001	0.0056	0.0069
87	0.01	0.0078	0.0001	0.007	0.0096	87	0.0048	0.01	0.0001	0.0051	0.0067
88	0.01	0.0083	0.0003	0.0063	0.0127	88	0.0046	0.	0.0003	-0.0004	0.0059
89	0.01	0.0076	0.0002	0.0058	0.0111	89	0.0043	0.	0.0003	-0.0006	0.0049
90	0.	0.0075	0.0002	0.0061	0.0102	90	0.0041	0.	0.0003	-0.0006	0.0038
91	0.	0.0058	0.0002	0.0043	0.0082	91	0.0038	0.	0.0003	-0.0007	0.0043
92	0.	0.0059	0.0002	0.0046	0.008	92	0.0034	0.	0.0003	-0.0007	0.0039
93	0.	0.0059	0.0001	0.0047	0.0075	93	0.0031	0.	0.0002	-0.0007	0.0036
94	0.	0.0062	0.0001	0.0056	0.007	94	0.0026	0.	0.0001	-0.0002	0.002
95	0.	0.0039	0.0001	0.0031	0.0047	95	0.0021	0.	0.0002	-0.0004	0.0029
96	0.	0.0033	0.0001	0.002	0.0045	96	0.0015	0.	0.0002	-0.0004	0.0027
97	0.	0.0028	0.0001	0.0008	0.004	97	0.0009	0.	0.0001	-0.0004	0.0025
98	0.	0.0021	0.0002	-0.0007	0.0037	98	0.0002	0.	0.0002	-0.0004	0.0028
99	0.	0.0015	0.0002	-0.0025	0.0037	99	-0.0003	0.	0.0001	-0.0004	0.0028
100	0.	0.0009	0.0003	-0.0044	0.0038	100	-0.0006	0.	0.0002	-0.0003	0.003
101	0.	0.0003	0.0004	-0.0071	0.0039	101	-0.0005	0.	0.0002	-0.0003	0.0036
102	0.	-0.0004	0.0005	-0.0101	0.0043	102	0.0003	0.	0.0002	-0.0002	0.0041
103	0.	-0.0013	0.0007	-0.0147	0.0052	103	0.0019	0.	0.0002	-0.0002	0.0048
104	0.	-0.0027	0.0011	-0.025	0.006	104	0.0045	0.	0.0005	-0.0001	0.0191
105	-0.01	-0.0055	0.0018	-0.0418	0.0078	105	0.0084	0.	0.0007	-0.0005	0.017
106	-0.01	-0.013	0.0039	-0.0748	0.0089	106	0.0134	0.01	0.0008	-0.0014	0.0201

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107	-0.01	-0.0442	0.0133	-0.3992	0.0139	107	0.0196	0.01	0.0013	-0.014	0.023
108	-0.01	-0.0489	0.0105	-0.2054	0.0241	108	0.0266	0.	0.0022	-0.0044	0.0067
109	-0.01	0.0257	0.0129	-0.0164	0.0203	109	0.0342	0.	0.002	0.	0.0098

year t	$\hat{k}(t)$	$\overline{k}(t)$	$SE_{\overline{k}}$	$q^k_{0.05}$	$q^k_{0.95}$	**year t**	$\hat{k}(t)$	$\overline{k}(t)$	$SE_{\overline{k}}$	$q^k_{0.05}$	$q^k_{0.95}$
1960	-1.3	6.13	0.2827	3.7939	9.1613	1984	3.52	1.77	0.046	1.275	2.0539
1961	-0.19	2.27	0.2456	-0.2264	5.4831	1985	2.51	2.21	0.0595	1.5538	2.5709
1962	0.97	5.07	0.2502	2.9583	7.7963	1986	1.54	0.77	0.0425	0.2907	1.142
1963	2.18	8.83	0.2909	6.9551	10.9743	1987	0.64	-1.9	0.0682	-2.4862	-1.6056
1964	3.43	3.48	0.2165	1.4618	6.0399	1988	-0.15	-3.35	0.0989	-3.9594	-2.7361
1965	4.77	7.68	0.2547	6.0947	9.6073	1989	-0.84	-2.95	0.0894	-3.6052	-2.5014
1966	6.17	8.1	0.2533	6.5128	9.8498	1990	-1.45	-2.15	0.0789	-2.9361	-1.5984
1967	7.57	7.71	0.2383	6.2315	9.306	1991	-2.06	-3.08	0.0962	-3.8592	-2.4801
1968	8.91	13.83	0.3608	11.3772	14.7117	1992	-2.69	-5.63	0.155	-6.3793	-4.7443
1969	10.05	15.44	0.393	12.5087	16.2193	1993	-3.38	-4.15	0.1272	-5.0984	-3.3549
1970	10.91	13.8	0.3515	11.2195	14.5167	1994	-4.18	-6.26	0.1716	-7.1204	-5.3717
1971	11.49	13.07	0.3321	10.5792	13.7046	1995	-5.13	-5.75	0.1673	-6.8116	-4.799
1972	11.8	13.3	0.3345	10.6654	13.8111	1996	-6.28	-6.81	0.1914	-7.9086	-5.7525
1973	11.86	12.86	0.32	10.2208	13.2279	1997	-7.61	-9.69	0.2564	-10.6773	-8.273
1974	11.69	11.24	0.2776	8.9587	11.5507	1998	-9.12	-11.4	0.2969	-12.405	-9.6768
1975	11.32	14.51	0.3584	11.283	14.9026	1999	-10.77	-12.85	0.3311	-13.8902	-10.892
1976	10.75	11.9	0.2924	9.3141	12.2299	2000	-12.52	-15.22	0.3881	-16.1522	-12.7274
1977	10.03	7.76	0.1882	6.2152	8.0153	2001	-14.36	-18.32	0.4641	-19.1748	-15.1714
1978	9.23	9.74	0.237	7.5953	10.0861	2002	-16.27	-18.39	0.4674	-19.4001	-15.3639
1979	8.36	6.88	0.1643	5.4322	7.1851	2003	-18.24	-17.94	0.4564	-19.0352	-15.079
1980	7.46	6.69	0.1585	5.2148	7.0417	2004	-20.31	-23.99	0.6026	-24.864	-19.7944
1981	6.53	6.84	0.164	5.2401	7.2148	2005	-22.41	-24.57	0.6171	-25.5949	-20.299
1982	5.56	5.27	0.1244	4.0226	5.6316	2006	-24.54	-27.89	0.6963	-28.8291	-22.8863
1983	4.55	5.17	0.1244	3.8819	5.5131						

⋆	$\hat{l}(t-x)$	$\overline{l}(t-x)$	$SE_{\overline{l}}$	$q^l_{0.05}$	$q^l_{0.95}$	⋆	$\hat{l}(t-x)$	$\overline{l}(t-x)$	$SE_{\overline{l}}$	$q^l_{0.05}$	$q^l_{0.95}$
1851	-112.99	22.22	10.	-100.	100.01	1929	19.14	20.39	1.53	9.78	37.75
1852	-102.52	10.43	9.79	-100.	100.01	1930	17.7	19.61	1.53	8.98	37.
1853	-91.79	23.	9.77	-100.	100.01	1931	15.93	18.05	1.53	7.43	35.43
1854	-80.47	22.6	10.74	-100.	100.01	1932	13.86	15.62	1.53	5.02	32.98
1855	-68.42	21.76	11.16	-100.	100.01	1933	11.54	13.28	1.53	2.72	30.61
1856	-55.89	23.47	11.1	-100.	100.01	1934	9.04	10.9	1.53	0.33	28.21
1857	-41.43	24.41	11.04	-100.	100.01	1935	6.42	12.71	1.54	2.07	30.16
1858	-24.48	20.19	11.37	-100.	100.01	1936	3.73	7.26	1.53	-3.29	24.57
1859	-5.71	31.58	10.25	-100.	100.01	1937	1.05	5.45	1.53	-5.1	22.76
1860	14.9	33.75	10.58	-100.	100.01	1938	-1.52	3.02	1.52	-7.5	20.31
1861	35.42	25.72	11.33	-100.	100.01	1939	-3.95	2.85	1.53	-7.74	20.22
1862	54.11	26.51	11.05	-100.	100.01	1940	-6.23	3.22	1.54	-7.42	20.64
1863	70.03	27.99	11.04	-100.	100.01	1941	-8.38	-2.59	1.53	-13.15	14.71
1864	82.67	19.69	11.6	-100.	100.01	1942	-10.36	-0.04	1.55	-10.68	17.4
1865	92.	20.43	11.4	-100.	99.53	1943	-12.2	-3.26	1.54	-13.9	14.16
1866	98.35	11.73	10.88	-100.	90.32	1944	-13.92	-1.16	1.56	-11.88	16.37
1867	102.22	17.72	11.6	-100.	100.01	1945	-15.52	-3.09	1.56	-13.86	14.47
1868	104.14	9.2	10.65	-100.	86.25	1946	-16.98	-11.52	1.54	-22.26	6.02
1869	104.58	10.58	8.89	-89.18	75.4	1947	-18.21	-2.18	1.58	-13.29	15.65
1870	103.96	6.51	10.38	-100.	81.31	1948	-19.28	-9.96	1.57	-21.05	7.94
1871	102.56	12.98	8.39	-86.6	71.99	1949	-20.15	-9.31	1.57	-20.41	8.54
1872	100.62	13.43	8.15	-86.31	69.13	1950	-20.85	-9.88	1.59	-21.32	8.24
1873	98.31	66.27	1.17	57.03	79.73	1951	-21.43	-8.05	1.6	-19.6	10.22
1874	95.79	65.26	1.18	56.43	78.92	1952	-21.98	-9.38	1.6	-21.04	8.98
1875	93.16	59.67	1.15	51.02	73.14	1953	-22.57	-9.94	1.6	-21.51	8.55
1876	90.51	62.26	1.28	52.9	77.14	1954	-23.27	-9.87	1.6	-21.46	8.61
1877	87.88	57.71	1.24	48.65	72.19	1955	-24.15	-9.72	1.63	-21.57	9.1
1878	85.3	57.27	1.26	48.05	72.08	1956	-25.26	-10.04	1.64	-22.06	9.06
1879	82.81	54.97	1.23	46.01	69.52	1957	-26.64	-11.33	1.63	-23.31	7.78
1880	80.41	53.75	1.23	44.81	68.26	1958	-28.3	-13.71	1.64	-25.84	5.57
1881	78.14	50.81	1.25	41.79	65.54	1959	-30.22	-15.33	1.65	-27.54	4.21
1882	76.03	50.21	1.26	41.08	65.07	1960	-32.36	-16.8	1.63	-28.91	2.66
1883	74.08	50.23	1.27	41.1	65.12	1961	-34.69	-16.59	1.64	-28.76	3.22
1884	72.28	48.62	1.28	39.39	63.68	1962	-37.16	-21.55	1.67	-34.05	-1.46
1885	70.62	47.93	1.28	38.74	62.95	1963	-39.69	-26.62	1.71	-39.43	-6.21
1886	69.09	47.78	1.28	38.61	62.85	1964	-42.18	-26.48	1.72	-39.38	-5.59
1887	67.7	45.6	1.29	36.33	60.75	1965	-44.56	-31.51	1.76	-44.73	-10.34
1888	66.46	46.12	1.3	36.81	61.44	1966	-46.79	-32.06	1.78	-45.44	-10.71
1889	65.39	44.11	1.3	34.75	59.41	1967	-48.82	-33.57	1.8	-47.11	-11.93
1890	64.5	44.23	1.33	34.73	59.79	1968	-50.65	-37.79	1.84	-51.65	-16.06
1891	63.8	44.64	1.34	35.09	60.32	1969	-52.27	-38.72	1.86	-52.68	-16.8
1892	63.27	43.97	1.35	34.3	59.8	1970	-53.7	-38.27	1.87	-52.33	-16.13
1893	62.88	43.13	1.36	33.38	59.04	1971	-55.03	-37.87	1.88	-52.06	-15.64
1894	62.62	44.78	1.39	34.93	60.96	1972	-56.35	-40.38	1.9	-54.76	-17.82
1895	62.42	44.94	1.39	35.08	61.16	1973	-57.74	-39.81	1.91	-54.31	-17.09
1896	62.23	44.44	1.41	34.47	60.81	1974	-59.25	-40.03	1.92	-54.65	-17.02
1897	62.	45.3	1.42	35.27	61.78	1975	-60.9	-45.33	1.96	-60.25	-22.06
1898	61.7	44.68	1.43	34.62	61.2	1976	-62.66	-47.13	1.98	-62.16	-23.39
1899	61.27	44.61	1.43	34.49	61.2	1977	-64.49	-49.48	2.	-64.71	-25.13
1900	60.7	45.41	1.44	35.26	62.03	1978	-66.35	-51.82	2.02	-67.19	-27.34
1901	59.95	43.84	1.44	33.67	60.49	1979	-68.2	-52.93	2.04	-68.42	-28.06
1902	59.03	44.05	1.45	33.85	60.76	1980	-70.02	-55.21	2.06	-70.89	-30.03
1903	57.92	42.71	1.45	32.5	59.41	1981	-71.76	-58.31	2.08	-74.15	-32.86
1904	56.63	41.79	1.45	31.57	58.52	1982	-73.39	-60.33	2.1	-76.36	-34.52
1905	55.19	41.21	1.45	30.95	57.97	1983	-74.88	-63.26	2.13	-79.51	-37.15

Continued on next page . . .

1906	53.59	40.03	1.45	29.77	56.78	1984	-76.23	-62.78	2.13	-79.02	-36.41
1907	51.84	39.77	1.46	29.49	56.55	1985	-77.49	-65.53	2.16	-82.01	-38.81
1908	49.94	38.55	1.46	28.23	55.37	1986	-78.69	-65.	2.16	-81.49	-38.18
1909	47.92	36.84	1.46	26.53	53.61	1987	-79.9	-64.36	2.16	-80.92	-37.3
1910	45.81	34.82	1.45	24.51	51.56	1988	-81.18	-65.9	2.17	-82.55	-38.59
1911	43.68	33.19	1.45	22.87	49.93	1989	-82.56	-68.35	2.2	-85.16	-40.81
1912	41.56	32.14	1.46	21.8	48.92	1990	-84.01	-70.28	2.22	-87.29	-42.53
1913	39.49	30.63	1.46	20.26	47.39	1991	-85.48	-72.61	2.23	-89.68	-44.63
1914	37.5	28.9	1.45	18.57	45.56	1992	-86.9	-75.12	2.26	-92.4	-46.51
1915	35.59	32.55	1.46	22.24	49.38	1993	-88.19	-76.84	2.27	-94.23	-48.11
1916	33.76	22.79	1.45	12.45	39.47	1994	-89.29	-78.63	2.29	-96.15	-49.54
1917	32.12	27.82	1.47	17.4	44.7	1995	-90.15	-79.68	2.31	-97.36	-50.37
1918	30.66	24.69	1.47	14.26	41.52	1996	-90.74	-79.82	2.31	-97.52	-50.38
1919	29.4	15.72	1.44	5.38	32.33	1997	-91.06	-79.19	2.31	-96.99	-49.5
1920	28.36	33.02	1.52	22.39	50.4	1998	-91.15	-80.87	2.34	-98.85	-50.81
1921	27.3	23.22	1.49	12.71	40.22	1999	-91.01	-79.57	2.33	-97.63	-49.43
1922	26.24	23.27	1.49	12.73	40.34	2000	-90.69	-79.25	2.34	-97.39	-48.82
1923	25.18	21.63	1.49	11.09	38.7	2001	-90.23	-76.91	2.34	-95.01	-46.32
1924	24.15	20.56	1.5	10.02	37.64	2002	-89.68	-79.47	2.36	-97.84	-48.55
1925	23.17	20.1	1.5	9.54	37.21	2003	-89.03	-78.51	2.36	-96.83	-47.64
1926	22.24	18.77	1.5	8.23	35.9	2004	-88.3	-76.	2.36	-94.46	-44.75
1927	21.31	18.58	1.51	8.03	35.75	2005	-87.53	-76.35	2.36	-94.86	-45.06
1928	20.32	19.78	1.52	9.21	37.07	2006	-86.75	-74.7	2.36	-93.22	-43.22

Table A.4: Parameter estimates for the RH06 model based on German male period life tables 1956-2006 from the Human Mortality Database (2009) ($\star$ denotes birth in year $t - x$).

APPENDIX to Chapter 7

B.1 Calculations for the Hull-White interest rate model

For the Hull-White model

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma_{spot}\, dW_t \tag{B.1}$$

the constants a and σ_{spot} are positive and $\theta(\,.\,)$ is chosen to fit the market forward rate term structure. Let $f^M(0,t)$ denote the market instantaneous forward rate at time 0 with maturity t defined as

$$f^M(0,t) = -\tfrac{d}{dt} \ln \left(D^M(0,t) \right)$$

where $D^M(0,t)$ denotes the market discount factor with the same maturity t. Thus, if real market values should be depicted, θ has to be chosen as

$$\theta(t) = \tfrac{d}{dt} f^M(0,t) + a f^M(0,t) + \tfrac{\sigma^2}{2a} \left(1 - e^{-2at} \right). \tag{B.2}$$

A derivation of Equation (B.2) is given in Lemma B.1.1.

Lemma B.1.1. *The moments of the random discount factor can be obtained analytically and expressed in terms of pure discount bonds as follows*

(i) $E\left[\tfrac{1}{\beta_{s,t}} \middle| \mathcal{I}_s \right] = D(s,t) = e^{A(s,t) - B(s,t) r_s}$

(ii) $E\left[\tfrac{1}{\beta_{s,t}^2} \middle| \mathcal{I}_s \right] = E\left[\tfrac{1}{\beta_{s,t}} \middle| \mathcal{I}_s \right]^2 e^{Var[I(s,t)|\,\mathcal{I}_s]} = \left(D(s,t) \right)^2 e^{Var[I(s,t)|\,\mathcal{I}_s]}$

(iii) $E\left[\tfrac{1}{\beta_{s,t}\beta_{s,u}} \middle| \mathcal{I}_s \right] = E\left[\tfrac{1}{\beta_{s,t}} \middle| \mathcal{I}_s \right] E\left[\tfrac{1}{\beta_{s,u}} \middle| \mathcal{I}_s \right] e^{Var[I(s,t)|\mathcal{I}_s]}$

$$= D(s,t)\, D(s,u)\, e^{Var[I(s,t)|\mathcal{I}_s]}$$

for $s < t < u$ with

$$\beta_{s,t} = \exp\left(I(s,t)\right) = \exp\left(\int_s^t r_u \, du\right),$$

$$\begin{aligned} V(s,t) &:= Var[I(s,t)|\mathcal{I}_s] \\ &= \frac{\sigma_{spot}^2}{a^2}\left((t-s) - \frac{2}{a}\left(1-e^{-a(t-s)}\right) + \frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right), \end{aligned} \tag{B.3}$$

$$\begin{aligned} A(s,t) &= \int_s^t \theta(u)\, B(u,t)\, du + \frac{\sigma_{spot}^2}{2}\int_s^t B^2(u,t)\, du, \\ &= \ln\left(\frac{D^M(0,t)}{D^M(0,s)}\right) + B(s,t)\, f^M(0,s) - \frac{\sigma_{spot}^2}{4a} B^2(s,t)\left(1-e^{-2as}\right) \\ B(s,t) &= \frac{1}{a}\left(1-e^{-a(t-s)}\right). \end{aligned}$$

Proof.

(i) Since the Hull-White dynamics (B.1) forms an Itô process, the application of Itô's lemma for the process $e^{at}\, r_t$ and subsequent integration on $[s,t]$ yields the solution

$$r_t = r_s\, e^{-a(t-s)} + \int_s^t e^{-a(t-v)}\theta(v)\, dv + \sigma_{spot}\int_s^t e^{-a(t-v)} dW_v. \tag{B.4}$$

From the definition of $I(s,t)$ an easy calculation shows that

$$\begin{aligned} I(s,t) &= B(s,t)r_s + \int_s^t\int_s^u \theta(v)e^{-a(u-v)}\, dvdu + \sigma_{spot}\int_s^t B(s,u)dW_u \\ &= B(s,t)r_s + \int_s^t f^M(0,u) - f^M(0,s)e^{-a(u-s)} \\ &- \frac{\sigma_{spot}^2}{2a^2}\left(e^{-a(u-s)} - e^{-2au} + e^{-2a(u+s)} - 1\right) du + \sigma_{spot}\int_s^t B(s,u)dW_u \\ &= B(s,t)r_s + \int_s^t \alpha(u) - \alpha(s)e^{-a(u-s)}\, du + \sigma_{spot}\int_s^t B(s,u)dW_u \end{aligned}$$

for $\alpha(s) = f^M(0,s) + \frac{\sigma_{spot}^2}{2a^2}\left(1-e^{-as}\right)^2$, i.e. for given Information $\mathcal{I}_s$ the random variable $I(s,t)$ is normally distributed with

$$Var\left[I(s,t)|\,\mathcal{I}_s\right] = E\left[\left(\sigma_{spot}\int_s^t B(s,u)dW_u\right)^2\middle|\,\mathcal{I}_s\right]$$

$$
\begin{aligned}
&= E\left[\sigma_{spot}^2 \int_s^t B(s,u)^2\,du \middle| \mathcal{I}_s\right] = \frac{\sigma_{spot}^2}{a^2}\int_s^t \left(1-2e^{-a(t-s)}+e^{-2a(t-s)}\right)du \\
&= \frac{\sigma_{spot}^2}{a^2}\left((t-s)+\frac{2}{a}e^{-a(t-s)}-\frac{2}{a}-\frac{1}{2a}e^{-2a(t-s)}+\frac{1}{2a}\right) \\
&= \frac{\sigma_{spot}^2}{a^2}\left((t-s)-\frac{2}{a}\left(1-e^{-a(t-s)}\right)+\frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[I(s,t)|\mathcal{I}_s\right] = B(s,t)\,r_s + \int_s^t \alpha(u)-\alpha(s)e^{-a(u-s)}\,du \\
&= B(s,t)\,r_s + \ln\left(\frac{D^M(0,s)}{D^M(0,t)}\right) \\
&\quad + \frac{\sigma_{spot}^2}{2a^2}\left(t-s+\frac{2}{a}\left(e^{-2at}-e^{-2as}\right)-\frac{1}{2a}\left(e^{-2at}-e^{-2as}\right)\right)-\alpha(s)B(s,t) \\
&= B(s,t)\left(r_s-\alpha(s)\right)+\ln\left(\frac{D^M(0,s)}{D^M(0,t)}\right)+\frac{1}{2}\left(V(0,t)-V(0,s)\right).
\end{aligned}
$$

Using that for $X \sim \mathrm{N}\left(\mu_x,\sigma_x^2\right)$ we have $E\left[e^X\right] = e^{\mu_x+\frac{1}{2}\sigma_x^2}$ and

$$
\begin{aligned}
&E\left[\frac{1}{\beta_{s,t}}\middle|\mathcal{I}_s\right] = e^{E[-I(s,t)|\mathcal{I}_s]+\frac{1}{2}Var[I(s,t)|\mathcal{I}_s]} \\
&= \exp\left[-B(s,t)\left(r_s-\alpha(s)\right)+\ln\left(\frac{D^M(0,t)}{D^M(0,s)}\right)-\frac{1}{2}\left(V(0,t)-V(0,s)\right)\right. \\
&\qquad \left.+\frac{\sigma_{spot}^2}{2a^2}\left(t-s-2B(s,t)+\frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right)\right] \\
&= \exp\left[-B(s,t)r_s+\alpha(s)B(s,t)+\ln\left(\frac{D^M(0,t)}{D^M(0,s)}\right)-\frac{1}{2}\left(V(0,t)-V(0,s)\right)\right. \\
&\qquad \left.+\frac{\sigma_{spot}^2}{2a^2}\left(t-s-2B(s,t)+\frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right)\right] \\
&= e^{-B(s,t)r_s+A(s,t)}
\end{aligned}
$$

with

$$
\begin{aligned}
A(s,t) &= \alpha(s)B(s,t)+\ln\left(\frac{D^M(0,t)}{D^M(0,s)}\right)-\frac{1}{2}\left(V(0,t)-V(0,s)\right) \\
&\quad +\frac{\sigma_{spot}^2}{2a^2}\left(t-s-2B(s,t)+\frac{1}{2a}\left(1-e^{-2a(t-s)}\right)\right) \\
&= B(s,t)\,f^M(0,s)+\ln\left(\frac{D^M(0,t)}{D^M(0,s)}\right)-\frac{\sigma_{spot}^2}{4a}B^2(s,t)\left(1-e^{-2as}\right).
\end{aligned}
$$

(ii) In analogy to (i) it follows

$$
\begin{aligned}
&E\left[\frac{1}{\beta_{s,t}^2}\middle|\mathcal{I}_s\right] = E\left[e^{-2I(s,t)}\middle|\mathcal{I}_s\right] \\
&= e^{E[-2I(s,t)|\mathcal{I}_s]+\frac{1}{2}Var[2I(s,t)|\mathcal{I}_s]} = e^{2E[-I(s,t)|\mathcal{I}_s]+\frac{1}{2}Var[I(s,t)|\mathcal{I}_s]}\,e^{Var[I(s,t)|\mathcal{I}_s]} \\
&= \left(E\left[\frac{1}{\beta_{s,t}}\middle|\mathcal{I}_s\right]\right)^2 e^{Var[I(s,t)|\mathcal{I}_s]} = \left(D(s,t)\right)^2\,e^{Var[I(s,t)|\mathcal{I}_s]}.
\end{aligned}
$$

(iii) Finally we have (for $s < t < u$)

$$
\begin{aligned}
& E\left[e^{-I(s,t)}\, e^{-I(s,u)} \middle| \mathcal{I}_s\right] \\
&= e^{E[-I(s,t)|\mathcal{I}_s]+E[-I(s,u)|\mathcal{I}_s]}\, e^{\frac{1}{2}Var[I(s,t)+I(s,u)|\mathcal{I}_s]} \\
&= e^{E[-I(s,t)|\mathcal{I}_s]+E[-I(s,u)|\mathcal{I}_s]}\, e^{\frac{1}{2}Var[2I(s,t)+I(t,u)|\mathcal{I}_s]} \\
&= e^{E[-I(s,t)|\mathcal{I}_s]}\, e^{E[-I(s,u)|\mathcal{I}_s]}\, e^{\frac{1}{2}(3Var[I(s,t)|\mathcal{I}_s]+Var[I(s,u)|\mathcal{I}_s])} \\
&= e^{E[-I(s,t)|\mathcal{I}_s]}\, e^{E[-I(s,u)|\mathcal{I}_s]}\, e^{\frac{1}{2}Var[I(s,t)|\mathcal{I}_s]}\, e^{\frac{1}{2}Var[I(s,u)|\mathcal{I}_s]}\, e^{Var[I(s,t)|\mathcal{I}_s]} \\
&= E\left[e^{-I(s,t)} \middle| \mathcal{I}_s\right]\, E\left[e^{-I(s,u)} \middle| \mathcal{I}_s\right]\, e^{Var[I(s,t)|\mathcal{I}_s]} \\
&= D(s,t)\, D(s,u)\, e^{Var[I(s,t)|\mathcal{I}_s]}.
\end{aligned}
$$

□

The derivation of the time-dependent Drift Expression (B.2) implies a comparison of model and market discount rates. Due to the fundamental partial differential equation for zero coupon prices the functions $A(s,t)$ and $B(s,t)$ are solutions to the following differential equation system

$$
\begin{aligned}
\tfrac{d}{ds}A(s,t) + B(s,t)\,\theta(s) + \tfrac{1}{2}B^2(s,t)\,\sigma_{spot}^2 &= 0 \\
\tfrac{d}{ds}B(s,t) - a\,B(s,t) - 1 &= 0 \\
A(t,t) \;=\; B(t,t) &= 0.
\end{aligned}
$$

where the first equation forms a Riccati differential equation with closed-form solution in case of the Hull-White model. A simple calculation provides that

$$
\begin{aligned}
B(s,t) &= \tfrac{1}{a}\left(1 - e^{-a(t-s)}\right), \\
A(s,t) &= \int_s^t \theta(u)\, B(u,t)\, du + \tfrac{\sigma_{spot}^2}{2} \int_s^t B^2(u,t)\, du.
\end{aligned}
$$

Differentiating w.r.t. time t yields

$$
\begin{aligned}
\tfrac{d}{dt}B(0,t) &= e^{-at} \quad \text{and thus} \\
\tfrac{d}{dt}A(0,t) &= \tfrac{d}{dt}\left(\int_0^t \theta(u)\, B(u,t)\, du + \tfrac{\sigma_{spot}^2}{2}\int_0^t B^2(u,t)\, du\right) \\
&= \int_0^t \tfrac{d}{dt}\tfrac{\theta(u)}{a}\, du - \int_0^t \tfrac{d}{dt}\tfrac{\theta(u)}{a}e^{-a(t-u)}\, du + \tfrac{\sigma_{spot}^2}{2a^2}\int_0^t \tfrac{d}{dt}\left(1 - e^{-a(t-u)}\right)^2 du \\
&= \int_0^t \theta(u)e^{-a(t-u)}\, du + \tfrac{\sigma_{spot}^2}{a}\int_0^t \left(1 - e^{-a(t-u)}\right) e^{-a(t-u)}\, du
\end{aligned}
$$

$$= \int_0^t \theta(u)\, e^{-a(t-u)}\, du + \tfrac{\sigma_{spot}^2}{a^2} \left(\tfrac{1}{2} \left(1 - e^{-2at}\right) - \left(1 - e^{-at}\right)\right).$$

The initial yield curve is matched if and only if

$$\begin{aligned} f^M(0,t) &= -\tfrac{d}{dt} \ln \left(D^M(0,t)\right) \stackrel{!}{=} -\tfrac{d}{dt} A(0,t) + \tfrac{d}{dt} B(0,t)\, r_0 \\ &= -\int_0^t \theta(u)\, e^{-a(t-u)}\, du - \tfrac{\sigma_{spot}}{a^2} \left(\tfrac{1}{2} \left(1 - e^{-2at}\right) - \left(1 - e^{-at}\right)\right) + e^{-at}\, r_0. \end{aligned}$$

A derivation w.r.t. time t results in

$$\begin{aligned} \tfrac{d}{dt} f^M(0,t) &= \theta(t) + a \int_0^t \theta(u)\, e^{-a(t-u)}\, du - \tfrac{\sigma_{spot}}{a} \left(e^{-2at} - e^{-at}\right) - a\, e^{-at}\, r_0 \\ &= \theta(t) - a\, f^M(0,t) - \tfrac{\sigma_{spot}}{2a} \left(1 - e^{-2at}\right). \end{aligned} \tag{B.5}$$

Solving the Equation (B.5) for $\theta(t)$ gives expression (B.2).

B.2 Calculations for the mortality rate models

B.2.1 The non mean reverting affine mortality intensity jump process (Model (I))

LEMMA B.2.1. *For the non-mean reverting mortality rate jump process*

$$d\mu_{x+t}(t) = \kappa_1 \mu_{x+t}(t) dt + dJ(t), \quad (\kappa_1 > 0)$$

it holds

$$_{T-t}p_{x+t}^{(I)}(t) = \exp\left(A_1(T-t) + B_1(T-t)\, \mu_{x+t}(t)\right), \tag{B.6}$$

where

$$\begin{aligned} A_1(T-t) &= \frac{\lambda \varphi_1}{\eta_1 - \kappa_1} \left(-\kappa_1 (T-t) + \log\left[1 - \eta_1\, B(T-t)\right]\right) \\ &\quad + \frac{\lambda \varphi_2}{\eta_2 + \kappa_1} \left(\kappa_1 (T-t) - \log\left[1 + \eta_2\, B(T-t)\right]\right) - \lambda (T-t), \\ B_1(T-t) &= \frac{1}{\kappa_1} \left(1 - e^{\kappa_1 (T-t)}\right) \\ & \eta_1, \eta_2 > 0\,,\ \lambda \geq 0\,,\ -\frac{1}{\eta_1} < B_1(T-t) < \frac{1}{\eta_2}. \end{aligned}$$

The deterministic counterpart of Model (I) is defined by

$$\mu_x = -\lambda\left(\frac{\varphi_1}{1-\frac{\eta_1}{\kappa_1}\left(1-e^{\kappa_1 x}\right)} + \frac{\varphi_2}{1+\frac{\eta_2}{\kappa_1}\left(1-e^{\kappa_1 x}\right)} - 1\right) + \mu_0(0)\, e^{\kappa_1 x}.$$

Proof. For simplification we set $T-t=t$ and give a solution for the square-root diffusion process (7.4) with a double-exponential jump term under a compound Poisson jump process with counting intensity λ and jump size density

$$f(t) = \frac{\varphi_1}{\eta_1} e^{-\frac{t}{\eta_1}}\, \mathbb{1}_{\{t\geq 0\}} + \frac{\varphi_2}{\eta_2} e^{\frac{t}{\eta_2}}\, \mathbb{1}_{\{t<0\}}.$$

According to Duffie and Singleton (2003) the expression

$${}_tp_x^{(I)}(0) = E\left[e^{-\int_0^T \mu_{x+u}(u)du}\middle|\, \mathcal{M}_0\right]$$

has a solution $\exp\left(A_1(t)+B_1(t)\mu_x(0)\right)$ under certain technical conditions, where $A_1(t)$ and $B_1(t)$ follow generalized Ricatti ordinary differential equations

$$\frac{dA_1(t)}{dt} = \lambda\left(\Psi\left(B(t)\right)-1\right), \qquad A_1(0)=0 \tag{B.7}$$

$$\frac{dB_1(t)}{dt} = \kappa_1 B_1(t) - 1, \qquad B_1(0)=0 \tag{B.8}$$

$$\Psi(B_1(t)) = \int_{-\infty}^{\infty} e^{B_1(t)u} df(u) \qquad \Psi(B_1(0)) = 1. \tag{B.9}$$

The jump transform equation (B.9) is calculated as follows

$$\begin{aligned}
\Psi(B_1(t)) &= \int_{-\infty}^{\infty} e^{B_1(t)u} f(u)du \\
&= \int_0^{\infty} \frac{\varphi_1}{\eta_1} e^{-\frac{u}{\eta_1}} e^{B_1(t)u} du + \int_{-\infty}^{0} \frac{\varphi_2}{\eta_2} e^{\frac{u}{\eta_2}} e^{B_1(t)u} du \\
&= \frac{\varphi_1}{\eta_1} \int_0^{\infty} e^{-\left(\frac{1}{\eta_1}-B_1(t)\right)u} du + \frac{\varphi_2}{\eta_2} \int_{-\infty}^{0} e^{\left(\frac{1}{\eta_2}+B_1(t)\right)u} du \\
&= \frac{\varphi_1}{\eta_1}\left[-\frac{1}{\frac{1}{\eta_1}-B_1(t)} e^{-\left(\frac{1}{\eta_1}-B_1(t)\right)u}\right]_0^{\infty} + \frac{\varphi_2}{\eta_2}\left[\frac{1}{\frac{1}{\eta_2}+B_1(t)} e^{-\left(\frac{1}{\eta_2}+B_1(t)\right)u}\right]_{-\infty}^{0} \\
&= \frac{\varphi_1}{1-\eta_1 B_1(t)} + \frac{\varphi_2}{1+\eta_2 B_1(t)}
\end{aligned} \tag{B.10}$$

The solution to Equation (B.8) is $B_1(t) = \frac{1}{\kappa_1}\left(e^{-\kappa_1 t}+1\right)$. For Equation (B.7) we apply the Expression (B.10)

$$\frac{dA_1(t)}{dt} = \lambda\left(\frac{\varphi_1}{1-\eta_1 B_1(t)} + \frac{\varphi_2}{1+\eta_2 B_1(t)} - 1\right)$$

which has solution

$$\begin{aligned} A_1(t) = & \frac{\lambda\varphi_1}{\eta_1+\kappa_1}\left(\kappa_1 t + \log\left(1-\eta_1 B_1(t)\right)\right) \\ & + \frac{\lambda\varphi_2}{\eta_2+\kappa_1}\left(\kappa_1 t + \log\left(1+\eta_2 B_1(t)\right)\right) - \lambda t. \end{aligned}$$

For the deterministic force of mortality it holds

$$\begin{aligned} \mu_x &= \lim_{t\to 0}\frac{P(x<\tau_0^{(i)}\le x+t \,|\, \tau_0^{(i)}>x)}{t} = \lim_{t\to 0}\frac{1}{t}\left(1-\frac{{}_{x+t}p_0(0)}{{}_x p_0(0)}\right) \\ &= \frac{-\frac{d}{dx}{}_x p_0(0)}{{}_x p_0(0)} = -\frac{d}{dx}\log\left({}_x p_0(0)\right) = -\frac{d}{dx}\log\left(e^{A_1(x)+\mu_0(0)B_1(x)}\right) \\ &= -\frac{d}{dx}A_1(x) - \mu_0(0)\frac{d}{dx}B_1(x) \\ &= -\lambda\left(\frac{\varphi_1}{1-\frac{\eta_1}{\kappa_1}\left(1-e^{\kappa_1 x}\right)} + \frac{\varphi_2}{1+\frac{\eta_2}{\kappa_1}\left(1-e^{\kappa_1 x}\right)} - 1\right) + \mu_0(0)\, e^{\kappa_1 x}. \end{aligned}$$

□

B.2.2 The non mean-reverting square-root diffusion process (mortality Model (II))

LEMMA B.2.2. *For the non mean-reverting square-root diffusion process*

$$d\mu_{x+t}(t) = \kappa_2\mu_{x+t}(t)dt + \sigma^{\mu}\sqrt{\mu_{x+t}(t)}dW_t^{\mu}, \quad (\kappa_2>0, \sigma^{\mu}\ge 0) \tag{B.11}$$

it holds

$${}_{T-t}p_{x+t}^{(II)}(t) = \exp\left(A_2(T-t) + B_2(T-t)\,\mu_{x+t}(t)\right),$$

where

$$\begin{aligned} & A_2(T-t) = 0, && B_2(T-t) = \frac{1-e^{b^{\mu}(T-t)}}{c^{\mu}+d^{\mu}\, e^{b^{\mu}(T-t)}}, \\ & b^{\mu} = -\sqrt{\kappa_2^2 + 2\left(\sigma^{\mu}\right)^2}, && c^{\mu} = \frac{b^{\mu}+\kappa_2}{2}, \quad d^{\mu} = \frac{b^{\mu}-\kappa_2}{2}, \\ & b^{\mu},\, c^{\mu},\, d^{\mu} < 0 && \textit{and} \quad \eta - c^{\mu} - (d^{\mu}+\eta)e^{b^{\mu}(T-t)} > 0. \end{aligned}$$

The deterministic counterpart of Model (II) is defined by

$$\mu_x = \frac{4\mu_0(0)(b^\mu)^2\, e^{b^\mu x}}{((\kappa_2 + b^\mu)(b^\mu - \kappa_2)e^{b^\mu x})^2}.$$

Proof. Analogous to Lemma B.2.1 and due to the affine form of (B.11) the coupled set of Ricatti equations is given by

$$\frac{dA_2(t)}{dt} = 0, \qquad A_1(0) = 0 \tag{B.12}$$

$$\frac{dB_2(t)}{dt} = \kappa_2 B_2(t) + \frac{1}{2}\sigma^\mu B_2(t)^2 - 1, \qquad B_1(0) = 0 \tag{B.13}$$

with solutions

$$A_2(t) = 0, \quad B_2(t) = \frac{1 - e^{b^\mu t}}{c^\mu + d^\mu\, e^{b^\mu t}}.$$

For the deterministic force of mortality it holds

$$\begin{aligned}\mu_x &= \frac{-\frac{d}{dx}\,{}_xp_0(0)}{{}_xp_0(0)} = -\frac{d}{dx}\log\left({}_xp_0(0)\right) = -\frac{d}{dx}\log\left(e^{A_1(x)+\mu_0(0)B_1(x)}\right)\\ &= -\frac{d}{dx}A_2(x) - \mu_0(0)\frac{d}{dx}B_2(x) = \frac{4\mu_0(0)(b^\mu)^2\, e^{b^\mu x}}{((\kappa_2 + b^\mu)(b^\mu - \kappa_2)e^{b^\mu x})^2}.\end{aligned}$$

□

B.2.3 Sensitivity analysis for the survival function of the mortality Models (I) and (II)

Proposition B.2.3. *W.l.o.g. we assume $s = 0$. The survival function ${}_tp_x(0)$ $= P(\tau_x \geq t)$*

(i) has negative directional derivative with regard to the speed factors κ_i ($i =$ 1,2), i.e.

$$\frac{\partial\, {}_tp_x(0)}{\partial \kappa_i} = {}_tp_x(0)\left(\frac{\partial A_i(t)}{\partial \kappa_i} - \left(\frac{\partial B_i(t)}{\partial \kappa_i}\cdot \mu_x(0) + B_i(t)\frac{\partial \mu_x(0)}{\partial \kappa_i}\right)\right) < 0.$$

(ii) has positive directional derivatives with regard to the "noise" factors λ, σ^μ, i.e.

$$\frac{\partial\, {}_tp_x(0)}{\partial \lambda} = {}_tp_x(0)\left(\frac{\partial A_i(t)}{\partial \lambda} - \left(\frac{\partial B_i(t)}{\partial \lambda}\cdot \mu_x(0) + B_i(t)\frac{\partial \mu_x(0)}{\partial \lambda}\right)\right) > 0,$$

$$\frac{\partial\, {}_tp_x(0)}{\partial \sigma^\mu} = {}_tp_x(0)\left(\frac{\partial A_i(t)}{\partial \sigma^\mu} - \left(\frac{\partial B_i(t)}{\partial \sigma^\mu}\cdot \mu_x(0) + B_i(t)\frac{\partial \mu_x(0)}{\partial \sigma^\mu}\right)\right) > 0$$

For a reasonable choice of the model parameters we also have

$$\frac{\partial\, {}_t\tilde{p}_x(0)}{\partial \kappa_i} < 0 \quad \left(\frac{\partial\, {}_t\tilde{p}_x(0)}{\partial \lambda} > 0 \;, \;\; \frac{\partial\, {}_t\tilde{p}_x(0)}{\partial \sigma^\mu} > 0 \right)$$

such that for $PV_0 \in \{X^{\Pi}, Z_0^{\Pi}\}$

$$\frac{\partial Var\left[PV_0\right]}{\partial \kappa_i} < 0 \quad \left(\frac{\partial Var\left[PV_0\right]}{\partial \lambda} > 0, \; \frac{\partial Var\left[PV_0\right]}{\partial \sigma^\mu} > 0 \right).$$

Proposition B.2.3 is left unproven since the derivation of the functions $A_i(t)$ and $B_i(t)$ is complex. An illustration of the numerical directional derivatives is given in Figure B.1.

Figure B.1 illustrates that an increasing speed factor κ_i induces a decrease in the survival probabilities and therefore a reduced portfolio value variance. On the other hand, an increase in the “noise” factors like the jump intensity λ or the diffusion coefficient σ^μ causes an increase in the multi-annual survival probabilities and consequently an upward shift in the present portfolio variance values.

B.3 Calculations for the expected discounted portfolio values

B.3.1 Moment calculation for the number of survivors

W.l.o.g. we assume $t = 0$ for simplification reasons. Due to the definition of (7.2) we obtain

$$E\left[N_i\right] = E\left[\sum_{n=1}^{N_0} 1_{\{\tau_x^{(n)} > i\}}\right]$$

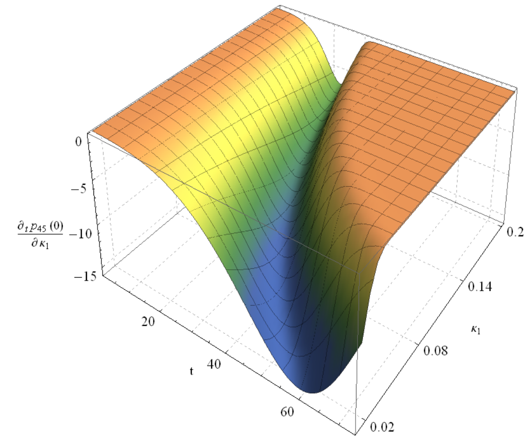
and

$$\begin{aligned} E\left[(N_i)^2\right] &= E\left[\left(\sum_{n=1}^{N_0} 1_{\{\tau_x^{(n)} > i\}}\right)^2\right] \\ &= \sum_{n=1}^{N_0} E\left[1_{\{\tau_x^{(n)} > i\}}\right] + 2\sum_{n=1}^{N_0-1}\sum_{k=n+1}^{N_0} E\left[1_{\{\tau_x^{(n)} > i\}} 1_{\{\tau_x^{(k)} > i\}}\right] \end{aligned}$$

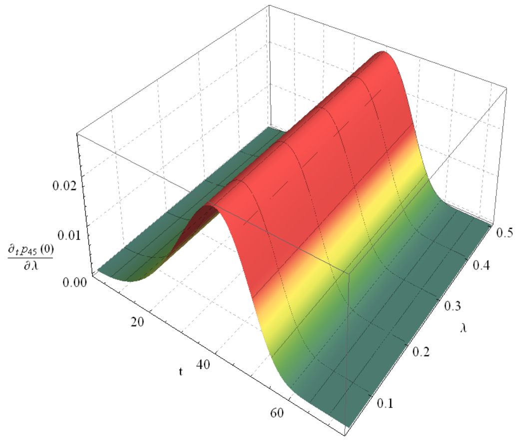
for times $i = 0,...,\bar{T}$. First, consider that for $k, n \in \{1,...,N_0\}$

$$E\left[1_{\{\tau_x^{(n)} > i\}}\right] = E\left[E\left[1_{\{\tau_x^{(n)} > i\}} \middle| \mathcal{M}_i\right]\right] = E\left[{}_i\tilde{p}_x(0)\right] = {}_ip_x(0).$$

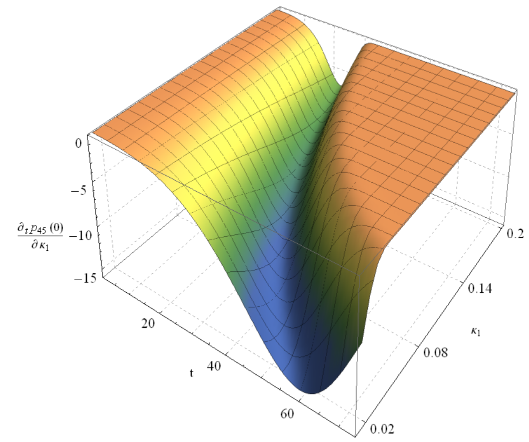
Sensitivity analysis of the survival function



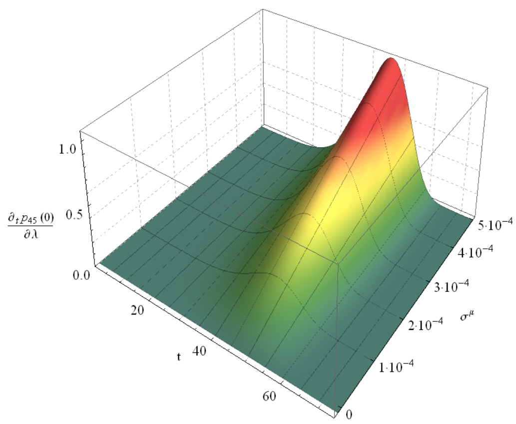
(B.1.1) speed factor κ_1 (Model (I))



(B.1.2) jump intensity λ (Model (I))



(B.1.3) speed factor κ_2 (Model (II))



(B.1.4) diffusion coefficient σ^μ (Model (II))

Figure B.1: Sensitivity analysis of the survival function.

Now consider the cross moments

$$E\left[1_{\{\tau_x^{(n)}>i\}}1_{\{\tau_x^{(k)}>i\}}\right] = E\left[E\left[1_{\{\tau_x^{(n)}>i\}}1_{\{\tau_x^{(k)}>i\}}\middle|\, \mathcal{M}_i\right]\right] = E\left[\left({}_i\tilde{p}_x(0)\right)^2\right].$$

Using the previous results, we have

$$\begin{aligned}
E\left[N_i\right] &= N_0\ {}_ip_x(0)\\
E\left[(N_i)^2\right] &= N_0\ {}_ip_x(0) + N_0\ (N_0-1)\ E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\\
&= N_0\ {}_ip_x(0) + N_0\ (N_0-1)\ \left({}_ip_x(0)\right)^2\ E\left[\left(\frac{{}_i\tilde{p}_x(0)}{{}_ip_x(0)}\right)^2\right]\\
&= N_0\ {}_ip_x(0)\left(1 - {}_ip_x(0)\ E\left[\left(\frac{{}_i\tilde{p}_x(0)}{{}_ip_x(0)}\right)^2\right]\right) + N_0^2\ E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\\
Var\left[N_i\right] &= E\left[(N_i)^2\right] - \left(E\left[N_i\right]\right)^2
\end{aligned}$$

$$= N_0 \; {}_ip_x(0) \left(1 - {}_ip_x(0) \; E\left[\left(\frac{{}_i\tilde{p}_x(0)}{{}_ip_x(0)}\right)^2\right]\right)$$
$$+N_0^2 \; \left(E\left[\left({}_i\tilde{p}_x(0)\right)^2\right] - \left({}_ip_x(0)\right)^2\right).$$

Notice that in case of deterministic mortality, we have $\tilde{p} = p$ and thus

$$\begin{aligned} E\left[(N_i)^2\right] &= N_0 \; {}_ip_x(0) + N_0^2 \; \left({}_ip_x(0)\right)^2 - N_0 \; \left({}_ip_x(0)\right)^2 \\ &= N_0 \; {}_ip_x(0) \; (1 - {}_ip_x(0)) + N_0^2 \; \left({}_ip_x(0)\right)^2. \end{aligned}$$

Furthermore, consider (for $k = 0,...,\bar{T}$)

$$\begin{aligned} E\left[N_i \; N_k\right] &= E\left[\sum_{n=1}^{N_0} 1_{\left\{\tau_x^{(n)}>i\right\}} \; \sum_{l=1}^{N_0} 1_{\left\{\tau_x^{(l)}>k\right\}}\right] \\ &= E\left[E\left[\sum_{n=1}^{N_0} 1_{\left\{\tau_x^{(n)}>i\right\}} \; \sum_{l=1}^{N_0} 1_{\left\{\tau_x^{(l)}>k\right\}} \middle| \; \mathcal{M}_{\bar{T}}\right]\right] \\ &= E\left[E\left[\sum_{n=1}^{N_0} 1_{\left\{\tau_x^{(n)}>\max\{i,k\}\right\}} + \sum_{n=1}^{N_0}\sum_{l\neq n}^{N_0} 1_{\left\{\tau_x^{(n)}>i\right\}} 1_{\left\{\tau_x^{(l)}>k\right\}} \middle| \mathcal{M}_{\bar{T}}\right]\right] \\ &= N_0 \; {}_{\max\{i,k\}}p_x(0) + E\left[\sum_{n=1}^{N_0}\sum_{l\neq n}^{N_0} \; E\left[1_{\left\{\tau_x^{(n)}>i\right\}} 1_{\left\{\tau_x^{(l)}>k\right\}} \middle| \mathcal{M}_{\bar{T}}\right]\right] \\ &= N_0 \; {}_{\max\{i,k\}}p_x(0) + E\left[N_0(N_0-1) \; {}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)\right] \\ &= N_0 \; {}_{\max\{i,k\}}p_x(0) + N_0^2 \; E\left[{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)\right] - N_0 \; E\left[{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)\right] \\ &= N_0 \; {}_kp_x(0) - N_0 \; {}_kp_x(0) \; E\left[\frac{{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)}{{}_kp_x(0)}\right] + N_0^2 \; E\left[{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)\right] \\ &= N_0 \; {}_kp_x(0) \left(1 - E\left[\frac{{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)}{{}_kp_x(0)}\right] + N_0^2 \; E\left[{}_i\tilde{p}_x(0) \; {}_k\tilde{p}_x(0)\right]\right). \end{aligned}$$

B.3.2 Moment calculation for the expected discounted portfolio values

W.l.o.g. we assume that the dynamics of the zero bond is given by Equation (7.6). Recall that with the binomial theorem over multiple addends and the independence of spot and mortality rate evolution we calculate the first moments for the present portfolio premium income and benefit value. Therefore, we obtain

$$E[X_0^{\Pi}] = E\left[\sum_{i=0}^{T-1} \pi_i \frac{N_i}{\beta_{0,i}}\right] = \sum_{i=0}^{T-1} \pi_i E\left[\frac{1}{\beta_{0,i}}\right] E\left[N_i\right] = \sum_{i=0}^{T-1} \pi_i N_0 \; D(0,i) \; {}_ip_x(0), \tag{B.14}$$

$$E\left[\left(X_0^{\Pi}\right)^2\right] = E\left[\left(\sum_{i=0}^{T-1}\pi_i\frac{N_i}{\beta_{0,i}}\right)^2\right]$$
$$= \sum_{i=0}^{T-1}\pi_i^2 E\left[N_i^2\right] E\left[\frac{1}{\beta_{0,i}^2}\right] + 2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2 E\left[N_i\, N_k\right] E\left[\frac{1}{\beta_{0,i}\,\beta_{0,k}}\right] \tag{B.15}$$
$$= \sum_{i=0}^{T-1}\pi_i^2 E\left[N_i^2\right]\; E\left[\frac{1}{\beta_{0,i}^2}\right] + 2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2 E\left[N_i\, N_k\right] D(0,i)\, D(0,k)\, e^{Var[I(0,i)]} \tag{B.16}$$

and

$$E[Z_0^{\Pi}] = E\left[c\sum_{i=T}^{\bar{T}-1}\frac{N_i}{\beta_{0,i}}\right] = c\sum_{i=T}^{\bar{T}-1}E\left[\frac{1}{\beta_{0,i}}\right]\; E\left[N_i\right] = cN_0\sum_{i=T}^{\bar{T}-1}D(0,i)\,{}_ip_x(0), \tag{B.17}$$

$$E\left[\left(Z_0^{\Pi}\right)^2\right] = E\left[\left(c\sum_{i=T}^{\bar{T}-1}\frac{N_i}{\beta_{0,i}}\right)^2\right]$$
$$= c^2\sum_{i=T}^{\bar{T}-1}E\left[N_i^2\right] E\left[\frac{1}{\beta_{0,i}^2}\right] + 2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1}E\left[N_iN_k\right] E\left[\frac{1}{\beta_{0,i}\,\beta_{0,k}}\right] \tag{B.18}$$
$$= c^2\sum_{i=T}^{\bar{T}-1}E\left[N_i^2\right] E\left[\frac{1}{\beta_{0,i}^2}\right] + 2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1}E\left[N_iN_k\right] D(0,i)\, D(0,k)\, e^{Var[I(0,i)]}. \tag{B.19}$$

Using (B.14) and (B.16), the variance becomes

$$Var\left[X_0^{\Pi}\right] = E\left[\left(\sum_{i=0}^{T-1}\pi_i\frac{N_i}{\beta_{0,i}}\right)^2\right] - \left(E\left[\sum_{i=0}^{T-1}\pi_i\frac{N_i}{\beta_{0,i}}\right]\right)^2$$
$$= \sum_{i=0}^{T-1}\pi_i^2 E\left[N_i^2\right]\; E\left[\tfrac{1}{\beta_{0,i}^2}\right] + 2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2 E\left[N_i\, N_k\right]\; E\left[\tfrac{1}{\beta_{0,i}\,\beta_{0,k}}\right]$$
$$-\sum_{i=0}^{T-1}\pi_i^2\left(E\left[N_i\right]\; E\left[\tfrac{1}{\beta_{0,i}}\right]\right)^2 - 2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2 E\left[N_i\right] E\left[N_k\right] E\left[\tfrac{1}{\beta_{0,i}}\right] E\left[\tfrac{1}{\beta_{0,k}}\right]$$
$$= \sum_{i=0}^{T-1}\pi_i^2\left(N_0\,{}_ip_x(0)\left(1 - {}_ip_x(0)\, E\left[\left(\frac{{}_i\tilde{p}_x(0)}{{}_ip_x(0)}\right)\right]\right) + N_0^2\; E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\right)$$
$$\cdot\left(D(0,i)\right)^2\; e^{Var[I(0,i)]}$$
$$+\;\; 2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2\left(N_0\,{}_ip_x(0)\left(1 - E\left[\frac{{}_i\tilde{p}_x(0)\,{}_k\tilde{p}_x(0)}{{}_kp_x(0)}\right]\right) + N_0^2\; E\left[{}_i\tilde{p}_x(0)\,{}_k\tilde{p}_x(0)\right]\right)$$
$$\cdot D(0,i)\, D(0,k)\, e^{Var[I(0,i)]} - N_0^2\sum_{i=0}^{T-1}\left({}_ip_x(0)\, D(0,i)\right)^2$$

$$- \quad 2N_0^2 \sum_{i=0}^{T-2} \sum_{k=i+1}^{T-1} \pi_i^2 \, {}_ip_x(0) {}_kp_x(0) D(0,i) D(0,k). \tag{B.20}$$

In the same way, the present benefit variance follows from (B.17) and (B.19)

$$\begin{aligned}
Var\left[Z_0^{\Pi}\right] &= E\left[\left(c\sum_{i=T}^{\bar{T}-1}\frac{N_i}{\beta_{0,i}}\right)^2\right] - \left(E\left[c\sum_{i=T}^{\bar{T}-1}\frac{N_i}{\beta_{0,i}}\right]\right)^2 \\
&= c^2\sum_{i=T}^{\bar{T}-1} E\left[N_i^2\right]\ E\left[\tfrac{1}{\beta_{0,i}^2}\right] + 2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1} E\left[N_i\,N_k\right]\ E\left[\tfrac{1}{\beta_{0,i}\,\beta_{0,k}}\right] \\
&\quad -c^2\sum_{i=T}^{\bar{T}-1}\left(E\left[N_i\right]\ E\left[\tfrac{1}{\beta_{0,i}}\right]\right)^2 - 2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1} E\left[N_i\right] E\left[N_k\right] E\left[\tfrac{1}{\beta_{0,i}}\right] E\left[\tfrac{1}{\beta_{0,k}}\right] \\
&= c^2\sum_{i=T}^{\bar{T}-1}\left(N_0\,{}_ip_x(0)\left(1 - {}_ip_x(0)\,E\left[\left(\frac{{}_i\tilde{p}_x(0)}{{}_ip_x(0)}\right)\right]\right) + N_0^2\ E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\right) \\
&\quad \cdot\left(D(0,i)\right)^2\ e^{Var[I(0,i)]} \\
&\quad + \ 2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1}\left(N_0\,{}_ip_x(0)\left(1 - E\left[\frac{{}_i\tilde{p}_x(0)_k\tilde{p}_x(0)}{{}_kp_x(0)}\right]\right) + N_0^2\ E\left[{}_i\tilde{p}_x(0)_k\tilde{p}_x(0)\right]\right) \\
&\qquad \cdot D(0,i)\,D(0,k)\,e^{Var[I(0,i)]} - N_0^2\sum_{i=T}^{\bar{T}-1}\left({}_ip_x(0)\,D(0,i)\right)^2 \\
&\quad - \ 2c^2N_0^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1} {}_ip_x(0)_kp_x(0)D(0,i)D(0,k).
\end{aligned} \tag{B.21}$$

B.3.3 Systematic variance per policy

By means of expressions (B.20) and (B.21) we calculate the systematic risk per policy for the portfolio premium income

$$\begin{aligned}
\lim_{N_0\to\infty}\frac{Var\left[X_0^{\Pi}\right]}{N_0^2} &= \sum_{i=0}^{T-1}\pi_i^2\left(D(0,i)\right)^2\left(E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\ e^{Var[I(0,i)]} - \left({}_ip_x(0)\right)^2\right) \\
&\quad +2\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\pi_i^2 D(0,i)D(0,k) \\
&\qquad \cdot\left(E\left[{}_i\tilde{p}_x(0)_k\tilde{p}_x(0)\right]\ e^{Var[I(0,i)]} - {}_ip_x(0)_kp_x(0)\right)
\end{aligned} \tag{B.22}$$

and benefit value

$$\begin{aligned}
\lim_{N_0\to\infty}\frac{Var\left[Z_0^{\Pi}\right]}{N_0^2} &= c^2\sum_{i=T}^{\bar{T}-1}\left(D(0,i)\right)^2\left(E\left[\left({}_i\tilde{p}_x(0)\right)^2\right]\ e^{Var[I(0,i)]} - \left({}_ip_x(0)\right)^2\right) \\
&\quad +2c^2\sum_{i=T}^{\bar{T}-2}\sum_{k=i+1}^{\bar{T}-1} D(0,i)D(0,k)
\end{aligned}$$

$$\cdot \Big(E\left[{}_i\tilde{p}_x(0)_k\tilde{p}_x(0)\right] e^{Var[I(0,i)]} - {}_ip_x(0)_kp_x(0) \Big) \qquad \text{(B.23)}$$

which is positive for $Var[I(0,i)] > 0$ ($i = 0,...,\bar{T}$) and/or $\tilde{p} \neq p$, i.e. any uncertainty arising from mortality and/or interest rates.

B.4 Sensitivity analysis for the present portfolio benefit variance

In particular, Subsection B.3.2 validates the intuition that the variance of Z_s^{Π} is increasing in the interest rate volatility as $Var[I(s,t)|\mathcal{I}_s]$ is. A verification for this assumption is given in the following lemma.

PROPOSITION B.4.1. *W.l.o.g. we assume $s = 0$. Standard derivative calculus yields*

$$\frac{\partial Var\left[I(0,t)\right]}{\partial \sigma_{spot}} = \frac{2\sigma_{spot}}{a^2}\left(t + \frac{2}{a}e^{-at} - \frac{1}{2a}e^{-2at} - \frac{3}{2a}\right) = \frac{2}{\sigma_{spot}} Var\left[I(0,t)\right] > 0 \qquad \text{(B.24)}$$

$$\frac{\partial Var\left[I(0,t)\right]}{\partial a} = -\frac{\sigma_{spot}^2}{2a^4}e^{-2at}\left(e^{2at}(-9+4at) + 4e^{at}(3+at) - 3 - 2at\right) < 0 \qquad \text{(B.25)}$$

whereas the latter inequality sufficiently holds in parameter cases $a < \frac{3}{2t}$ and $t \leq \omega + 1$.

LEMMA B.4.2. *The directional derivatives of the present portfolio benefit variance $Var\left[Z_0^{\Pi}\right]$ are calculated as*

$$\frac{\partial Var\left[Z_0^{\Pi}\right]}{\partial \sigma_{spot}} = c^2 \sum_{i=T}^{\bar{T}-1} E\left[(N_i)^2\right] (D(0,i))^2 \, e^{Var[I(0,i)]} \, \frac{\partial Var\left[I(0,i)\right]}{\partial \sigma_{spot}}$$
$$+ \quad 2c^2 \sum_{i=T}^{\bar{T}-2} \sum_{k=i+1}^{\bar{T}-1} E\left[N_i N_k\right] D(0,i)\, D(0,k)\, e^{Var[I(0,i)]} \, \frac{\partial Var\left[I(0,i)\right]}{\partial \sigma} > 0 \qquad \text{(B.26)}$$

as well as

$$\frac{\partial Var\left[Z_0^{\Pi}\right]}{\partial a} = c^2 \sum_{i=T}^{\bar{T}-1} E\left[(N_i)^2\right] (D(0,i))^2 \, e^{Var[I(0,i)]} \, \frac{\partial Var\left[I(0,i)\right]}{\partial a}$$
$$+ \quad 2c^2 \sum_{i=T}^{\bar{T}-2} \sum_{k=i+1}^{\bar{T}-1} E\left[N_i N_k\right] D(0,i)\, D(0,k)\, e^{Var[I(0,i)]} \, \frac{\partial Var\left[I(0,i)\right]}{\partial a} < 0. \qquad \text{(B.27)}$$

Expressions for directional derivatives of the present portfolio premium income value X_0^{Π} are obtained in a completely analogous way.

Proof. (B.26) is a conclusion of (B.24), (B.25) and Subsection (B.3.2). □

Proposition B.4.1 is quite intuitive in the way that the greater interest rate diffusion the greater deviations of the interest rate integral $I(0,.)$ turn out. According to Subsections B.3.2 and B.3.3 the impact on the portfolio values is therefore noticeable. On the other hand, the greater the speed of mean reversion a (for fixed volatility σ_{spot}) the lesser the portfolio value variance since the mean reversion revel is reached comparatively earlier. Figure B.2 highlights these relations and makes the chosen interest Model (7.8) appear meaningful according to the criteria catalogue of Chapter 4. A sensitivity analysis concerning the two distinct mortality Models (I) and (II) is given in Subsection B.2.3.

Sensitivity analysis of the portfolio benefit variance

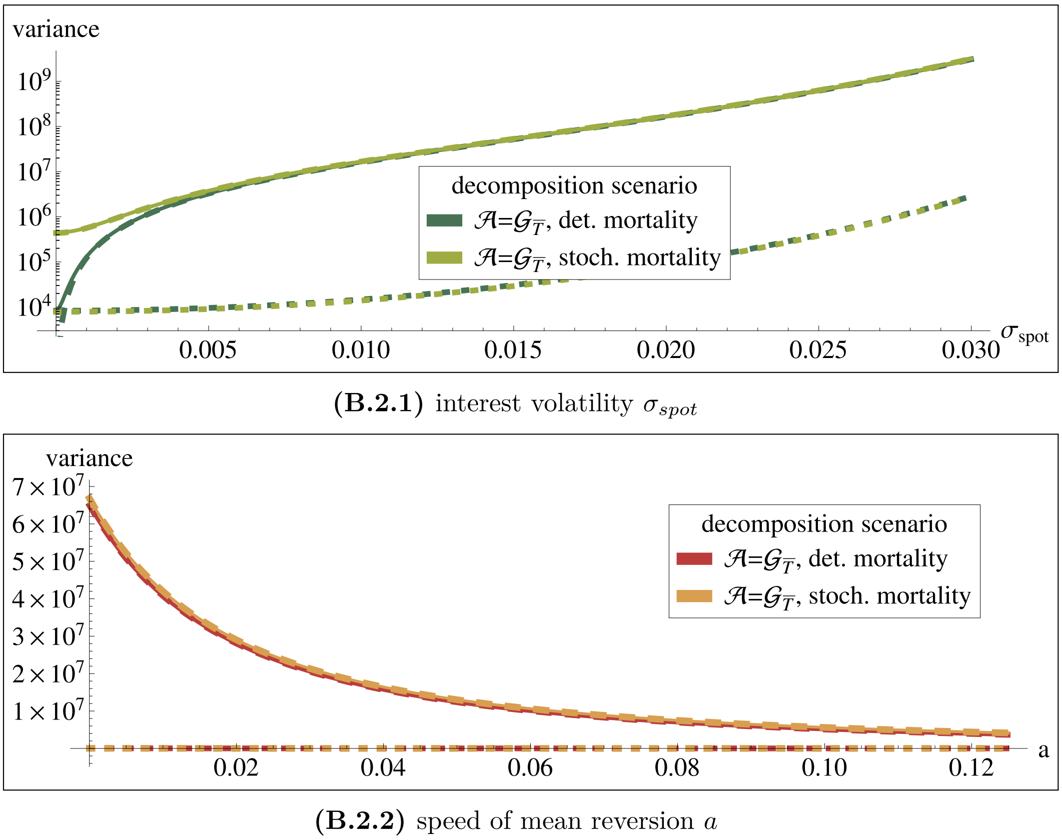


Figure B.2: Sensitivity analysis of the portfolio benefit variance parts w.r.t. mortality Model (II), a portfolio of $N_0 = 100$ individuals aged $x = 45$ years and a contract deferment period $T = 15$. In accordance with (7.17) the total variance (solid) equals the pooling variance part (dotted) and the non–pooling variance part (dashed). The simulations are based on $5 \cdot 10^5$ iterations.

APPENDIX to Chapter 8

C.1 Calculations for the non mean reverting Brownian Gompertz-Makeham mortality rate model

C.1.1 Proof of the normal distribution assumption concerning the stochastic mortality reduction factor process

Since Y_t forms an Itô integral we can define it as a limiting Riemann sum

$$Y_t = \int_0^t e^{-\kappa u} dW_u^\mu = \lim_{n\to\infty} \sum_{i=1}^n e^{-\kappa t_i} \left(W_{t_i}^\mu - W_{t_{i-1}}^\mu \right)$$

for $t_0 = 0 < t_1 < ... < t_{n-1} < t_n = t$ partitioned into n subintervals. Since for a Brownian motion W^μ the difference $W_t^\mu - W_s^\mu$ is normally $N(0,t-s)$-distributed for $0 \leq s \leq t$ it follows

$$\begin{aligned} E\left[Y_t\right] &= E\left[\int_0^t e^{-\kappa u} dW_u^\mu\right] = \lim_{n\to\infty} \sum_{i=1}^n e^{-\kappa t_i} E\left[W_{t_i}^\mu - W_{t_{i-1}}^\mu\right] = 0, \\ E\left[(Y_t)^2\right] &= E\left[\left(\int_0^t e^{-\kappa u} dW_u^\mu\right)^2\right] = E\left[\int_0^t e^{-2\kappa u} du\right] \\ &= \left[-\frac{1}{2\kappa} e^{-2\kappa u}\right]_0^t = \frac{1 - e^{-2\kappa t}}{2\kappa} \end{aligned}$$

where the second moment results from Itô isometry. Furthermore, for the logarithmised mortality rate (8.14) it holds that

$$\begin{aligned} E\left[\ln\left(\mu_{x+t}(t)\right)\right] &= E\left[\ln\left(\mu_{x+t}(0)\right) + \left(\alpha + \beta(x+t)\right) t + \gamma Y_t\right] \\ &= \ln\left(\mu_{x+t}(0)\right) + \left(\alpha + \beta(x+t)\right) t \\ Var\left[\ln\left(\mu_{x+t}(t)\right)\right] &= Var\left[\ln\left(\mu_{x+t}(0)\right) + \left(\alpha + \beta(x+t)\right) t + \gamma Y_t\right] = Var\left[\gamma Y_t\right] \\ &= \frac{\gamma^2}{2\kappa} \left(1 - e^{-2\kappa t}\right). \end{aligned}$$

□

C.1.2 Proof of Proposition 8.3.1

Along the lines of Milevsky and Promislow (2001) and due to Itô's lemma we obtain

$$\begin{aligned} d\mu_{x+t}(t) &= \frac{\partial \mu_{x+t}(t)}{\partial t}dt + \frac{\partial \mu_{x+t}(t)}{\partial Y_t}dY_t + \frac{1}{2}\frac{\partial^2 \mu_{x+t}(t)}{(\partial Y_t)^2}d\left\langle Y\right\rangle_t \\ &= \left(\alpha + \beta(x+2t)\right)\mu_{x+t}(t)\,dt + \gamma\mu_{x+t}(t)\,dY_t + \tfrac{1}{2}\gamma^2\mu_{x+t}(t)\,dt \\ &= \left(\alpha + \beta(x+2t)\right)\mu_{x+t}(t)\,dt + \gamma\mu_{x+t}(t)\left(-\kappa Y_t\,dt + dW_t^\mu\right) + \tfrac{1}{2}\gamma^2\mu_{x+t}(t)\,dt \\ &= \left(\alpha + \beta(x+2t)\right)\mu_{x+t}(t)\,dt \\ &\quad + \gamma\mu_{x+t}(t)\left(-\frac{\kappa}{\gamma}\left(\ln\left(\frac{\mu_{x+t}(t)}{\mu_{x+t}(0)}\right) - (\alpha+\beta(x+t)t)\right)dt + dW_t^\mu\right) \\ &\quad + \tfrac{1}{2}\gamma^2\mu_{x+t}(t)\,dt \\ &= \Big(\alpha + \beta(x+2t) + \tfrac{1}{2}\gamma^2 + \kappa\ln\left(\mu_{x+t}(0)\right) + \kappa(\alpha+\beta(x+t))t\Big)\,\mu_{x+t}(t)\,dt \\ &\quad - \kappa\ln\left(\mu_{x+t}(t)\right)\mu_{x+t}(t)\,dt + \gamma\,\mu_{x+t}(t)\,dW_t^\mu \\ &=: \zeta(\mu_{x+t}(t),t)\,\mu_{x+t}(t)\,dt + \gamma\,\mu_{x+t}(t)\,dW_t^\mu \end{aligned}$$

which proofs the drift expression of Proposition 8.3.1. □

C.2 Existence and uniqueness of the fair percentage charge

C.2.1 Proof of the fund value Equation (8.1)

By means of mathematical induction we show that for $i = 1,\dots,N-1$

$$A(t_i,\pi,\varphi) := \sum_{j=0}^{i-1}\pi_{t_j}^I\,\frac{S_{t_i}}{S_{t_j}} = \sum_{j=0}^{i-1}(1-\varphi)^{i-j}\,\pi_{t_j}^N\,\frac{S_{t_i}}{S_{t_j}} \tag{C.1}$$

for investment premiums $\pi_{t_j}^I = (1-\varphi)\pi_{t_j}^N - \varphi\cdot A(t_j,\pi,\varphi)$.

For $i=1$ Equation (C.1) holds since

$$\sum_{j=0}^{1-1}\pi_{t_j}^I\frac{S_{t_1}}{S_{t_j}} = (1-\varphi)\pi_{t_0}^N\frac{S_{t_1}}{S_{t_0}} = \sum_{j=0}^{1-1}(1-\varphi)^{1-j}\pi_{t_j}^N\frac{S_{t_1}}{S_{t_j}}$$

which constitutes the basic step. Assume that (C.1) holds for arbitrary $i-1 \leq N-2$, i.e.

$$\sum_{j=0}^{i-2}\pi_{t_j}^I\,\frac{S_{t_i}}{S_{t_j}} = \sum_{j=0}^{i-2}(1-\varphi)^{i-1-j}\,\pi_{t_j}^N\,\frac{S_{t_i}}{S_{t_j}}.$$

Hence, for the inductive step it follows

$$\begin{aligned}\sum_{j=0}^{i-1}\pi^I_{t_j}\frac{S_{t_i}}{S_{t_j}} &= \pi^I_{t_{i-1}}\frac{S_{t_i}}{S_{t_{i-1}}}+\sum_{j=0}^{i-2}\pi^I_{t_j}\frac{S_{t_i}}{S_{t_j}}\\ &= (1-\varphi)\pi^N_{t_{i-1}}\frac{S_{t_i}}{S_{t_{i-1}}}+(1-\varphi)\sum_{j=0}^{i-2}(1-\varphi)^{i-1-j}\pi^N_{t_j}\frac{S_{t_i}}{S_{t_j}}\\ &= \sum_{j=0}^{i-1}(1-\varphi)^{i-j}\pi^N_{t_j}\frac{S_{t_i}}{S_{t_j}}\end{aligned}$$

which completes induction. □

C.2.2 Proof of Proposition 8.4.5

Let η denote the repayment level and assume that the guarantee levels g_D and g_I are set to zero such that the call options (8.2) and (8.6) reduce to pure unit-linked benefits. Then the fair charge φ^* is a solution to

$$\begin{aligned}&EDPV^{RF}(\varphi)=\sum_{i=0}^{N-1}\varphi\sum_{j=0}^{i}(1-\varphi)^{i-j}\,\pi^N_{t_j}\;D(0,t_j)\;{}_{t_j-t_i}p_{x+t_i}(t_i)\\ &\overset{!}{=} EDPV^{GMDB}_\eta(\varphi)+EDPV^{GMIB}_\eta(\varphi)\\ &=\eta\int_0^T D(0,u)\,E_{P^u}\left[\sum_{j=0}^{n^*(u)}(1-\varphi)^{i-j-1}\pi^N_{t_j}\frac{S_u}{S_{t_j}}\middle|\mathcal{I}_0\right]\tfrac{d}{du}({}_uq_x(0))\;du\\ &\quad+\eta\,D(0,T)\,E_{P^{t_i}}\left[\sum_{j=0}^{N-1}(1-\varphi)^{N-j-1}\pi^N_{t_j}\frac{S_T}{S_{t_j}}\middle|\mathcal{I}_0\right]{}_Tp_x(0)\\ &\quad-\int_0^T D(0,u)\sum_{i=0}^{n^*(u)}\pi^N_{t_i}\,E_{P^u}\left[\frac{S_u}{S_{t_i}}\middle|\mathcal{I}_0\right]\tfrac{d}{du}({}_uq_x(0))\;du\\ &\quad-\;D(0,T)\sum_{i=0}^{N-1}\pi^N_{t_i}\,E_{P^T}\left[\frac{S_T}{S_{t_i}}\middle|\mathcal{I}_0\right]{}_Tp_x(0)\\ &=\eta\int_0^T\sum_{j=0}^{n^*(u)}(1-\varphi)^{i-j-1}\pi^N_{t_j}D(0,t_j)\,\tfrac{d}{du}({}_uq_x(0))\;du\\ &\quad-\int_0^T\sum_{i=0}^{n^*(u)}\pi^N_{t_i}\,D(0,t_i)\,\tfrac{d}{du}({}_uq_x(0))\;du\\ &\quad+\eta\sum_{j=0}^{N-1}(1-\varphi)^{N-j-1}\pi^N_{t_j}D(0,t_j)\,{}_Tp_x(0)-\;D(0,T)\sum_{i=0}^{N-1}\pi^N_{t_i}\,D(0,t_i)\,{}_Tp_x(0).\end{aligned}$$

For $\varphi = 0$ the lefthandside of the above equation equals zero. Then, the righthandside can only be zero for $\eta = 1$ since

$$A(t_i,\pi,0) = \sum_{j=0}^{n^*(t_i)} \pi_{t_j}^N \frac{S_{t_i}}{S_{t_j}} \quad \text{for all} \quad i \in \underline{T} \quad \text{and} \quad A(T,\pi,0) = \sum_{i=0}^{N-1} \pi_{t_i}^N \frac{S_T}{S_{t_i}}.$$

The pension is solely affected by mortality risk since there are no financial guarantees given ($g_D(t) = g_I(t,x) = 0$). Thus, the equivalence principle holds. Otherwise, for $\varphi < 0$, the lefthandside is negative which implies that η has to be less than 1. The negative charge compensates the insured for the reduced option payoff. □

C.2.3 Proof of Theorem 8.4.7

Considering the lefthandside of Definition 8.4.4 and rewriting the difference as a function of φ leads to

$$f(\varphi) := EDPV^{RF}(\varphi) - EDPV_{\eta=1}^{GMDB}(\varphi) - EDPV_{\eta=1}^{GMIB}(\varphi)$$

Then a fair percentage charge φ^* is a solution of $f(\varphi_*) = 0$. The case of a VA contract without guarantees described in Proposition 8.4.5 implies

$$\lim_{\varphi \to 0} \frac{f(\varphi)}{\varphi} = -\infty \tag{C.2}$$

and for $\varphi \to 1$ together with assumption (8.23) it holds

$$\lim_{\varphi \to 1} \frac{f(\varphi)}{\varphi} > 0,. \tag{C.3}$$

Under the contract specifications made in Sec. 8.2, the guarantees for each time t divided by φ are strictly decreasing and continuous in φ with

$$\lim_{\varphi \to \infty} \frac{g_D(t)}{\varphi} = 0, \qquad \lim_{\varphi \to 0} \frac{g_D(t)}{\varphi} = \infty,$$
$$\lim_{\varphi \to \infty} \frac{E_{P^t}[g_I(t,x)|\,\mathcal{F}_0]}{\varphi} = 0, \qquad \lim_{\varphi \to 0} \frac{E_{P^t}[g_I(t,x)|\,\mathcal{F}_0]}{\varphi} = \infty.$$

Nevertheless, the expectations of the overall guarantees G_D and G_I divided by φ are calculated as

$$E_{P^t}\left[\frac{G_D(t)}{\varphi}\,\middle|\,\mathcal{F}_0\right] = \frac{g_D(t)}{\varphi} + E_{P^t}\left[\frac{1}{\varphi}\left[A(t,\pi,\varphi) - g_D(t)\right]^+\,\middle|\,\mathcal{F}_0\right],$$

$$E_{P^t}\left[\frac{G_I(t,x)}{\varphi}\middle|\,\mathcal{F}_0\right] = \frac{E_{P^t}[g_I(t,x)|\,\mathcal{F}_0]}{\varphi} + E_{P^t}\left[\frac{1}{\varphi}\left[A(t,\pi,\varphi) - g_I(t,x)\right]^+\middle|\,\mathcal{F}_0\right].$$

With respect to the convexity property of the embedded call options, both expectations are strictly increasing in φ. Furthermore, the expected value of the total charge withdrawn by the insurer divided by φ is

$$\sum_{i=0}^{N-1}\sum_{j=0}^{i}(1-\varphi)^{i-j}\,\pi_{t_j}^N\,D(0,t_j)\;{}_{t_j-t_i}p_{x+t_i}(t_i)$$

and strictly decreasing in φ. Altogether, this implies that $\frac{f(\varphi)}{\varphi}$ is strictly increasing and continuous in φ. By the mean value theorem there exists a unique fair charge φ^* with $\frac{f(\varphi_*)}{\varphi_*} = 0$. □