Fault detection in nonlinear systems: An observer-based approach

Von der der Fakultät für Ingenieurwissenschaften der
Universität Duisburg-Essen
zur Erlangung des akademischen Grades eines

Doktor-Ingenieurs (Dr.-Ing.)

genehmigte Dissertation

von

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Tag der mündlichen Prüfung: July 16, 2010

Acknowledgments

All praise to Allah who created me, gifted me with a healthy body and brain to work and think and provided me with the opportunities to excel in the life.

This work is carried out during my stay at the Institute of Automatic Control and Complex Systems (AKS), University of Duisburg-Essen for acquiring my Ph.D degree. I owe my deepest gratitude to Prof. Dr.-Ing. Steven X. Ding who let me the opportunity to join this excellent institute. His perpetual encouragement, guidance and support enabled me to complete the work. I am also grateful to Prof. Dr. Michel Kinnaert for his interest in my work and his suggestions, corrections and constructive comments.

I would like to acknowledge my group mates M.Sc. Abdul Qayyum Khan and M.Sc. Wei Chen for their support and long hour discussions. I am obliged to my senior colleague Dr.-Ing. Ibrahim Al-Salami who not only helped me in research but also in solving social problems during my initial days in Duisburg. I am also indebted to my lab mates for their discussions and creating an inspiring and friendly atmosphere in the lab. Many thanks to to all my colleagues at AKS for their suggestions, comments and support during my research at the institute.

I would like to thank my wife for her cooperation and to my parents for their love and affection which helped me to withstand all the difficult stages in my life.

Finally, I wish to acknowledge Higher Education Commission (HEC) of Pakistan and German Academic Exchange Program (DAAD) for providing me financial assistance to carry out this research.

Muhammad Abid
Duisburg, July 16, 2010

To my parents

Contents

Abbreviations and notations

Abbreviations

Abbreviation	Meaning
FAR	false alarm rate
FD	fault detection
FDF	fault detection filter
FDI	fault detection and isolation
FDR	fault detection rate
HJ	Hamilton-Jacobi
HJE	Hamilton-Jacobi equation
HJI	Hamilton-Jacobi-Isaacs
HOT	higher order terms
inf	infimum
LTI	linear time invariant
LMI	linear matrix inequality
min	minimum
RMS	root mean square
SDF	set of detectable fault
SDFA	set of disturbances that cause false alarms
sup	supremum

Notations

Symbol	Description
$=$	equal to
$\neq$	not equal to
$\triangleq$	equal by definition
$\in$	belongs to
$\subseteq$	is subset of
γ	Lipschitz constant

$\leq$	element-wise less than or equal to
$\Rightarrow$	implies
$\otimes$	Kroneker product
$\square$	end of proof
δ_d	upper bound on $\mathcal{L}_2$ norm of disturbance
$\delta_{f,min}$	the smallest size of fault to be detected
Σ	a nonlinear system
Σ^{-1}	inverse of a nonlinear system
$\mathrm{D}\Sigma$	Fréchet derivative of Σ
$\lvert\cdot\rvert$	absolute value of a scaler, element-wise absolute value of a vector or matrix
$\lVert\cdot\rVert_-$	$\mathcal{H}_-$ index for a systems
$\lVert\cdot\rVert_\infty$	$\mathcal{L}_\infty$ norm of a signal or $\mathcal{H}_\infty$ norm of a system
$\lVert\cdot\rVert_2$	$\mathcal{L}_2$-norm of a signal
$\overline{\sigma}(\cdot)$	upper principal gain, maximum singular value
$\underline{\sigma}(\cdot)$	lower principal gain, minimum singular value
$\frac{\partial a}{\partial x}$	partial derivative of a vector a w.r.t. a vector x
a^T	transpose of a vector a
A^T	transpose of a matrix A
A^{-1}	inverse of a matrix A
$A > 0 (A \geq 0)$	the symmetric matrix A is positive (semi) definite
$A < 0 (A \leq 0)$	the symmetric matrix A is negative (semi) definite
f	fault vector, the time argument is omitted
d	unknown input vector, the time argument is omitted
$\mathcal{D}^+$	generalized or pseudo inverse of $\mathcal{D}$
I_n	identity matrix of dimensions $n \times n$
$I_{n\times m}$	identity matrix of dimensions $n \times m$
J_{th}	threshold
L	observer (filter) gain matrix
O_n	zero matrix of dimensions $n \times n$
$O_{n\times m}$	zero matrix of dimensions $n \times m$
$\mathbb{R}^n$	space of real valued n-dimensional vector
$\mathbb{R}^{n\times m}$	space of real valued $n \times m$ matrices
r	residual signal, the time argument is omitted
$\tilde{r}$	modified residual signal
V_x	row vector containing the partial derivative of scaler function V w.r.t. a vector x
V_{xx}	second partial derivative of scaler function V w.r.t. a vector x
V_t	partial derivative of scaler function V w.r.t. a scalar t
u	input vector, the time argument is omitted
x, ξ	state vector, the time argument is omitted
$\hat{x}$	estimation of the state vector x
y	output vector, the time argument is omitted
$\hat{y}$	estimation of the output vector y

Abstract

An un-permitted deviation of at least one characteristic property or parameter of a system from standard condition is referred as a fault. Faults result in reduced efficiency of the system, reduced quality of the product, and sometimes complete breakdown of the process. This not only causes economic losses but may also result in fatalities. An early detection of faults can assist to avert these losses. Therefore, fault detection and process monitoring is becoming an essential part of modern control systems. Fault detection in linear dynamical systems has been extensively studied and well established techniques exist in the literature. However, fault detection for nonlinear dynamical systems is yet an active field of research. This work is motivated by the fact that most of real systems are nonlinear in nature and there is a need to develop fault detection techniques for nonlinear systems. Observer-based methods for fault detection have proven to be among the most capable approaches, therefore, this research is focused towards these methods.

The first step in observer-based fault detection is to generate a symptom signal, called the residual signal, which carries the information of faults. This is done by comparing the measurements from the process to their estimates generated by an observer (filter). It is desired that the residual signal is sensitive to faults and robust against disturbances. This research presents new methods for designing observer (filter) to generate residual signal which is sensitive to faults and robust against disturbances. Three types of filters are proposed in this dissertation; these include a fault sensitive filter, disturbance attenuating filter, and a filter to achieve simultaneous attenuation of disturbances and amplification of faults.

Despite the disturbance attenuation property of the proposed filters, the residual signal is not completely decoupled from the effect of disturbances and uncertainties. Therefore, a threshold is needed to care for the effect of disturbances and uncertainties. Selection of threshold plays an important role in the performance of the fault detection system. If it is selected too high, some faults will not be detected. Conversely, if it is selected too low, disturbances and uncertainties will result in false alarms. This research presents a new method to determine the threshold to avoid false-alarms and to minimize missed-detections. A threshold generator is proposed which is itself a dynamic system and produces a variable threshold. This threshold changes with the effects of uncertainties and disturbances and fits more tightly to the fault-free residual signal and, hence, the performance of fault detection system is improved.

In addition to the residual generation stage, the efficiency of a fault detection system can also be optimized by post-filtering. A further contribution of this research is in proposing a post-filter which operates on the residual signal to generate a modified residual signal. This modified residual signal is simultaneously sensitive to faults and robust against disturbances. Together with this post-filter, a strategy is adopted to select a threshold which maximizes the fault detectability and minimizes the number of false-alarms.

Chapter 1

Introduction

This chapter gives a brief description of the motivations and objectives of this study. The outline of the dissertation and its contribution are also presented in this chapter.

1.1 Motivations

The topic of research in this dissertation is observer-based fault detection in nonlinear systems. The motivations of carrying out this study can be elucidated by the following three questions: (1) Why fault detection? (2) Why observer-based methods? and (3) Why nonlinear systems? These questions are answered below.

Why fault detection? Due to increasing demand in high degree of sophistication and automation, there is an increasing trend in the complexity of technical processes. With increased complexity, the probability of occurrence of faults is also increased. Faults can occur in the process components, sensors or actuators, for example, short circuiting or overheating of electrical components, breakage in bearings due to mechanical stresses, leakages in pipes, sticking of valves, cracks in tanks, drifting of sensors etc. Faults can cause the process to operate far away from the optimal operating points and hence, can reduce the efficiency of the process, quality of the product and if grown large enough, may result in complete failure of the process which requires additional costs for maintenance. Vedam and Venkatasubramanian [1] claim that only in US petrochemical industry, 20 billion dollars per year is lost due to poor abnormal situation management. In safety-critical processes such as aircrafts, nuclear reactors etc., faults may result in fatalities, a few incidents are listed below:

- Boeing 747-200F lost both engines on taking off from Schiphol Airport in Amsterdam. After 15 minutes, the crew lost the control and the plane crashed into a building with a considerable loss of life. Maciejowski [2] has shown that the incident could have been avoided by reconfiguration of the controller.

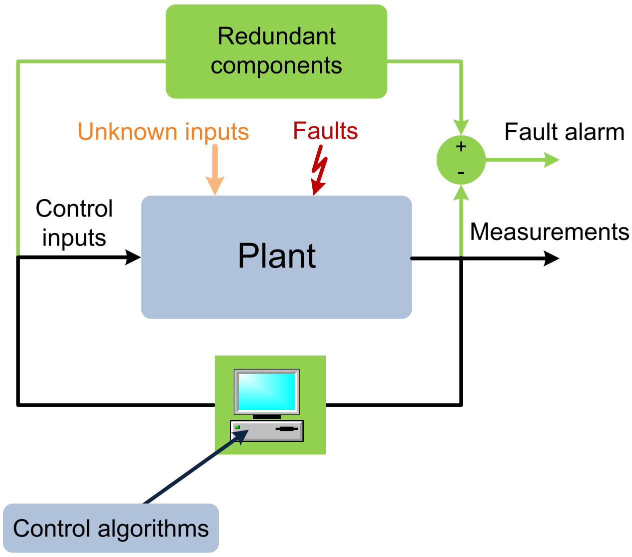
- The American Airline DC10 crashed at Chigao-O'Hare International Airport. The pilot had the indication of fault only 15 seconds prior to the accident. Later studies showed that the crash could have been avoided [3].

- An explosion happened in a huge nuclear power plant in the town of Chernobyl in 1986. The main cause for this tragedy was the faulty outdated technology and the lack of a fault handling mechanism [4, 5].
- Due to complete loss of flying surface in tail, Japan Airlines Flight 123 was crashed on 12 August 1984 resulting in 520 casualties [6].
- In Delta flight 1080 from San Diego to Los Angeles, the elevator became jammed at 19 degrees up and the pilot was given no indication of the failure. However, the pilot manged to reconfigure the lateral control elements and managed to land safely [3, 5, 7].

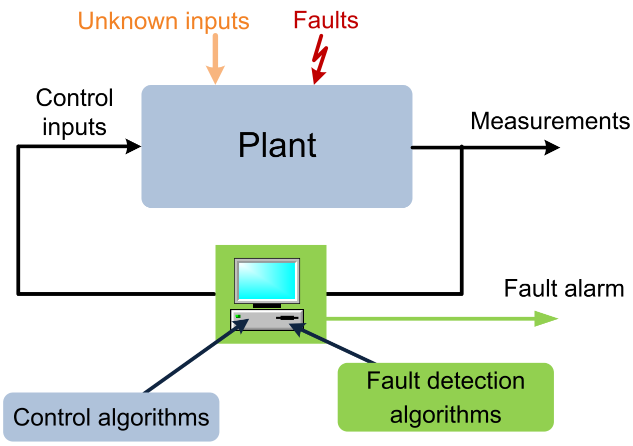
Timely detection of faults can avoid, or at least, minimize the severity of economic losses and fatalities by reconfiguration of controllers or safely switching off the process for maintenance. For example, the later studies carried out for the above mentioned incidents proved that many incident could have been avoided if there were a suitable monitoring system. Engineers and researchers have realized the usefulness of fault detection and isolation (FDI) and fault tolerant control (FTC) as indicated by several research papers, invited tutorials, plenary talks in well reputed conferences such as ACC, CDC, IFAC SAFEPROCESS, ECC and IFAC World Congress, and research papers in high ranked journals.

Why observer-based methods? Fault detection and isolation (FDI) can be achieved by adding either hardware (or physical) redundancy or analytical redundancy in the process. Hardware redundancy means that additional (redundant) components are used in parallel to the process components. If the behavior of a process component is different from that of the redundant component, it gives an indication of the occurrence of a fault. Hardware redundancy has the advantages of high reliability and direct fault isolation but has associated disadvantages of additional cost, additional space required to accommodate the components and additional weight [8–10]. The analytical redundancy based approaches give indication of faults by comparing the measured outputs of the process to their estimations. Analytical redundancy based fault detection algorithms can be implemented on some digital computer and hence avoid the disadvantages related to the hardware redundancy based fault detection techniques. Due to associated advantages, analytical redundancy based approaches are becoming more and more popular. Figure 1.1 depicts the concepts of hardware and analytical redundancy based fault detection, the analytical redundancy based fault detection algorithms can be implemented in the same processor which implements the control algorithms, thus no additional hardware is needed.

Among the analytical redundancy based fault detection schemes, analytical model-based techniques use the deepest knowledge of the monitored process and, therefore, are the most capable approaches for fault detection [11, 12]. There are three types of analytical model-based approaches, these include: the observer-based approach, parity-space approach and the parameter identification approach. In the past few decades, observer-based methods have received considerable interests. There are possibly three reasons for this particular attention to observer-based methods. Firstly, due to associated advantages of observer-based approaches, e.g., quick detection, requiring no excitation signal, possibility of on-line implementation etc. Secondly, other model-based approaches which include parity-space approach and parameter identification approach are, under certain conditions/assumptions,



(a) Hardware redundancy



(b) Analytical redundancy

Figure 1.1: Hardware redundancy vs analytical redundancy: (a) shows the hardware redundancy based fault detection, additional redundant components are used to analyze the presence of faults, (b) shows the analytical redundancy based fault detection, FDI algorithms can be implemented in the same processor which implements the control algorithms, thus no additional hardware is needed.

a specific form of the observer-based approaches, and thirdly, control engineers are more familiar with the concepts of observer design.

Why for nonlinear systems? Fault detection for linear dynamical systems has been well studied and quite a large number of methods exist in literature. One can find a detailed study of these methods in recent books [8, 13–19] and survey papers [9, 11, 20–25]. Recall that most of the systems are nonlinear in nature, one method for fault detection in nonlinear systems is to linearize them at some operating point and use the techniques developed for linear systems. The linearization errors can be modeled as unstructured uncertainties and their effects can be taken care of by utilizing the robust methods [8, 10]. However, if the process has high nonlinearities, or the operating region is too wide, the linearization error will be too large to be handled by the robust linear fault detection techniques. Therefore, there is a need to study fault detection techniques for nonlinear systems. Fault detection in nonlinear dynamical systems is still an active field of research [26–30] and there is enough space for further improvements.

1.2 Objectives

The idea of observer-based fault detection is to generate estimations of measured signals using model of the monitored process, and compare the measurements with their estimations to generate a symptom signal, called the *residual signal*, which carries the information of faults. In ideal situations, when there are no disturbances and modeling uncertainties, the estimations will completely match with the measurements in fault-free case and the residual signal will be zero. Any deviation of residual signal from zero will give an indication of faults. However, the presence of modeling uncertainties and disturbances is inevitable. Therefore, the aim is to design observers such that the affect of disturbances and uncertainties on the residual signal is reduced while the affect of faults is considerably increased. Now, instead of setting deviation of residual from zero as indicator of faults, a threshold which cares for the effect of disturbances and uncertainties, should be selected and if the residual exceeds the selected threshold, it gives an indication of the presence of faults. The selection of threshold plays a very important role in the performance of a fault detection system, if it is selected too low, some of the disturbances and uncertainties will cause the residual to cross the threshold and appear as faults, this is definitely not desired. Conversely, if the threshold is selected too high, some of the faults will not enable the residual to cross the threshold, and hence will remain undetected. One solution is to select, instead of a constant threshold, a variable threshold which changes with the variations in the effects of uncertainties and disturbances on the residual signal, more the affect of uncertainties and disturbances on residual signal, higher is the threshold and vice versa. This can, to a large extent, improve the performance of fault detection system. In brief, the main objectives in the design of a fault detection system are to design an observer such that the residual signal is minimally influenced by disturbances and uncertainties and is maximally influenced by faults, and to select a threshold which is as close to fault-free residual signal as possible. These design objectives have been solved for linear dynamical systems, comprehensive study can be found in recent monograms [8, 13–15]. The objectives of this dissertation is to present techniques for optimal residual generation and threshold computation for nonlinear dynamical systems. These objectives can be summarized in the following two points

1. Generation of residual signal sensitive to faults and robust against disturbances. To discuss the sensitivity and the robustness level, the worst case situation should be considered, i.e., the faults which have minimum effect on residual and disturbances which have maximum effect on residual should be considered.

2. To avoid too conservative threshold, and hence to improve the performance of fault detection system, extension of the concept of variable threshold to nonlinear dynamical systems.

If the above mentioned objectives are achieved, faults will be detected more quickly, while there will be less number of false alarms. From the previous statement, one would instantly realize that the real goals of fault detection system are high fault detection rate (FDR) and low false alarm rate (FAR) , optimal residual generation and threshold computation are only tools to achieve these goals. However, high FDR and low FAR are two conflicting goals, therefore, an optimal trade-off should be made between the two. An excellent work to find this optimal trade-off design for linear systems can be found in [13, 31]. The third objective of this dissertation is to solve the optimal trade-off design problem for nonlinear systems, which is summarized in the following point

3. Proposing a strategy for nonlinear systems which achieves an optimal trade-off between high fault detection rate and low false alarm rate.

1.3 Outline and contribution of the thesis

This thesis is divide into seven chapters. The first and the second chapters serve as introductory material. The rest of the chapters summarize the contribution and research results of this study.

The first chapter describes motivations, objectives and contribution of the thesis.

Chapter 2 gives an overview of fault detection and isolation (FDI) and presents some existing methods for fault detection in nonlinear systems. It begins with the definitions of basic concepts such as faults, failures, fault detection etc. A classification of fault detection techniques, with a brief discussion on each approach, is also presented in this chapter. An appropriate attention is paid to observer-based methods, their robustness and sensitivity issues are elaborated. The chapter also presents fault detection methods for nonlinear systems, most commonly used observers for fault detection in nonlinear systems are described in a bit details. State of the art methods for residual evaluation in nonlinear systems are also presented.

Residual generation is the first step in an observer-based fault detection scheme. It is desired that the residual signal should be sensitive to faults and robust against disturbances. Over the past, $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index have been used to measure the disturbance attenuation and fault sensitivity property of residual generator. Based on the standard tools from game theory, Chapter 3 presents approaches for designing $\mathcal{H}_-$ fault sensitive residual generator, $\mathcal{H}_\infty$ disturbance attenuating residual generator and $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective residual generator for nonlinear systems. The chapter begins with some examples introducing concepts of $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index and their significance in fault detection system design. Then sufficient conditions for $\mathcal{H}_-$ fault sensitivity of residual signal are derived. A delicate difference between $\mathcal{H}_\infty$ filtering problem and $\mathcal{H}_\infty$ fault detection filtering problem is explained, an example is provided to highlight the difference. It is explicated that although the $\mathcal{H}_\infty$ filtering problem has been extensively studied, $\mathcal{H}_\infty$ fault detection filtering problem for nonlinear systems has not been discussed. Sufficient conditions for $\mathcal{H}_\infty$ fault detection filtering problem are derived for nonlinear systems. Simultaneous attenuation of disturbances and amplification of faults is desired feature of a residual generator, therefore, the $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective fault detection filtering problem is discussed. In all the three filters, both finite horizon and infinite horizon cases are handled. The proposed methods are demonstrated by simulation examples.

With the fault detection filters mentioned above, the effect of disturbances on the residual signal is not completely eliminated, therefore, there is a need to use a threshold. Selection of threshold is significantly important in the design of a fault detection system, preferably a variable threshold can achieve a better performance. Chapter 4 begins with explaining the concept of variable thresholds. Based on the recently proposed approach of dynamic threshold generation for linear systems, Chapter 4 presents an approach to generate dynamic threshold for a class of nonlinear systems. A dynamic upper bound on the modulus of error dynamics is obtained which is used to generate a dynamic threshold. With a simulation example of a simple second order system, it is demonstrated that the proposed method of dynamic threshold generation can successfully avoid false alarms and can detect small faults.

The residual generation methods presented in Chapter 3 and threshold selection technique presented in Chapter 4 are tools to achieve a trade-off between high FDR and low FAR. This optimal trade-off can also be reached by designing a dynamic post-filter and a determining a threshold. With the assumption that a stable residual generator is available, Chapter 5 presents an approach to design a post-filter and determine a threshold to attain an optimal trade-off between high FDR and low FAR. The optimal trade-off design problem is formulated into two optimization problems; these are the maximization of fault detectability for an allowed FAR and the minimization of false alarms for a required FDR. For the first optimization problem, a threshold is selected which guarantees that the FAR is not more than the allowed one. Then based on the selected threshold and utilizing the factorization approach, a post-filter is proposed which maximizes fault detectability. Similarly for the second optimization problem, a threshold is selected to ensure the required FAR and the post-filter minimizes the number of false alarms. Furthermore, it is demonstrated that after using the post-filer, which acts on the residual signal to generate a modified residual signal, the modified residual signal is sensitive to faults and robust against disturbances, i.e. the proposed post-filter also gives optimization in the sense of $\mathcal{H}_-/\mathcal{H}_\infty$ optimization. A simple example is provided to illustrate the proposed methods and to show its effectiveness.

Chapter 6 gives the application of the optimal trade-off design method developed in Chapter 5 to the three tank system. The three tank system has high nonlinearities and is, therefore, widely used as a benchmark to test nonlinear control and FDI algorithms. After describing the three tank system, a fault detection filter is used to generate residual signal. Then a post-filter is designed and threshold is computed to give and optimal trade-off between high FDR and low FAR. The simulation results are presented which show that all the faults including sensor faults, actuator faults and component faults are successfully detected. Both abrupt and incipient fault situations are presented.

The concluding remarks of this research and some future recommendations for possible extension of the work are presented in the last chapter.

Chapter 2

Background and state of the art

This chapter introduces to the basic concepts in fault detection and isolation (FDI) and presents some existing methods for fault detection in nonlinear systems. Fundamental concepts, such as faults, failures, fault detection, fault isolation etc. are defined. Different types of faults and their effects on the performance of processes are explained. Several methods for fault detection exist in literature; a widely accepted classification of these methods is presented with a particular focus on the model-based fault detection techniques. Observer-based fault detection in nonlinear systems is of major interest in this thesis, therefore, state of the art observer-based residual generation methods for nonlinear systems are presented in a bit details. Most commonly used evaluation functions and threshold selection approaches are described.

2.1 Some basic concepts

The terminologies used in FDI has been fairly standardized after the suggestions from SAFEPROCESS Technical Committee [25]. Throughout the text a *fault* means an unpermitted deviation of at least one characteristic property or parameter of a system from the acceptable/usual/standard condition. A very related term is *failure* which is a permanent interruption of the system's ability to perform a required function under specified operating conditions. There is a slight difference between fault and failure, failure means complete breakdown of a component, whereas fault is only deviation from normal characteristics. As far as detection is concerned, faults and failures can be treated alike. In sequel, we will use the term fault to encompass failure as well.

Faults can be described as external inputs or parameter deviations which change the behavior of the process. Like faults, *disturbances* and *uncertainties* can also be modeled as external inputs. Furthermore, disturbances and uncertainties have effects on the process similar to that of faults. But as compared to faults, disturbances are unavoidable and are present even during the normal operation of the process. Moreover, the controller is designed so that it can perform well in the presence of disturbances. Faults, on the other hand, are more severe changes and their affects can not be overcome by a fixed controller and, therefore, must be detected.

The purpose of *fault diagnosis* is to detect faults and to determine their locations and

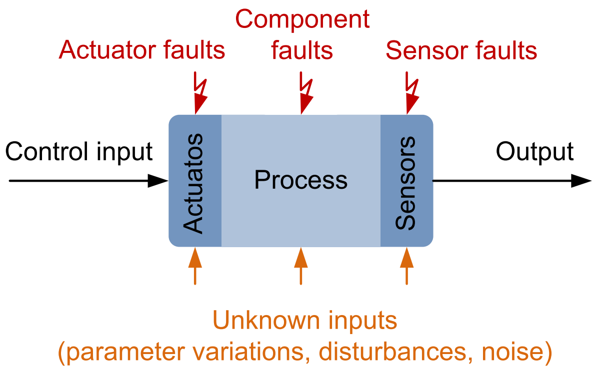


Figure 2.1: Representation of different kinds of external inputs to a system, including faults, disturbances and parameter variations

significance. The procedure of fault diagnosis consists of three steps namely fault detection, fault isolation and fault identification. *Fault detection* (FD) is the process of determining the presence of faults and the time of their occurrence. The function of *fault isolation* is to exactly locate the reason or the origin of fault. Once the fault has been detected and isolated, the step of *fault identification* starts that aims to find an approximate time behavior of the fault. The conditions for the isolation of faults are quite harder and that of fault identification or even more stringent (see [13, Chapter 4]) which makes it, in most situations, impossible to isolate and identify the faults. The scope of the thesis is limited to the detection of faults.

2.1.1 Types of faults

Fault in a system is an external input that causes a deviation from the normal behavior of the system. Faults can occur in the actuators, process components or the sensors as shown in Figure 2.1, and are categorized accordingly. Each of these faults and their effects are briefly described below.

Component faults

These are the faults which appear in the components of plant. Component faults alter the physical parameters of the plant which, in turn, results in change of its dynamical properties. The common reason for these faults is usually wear and tear, aging of components etc. Some examples of component faults are leakages in tanks, breakages or cracks in gearbox system, change in friction due to lubricant deterioration etc. Component faults may result in instability of the process, therefore, it is extremely important to detect these faults.

Actuator faults

Actuators are needed to transform control signals into proper actuation signals such as torques and forces to drive the system. A fault in an actuator may result in higher energy consumption to total loss of control [28]. Examples of actuator faults include stuck-up of control valves, faults in pumps, motors etc. Some common actuator faults in servomotors

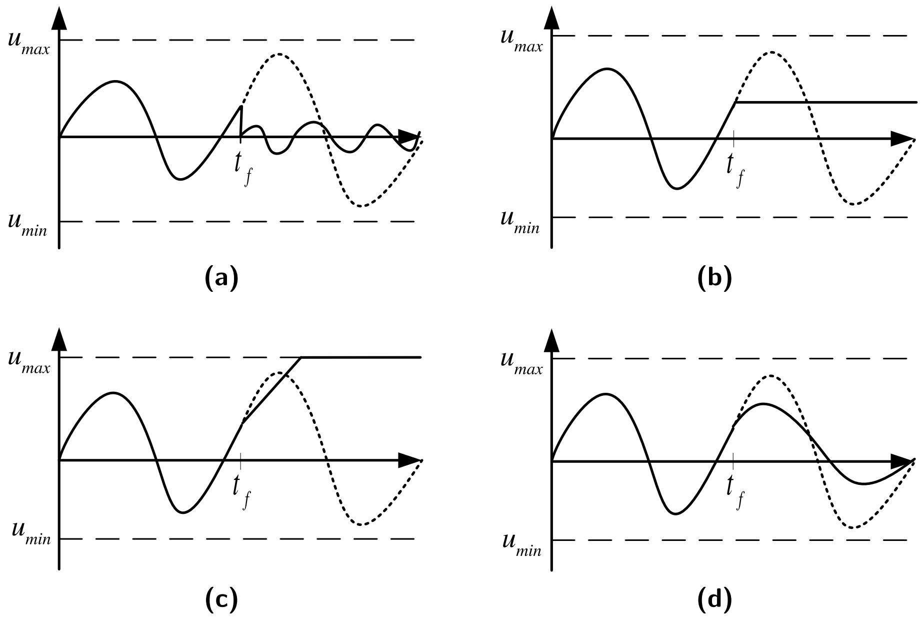


Figure 2.2: Graphical representation of common types of actuator faults in servomotors [32]. Dotted lines show the desired value of actuator and the solid lines show actual value. (a) floating around trim, (b) lock-in-place, (c) hard-over failure and (d) loss of effectiveness

are lock-in-place, float around trim, hard-over failure and loss of effectiveness [28, 32] as shown in Figure 2.2.

Sensor faults

In closed loop systems, the measurements obtained by sensors are used to generate the control inputs and any fault in sensors can cause operating points that are far from the optimal ones [14]. This results in degradation in the performance of the system. It is therefore, very important to detect these faults. Typical examples of sensor faults are: bias, drift, performance degradation (or loss of accuracy), sensor freezing and calibration error [28, 32, 33] as illustrated in Figure 2.3.

Faults can also be categorized according to whether these have developed slowly in the system (*Incipient faults*) , arisen suddenly like a step change (*Abrupt faults*) or occurred in discrete intervals (*Intermittent faults*) as shown in Figure 2.4. Abrupt faults have more severe affects and may result in damage of equipments. However, fortunately abrupt faults are easier to detect. Incipient faults grow slowly and result in degradation of equipments. Their slowly changing behavior makes it difficult to detect them.

Faults may also be classified into *additive faults* and *multiplicative faults* according to the way in which these are modeled. Actuator and sensor faults are more easily modeled as additive faults, whereas component faults are modeled as multiplicative faults.

2.1.2 Desired features of fault detection schemes

Advanced methods of fault detection should satisfy the following requirements [15]

- early detection of abrupt and incipient faults

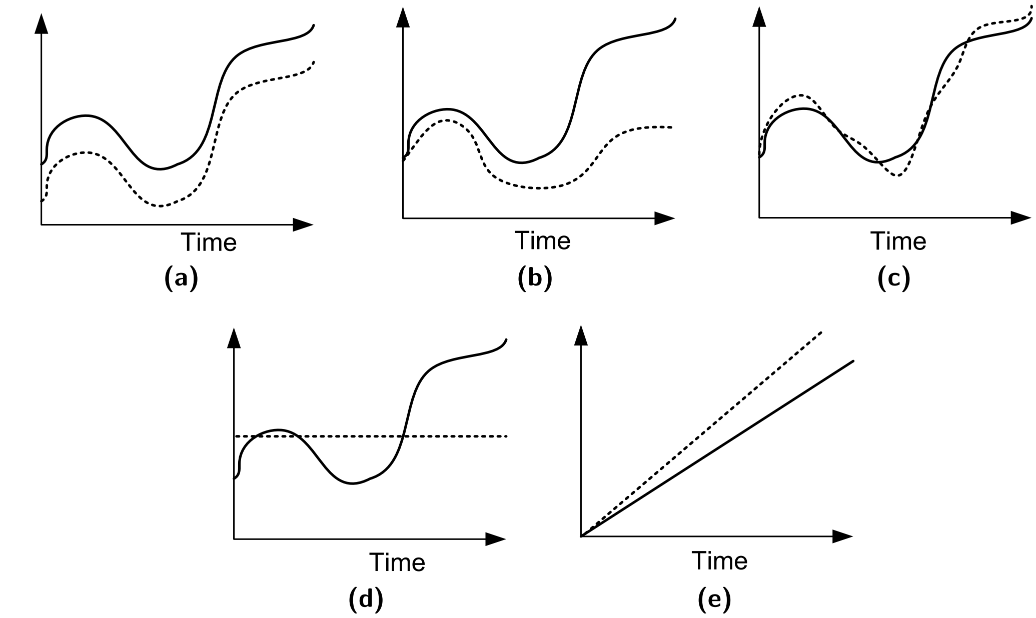


Figure 2.3: Graphical depiction of different kinds of sensor faults [33]. Solid lines show the actual values whereas the dotted lines show the measured values. (a) Bias, (b) Drift, (c) Loss of accuracy, (d) Freezing and (e) Calibration error

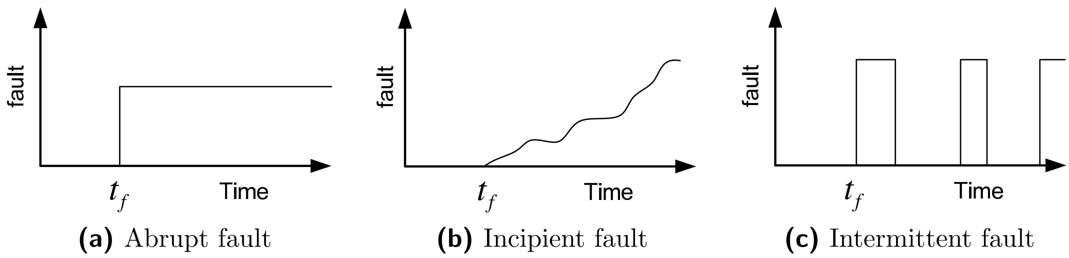


Figure 2.4: Graphical illustration of abrupt, incipient and intermittent faults occurring at time t_f

- detection of actuator, component and sensor faults
- detection of faults in closed loop
- supervision of processes in transient states

Other than the above mentioned features, a fault detection technique should consume less computational cost so that on-line implementation is easily achieved. Furthermore, the design procedure should be simple.

2.2 Classification of fault detection schemes

The importance of fault detection has been realized since the invention of machines. The earliest way of detecting faults was biological senses, such as looking for changes in color and shape, listening to sounds unusual in pitch and loudness, touching to feel heat or vibration, and smelling for fumes because of leakage or overheating [16]. However, with industrial revolution, there was a need to autonomously detect faults without human intervention. This was to save extra labor, to achieve more precise and quick detection of faults and the fact that some parts or location may not be accessible to, or dangerous for human beings.

Classical way of fault detection is by limit checking which is done by setting an upper and lower limit for the measured variables. If the measured variable exceeds the limit, it gives indication of fault. There has to be a compromise in selecting the limit bound; if selected too narrow some fluctuations and disturbances will cause an alarm of fault and if selected too wide, some of the small magnitude faults may not be detected. These methods are suitable for processes with steady state behavior. The advantage of these methods is simplicity and reliability for steady-state situations [15, 34]. The disadvantage of limit checking methods is that faults can be detected only when these grow large enough to cross the limit. This may cause more damage to the process as compared to that if it was detected earlier and suitable remedies had been taken. Another disadvantage is that these methods fail when the monitored process has dynamic properties or the operating points are changing rapidly. Therefore advanced methods of supervision must be used. These methods can be classified into i) Plausibility test, ii) Signal-based methods and iii) Model-based methods. These are briefly described in the following subsections.

2.2.1 Plausibility test

The idea is based on checking the plausibility of measured values. This means that the measurements are compared with their rough behavior under normal operation, for example, the sign and size of the measurements. The plausibility test can be implemented by simple logical gates. It has the drawback that it is less efficient in detecting faults and becomes impossible in complex plants.

2.2.2 Signal-based fault detection

In signal-based approaches, one gets the information of faults by collecting some properties of the measured signals. Examples of these properties are the magnitudes of the time function, trend checking from the derivative, mean and variance, spectral power densities, correlation coefficients, etc., of the measured signals. Figure 2.5 shows the conceptual diagram for signal-based fault detection schemes.

Limit checking of absolute value of the measurements and the limit checking of derivative (trend) of the measurements are the two most simple and widely used approaches for fault detection. In limit checking of absolute value of the measurements, suitable upper and lower limits are set based on the knowledge of the plant, and if the magnitude of the measurement crosses the limit, it gives an indication of a fault, i.e.,

$$\begin{aligned} Y_{min} \leq y(t) \leq Y_{max} &\Rightarrow \text{fault-free} \\ y(t) < Y_{min} \text{ or } y(t) > Y_{max} &\Rightarrow \text{faulty} \end{aligned}$$

Similarly, in limit checking of the trend of the measurements, if the derivative of the measurements crosses the pre-defined upper and lower limits, a fault alarm is released, i.e.,

$$\begin{aligned} \dot{Y}_{min} \leq \dot{y}(t) \leq \dot{Y}_{max} &\Rightarrow \text{fault-free} \\ \dot{y}(t) < \dot{Y}_{min} \text{ or } \dot{y}(t) > \dot{Y}_{max} &\Rightarrow \text{faulty} \end{aligned}$$

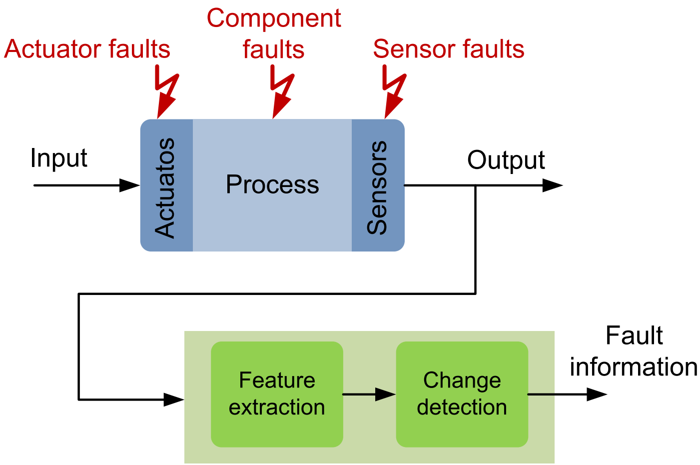


Figure 2.5: Schematic diagram of signal-based fault detection scheme

Limit checking approach is simple and can be easily implemented. However, the drawback is that fault can only be detected when it grows enough to cross the limits. Moreover, these approaches are not suitable for the dynamic systems with transient behavior.

2.2.3 Model-based fault detection

The idea of model-based fault detection schemes is to compare the behavior of actual process to that of the nominal fault-free model of the process driven by the same input. Model-based approaches are more powerful than the signal processing-based approaches [11, 12], because these use more information about the process. Figure 2.6 shows the schematic diagram of a model-based fault detection scheme. It consists of two main stages; residual generation and residual evaluation. The objective of residual generation is to produce a signal, called residual signal, by comparing the measurements with their estimates and the purpose of residual evaluation is to inspect the residual signal for possible presence of faults.

Based on the model used for the purpose of residual generation, model-based fault detection schemes can further be divided into two categories. The model can be an analytical model represented by set of differential equations or it can be knowledge-based model represented by, for example, neural networks, petri nets, experts systems, fuzzy rules etc. Knowledge-based model approaches do not need full analytical modeling, therefore, are more suitable in information-poor systems or in situations where the mathematical model of the process is difficult to obtain or is too complex. This is the case, for example, in chemical processes which are difficult to model analytically. A comprehensive study of these methods can be found in survey papers [11, 35–38] and recent books [17, 38]. Recently, hybrid approaches which simultaneously use the mathematical model and neural networks have also been proposed [28].

In analytical model-based approaches, residual signal is generated using the mathematical model of the system. The most commonly used analytical model-based approaches for residual generation include i) Observer-based approach, ii) Parity space approach and iii) Parameter estimation-based approach, these are described below.

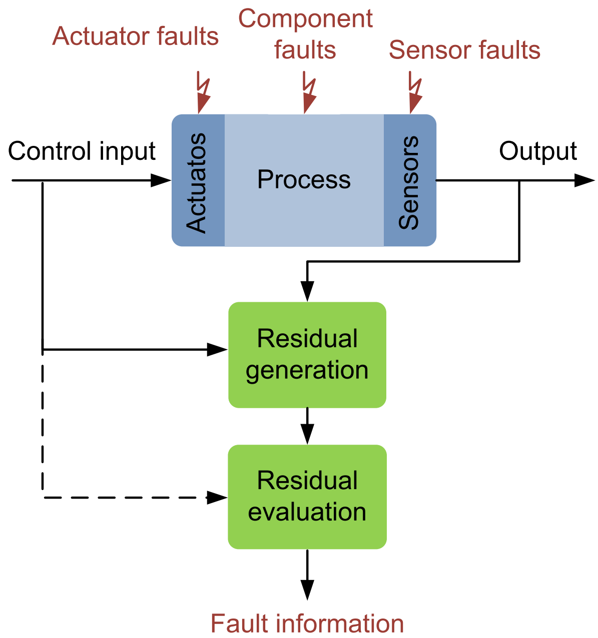


Figure 2.6: Schematic diagram of model-based fault detection schemes

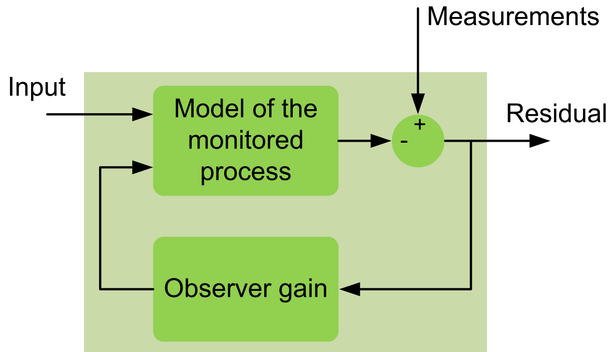


Figure 2.7: Schematic diagram of observer-based residual generator

Observer-based approach

In observer-based approaches, residual signal is generated by comparing measurements from process with their estimates generated by an observer (a filter). It should be noted that there is a difference between observers used for control purposes and observers for FD. The observers needed for control are state observers, i.e., they estimate states which are not measured. In contrary, the observers needed for FD are output observer, i.e., these observers generate estimation of the measurements. A special form is the fault detection filter, which generates estimation of all the states, irrespective of whether they are measured or not. In this case, these can be used both for control and FDI. Figure 2.7 shows the schematic diagram of an observer-based residual generation scheme.

The idea of using observers for residual generation goes back to 1970s when Beard proposed a detection filter which was later modified by Jones to the so called Beard-Jones detection filter. In parallel to the Beard-Jones detection filter, Kalman filter was used in stochastic setting.

The robustness of residual signal against unknown inputs has been widely discussed in literature and several approaches have been proposed to tackle this problem, a survey of these approaches can be found in [39]. The first attempt to improve the robustness of observer-based instrument fault detection scheme was made by [40]. Robust fault detection using unknown input observers was a focus of research in 1980's. With a pioneering work of [41], a considerable contribution was made in literature by [42–45]. The idea is to completely decouple the state estimation from unknown inputs (disturbances) using the information of the unknown input distribution matrix. If the states are decoupled from the unknown inputs, residual is also independent. A related approach for robust residual generation is use of eigenstructure approach which also decouples residual from unknown inputs. The existence conditions for the eigenstructure assignment approach are more relaxed compared to unknown input observers. In this approach, instead of decoupling state estimations from unknown inputs, the residual signal is made independent of unknown inputs. The approach was initially proposed in [46] and further developments were made in [47, 48]. Another approach for disturbance decoupling was proposed in [49] which involves geometric approach for decoupling the effect of disturbance. Unknown input observers, eigenstructure assignment approach and the geometric approach are all based upon eliminating the effect of disturbance. However, these approaches can not properly handle modeling uncertainties. One way to handle the uncertainties is to model them as external unknown inputs and then utilize the disturbance decoupling approaches. The existence conditions for disturbance decoupling approaches are quite stringent which restricts their application. Furthermore, if the fault lies in the same space as the disturbances, its effect on the residual is also decoupled and cannot be detected. Therefore, instead of completely decoupling unknown inputs, much focus has also been paid to design observers which attenuate the effect of unknown inputs and amplify the effect of faults on the residual signal. Following the standard results from control theory, several approaches [43, 50, 51] were proposed utilizing the $\mathcal{H}_\infty$, $\mathcal{H}_2$ indices to design observer so as to attenuate the effect of disturbance on the residual signal. In parallel to robustness problem, much of attention is also paid to fault sensitivity problems [52–55]. With simultaneous consideration to robustness and sensitivity proposed in [56], there were a series of articles which solved the sensitivity and robustness problem and LMI solutions were presented to solve $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective optimization [53, 55, 57–60]. The unified solution was presented in [61, 62] which gives a simultaneous solution of the multi-objective $\mathcal{H}_i/\mathcal{H}_\infty$ optimization and involves less computations. A comprehensive study of the unified approach can be found in [13].

Observer-based approach for fault detection in nonlinear systems is the main interest of this thesis. A survey of some observer-based techniques for residual generation in nonlinear systems will be presented in Section 2.4.1.

Parity space approach

The *parity space* approach uses the check of parity of mathematical equations of the system by using the measurements. Chow and Willsky [63] first proposed parity equations for state space model of the system, later contributions were made using the transfer functions in [64–66]. Figure 2.8 shows the configuration of parity space-based residual gneration [11].

Parity space approach and observer-based approach are similar as shown in [12, 67, 68] and there exists a one-to-one mapping between the design parameters of observer and

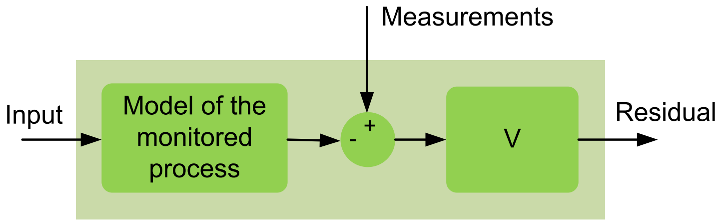


Figure 2.8: Schematic diagram of parity space approach for residual generator

parity relation based residual generator. Two theorems are presented in [13] that show how to calculate parity vector corresponding to observer-based residual generator and vice versa. Thus we can design residual generator in parity space and then can transform the parity vector into diagnostic observer parameters for on-line implementation. The implementation of the parity relation based residual generator uses a non-recursive form, while the observer-based residual generator represents a recursive form. Thus it is usual to design in parity space and to realize in observer-based structure.

Parity relation based fault detection schemes have also been utilized for nonlinear systems. Krishnaswami and Rizzoni [69] presented a parity space approach based on the inverse model of input output nonlinear systems. Recently, [70, 71] generalized the parity space approach for linear systems to nonlinear systems described by TS fuzzy models. A comprehensive study of parity space approach for nonlinear processes can be found in [15]. For multi-input single-output nonlinear systems, the relationship between the parity relations and high gain observers has been shown in [72, 73].

Parameter estimation based approach

The *parameter estimation* approach for fault detection was first proposed in [74–76] and is based on the assumption that the faults are reflected in the physical parameters of systems. With this assumption, parameters of system are estimated on-line repeatedly, if there is a discrepancy in the estimated parameters and the actual parameters, it gives indication of faults. An advantage of parameter estimation approach is that with only one input and one output signal, several parameters can be estimated which give a detailed picture on internal process quantities [15]. Another advantage of the method is that it yields the size of the deviations which is important for fault analysis [11]. Parameter estimation based approach is useful for component fault detection, although it can also detect sensor and actuator faults. A disadvantage is that an excitation is always needed in order to estimate the parameters which may result in problems if the precess is operating at a stationary points [11]. There are several parameter estimation techniques, among them are methods of least squares (LS), recursive least squares (RLS), extended least squares (ELS), etc.

Parameter estimation techniques have also been applied to fault detection in nonlinear systems, study of parameter estimation based fault detection in nonlinear systems can be found in [15] and application to a nonlinear satellite model in [77]. There is a close relationship between parameter identification based fault detection and the observer-based fault detection approach as demonstrated in [78, 79].

2.3 A comparison of different fault detection methods

It is quite difficult to compare different fault detection schemes. The decision upon which FD method should be used depends on several factors, among them are the availability of mathematical model, information about the process, type of disturbances and uncertainties, nonlinearities, closed loop or open loop etc. For example, in electrical and mechanical systems, it is relatively easy to obtain a mathematical model so analytical model-based approaches are preferred. In contrary, chemical and industrial processes are difficult to model, or even if a mathematical model can be obtained, it is quite complex. In that case, qualitative model-based approaches or signal-based approaches should be applied. Analytical model-based approaches are usually faster and on-line implementation is easier hence more suitable for processes with fast dynamics.

Among the analytical model-based approaches, an interesting comparison is presented in [15] which could be used as a guideline for selecting the FD method. Parameter estimation based approaches need only the structure of the process as against parity space approaches and observer-based approaches which need not only the structure of the process but also the parameters. Both the parity space approach and the observer-based approach do not need an input signal change to detect faults whereas parameter estimation approach requires input excitation. To detect multiplicative faults, parameter estimation approach is a better choice as compared to observers and parity equations. Parity space approaches are more sensitive to measurement noise as compared to observer-based approaches and parameter estimation approaches. It is also possible to combine different approaches to collect the advantages of each approach. A few combining strategies are enlisted in [15].

2.4 State of the art nonlinear FD techniques

Most of the real systems are nonlinear in nature. One approach for the fault detection of nonlinear systems is to linearize them at operating points and use the well established theory of linear FDI. The effect of linearization errors is handled by applying robust FDI schemes. In certain cases when the linearized model does not deviate too much from the nonlinear model, these approaches can perform well. However, in situations where the monitored process has high nonlinearities, or the operating region of the plant is large enough, linearization about a point may result into high modeling errors. Using the linear model-based approaches will result into high false alarm rate with most of the faults undetected [22]. These limitations of linear FD methods motivated the researchers to study the nonlinear fault detection techniques.

Several approaches for fault detection of nonlinear systems have been proposed. These include analytical model-based techniques [8, 80], neural networks [81], fuzzy systems [11], data driven methods [82], statistical techniques etc. Among these techniques, the analytical model-based approaches use the deepest knowledge of the process, and therefore, are the most capable approaches for fault detection provided that the mathematical model of the process is available. Three types of analytical model-based residual generation approaches are classified in the last section, these are the observer-based approach, parity space approach and parameter identification approach. The major focus of this thesis is observer-based approaches because of two reasons. Firstly, it has been shown that parity relations are a special form of observers, called the dead-beat observers (i.e. observers having all the poles at origin). Likewise, parameter identification approaches have much similarities to observer-based approaches. So, it does not cause any loss of generality to focus on observer-based approaches [12]. Secondly, observers are more familiar to the control community as compared to parity relations and parameter identification approaches. In the next subsection, some state of the art methods for observer-based residual generation in nonlinear systems are explained. Residual evaluation techniques will be presented in subsection 2.4.2.

2.4.1 Observer-based residual generation

Over the past, several observer-based approaches for residual generation in nonlinear systems have been proposed, see [8, 22, 27, 30, 39, 80, 83] for survey of these approaches. Some of them are described in below:

Extended Luenberger observer

The Luenberger observer was used for fault detection in linear systems in [84]. For fault detection in nonlinear systems, one can linearize the nonlinear model at an operating point and apply the Luenberger observer. However, if the operating region is too wide, the linearized model will deviate largely from the nonlinear model, particularly, if the system is operating away from the linearizing point. The idea of the extended Luenberger observer is to linearize the model around current estimate of states $\hat{x}(t)$, instead of a fix point (e.g. $x = 0$), and then apply the Luenberger observer. Consider, for example, the nonlinear system

$$\begin{aligned} \dot{x} &= a(x,u), \quad x(0) = x_0 \\ y &= c(x,u) \end{aligned}$$

Then a nonlinear observer is

$$\begin{aligned} \dot{\hat{x}} &= a(\hat{x},u) + L(\hat{x},u)(y - c(\hat{x},u)), \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} &= c(\hat{x},u) \end{aligned}$$

where $L(\hat{x},u)$ is the observer gain which is computed at each time instant in such a way that the eigenvalues of $\left(\frac{\partial a(x,u)}{\partial x} - L(\hat{x},u)\frac{\partial c(x,u)}{\partial x}\right)$ are stable. The detailed study can be found in [85]. A similar approach for state estimation and its application to fault detection has been proposed in [86]. Because of the requirements of repetitive calculation of observer gain (which means more on-line computations) and the linearization errors, the extended Luenberger observer is rarely used in practice.

In stochastic settings, the counterpart of the extended Luenberger observer is the extended Kalman filter (EKF) . Similar to extended Luenberger observer, the basic idea of extended Kalman filter is to linearize the system around the current estimation of states and apply the linear Kalman filter.

The Thau observer approach

The Thau observer was developed in [87] for the state estimation of a class of nonlinear systems. It has also been applied to fault detection in [88]. The class of nonlinear systems examined in this approach is described by

$$\dot{x} = Ax + Bu + g(x,u)$$
$$y = Cx$$

It is assumed that the pair (C, A) is observable, the nonlinear part $g(x,u)$ is continuously differentiable and satisfies the Lipschitz condition locally, i.e.

$$\|g(x,u) - g(\hat{x},u)\| \leq \gamma \|x - \hat{x}\|$$

The structure of the Thau observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + g(\hat{x},u) + L(y - \hat{y})$$
$$r = y - C\hat{x}$$

and the observer gain $L = P^{-1}C^T$, P is the solution to the Lyapunov equation

$$A^T P + PA - C^T C + \theta P = 0 \tag{2.1}$$

where θ is a positive parameter which is chosen in such a way that it ensures the solution of the Lyapunov equation (2.1).

Nonlinear identity observer approach

The nonlinear identity observer was first proposed in [89] for detection and isolation of component faults. Further contributions were presented in [21, 86]. The nonlinear system considered is given as

$$\dot{x} = a(x,u), \quad x(0) = x_0$$
$$y = c(x,u)$$

The nonlinear observer is then given by

$$\dot{\hat{x}} = a(\hat{x},u) + L(\hat{x},u)(y - \hat{y})$$
$$r = y - c(\hat{x},u)$$

Defining the estimation error $e = x - \hat{x}$, the error dynamics can be written as

$$\dot{e} = A(\hat{x},u)e - L(\hat{x},u)C(\hat{x},u)e + HOT$$
$$r = C(\hat{x},u)e + HOT$$

where

$$A(\hat{x},u) = \left.\frac{\partial a(x,u)}{\partial x}\right|_{x=\hat{x}}, \quad C(\hat{x},u) = \left.\frac{\partial c(x,u)}{\partial x}\right|_{x=\hat{x}}$$

and HOT represents the higher order terms. Neglecting these HOT, the gain matrix $L(\hat{x},u)$ can be found in such a way that the error dynamics are asymptotically stable. In some situations, for example in Lipschitz nonlinear systems, a constant matrix will guarantee the stability[80]. For the case $C(x,u) = Cx$, the matrix $L(\hat{x},u)$ has the form [80]

$$L(\hat{x},u) = P^{-1}\hat{A}(\hat{x},u)C^T Q$$

where the symmetric positive definite matrix P should be such that

$$K^T P \frac{\partial a(x,u)}{\partial x}\bigg|_{x=\hat{x}} < 0$$

where K is the highest rank right orthogonal matrix to C and

$$\hat{A}(\hat{x},u) = \text{diag}\left(\frac{1}{2}\sum_{j=1}^{n}|\psi_{ij}+\psi_{ji}|\right),\ i = 1,\cdots,n$$

where ψ_{ij} is the ij^{th} element of the matrix $P\frac{\partial a(x,u)}{\partial x}\Big|_{x=\hat{x}}$ and Q is a matrix satisfying $C^TQC - I \geq 0$.

Nonlinear unknown input observer approach

As discussed earlier, the unknown input observer completely cancels the affect of disturbances on the residual signals. The unknown input observer was first applied for fault detection in [41] and later contributions were introduced in [44, 67] for linear systems. The approach was extended to a class of nonlinear systems in [67]. This approach can be applied to the class of nonlinear systems modeled by

$$\begin{aligned}\dot{x} &= Ax + B(y,u) + E_d d + E_f(x,u)f_f\\ y &= Cx + F_f(x,u)f_s\end{aligned}$$

where d represents the unknown inputs, f_f represents the component or the actuator faults and f_s represents the sensor faults. Then the observer is given by

$$\begin{aligned}\dot{\hat{x}} &= Fz + J(y,u) + Gy\\ r &= L_1\hat{x} + L_2 y\end{aligned}$$

For the observer to be decoupled from the unknown inputs, following conditions should be satisfied [80].

$$\begin{aligned}TA - FT &= GC, \quad F \text{ stable}\\ J(y,u) &= TB(y,u)\\ L_1T + L_2C &= 0, \quad TE_d = 0\\ \text{rank}(TE_f(x,u)) &= \text{rank}(E_f(x,u))\\ \text{rank}\left(\begin{bmatrix} G\\ L_2\end{bmatrix}\right) &= \text{rank}(F_f(x,u))\end{aligned}$$

Provided the above requirements are fulfilled, defining the estimation error $e = Tx - \hat{x}$, the dynamics of the residual are governed by

$$\begin{aligned}\dot{e} &= Fe - GE_f(x,u)f_f + TF_f(x,u)f_s\\ r &= L_1e + L_2F_f(x,u)f_s\end{aligned}$$

A drawback of the unknown input observer approach is that the class of the systems covered by this technique is very limited. There are some methods which can transform other nonlinear model to the form suitable for unknown input observer design approach, however, the existence conditions for such transformations are very restrictive. Even if the existence conditions are satisfied, finding the transformations involves the solution of higher order partial differential equations [8].

Disturbance decoupling nonlinear observer approach

To relax the existence conditions for the nonlinear unknown input observers, the idea was extended by [90, 91] to disturbance decoupling nonlinear observers. A more general class of nonlinear systems was considered in this approach.

$$\begin{aligned} \dot{x} &= a(x,u) + E_d(x)d + E_f(x,u)f \\ y &= c(x,u) \end{aligned}$$

To decouple the states from disturbances, a nonlinear state transformation $z = T(x)$ is used and the transformed system becomes

$$\dot{T}(x) = \frac{\partial T(x)}{\partial x}\Big(a(x,u) + E_d(x)d + E_f(x,u)f \Big)$$

The transformation should be such that the transformed system becomes unaffected by disturbances but still reflects the effect of faults. The desired transformation can be selected as

$$\frac{\partial T(x)}{\partial x} E_d(x) = 0 \tag{2.2}$$

If such a transformation exists, the transformed system is given by

$$\begin{aligned} \dot{z} &= \frac{\partial T(x)}{\partial x}\Big(a(x,u) + E_f(x,u)f_f \Big) \\ y^* &= c^*(z,u,y) \end{aligned}$$

which no longer has the effect of disturbances. Seliger and Frank [91] have also discussed a special case when the disturbance distribution matrix is also dependent on u. In that case the required transformation will also depend on u. Conditions for the existence of transformation (2.2) are also derived in [91]. If the existence conditions are satisfied, the problem remains solving the partial differential equations (2.2). After the transformation is achieved, an observer can be designed using any observer design method, e.g., it can be designed using nonlinear identity observer approach etc.

High-gain observer approach

High gain observer was initially proposed in [92–94] and was applied to fault detection in [95–98]. The class of nonlinear systems covered by the approach is represented by

$$\begin{aligned} \dot{x} &= Ax + g(x) + \sum_{i=1}^{m} u_i \psi_i(x) \\ y &= Cx \end{aligned} \tag{2.3}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_n(x) \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix},$$

$$\psi_i(x) = [\ \psi_{i1}(x) \quad \psi_{i2}(x) \quad \cdots \quad \psi_{in}(x)\ ]^T$$

Note that the class of systems represented by (2.3) is fairly general, as most of the affine nonlinear systems can be transformed into (2.3). The required transformation can be found in [95]. With the assumption that $g(x)$ and ψ_i are globally Lipschitz, an observer for (2.3) is of the form

$$\dot{\hat{x}} = A\hat{x} + g(\hat{x}) + \sum_{i=1}^{m} u_i \psi_i(\hat{x}) - S_\theta^{-1} C^T (C\hat{x} - y)$$

where S_θ is the unique solution of the Lyapunov algebraic equation

$$\theta S_\theta + A^T S_\theta + S_\theta A = C^T C \tag{2.4}$$

The high gain observer design approach was extended to a more general class of nonlinear systems in [95]. The class of systems is

$$\begin{aligned} \dot{x} &= A(t)x + \psi(t,u,x) \\ y &= Cx \end{aligned} \tag{2.5}$$

where

$$A(t) = \begin{pmatrix} 0 & a_1(t) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & a_{n-1}(t) \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix},$$

and the ith component $\psi_i(t,u,x)$ is such that $\psi_i(t,u,x) = \psi_i(t,u,x_1,\cdots,x_i)$. Furthermore, the following two assumptions are satisfied

- ψ is globally Lipschitz w.r.t. x and t, locally w.r.t. u
- a_i, $i = 1,\cdots,n-1$ are known differentiable functions with unknown derivatives, and there exist $\epsilon > 0, M > 0, M' > 0$ such that, for every $t \geq 0$, $\epsilon \leq |a_i(t)| \leq M$ and $|\frac{d}{dt}a_i(t)| \leq M'$ for $i = 1,\cdots,n-1$

then an observer for (2.5) is of the form

$$\dot{\hat{x}} = A(t)\hat{x} + \psi(t,u,\hat{x}) - \Lambda^{-1}(t) S_\theta^{-1} C^T (C\hat{x} - y)$$

where S_θ is the solution of (2.4) and

$$\Lambda(t) = \mathrm{diag}\{1, a_1(t), a_1(t)a_2(t), \cdots, a_1(t)\cdots a_{n-1}(t)\}$$

Sliding mode observer approach

Sliding mode observers have been vastly applied to fault detection in linear systems [99–102] as well as in nonlinear systems [103–107]. The inherent property of sliding mode observers of being robust to uncertainties and disturbances makes them suitable for state estimation and fault detection. Designing a sliding mode observer consists of two steps, construction of a sliding surface and designing a control law which drives the system trajectories to the sliding surface in finite time. As the trajectories reach to the sliding surface, they become insensitive to the external disturbances. In below, we describe the major steps involved in the design of sliding mode observer. The discussion is based on the results from [104]. Consider the class of nonlinear systems described by

$$\begin{aligned} \dot{x} &= Ax + g(x,u) + E\psi(x,u,t) \\ y &= Cx \end{aligned} \tag{2.6}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the sate vector, input vector and the output vector, respectively. $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times r}$, $D \in \mathbb{R}^{n\times q}$ and $C \in \mathbb{R}^{p\times n}$ are the constant matrices. The unknown nonlinear term $g(x,u)$ is Lipschitz with respect to x, the unknown nonlinear term $\psi(x,u,t)$ represents the modeling uncertainties and disturbances and is bounded by a known Lipschitz function $\xi(x,u,t)$, i.e.,

$$\|\psi(x,u,t)\| \leq \xi(x,u,t)$$

Under the assumption that $rank(C[E\ D]) = rank([E\ D])$, there exists a coordinate system in which the triple $(A,[E\ D],C)$ has the following structure

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} 0_{(n-p)\times r} & 0_{(n-p)\times q} \\ E_2 & D_2 \end{bmatrix}, \begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix},\right)$$

where $A_1 \in \mathbb{R}^{(n-p)\times(n-p)}$, $C_2 \in \mathbb{R}^{p\times p}$ are nonsingular. Hence, the system (2.6) can be represented as

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_2x_2 + g_1(x,u) \\ \dot{x}_2 &= A_3x_1 + A_4x_2 + g_2(x,u) + E_2\psi(x,u,t) \\ y &= C_2x_2 \end{aligned} \tag{2.7}$$

If all the invariant zeros of the matrix triple $(A,[E\ D],C)$ lie in the left half plane, then there exists a matrix L which has the structure $L = [L_1\ 0]$ such that $A_1 + LA_3$ is stable. Applying the coordinate transformation

$$z = \begin{bmatrix} I_{n-p} & L \\ 0 & I_p \end{bmatrix} x$$

to the system (2.7) results into

$$\begin{aligned} \dot{z}_1 &= (A_1 + LA_3)z_1 + (A_2 + LA_4 - (A_1 + LA_3)L)\, z_2 + [I_{n-p}\ L]g(T^{-1}z,u) \\ \dot{z}_2 &= A_3z_1 + (A_4 - A_3L)z_2 + g_2(T^{-1}z,u) + E_2\psi(T^{-1}z,u,t) \\ y &= C_2z_2 \end{aligned} \tag{2.8}$$

Then a sliding mode observer for (2.8) is given as

$$\begin{aligned} \dot{\hat{z}}_1 &= (A_1 + LA_3)\hat{z}_1 + (A_2 + LA_4 - (A_1 + LA_3)L)C_2^{-1}y + [I_{n-p}\ L]g(T^{-1}\hat{z},u) \\ \dot{\hat{z}}_2 &= A_3\hat{z}_1 + (A_4 - A_3L)\hat{z}_2 - K(y - C_2\hat{z}_2) + g_2(T^{-1}\hat{z},u) + \nu(t,u,y,\hat{y},\hat{z}) \\ \hat{y} &= C_2\hat{z}_2 \end{aligned} \tag{2.9}$$

The gain matrix K is chosen such that $C_2(A_2 - A_3L)C_2^{-1} + C_2K$ is symmetric negative definite. The function ν is defined by

$$\nu = k(\cdot)C_2^{-1}\frac{y - \hat{y}}{\|y - \hat{y}\|} \quad \text{if } y - \hat{y} \neq 0 \tag{2.10}$$

where $k(\cdot)$ is a positive scalar function to be determined. With the state estimation error defined by $e_1 = z_1 - \hat{z}_1$ and $e_y = y - \hat{y} = C_2(z_2 - \hat{z}_2)$, the motion of estimation error dynamics associated with the the sliding surface defined by

$$\mathcal{S} = \{(e_1, e_y)|e_y = 0\}$$

is stable if the matrix inequality

$$\bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \frac{1}{\epsilon}\bar{P}\bar{P}^T + \epsilon\gamma_g^2 I_{n-p} + \alpha P < 0$$

is solvable for $\bar{P}$. Where $\bar{P} = P[I_{n-p}\ L]$, $\bar{A} = [A_1\ A_3]^T$, $P > 0$, α and ϵ are positive constants and γ_g is the Lipschitz constant for the $g(x, u)$ with respect to x. The scalar function $k(\cdot)$ is chosen to satisfy

$$k(t, u, y, \hat{z}) \geq (\|C_2A_3\| + \|C_2\|\gamma_g + \|C_2E\|\gamma_\xi)\hat{w} + \|C_2E_2\|\xi(T^{-1}\hat{z}, u, t) + \|C_2D_2\|\rho(y, u, t) + \eta$$

where η is a positive constant and $\hat{w}$ is the solution to the differential equation

$$\dot{\hat{w}} = -\frac{1}{2}\alpha\hat{w}(t)$$

Geometric approach

The nonlinear geometric approach for fault detection, proposed in [108], is nonlinear extension to the detection filter proposed by Massoumnia [109]. The idea is based on constructing a subsystem which is affected by faults and is decoupled from disturbances. This is done by finding an unobservability subspace in which all disturbances are unobservable. Once this subspace has been determined, a simple asymptotic observer is designed for the subsystem, and this guarantees that the disturbances are decoupled. The geometric approach can also be used for fault isolation. In this case all faults, except the one to be isolated, are treated as disturbances and are rendered unobservable to the subsystem. Persis and Isidori [108] proposed recursive algorithm to find the unobservability subspace.

The geometric approach suffers from the same disadvantages as the unknown input observer techniques. For certain systems, such an unobservability subspace may not exist or faults may also belong to the same subspace as the disturbances.

Game theoretic approach for observer design

Game theory has been utilized to design observers for linear [110–112] as well as nonlinear systems [113–118], specifically in the context of fault detection [50, 119, 120]. The advantage of applying game theory is that extreme case scenario can be treated very easily and the $\mathcal{H}_\infty$ disturbance attenuating observers can be designed for nonlinear systems. Another advantage is that more general class of nonlinear systems can be handled.

To design $\mathcal{H}_\infty$ disturbance attenuation filter using game theory, the fundamental idea is to find out the worst possible disturbance and then to design an observer gain which achieves the desired attenuation of the disturbance. If the designed observer can achieve a desired level of attenuation α to the worst possible disturbance, all other disturbances which are not the worst disturbance will be clearly attenuated more than that attenuation level α. Below we present game theoretic observer for nonlinear systems. It should be noted that this is not a fault detection filter, rather a state estimation filter. The objective to present it here is that our approach presented in Chapter 2 is base on similar ideas.

For the class of nonlinear systems described by

$$\begin{aligned}\dot{x} &= a(x,t) + E_d(x,t)d \\ y &= c(x,t) + F_d(x,t)d\end{aligned}$$

which satisfies the following assumptions

- a, E_d and F_d are smooth functions in x for every t, and continuous in t
- a has an equilibrium point x_0, i.e., $a(x_0,t) = 0 \ \forall t$
- $c(x_0,t) = 0 \ \forall t$
- x_0 is asymptotically stable point of f
- $E_d(x,t)F_d^T(x,t) = 0 \quad \forall t$

Berman [114] proposed the following observer to ensure the $\mathcal{H}_\infty$ disturbance attenuation level greater than or equal to α

$$\begin{aligned}\dot{\hat{x}} &= a(\hat{x},t) + L(\hat{x},t)[y - c(\hat{x})] \\ \hat{y} &= c(\hat{x},t)\end{aligned}$$

The filter gain matrix $L(\hat{x},t)$ is the solution of the following equation

$$V_{\hat{x}}L(\hat{x},t) = -2\alpha^2[c(x,t) - c(\hat{x},t)]^T(F_d(x,t)F_d^T(x,t))^{-1}$$

where $V_{\hat{x}} = \frac{\partial V}{\partial \hat{x}}$ is the solution of the following partial differential inequality

$$\begin{aligned}\frac{\partial V}{\partial t} &+ V_x a(x,t) + V_{\hat{x}} a(\hat{x},t) \\ &- \alpha^2[c(x,t) - c(\hat{x},t)](F_d(x,t)F_d^T(x,t))^{-1}[c(x,t) - c(\hat{x},t)] \\ &+ [c(x,t) - c(\hat{x},t)][c(x,t) - c(\hat{x},t)]^T + \frac{1}{4\alpha^2}V_x(E_d(x,t)E_d^T(x,t))V_x^T \leq 0\end{aligned}$$

An advantage of the above approach is that more general class of systems can be treated. An associated disadvantage is that the approach is restricted by the demand to solve nonlinear partial differential equation (inequality).

Another very interesting approach utilizing the game theory for optimal sensitive fault detection filter design was introduced in [120]. To describe the method, consider a class of nonlinear systems described by

$$\begin{aligned}\dot{x}_1 &= a_1(x_1,u) + a_2(x_1,u)x_2 + E_f(x_1,u)f \\ y_1 &= c(x_1) + d_1 \\ y_2 &= x_2 + d_2\end{aligned}$$

It is desired to find a filter such that for some fixed positive number α and a given choice of positive definite matrices Q, M, N, V, Π_0, the cost function

$$J(f, d_1, d_2) = \int_0^{t_1} [|c(x_1) - c(\hat{x}_1)|_Q^2 + |f|_{N^{-1}}^2 - \alpha^2(|d_2|_{N^{-1}}^2 + |d_1|_{V^{-1}}^2)]dt - |x_1(0)|_{\alpha^2\Pi_0}^2 \quad (2.11)$$

is such that

$$\sup_{d_1, d_2} \inf_f J(f, d_1, d_2) \leq 0 \quad (2.12)$$

Based upon the above cost function, the following filter is proposed

$$\dot{\hat{x}}_1 = a_1(\hat{x}_1, u) + a_2(\hat{x}_1, u)y_2 + 2\alpha^2 Y_{x_1x_1}^{-1}(\hat{x}_1, t)\left(\frac{\partial c}{\partial x_1}(\hat{x}_1)\right)V^{-1}(y_1 - c(\hat{x}_1))$$

where $Y(x_1, t)$, defined on $\mathbb{R}^n \times [0, t_1] \to \mathbb{R}$, is twice continuously differential function with respect to both x_1 and t and $Y_{x_1x_1}(\hat{x}_1, t)$ is nonsingular for all $r \in [0, t_1]$. Furthermore, $Y(x_1, t)$ satisfies the following properties:

1. $Y(x_1, t) \geq 0$ for all $(x_1, t) \in \mathbb{R}^n \times [0, t_1]$, and $Y(x_1, 0) = |x_1|_{\alpha^2\Pi_0}^2$
2. there exists a function $\hat{x}_1 : [0, t_1] \to \mathbb{R}^n$ which is unique minimum of $Y(x_1, t)$ with respect to x_1 for each fixed t
3. $Y(x_1, t)$ satisfies the following partial differential equation

$$\begin{aligned} &Y_t(x_1, t) + Y(x_1, t)\left(a_1(u, x_1) + a_2(u, x_1)y_2\right) \\ &\quad + \frac{1}{\alpha^2}Y_{x_1}\left(a_2(u, x_1)Ma_2^T(u, x_1) - \alpha^2 E_f(u, x_1)NE_f^T(u, x_1)\right)Y_{x_1}^T(x_1, t) \\ &\quad\quad + |c(x_1) - c(\hat{x}_1)|_Q^2 - \alpha^2|y_1 - c(x_1)|_{V^{-1}}^2 + \alpha^2|y_1 - c(\hat{x}_1)|_{V^{-1}}^2 = 0 \end{aligned}$$

   for all $t \in [0, t_1]$

From the cost functions defined in (2.11) and (2.12), it can be seen that the disturbance attenuation problem in the presence of worst-case fault (the fault to which the residual is minimally sensitive) is solved. So, this filter should not be understood as a fault sensitive filter, rather only disturbance attenuation property is achieved by this filter.

2.4.2 Residual evaluation methods

After residual generation, the second step in model-based fault detection scheme is residual evaluation. In this step, the residual signal is manipulated to indicate the occurrence of fault. In ideal situations when there is no disturbances or their effect on the residual signal is completely eliminated, there are no modeling uncertainties and the initial conditions of the observer are the same as that of the process, the residual signal is zero in fault-free case. In that case, any deviation of residual from zero will indicate the presence of faults. However, these ideal situations are never attained and there are always modeling errors, initial conditions of the observer may be different from that of the process. This causes the residual signal to deviate from zero even in the absence of faults. The purpose of residual evaluation is to decide about the occurrence of faults even in the presence of disturbances and uncertainties.

As shown in Figure 2.9, residual evaluation consists of three stages; these are residual processing, threshold selection and decision making. These are described below.

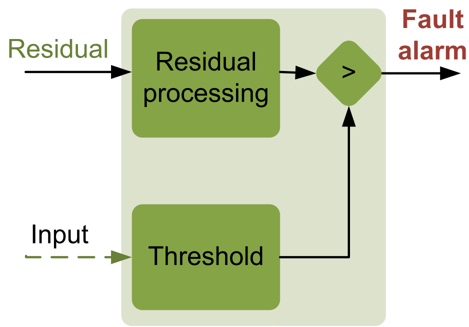


Figure 2.9: Residual evaluation

Residual processing

Based on the type of the monitored system, two strategies for residual processing have been used. For deterministic systems, norm-based residual processing strategy is preferred and for stochastic systems, statistical methods are adopted. Several different evaluation functions have been proposed in literature. A detailed study can be found in [13]. In deterministic settings, $\mathcal{L}_2$-norm is the most commonly used evaluation function and is defined as

$$\|r\|_2 = \sqrt{\int_0^\infty r^T r dt}$$

The implementation of $\mathcal{L}_2$-norm is not feasible, since the value of $\|r\|_2$ is not known till $t = \infty$. Therefore, it is actually implemented over a time window as

$$\|r\|_{2,[t-\tau,t]} = \sqrt{\int_{t-\tau}^{t} r^T r dt}$$

The fault detection systems are designed based on $\|r\|_2$ and realized based on $\|r\|_{2,[t-\tau,t]}$, this results in loss of the optimality of fault detection system. A detailed study on the influence of using $\|r\|_{2,[t-\tau,t]}$ instead of $\|r\|_2$ on the performance of fault detection system has been studied in [121]. In some cases, the RMS value of the residual signal is used as evaluation function

$$\|r\|_{RMS} = \sqrt{\frac{1}{\tau}\int_0^\tau r^T r dt}$$

Another evaluation function proposed in [122] is described as

$$\mathscr{S}_v r \triangleq \int_0^t v(t-\tau)|r|dt$$

where v is a weighting function to increase the influence of the most recent data. Further discussion on this evaluation function will be made in Chapter 4. A generalized version of the above evaluation function is given as

$$\mathscr{S}_v^p r \triangleq \left(\int_0^t v(t-\tau)|r|^p dt \right)^{1/p}$$

Other commonly used evaluation functions for deterministic systems include absolute value, peak value, average value or moving average etc. In stochastic settings, frequently used evaluation functions are mean, variance, likelihood ratio (LR), generalized likelihood ratio (GLR) etc. This thesis is restricted to only deterministic nonlinear systems, further details about stochastic evaluation functions could be found in [13, 14, 123, 124].

Threshold selection

Selection of threshold is important for a fault detection system. If threshold is selected too low, it will result in false alarms, i.e. some of disturbances will cause the residual to cross the threshold and result in an alarm. If the threshold is selected too high, small faults will not be detected. A detailed study of different threshold selection methods and their computation details for linear systems can be found in [13]. Although the residual generation for nonlinear systems has been extensively studied, only a little attention has been paid to threshold computation for nonlinear systems [125–127]. In deterministic settings, threshold is usually selected slightly higher than the supremum value of evaluated residual signal in fault-free case. For example, in the case when $\mathcal{L}_2$ is used as evaluation function, the threshold J_{th} will be defined by

$$J_{th} = \alpha\delta_d$$

where $\alpha = sup_{d\neq 0, f=0}\frac{\|r\|_2}{\|d\|_2}$ and $\|d\|_2 \leq \delta_d$, d and f represent the disturbance and fault respectively. Computation of α for nonlinear systems requires the solution of partial differential equations. For Lipschitz nonlinear systems, [125] has shown that these partial differential equation can be reduced to differential equation and can be formulated into LMI which can be easily solved using MATLAB. Selection of supremum value of evaluated residual signal in fault free case results into a conservative threshold which can cause miss-detection of faults, therefore, some trade-off approaches for threshold selection can be used which will be discussed in Chapter 5. Instead of using a constant threshold, a variable threshold can achieve better fault detection. In Chapter 4, we will develop a variable threshold scheme for a class of nonlinear systems.

Decision logic

The third step in residual evaluation is the decision logic. The simplest decision logic is to compare the evaluated residual signal with the threshold, if the evaluated residual exceeds the threshold, the fault-alarm is released, i.e.,

$$\begin{aligned} J > J_{th} &\Rightarrow \text{ fault} \\ J \leq J_{th} &\Rightarrow \text{ fault-free} \end{aligned}$$

There are also some approaches which use fuzzy logic or neural networks for residual evaluation, see e.g. [11, 83] for an introductory study. These approaches can be used to perform either all the three steps of residual evaluation or only one of these steps.

2.5 Summary

This chapter introduced to the fundamental concepts in fault detection with focus on fault detection of nonlinear systems. Definitions of elementary nomenclature such as fault, failure, fault detection, fault identification and fault isolation were provided. The difference between faults and disturbances was elaborated. A classification of fault detection schemes was presented elaborating the main features of each approach. A particular attention was paid to observer-based fault detection schemes, their robustness properties were discussed and several approaches developed over the past for robust residual generation were introduced. At the end, some state of the art fault detection techniques for nonlinear systems were presented. Commonly used nonlinear observers for residual generation were explained. A brief description of residual evaluation and threshold computation for nonlinear systems was described.

Chapter 3

Residual generation with $\mathcal{H}_-$, $\mathcal{H}_\infty$ and $\mathcal{H}_-/\mathcal{H}_\infty$ optimizations

The contribution of this chapter includes the design of fault detection filters for residual generation in nonlinear systems. Sensitivity to faults and robustness against disturbances are desired features of a residual generation scheme. Utilizing the game theoretic approach, $\mathcal{H}_-$ index based fault sensitive, $\mathcal{H}_\infty$ norm based disturbance attenuating and $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective fault detection filters are designed. The proposed methods and their effectiveness are illustrated by an academic example.

Residual generation is an important step in the design of observer-based fault detection system. It is desired that the residual signal is significantly sensitive to faults and robust against all other inputs (including unknown inputs such as disturbances and measurement noise, and known inputs such as the reference input). To achieve these desired aspects, several techniques for designing observers for fault detection in nonlinear systems have been proposed in literature, some of them were presented in Chapter 2. Based on the disturbance handling property, these techniques can broadly be classified in two categories – the first category of the approaches aims at complete decoupling of the effect of disturbances on residual signal. Nonlinear unknown input observer, disturbance decoupling nonlinear observer, geometric approach for nonlinear observer design belong to this class. The second category of approaches does not make complete decoupling of disturbances, rather, the effect of disturbances on the residual signal is attenuated. High-gain observers, game-theoretic approach based observers belong to this second category. Due to quite stringent existence conditions for disturbance decoupling approaches, the second category of approaches are getting more attention in recent years [114, 120]. The techniques proposed in this chapter also belong to this second category of approaches, and disturbance attenuation rather than complete decoupling is addressed. Together with the disturbance attenuation, we also focus on fault sensitivity of the residual signal. To measure the degree of disturbance attenuation, $\mathcal{H}_\infty$ norm is extensively used. Likewise, the so called $\mathcal{H}_-$ index have been utilized for linear systems to measure the sensitivity to the faults [13, 54–57]. The fault detection filters presented in this chapter are also based on $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index, therefore, it is useful to describe these concepts. We take some examples from linear systems to elaborate $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index and their significance in fault detection.

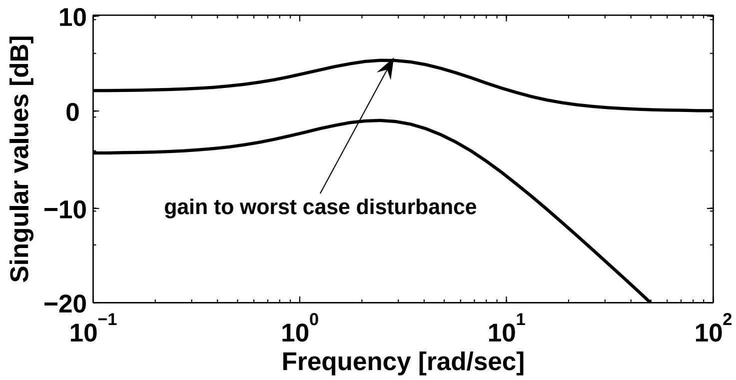


Figure 3.1: The singular value plot for the example system

Example 3.1. *Consider a stable linear system described by*

$$\begin{aligned}\dot{x} &= Ax + E_d d \\ y &= Cx + F_d d\end{aligned} \tag{3.1}$$

with

$$A = \begin{pmatrix} -6 & -8 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -6 & -4 \\ 0 & 0 & 2 & 0 \end{pmatrix}, E_d = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2.5 & 1 & 0 & 0.25 \\ 0 & 0.5 & 2.5 & 1.25 \end{pmatrix}, F_d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The related transfer function is $G_d(s) = C(sI-A)^{-1}E_d+F_d$. The $\mathcal{L}_2$ gain from disturbance d to the signal y changes with the direction and frequency of the disturbance vector. Figure 3.1 shows the plot of principal gains (gain to two extreme cases of disturbance vector directions) for the system. It can be seen that for the disturbance vector in certain direction and of specific frequency, the gain $\|y\|_2/\|d\|_2$ is the maximum, this specific disturbance is called the worst case disturbance (because it has the maximum effects on y) and the corresponding gain is referred to as the $\mathcal{H}_\infty$ norm, i.e.

$$\|G_d\|_\infty = \sup_\omega \bar{\sigma}(G_d(j\omega)) = \sup_{d \neq 0} \frac{\|y\|_2}{\|d\|_2}$$

where $\bar{\sigma}(G_d(j\omega))$ represents the upper principal gain of $G_d(j\omega)$ and $\|\cdot\|_2$ represents the $\mathcal{L}_2$ norm of a signal.

It is clear that if the fault detection filter is so designed to achieve a specific level of attenuation α to the worst case disturbance, surely any disturbance will have attenuation greater than or equal to α. The concept of worst case disturbance and hence $\mathcal{H}_\infty$ disturbance attenuation can be generalized to nonlinear systems. Consider a nonlinear system described by

$$\Sigma_{\mathscr{D}} : \begin{cases} \dot{x} = a(x) + E_d(x)d \\ y = c(x) + F_d(x)d \end{cases} \tag{3.2}$$

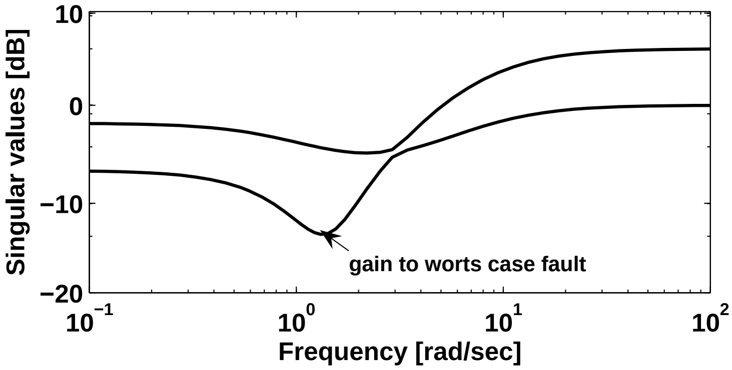


Figure 3.2: The singular value plot for the example system

then the $\mathcal{H}_\infty$ norm for the nonlinear system $\Sigma_{\mathscr{D}}$ is given as

$$\|\Sigma_{\mathscr{D}}\|_\infty = \sup_{d\neq 0} \frac{\|y\|_2}{\|d\|_2}$$

To explain the concept of $\mathcal{H}_-$ fault sensitivity, we examine an example of a linear system.

Example 3.2. *Consider a linear system described by*

$$\begin{aligned} \dot{x} &= Ax + E_f f \\ y &= Cx + F_f f \end{aligned} \tag{3.3}$$

$$A = \begin{pmatrix} -9 & -4 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -9 & -4 \\ 0 & 0 & 2 & 0 \end{pmatrix}, E_f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & -0.5 & 0 & 0.125 \\ 0 & 0.25 & -4 & -1.5 \end{pmatrix}, F_f = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The related transfer function is $G_f(s) = C(sI - A)^{-1}E_f + F_f$. The $\mathcal{L}_2$ gain from fault f to the signal y changes with the direction and frequency of the fault vector. Figure 3.2 shows the plot of principal gains for the system. It can be seen that for the fault vector in certain direction and specific frequency, the gain $\|y\|_2/\|f\|_2$ is the minimum, this specific fault is called the worst case fault (because it has the minimum effects on y and is, therefore difficult to be detected as compared to a fault of same magnitude with different direction) and the corresponding gain is referred to as the $\mathcal{H}_-$ index, i.e.

$$\|G_f\|_- = \inf_{\omega} \underline{\sigma}(G_f(j\omega)) = \inf_{f\neq 0} \frac{\|y\|_2}{\|f\|_2}$$

where $\underline{\sigma}(G_f(j\omega))$ represents the lower principal gain of $G_f(j\omega)$.

It is obvious that if the worst fault has an $\mathcal{L}_2$ gain $\|y\|_2 / \|f\|_2$ greater than or equal to a given positive constant β, then all the faults will have gain greater than or equal to β. The concept of $\mathcal{H}_-$ can be generalized to nonlinear systems. For a nonlinear system described by

$$\Sigma_{\mathscr{F}} : \begin{cases} \dot{x} = a(x) + E_f(x)f \\ y = c(x) + F_f(x)f \end{cases} \tag{3.4}$$

$\mathcal{H}_-$ index for the system $\Sigma_{\mathscr{F}}$ can be defined as

$$\|\Sigma_{\mathscr{F}}\|_- = \inf_{f \neq 0} \frac{\|y\|_2}{\|f\|_2}$$

For a more practical application, it is useful to define $\mathcal{H}_\infty$ norm and $\mathcal{H}_-$ index over a finite horizon $[0, t_1]$ as described in the following definitions.

Definition 3.1. *Given nonlinear system* (3.2)*, the $\mathcal{H}_\infty$ norm of the system $\Sigma_{\mathscr{D}}$ in finite horizon case represented by $\|\Sigma_{\mathscr{D}}\|_{\infty,[0,t_1]}$ is defined as*

$$\|\Sigma_{\mathscr{D}}\|_{\infty,[0,t_1]} = \sup_{d \neq 0} \frac{\|y\|_{2,[0,t_1]}}{\|d\|_{2,[0,t_1]}}$$

where $\|y\|_{2,[0,t_1]} = \sqrt{\int_0^{t_1} y^T y \, dt}$

Definition 3.2. *Given nonlinear system* (3.4)*, the $\mathcal{H}_-$ index of the system $\Sigma_{\mathscr{F}}$ in finite horizon case represented by $\|\Sigma_{\mathscr{F}}\|_{-,[0,t_1]}$ is defined as*

$$\|\Sigma_{\mathscr{F}}\|_{-,[0,t_1]} = \inf_{f \neq 0} \frac{\|y\|_{2,[0,t_1]}}{\|f\|_{2,[0,t_1]}}$$

$\mathcal{H}_-$ fault sensitive filter design has received considerable interests for linear systems [53–55]. As far as nonlinear systems are concerned, $\mathcal{H}_-$ fault sensitivity problem has not been addressed. The $\mathcal{H}_-$ fault sensitive FDF design problem for nonlinear systems will be formulated in Section 3.1 and a solution will be presented in Section 3.2.

There has been an extensive research to design $\mathcal{H}_\infty$ disturbance attenuation filter, both for linear [50, 128–131] as well as nonlinear systems [114, 117, 120, 132]. However, that research is mainly focused on state estimation rather than output estimation. In Section 3.3, the difference between the two estimation problems will be highlighted. For the purpose of fault detection, output estimation is of importance and state estimation is of little use. To our knowledge, the $\mathcal{H}_\infty$ disturbance attenuation problem for fault detection in nonlinear systems has not been studied. The problem of $\mathcal{H}_\infty$ disturbance attenuation FDF will be formulated in Section 3.1 and a solution to this problem will be presented in Section 3.3.

The $\mathcal{H}_\infty$ based fault detection filter attenuates the effects of disturbances on the residual signal, but at the same time, may also result in the attenuation of the effects of faults on the residual signal, which is not desired. Similarly, the $\mathcal{H}_-$ index based fault detection filter makes the residual more sensitive to faults, but at the same time may also result in amplification of the effect of disturbances on the residual signal, this is also not desired. To attain simultaneous $\mathcal{H}_-$ fault sensitivity and $\mathcal{H}_\infty$ disturbance attenuation, the multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ index was proposed in [56]. The multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ index based fault detection filter for linear systems has been widely studied and LMI solutions are presented [13, 55]. This problem for nonlinear systems is formulated in Section 3.1 and the solution is provided in Section 3.4.

3.1 Problem formulation

Consider the affine nonlinear systems described by

$$\begin{cases} \dot{x} = a(x) + E_d(x)d + E_f(x)f \\ y = c(x) + F_d(x)d + F_f(x)f \end{cases} \tag{3.5}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the measurement vector, $d \in \mathbb{R}^{k_d}$ is the vector of unknown inputs (which include disturbances and measurement noise) and $f \in \mathbb{R}^{k_f}$ is the fault vector to be detected, the time argument of these vectors is omitted for notational simplicity. All the elements in the vectors $a(x) \in \mathbb{R}^n$, $c(x) \in \mathbb{R}^m$ and the matrices $E_f(x) \in \mathbb{R}^{n\times k_f}$, $E_d(x) \in \mathbb{R}^{n\times k_d}$, $F_f(x) \in \mathbb{R}^{m\times k_f}$, $F_d(x) \in \mathbb{R}^{m\times k_d}$ are smooth functions of their arguments. For the purpose of fault detection, we apply the following nonlinear FDF

$$\begin{cases} \dot{\hat{x}} = a(\hat{x}) + L(\hat{x},t)(y - \hat{y}) \\ \hat{y} = c(\hat{x}) \\ r = y - \hat{y} \end{cases} \tag{3.6}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of the state vector, $\hat{y} \in \mathbb{R}^m$ is the estimate of the measurement vector, $L(\hat{x},t) \in \mathbb{R}^{n\times m}$ is the filter gain matrix and $r \in \mathbb{R}^m$ is residual vector which is used to decide about the occurrence of faults. Combining the the system state vector and the filter state vector, the dynamics of the residual generator become

$$\Sigma_{\mathscr{A}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_d(\tilde{x})d + \tilde{E}_f(\tilde{x})f \\ r = \tilde{c}(\tilde{x}) + F_d(x)d + F_f(x)f \end{cases} \tag{3.7}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \tilde{a}(\tilde{x}) = \begin{bmatrix} a(x) \\ a(\hat{x}) + L(\hat{x},t)\tilde{c}(\tilde{x}) \end{bmatrix}$$

$$\tilde{E}_f(\tilde{x}) = \begin{bmatrix} E_f(x) \\ L(\hat{x},t)F_f(x) \end{bmatrix}, \tilde{E}_d(\tilde{x}) = \begin{bmatrix} E_d(x) \\ L(\hat{x},t)F_d(x) \end{bmatrix}, \tilde{c}(\tilde{x}) = c(x) - c(\hat{x})$$

To achieve the sensitivity and robustness features of the fault detection filter, the following three filtering problems are formulated.

Problem 3.1. $\mathcal{H}_-$ fault detection filter design
Given system (3.7) *and a constant $\beta > 0$, find a filter gain matrix $L(\hat{x},t)$ such that the $\mathcal{L}_2$ gain from fault to the residual is greater than or equal to β, i.e.,*

$$\|\Sigma_{\mathscr{A}}\|_{-,[0,t_1]} = \inf_{d=0,f\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|f\|_{2,[0,t_1]}} \geq \beta \tag{3.8}$$

Problem 3.2. $\mathcal{H}_\infty$ fault detection filter design
Given system (3.7) *and a constant $\alpha > 0$, find a filter gain matrix $L(\hat{x},t)$ such that the $\mathcal{L}_2$ gain from disturbance to the residual is less than or equal to α, i.e.,*

$$\|\Sigma_{\mathscr{A}}\|_{\infty,[0,t_1]} = \sup_{f=0,d\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|d\|_{2,[0,t_1]}} \leq \alpha \tag{3.9}$$

Problem 3.3. $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection filter design
Given system (3.7) *and two scalars $\alpha > 0$ and $\beta > 0$, find a filter gain matrix $L(\hat{x},t)$ such that*

1. *The $\mathcal{L}_2$ gain from disturbance to the residual is less than or equal to α, i.e.,*

$$\|\Sigma_{\mathscr{A}}\|_{\infty,[0,t_1]} = \sup_{f=0,d\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|d\|_{2,[0,t_1]}} \leq \alpha \tag{3.10}$$

2. *The $\mathcal{L}_2$ gain from fault to the residual is greater than or equal β, i.e.,*

$$\|\Sigma_{\mathscr{A}}\|_{-,[0,t_1]} = \inf_{d=0,f\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|f\|_{2,[0,t_1]}} \geq \beta \tag{3.11}$$

Remark 3.1. *In the above formulation, the effect of known inputs u has not been taken into account. However, in any fault detection system, it is desired that, additional to robustness against unknown inputs, the residual signal should be robust against the known inputs as well. In linear systems, the effect of known inputs on the residual generator dynamics is canceled out, but this is not the situation in nonlinear systems. Thus we should examine the following augmented system*

$$\begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{B}(\tilde{x})u + \tilde{E}_d(\tilde{x})d + \tilde{E}_f(\tilde{x})f \\ r = \tilde{c}(\tilde{x}) + \tilde{D}(\tilde{x})u + F_d(x)d + F_f(x)f \end{cases}$$

Since it is desired that the residual signal should be robust against known inputs u, we may treat them in the same way as the unknown inputs d, i.e., for the above augmented system, we may write as

$$\begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \begin{bmatrix} \tilde{B}(\tilde{x}) & \tilde{E}_d(\tilde{x}) \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} + \tilde{E}_f(\tilde{x})f \\ r = \tilde{c}(\tilde{x}) + \begin{bmatrix} \tilde{D}(\tilde{x}) & F_d(x) \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} + F_f(x)f \end{cases} \tag{3.12}$$

defining

$$w = \begin{bmatrix} u \\ d \end{bmatrix}, \tilde{E}_w(\tilde{x}) = \begin{bmatrix} \tilde{B}(\tilde{x}) & \tilde{E}_d(\tilde{x}) \end{bmatrix}, F_w(\tilde{x}) = \begin{bmatrix} \tilde{D}(\tilde{x}) & F_d(x) \end{bmatrix}$$

the system (3.12) *is transformed into*

$$\begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_w(\tilde{x})w + \tilde{E}_f(\tilde{x})f \\ r = \tilde{c}(\tilde{x}) + F_w(\tilde{x})w + F_f(x)f \end{cases} \tag{3.13}$$

which is in the same form as (3.7) *and we may perform a similar analysis as will be done for* (3.7).

The approach to solve the above stated filtering problems will be based on two-players zero-sum differential game theory, a brief description of two-players zero-sum differential games is presented in Appendix A. For a more detailed study, the reader is referred to [133], and to [134] for particular focus to control problems.

In next section, a game-theoretic approach to provide a solution of the $\mathcal{H}_-$ FDF design problem is presented.

3.2 $\mathcal{H}_-$ fault sensitive FDF

The minimum sensitivity problem defined in Problem 3.1 is satisfied if the functional defined by

$$\mathscr{J}(L,f) \triangleq \int_0^{t_1} \left\{ r^T r - \beta^2 f^T f \right\} dt \quad ; d = 0 \tag{3.14}$$

is greater than or equal to zero for each possible fault. This can be viewed as a two player zero-sum differential game with the cost functional (3.14). The minimizing player tries to minimize the functional through f and the maximizing player maximizes the functional through L. Below we present a theorem which provides a solution to the sensitivity problem. V_x and $V_{\hat{x}}$ are row vectors of first partial derivatives of V with respect to x and $\hat{x}$, respectively.

Theorem 3.1. *Given system* (3.7)*, a positive constant* β *and assume*
A1. $R_f \triangleq F_f^T(x)F_f(x) - \beta^2 I_{k_f} > 0$
A2. $F_f(x)$ *is a square matrix*
If there exists a function $V(\tilde{x},t)$ *with* $V(\tilde{x}(0),0) = 0$ *that satisfies the following Hamilton-Jacobi-Isaacs (HJI) equation*

$$\begin{aligned} -\frac{\partial V(\tilde{x},t)}{\partial t} &= V_x(\tilde{x},t)a(x) + V_{\hat{x}}(\tilde{x},t)a(\hat{x}) \\ &\quad + \tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1}\tilde{c}(\tilde{x}) - \tilde{c}^T(\tilde{x})\tilde{c}(\tilde{x}) \\ &\quad - V_x(\tilde{x},t)E_f(\tilde{x})(F_f^T(x)F_f(x))^{-1}F_f^T(x)\tilde{c}(\tilde{x}) \end{aligned} \tag{3.15}$$

then $L^\star(\hat{x},t)$ *satisfying*

$$\begin{aligned} V_{\hat{x}}(\tilde{x},t)L^\star(\hat{x},t) &= -V_x(\tilde{x},t)E_f(x)(F_f^T(x)F_f(x))^{-1}F_f^T(x) \\ &\quad + 2\tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \tag{3.16}$$

solves the sensitivity problem (3.8)*.*

Proof. For the augmented system (3.7) with the cost functional (3.14), utilizing the standard concepts from dynamic game theory we conclude that, if there exists continuously differentiable function $V(\tilde{x},t)$ on $[0,t_1] \times \mathbb{R}^n$, satisfying the following Hamilton-Jacobi-Isaacs equation

$$\begin{aligned} -\frac{\partial V(\tilde{x},t)}{\partial t} &= \max_{L \in \mathbb{R}^{n\times m}} \min_{f \in \mathbb{R}^k} \left\{ \frac{\partial V(\tilde{x},t)}{\partial \tilde{x}} \dot{\tilde{x}} + r^T r - \beta^2 f^T f \right\} \\ &= \max_{L \in \mathbb{R}^{n\times m}} \min_{f \in \mathbb{R}^k} \Big\{ V_x(\tilde{x},t)a(x) + V_x(\tilde{x},t)E_f(x)f + V_{\hat{x}}(\tilde{x})a(\hat{x}) \\ &\qquad + V_{\hat{x}}(\tilde{x},t)L(\hat{x},t)\tilde{c}(\tilde{x}) + V_{\hat{x}}(\tilde{x},t)L(\hat{x},t)F_f(x)f \\ &\qquad + r^T r - \beta^2 f^T f \Big\} \\ &= V_x(\tilde{x},t)a(x) + V_x(\tilde{x},t)E_f(x)f^\star + V_{\hat{x}}(\tilde{x},t)a(\hat{x}) \\ &\qquad + V_{\hat{x}}(\tilde{x},t)L^\star(\hat{x},t)\tilde{c}(\tilde{x}) + V_{\hat{x}}(\tilde{x},t)L^\star(\hat{x},t)F_f(x)f^\star \end{aligned}$$

$$+ r^T(f^\star)r(f^\star) - \beta^2 f^{\star T} f^\star \qquad (3.17)$$

then the pair of strategies $(L^\star, f^\star)$ provides a saddle-point solution, i.e. $\mathscr{J}(L, f^\star) \le \mathscr{J}(L^\star, f^\star) \le \mathscr{J}(L^\star, f)$. Furthermore, saddle-point value of the game is $\mathscr{J}(L^\star, f^\star) = V(\tilde{x}(0), 0)$. This ensures $\mathscr{J}(L^\star, f) \ge V(\tilde{x}(0), 0) = 0$, which in turn guarantees that $L^\star$ solves the sensitivity problem (3.8).

Henceforth, for the sake of simplicity of notation, we will not write the arguments of the functions except where it makes ambiguity. Rest of the proof consists of finding $f^\star$ and $L^\star$ and substituting them into (3.17). For that purpose, we define the following Hamiltonian function

$$\begin{aligned}\mathscr{H}(L, f) &\triangleq V_{\tilde{x}}\dot{\tilde{x}} + r^T r - \beta^2 f^T f \\ &= V_x a(x,t) + V_x E_f f + V_{\hat{x}} a(\hat{x}, t) + V_{\hat{x}} L\tilde{c} + V_{\hat{x}} L F_f f + r^T r - \beta^2 f^T f \qquad (3.18)\end{aligned}$$

To find $f^\star$ (the fault which has the minimum effect on the residual), we differentiate $\mathscr{H}$ with respect to f (standard results from matrix calculus are applied, Appendix B gives a table of these results, details can be found in any reference for matrix calculus). It turns out

$$\begin{aligned}\frac{\partial \mathscr{H}(L,f)}{\partial f} &= V_x E_f + V_{\hat{x}} L F_f + 2r^T \frac{\partial r}{\partial f} - 2\beta^2 f^T \\ &= V_x E_f + V_{\hat{x}} L F_f + 2r^T F_f - 2\beta^2 f^T \\ &= V_x E_f + V_{\hat{x}} L F_f + 2\tilde{c}^T F_f + 2 f^T F_f^T F_f - 2\beta^2 f^T\end{aligned}$$

For $f^\star$ to be the critical point, $\left.\frac{\partial \mathscr{H}(L,f)}{\partial f}\right|_{f=f^\star} = 0$. Hence, we get an expression for $f^\star$

$$\begin{aligned} & V_x E_f + V_{\hat{x}} L F_f + 2\tilde{c}^T F_f + 2 f^{\star T} F_f^T F_f - 2\beta^2 f^{\star T} = 0 \\ \Rightarrow \quad & V_x E_f + V_{\hat{x}} L F_f + 2\tilde{c}^T F_f + 2 f^{\star T}(F_f^T F_f - \beta^2 I_{k_f}) = 0 \\ \Rightarrow \quad & f^\star = -\frac{1}{2} R_f^{-1}\left(E_f^T V_x^T + F_f^T L^T V_{\hat{x}}^T + 2F_f^T \tilde{c}\right)\end{aligned}$$

where I_{k_f} represents the identity matrix of dimensions $k_f \times k_f$ and $R_f = (F_f^T F_f - \beta^2 I_{k_f})$. It follows from Assumption $A1$ that $\left.\frac{\partial^2 \mathscr{H}(L,f)}{\partial f^2}\right|_{f=f^\star} = (2F_f^T F_f - 2\beta^2 I_{k_f}) > 0$, which means $f^\star$ really minimizes the Hamiltonian function. Substitute $f = f^\star$ in (3.18), we obtain the following expression

$$\begin{aligned}\mathscr{H}(L, f^\star) = V_x a(x) + V_x E_f f^\star + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L\tilde{c} + V_{\hat{x}} L F_f f^\star \\ + (\tilde{c} + F_f f^\star)^T(\tilde{c} + F_f f^\star) - \beta^2 f^{\star T} f^\star\end{aligned}$$

after simplifications,

$$\begin{aligned}\mathscr{H}(L, f^\star) = V_x a(x) - \frac{1}{4} V_x E_f R_f^{-1} E_f^T V_x^T - \frac{1}{4} V_x E_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_x E_f R_f^{-1} F_f^T \tilde{c} + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L\tilde{c} - \frac{1}{4} V_{\hat{x}} L F_f R_f^{-1} E_f^T V_x^T \\ - \frac{1}{4} V_{\hat{x}} L F_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T - V_{\hat{x}} L F_f R_f^{-1} F_f^T \tilde{c} + \tilde{c}^T \tilde{c} - \tilde{c}^T F_f R_f^{-1} F_f^T \tilde{c} \qquad (3.19)\end{aligned}$$

Since all the terms in above expression are scalars, for the purpose of simplification we may replace some terms by their transpose. As a result:

$$\begin{aligned}\mathscr{H}(L,f^\star) = V_x a(x) - \frac{1}{4}V_x E_f R_f^{-1} E_f^T V_x^T - \frac{1}{2}V_x E_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_x E_f R_f^{-1} F_f^T \tilde{c} + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L\tilde{c} - \frac{1}{4}V_{\hat{x}} L F_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_{\hat{x}} L F_f R_f^{-1} F_f^T \tilde{c} + \tilde{c}^T\tilde{c} - \tilde{c}^T F_f R_f^{-1} F_f^T \tilde{c}\end{aligned}$$

To find $L^\star$, set $\frac{\partial \mathscr{H}(L,f^\star)}{\partial L}\Big|_{L=L^\star} = 0$ to get:

$$V_{\hat{x}}^T \tilde{c}^T - \frac{1}{2}V_{\hat{x}}^T V_x E_f R_f^{-1} F_f^T - \frac{1}{2}V_{\hat{x}}^T V_{\hat{x}} L^\star F_f R_f^{-1} F_f^T - V_{\hat{x}}^T \tilde{c}^T F_f R_f^{-1} F_f^T = 0$$

from which we obtain an expression for $L^\star$

$$V_{\hat{x}} L^\star = -V_x E_f (F_f^T F_f)^{-1} F_f^T + 2\tilde{c}^T (F_f R_f^{-1} F_f^T)^{-1} - 2\tilde{c}^T$$

It remained to prove by taking the second derivative of Hamiltonian, that $L^\star$ maximizes the Hamiltonian. So

$$\frac{\partial^2 \mathscr{H}(L,f^\star)}{\partial L^2}\Big|_{L=L^\star} = -\frac{1}{2}(F_f R_f^{-1} F_f^T \otimes V_{\hat{x}}^T V_{\hat{x}})^T$$

where $\otimes$ represents the Kronecker product of matrices. Since R_f is positive definite by Assumption $A1$, $\frac{\partial^2 \mathscr{H}(L,f^\star)}{\partial L^2}\Big|_{L=L^\star}$ will be negative definite, meaning that $L^\star$ will be maximizing as required. Furthermore, Assumptions $A1$ and $A2$ also ensure that $(F_f^T F_f)^{-1}$ and $(F_f R_f^{-1} F_f^T)^{-1}$ exist. By substituting $f^\star$ and $L^\star$ into (3.17) and performing some simplifications:

$$-\frac{\partial V}{\partial t} = V_x a(x) + V_{\hat{x}} a(\hat{x}) + \tilde{c}^T (F_f R_f^{-1} F_f^T)^{-1}\tilde{c} - \tilde{c}^T\tilde{c} - V_x E_f (F_f^T F_f)^{-1} F_f^T \tilde{c}$$

This completes the proof. □

Remark 3.2. *The Assumption $A1$ is not specific to nonlinear systems only. A similar assumption is also made for linear systems when $\mathcal{H}_-$ index problem is formulated. Since we are only interested in nonzero $\mathcal{H}_-$ index, this assumption ensures that it is greater than zero.*

Remark 3.3. *It should be noted that Assumption $A2$ does not cause any loss of generality. In the case that Assumption $A2$ is not satisfied, some fictitious fault variables can be added to make $F_f(x)$ a square matrix.*

Remark 3.4. *If we look at the expression* (3.16) *very carefully, we notice that, for a general nonlinear system, it would be quite difficult to find an L which is independent of states of the system. However, if we could somehow transform the equality in* (3.15) *to inequality, there would be more flexibility in the selection of $V(\tilde{x},t)$ and, hence, more freedom in selection of L [114]. In that case, it will become more easier to select a state independent filter gain matrix. In this regard, the next remark is useful.*

Remark 3.5. *The minimum sensitivity problem* (3.8) *is satisfied if*

$$\int_0^{t_1} r^T r dt - \beta^2 \int_0^{t_1} f^T f dt \geq 0$$

the above inequality is also satisfied if for a negative semidefinite $Y(\tilde{x}(t),t)$ *with* $Y(\tilde{x}(0),0) = 0$, *the following inequality holds:*

$$\int_0^{t_1} r^T r dt - \beta^2 \int_0^{t_1} f^T f dt + Y(\tilde{x}(t_1),t_1) \geq 0$$

or

$$\int_0^{t_1} r^T r dt - \beta^2 \int_0^{t_1} f^T f dt + \int_0^{t_1} \frac{d}{dt} Y(\tilde{x},t) dt \geq 0$$

which is satisfied if

$$r^T r - \beta^2 f^T f + \frac{d}{dt} Y(\tilde{x},t) \geq 0$$

or

$$-\frac{\partial Y(\tilde{x},t)}{\partial t} \leq \frac{\partial Y(\tilde{x},t)}{\partial \tilde{x}} \dot{\tilde{x}} + r^T r - \beta^2 f^T f \tag{3.20}$$

Note that (3.20) *is similar to* (3.17) *except the inequality sign and that* $Y(\tilde{x},t)$ *must be negative semidefinite. On one hand* (3.20) *would be better because the inequality is easy to deal with (see Remark 3.4), on the other hand, it provides less freedom by restricting* $Y(\tilde{x},t)$ *to be negative semidefinite.*

Here a question arises, "why to use game theory if we can obtain the sufficient conditions for the fault sensitivity, as in (3.20), without applying it?" The answer to this question is that we do not know f, therefore, we have to use its worst-case value. Game theory helps to find the worst-case fault and the corresponding best case filter gain.

Combining the result from Theorem 3.1 and Remark 3.5, we have the following corollary.

Corollary 3.1. *Given system* (3.7), *a positive constant* β *and assume*
A1. $R_f \triangleq F_f^T(x)F_f(x) - \beta^2 I_{k_f} > 0$
A2. $F_f(x)$ *is a square matrix*
If there exists a negative semidefinite function $V(\tilde{x},t)$ *with* $V(\tilde{x}(0),0) = 0$ *that satisfies the following HJI inequality*

$$\begin{aligned} -\frac{\partial V(\tilde{x},t)}{\partial t} \leq\; & V_x(\tilde{x},t)a(x) + V_{\hat{x}}(\tilde{x},t)a(\hat{x}) + \tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1}\tilde{c}(\tilde{x}) \\ & - \tilde{c}^T(\tilde{x})\tilde{c}(\tilde{x}) - V_x(\tilde{x},t)E_f(\tilde{x})(F_f^T(x)F_f(x))^{-1}F_f^T(x)\tilde{c}(\tilde{x}) \end{aligned} \tag{3.21}$$

then $L^\star(\hat{x},t)$ *satisfying*

$$\begin{aligned} V_{\hat{x}}(\tilde{x},t)L^\star(\hat{x},t) =\; & -V_x(\tilde{x},t)E_f(x)(F_f^T(x)F_f(x))^{-1}F_f^T(x) \\ & + 2\tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \tag{3.22}$$

solves the sensitivity problem (3.8).

Proof. The proof follows directly from Theorem 3.1 and Remark 3.5. □

In the case of infinite-horizon filtering problem, i.e., when $t_1 \to \infty$, and if we are interested in time-invariant gain for the filter, the function V can be selected independent of time. In that case, we can summarize the results in the following corollary.

Corollary 3.2. *Given system* (3.7)*, a positive constant* β *and assume*
A1. $R_f \triangleq F_f^T(x)F_f(x) - \beta^2 I_{k_f} > 0$
A2. $F_f(x)$ *is a square matrix*
If there exists a negative semidefinite function $V(\tilde{x})$ *with* $V(\tilde{x}(0)) = 0$ *that satisfies the following HJI inequality*

$$\begin{aligned} 0 \le V_x(\tilde{x})a(x) + V_{\hat{x}}(\tilde{x})a(\hat{x}) + \tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1}\tilde{c}(\tilde{x}) \\ - \tilde{c}^T(\tilde{x})\tilde{c}(\tilde{x}) - V_x(\tilde{x})E_f(\tilde{x})(F_f^T(x)F_f(x))^{-1}F_f^T(x)\tilde{c}(\tilde{x}) \end{aligned} \quad (3.23)$$

then $L^\star(\hat{x})$ *satisfying*

$$\begin{aligned} V_{\hat{x}}(\tilde{x})L^\star(\hat{x}) = -V_x(\tilde{x})E_f(x)(F_f^T(x)F_f(x))^{-1}F_f^T(x) \\ + 2\tilde{c}^T(\tilde{x})(F_f(x)R_f^{-1}F_f^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \quad (3.24)$$

solves the sensitivity problem (3.8) *in infinite horizon sense.*

Proof. The proof follows from Corollary 3.1 by substituting a time invariant function $V(\tilde{x})$. □

Note that the $\mathcal{H}_-$ fault sensitivity conditions derived above do not guarantee the stability of the filter. This is different from the $\mathcal{H}_\infty$ disturbance attenuation problem, as will be discussed in the next section, where the disturbance attenuation condition also pledges the stability of the filter. Therefore, in designing the $\mathcal{H}_-$ fault sensitive filter, the stability issue should be separately addressed. This can be done by selecting a positive definite Lyapunov function and showing the negative definiteness of its time derivative along the augmented system trajectories. Another common approach is to design using multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ optimization, as will be described in Section 3.4. In this situation, the $\mathcal{H}_-$ index gives the required level of fault sensitivity, and the stability is guaranteed by the $\mathcal{H}_\infty$ optimization.

Remark 3.6. *Our solution to* $\mathcal{H}_-$ *fault detection filter design involves solving an HJI inequality. We come across such HJI inequalities (equations) mostly in dealing with non-linear systems. The analytical solution of such (inequalities) equations is, in general, hard to obtain and some numerical approach should be utilized.*

In case of linear systems, i.e.

$$\begin{aligned} \dot{x} &= Ax + E_f f + E_d d \\ y &= Cx + F_f f + F_d d \end{aligned} \quad (3.25)$$

with a linear fault detection filter

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L(\hat{x})(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \quad (3.26)$$

The simplest negative definite function to choose is $V(\tilde{x}) = (x-\hat{x})^T P(x-\hat{x})$ with $P \le 0$. In this case

$$V_x(\tilde{x}) = 2(x-\hat{x})^T P = -V_{\hat{x}}(\tilde{x})$$

and (3.23) is simplified as:

$$\begin{aligned}2(x-\hat{x})^T PAx - 2(x-\hat{x})^T PA\hat{x} + (x-\hat{x})C^T(F_f R_f^{-1} F_f^T)^{-1}C(x-\hat{x})\\ -(x-\hat{x})C^T C(x-\hat{x}) - 2(x-\hat{x})^T PE_f(F_f^T F_f)^{-1}F_f C(x-\hat{x}) \ge 0\end{aligned}$$

$$\begin{aligned}(x-\hat{x})^T PA(x-\hat{x}) + (x-\hat{x})^T A^T P(x-\hat{x}) + (x-\hat{x})C^T(F_f R_f^{-1} F_f^T)^{-1}C(x-\hat{x})\\ -(x-\hat{x})C^T C(x-\hat{x}) - 2(x-\hat{x})^T PE_f(F_f^T F_f)^{-1}F_f C(x-\hat{x}) \ge 0\end{aligned}$$

$$PA + A^T P + C^T(F_f R_f^{-1} F_f^T)^{-1}C - C^T C - 2PE_f(F_f^T F_f)^{-1}F_f C \ge 0 \tag{3.27}$$

and (3.24) simplifies to

$$\begin{aligned}-2(x-\hat{x})^T PL^\star = -2(x-\hat{x})^T PE_f(F_f^T F_f)^{-1}F_f^T\\ +2(x-\hat{x})^T C^T(F_f R_f^{-1} F_f^T)^{-1} - 2(x-\hat{x})^T C^T\end{aligned}$$

$$PL^\star = PE_f(F_f^T F_f)^{-1}F_f^T - C^T(F_f R_f^{-1} F_f^T)^{-1} + C^T \tag{3.28}$$

The design of $\mathcal{H}_-$ fault sensitive FDF is an iterative procedure. One can begin with the highest possible value of β permitted by assumption $A1$, and check for possible existence of sufficient conditions (3.21),(3.22) ((3.23), (3.24) for infinite horizon case). If these are not satisfied then reduce β until the sufficient conditions are fulfilled. For linear systems, these sufficient conditions are usually formulated into LMIs and the iterations are performed with efficient algorithms in MATLAB R LMI toolbox.

3.3 $\mathcal{H}_\infty$ disturbance attenuating FDF

The $\mathcal{H}_\infty$ nonlinear filtering has been extensively studied in literature, some references are mentioned at the beginning of this chapter. It should be explicated here that there is a delicate difference in the $\mathcal{H}_\infty$ filtering problem and the $\mathcal{H}_\infty$ FDF. In the former situation, the objective is to minimize

$$\sup_{d\neq 0} \frac{\|c(x)-c(\hat{x})\|_2}{\|d\|_2} \tag{3.29}$$

while in the later case, it is desired to minimize

$$\sup_{d\neq 0} \frac{\|y-c(\hat{x})\|_2}{\|d\|_2} = \sup_{d\neq 0} \frac{\|c(x)-c(\hat{x})+F_d(x)d\|_2}{\|d\|_2} \tag{3.30}$$

For fault detection purpose, the later optimization is of more interests. To demonstrate the difference between the two optimization problems and to illustrate that satisfying (3.29) does not imply that (3.30) is also fulfilled, we give an example from linear systems.

Example 3.3. *Consider a linear system described by*

$$\dot{x} = Ax + E_d d$$
$$y = Cx + F_d d$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \; E_d = F_d = C = I_2$$

we apply the following filter for the above system

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

the error dynamics are

$$\dot{e} = (A - LC)e + (E_d - LF_d)d \tag{3.31}$$
$$z = Ce \tag{3.32}$$
$$r = Ce + F_d d \tag{3.33}$$

The disturbance attenuation filtering problem (3.29) *is satisfied if an L satisfies the following inequality for a positive definite P*

$$P(A - LC) + (A - LC)^T P + \frac{1}{\alpha^2} P(E - LF)(E - LF)^T P + C^T C \leq 0$$

Note that the above inequality is satisfied for $\alpha = 1$ and

$$L = \begin{bmatrix} 1.2011 & 0 \\ 0 & 0.5344 \end{bmatrix}, \; P = \begin{bmatrix} 1.1454 & 0 \\ 0 & 1.1454 \end{bmatrix}$$

Figure 3.3 shows the singular value plot of the system defined by (3.31) *and* (3.32). *It can be seen, as guaranteed by $\mathcal{H}_\infty$ filtering problem, that $\sup_d \frac{\|z\|_2}{\|d\|_2} = \sup_w \bar{\sigma}(w)$ is less than the predefined value of α. However, as illustrated by Figure 3.4 which shows the singular value plot for the system* (3.31) *and* (3.33), *the same is not guaranteed for $\sup_d \frac{\|r\|_2}{\|d\|_2}$. This demonstrates the difference between the two filtering problems.*

The above example demonstrates that there is a difference between the $\mathcal{H}_\infty$ filtering problems and $\mathcal{H}_\infty$ fault detection filtering problems. The former have been extensively studied and some references were provided in the introduction of this chapter, while to our knowledge, the later case have not been discussed for nonlinear systems. This section is dedicated to the study of $\mathcal{H}_\infty$ fault detection filtering problem.

Similar to the $\mathcal{H}_-$ fault sensitivity problem, we can also formulate the $\mathcal{H}_\infty$ disturbance attenuation problem as a two-players zero-sum differential game with disturbance d and filter gain L as two players. Thus defining the cost functional

$$\mathscr{J}(L, d) \triangleq \int_0^{t_1} (r^T r - \alpha^2 d^T d)\mathrm{dt} \quad (f = 0) \tag{3.34}$$

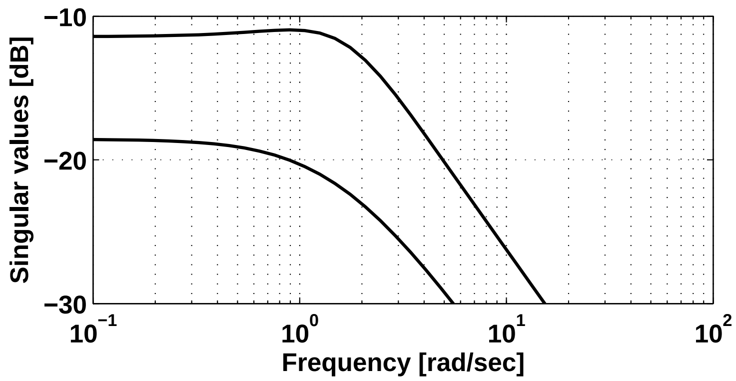


Figure 3.3: Singular value plot for the system defined by (3.31) and (3.32)

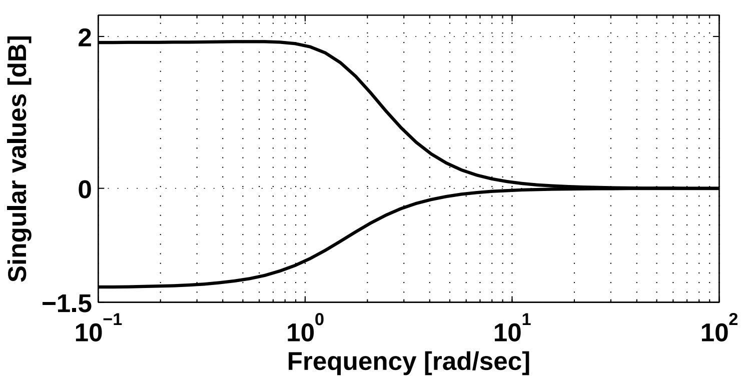


Figure 3.4: Singular value plot for the system defined by (3.31) and (3.33)

the disturbance attenuation problem (3.10) is satisfied if

$$\inf_{L} \sup_{d \neq 0} \mathscr{J}(L, d) \leq 0$$

The following theorem gives a solution to the disturbance attenuation problem using the results from game theory.

Theorem 3.2. *Given system* (3.7)*, a positive constant* α *and assume*
A1. $R_d \triangleq F_d^T(x) F_d(x) - \alpha^2 I_{k_d} < 0$
A2. $F_d(x)$ *is a square matrix*
If there exists a function $Y(\tilde{x}, t)$ *with* $Y(\tilde{x}(0), 0) = 0$ *that satisfies the following HJI equation*

$$\begin{aligned} -\frac{\partial Y(\tilde{x}, t)}{\partial t} = {} & Y_x(\tilde{x}, t) a(x) + Y_{\hat{x}}(\tilde{x}, t) a(\hat{x}) \\ & + \tilde{c}^T(\tilde{x})(F_d(x) R_d^{-1} F_d^T(x))^{-1} \tilde{c}(\tilde{x}) - \tilde{c}^T(\tilde{x}) \tilde{c}(\tilde{x}) \\ & - Y_x(\tilde{x}, t) E_d(\tilde{x})(F_d^T(x) F_d(x))^{-1} F_d^T(x) \tilde{c}(\tilde{x}) \end{aligned} \tag{3.35}$$

then $L^\star(\hat{x}, t)$ *satisfying*

$$\begin{aligned} Y_{\hat{x}}(\tilde{x}, t) L^\star(\hat{x}, t) = {} & -Y_x(\tilde{x}, t) E_d(x)(F_d^T(x) F_d(x))^{-1} F_d^T(x) \\ & + 2\tilde{c}^T(\tilde{x})(F_d(x) R_d^{-1} F_d^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \tag{3.36}$$

solves the disturbance attenuation problem (3.10)*.*

Proof. (sketch: details which are similar to the proof of Theorem 3.1 are omitted) For the system (3.7) with no fault and the cost functional (3.34), if there exists a function $Y(\tilde{x}, t)$ on $[0, t_1] \times \mathbb{R}^n$ satisfying the following HJI equation

$$\begin{aligned} -\frac{\partial Y(\tilde{x}, t)}{\partial t} & = \min_{L \in \mathbb{R}^{n \times m}} \max_{d \in \mathbb{R}^{k_d}} \left\{ \frac{\partial Y(\tilde{x}, t)}{\partial \tilde{x}} \dot{\tilde{x}} + r^T r - \alpha^2 d^T d \right\} \\ & = \min_{L \in \mathbb{R}^{n \times m}} \max_{d \in \mathbb{R}^{k_d}} \Big\{ Y_x(\tilde{x}, t) a(x) + Y_x(\tilde{x}, t) E_d(x) d + Y_{\hat{x}}(\tilde{x}) a(\hat{x}) \\ & \qquad + Y_{\hat{x}}(\tilde{x}, t) L(\hat{x}, t) \tilde{c}(\tilde{x}) + Y_{\hat{x}}(\tilde{x}, t) L(\hat{x}, t) F_d(x) d \\ & \qquad + r^T r - \alpha^2 d^T d \Big\} \\ & = Y_x(\tilde{x}, t) a(x) + Y_x(\tilde{x}, t) E_d(x) d^\star + Y_{\hat{x}}(\tilde{x}, t) a(\hat{x}) \\ & \qquad + Y_{\hat{x}}(\tilde{x}, t) L^\star(\hat{x}, t) \tilde{c}(\tilde{x}) + Y_{\hat{x}}(\tilde{x}, t) L^\star(\hat{x}, t) F_d(x) d^\star \\ & \qquad + r^T(d^\star) r(d^\star) - \alpha^2 d^{\star T} d^\star \end{aligned} \tag{3.37}$$

then the pair of strategies $(L^\star, d^\star)$ provides a saddle-point solution, i.e. $\mathscr{J}(L^\star, d) \leq \mathscr{J}(L^\star, d^\star) \leq \mathscr{J}(L, d^\star)$. Furthermore, saddle-point value of the game is $\mathscr{J}(L^\star, d^\star) = Y(\tilde{x}(0), 0)$. This ensures $\mathscr{J}(L^\star, d) \leq V(\tilde{x}(0), 0) = 0$, which in turn guarantees that $L^\star$ solves the disturbance attenuation problem (3.10).

Rest of the proof consists of finding $d^\star$ and $L^\star$ and substituting them into (3.37). For that purpose, we define the following Hamiltonian function

$$\begin{aligned} \mathscr{H}(L, f) & \triangleq V_{\tilde{x}} \dot{\tilde{x}} + r^T r - \alpha^2 d^T d \\ & = V_x a(x) + V_x E_d d + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L \tilde{c} + V_{\hat{x}} L F_d d + r^T r - \alpha^2 d^T d \end{aligned} \tag{3.38}$$

Similar to the case of fault sensitivity problem, differentiate the Hamiltonian with respect to d, set the derivative equal to zero to find the worst case disturbance

$$d^\star = R_d^{-1}(E_d^T Y_x^T + F_d^T L^T Y_{\hat{x}}^T + 2F_d^T \tilde{c})$$

Substitute the worst case disturbance in the defined Hamiltonian, differentiate it with respect to L and set the derivative equal to zero to obtain an expression for $L^\star$ as given in (3.36). By similar arguments as in the proof of fault sensitivity condition, it can be shown that $d^\star$ maximizes the Hamiltonian and $L^\star$ minimizes the Hamiltonian, as required. Substitute $d^\star$ and $L^\star$ in (3.37) to get (3.35). □

As it has been discussed, an inequality version of HJI in (3.35) would be easier to solve, therefore, it is desired to convert it into HJI inequality. This can be done by using similar arguments as used in $\mathcal{H}_-$ sensitivity condition. These arguments are analogous to the dissipativity approach and can be stated as under:

The system (3.7) is said to be dissipative with respect to supply rate of

$$\alpha^2 \|d\|_2^2 + \|r\|_2^2 \tag{3.39}$$

if there exists a positive function $Y(\tilde{x},t)$ that satisfies

$$\frac{\partial Y}{\partial t} + Y_{\tilde{x}}\dot{\tilde{x}} - \alpha^2 d^T d + r^T r \le 0 \tag{3.40}$$

and from the definition of dissipativity, it follows that

$$\|r\|_2^2 \le \alpha^2 \|d\|_2^2$$

Furthermore, we known that if a system is dissipative with respect to supply rate given in (3.39), then it is finite gain $\mathcal{L}_2$ stable [135, 136]. Thus, we can summarize the results in the form of following corollary.

Corollary 3.3. *Given system* (3.7) *with* $f = 0$*, a positive constant* α *and assume*
A1. $R_d \triangleq F_d^T(x)F_d(x) - \alpha^2 I_{k_d} < 0$
A2. $F_d(x)$ *is a square matrix*
If there exists a positive definite function $Y(\tilde{x},t)$ *with* $Y(\tilde{x}(0),0) = 0$ *that satisfies the following HJI inequality*

$$\begin{aligned} -\frac{\partial Y(\tilde{x},t)}{\partial t} \ge\ & Y_x(\tilde{x},t)a(x) + Y_{\hat{x}}(\tilde{x},t)a(\hat{x}) + \tilde{c}^T(\tilde{x})(F_d(x)R_d^{-1}F_d^T(x))^{-1}\tilde{c}(\tilde{x}) \\ & - \tilde{c}^T(\tilde{x})\tilde{c}(\tilde{x}) - Y_x(\tilde{x},t)E_d(\tilde{x})(F_d^T(x)F_d(x))^{-1}F_d^T(x)\tilde{c}(\tilde{x}) \end{aligned} \tag{3.41}$$

and $L^\star(\hat{x},t)$ *satisfying*

$$\begin{aligned} Y_{\hat{x}}(\tilde{x},t)L^\star(\hat{x},t) =\ & -Y_x(\tilde{x},t)E_d(x)(F_d^T(x)F_d(x))^{-1}F_d^T(x) \\ & + 2\tilde{c}^T(\tilde{x})(F_d(x)R_d^{-1}F_d^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \tag{3.42}$$

then the system (3.7) *is dissipative with respect to supply rate of* (3.39) *and, therefore, solves the disturbance attenuation problem* (3.10).

Note that with the dissipativity theory, we have obtained an inequality version of equation (3.35) which is easier to solve (see Remark 3.4). Furthermore, the dissipativity property also guarantees the stability of the filter. Now question arises why do we use the game theory? The answer is that with the game theoretic approach, the worst case disturbance and the corresponding optimal filter gain can be found.

In the case when $t_1 \to \infty$, and we are interested in time invariant solutions, we can take a time independent function $Y(\tilde{x})$. In this situation, we have the following results.

Corollary 3.4. *Given system* (3.7) *with* $f = 0$*, a positive constant* α *and assume*
A1. $R_d \triangleq F_d^T(x)F_d(x) - \alpha^2 I_{k_d} < 0$
A2. $F_d(x)$ *is a square matrix*
If there exists a positive definite function $Y(\tilde{x})$ *with* $Y(\tilde{x}(0)) = 0$ *that satisfies the following HJI inequality*

$$\begin{aligned} 0 \geq Y_x(\tilde{x})a(x) + Y_{\hat{x}}(\tilde{x})a(\hat{x}) \\ + \tilde{c}^T(\tilde{x})(F_d(x)R_d^{-1}F_d^T(x))^{-1}\tilde{c}(\tilde{x}) - \tilde{c}^T(\tilde{x})\tilde{c}(\tilde{x}) \\ - Y_x(\tilde{x})E_d(\tilde{x})(F_d^T(x)F_d(x))^{-1}F_d^T(x)\tilde{c}(\tilde{x}) \end{aligned} \tag{3.43}$$

and $L^\star(\hat{x})$ *satisfying*

$$\begin{aligned} Y_{\hat{x}}(\tilde{x})L^\star(\hat{x}) = -Y_x(\tilde{x})E_d(x)(F_d^T(x)F_d(x))^{-1}F_d^T(x) \\ + 2\tilde{c}^T(\tilde{x})(F_d(x)R_d^{-1}F_d^T(x))^{-1} - 2\tilde{c}^T(\tilde{x}) \end{aligned} \tag{3.44}$$

then the systems (3.7) *is dissipative with respect to supply rate of* (3.39) *and, therefore, solves the disturbance attenuation problem* (3.10) *in infinite horizon sense.*

Proof. The proof follows by substituting a time independent function $Y(\tilde{x})$ in the results of Corollary 3.3. □

For the case of linear systems defined in (3.25) and the corresponding linear fault detection filter (3.26), we can select $Y(\tilde{x}) = (x-\hat{x})^T Q(x-\hat{x})$ with $Q \geq 0$. In this case (3.43) and (3.44) are simplified to

$$QA + A^TQ + C^T(F_dR_d^{-1}F_d^T)^{-1}C - C^TC - 2QE_d(F_d^TF_d)^{-1}F_dC \leq 0 \tag{3.45}$$

$$QL^\star(\hat{x}) = QE_d(F_d^TF_d)^{-1}F_d^T - C^T(F_dR_d^{-1}F_d^T)^{-1} + C^T \tag{3.46}$$

Similar to the case of $\mathcal{H}_-$ fault sensitive FDF, the $\mathcal{H}_\infty$ disturbance attenuation FDF design is also an iterative procedure. Starting with the smallest possible value of α permitted by $A1$, one can successively increase α until (3.41),(3.42) are satisfied ((3.43), (3.44) in infinite horizon case are satisfied).

As compared to $\mathcal{H}_-$ fault sensitive FDF which guarantees a required level of sensitivity of residual to the faults, the $\mathcal{H}_\infty$ disturbance attenuating FDF ensures that residual is robust against disturbances. In addition to the disturbance attenuation property, $\mathcal{H}_\infty$ disturbance attenuating FDF also helps to set up the threshold. Thus, if $\mathcal{L}_2$ norm of residual signal is used as evaluation function, the maximum possible effect of disturbances on the evaluated residual is not greater than $\alpha\delta_d$, where $\|d\|_2 \leq \delta_d$. One can set threshold equal to $\alpha\delta_d$ to avoid any possible false alarms. Further discussions on residual evaluation and threshold settings will be provided in Chapters 4 and 5.

3.4 $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF

In the last two sections, two fault detection filtering problems were discussed. In the first case, $\mathcal{H}_-$ fault detection filter was presented which guaranteed a fault sensitivity greater than a given constant. However, this filter does not give any indication of the disturbance attenuation property of the filter. It is possible that, in addition to making residual sensitive to faults, the filter also makes residual sensitive to disturbances. This is certainly not a desired feature of $\mathcal{H}_-$ filter. Similar is the situation for the second FDF, the $\mathcal{H}_\infty$ fault detection filter guarantees the disturbance attenuation greater than a given constant but does not give any indication of fault sensitivity. In addition to disturbance attenuation, the filter may also result in attenuating the effect of faults. A better fault detection filter should be simultaneously robust against disturbance and sensitive to faults. For that purpose, multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ index can be used. The multi-objective index was first proposed in [56] and later several techniques were proposed for linear systems. This section presents $\mathcal{H}_-/\mathcal{H}_\infty$ filtering problem for nonlinear systems.

In $\mathcal{H}_-/\mathcal{H}_\infty$ based design, the objective is to maximize the ratio β/α, rather than individually maximizing β or minimizing α. Thus an $\mathcal{H}_-/\mathcal{H}_\infty$ based filter may not be the optimal in the sense of $\mathcal{H}_-$ index, i.e., another filter gain may exist which gives a higher fault to residual gain than that delivered by the $\mathcal{H}_-/\mathcal{H}_\infty$ based designed filter gain. A similar comparison of $\mathcal{H}_\infty$ based FDF and multi-objective index based FDF can be presented, i.e., the disturbance to residual gain for the former FDF may be smaller than that for the later FDF.

The $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF problem can be formulated as two two-players nonzero-sum differential games with following two cost functionals:

$$\mathscr{J}_1(L,f) = \int_0^{t_1} (r^T r - \beta f^T f)\ \mathrm{dt} \quad ; d = 0 \tag{3.47}$$

$$\mathscr{J}_2(L,d) = \int_0^{t_1} (r^T r - \alpha d^T d)\ \mathrm{dt} \quad ; f = 0 \tag{3.48}$$

The first functional is associated with $\mathcal{H}_-$ fault sensitivity and the second cost functional is associated with the $\mathcal{H}_\infty$ disturbance attenuation. Making $\mathscr{J}_1 \geq 0$ guarantees $\|\Sigma_{\mathscr{A}}\|_{-,[0,t_1]} = \inf_{f\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|f\|_{2,[0,t_1]}} \geq \beta$ and making $\mathscr{J}_2 \leq 0$ guarantees $\|\Sigma_{\mathscr{A}}\|_{\infty,[0,t_1]} = \sup_{d\neq 0} \frac{\|r\|_{2,[0,t_1]}}{\|d\|_{2,[0,t_1]}} \leq \alpha$. The following theorem now gives the sufficient conditions for the solvability of the problem.

Theorem 3.3. *Consider the nonlinear system* (3.7)*, positive constants α and β and assume*
A1. $R_f \triangleq F_f^T(x)F_f(x) - \beta^2 I_{k_f} > 0$
A2. $R_d \triangleq F_d^T(x)F_d(x) - \alpha^2 I_{k_d} < 0$
A2. $F_f(x)$, $F_d(x)$ *are square matrices*
Suppose that there exists a pair of functions $V(\tilde{x},t)$ with $V(\tilde{x}(0),0) = 0$ and $Y(\tilde{x},t)$ with $V(\tilde{x}(0),0) = 0$ and the filter gain $L(\hat{x},t)$ satisfying the following partial differential equa-

tions,

$$V_x a(x) - \frac{1}{4} V_x E_f R_f^{-1} E_f^T V_x^T - \frac{1}{2} V_x E_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_x E_f R_f^{-1} F_f^T \tilde{c} + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L \tilde{c} - \frac{1}{4} V_{\hat{x}} L F_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_{\hat{x}} L F_f R_f^{-1} F_f^T \tilde{c} + \tilde{c}^T \tilde{c} - \tilde{c}^T F_f R_f^{-1} F_f^T \tilde{c} = -\frac{\partial V}{\partial t} \quad (3.49)$$

$$Y_x a(x) - \frac{1}{4} Y_x E_d R_d^{-1} E_d^T Y_x^T - \frac{1}{2} Y_x E_d R_d^{-1} F_d^T L^T Y_{\hat{x}}^T \\ - Y_x E_d R_d^{-1} F_d^T \tilde{c} + Y_{\hat{x}} a(\hat{x}) + Y_{\hat{x}} L \tilde{c} - \frac{1}{4} Y_{\hat{x}} L F_d R_d^{-1} F_d^T L^T Y_{\hat{x}}^T \\ - Y_{\hat{x}} L F_d R_d^{-1} F_d^T \tilde{c} + \tilde{c}^T \tilde{c} - \tilde{c}^T F_d R_d^{-1} F_d^T \tilde{c} = -\frac{\partial Y}{\partial t} \quad (3.50)$$

then $L(\hat{x}, t)$ solves the multi-objective filtering problem defined in Problem 3.3.

Proof. (sketch: details which are similar to the proof of Theorems 3.1, 3.2 are omitted) The proof follows by defining Hamiltonian functions $\mathscr{H}_1$ and $\mathscr{H}_2$ as in (3.18) and (3.38), respectively. Find $f^\star$ and $d^\star$ by setting

$$\left.\frac{\partial \mathscr{H}_1}{\partial f}\right|_{f^\star} = 0, \qquad \left.\frac{\partial \mathscr{H}_2}{\partial d}\right|_{d^\star} = 0$$

and substitute in the Hamiltonians $\mathscr{H}_1$ and $\mathscr{H}_2$ to reach at the results. Then L which simultaneously satisfies (3.49) and (3.49) will satisfy both the sensitivity and robustness conditions. □

In previous sections, we derived sufficient conditions for $\mathcal{H}_-$ fault sensitivity and $\mathcal{H}_\infty$ disturbance attenuation fault detection filtering problems. These conditions were in form of an HJI and and explicit expression for the filter gain L. However, in $\mathcal{H}_-/\mathcal{H}_\infty$ based multi-objective filtering problems, we can not derive an explicit expression for L, because the same L has to satisfy both the sensitivity and robustness conditions. That is, if we derive an explicit expression for L based on (3.49), it may not satisfy (3.50), and vice versa.

It should be noted that the $\mathcal{H}_\infty$ disturbance robustness condition also ensures the finite gain $\mathcal{L}_2$ stability of the filter (see [135, 136] for finite-gain $\mathcal{L}_2$ stability).

Although the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization based FDF is better in the sense that it provides simultaneous disturbance attenuation and fault sensitivity, but the disadvantage is that it involves solving two coupled partial differential equations.

It can be shown (as in Section 3.2 and Section 3.3) that for negative semidefinite V and positive semidefinite Y, the '=' in (3.49) and (3.49) can be respectively changed to $\geq$ and $\leq$. In the case when $t_1 \to \infty$, and we are interested in time invariant filter gain $L(\hat{x})$, the functions V and Y can be taken independent of time. Hence we have the following corollary

Corollary 3.5. *Consider the nonlinear system* (3.7)*, positive constants α and β and assume*
A1. $R_f \triangleq F_f^T(x) F_f(x) - \beta^2 I_{k_f} > 0$

A2. $R_d \triangleq F_d^T(x)F_d(x) - \alpha^2 I_{k_d} < 0$
A2. $F_f(x)$, $F_d(x)$ *are square matrices*
Suppose that there exists a pair of negative definite function $V(\tilde{x})$ *with* $V(\tilde{x}(0)) = 0$ *and a positive definite function* $Y(\tilde{x})$ *with* $Y(\tilde{x}(0)) = 0$ *and the filter gain* $L(\hat{x})$ *satisfying the following partial differential equations,*

$$\begin{aligned} V_x a(x) - \frac{1}{4}V_x E_f R_f^{-1} E_f^T V_x^T - \frac{1}{2}V_x E_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_x E_f R_f^{-1} F_f^T \tilde{C} + V_{\hat{x}} a(\hat{x}) + V_{\hat{x}} L\tilde{c} - \frac{1}{4} V_{\hat{x}} L F_f R_f^{-1} F_f^T L^T V_{\hat{x}}^T \\ - V_{\hat{x}} L F_f R_f^{-1} F_f^T \tilde{c} + \tilde{c}^T \tilde{c} - \tilde{c}^T F_f R_f^{-1} F_f^T \tilde{c} \geq 0 \end{aligned} \quad (3.51)$$

$$\begin{aligned} Y_x a(x) - \frac{1}{4}Y_x E_d R_d^{-1} E_d^T Y_x^T - \frac{1}{2}Y_x E_d R_d^{-1} F_d^T L^T Y_{\hat{x}}^T \\ - Y_x E_d R_d^{-1} F_d^T \tilde{c} + Y_{\hat{x}} a(\hat{x}) + Y_{\hat{x}} L\tilde{c} - \frac{1}{4} Y_{\hat{x}} L F_d R_d^{-1} F_d^T L^T Y_{\hat{x}}^T \\ - Y_{\hat{x}} L F_d R_d^{-1} F_d^T \tilde{c} + \tilde{c}^T \tilde{c} - \tilde{c}^T F_d R_d^{-1} F_d^T \tilde{c} \leq 0 \end{aligned} \quad (3.52)$$

then $L(\hat{x})$ *solves the multi-objective filtering problem defined in Problem 3.3 in infinite horizon sense.*

Proof. The proof follows by substituting time invariant functions $V(\tilde{x})$ and $Y(\tilde{x})$ in the results of Theorem 3.3. □

For linear system defined in (3.25) and corresponding fault detection filter (3.26), the negative semidefinite function $V(\tilde{x})$ can be selected as $V(\tilde{x}) = (x-\hat{x})^T P(x-\hat{x})$ with $P \leq 0$ and the positive definite function $Y(\tilde{x}) = (x-\hat{x})^T Q(x-\hat{x})$ with $Q \geq 0$. In this case (3.51) is simplified as under:

$$\begin{aligned} &(x-\hat{x})^T PA(x-\hat{x}) + (x-\hat{x})^T A^T P(x-\hat{x}) - (x-\hat{x})^T PE_f R_f^{-1} E_f^T P(x-\hat{x}) \\ &+ (x-\hat{x})^T PE_f R_f^{-1} F_f^T L^T P(x-\hat{x}) - (x-\hat{x})^T PE_f R_f^{-1} F_f^T C(x-\hat{x}) \\ &- 2(x-\hat{x})^T PLC(x-\hat{x}) + (x-\hat{x})^T PLF_f R_f^{-1} E_f^T P(x-\hat{x}) \\ &- (x-\hat{x})^T PLF_f R_f^{-1} F_f^T L^T P(x-\hat{x}) + (x-\hat{x})^T PLF_f R_f^{-1} F_f^T C(x-\hat{x}) \\ &+ C^T(x-\hat{x})^T C(x-\hat{x}) - (x-\hat{x})^T PE_f R_f^{-1} F_f^T C(x-\hat{x}) \\ &+ (x-\hat{x})^T PLF_f R_f^{-1} F_f^T C(x-\hat{x}) - C^T(x-\hat{x})^T F_f R_f^{-1} F_f^T C(x-\hat{x}) \geq 0 \end{aligned}$$

Since all terms in above equation are scalars which may be transposed, the above expression becomes

$$\begin{aligned} &PA + A^T P - PE_f R_f^{-1} E_f^T P + PE_f R_f^{-1} F_f^T L^T P - PE_f R_f^{-1} F_f^T C - PLC \\ &- C^T L^T P + PLF_f R_f^{-1} E_f^T P - PLF_f R_f^{-1} F_f^T L^T P + PLF_f R_f^{-1} F_f^T C \\ &+ C^T C - C^T F_f R_f^{-1} E_f^T P + C^T F_f R_f^{-1} F_f^T L^T P - C^T F_f R_f^{-1} F_f^T C \geq 0 \end{aligned}$$

After a few further simplifications

$$\begin{aligned} &P(A - LC - (E_f - LF_f)R_f^{-1} F_f^T C) + (A - LC - (E_f - LF_f)R_f^{-1} F_f^T C)^T P \\ &- P(E_f - LF_f)R_f^{-1}(E_f - LF_f)^T P + C^T(I_{k_f} - F_f R_f^{-1} F_f^T)C \geq 0 \end{aligned}$$

Likewise, after similar simplifications, (3.52) is reduced to

$$Q(A - LC - (E_d - LF_d)R_d^{-1}F_d^TC) + (A - LC - (E_d - LF_d)R_d^{-1}F_d^TC)^TQ \\ - Q(E_d - LF_d)R_d^{-1}(E_d - LF_d)^TQ + C^T(I_{k_d} - F_dR_d^{-1}F_d^T)C \geq 0$$

These are the well known result for case of linear systems (see e.g. [13, Chapter 7]). The above nonlinear matrix inequalities (NMI) can be reduced to linear matrix inequalities (LMI) and powerful tools are available to solve such LMIs.

3.5 An example

To illustrate our results for nonlinear systems, we consider an example of a simple second order system. The dynamics of the system are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 - cx_1 - k_n \sin(x_2) + k_g u + d \\ y &= x_1 + f + 0.5d \end{aligned} \tag{3.53}$$

x_1, x_2 are respectively the angular position and the angular velocity of the positioning system, y is the measurement, u is the input, d is the measurement noise and f is the sensor fault to be detected. The numerical values of the system parameters are: $b = 1$, $c = 0.4$, $k_1 = 0.1$ and $k_g = 1$. Note that in this example, reference input $u(t)$ is also present. Since the input matrix is independent of states, the influence of $u(t)$ on state estimation error is removed after using an FDF. We have the following system matrices:

$$a(x) = \begin{bmatrix} x_2 \\ -bx_2 - cx_1 - k_n \sin(x_2) \end{bmatrix}, E_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, F_f = \begin{bmatrix} 1 \end{bmatrix}, F_d = \begin{bmatrix} 0.5 \end{bmatrix}$$

$\mathcal{H}_-$ fault sensitive FDF

It follows from Corollary 3.2 that the fault sensitivity problem (3.8) with infinite horizon case is solved for the example system if we find an $L(\hat{x})$ and a negative definite function $V(\tilde{x})$ with $V(\tilde{x}(0))$ which satisfy (3.23) and (3.24). With

$$A_1 = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}, \; A_2(x) = \begin{bmatrix} 0 \\ -k_n \sin(x_2) \end{bmatrix}$$

equation (3.23) becomes

$$V_xA_1x + V_xA_2(x) + V_{\hat{x}}A_1\hat{x} + V_{\hat{x}}A_2(\hat{x}) \\ + \tilde{C}^T(\tilde{x})(F_fR_f^{-1}F_f^T)^{-1}\tilde{C}(\tilde{x}) - \tilde{C}^T(\tilde{x})\tilde{C}(\tilde{x}) - V_xE(F^TF)^{-1}F^T\tilde{C}(\tilde{x}) \geq 0 \tag{3.54}$$

We treat the stationary case ($t_1 \to \infty$) and select $V = (x - \hat{x})^TP(x - \hat{x})$, where P is a symmetric negative definite matrix, (3.54) becomes,

$$(x - \hat{x})^TPA_1(x - \hat{x}) + (x - \hat{x})^TA_1^TP(x - \hat{x}) \\ + (x - \hat{x})^TC^T(F_fR_f^{-1}F_f^T)^{-1}C(x - \hat{x}) - (x - \hat{x})^TC^TC(x - \hat{x}) \\ - 2(x - \hat{x})^TPE(F^TF)^{-1}F^TC(x - \hat{x}) + J_1 \geq 0 \tag{3.55}$$

where

$$\begin{aligned}
J_1 &= 2(x-\hat{x})^T P A_2(x) - 2(x-\hat{x})^T P A_2(\hat{x}) \\
&= -2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ \sin(x_2)-\sin(\hat{x}_2) \end{bmatrix} \\
&= -2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ 2\sin(\frac{x_2-\hat{x}_2}{2})\cos(\frac{x_2+\hat{x}_2}{2}) \end{bmatrix} \\
&= -2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\sin(\frac{x_2-\hat{x}_2}{2})}{\frac{(x_2-\hat{x}_2)}{2}} \cos(\frac{x_2+\hat{x}_2}{2}) \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x}) \\
&= -2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sinc}(\frac{x_2-\hat{x}_2}{2}) \cos(\frac{x_2+\hat{x}_2}{2}) \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x}) \\
&\geq 2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x})
\end{aligned}$$

The last inequality follows from the fact that the minimum value of $\mathrm{sinc}(\frac{x_1-\hat{x}_1}{2})\cos(\frac{x_1+\hat{x}_1}{2})$ is -1 and $P<0$. Hence, a sufficient condition for (3.55) becomes,

$$\begin{aligned}
&(x-\hat{x})^T P A_1(x-\hat{x}) + (x-\hat{x})^T A_1^T P(x-\hat{x}) \\
&\quad + (x-\hat{x})^T C^T (F_f R_f^{-1} F_f^T)^{-1} C(x-\hat{x}) - 2(x-\hat{x})^T P E_f (F_f^T F_F)^{-1} F_F^T C(x-\hat{x}) \\
&\quad - (x-\hat{x})^T C^T C(x-\hat{x}) + 2k_n(x-\hat{x})^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x}) \geq 0
\end{aligned}$$

or

$$\begin{aligned}
&PA_1 + A_1^T P + C^T (F_f R_f^{-1} F_f^T)^{-1} C - C^T C - 2PE_f (F_f^T F_f)^{-1} F_f^T C \\
&\quad + 2k_n P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \geq 0
\end{aligned}$$

a sufficient condition for the above inequality is

$$PA_1 + A_1^T P - C^T C - 2PE_f (F_f^T F_f)^{-1} F_f^T C + 2k_n P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \geq 0 \tag{3.56}$$

and the filter gain L is given by

$$\begin{aligned}
-2(x-\hat{x})^T P L &= -2(x-\hat{x})^T P E_f (F_f^T F_f)^{-1} F_f^T \\
&\quad + 2(x-\hat{x})^T C^T (F R^{-1} F^T)^{-1} - 2(x-\hat{x})^T C^T
\end{aligned}$$

or

$$L = E_f (F_f^T F_f)^{-1} F_f^T - P^{-1} C^T (F R^{-1} F^T)^{-1} + P^{-1} C^T \tag{3.57}$$

The inequality (3.56) is satisfied for

$$P = \begin{bmatrix} -4.2363 & -2.7369 \\ -2.7369 & -4.4638 \end{bmatrix}$$

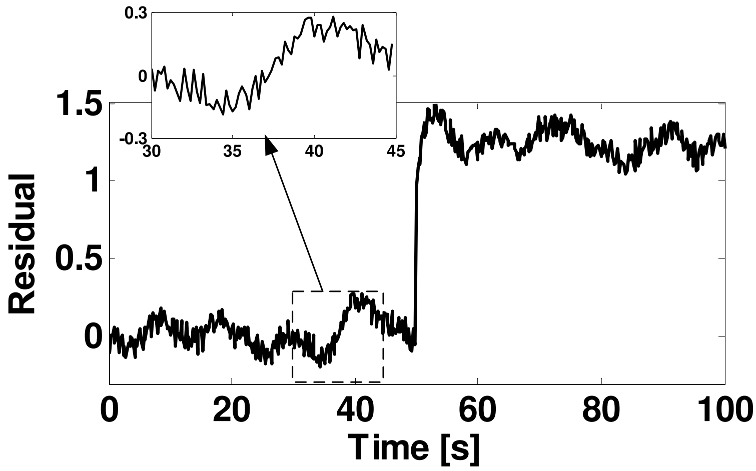


Figure 3.5: Residual signal for a unit step sensor fault which occurs at 30 s

Note that $F^T F = 1$, so according to Assumption $A1$, the maximum possible value of β is 1. We calculate the filter gain L for $\beta = 0.9$ to get:

$$L = \begin{bmatrix} -0.3166 \\ 0.1941 \end{bmatrix} \tag{3.58}$$

The example was simulated in Simulink^R for a unit step fault which occurs at 30 seconds in the sensor measuring the angular position of the system. The initial conditions of the positioning system were set as [0.2 0] in simulation. The simulation results are shown in the Figure 3.5. It can be observed that the residual signal is sufficiently sensitive to the fault. Some further discussion on this figure will be made in $\mathcal{H}_\infty$ based FDF design.

$\mathcal{H}_\infty$ disturbance attenuating FDF

Now we design an $\mathcal{H}_\infty$ disturbance attenuating FDF for the example system. It follows Corollary 3.4 that the $\mathcal{H}_\infty$ disturbance attenuation problem with infinite horizon case is achieved if there exits a positive definite function $Y(\tilde{x})$ with $Y(\tilde{x}(0))$ which satisfies (3.43), the filter gain $L(\hat{x})$ is then given by (3.44). With the matrices A_1 and $A_2(x)$ defined above, (3.43) becomes

$$\begin{aligned} Y_x A_1 x + Y_x A_2(x) + Y_{\hat{x}} A_1 \hat{x} + Y_{\hat{x}} A_2(\hat{x}) + \tilde{C}^T(\tilde{x})(F_d R_d^{-1} F_d^T)^{-1} \tilde{C}(\tilde{x}) \\ - \tilde{C}^T(\tilde{x})\tilde{C}(\tilde{x}) - Y_x E_d (F_d^T F_d)^{-1} F_d^T \tilde{C}(\tilde{x}) \leq 0 \end{aligned} \tag{3.59}$$

Select $Y(\tilde{x}) = (x - \hat{x})^T Q (x - \hat{x})$, with $Q \geq 0$, (3.59) reduces to

$$\begin{aligned} (x-\hat{x})^T Q A_1 (x-\hat{x}) + (x-\hat{x})^T A_1^T Q (x-\hat{x}) + (x-\hat{x})^T C^T (F_d R_d^{-1} F_d^T)^{-1} C (x-\hat{x}) \\ - (x-\hat{x})^T C C (x-\hat{x}) - 2(x-\hat{x})^T Q E_d (F_d^T F_d)^{-1} F_d^T C (x-\hat{x}) + J_2 \leq 0 \end{aligned} \tag{3.60}$$

where

$$J_2 = 2(x-\hat{x})^T P A_2(x) - 2(x-\hat{x})^T P A_2(\hat{x})$$

with similar simplifications as for J_1, we reach at

$$J_2 = -2k_n (x-\hat{x})^T Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{sinc}(\frac{x_2 - \hat{x}_2}{2}) \cos(\frac{x_2 + \hat{x}_2}{2}) \begin{bmatrix} 0 & 1 \end{bmatrix} (x - \hat{x})$$

Noting that $Q \geq 0$ and that $\text{sinc}(\frac{x_2-\hat{x}_2}{2})\cos(\frac{x_2+\hat{x}_2}{2})$ can have values in between -1 and 1, we conclude

$$-2k_n(x-\hat{x})^T Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x}) \leq J_2 \leq 2k_n(x-\hat{x})^T Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x})$$

Thus sufficient condition for (3.60) is

$$\begin{aligned} &(x-\hat{x})^T Q A_1 (x-\hat{x}) + (x-\hat{x})^T A_1^T Q (x-\hat{x}) \\ &\qquad + (x-\hat{x})^T C^T (F_d R_d^{-1} F_d^T)^{-1} C (x-\hat{x}) - (x-\hat{x})^T C C (x-\hat{x}) \\ &\qquad - 2(x-\hat{x})^T Q E_d (F_d^T F_d)^{-1} F_d^T C (x-\hat{x}) + 2k_n (x-\hat{x})^T Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} (x-\hat{x}) \leq 0 \end{aligned}$$

which simplifies to

$$\begin{aligned} &Q A_1 + A_1^T Q + C^T (F_d R_d^{-1} F_d^T)^{-1} C - CC \\ &\qquad - 2 Q E_d (F_d^T F_d)^{-1} F_d^T C + 2k_n Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \leq 0 \end{aligned} \tag{3.61}$$

and the expression for $L(\hat{x})$ is obtained from (3.44)

$$L = E_d (F_d^T F_d)^{-1} F_d^T - 2Q^{-1} C^T (F_d R_d^{-1} F_d^T)^{-1} - 2Q^{-1} C^T \tag{3.62}$$

The following positive definite Q solves (3.61) for $\alpha = 0.5$ (the minimum possible value of α),

$$Q = \begin{bmatrix} 2.8407 & 0.8441 \\ 0.8441 & 4.6377 \end{bmatrix}$$

and from (3.62), the filter gain L is

$$L = \begin{bmatrix} 0.3024 \\ 0.9450 \end{bmatrix}$$

Figure 3.6 shows the simulation results for a unit step sensor fault which occurs at 50 seconds. It can be seen that the effect of disturbances on the residual signal is considerably reduced. The disturbance and the sensor noise is bounded by $d \in [-0.2, 0.2]$.

Discussions

In this example, we have designed two filters for the example system: an $\mathcal{H}_-$ fault sensitive FDF and an $\mathcal{H}_\infty$ disturbance attenuation FDF. The inset figures in Figure 3.5 and Figure 3.6 show that the $\mathcal{H}_\infty$ disturbance attenuating filter has a better disturbance attenuation property. But at the same time we can notice that the $\mathcal{H}_-$ fault sensitive filter provides better sensitivity to the faults.

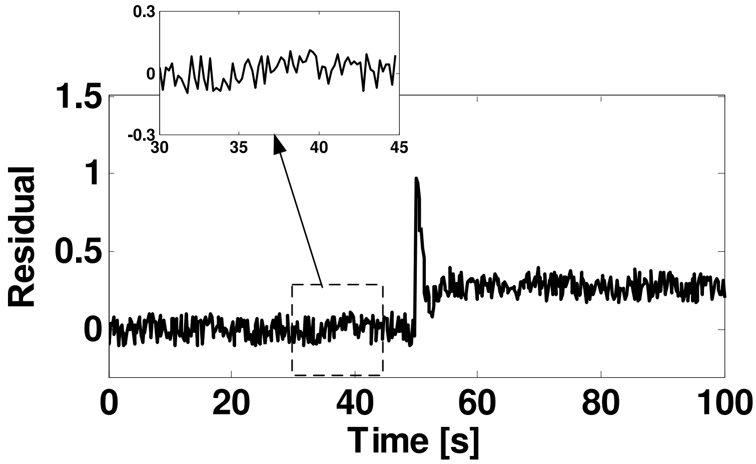


Figure 3.6: Residual signal for a unit step sensor fault which occurs at 30 s

3.6 Summary

Residual generation is an important step in observer-based fault detection of dynamic systems. The residual signal is obtained by comparing the measurements with their estimates generated by the observer (filter). For a better performance of the fault detection system, it is desired that the designed observer (filter) should have the properties such that the residual signal is sensitive to faults and robust against disturbances. To measure sensitivity to faults and robustness against disturbances, the worst case scenario is usually examined. For that purpose $\mathcal{H}_-$ index and $\mathcal{H}_\infty$ norm are widely used in literature. This chapter explained the basic idea of $\mathcal{H}_-$ index and $\mathcal{H}_\infty$ norm and their application in fault detection. To achieve fault sensitivity, $\mathcal{H}_-$ fault sensitive FDF was presented for nonlinear systems. The $\mathcal{H}_-$ fault sensitive FDF ensures that the $\mathcal{L}_2$ gain from worst case fault (the fault for which the residual is minimally sensitive) to the residual is greater than a known constant. To deal with the disturbance attenuation, $\mathcal{H}_\infty$ disturbance attenuating FDF was proposed. The difference between extensively studied $\mathcal{H}_\infty$ filtering problem and the $\mathcal{H}_\infty$ fault detection filtering problem was emphasized with an example. The later is of more interest for fault detection but has not been discussed for nonlinear systems. The proposed $\mathcal{H}_\infty$ disturbance attenuating filter guarantees that the $\mathcal{L}_2$ gain from worst case disturbance to the residual is less than a known constant. It was highlighted that, additional to fault sensitivity, the $\mathcal{H}_-$ fault sensitive FDF may also result in sensitivity to disturbances. Likewise, the $\mathcal{H}_\infty$ disturbance attenuating FDF may result in attenuation of faults additional to attenuation of disturbances. This is, off course, not desired. The simultaneous attenuation of disturbances and amplification of faults should be attained. To achieve this objective, the $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF was proposed. In all the three proposed fault detection filters, both finite horizon and infinite horizon cases were addressed. It was shown that for linear systems, our results reduce to well known results for linear systems. The proposed FDF methods were illustrated with simulation example of a simple second order system.

Chapter 4

Dynamic threshold computation

The main contribution of this chapter includes the presentation of a method for generating dynamic threshold for a class of nonlinear systems. For systems with Lipschitz nonlinearities and unstructured uncertainties, an inequality for the upper bound on the modulus of error dynamics is derived. This upper bound is utilized for designing a dynamic threshold. The effectiveness of the proposed technique is demonstrated by simulation results of an simple Lipschitz nonlinear system.

As discussed in Chapter 2, the process of model-based fault detection consists of two steps, residual generation and residual evaluation. In observer-based residual generation techniques, estimations of the measurements are generated using observers and the residual is the difference of the two. In ideal case, when there are no disturbances and modeling uncertainties, residual becomes zero in the absence of faults. A deviation from zero gives the indication of presence of fault. In reality, such ideal conditions never exist and there are always disturbances and uncertainties. This results in the residual signal being nonzero even in the absence of faults. The aim of residual evaluation is to decide whether a fault has occurred or not even in the presence of modeling uncertainties and disturbances. For that purpose, residual is evaluated and then compared to some threshold to decide about the presence of faults. Selection of threshold plays an important role in the performance of a fault detection system. If a constant threshold is used and is selected too high, it may result in mis-detection of faults, whereas if it is selected too low, some of disturbances may

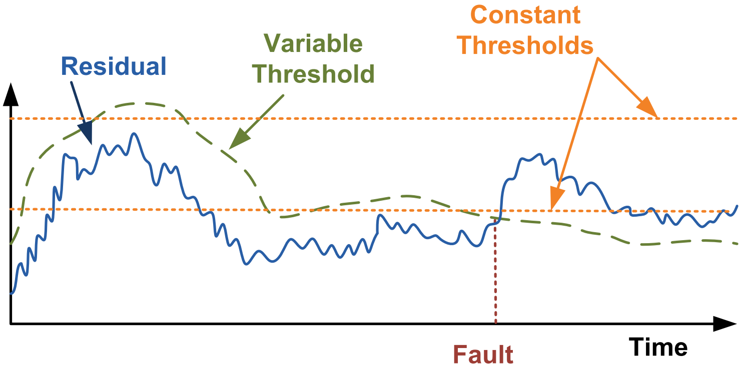
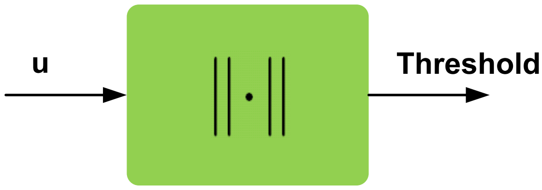
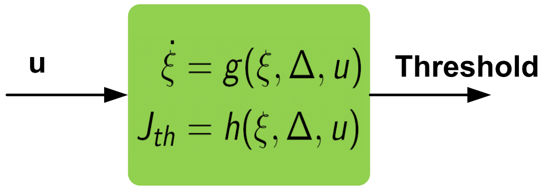


Figure 4.1: Comparison of a constant and a variable thresholds for fault detection



(a) Classical method for variable threshold generation: threshold is generated by multiplying the input with norm of uncertainties



(b) Dynamic threshold generation: the threshold generator is a dynamic system

Figure 4.2: Classical method for variable threshold generation vs dynamic threshold generation

cause the residual to cross the threshold and result in false-alarm. To avoid this difficulty, as proposed by [137], one can use variable or adaptive threshold which varies according to the control activity and uncertainties in the plant. Figure 4.1 compares a constant threshold to a variable threshold . It can be noticed that a constant threshold either results in miss-detection of faults (if selected too high) or some of the uncertainties appear as faults (if threshold is selected too low). Whereas, the variable threshold can successfully avoid false alarms generated by uncertainties and, thus improves the fault detectability.

Several techniques have been presented in the literature for designing adaptive threshold for linear systems. For example, [43] proposed frequency domain approach to design an adaptive threshold, [138] used the statistical approach to design adaptive threshold. Recently, [122], [139], [140] proposed the idea of dynamic threshold generation which not only varies according to control activity, but also possesses dynamic properties and hence, is more tighter to the fault-free residual signal resulting in better fault detection system. Figure 4.2 show the difference of a classical adaptive threshold generator and a dynamic threshold generator. In the domain of nonlinear systems, only a few articles address the problem of threshold generation[141], [126], [127]. In this chapter, a the time domain approach for dynamic threshold generation for Lipschitz nonlinear systems is proposed. The threshold generated is adaptive in nature and the on-line implementation does not involve complex mathematical computation.

4.1 Problem formulation

Consider Lipschitz nonlinear systems described by

$$\dot{x} = Ax + \phi(x,u) + \eta(x,u,t) + E_f f \quad (4.1)$$

$$y = Cx + F_f f \quad (4.2)$$

where $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the measurable output vector, $\phi : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ are smooth vector fields. E_f,

F_f are the fault input and output matrices, respectively, and C is the output matrix of appropriate dimensions. It is assumed that the modeling uncertainty η, which may also include external disturbances, is unstructured and is an unknown nonlinear function of u and t, but it is bounded by some known functional, i.e.,

$$|\eta(x,u,t)| \leq \bar{\eta}(u,t)$$

$\phi(x,u)$ is nonlinearity with a Lipschitz constant γ, i.e.,

$$|\phi(x,u) - \phi(x-e,u)| \leq \gamma|e|$$

As discussed in [142], the class of systems covered by (4.1) is fairly general and any nonlinear system of the form $\dot{x} = f(x,u)$ can be expressed in form (4.1) as long as $f(x,u)$ is differentiable with respect to x. In addition, most of the nonlinearities may be considered as Lipschitz at least locally.

For the purpose of fault detection, we need to generate a residual signal which carries the information about faults. To generate such a residual for system (4.1)-(4.2), we take the observer of the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \phi(\hat{x},u) + L[y - C\hat{x}] \\ \hat{y} &= C\hat{x} \end{aligned} \tag{4.3}$$

Several techniques for observer design for Lipschitz nonlinear systems exist in the literature. Some of the most commonly applied to fault detection have been described in Chapter 2. These include the Thau's observers, high gain observers [94], [98], [95], sliding mode observers, etc. The approach developed in Chapter 3 for fault sensitive detection filter design can also be utilized. Other important observer design approaches for Lipshcitz nonlinear systems are the distance to unobservability based observer [142, 143] and observer design for systems with large Lipschitz constant [144]. Here, we assume that such an observer is already designed.

The residual is simply defined as

$$r = y - \hat{y}$$

Introducing the variable $e = x - \hat{x}$, the dynamics of residual generator can be written as

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u) + \eta(x,u,t) + E_f f \\ \dot{e} &= (A - LC)e + \phi(x,u) - \phi(x-e,u) + \eta(x,u,t) + E_f f \\ r &= Ce + F_f f \end{aligned}$$

Using the simplified notation $\bar{A} = A - LC$ and $\Phi(x,e,u) = \phi(x,u) - \phi(x-e,u)$, the dynamics of the residual generator can be written as

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u) + \eta(x,u,t) + E_f f \\ \dot{e} &= \bar{A}e + \Phi(x,e,u) + \eta(x,u,t) + E_f f \\ r &= Ce + F_f f \end{aligned}$$

Due to the presence of disturbances and modeling uncertainties, the residual signal will not be zero even in the absence of faults. Residual evaluation and threshold setting serves to avoid the effect of uncertainties and disturbances. A decision on the occurrence of a fault is then made by comparing the evaluated residual to the threshold. Some standard evaluation functions are described in Chapter 2. As proposed in [122], the following evaluation function is used,

$$J = \|r\|_v \triangleq \int_0^t v(t-\tau)\,|r(\tau)|\,d\tau = (v * |r|)(t)$$

Where v is the weighting function which increases the influence of the most recent data. A simple choice of v could be an exponential function $\mu e^{-\mu t}$ which has the advantage that the convolution with v can be implemented by a simple first order transfer function. Two major advantages of using the above evaluation function are mentioned in [122]. Firstly, the absolute value of the residual is likely to be less sensitive to outliers than the square and secondly, it provides efficient ways of calculating the robust threshold. Threshold is then selected as

$$J_{th} = \sup_{\text{fault-free}} \|r\|_v \tag{4.4}$$

and the following decision logic is used to detect the occurrence of faults:

$$J > J_{th} \Rightarrow \text{fault alarm}$$
$$J \le J_{th} \Rightarrow \text{no alarm}$$

Designing an optimum threshold, which avoids false alarm with maximum probability of fault detection is the design problem. The main task is to develop a method for generating dynamic threshold for fault detection for the system (4.1) - (4.2).

4.2 Preliminaries and notations

Before we are able to proceed for further discussion, it will be useful to present some preliminary material. The convolution between two functions F and G will be represented as $\mathbf{F}G$, i.e.,

$$\mathbf{F}G \triangleq F * G$$

The element-wise inequalities between matrices and vectors will be represented by $\leq$ and $\geq$, for example, with

$$X \triangleq \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}, \; Y \triangleq \begin{bmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nm} \end{bmatrix}$$

then $X \leq Y$ means that $x_{ij} \leq y_{ij}$ for all $i \in 1 \cdots n$ and all $j \in 1 \cdots m$. $|\cdot|$ will represent the matrix modulus function, i.e. element-wise absolute value. Thus

$$|X| = \begin{bmatrix} |x_{11}| & \cdots & |x_{1m}| \\ \vdots & \ddots & \vdots \\ |x_{n1}| & \cdots & |x_{nm}| \end{bmatrix}$$

For functions, $|\cdot|$ will be interpreted point-wise, so that $|F|(t) \triangleq |F(t)|$. Inequalities between functions are also intended point-wise, i.e. $F \leq G$ means $F(t) \leq G(t)$ for all $t \in \mathbb{R}_+$. Next, a property and lemmas are presented that will be used for the derivation of the results.

Property 4.1. *[122] Let c be an arbitrary scalar and let A, B and C be matrices of compatible dimensions. Then*

a. $|A + B| \leq |A| + |B|$

b. $|AC| \leq |A||C|$

c. $|cA| = |c||A|$

Lemma 4.1 ([122]). *Let $F \in \mathbb{L}_{pe}^{n\times m}$ and $G, H \in \mathbb{L}_{qe}^{m\times r}$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Furthermore, let $J \in \mathbb{L}_{\infty e}^{r\times s}$ and define*

$$\bar{J}(t) \triangleq \sup_{\tau\in[0,t]} |J(\tau)|$$

Then

a. *If $F(t) \geq 0$ for all t and $H \geq G$ then $F * H \geq F * G$*

b. $|F * G| \leq |F| * |G|$

c. *If $F(t) \geq 0$ for all t then $F * |GJ| \leq (F * |G|)\bar{J}$*

and all the convolutions above are finite for all $t \geq 0$

Lemma 4.2 ([122]). *Let $G \in \mathbb{L}_{pe}^{n\times n}$, $1 \leq p \leq \infty$ and define the linear operator $\mathbf{G}$ by $\mathbf{G}F \triangleq G * F$. Let $\mathbf{T} \triangleq (I - \mathbf{G})^{-1} - I$ and define T as the function such that $\mathbf{T}F \triangleq T * F$. If $\|\mathbf{G}\|_p \leq 1$ and $G(t) \geq 0$ for all $t \geq 0$ then $\|\mathbf{T}\|_p < \infty$ and $T(t) \geq 0$ for all $t \geq 0$.*

In the above Lemmas, the space $\mathbb{L}_{pe}^{n\times m}$, for example, represents the set of functions from $\mathbb{R}_+$ to $\mathbb{R}^{n\times m}$ such that $\|\mathcal{P}_\tau x\|_p < \infty$ for all $\tau > 0$, where $\|\cdot\|$ is some matrix norm and the truncation operator $\mathcal{P}_\tau$ is defined as

$$(\mathcal{P}_\tau x)(t) = \begin{cases} x(t) & t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

4.3 Dynamic threshold generation

Below, a theorem is presented which gives an upper bound on the modulus of the error dynamics and will be applied to derive an expression for dynamic threshold generator.

Theorem 4.1. *Consider the Lipschitz nonlinear system described by*

$$\dot{e} = \bar{A}e + \Phi(x, e, u) + \eta(x, u, t) \tag{4.5}$$

where $\bar{A}$ is Hurwitz. Assume that the nonlinear term $\Phi(x, e, u) = \phi(x, u) - \phi(x - e, u)$ is Lipschitz with the Lipschitz constant γ, i.e.,$|\Phi(x, e, u)| \leq \gamma|e|$, and $\eta(x, u, t)$ is bounded by a known functional, i.e., $|\eta(x, u, t)| \leq \bar{\eta}(u, t)$. Let $G(t) \triangleq exp(\bar{A}t)$ and $H(t)$ be a function that satisfies $H(t) \geq |G(t)|$. If $\|\gamma\mathbf{H}\|_p < 1$, then $\|(I - \gamma\mathbf{H})^{-1}\|_p < \infty$ and

$$|e| \leq (I - \gamma\mathbf{H})^{-1}\mathbf{H}\bar{\eta}(u, t)$$

Proof. The solution to the error dynamics (4.5) can be expressed by

$$\begin{aligned} e &= \int_0^t exp(\bar{A}(t-\tau))\left\{\Phi(x(\tau),\hat{x}(\tau),u(\tau)) + \eta(x(\tau),u(\tau),\tau)\right\} d\tau \\ &= \int_0^t exp(\bar{A}(t-\tau))\Phi(x(\tau),\hat{x}(\tau),u(\tau))d\tau + \int_0^t exp(\bar{A}(t-\tau))\eta(x(\tau),u(\tau),\tau)d\tau \end{aligned}$$

The upper bound on the modulus of e can be derived as

$$\begin{aligned} |e| &= \left| \int_0^t exp(\bar{A}(t-\tau))\Phi(x(\tau),\hat{x}(\tau),u(\tau))d\tau + \int_0^t exp(\bar{A}(t-\tau))\eta(x(\tau),u(\tau),\tau)d\tau \right| \\ &\le \left| \int_0^t exp(\bar{A}(t-\tau))\Phi(x(\tau),\hat{x}(\tau),u(\tau))d\tau \right| + \left| \int_0^t exp(\bar{A}(t-\tau))\eta(x(\tau),u(\tau),\tau)d\tau \right| \end{aligned}$$

where the inequality follows Property 1(a). The above expression may be represented in a simplified manner using convolution operator,

$$\begin{aligned} |e| &\le |G(t) * \Phi(x,\hat{x},u)| + |G(t) * \eta(x,u,t)| \\ &\le |G(t)| * |\Phi(x,\hat{x},u)| + |G(t)| * |\eta(x,u,t)| \\ &\le \gamma\, |G(t)| * |e| + |G(t)| * |\eta(x,u,t)| \\ &\le \gamma\, |G(t)| * |e| + |G(t)| * \bar{\eta}(u,t) \end{aligned}$$

where the first inequality follows Lemma 4.1(b) and the second inequality follows using the Lipschitz condition. Using the assumption $H(t) \ge |G(t)|$, we have

$$|e| \le \gamma \mathbf{H}\, |e| + \mathbf{H}\bar{\eta}(u,t)$$

From the above equation, it is obvious that

$$(I - \gamma\mathbf{H})\, |e| \triangleq \zeta \le \mathbf{H}\bar{\eta}(u,t)$$

By defining the linear operator $\mathbf{T} \triangleq (I - \gamma\mathbf{H})^{-1} - I$, $|e|$ can be expressed as

$$|e| = (I - \gamma\mathbf{H})^{-1}\, \zeta = \mathbf{T}\zeta + \zeta$$

It is clear from Lemma 4.2 that $\mathbf{T}$ is bounded and $T(t) \ge 0$ for all $t \ge 0$. From Lemma 4.1, it follows that

$$\begin{aligned} |e| &= \mathbf{T}\zeta + \zeta \\ &\le \mathbf{T}\mathbf{H}\bar{\eta}(u,t) + \mathbf{H}\bar{\eta}(u,t) \\ &= \left((I - \gamma\mathbf{H})^{-1} - I\right)\mathbf{H}\bar{\eta}(u,t) + \mathbf{H}\bar{\eta}(u,t) \\ &= (I - \gamma\mathbf{H})^{-1}\,\mathbf{H}\bar{\eta}(u,t) \end{aligned}$$

□

Remark 4.1. *It should be noted that we have not considered the effect of initial conditions $e(0)$ in the above results. This may be included without any difficulty. However, for the purpose of fault detection, since we assume that the observer has already converged before the occurrence of fault, we may ignore the effect of initial conditions.*

To this point, it has not been said how to find an $\mathbf{H}$ with impulse response $H(t) \geq |G(t)|$. The following lemma provides a solution to find such an H for the case when $\bar{A}$ has n linearly independent eigenvectors and only real eigenvalues.

Lemma 4.3 ([122])**.** *Let $T(t) = Ce^{\bar{A}t}B$ where $\bar{A}$ has only real eigenvalues and is diagonalizable as $\bar{A} = SDS^{-1}$ where D is a diagonal matrix with the eigenvalues of $\bar{A}$ on the diagonal and the columns of S consists of n linearly independent eigenvectors of $\bar{A}$. Then*

$$|T(t)| \leq U(t) = |CS|e^{Dt}|S^{-1}B|$$

The above lemma provides a solution to find H, only when $\bar{A}$ has real eigenvalues. The above lemma is extended to a generalized case where the eigenvalues of $\bar{A}$ could also be complex.

Theorem 4.2. *Let $T(t) = Ce^{\bar{A}t}B$ where $\bar{A}$ is diagonalizable as $\bar{A} = SDS^{-1}$, D is a diagonal matrix with the eigenvalues of $\bar{A}$ on the diagonal and the columns of S consists of n linearly independent eigenvectors of $\bar{A}$. Furthermore, let D_{real} and D_{imag} be, respectively the diagonal matrices with the real and imaginary part of the eigenvalues of $\bar{A}$ on the diagonal. Then*

$$|T(t)| \leq U(t) = |CS|e^{D_{real}t}|S^{-1}B|$$

Proof.

$$\begin{aligned}
|Ce^{\bar{A}t}B| &= |CSe^{Dt}S^{-1}B| \\
&\leq |CS||e^{Dt}||S^{-1}B| \\
&= |CS||e^{D_{real}t}e^{D_{imag}t}||S^{-1}B| \\
&\leq |CS||e^{D_{real}t}||e^{D_{imag}t}||S^{-1}B| \\
&\leq |CS|e^{D_{real}t}|S^{-1}B|
\end{aligned}$$

□

With the upper bound on the modulus of the error dynamics determined in Theorem 4.1, the dynamic upper bound on the residual signal r is given as

$$\begin{aligned}
|r| &= |Ce| \\
&\leq |C|\,|e| \\
&\leq |C|\,(I - \gamma\mathbf{H})^{-1}\mathbf{H}\bar{\eta}(u,t)
\end{aligned}$$

Hence, the expression for threshold defined in (4.4) becomes

$$J_{th} = v * |C|\,(I - \gamma\mathbf{H})^{-1}\mathbf{H}\bar{\eta}(u,t) \tag{4.6}$$

It should be noted that in comparison to the fixed threshold, J_{th} in (4.6) is generated by a dynamic system with u as input. This nature of adaptiveness in threshold will reduce the false alarm rate and will increase the fault detection rate.

In the next section, an example is presented to illustrate the proposed technique.

4.4 An example

To illustrate the proposed method for threshold selection, the nonlinear system described in Example 3.5 is considered:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 - cx_1 - k_n \sin(x_2) + (k_g + \Delta_{k_g})u + d \\ y &= x_1 \end{aligned} \tag{4.7}$$

We assume that the parameter k_g is uncertain which results in modeling uncertainty. $d(t)$ is uniformly distributed external disturbance to the system with the interval $[-\delta_d, \delta_d]$. The system dynamics are of the form (4.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ k_g \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\phi(x) = \begin{bmatrix} 0 \\ -k_n \sin(x_1) \end{bmatrix}, \; \eta = \begin{bmatrix} 0 \\ \Delta_{k_g} u + d(t) \end{bmatrix}$$

Numerical values of the parameters are $b = 1$, $c = 0.4$, $k_n = 0.1$, $k_g = 1$. It is assumed that the uncertainty in k_g is at most 10%, which gives an upper bound on uncertainty as $\bar{\eta} = |0.1k_g u| + \delta_d$. We take $\delta_d = 0.3$ in this example. The Lipschitz constant for the nonlinear term ϕ, is $\gamma = 0.1$. For the purpose of residual generation, the observer structure of (4.3) is utilized, where A, B, C, ϕ are as defined above, and the filter gain L as computed in Example 3.5

$$L = \begin{bmatrix} 0.3024 \\ 0.9450 \end{bmatrix}$$

Then

$$A - LC = \begin{bmatrix} -0.30 & 1.00 \\ -1.34 & -1.00 \end{bmatrix}$$

The eigenvalues of $A - LC$ are complex and lie at $-0.65 \pm 1.10i$. Using Theorem 4.2, $H(t)$ is found as follows:

$$D = \begin{bmatrix} -0.65 + 1.10i & 0 \\ 0 & -0.65 - 1.10i \end{bmatrix},$$
$$S = \begin{bmatrix} -0.20 - 0.62i & -0.20 + 0.62i \\ 0.76 & 0.76 \end{bmatrix}$$

$$H(t) = |S| e^{D_{real} t} |S^{-1}| = \begin{bmatrix} 1.04e^{-0.65t} & 0.90e^{-0.65t} \\ 1.22e^{-0.65t} & 1.05e^{-0.65t} \end{bmatrix}$$

and

$$\mathbf{H}(s) = \begin{bmatrix} \frac{1.04}{s+0.65} & \frac{0.90}{s+0.65} \\ \frac{1.22}{s+0.65} & \frac{1.05}{s+0.65} \end{bmatrix}$$

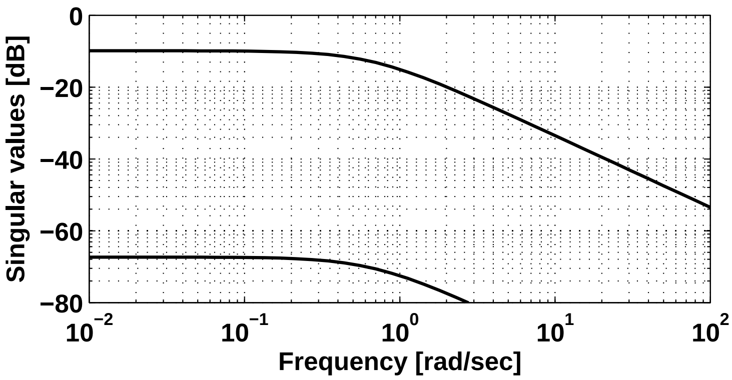


Figure 4.3: Singular value plot for $\mathbf{H}(s)$

Before using $H(t)$ for computation of threshold using (4.6), we need to show that $\|\gamma\mathbf{H}\|_p < 1$, as required by the Theorem 4.1. The easiest to find is for $p = \infty$, $\|\gamma\mathbf{H}\|_\infty \leq \bar{\sigma}(\gamma\mathbf{H})$ because efficient algorithms exist in MATLAB® for the computation of singular values of a transfer function matrix. Figure 4.3 shows the singular value plot for $\gamma\mathbf{H}$ using MATLAB®. We notice that $\bar{\sigma}(\gamma\mathbf{H}) < 1$ and hence, $\|\gamma\mathbf{H}\|_\infty < 1$. After a few algebraic simplifications, we get

$$(I - \gamma\mathbf{H})^{-1}\mathbf{H} = \frac{1}{s^2 + 1.12s + 0.30}\begin{bmatrix} 1.04s + 0.69 & 0.90s + 0.60 \\ 1.22s + 0.81 & 1.05s + 0.70 \end{bmatrix}$$

and

$$|C|\,(I - \gamma\mathbf{H})^{-1}\mathbf{H}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{0.9s + 0.60}{s^2 + 1.12s + 0.30}$$

We choose $\mu = 0.5$ so that $v(t) = 0.5e^{-0.5t}$. The choice of μ is arbitrary, smaller value of μ will result in more smoothened evaluated signal, and vice versa. Thus, the dynamic threshold defined in (4.6) for this example is:

$$J_{th} = \mathscr{L}^{-1}\left\{\frac{0.5}{s + 0.5}\frac{0.9s + 0.60}{s^2 + 1.12s + 0.30}\right\} * \bar{\eta}(u, t)$$

which can be implemented by a third order transfer function. The above example was simulated in Simulink® with an input signal shown in Figure 4.4a. A constant fault of magnitude 1 V occurs at $50s$ due to malfunctioning of the power electronics driving the positioning system. Figure 4.4c shows the residual and the corresponding threshold, it is seen that the variation in residual signal due to changes in control input are mimicked in dynamic threshold whereas the fault is timely detected.

4.5 Summary

In this chapter, a method for dynamic threshold selection for Lipschitz nonlinear systems with unstructured modeling uncertainty and external disturbances bounded by some known functional was presented. An upper bound on the modulus of error dynamics was derived.

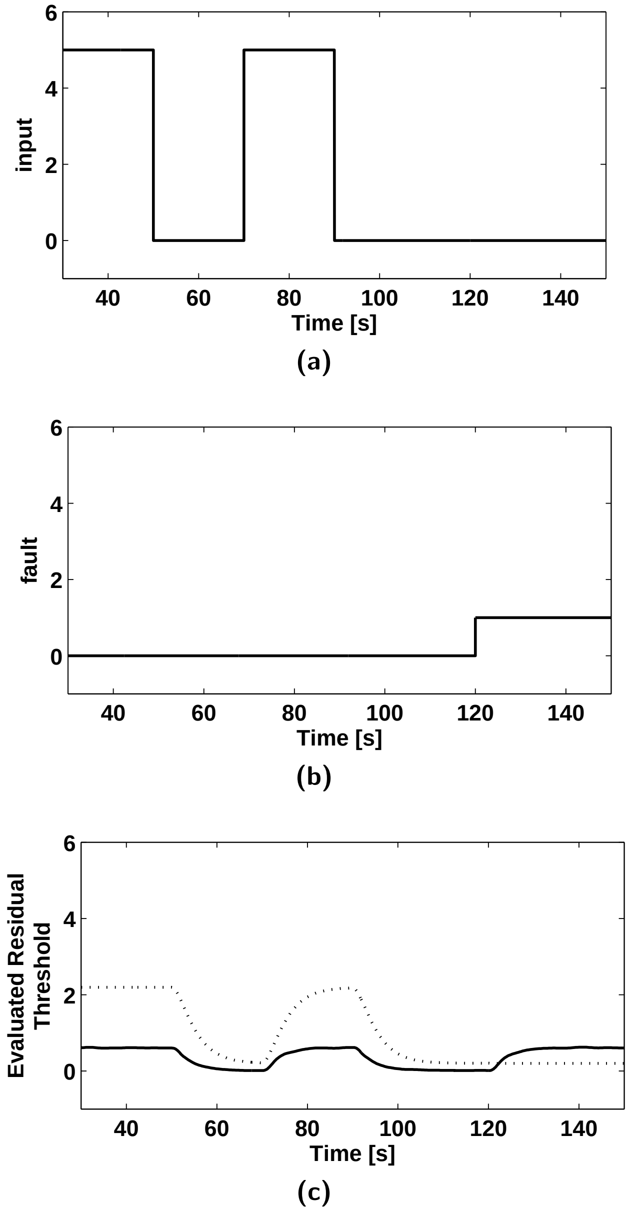


Figure 4.4: Simulation results for fault detection of the example system (a) input to the system (b) fault in the system occurring at 5s (c) residual and the corresponding dynamic threshold.

This upper bound was utilized to compute a threshold. Thus, threshold generator is itself a dynamic system which results in more tighter threshold, and hence, a better performance of the fault detection system. The proposed method was demonstrated by an example that demonstrated the effectiveness of the proposed method.

Chapter 5

Optimal trade-off design using post-filter and threshold

The problem of optimal trade-off design for fault detection in nonlinear systems is addressed in this chapter. Using the factorization approach, a post-filter is designed and a threshold is computed to achieve an optimal trade-off between fault detectability and number of false alarms. Furthermore, it is shown that the proposed post-filter also provides the $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective optimization. The effectiveness of the proposed method is demonstrated by an academic example.

As discussed earlier, an observer-based fault detection system consists of residual generation and residual evaluation. In residual generation step, objective is to make the residual signal sensitive to faults and robust against disturbances. In Chapter 2, some state of the art methods for observer-based residual generation in nonlinear systems were presented. In Chapter 3, three types of fault detection filters were proposed to achieve the fault sensitivity and disturbance robustness properties of the residual signal. Despite disturbance attenuation properties of these residual generation techniques, complete rejection of the effect of disturbances and uncertainties on the residual signal is not possible. Therefore, residual evaluation is performed to decide about the presence of faults. Conventionally, a residual evaluation process consists of taking some norm of the residual signal to obtain the so called *evaluated residual signal* and comparing it with a threshold. If the evaluated residual signal exceeds the threshold, a fault alarm is released. Selection of threshold plays a very important role in the performance of a fault detection system. If threshold is selected too low, disturbances and uncertainties will be able to cause the evaluated residual signal cross the threshold resulting in false alarms. If it is selected too high, some of the faults will not be detected. One approach for threshold selection is to take the maximum possible effect of disturbances on the evaluated residual signal. If $\mathcal{L}_2$ norm is used as evaluation function, then this maximum effect is equal to $\mathcal{L}_2$ norm of disturbance multiplied by $\mathcal{H}_\infty$ norm of the system describing the dynamics of residual generator. However, this will result in a very conservative threshold and most of the small faults will not be detected. To avoid this conservativeness, an approach to select a variable threshold was presented in Chapter 4 for a class of nonlinear systems. This chapter presents another approach for selecting a threshold to reduce the conservativeness of threshold and to make an optimal trade-off between fault detectability and the reduced number of false alarms.

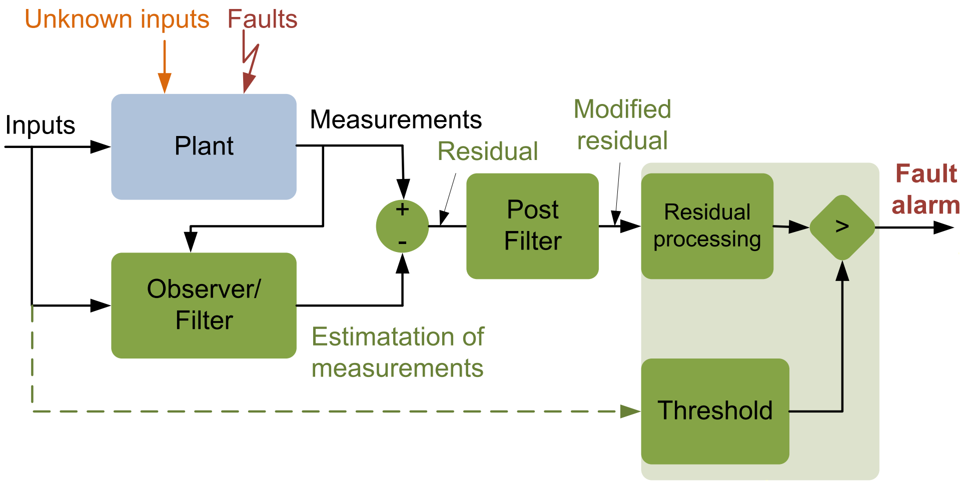


Figure 5.1: Schematic diagram of an observer-based fault detection scheme

Additional to taking norm of the residual signal, generating threshold and selecting a decision logic, this chapter also introduces a post-filter in the step of residual evaluation as shown in Figure 5.1. Utilizing the factorization approach, we propose a method to design this post-filter. The proposed post-filter servers two purposes, firstly, it operates on the residual signal to generate *modified residual signal*. The modified residual signal has increased sensitivity to faults and robustness against disturbances, i.e., the multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ optimization is achieved. Thus one can use any arbitrary stable fault detection filter and the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization will be guaranteed by the post-filter. Secondly, the post-filter together with the designed threshold makes an optimal trade-off between fault detectability and reduced number of false alarms. There can be two situations of the optimal trade-off between the fault detectability and the false alarm rate. These are: i) Minimization of false alarms for a required fault detection rate(FDR) ii) Maximization of fault detectability for an allowed false alarm rate (FAR) . We propose two solutions for the above mentioned optimal trade-off design problems. For the first situation, based on the required fault detection rate, a threshold is determined which ensures the required FDR, and the false alarm rate is minimized by designing the post-filter. The second situation is more realistic, in many practical situations allowable FAR is provided and it is desired to maximize the fault detectability. A threshold is selected to guarantee that the FAR is lower than the allowed one, and fault detectability is maximized by using the post-filter. Factorization approach is utilized to obtain solution to these optimization problems.

Section 5.1 presents some preliminary material which will be helpful for further discussions. The problem of finding a trade-off between high fault detectability and reduced number of false alarms is formulated in Section 5.2. Section 5.3 provides a solution to the formulated problems. The threshold selection is presented in Section 5.4. In Section 5.5, we compare our results to the existing optimization problems. An example is presented in Section 5.6 to demonstrate the effectiveness of the proposed approach. Conclusion is presented in the last section.

5.1 Preliminaries

To impart some useful concepts for our further discussion, we consider the affine nonlinear system described by

$$\Sigma : \begin{cases} \dot{x} = a(x) + B(x)u \\ y = c(x) + D(x)u \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is input vector and $y \in \mathbb{R}^p$ is the output vector. $a(x)$, $B(x)$, $c(x)$ and $D(x)$ are smooth functions of appropriate dimensions. We will use $\Sigma(u)$ to represent the output of system Σ with input u, i.e., $y = \Sigma(u)$. The series interconnection of two systems will be represented by the operator $\circ$, i.e., $\Sigma_2 \circ \Sigma_1$ means the output of system Σ_1 appears as input to system Σ_2. $\mathrm{D}\Sigma$ represents the Fréchet derivative of the system Σ (see [145] for details about Fréchet derivative). Σ^{-1} is obtained by interchanging the roles of inputs and outputs of Σ, i.e., with the assumption that $D(x)$ is invertible, Σ^{-1} is given by:

$$\Sigma^{-1} : \begin{cases} \dot{x} = a(x) - B(x)D^{-1}(x)c(x) + B(x)D^{-1}(x)y \\ u = -D^{-1}(x)c(x) + D^{-1}(x)y \end{cases}$$

Definition 5.1. *[146] The nonlinear system Σ defined in* (5.1) *is called weakly minimum phase if x_{eq} is a Lyapunov stable equilibrium point of $a(x) - B(x)D^{-1}(x)c(x)$ and is called strictly minimum phase if x_{eq} is an asymptotically stable equilibrium point of $a(x) - B(x)D^{-1}(x)c(x)$*

Definition 5.2. *[147] The nonlinear system Σ as in* (5.1) *is called inner if there exists a non-negative valued storage function $P(x)$ with $P(0) = 0$ such that*

$$P(x(t_2)) - P(x(t_1)) = \frac{1}{2}\int_{t_1}^{t_2} \left[u^T u - y^T y\right] dt$$

over all trajectories (y, x, u) of the system.

Definition 5.3. *[147] The nonlinear system Σ is called outer if it is an asymptotically stable, weakly minimum phase system.*

Lemma 5.1. *For nonlinear system $\Sigma : u \to y$, we have*

a) $\|y\|_2 \le \|\Sigma\|_\infty \|u\|_2$

b) $\|y\|_2 \ge \|\Sigma\|_- \|u\|_2$

Proof. Following the definitions

$$\|y\|_2 = \frac{\|y\|_2}{\|u\|_2}\|u\|_2 \le \sup_{u \ne 0} \frac{\|y\|_2}{\|u\|_2}\|u\|_2 = \|\Sigma\|_\infty \|u\|_2$$

$$\|y\|_2 = \frac{\|y\|_2}{\|u\|_2}\|u\|_2 \ge \inf_{u \ne 0} \frac{\|y\|_2}{\|u\|_2}\|u\|_2 = \|\Sigma\|_- \|u\|_2$$

□

Lemma 5.2. *For two nonlinear systems Σ_A: $y \mapsto z$ and $\Sigma_B : u \mapsto y$, and the assumption that for $u \neq 0$, $y = \Sigma_B(u) \neq 0$, we have*

a) $\|\Sigma_A \circ \Sigma_B\|_\infty \leq \|\Sigma_A\|_\infty \|\Sigma_B\|_\infty$

b) $\|\Sigma_A \circ \Sigma_B\|_- \leq \|\Sigma_A\|_\infty \|\Sigma_B\|_-$

c) $\|\Sigma_A \circ \Sigma_B\|_\infty \geq \|\Sigma_A\|_- \|\Sigma_B\|_\infty$

Proof. a)

$$\begin{aligned}
\|\Sigma_A \circ \Sigma_B\|_\infty &= \sup_{u \neq 0} \frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|u\|_2} \\
&= \sup_{u \neq 0} \left(\frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|\Sigma_B(u)\|_2} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \right) \\
&\leq \sup_{y \neq 0} \frac{\|\Sigma_A(y)\|_2}{\|y\|_2} \sup_{u \neq 0} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \\
&\leq \|\Sigma_A\|_\infty \|\Sigma_B\|_\infty
\end{aligned}$$

b)

$$\begin{aligned}
\|\Sigma_A \circ \Sigma_B\|_- &= \inf_{u \neq 0} \frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|u\|_2} \\
&= \inf_{u \neq 0} \left\{ \frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|\Sigma_B(u)\|_2} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \right\} \\
&= \inf_{u \neq 0} \left\{ \frac{\|\Sigma_A(y)\|_2}{\|y\|_2} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \right\} \\
&\leq \sup_{y \neq 0} \frac{\|\Sigma_A(y)\|_2}{\|y\|_2} \inf_{u \neq 0} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \\
&\leq \|\Sigma_A\|_\infty \|\Sigma_B\|_-
\end{aligned}$$

c)

$$\begin{aligned}
\|\Sigma_A \circ \Sigma_B\|_\infty &= \sup_{u \neq 0} \frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|u\|_2} \\
&= \sup_{u \neq 0} \left\{ \frac{\|\Sigma_A \circ \Sigma_B(u)\|_2}{\|\Sigma_B(u)\|_2} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \right\} \\
&= \sup_{u \neq 0} \left\{ \frac{\|\Sigma_A(y)\|_2}{\|y\|_2} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \right\} \\
&\geq \inf_{y \neq 0} \frac{\|\Sigma_A(y)\|_2}{\|y\|_2} \sup_{u \neq 0} \frac{\|\Sigma_B(u)\|_2}{\|u\|_2} \\
&\geq \|\Sigma_A\|_- \|\Sigma_B\|_\infty
\end{aligned}$$

□

The inner-outer factorization of a nonlinear systems Σ is represented as

$$\Sigma = \Sigma_i \circ \Sigma_o$$

The lossless system Σ_i is inner factor and stable minimum phase system Σ_o is the outer factor. The problem of inner-outer factorization (more precisely (j,J)-inner-outer factorization) has been widely discussed in literature (mainly by Ball and coworkers, see e.g., [146, 147]). On the assumption that $x_{eq} = 0$ is asymptotically stable equilibrium point of $\dot{x} = a(x)$, the outer factor for the system Σ defined in (5.1) is given by [147]

$$\Sigma_o : \begin{cases} \dot{\xi} = a(\xi) + B(\xi)u \\ \bar{y} = \bar{c}(\xi) + \bar{D}(\xi)u \end{cases}$$

where $m \times m$ smooth function $\bar{D}(\xi)$ with invertible values for all ξ is given by $D^T(\xi)D(\xi) = \bar{D}^T(\xi)\bar{D}(\xi) = E(\xi)$ and $\bar{c}(\xi) = \bar{D}(\xi)E^{-1}(\xi)\left\{D^T(\xi)c(\xi) + B^T(\xi)P_x^T(\xi)\right\}$. $P(\xi) \geq 0$ with $P(0) = 0$ is a smooth function which satisfies the following Hamilton-Jacobi equation

$$P_\xi(\xi)\left[a(\xi) - B(\xi)E^{-1}(\xi)D^T(\xi)c(\xi)\right] + \frac{1}{2}c^T(\xi)\left[I - D(\xi)E^{-1}(\xi)D^T(\xi)\right]c(\xi) - \frac{1}{2}P_\xi(\xi)B(\xi)E^{-1}(\xi)B^T(\xi)P_\xi^T(\xi) = 0$$

Furthermore, the inner is conservative, i.e.,

$$[D\Sigma_i]^T \circ [\Sigma_i(\bar{y})] = \bar{y} \tag{5.2}$$

It has been shown in [148] that if Σ_i satisfies (5.2), then Σ_i is norm preserving, i.e. $\|y\|_2 = \|\bar{y}\|_2$. In other words, we can say

$$\|\Sigma_i\|_\infty = \|\Sigma_i\|_- = 1$$

For the purpose of FDI, the problem of co-inner-outer factorization is of more interest and is represented as:

$$\Sigma = \Sigma_{co} \circ \Sigma_{ci}$$

where Σ_{co} is an outer system and Σ_{ci} is an inner system. The co-inner-outer factorization is a dual problem to the inner-outer factorization. By proceeding in a parallel way to inner-outer factorization in [147], we have the following theorem for the co-inner-outer factorization.

Theorem 5.1. *Suppose that for the system Σ defined in* (5.1), $x_{eq} = 0$ *is asymptotically stable equilibrium point of $\dot{x} = a(x)$ and there exists a solution $P(\xi) \geq 0$ with $P(0) = 0$ to the following Hamilton-Jacobi equation*

$$P_\xi(\xi)\left[a(\xi) - B(\xi)D^T(\xi)E^{-1}(\xi)c(\xi)\right] + \frac{1}{4}P_\xi(\xi)B(\xi)\left[I - D^T(\xi)E^{-1}(\xi)D(\xi)\right]B^T(\xi)P_\xi^T(\xi) - c^T(\xi)E^{-1}(\xi)c(\xi) = 0 \tag{5.3}$$

Then the co-outer factor Σ_{co} for Σ is given by

$$\Sigma_{co} : \begin{cases} \dot{\xi} = a(\xi) + \bar{B}(\xi)\bar{y} \\ y = c(\xi) + \bar{D}(\xi)\bar{y} \end{cases} \tag{5.4}$$

where $p \times p$ smooth function $\bar{d}(\xi)$ with invertible values for all ξ is given by $d(\xi)d^T(\xi) = \bar{d}(\xi)\bar{d}^T(\xi) = E(\xi)$ and

$$P_\xi(\xi)\bar{B}(\xi) = \left[P_\xi(\xi)B(\xi)D^T(\xi) + 2c^T(\xi)\right]E^{-1}(\xi)\bar{D}(\xi) \tag{5.5}$$

Proof. The proof follows in same lines as for inner-outer factorization in [147]. □

The factorization of nonlinear systems involves solving a Hamilton-Jacobi equations (5.3). Analytical solution of such equation is difficult to obtain, however, there exist some methods to obtain the approximate solution for the Hamilton-Jacobi equations, see e.g. [149–152] and references therein for approximate solutions of Hamilton-Jacobi equations. Two of these methods are outlined in Appendix C.

5.2 Problem formulation

If we put bird's eye view on a fault detection system, it appears as a box with measurements as inputs and the 'fault alarm' as output. It is desired that the fault detection system should release a fault alarm as soon as fault occurs in the system. The presence of uncertainties and disturbances can also appear to the fault detection system as faults and may cause to release alarm, this is called a false alarm. Definitely, the presence of false alarms is not desired in any fault detection system. To measure how frequently a false alarm occurs in a fault detection system, false alarm rate (FAR) was defined in statistical settings [123]. Similarly, to measure the fault detectability, fault detection rate (FDR) was defined [123]. Later, these concepts were generalized to the norm based context [13, 31]. We know that high FDR and low FAR are the design objectives of fault detection systems. But these are the conflicting requirements and therefore, to find an optimal trade-off between high FDR and low FAR is the challenge in designing fault detection systems. Note that, residual sensitivity to faults and robustness to disturbances are only tools to achieve the objective of the optimal trade-off between high FDR and low FAR. In this section, first we give the norm based definitions of FDR and FAR, and then based on norm based definitions, we formulate the optimal trade-off design problem into finding a post-filter and a threshold.

We consider again the affine nonlinear system described by (3.5) and the corresponding fault detection filter structure defined in (3.6). We assume that a stable fault detection filter is already designed. The dynamics of augmented system are then described by (3.7). For easy reference, the dynamics of the augmented systems are again described here

$$\Sigma_{\mathscr{A}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_f(\tilde{x})f + \tilde{E}_d(\tilde{x})d \\ r = \tilde{c}(\tilde{x}) + \tilde{F}_f(\tilde{x})f + \tilde{F}_d(\tilde{x})d \end{cases} \tag{5.6}$$

where $f \in \mathbb{R}^{k_f}$ is the fault vector that has to be detected, $d \in \mathbb{R}^{k_d}$ is vector of disturbances, the vectors $\tilde{x}$, $\tilde{a}(\tilde{x})$, $\tilde{c}(\tilde{x})$ and the matrices $\tilde{E}_f(\tilde{x})$, $\tilde{E}_f(\tilde{x})$, $\tilde{F}_f(\tilde{x})$, $\tilde{F}_d(\tilde{x})$ are as defined in Section 3.1.

As shown in Figure 5.1, the residual signal r acts as input to the post-filter to generate a signal, we call the *modified residual signal* $\tilde{r}$, i.e.

$$\tilde{r} = \Sigma_{\mathscr{R}}(r)$$

$\Sigma_{\mathscr{R}}$ is a dynamic nonlinear system which is to be designed to achieve the optimal trade-off design problems (to be stated in this Section).

Note that in the above formulation, we have not taken into account the effect of known inputs u. The known inputs can be treated in similar way as the disturbances d (please see Remark 3.1 for more details).

For the purpose of analysis, we define $\Sigma_{\mathscr{D}}$ to represent the fault-free residual generator dynamics, i.e.,

$$\Sigma_{\mathscr{D}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_d(\tilde{x})d \\ r = \tilde{c}(\tilde{x}) + \tilde{F}_d(\tilde{x})d \end{cases} \tag{5.7}$$

Similarly, we define $\Sigma_{\mathscr{F}}$ to represent the residual generator dynamics with $d = 0$, i.e.,

$$\Sigma_{\mathscr{F}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_f(\tilde{x})f \\ r = \tilde{c}(\tilde{x}) + \tilde{F}_f(\tilde{x})f \end{cases} \tag{5.8}$$

$\mathcal{L}_2$ norm is commonly used as evaluation function, we will also use $\mathcal{L}_2$ norm of the modified residual signal $\tilde{r}$ for the purpose of residual evaluation, i.e.,

$$J = \|\tilde{r}\|_2 = \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{A}}(f, d)\|_2 \tag{5.9}$$

The corresponding threshold is denoted by J_{th} and the following simple decision logic is used to decide about the occurrence of faults

$$\begin{aligned} J - J_{th} > 0 &\Rightarrow \text{fault} \\ J - J_{th} \leq 0 &\Rightarrow \text{ fault-free} \end{aligned} \tag{5.10}$$

Note that a false-alarm will be created if in a fault-free case

$$\|\tilde{r}\|_2 = \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{A}}(d)\|_2 > J_{th} \quad \text{false-alarm condition} \tag{5.11}$$

Likewise, a fault will be detected if

$$\|\tilde{r}\|_2 = \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{A}}(f, d)\|_2 > J_{th} \quad \text{fault detection condition} \tag{5.12}$$

To measure how frequently a false alarm is released and how frequently a fault is detected, false alarm rate (FAR) and fault detection rate (FDR) were originally defined in stochastic settings [123]. Later, the concept was generalized to deterministic context in [13]. Below we explain the deterministic definition of FAR and FDR.

In fault free case,

$$\sup_{d \neq 0, f=0} \|\tilde{r}\|_2 = \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty \delta_d = \alpha\delta_d$$

It can be noticed that setting the threshold equal to $\alpha\delta_d$ will guarantee a zero false alarm. With this observation, [13] defined FAR in norm based framework as

$$FAR = 1 - \frac{J_{th}}{\alpha\delta_d} \tag{5.13}$$

The set of disturbances that cause false alarms (SDFA) is denoted by $\Omega_{FA}(\Sigma_{\mathscr{R}}, J_{th})$, i.e.,

$$\Omega_{FA}(\Sigma_{\mathscr{R}}, J_{th}) = \{d | (5.11) \text{ is satisfied}\} \tag{5.14}$$

The size of SDFA gives an indication of the number of possible false alarms - smaller the set Ω_{FA}, lower is the false alarm number and vice versa.

Likewise, for no disturbance,

$$\inf_{f\neq 0, d=0} \|\tilde{r}\|_2 = \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_- \delta_{f,min} = \beta\delta_{f,min}$$

and setting the threshold equal to $\beta\delta_{f,min}$ will give 100 % detection of faults whose size is larger than $\delta_{f,min}$. Hence, FDR in norm based framework can be defined as [13]

$$FDR = \frac{\beta\delta_{f,min}}{J_{th}} \tag{5.15}$$

We denote the set of detectable faults (SDF) by $\Omega_{DE}(\Sigma_{\mathscr{R}}, J_{th})$, i.e.,

$$\Omega_{DE}(\Sigma_{\mathscr{R}}, J_{th}) = \{f| \text{ (5.12) is satisfied}\} \tag{5.16}$$

the size of SDF gives the measure of the performance of fault detection system regarding fault detectability – larger is the set Ω_{DE}, higher will be the fault detectability and vice versa.

So, the design objective of finding a trade-off between FDR and FAR can be expressed in terms of the following two optimization problems.

Problem 5.1 (Maximizing SDF under a given FAR (PMax-SDF))**.** *Given an FAR, find a stable post-filter $\Sigma_{\mathscr{R}_{opt}}$ so that*

$$\Omega_{DE}(\Sigma_{\mathscr{R}}, J_{th}) \subseteq \Omega_{DE}(\Sigma_{\mathscr{R}_{opt}}, J_{th}) \tag{5.17}$$

Problem 5.2 (Minimizing SDFA under a given FDR (PMin-SDFA))**.** *Given FDR, find a stable post-filter $\Sigma_{\mathscr{R}_{opt}}$ so that*

$$\Omega_{FA}(\Sigma_{\mathscr{R}}, J_{th}) \supseteq \Omega_{FA}(\Sigma_{\mathscr{R}_{opt}}, J_{th}) \tag{5.18}$$

In next section, a threshold selection method and post-filter design to give solutions to these problems is proposed.

5.3 Post-filter design

Using the co-inner-outer factorization, we present two theorems which propose threshold selection and post-filter design methods to provide solutions to the problems formulated in the last section.

Theorem 5.2. *Assume that $\Sigma_{\mathscr{D}}$ can be factorized into co-inner-outer factors as $\Sigma_{\mathscr{D}} = \Sigma_{\mathscr{D}_{co}} \circ \Sigma_{\mathscr{D}_{ci}}$, then*

$$\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1} \tag{5.19}$$

solves the optimization problem PMax-SDF defined in (5.17)*.*

Proof. For a given FAR, the threshold is given according to (5.13) as

$$J_{th} = (1 - FAR) \left\| \Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d$$

Let

$$\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1}$$

where $\Sigma_{\mathscr{Q}}$ is any arbitrarily selectable stable system. Then from the fault detection condition (5.12), fault is detected if

$$\left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR) \left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d > 0 \tag{5.20}$$

Note that

$$\left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}} \right\|_\infty = \left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{ci}} \right\|_\infty = \left\| \Sigma_{\mathscr{Q}} \right\|_\infty$$

The first equality follows by cancellation of the co-outer part of $\Sigma_{\mathscr{D}}$ and the second by the fact that $\left\| \Sigma_{\mathscr{D}_{ci}} \right\|_\infty = 1$. Also utilizing Lemma 5.1(a), we have

$$\left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 \leq \left\| \Sigma_{\mathscr{Q}} \right\|_\infty \left\| \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2$$

So that detection condition (5.20) becomes

$$\begin{aligned} 0 &< \left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR) \left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d \\ &\leq \left\| \Sigma_{\mathscr{Q}} \right\|_\infty \left(\left\| \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR)\delta_d \right) \end{aligned}$$

Therefore

$$\left\| \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR)\delta_d > 0 \tag{5.21}$$

is a necessary condition for fault to be detected. Setting $\Sigma_{\mathscr{Q}} = I$ and therefore, $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$ leads to

$$\begin{aligned} &\left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR) \left\| \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d \\ &\qquad = \left\| \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{A}}(f,d) \right\|_2 - (1 - FAR)\delta_d \end{aligned}$$

which means that (5.21) is also sufficient condition for f to be detectable provided that $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$. Thus,

$$\Omega_{DE}(\Sigma_{\mathscr{R}}, J_{th}) \subseteq \Omega_{DE}(\Sigma_{\mathscr{R}_{opt}}, J_{th})$$

which means that the set of detectable faults with post-filter $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$ is superset to the set of detectable faults with any other post-filter, that is, the set of detectable faults is maximized with the optimal post-filter. Furthermore, as the co-outer is weakly minimum phase, hence its inverse is stable. This completes the proof. □

Using a similar approach, the following theorem provides a solution to PMin-SDFA problem defined in (5.18).

Theorem 5.3. *Assume that* $\Sigma_{\mathscr{F}}$ *can be factorized into co-inner-outer factors as* $\Sigma_{\mathscr{F}} = \Sigma_{\mathscr{F}_{co}} \circ \Sigma_{\mathscr{F}_{ci}}$*, then*

$$\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1} \tag{5.22}$$

solves the optimal trade-off design problem PMin-SDFA defined in (5.18)

Proof. For a given FDR, the threshold is given according to (5.15) as

$$J_{th} = \frac{\delta_{f,min}}{FDR} \|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_{-}$$

Let

$$\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1}$$

where $\Sigma_{\mathscr{Q}}$ is any arbitrarily selectable stable system. Then from (5.11), false alarm will be generated if

$$\left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR} \left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\right\|_{-} > 0 \tag{5.23}$$

Note that

$$\left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\right\|_{-} = \|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{ci}}\|_{-} = \|\Sigma_{\mathscr{Q}}\|_{-}$$

The first equality follows by cancellation of the co-outer part $\Sigma_{\mathscr{F}}$ and the second by noting that that $\|\Sigma_{\mathscr{F}_{ci}}\|_{-} = 1$. Utilizing Lemma 5.1(b), we have

$$\left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 \geq \|\Sigma_{\mathscr{Q}}\|_{-} \left\|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2$$

then the false alarm condition (5.23) becomes

$$\begin{aligned} 0 &< \|\Sigma_{\mathscr{Q}}\|_{-} \left(\left\|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR}\right) \\ &\leq \left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR} \left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\right\|_{-} \end{aligned}$$

Hence,

$$\|\Sigma_{\mathscr{Q}}\|_{-} > 0, \quad \left\|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR} > 0 \tag{5.24}$$

is sufficient condition for a false alarm. Setting $\Sigma_{\mathscr{Q}} = I$ and therefore, $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1}$ leads to

$$\left\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR} \|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{ci}}\|_{-} = \|\Sigma_{\mathscr{Q}}\|_{-} \left\|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}(d)\right\|_2 - \frac{\delta_{f,\min}}{FDR} > 0$$

which means that (5.24) is also necessary condition for d to cause false alarm provided $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1}$. So

$$\Omega_{FA}(\Sigma_{\mathscr{R}_{opt}}, J_{th}) \subseteq \Omega_{FA}(\Sigma_{\mathscr{R}}, J_{th})$$

which means that the set of disturbances that cause false-alarms with post-filter $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1}$ is subset to the set of disturbances that cause false-alarms with any other post-filter, that is, the set of disturbances that cause false alarms is minimized with the optimal post-filter. Furthermore, as the co-outer is weakly minimum phase, hence its inverse is stable. This completes the proof. □

The proposed post-filter design approach not only provides optimization in the sense of FDR and FAR but, as will be shown in Section 5.5, also gives optimization in the sense of multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ trade-off design. As compared to the $\mathcal{H}_-/\mathcal{H}_\infty$ trade-off design, where two coupled partial differential equations with two unknown functions (L, P) are to be solved [153], the proposed approach has the advantage that it needs the solution of only one partial differential equation with one unknown function (P), although the order of the equation is doubled.

The approach for multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ trade-off design is to assume a constant post filter and to make optimization over the filter gain L. In our analysis, it is easier to take a stabilizing filter gain and make optimization over the post-filter. This has the drawback that two time effort is to be made, once for designing a stabilizing filter gain matrix and the other for optimal post-filter design. However, this approach has more advantage in situations when the observer has already been designed, e.g. for control purposes, and we can not change the observer properties.

5.4 Threshold selection

Threshold selection is an important step in the design of fault detection scheme. The effect of disturbances and modeling uncertainties on the residual signal can not be eliminated in most of the cases, therefore, a proper threshold has to be selected to decide the presence of faults. A high threshold will result in low number of false alarms but at the same time will result in low fault detectability. Likewise, a low threshold selection will give a good fault detectability but with increased false alarm rate. In last section, we proposed that threshold can be selected to guarantee a required fault detection rate or allowable false alarm rate. Based on these threshold selections, we derived post-filters to minimize the false alarm rate and maximize the fault detectability. With these post filters, the expression for threshold is considerably simplified and can be easily computed. Thus for a given FAR, and post-filter settings according to $\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{D}_{co}}^{-1}$, threshold is obtained from (5.13) as

$$\begin{aligned}
J_{th} &= (1 - FAR) \left\| \Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d \\
&= (1 - FAR) \left\| \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}} \right\|_\infty \delta_d \\
&= (1 - FAR) \left\| \Sigma_{\mathscr{D}_{ci}} \right\|_\infty \delta_d \\
&= (1 - FAR) \delta_d
\end{aligned} \tag{5.25}$$

Likewise, for a given FDR and post filter settings according to $\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{F}_{co}}^{-1}$, threshold is determined from (5.15)

$$\begin{aligned}
J_{th} &= \frac{\delta_{f,min}}{FDR} \left\| \Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}} \right\|_- \\
&= \frac{\delta_{f,min}}{FDR} \left\| \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{F}} \right\|_- \\
&= \frac{\delta_{f,min}}{FDR} \left\| \Sigma_{\mathscr{F}_{ci}} \right\|_- \\
&= \frac{\delta_{f,min}}{FDR}
\end{aligned} \tag{5.26}$$

5.5 The relationship with optimal residual generators

Residual generation in nonlinear systems has received considerable interests among the researchers. Some of the well known observer-based residual generation methods for nonlinear systems were presented in Chapter 2. In Chapter 3 we proposed observer-based residual generation techniques for nonlinear systems utilizing the game theoretic approach. Three types of observers were presented, these include $\mathcal{H}_-$ fault sensitive FDF, $\mathcal{H}_\infty$ disturbance attenuating FDF and $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF. The $\mathcal{H}_-$ fault sensitive FDF ensures that the minimum $\mathcal{L}_2$ gain from fault to residual is not less than a give constant, and $\mathcal{H}_\infty$ disturbance attenuating FDF guarantees that in worst case $\mathcal{L}_2$ gain from disturbance to residual is not greater than a give constant. However, these two FDF have associated disadvantages. For example, besides amplifying the effect of faults, the $\mathcal{H}_-$ fault sensitive FDF may also amplify the disturbances. Likewise, $\mathcal{H}_\infty$ disturbance attenuating FDF may also attenuate the effect of faults along with attenuating the effect of disturbances. Therefore, $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF is the best in the sense that it simultaneously reduces the effect of disturbances and amplifies the effect of faults (see Chapter 3 for more details). In this section, we show that the post-filters design methods proposed in this chapter not only deliver optimization in the sense of optimal trade-off between FAR and FDR but also achieve $\mathcal{H}_-/\mathcal{H}_\infty$ optimization.

From previous discussions, we know that the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization problem is to maximize

$$\frac{\inf_{f\neq 0} \|r\|_2 / \|f\|_2}{\sup_{d\neq 0} \|r\|_2 / \|d\|_2} = \frac{\|\Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{D}}\|_\infty}$$

If the post-filter is used as shown in Figure 5.1, the residual signal r acts as input to the post-filter to generate the modified residual signal $\tilde{r}$, and the multi-objective optimization problem is then represented as

$$\frac{\inf_{f\neq 0} \|\tilde{r}\|_2 / \|f\|_2}{\sup_{d\neq 0} \|\tilde{r}\|_2 / \|d\|_2} = \frac{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty}$$

Next two theorems show that using the post-filters proposed in Section 5.3 will optimize the above index. Thus one can use any stable fault detection filter for residual generation and the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization will be achieved by the proposed post-filter.

Theorem 5.4. *Assume that $\Sigma_{\mathscr{D}}$ can be factorized into co-inner-outer factors as $\Sigma_{\mathscr{D}} = \Sigma_{\mathscr{D}_{co}} \circ \Sigma_{\mathscr{D}_{ci}}$, then*

$$\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1} \tag{5.27}$$

solves the optimal residual generator design problem according to the $\mathcal{H}_-/\mathcal{H}_\infty$ index, i.e.,

$$\Sigma_{\mathscr{R}_{opt}} = \arg \sup_{\Sigma_{\mathscr{R}}} \frac{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty} = \Sigma_{\mathscr{D}_{co}}^{-1}$$

Proof. Let post-filter

$$\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1}$$

where $\Sigma_{\mathscr{Q}}$ is any arbitrarily selectable stable system. Recall that $\Sigma_{\mathscr{F}}$ and $\Sigma_{\mathscr{D}}$ represent the augmented system $\Sigma_{\mathscr{A}}$ for case $f = 0$ and $d = 0$, respectively, then using Lemma 5.2(b) and noting that $\|\Sigma_{\mathscr{D}_{ci}}\|_\infty = 1$, we get

$$\begin{aligned}
\frac{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty} &= \frac{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty} \\
&= \frac{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{ci}}\|_\infty} \\
&= \frac{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{Q}}\|_\infty} \\
&\leq \frac{\|\Sigma_{\mathscr{Q}}\|_\infty \|\Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{Q}}\|_\infty} \\
&\leq \|\Sigma_{\mathscr{D}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-
\end{aligned}$$

The equalities can be achieved when $\Sigma_{\mathscr{Q}} = I$, so that the optimal post-filter is $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$. This completes the proof. □

By following similar steps, it can be shown that (5.27) also solves the optimal residual generator design problem according to the $\mathcal{H}_\infty/\mathcal{H}_\infty$ multi-objective performance index.

Theorem 5.5. *Assume that $\Sigma_{\mathscr{F}}$ can be factorized into co-inner-outer factors as $\Sigma_{\mathscr{F}} = \Sigma_{\mathscr{F}_{co}} \circ \Sigma_{\mathscr{F}_{ci}}$, then*

$$\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1} \tag{5.28}$$

solves the optimal residual generator design problem according to the $\mathcal{H}_-/\mathcal{H}_\infty$ index, i.e.,

$$\Sigma_{\mathscr{R}_{opt}} = \arg\sup_{\Sigma_{\mathscr{R}}} \frac{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty} = \Sigma_{\mathscr{F}_{co}}^{-1}$$

Proof. Let post-filter

$$\Sigma_{\mathscr{R}} = \Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1}$$

where $\Sigma_{\mathscr{Q}}$ is any arbitrarily selectable stable system. Recall that $\Sigma_{\mathscr{F}}$ and $\Sigma_{\mathscr{D}}$ represent the augmented system $\Sigma_{\mathscr{A}}$ for case $f = 0$ and $d = 0$, using Lemma 5.2(c) and noting that $\|\Sigma_{\mathscr{F}_{ci}}\|_- = 1$, we get

$$\begin{aligned}
\frac{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{R}} \circ \Sigma_{\mathscr{D}}\|_\infty} &= \frac{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{F}}\|_-}{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty} \\
&= \frac{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{ci}}\|_-}{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty} \\
&= \frac{\|\Sigma_{\mathscr{Q}}\|_-}{\|\Sigma_{\mathscr{Q}} \circ \Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty} \\
&\leq \frac{\|\Sigma_{\mathscr{Q}}\|_-}{\|\Sigma_{\mathscr{Q}}\|_- \|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty} \\
&\leq \frac{1}{\|\Sigma_{\mathscr{F}_{co}}^{-1} \circ \Sigma_{\mathscr{D}}\|_\infty}
\end{aligned}$$

The equalities can be achieved when $\Sigma_{\mathscr{Q}} = I$, so that the optimal post-filter is $\Sigma_{\mathscr{R}_{opt}} = \Sigma^{-1}_{\mathscr{F}_{co}}$. This completes the proof. □

5.6 An Example

To demonstrate the effectiveness of the proposed method, we present here a simple example of a nonlinear system described by

$$\begin{aligned} \dot{x} &= -x - x^3 + u + d + f_a \\ y &= x + \eta + f_s \end{aligned}$$

For residual generation, we apply the following nonlinear fault detection filter

$$\begin{aligned} \dot{\hat{x}} &= -\hat{x} - \hat{x}^3 + u + L(y - \hat{y}) \\ \hat{y} &= \hat{x} \end{aligned}$$

where $L = 3$ is the filter gain. The dynamics of residual generator are;

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} -x - x^3 \\ -\hat{x} - \hat{x}^3 + L(x - \hat{x}) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} d \\ \eta \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} f_a \\ f_s \end{bmatrix} \\ r &= x - \hat{x} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ \eta \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} f_a \\ f_s \end{bmatrix} \end{aligned}$$

Post-filter and threshold to solve PMaxSDF

Recall that the optimal post-filter which provides the solution to PMaxSDF problem is the inverse of the co-outer part of $\Sigma_{\mathscr{D}}$. For the example system,

$$\Sigma_{\mathscr{D}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_d d \\ r = \tilde{c}(\tilde{x}) + \tilde{F}_d d \end{cases} \tag{5.29}$$

with

$$\tilde{a}(\tilde{x}) = \begin{bmatrix} -x - x^3 \\ -\hat{x} - \hat{x}^3 + L(x - \hat{x}) \end{bmatrix}, \ \tilde{E}_d = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix}, \ \tilde{c}(\tilde{x}) = x - \hat{x}, \ \tilde{F}_d = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

To find the co-outer for $\Sigma_{\mathscr{D}}$, we need to solve the following equation which comes from (5.3),

$$\begin{aligned} & P_\xi(\xi)[\tilde{a}(\xi) - \tilde{E}_d \tilde{F}_d^T (\tilde{F}_d \tilde{F}_d^T)^{-1} \tilde{c}(\xi)] \\ & \quad + \frac{1}{4} P_\xi(\xi) \tilde{E}_d [I - \tilde{F}_d^T (\tilde{F}_d \tilde{F}_d^T)^{-1} \tilde{F}_d] \tilde{E}^T P_\xi^T(\xi) - \tilde{c}^T(\xi) (\tilde{F}_d \tilde{F}_d^T)^{-1} \tilde{c}(\xi) = 0 \end{aligned} \tag{5.30}$$

We utilize the method proposed in [151] (see Appendix C) for solving (5.30). First, we transform (5.30) in the form of (C.1). For that purpose, we define

$$\phi(\xi) = \tilde{a}(\xi) - \tilde{E}_d (\tilde{F}_d \tilde{F}_d^T)^{-1} \tilde{F}_d^T \tilde{c}(\xi) = \begin{bmatrix} -\xi_1 - \xi_1^3 \\ -\xi_2 - \xi_2{}^3 \end{bmatrix}$$

and

$$\mathcal{D} = \frac{1}{2}\tilde{E}_d[I - \tilde{F}_d^T(\tilde{F}_d\tilde{F}_d^T)^{-1}\tilde{F}_d]\tilde{E}_d^T = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathcal{D}^+ = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

then from (C.2),

$$\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} - \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} - 2(\xi_1 - \xi_2)^T(\xi_1 - \xi_2) = 0$$

or

$$\psi_1^2 - (\xi_1 - \xi_1^3)^2 - (\xi_1 - \xi_2)^2 = 0$$

which gives

$$\psi_1 = \sqrt{(\xi_1 - \xi_1^3)^2 + (\xi_1 - \xi_2)^2}$$

and from (C.3)

$$\begin{aligned} P_\xi(\xi) &= -\begin{bmatrix} (\phi_1 - \psi_1) & (\phi_2 - \psi_2) \end{bmatrix}\mathcal{D}^+ \\ &= -2\begin{bmatrix} (\phi_1 - \psi_1) & 0 \end{bmatrix} \end{aligned}$$

or

$$P_\xi(\xi) = \begin{bmatrix} 2\xi_1 + 2\xi_1^3 + 2\sqrt{(\xi_1 - \xi_2)^2 + (\xi_1 + \xi_1^3)^2} & 0 \end{bmatrix}$$

Further, we can show that $P_\xi(\xi)$ satisfies the symmetry conditions (C.4). Then the co-outer $\Sigma_{\mathscr{D}_{co}}$ is obtained from (5.4) as;

$$\Sigma_{\mathscr{D}_{co}} : \begin{cases} \dot{\xi} = \begin{bmatrix} -\xi_1 - \xi_1^3 \\ -\xi_2 - \xi_2^3 \end{bmatrix} + \bar{B}(\xi)\tilde{r} \\ r = \xi_1 - \xi_2 + \tilde{r} \end{cases}$$

Where $\bar{B}(\xi)$ is as defined in (5.5). The inverse of the co-outer system is the optimal post-filter and is given as

$$\begin{aligned} \Sigma_{\mathscr{R}_{opt}} &= \Sigma_{\mathscr{D}_{co}^{-1}} \\ &: \begin{cases} \dot{\xi} = \begin{bmatrix} -\xi_1 - \xi_1^3 \\ -\xi_2 - \xi_2^3 \end{bmatrix} - \bar{B}(\xi)(\xi_1 - \xi_2) + \bar{B}(\xi)r \\ \tilde{r} = -\xi_1 + \xi_2 + r \end{cases} \end{aligned} \tag{5.31}$$

The post filter (5.31) is used to generate the modified residual signal for fault detection. The example is simulated in Simulink R for $\delta_d \leq 0.2$ and a unit step sensor fault occurring at $t = 5s$. The simulation results are shown in Figure 5.2. It should be noted that $\mathcal{L}_2$ norm of a signal can not be implemented in reality, and a practical solution is to approximate it with RMS value of the signal over a finite length moving window. The length of window can be arbitrarily selected, a large window gives better approximation of $\mathcal{L}_2$ norm but

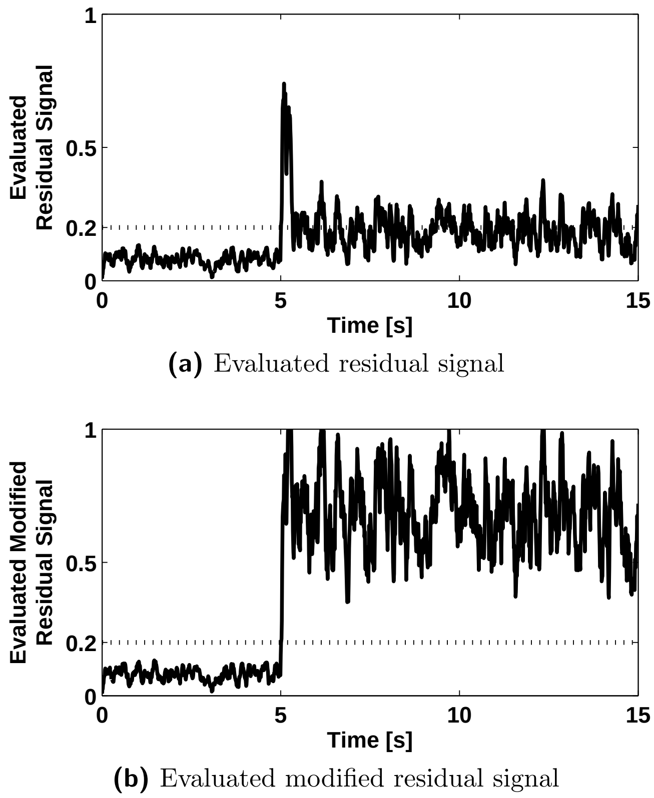
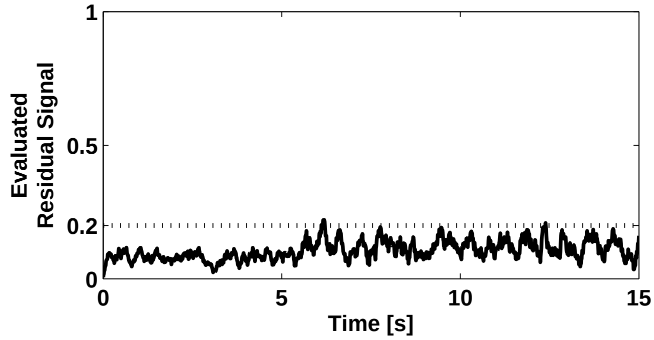
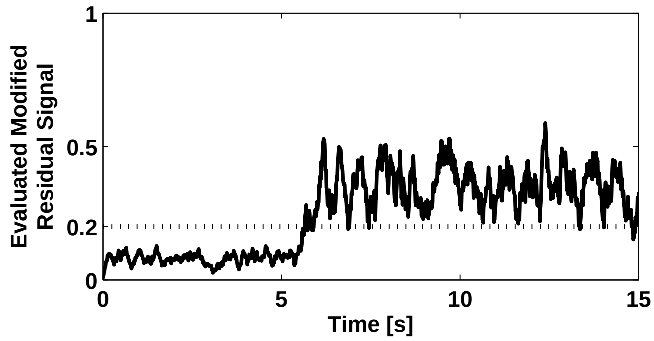


Figure 5.2: Superiority of using the proposed optimal trade-off design method. (a) evaluated residual signal r, (b) evaluated modified residual signal $\tilde{r}$ (after using the proposed post-filter) and the corresponding threshold for a sensor fault occurring at $t = 5$ seconds.



(a) Evaluated intermediate signal



(b) Evaluated residual signal

Figure 5.3: Superiority of using the proposed optimal trade-off design method. (a) evaluated residual signal r, (b) the evaluated modified residual signal $\tilde{r}$ (after using the proposed post-filter) and the corresponding threshold for an actuator fault occurring at $t = 5$ seconds.

with a delayed detection of fault. In simulation, we selected a window length equal to 0.15 seconds. Threshold can be selected according to (5.25) which results in $J_{th} = 0.2$ for a zero FAR. To demonstrate the effectiveness of the proposed method, the evaluated residual signal r (i.e. without using the post-filter) is compared with evaluated modified residual signal obtained after applying the post-filter. It becomes obvious that after using the post-filter, the effect of faults is considerably increased while the disturbances are attenuated. Thus increasing the fault detectability with zero false alarm rate.

The example is also simulated for a unit step actuator fault occurring at $t = 5$ seconds. The simulation results are shown in Figure 5.3. It is clear that if threshold is set according to zero FAR, the fault can not be detected if the post-filter is not used (see Figure 5.3a). However, after applying the proposed post-filter, fault is quickly detected (see Figure 5.3b).

Post-filter and threshold to solve PMinSDFA

The optimal post-filter which solves the PMinSDFA problem is given by the inverse of co-outer part of $\Sigma_{\mathscr{F}}$. For the example system,

$$\Sigma_{\mathscr{F}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_f f \\ r = \tilde{c}(\tilde{x}) + \tilde{F}_f f \end{cases} \tag{5.32}$$

with

$$\tilde{a}(\tilde{x}) = \begin{bmatrix} -x - x^3 \\ -\hat{x} - \hat{x}^3 + L(x - \hat{x}) \end{bmatrix}, \; \tilde{E}_f = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix}, \; \tilde{c}(\tilde{x}) = x - \hat{x}, \; \tilde{F}_f = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

We can proceed in similar steps as for finding the co-outer of $\Sigma_{\mathscr{D}}$ to obtain the co-outer for $\Sigma_{\mathscr{F}}$. Then the optimal post-filter will be $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1}$. For the example system, $\Sigma_{\mathscr{F}}$ and $\Sigma_{\mathscr{D}}$ are identical, therefore, we will have the same post-filter as was designed for the PMax-SDF problem. And the threshold will be given as:

$$J_{th} = \frac{\delta_{f,min}}{FDR}$$

If we are interested in detecting the faults with magnitude greater than 0.2 and the desired FDR is, for example, 80%, then the threshold will be 0.25.

5.7 Summary

In this chapter, a method to design post-filter and to select threshold for achieving an optimal trade-off between fault detectability and number of false alarms was proposed. The trade-off design was formulated as two optimization problems. These include the maximization of fault detectability under an allowed false alarm rate and the minimization of false alarm number under a required fault detection rate. To solve these optimization problems, threshold was selected according to either the allowed FAR or the required FDR, then utilizing the co-inner-outer factorization, the post-filter was obtained either to maximize fault detectability or to minimize number of false alarms. It comes out that the optimal post-filter is the inverse of the co-outer part of fault-free residual generator dynamics for the former case and the inverse of co-outer part of disturbance-free residual generator dynamics for the later case. It was proved that the designed post-filters not only achieve optimization in the sense of FAR and FDR but also result in the so called $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective optimization. The proposed method was illustrated by an example.

Chapter 6

Application to the three tank benchmark

The three tank system DTS200 is an experimental setup with typical characteristics of chemical processes, such as tanks, pipes, pumps and has been adopted as a standard benchmark for fault detection and isolation [154]. The mathematical model of the three tanks system has high nonlinearities and is, therefore, suitable to evaluate nonlinear fault detection techniques. This chapter implements the optimal trade-off design using post-filter technique proposed in Chapter 5 to the nonlinear model of the three tank benchmark.

For the experimental three tank benchmark system, all the states are measurable; consequently, a nonlinear observer can be designed to deliver linear estimation error dynamics over the whole operating region [155]. This is, off course, a very appropriate method for fault detection of three tank system. Alongside, the three tank system has high nonlinearities; therefore, it serves as an excellent benchmark to test the nonlinear FDI algorithms. Several FDI algorithms have been implemented on both linearized and nonlinear model of the three tank system. For example, Koenig et al. [156] designed a linear observer to detect and isolate component and actuator faults around and operating point, $\mathcal{H}_\infty$ based FDI for linearized model was presented in [157], multi-objective approach for linearized model was discussed in [158], sliding mode observer for fault detection was designed in [159], model interval approach was implemented in [160], parameter identification approach was applied in [161, 162], observer-based schemes in [22, 155, 159, 163–165], structured augmented state models in [29], FDI over networks in [166]. Noura et al. [167] applied the unknown input observer-based FDI algorithms to the three tank system, utilizing both the model linearized at an operating point and model linearized using input-output linearization method. A comparison of different fault detection techniques for linearized model of three tank system is presented in [168]. This chapter realizes the approach presented in Chapter 5 to the nonlinear model of the three tank system. Simulation results show the successful detection of both abrupt and incipient faults in sensors, actuators and components of the system.

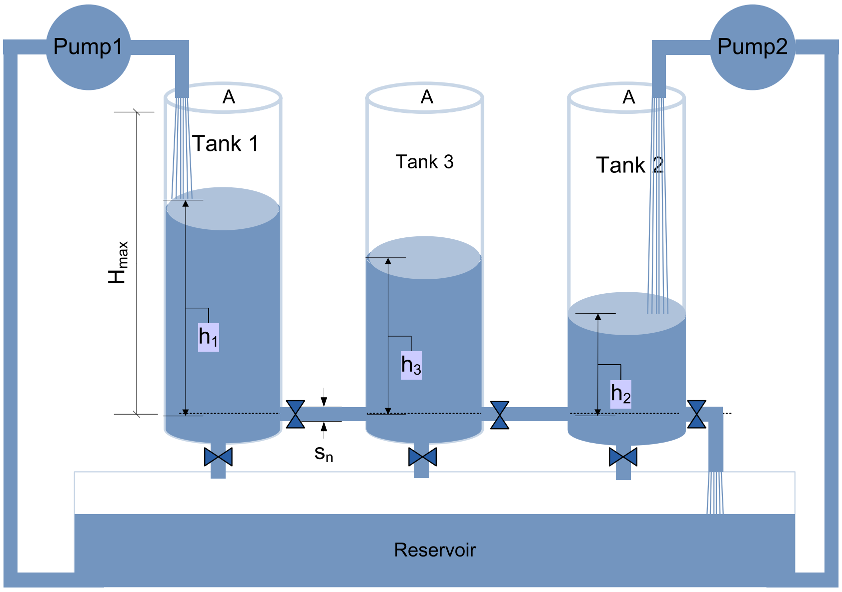


Figure 6.1: schematic diagram of the three tank system

6.1 Description of the three tank system

The schematic diagram for the three tank experimental setup DTS200 is shown in Figure 6.1. It consists of three water tanks, each of cross sectional area A_c and height H_{max}. The central tank T3 is connected to tanks T1 and T2 through pipes of cross sectional areas s_{13}, s_{23} respectively. Water can flow out from T2 to a central reservoir through a pipe of cross-section area s_0. There are two pumps with flow rates Q_1 and Q_2 which pump the water from the central reservoir to the tanks T1 and T2, respectively. The flow rates Q_1 and Q_2 can be varied through a computer interface, the maximum being equal to $Q_{1_{max}}$ and $Q_{2_{max}}$. The water levels in the tanks are measured through sensors, which could be read by an interfaced computer. There are additional valves at the bottom of each tank to simulate the leakage in the tanks [169]. Different kinds of faults can be simulated, these include:

- **Actuator faults:** Actuator faults can be simulated in three tank system by changing the flow rates of the two pumps. This can be done by using potentiometers which reduce the control signals to the pumps. These fault are represented as f_{a_1} and f_{a_2}.

- **Component faults:** The component faults in the three tank system can be simulated by leakage in the tanks and are represented as f_{l_1}, f_{l_2} and f_{l_3}. In DTS200, there are extra valves at the bottom of each tank to mimic the leakages.

- **Sensor faults:** In each tank of DTS200, there are sensors to measure the water levels in the tanks. Faults in the sensors are represented by f_{s_1}, f_{s_2} and f_{s_3}. In DTS200, sensor faults can be introduced by potentiometers which scale the measurements from sensors.

Due to water inflow from pumps to the tanks, the water levels in the tanks fluctuate and cannot be accurately measured. This is the source of sensor noise. The quantization errors and uncertainties in the parameters are treated as deterministic disturbances.

Table 6.1: Parameters of the three tank system [13]

Parameters	Symbol	Value	Unit
cross section area of tanks	A_c	154	cm^2
cross section area of pipes	s_{13}, s_{23}, s_0	0.5	cm^2
max. height of tanks	H_{max}	62	cm^2
max. flow rate of pump 1	$Q_{1_{max}}$	100	cm^3/sec
max. flow rate of pump 2	$Q_{2_{max}}$	100	cm^3/sec
coeff. of flow for pipe 1	a_1	0.46	
coeff. of flow for pipe 2	a_2	0.60	
coeff. of flow for pipe 3	a_3	0.45	
gravitational constant	g	980	cm/sec^2

6.2 Modeling of the three tank system

Using the incoming and outgoing mass flows under consideration of Torricelli's Law [170], the dynamics of the three tank system is modeled by [13, 169]:

$$
\begin{aligned}
A_c\dot{h}_1 &= -a_1 s_{13}\mathrm{sgn}(h_1 - h_3)\sqrt{2g|h_1 - h_3|} + Q_1 + f_{a_1} + f_{l_1} + d_1 \\
A_c\dot{h}_2 &= a_3 s_{23}\mathrm{sgn}(h_3 - h_2)\sqrt{2g|h_3 - h_2|} - a_2 s_0\sqrt{2gh_2} + Q_2 + f_{a_2} + f_{l_2} + d_2 \\
A_c\dot{h}_3 &= a_1 s_{13}\mathrm{sgn}(h_1 - h_3)\sqrt{2g|h_1 - h_3|} - a_3 s_{23}\mathrm{sgn}(h_3 - h_2)\sqrt{2g|h_3 - h_2|} + f_{l_3} + d_3 \\
y_1 &= h_1 + f_{s_1} + \eta_1 \\
y_2 &= h_2 + f_{s_2} + \eta_2 \\
y_3 &= h_3 + f_{s_3} + \eta_3
\end{aligned}
$$

where A_c is the cross-sectional area of the tanks, s_{13}, s_{23} and s_0 are respectively the cross-sectional areas of the pipe connecting tanks T1 and T3, the pipe connecting tanks T2 and T3 and the outlet pipe in tank T2. a_1, a_3 and a_2 are the corresponding coefficients of flow. These co-efficients depend, among the other things, on the viscosity of the liquid (water in this study), ambient temperature and friction of the pipes. Table 6.2 shows the estimated values of these coefficients. g is the gravitational constant, sgn represents the signum function, which is defined as

$$
\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}
$$

Defining

$$
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},
$$

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix} = \begin{pmatrix} f_{a_1} + f_{l_1} \\ f_{a_2} + f_{l_2} \\ f_{l_3} \\ f_{s_1} \\ f_{s_2} \\ f_{s_3} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

we have the following state space model for the three tank system

$$\begin{aligned} \dot{x} &= a(x) + Bu + E_f f + E_d d \\ y &= Cx + F_f f + F_d d \end{aligned} \tag{6.1}$$

where

$$a(x) = \begin{pmatrix} -\frac{a_1 s_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(x_1 - x_3)\sqrt{|x_1 - x_3|} \\ \frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(x_3 - x_2)\sqrt{|x_3 - x_2|} - \frac{a_2 s_0\sqrt{2g}}{A_c}\sqrt{x_2} \\ \frac{a_1 S_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(x_1 - x_3)\sqrt{|x_1 - x_3|} - \frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(x_3 - x_2)\sqrt{|x_3 - x_2|} \end{pmatrix}$$

$$B = \frac{1}{A_c}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = I_3$$

$$E_f = E_d = \begin{pmatrix} I_3 & O_3 \end{pmatrix}, \quad F_f = F_d = \begin{pmatrix} O_3 & I_3 \end{pmatrix}$$

where I_3 and O_3 respectively represent an identity matrix and a zero matrix of dimensions 3×3. Based on the above mathematical model, the next section deals with the optimal trade-off design for the three tank system.

6.3 Optimal trade-off design for the three tank system

In Chapter 5, two optimal trade-off design problems were formulated, these include the maximization of fault detectability for an allowed FAR (PMax-SDF) and minimization of false alarm numbers for a required FDR (PMin-SDFA). Solution to these problems were provided in terms of designing post-filters and selecting thresholds. The effectiveness of the proposed trade-off design problems was illustrated by an academic example. In this section, we apply these result to the fault detection of nonlinear model of the three tank system.

For the purpose of residual generation, the following fault detection filter is applied

$$\begin{aligned} \dot{\hat{x}} &= a(\hat{x}) + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \\ r &= y - \hat{y} \end{aligned}$$

It was proved in the last chapter that any stable filter gain L can be selected and the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization is achieved by the post-filter. Thus, we select arbitrarily, the following constant filter gain

$$L = \begin{pmatrix} 0.0833 & 0.0250 & 0.0833 \\ 0.0417 & 0.1667 & 0.0833 \\ 0.0833 & 0.0833 & 0.1667 \end{pmatrix} \tag{6.2}$$

The augmented system, as described in (5.6) becomes

$$\Sigma_{\mathscr{A}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{B}u + \tilde{E}_f f + \tilde{E}_d d \\ r = \tilde{c}(\tilde{x}) + F_f f + F_d d \end{cases} \tag{6.3}$$

where $\tilde{x}, \tilde{a}(\tilde{x}), \tilde{B}, \tilde{E}_f, \tilde{E}_d, \tilde{c}(\tilde{x})$ are as defined in Section 5.2. In next two subsections, the solution to the two optimal trade-off design problems for the three tank system are provided.

6.3.1 Solving PMax-SDF for three tank system

Recall from the theory of Chapter 5 that the optimal post filter that solves PMax-SDF is the inverse of the co-outer part of $\Sigma_{\mathscr{D}}$, i.e., $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$, where $\Sigma_{\mathscr{D}}$ is obtained by setting $f = 0, u = 0$ in (6.3) and is given by

$$\Sigma_{\mathscr{D}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_d d \\ r = \tilde{c}(\tilde{x}) + F_d d \end{cases} \tag{6.4}$$

To find the co-outer for (6.4), following HJ equation needs to be solved

$$P_\xi(\xi)\left[\tilde{a}(\xi) - \tilde{E}_d F_d^T (F_d F_d^T)^{-1} \tilde{c}(\xi)\right] + \frac{1}{4} P_\xi(\xi) \tilde{E}_d \left[I - F_d^T (F_d F_d^T)^{-1} F_d\right] \tilde{E}_d^T P_\xi^T(\xi) - \tilde{c}^T(\xi)(F_d F_d^T)^{-1} \tilde{c}(\xi) = 0 \tag{6.5}$$

Next, the solution of the above HJ equation is described.

Solving the HJE for three tank system

To solve the HJ equation (6.5), we use the method proposed in [151, 171] (see Appendix C.1). To apply the method, (6.5) needs to be brought in the form of (C.1). For that purpose, define

$$\begin{aligned} \phi(\xi) &= \tilde{a}(\xi) - \tilde{E}_d F_d^T (F_d F_d^T)^{-1} \tilde{c}(\xi) \\ h(\xi) &= \tilde{c}(\xi) \\ \mathcal{D} &= \frac{1}{2} \tilde{E}_d \left(I - F_d^T (F_d F_d^T)^{-1} F_d\right) \tilde{E}_d^T \end{aligned}$$

then (6.5) becomes

$$P_\xi(\xi)\phi(\xi) + \frac{1}{2} P_\xi(\xi) \mathcal{D} P_\xi^T(\xi) - h^T(\xi)(dd^T)^{-1} h(\xi) = 0$$

which is similar to (C.1). Further calculations for the solution of HJE are as follows

$$F_d F_d^T = (F_d F_d^T)^{-1} = I_{3\times 3}$$

$$\begin{aligned}
\tilde{E}_d\tilde{E}_d^T &= \begin{pmatrix} I_3 & O_3 \\ O_3 & LL^T \end{pmatrix} \\
\tilde{E}_d F_d^T (F_d F_d^T)^{-1} F_d \tilde{E}_d^T &= \begin{pmatrix} I_3 & O_3 \\ O_3 & L \end{pmatrix}\begin{pmatrix} O_3 \\ I_3 \end{pmatrix} I_3 \begin{pmatrix} O_3 & I_3 \end{pmatrix}\begin{pmatrix} I_3 & O_3 \\ O_3 & L^T \end{pmatrix} \\
&= \begin{pmatrix} O_3 & O_3 \\ O_3 & LL^T \end{pmatrix} \\
\mathcal{D} &= \frac{1}{2}\left(\tilde{E}_d\tilde{E}_d^T - \tilde{E}_d F_d^T (F_d F_d^T)^{-1} F_d \tilde{E}_d^T\right) = \frac{1}{2}\begin{pmatrix} I_3 & O_3 \\ O_3 & O_3 \end{pmatrix} \\
\mathcal{D}^+ &= 2\begin{pmatrix} I_3 & O_3 \\ O_3 & O_3 \end{pmatrix}
\end{aligned}$$

where $\mathcal{D}^+$ is the generalized inverse of $\mathcal{D}$. Further,

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} = \tilde{a}(\xi) - \tilde{E}_d F_d^T (F_d F_d^T)^{-1}\tilde{c}(\xi)$$

$$= \begin{pmatrix}
-\frac{a_1 s_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_1-\xi_3)\sqrt{|\xi_1-\xi_3|} \\
\frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_3-\xi_2)\sqrt{|\xi_3-\xi_2|} - \frac{a_2 s_0\sqrt{2g}}{A_c}\sqrt{\xi_2} \\
\frac{a_1 S_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_1-\xi_3)\sqrt{|\xi_1-\xi_3|} - \frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_3-\xi_2)\sqrt{|\xi_3-\xi_2|} \\
-\frac{a_1 s_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_4-\xi_6)\sqrt{|\xi_4-\xi_6|} \\
\frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_6-\xi_5)\sqrt{|\xi_6-\xi_5|} - \frac{a_2 s_0\sqrt{2g}}{A_c}\sqrt{\xi_5} \\
\frac{a_1 S_{13}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_4-\xi_6)\sqrt{|\xi_4-\xi_6|} - \frac{a_3 s_{23}\sqrt{2g}}{A_c}\mathrm{sgn}(\xi_6-\xi_5)\sqrt{|\xi_6-\xi_5|}
\end{pmatrix}$$

Then using (C.2), we have

$$\psi_1^2 + \psi_2^2 + \psi_3^2 - \phi_1^2 - \phi_2^2 - \phi_3^2 - (\xi_1-\xi_4)^2 - (\xi_2-\xi_5)^2 - (\xi_3-\xi_6)^2 = 0 \tag{6.6}$$

Also, from (C.5), we have following conditions

$$\begin{aligned}
\phi_{1,\xi_2} \pm \psi_{1,\xi_2} &= \phi_{2,\xi_1} \pm \psi_{2,\xi_1} \\
\phi_{1,\xi_3} \pm \psi_{1,\xi_3} &= \phi_{3,\xi_1} \pm \psi_{3,\xi_1} \\
\phi_{2,\xi_3} \pm \psi_{2,\xi_3} &= \phi_{3,\xi_2} \pm \psi_{3,\xi_2}
\end{aligned} \tag{6.7}$$

Setting

$$\begin{aligned}
\psi_1 &= \phi_1 \\
\psi_2 &= \phi_2 \\
\psi_3^2 &= \phi_3^2 + (\xi_1-\xi_4)^2 + (\xi_2-\xi_5)^2 + (\xi_3-\xi_6)^2
\end{aligned} \tag{6.8}$$

satisfies the conditions (6.6) and (6.7). Therefore, from (C.3), following solution for the HJE in (6.5) is obtained

$$\begin{aligned} P_\xi(\xi) &= -(\phi^T \pm \psi^T)\mathcal{D}^+ \\ &= 2\begin{pmatrix} -\phi_1 - \psi_1 & -\phi_2 - \psi_2 & -\phi_3 - \psi_3 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{6.9}$$

Note that for the co-outer, we do not need an expression for $P(\xi)$, rather knowledge of $P_\xi(\xi)$ is sufficient.

Optimal post filter

Once we have solved the HJE, according to (5.4), the co-outer for $\Sigma_{\mathscr{D}}$ is given by

$$\Sigma_{\mathscr{D}_{co}} : \begin{cases} \dot{\xi} = \tilde{a}(\xi) + \bar{B}(\xi)\tilde{r} \\ r = \tilde{c}(\xi) + \bar{D}\tilde{r} \end{cases} \tag{6.10}$$

where $F_d F_d^T = \bar{D}\bar{D}^T = I$ and according to (5.5)

$$P_\xi(\xi)\bar{B}(\xi) = P_\xi(\xi)\tilde{E}_d F_d^T + 2\tilde{c}^T(\xi)$$

Then the optimal post-filter is the inverse of the co-outer part of $\Sigma_{\mathscr{D}}$, i.e.,

$$\begin{aligned} \Sigma_{\mathscr{R}_{opt}} &= \Sigma_{\mathscr{D}_{co}}^{-1} \\ &= \begin{cases} \dot{\xi} = \tilde{a}(\xi) - \bar{B}(\xi)\tilde{c}(\xi) + \bar{B}(\xi)r \\ \tilde{r} = -\tilde{c}(\xi) + r \end{cases} \end{aligned} \tag{6.11}$$

Threshold selection

For a required FAR and the post-filter settings according to $\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{D}_{co}}^{-1}$, the threshold is obtained from (5.25) as

$$J_{th} = (1 - FAR)\delta_d \tag{6.12}$$

where δ_d is a bound on the $\mathcal{L}_2$ norm of disturbance, i.e., $\|d\|_2 \le \delta_d$.

6.3.2 Simulation results

To validate the results, nonlinear model of the three tank system given in (6.1) with the fault detection filter and the post filter designed in the last section is implemented in Simulink R . The simulation results for the various fault scenario is described in the following subsections. To indicate the superiority of the proposed method, we also plot the evaluated residual signal r (which is the signal obtained without using the post filter). By observing the simulation results, we can immediately notice that after using the post-filter, FDR is considerably increased while the selected threshold guarantees that the FAR is lower than the allowed one.

The simulations were performed with a constant inflow of pumps $Q_1 = 25.6$ cm^3/sec, $Q_2 = 39.5$ cm^3/sec, uniform disturbance $d \in [-0.5, 0.5]$ and a constant filter gain L given by

$$L = \begin{pmatrix} 0.0833 & 0.0250 & 0.0833 \\ 0.0417 & 0.1667 & 0.0833 \\ 0.0833 & 0.0833 & 0.1667 \end{pmatrix} \tag{6.13}$$

The $\mathcal{L}_2$ norm of the modified residual signal is approximated by its RMS value over a moving window. The length of window is selected as 10 seconds (see discussion in Example in Chapter 5 for selection of window length).

With $\delta_d = 0.5$ and zero percent FAR, the threshold comes out to be 0.5 according to expression (6.12).

Actuator fault detection

Actuator faults i.e., faults in pumps in three tank system are introduced by two additional potentiometers which reduce the control signal up to 0% of its value, i.e. the faulty flow rate of the pump becomes $Q_f = Q - \theta Q$, where Q_f is the faulty flow rate, Q is the desired flow rate and θ is the gain of the potentiometer. Simulation results for actuator faults in pumps P1 and P2 occurring at 200 seconds are respectively shown in Figures 6.2 and 6.3. With a fault in pump P1, the dynamic behavior of water levels in tanks T2 and T3 are also affected. Similarly, the fault in pump P2 also shows off in water levels of tanks T1 and T3. From figures, it can be observed that after using the post-filter, the residual signal becomes more sensitive to faults resulting in convenience of fault detection. The simulations were also performed for the case when there is an increase in the flow rate, i.e., $Q_f = Q + \theta Q$. In that case the fault was also successfully detected (simulation results for this case are not shown here). It can be observed from the figures that it is easier to detect a fault of same magnitude in P1 as compared to fault in P2. The reason is the difference in co-efficient of outflows of the two tanks, the same change in the pump inflows causes a different change in the levels in the tanks, and hence different fault detectability.

Component fault detection

Leakages in tanks are typical source of component faults in chemical processes. To simulate the leakages in the three tank system, there are extra outlets controlled by valves at the bottom of each tank. The magnitude of the outflow from a tank due to leakage is a function of the water level in that tank. However, for the purpose of simulations, we will assume a constant leakage in tanks. The simulation results for the leakages in tanks T1, T2 and T3 are shown in Figures 6.4, 6.5 and 6.6. The effect of leakage in tanks T1 and T2 is similar to the actuator fault in pumps P1 and P2, because both the pumps fault and the leakage fault cause a drop in net inflow of water into the tanks. Furthermore, a leakage in a tank not only effects the level in corresponding tank but the water levels in other tanks are also affected. From the simulation results, it can be noticed that after using post-filter, there is considerable increase in fault detection rate. It is observed from the simulations that the modified residual signal is more sensitive to leakage in tank T1 as compared to leakages in tanks T2 and T3. The reason is that the coefficient of outflow of T1, i.e., a_1 is smaller compared to a_2 and a_3, so the same size of leakage in T1 results into a relatively large change in water levels in the tanks and hence better detectability.

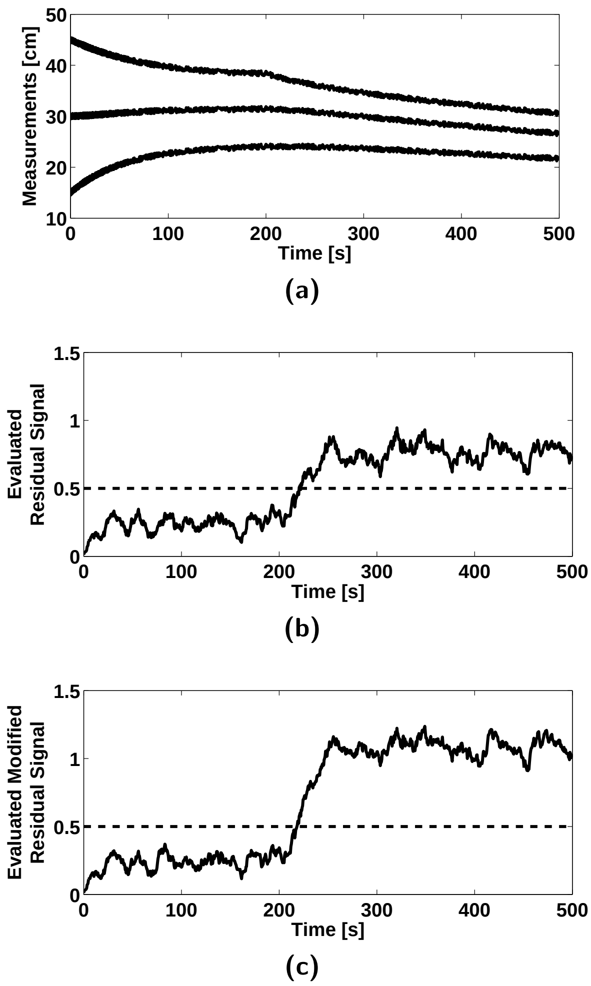


Figure 6.2: Actuator fault detection. A fault of 25% of the flow rate of pump P1 occurs at $t = 200$ seconds ($f_{a_1} = 0.25Q_1$). Figure (a) shows the measurements of water levels in the three tanks, the fault is directly reflected in water level of tank T1 and indirectly in water levels of tanks T2 and T3. Figure (b) shows the evaluated residual signal. In Figure (c) the solid line shows the evaluated modified residual signal and the dotted line is the threshold to guarantee the allowed FAR. It becomes obvious that after using the post-filter, fault detection is quite easily achieved

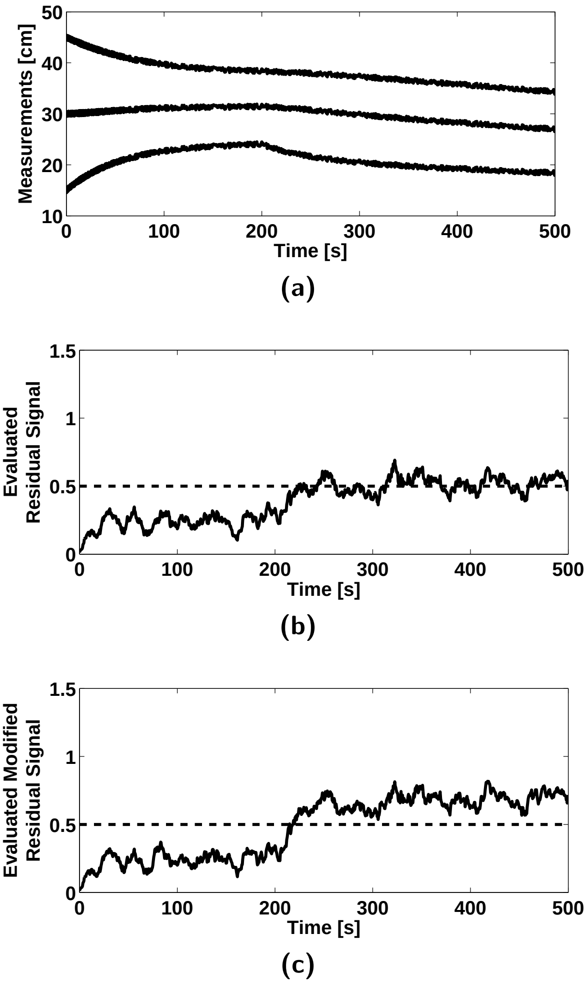
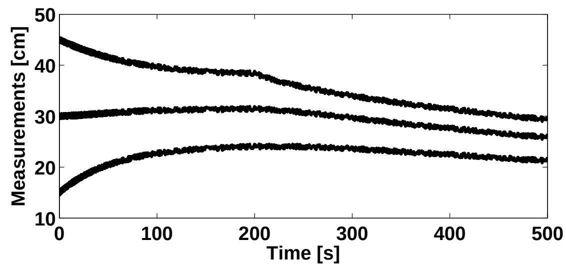
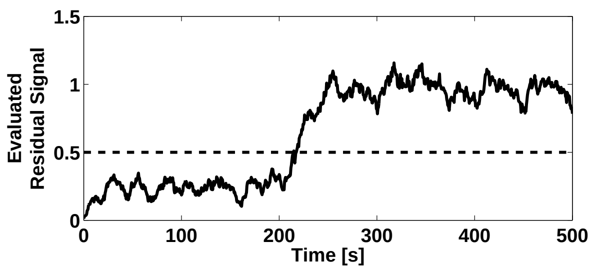


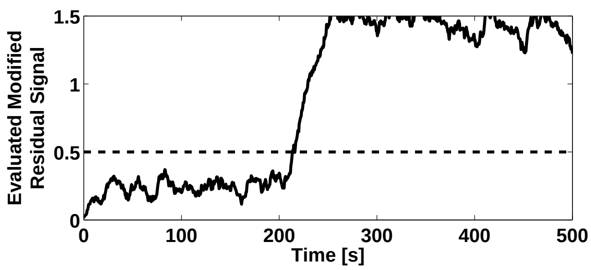
Figure 6.3: Actuator fault detection. A fault of 25% of the flow rate of pump P2 occurrs at $t = 200$ seconds ($f_{a_2} = 0.25Q_2$). Figure (a) show the measurements of water levels in the three tanks, water level in tank T2 is directly effected by the fault. Figure (b) shows the evaluated residual signal. In Figure (c), the solid line shows the evaluated modified residual signal and the dotted line shows the threshold which is selected according to the allowed FAR



(a)



(b)



(c)

Figure 6.4: Component fault detection. A leakage in tank T1, simulation is for a constant leakage of magnitude $f_{l_1} = -0.06$ cm^3/sec. Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal, Figure (c) shows the evaluated modified residual signal and the dotted lines shown the threshold to guarantee the FAR lower than the allowed one.

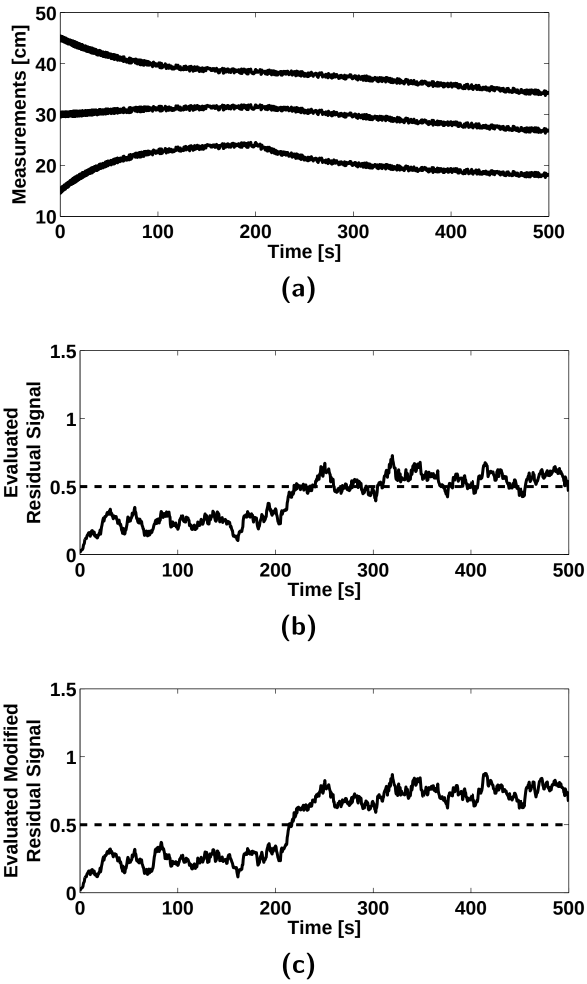


Figure 6.5: Component fault detection. A leakage in tank T2, simulation is for a constant leakage of magnitude $f_{l_2} = -0.06$ cm^3/sec. Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

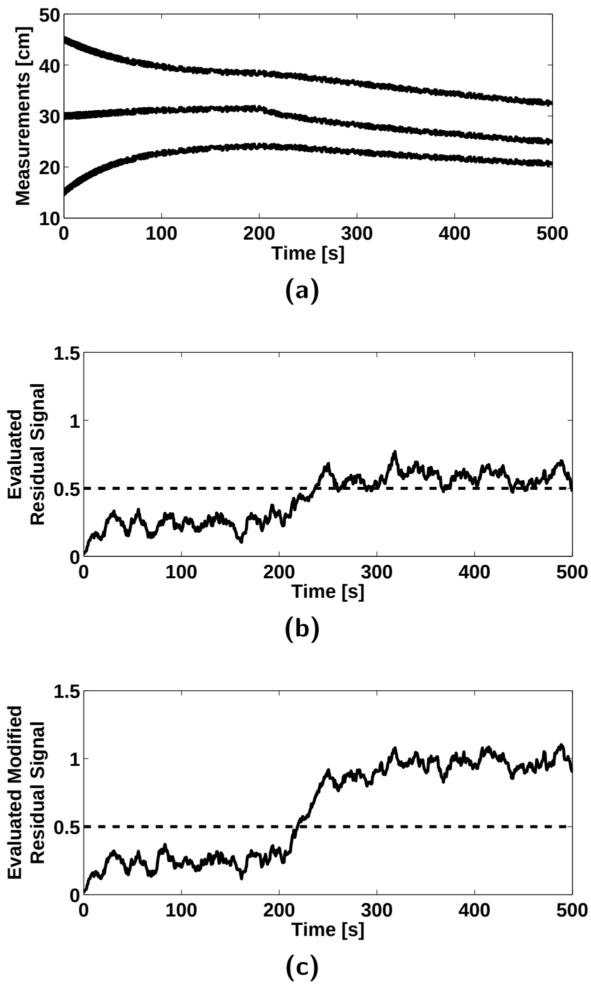


Figure 6.6: Component fault detection. A leakage in tank T3, simulation is for a constant leakage of magnitude $f_{l_3} = -0.06$ cm^3/sec. Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

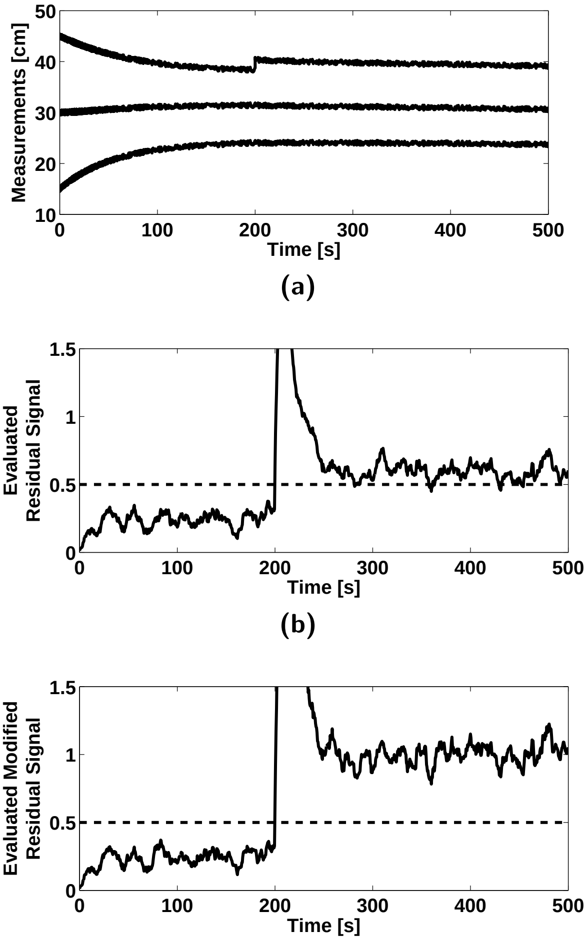


Figure 6.7: Sensor fault detection. A drift of 1 cm occurs in the sensor measuring the water level in tank T1 at $t = 200$ seconds ($f_{s_1} = 1$ cm). Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

Sensor fault detection

There are sensors in each tank of the three tank system to measure water levels in the tanks. Figures 6.7, 6.8 and 6.9 show the simulation results for a 1 centimeter drift occurring at 200 seconds in the sensors. From simulation results, we can observe that after using the post filter, the fault detection rate is considerably increased.

Incipient fault detection

Most of the faults in real systems develop slowly by the time with the aging of components, sensors or actuators. Such slowly growing faults are called the incipient faults. These faults are usually less severe but are more difficult to detect. Figure 6.10 shows the simulation results for an incipient fault in sensor measuring the water level in tank T1. The fault develops slowly with a slope 0.01cm/sec and saturates at a value of 2 cm. In Figure 6.10 only the results for sensor fault in tank T1 are shown, other two sensor faults have the similar behavior and can be successfully detected.

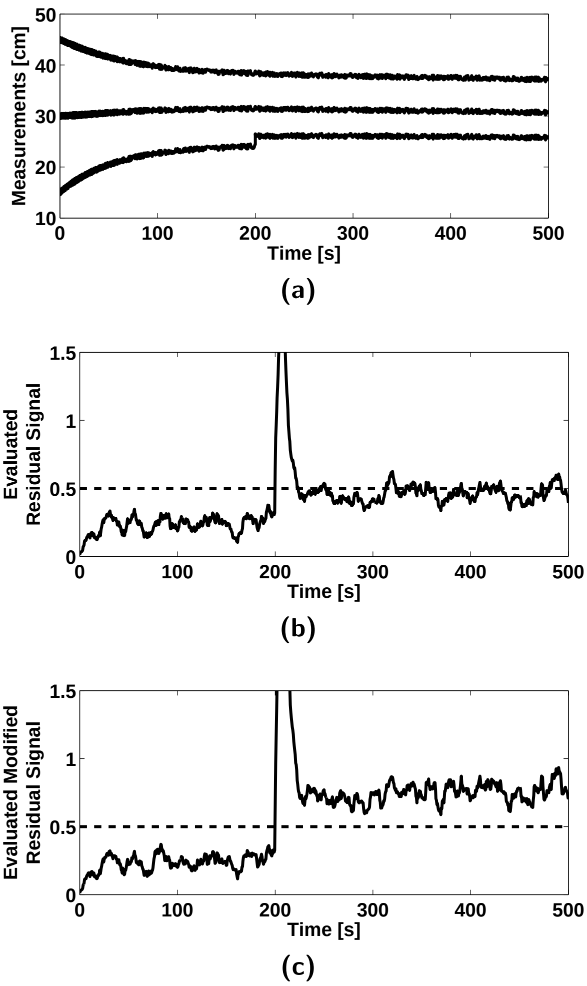


Figure 6.8: Sensor fault detection. A drift of 1 cm occurs in the sensor measuring the water level in tank T2 at $t = 200$ seconds ($f_{s_2} = 1$ cm). Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

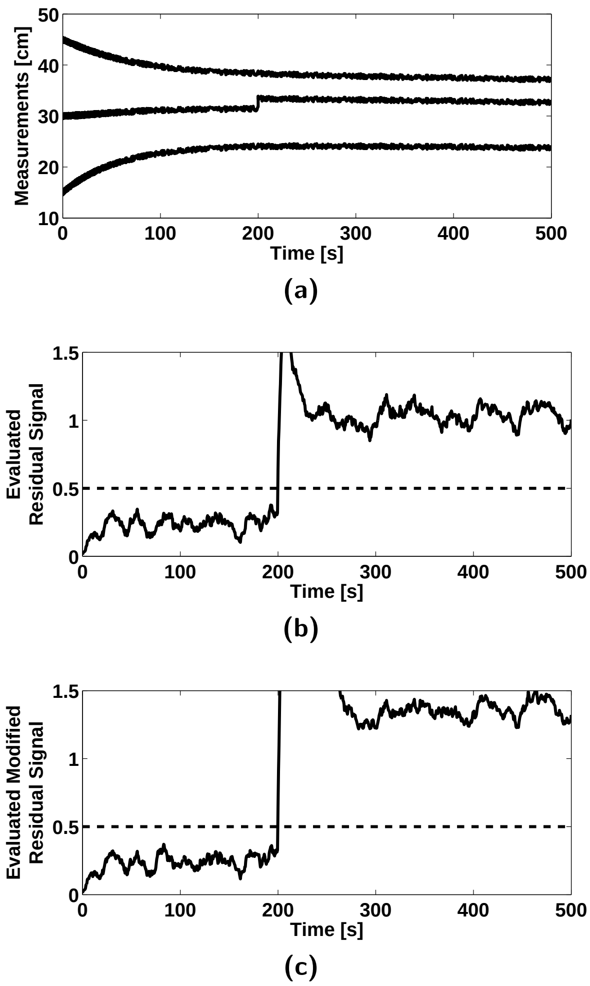


Figure 6.9: Sensor fault detection. A drift of 1 cm occurs in the sensor measuring the water level in tank T3 at $t = 200$ seconds ($f_{s_3} = 1$ cm). Figure (a) shows the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

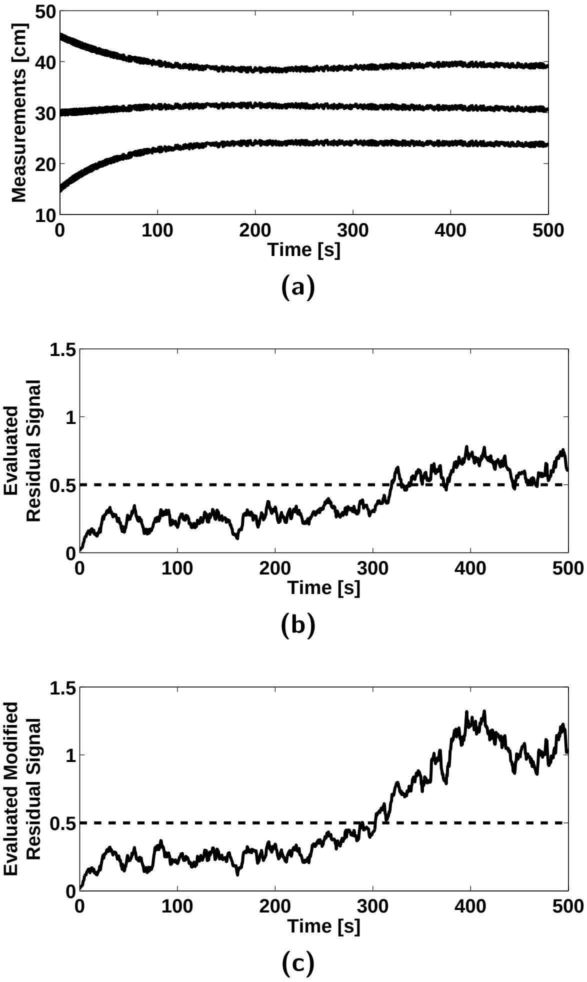


Figure 6.10: Incipient sensor fault detection. An incipient fault with slope $0.01cm/sec$ occurs in the sensor measuring the water level in tank T3 at $t = 200$ seconds. Figure (a) show the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

Figure 6.11 shows a successful detection of incipient component fault. A slowly growing leakage occurs in tank T1 at $t = 200$ seconds. Incipient leakages in tanks T2 and T3 were also simulated and the results (not shown here) show a successful detection.

Due to aging of the pumps, the flow rate of the pumps may slowly decrease. Figure 6.12 shows the successful detection of incipient fault in pump P1. The fault starts at $t = 200$ seconds with a slope of $-0.77\text{cm}^3/\text{second}$. The simulation results show the successful detection of fault. Incipient fault in pump P1 was also simulated and results show the incipient fault in pump P1 is also successfully detected.

6.3.3 Solving PMin-SDFA for three tank system

The problem of minimizing false alarms for a required FDR (PMin-SDFA) was formulated in Chapter 5 and solution was provided in terms of designing an optimal post-filter and selecting a threshold. The optimal post-filter which solves PMin-SDFA is the inverse of

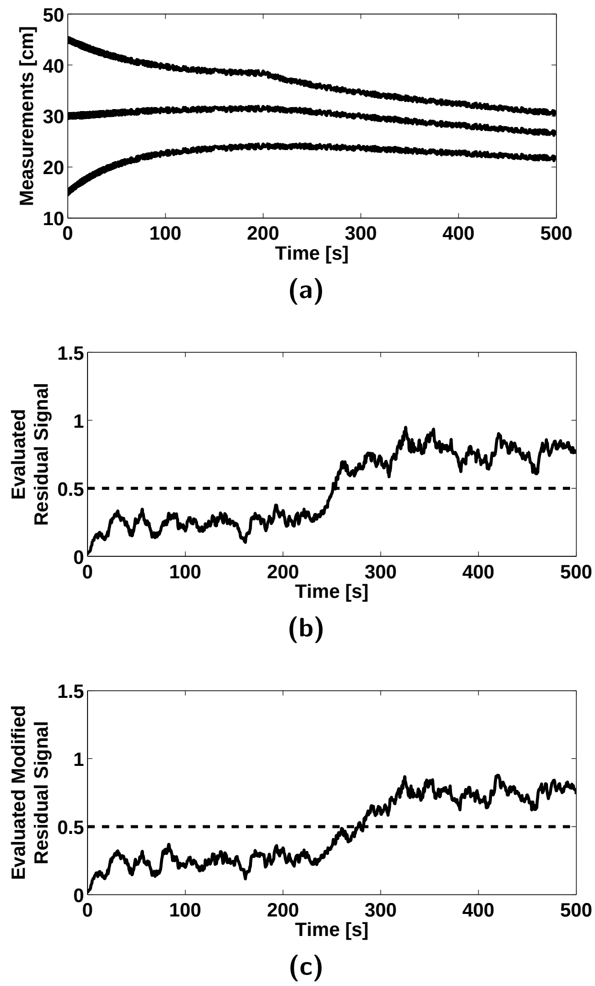


Figure 6.11: Incipient component fault detection. A slowly growing leakage (with slope -0.001 cm^3/sec) occurs at $t = 200$ seconds in tank T1. Figure (a) show the measurements of water levels in the three tanks, Figure (b) shows the evaluated residual signal and Figure (c) shows the evaluated modified residual signal

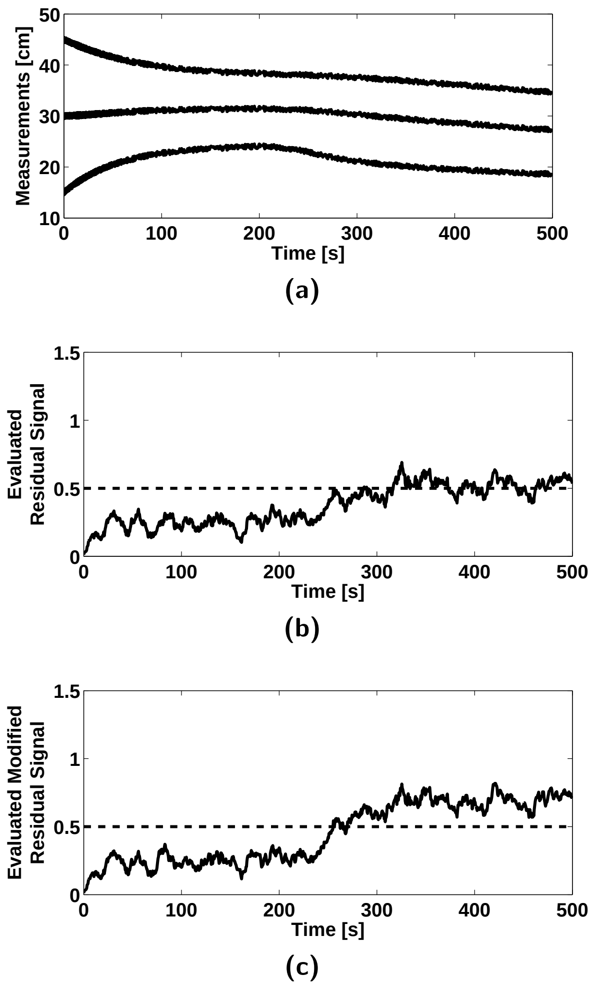


Figure 6.12: Incipient actuator fault detection. The flow from pump P2 slowly decreases from 38.5 cm^3/sec to 28.8 cm^3/sec in 50 seconds. Figure (a) show the measurements of water levels in the three tanks, (b) shows the evaluated residual signal and (c) shows the evaluated modified residual signal.

the co-outer part of $\Sigma_{\mathscr{F}}$, i.e.,

$$\Sigma_{\mathscr{R}_{opt}} = \Sigma_{\mathscr{F}_{co}}^{-1} \tag{6.14}$$

where $\Sigma_{\mathscr{F}}$ is obtained by setting $d = 0$, $u = 0$ in (6.3)

$$\Sigma_{\mathscr{F}} : \begin{cases} \dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{E}_f f \\ r = \tilde{c}(\tilde{x}) + F_f f \end{cases} \tag{6.15}$$

We notice that for three tank system, $\tilde{E}_f = \tilde{E}_d$ and $E_f = F_d$. Therefore the co-outer part of $\Sigma_{\mathscr{F}}$ will be the same as the co-outer for $\Sigma_{\mathscr{D}}$.

For a required FDR and post-filter setting according (6.14), the threshold should be

$$J_{th} = \frac{\delta_{f,min}}{FDR}$$

where $\delta_{f,min}$ is the minimum fault to be detected. For a 100% FDR and $\delta_{f,min} = 0.5$, the threshold will be equal to 0.5.

6.4 Summary

The mathematical model of the three tank benchmark can be obtained by applying the Torricelli's Law. The high nonlinearities in the mathematical model of the three tank system make it very suitable benchmark to test nonlinear FDI schemes. This chapter demonstrated the application of optimal trade-off design approach developed in last chapter to the three tank system. The solution to the two optimization problem PMax-SDF and PMin-SDFA results in the design of post-filter and computation of thresholds, these computations were made for the three tank system. Simulations were performed in Simulink R which showed the effectiveness of the proposed approach. Different kinds of faults which include sensor fault, actuator faults and component faults are simulated and plot of evaluated modified residual signal shows successful detection. It has been shown that both abrupt and incipient faults can be detected.

Chapter 7

Conclusions and future directions

To improve the performance of nonlinear fault detection systems, some new methods for residual generation and residual evaluation were proposed in the preceding chapters. This chapter presents some concluding remarks and indicates a few future directions for possible extension of this work.

7.1 Conclusions

The major focus of this thesis is on observer-based fault detection for nonlinear systems. The goals of this thesis were precisely stated in the first chapter. To achieve these goals, some new approaches were proposed, the performance of these approaches was proved mathematically and their effectiveness was illustrated by simulation results.

The first objective of this thesis was generation of residual signal which is sensitive to faults and robust against disturbances. To achieve this objective, game theoretic approach was utilized to propose three fault detection filters, these include the $\mathcal{H}_-$ fault sensitive FDF, the $\mathcal{H}_\infty$ disturbance attenuating FDF and the $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF. The concepts $\mathcal{H}_-$ index and $\mathcal{H}_\infty$ norm and their significance in fault detection were explained by numerical examples. The difference between $\mathcal{H}_\infty$ filtering problem and the $\mathcal{H}_\infty$ fault detection filtering problem was highlighted. These fault detection filters were obtained for both finite horizon and infinite horizon cases. Examples were provided to show that the desired objective is achieved. From the discussions presented therein, we conclude

1. There is a difference between $\mathcal{H}_\infty$ filtering problem and $\mathcal{H}_\infty$ fault detection filtering problem. This difference was highlighted by Example 3.3.

2. The $\mathcal{H}_\infty$ filtering problem for nonlinear systems has been discussed in literature, but $\mathcal{H}_\infty$ fault detection filtering problem has not received particular attention.

3. $\mathcal{H}_-$ fault sensitive FDF guarantees a level of fault sensitivity, but does not give any indication of disturbance attenuation property of the filter. It is possible that the $\mathcal{H}_-$ fault sensitive FDF may also make the residual signal sensitive to disturbances.

4. A similar conclusion can be drawn for the $\mathcal{H}_\infty$ disturbance attenuating FDF; besides attenuating the effect of disturbances on the residual signal, the effect of faults might also be reduced.

5. The multi-objective $\mathcal{H}_-/\mathcal{H}_\infty$ FDF simultaneously makes the residual sensitive to faults and robust against disturbances. It is possible that $\mathcal{H}_-/\mathcal{H}_\infty$ FDF generates a residual signal which is less sensitive to faults compared to the residual signal generated by $\mathcal{H}_-$ FDF. Similarly, $\mathcal{H}_-/\mathcal{H}_\infty$ FDF generates a residual signal which could be less robust against disturbances as compared to the residual signal generated by $\mathcal{H}_\infty$ FDF. However, the ratio of fault sensitivity and disturbances robustness is maximized by $\mathcal{H}_-/\mathcal{H}_\infty$ FDF.

After successful residual generation, residual evaluation is performed to make decision about the occurrence of faults. Residual evaluation consists of taking the norm of the residual signal, to obtain an evaluated residual signal, and comparing the evaluated residual signal with a threshold. Usually the maximum possible influence of disturbances on the evaluated residual signal is taken as a threshold, i.e., if $\mathcal{L}_2$ norm of the residual signal is used to obtain the evaluated residual signal, the maximum possible effect of disturbances on the evaluated residual signal is $\alpha\delta_d$, where α is the $\mathcal{H}_\infty$ norm of the system describing the dynamics of residual generator and δ_d is the upper bound on $\mathcal{L}_2$ norm of the disturbance. This gives a very conservative threshold and most of the faults remain undetected. To avoid this over conservativeness of the threshold, a variable threshold, which changes with uncertainties, can be selected. To design such a variable threshold was the second objective of this dissertation. This objective was achieved by determining a dynamic system which gives an upper bound on modulus of residual generator dynamics, and hence an upper bound on the modulus of the residual signal. This upper bound was utilized to compute a variable threshold. Since this threshold is generated by a dynamic system, it has the capability to fit more tightly to a fault-free residual signal. An example was provided to show that the designed threshold can successfully detect faults while avoiding false alarms. Following are the conclusions which can be drawn from the discussions

6. A variable threshold can improve the performance of fault detection system.
7. The conventional approach for generating a variable threshold is to multiply the input with some norm of uncertainties. The dynamic threshold generator possesses the properties of a dynamic system and can better fit to a fault-free residual signal, improving the performance of fault detection system.

The third objective of this thesis was to devise a strategy which achieves an optimal trade-off between high fault detection rate and low false alarm rate. Two trade-off design problems were formulated which include the minimization of number of false alarms for a required FDR and maximization of fault detectability for an allowed FAR. The solutions to these problems were achieved by proposing post-filters and thresholds. This post-filtering and threshold selection scheme not only provided the desired optimal trade-off but also solved the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization problem. An example was presented to demonstrate the effectiveness of the proposed technique. The proposed method for optimal trade-off was also applied to the nonlinear model of a three tank system. The simulation results showed that both abrupt and incipient faults in sensors, actuators and components of three tank system were successfully detected and the false alarms were reduced. Following are conclusions from the optimal trade-off design method.

8. The optimal trade-off between low FAR and high FDR is the true objective in the design of fault detection system, optimal residual generation and optimal residual evaluation are only the tools to attain the objective.

9. The optimal trade-off was achieved by proposing post-filter and threshold selection scheme.

10. The proposed post-filter also solves the $\mathcal{H}_-/\mathcal{H}_\infty$ optimization problem.

7.2 Future directions

The last section summarized the results presented in this dissertation. The proposed techniques and their applications to improve the fault detection of nonlinear systems were briefly described. Besides the admired features of the proposed methods, there is a room for further improvements. In below, we outline a few possible directions for possible extension of the work.

- Chapter 3 proposed solutions to $\mathcal{H}_-$ fault sensitive FDF, $\mathcal{H}_\infty$ disturbance attenuating FDF and $\mathcal{H}_-/\mathcal{H}_\infty$ multi-objective FDF. In the proposed methods for the design of these filters, a difficulty sometimes arises in finding a filter gain independent of system states (this was discussed in Remark 3.4). To overcome this difficulty, some special structure of filter, for example the one recently proposed in [117], can be studied as a future extension of the work to make it more easier to obtain a state independent filter gain.

  As we know from the previous discussions in Chapter 3, it is desired that the residual signal should only be sensitive to faults and robust to all other inputs (which include disturbances d and the known inputs u). Following the traditional approach, we have summed up these inputs into one vector w, i.e., $w = \begin{bmatrix} d \\ u \end{bmatrix}$ and the attenuation $\frac{\|r\|_2}{\|w\|_2} \leq \alpha$ is achieved. This strategy has a drawback: the information of the input u is available on-line and could be utilized to relax the over conservative situation which arises by using the upper bound and worst scenario of the input vector u. This issue has not been addressed for nonlinear filtering problems and is very rarely discussed in literature for linear uncertain systems. To our knowledge, only [110, 172] have proposed utilizing the information of known inputs in linear filtering problems. A possible future direction to extend our results is to include the information of known inputs in solving the fault detection filtering problems (or even the filtering problems).

- We have presented a dynamic threshold generation method for a class of nonlinear systems with Lipschitz nonlinearities. Many systems satisfy the Lipschitz condition globally, for example, the robotic systems etc. There are many other systems which can be converted into a form which is, at least, locally Lipschitz. A possible future direction is to extend the proposed approach of dynamic threshold generation to a more general class of nonlinear systems. This will have two advantages, firstly, more wider set of systems would be covered. Secondly, the computation of Lipschitz constant which involves numerical computations (see [173] for calculation of Lipschitz constant) could be avoided.

- The optimal trade-off design problem presented in this thesis involves the design of a post-filter and computation of a threshold. As demonstrated by an academic example and simulations results of three tank systems, the proposed trade-off design approach achieves overwhelming results. However, one imperfection in the proposed approach is the order of the post-filter, it comes out to be of the same dimensions as the dynamics of the augmented system, i.e, $2n$. To avoid this imperfection, two directions for future extension of the work are suggested:

- One possible future extension of the work is to obtain a reduced order post-filter. This can be probably achieved by system order reduction techniques.
- Another possible direction for future extension is to achieve the optimal trade-off by using filter gain L instead of the post-filter. For linear systems, it is done as follows: an expression for the post-filter is derived to deliver an optimal trade-off between low FAR and high FDR, analogous to the nonlinear systems, it comes out to be the inverse of co-outer part of $\Sigma_{\mathscr{D}}$ (or of $\Sigma_{\mathscr{F}}$) as defined in (5.7) (or in (5.8)). Then the filter gain is selected in such a way (see (12.65) in [13, Chapter 12]) that the co-outer of $\Sigma_{\mathscr{D}}$ (or of $\Sigma_{\mathscr{F}}$) is identity system, which means that the optimal post-filter should be identity system and hence no more required. In other words, the problem of finding optimal post-filter is transformed into computing the optimal filter gain. This has the advantage that the use of high dimensional post-filter is avoided, which means computational cost is substantially reduced. A similar procedure for nonlinear systems to avoid the use of high dimensional post-filter could be a future direction for possible extension of the results proposed in this thesis.

Appendix A

Two-players zero sum differential game

In game theory, the zero-sum game means that the sum of the cost functions of each of the players is identically zero [133]. What one player gets is at the cost of what the other player looses, and the total sum is zero. Consider a two player zero-sum differential game with cost functional $J = J(u_1, u_2)$ which is to be maximized by u_1 and minimized by u_2. If there exists a pair $(u_1^\star, u_2^\star)$ such that

$$J(u_1, u_2^\star) \le J(u_1^\star, u_2^\star) \le J(u_1^\star, u_2)$$

then the pair $(u_1^\star, u_2^\star)$ is called a saddle-point solution. In this case, $J(u_1^\star, u_2^\star)$ is called the saddle-point value of the game.

Below is a theorem which gives necessary and sufficient condition for the solution of a two person zero-sum differential game defined by the state equations

$$\dot{x} = a(t, x, u_1, u_2) \quad x(0) = x_0 \tag{A.1}$$

with the cost functional

$$J(u_1, u_2) = \int_0^T g(t, x, u_1, u_2) dt \tag{A.2}$$

Theorem A.1. *[133] For a two-person zero-sum differential game of prescribed duration [0,T], and under either MPS or CLPS information pattern, a pair of strategies* $u_i^\star \in S_i; i = 1, 2$ *provides a feedback saddle-point solution if there exists a function*

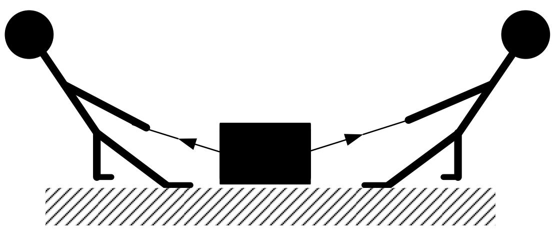


Figure A.1: Two-players zero-sum game

$V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following partial differential equation

$$
\begin{aligned}
-\frac{\partial V(x,t)}{\partial t} &= \min_{u_1 \in S_1} \max_{u_2 \in S_2} \left\{ \frac{\partial V(x,t)}{\partial x} a(t,x,u_1,u_2) + g(t,x,u_1,u_2) \right\} \\
&= \max_{u_1 \in S_1} \min_{u_2 \in S_2} \left\{ \frac{\partial V(x,t)}{\partial x} a(t,x,u_1,u_2) + g(t,x,u_1,u_2) \right\} \\
&= \left\{ \frac{\partial V(x,t)}{\partial x} a(t,x,u_1^\star,u_2^\star) + g(t,x,u_1^\star,u_2^\star) \right\}
\end{aligned}
$$

Every such saddle-point solution is strongly time consistent, and the unique saddle-point value of the game is $V(0,x(0))$.

The above partial differential equation is referred as Hamiltonian-Jacobi-Isaacs (HJI) equation or simply Isaacs equation. In Theorem A.1, two terms MPS and CLPS are used, which are explained below: In a two-player continuous time differential game of prescribed duration $[0,T]$, we say that player P_i's information set $\eta_i(t)$ is

- closed loop perfect state (CLPS) pattern if $\eta_i(t) = \{x(s), 0 \le s \le t\},\ t \in [0,T]$
- memoryless perfect state (MPS) pattern if $\eta_i(t) = \{x(0), x(t)\},\ t \in [0,T]$

If we are only interested in time-invariant solutions, then the above Isaacs equation reduces to

$$
\begin{aligned}
0 &= \min_{u_1 \in S_1} \max_{u_2 \in S_2} \left\{ \frac{\partial V(x)}{\partial x} a(x,u_1,u_2) + g(x,u_1,u_2) \right\} \\
&= \max_{u_1 \in S_1} \min_{u_2 \in S_2} \left\{ \frac{\partial V(x)}{\partial x} a(x,u_1,u_2) + g(x,u_1,u_2) \right\}
\end{aligned}
$$

Appendix B

Matrix Calculus

The results derived in Chapter 3 involved extensive use matrix calculus. Derivatives of matrix functions with respect to vectors or matrices are very puzzling. It is, therefore, useful to provide here some of the most commonly used formulas [174].

The derivative of a scalar function with respect to a column vector is taken as a row vector [1].

$$
\begin{aligned}
&\frac{d}{dx}x^T a = \frac{d}{dx}a^T x = a^T \\
&\frac{d}{dx}Ax = \frac{d}{dx}x^T A = A \\
&\frac{d}{dx}x^T x = 2x^T \\
&\frac{d}{dx}x^T Ax = x^T(A + A^T) \\
&\frac{d}{dx}(Ax + b)^T(Ax + b) = 2(Ax + b)^T A \\
&\frac{d}{dx}(Ax + b)^T(Dx + e) = (Ax + b)^T D + (Dx + e)^T A \\
&\frac{d}{dx}(Ax + b)^T C(Dx + e) = (Ax + b)^T CD + (Dx + e)^T C^T A
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dX}a^T Xb = ba^T \\
&\frac{d}{dX}a^T X^T b = ab^T \\
&\frac{d}{dX}a^T X^T Xa = 2Xaa^T \\
&\frac{d}{dX}a^T X^T Xb = X(ba^T + ab^T)
\end{aligned}
$$

[1] There is another convention according to which, the derivative of a scalar function with respect to a column vector is a column vector.

$$\frac{d}{dX}a^T X^T A X b = b a^T X^T A + a b^T X^T A^T$$
$$\frac{d}{dX}a^T X^T A X a = 2 a a^T X^T A$$
$$\frac{d}{dX}(Xa + b)^T (Xa + b) = a(Xa + b)^T (C + C^T)$$
$$\frac{d}{dX}a^T X^{-1} b = -X^{-1} b a^T X^{-1}$$

The Chain Rule: If Z is a function of Y which is itself a function of X, then

$$\frac{dZ}{dX} = \frac{dZ}{dY}\frac{dY}{dX}$$

Appendix C

Solution of Hamilton-Jacobi equations (inequalities)

In dealing with nonlinear optimal control problems, optimal filtering problems and inner-outer factorization problems, one encounters with Hamilton-Jacobi (HJ), Hamilton-Jacobi-Isaacs (HJI) and Hamilton-Jacobi-Bellman (HJB) equations. The optimal residual generation problem in Chapter 3 and the integrated design of nonlinear fault detection systems in Chapter 5 depend on the solution of partial differential equations which are similar to Hamilton-Jacobi (HJ) equations. The analytical solution of these equations (inequalities) is in general not possible, therefore there has been active research to find the approximate solutions. A detailed study of some methods can be found in [149, 151, 152, 171, 175–179] and the references therein. In this thesis, we have utilized the methods proposed in [151, 171, 176] and in [152] to solve the simulation examples and the benchmark studies. In next two sections, the computational algorithm of these two methods will be described.

C.1 Computational details of the approach proposed by Aliyu for solving HJI equations (inequalities)

Consider the partial differential equations described by

$$P_x(x)\phi(x) + \frac{1}{2}P_x(x)\mathcal{D}(x)P_x^T(x) - h^T(x)h(x) = 0, \quad P(0) = 0 \quad \forall x \in N \tag{C.1}$$

Define the matrix $\mathcal{D}^+(x)$ as the generalized inverse of $\mathcal{D}(x)$ and suppose that there exists a vector field ψ such that

$$\psi^T(x)\mathcal{D}^+(x)\psi(x) - \phi^T(x)\mathcal{D}^+(x)\phi(x) - 2h^T(x)h(x) = 0 \tag{C.2}$$

then

$$P_x(x) = -(\phi(x) \pm \psi(x))^T\mathcal{D}^+(x) \tag{C.3}$$

satisfies (C.1). If the equality in (C.2) is replaced with inequality, (C.3) satisfies the inequality (C.1). Furthermore, $P(x)$ is symmetric if

$$\left(\frac{\partial\phi(x)}{\partial x} \pm \frac{\partial\psi(x)}{\partial x}\right)^T \mathcal{D}^+(x) + \left(I_n \otimes (\phi(x) \pm \psi(x))^T\right) \frac{\partial\mathcal{D}^+(x)}{\partial x}$$
$$= \mathcal{D}^+(x) \left(\frac{\partial\phi(x)}{\partial x} \pm \frac{\partial\psi(x)}{\partial x}\right) + \frac{\partial\mathcal{D}^+(x)}{\partial x} \left(I_n \otimes (\phi(x) \pm \psi(x))\right) \quad \text{(C.4)}$$

If $\mathcal{D}^+(x)$ is a constant matrix, (C.4) reduces to

$$\left(\frac{\partial\phi(x)}{\partial x} \pm \frac{\partial\psi(x)}{\partial x}\right)^T \mathcal{D}^+ = \mathcal{D}^+ \left(\frac{\partial\phi(x)}{\partial x} \pm \frac{\partial\psi(x)}{\partial x}\right) \quad \text{(C.5)}$$

For $P(x)$ to be positive semidefinite, following inequality must hold

$$\left(\frac{\partial\phi(x)}{\partial x} \pm \frac{\partial\psi(x)}{\partial x}\right)^T \mathcal{D}^+(x) + \left(I_n \otimes (\phi(x) \pm \psi(x))^T\right) \frac{\partial\mathcal{D}^+(x)}{\partial x} \leq 0 \quad \text{(C.6)}$$

C.2 Successive approximation of solution of HJI equation proposed by Wise

Consider the partial differential equation described by (5.3) and suppose that the nonlinearities are confined to the system function $a(x)$ and is represented as

$$a(x) = Ax + \Delta a(x)$$

then (5.3) reduces to

$$P_x(x)\left(Ax + \Delta a(x) - BD^T(DD^T)^{-1}Cx\right)$$
$$+ \frac{1}{4}P_x(x)B\left(I - D^T(DD^T)^{-1}D\right)B^T P_x^T + x^T C^T (DD^T)^{-1} Cx = 0 \quad \text{(C.7)}$$

The solution of (C.7) can be approximated by the sum of solution of the linear part and the solution of the nonlinear part of (C.7), i.e.,

$$P(x) = x^T P x + \Delta P(x)$$

The linear part of (C.7) is an ARE and is given by

$$P(A - BD^T(DD^T)^{-1}C) + PB\left(I - D^T(DD^T)^{-1}D\right)B^T P + C^T(DD^T)^{-1}C = 0$$

To obtain the solution of the nonlinear part, define

$$\begin{aligned}
\tilde{A} &= A - BD^T(DD^T)^{-1}C \\
\tilde{Q} &= C^T(DD^T)^{-1}C \\
\tilde{R} &= B\left(I - D^T(DD^T)^{-1}D\right)B^T \\
\tilde{F} &= \tilde{A} + \tilde{R}P \\
q(z,p) &= -2\Delta a_z^T(z)Pz - \Delta a_z^T(z)p - 2P\Delta a(z) \\
J(t) &= \int_0^t e^{\tilde{F}\tau}(\frac{1}{2}\tilde{R})e^{\tilde{F}^T\tau}d\tau
\end{aligned}$$

where $a_z(z)$ represents the partial derivative of $a(z)$ with respect to z.

$$z(t) = e^{\tilde{F}t}z(0) + J(t)p(t) + e^{\tilde{F}t}\int_0^t e^{-\tilde{F}\tau}\left\{\Delta a(z(\tau)) - J(\tau)q(z(\tau),p(\tau))\right\}d\tau \tag{C.8}$$

$$p(t) = -e^{\tilde{F}^T\tau q(z(\tau),p(\tau))d\tau} \tag{C.9}$$

Then the successive approximation algorithm begins with setting

$$z^{(0)}(t) = e^{\tilde{F}t}z$$
$$p^{(0)}(t) = 0$$

and substituting into (C.8) to get the first approximation. The repeated approximations are obtained by plugging the latest computations of $z(t)$ and $p(t)$ into (C.8). The solution to the nonlinear part of (C.7) is then given by

$$\Delta P_x(x) = p(0) \tag{C.10}$$

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Biography

Muhammad Abid was born on 2 March 1980 in Haripur, Pakistan. He received his secondary education from Government Postgraduate College, Haripur. He was awarded with merit scholarship for B.Sc. Electrical Engineering which was completed with honors in 2002 from NWFP University of Engineering and Technology Peshawar. In 2002, he was awarded with a fellowship from Pakistan Institute of Engineering and Applied Sciences (PIEAS), Islamabad for masters in Systems Engineering. After completing his M.Sc. Systems Engineering in September 2004 from Department of Electrical Engineering PIEAS, he joined the same department as a lecturer. Since April 2007, he is a PhD candidate at Institute of Automatic Control and Complex Systems, University of Duisburg-Essen with a joint funding from Higher Education Commission (HEC) of Pakistan and German Academic Exchange Program (DAAD). His research interests include fault detection in nonlinear systems, passivity based control and robust control of uncertain systems.