Partial isomorphism

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Für meine Töchter.

In memoriam
Daniel
Rüdiger Göbel

Contents

Introduction

A classical aim in abelian group theory is the classification of groups with the help of numerical invariants. The first important result in this area is the famous theorem by Ulm [U] from 1933. It states that two countable abelian p-groups are isomorphic if and only if their numerical invariants, the so-called Ulm-Kaplansky invariants, coincide. For uncountable p-groups the theorem is false, but Hill [H] and Walker [Wal] proved that it still holds for the class of totally projective p-groups, which is the greatest natural class of abelian p-groups such that every member is completely determined by its Ulm-Kaplansky invariants. Passing to torsion-free and mixed modules it was then Warfield [War] who extended Ulm's theorem to the class of Warfield modules introducing new numerical invariants, the so-called Warfield invariants.

Taking a completely different point of view, 1954 it was Szmielew [S] who first considered abelian groups using model-theoretical techniques. She gave simple group theoretic conditions for the case that two abelian groups satisfy the same sentences of the lower predicate calculus, i.e. $L_{\omega\omega}$. But although the axioms for abelian group theory can be formulated in $L_{\omega\omega}$, the compactness theorem shows that, for example, torsion groups cannot be characterized in $L_{\omega\omega}$ and a language with infinite expressions is needed, e.g. $L_{\omega_\alpha\omega}$ for $\alpha > 0$.

In 1970 Barwise and Eklof [BE] took up Szmielew's approach and proved a model-theoretic generalization of Ulm's theorem for torsion groups of any cardinality which implies the original theorem as a corollary. Their proof makes use of a certain sequence of cardinals which is closely related to the classical Ulm-Kaplansky invariants. As a corollary they also obtain an upward Löwenheim-Skolem theorem for torsion groups.

The natural question of extending the Barwise-Eklof result to Warfield modules was considered independently first by Jacoby in her unpublished Ph.D. thesis [J1] and later, following and building on the diploma thesis of the author, by Göbel, Loth, Strüngmann and the author [GLLS]. The result of the latter was a classification theorem for $\mathbb{Z}_{(p)}$-modules with nice decomposition bases which implies the classical theorem for countable Warfield modules. Again, additional numerical invariants were defined, deduced from the original Warfield invariants.

After learning of each other's works, Jacoby, Loth, Strüngmann and the author published a joined paper [JLLS] in which they pass to L_δ $(= L^\delta_{\infty\omega})$ and prove an analogous global result for modules with partial decomposition bases.

Nearly all the proofs, both of the original classification theorems and the model-theoretic generalizations, consider sets of height-preserving (partial) isomorphisms, albeit the isomorphisms in the generalizations are sometimes only required to preserve heights up to a certain limit. These partial isomorphisms then are extended according to the well-known back-and-forth property. The isomorphism thus achieved is called partial isomorphism, which, for countable or countably generated structures, is equivalent to general isomorphism [B]. Partial isomorphism as introduced by Barwise is described as "the strongest possible absolute notion of isomorphism", and "one of which mathematicians should be aware". Karp [K] provides the linking result between abelian isomorphism and model-theoretic isomorphism (infinitary equivalence), namely that two structures for a language L are partially isomorphic if and only if they are L_∞-equivalent, therefore both serve as models for the same sentences of L_∞. This result was later refined for the case that only sentences up to a certain complexity (quantifier rank) are valid in both models. Modifications of this refined result were also used in the works of [BE], [J1] and [GLLS].

The second part of this thesis deals with the question how closely related partially isomorphic structures are in the uncountable case. With the help of L_∞-equivalence we will construct a class of partially isomorphic modules, solely characterized by their generalized Ulm-Kaplansky invariants. This construction will be realized in the universe V=L where it is classical to use the Diamond Principle and a Step-Lemma. Constructions in this fashion have been done before to realize modules with a prescribed endomorphism ring [DG] and the techniques used in this work are close to those applied there. However, in this work we take special care of the fact that at each point in the construction we have complete control over the Ulm-Kaplansky invariants of the achieved modules. Thus we are able to construct modules with prescribed sequences of Ulm-Kaplansky invariants. We then receive large classes of partially isomorphic modules which contain modules which posess a prescribed endomorphism ring, as well as modules which do not. Thus it is shown that "the strongest absolute notion of isomorphism" is not so strong as could have been hoped for in the uncountable case.

This thesis then is structured as follows:

Chapter 1 contains three sections which provide the necessary tools to investigate possible invariants of p-groups, Warfield modules and Warfield groups. The classical, abelian background is given in Section 1.1, the model-theoretic background is provided in Section 1.2. Section 1.3 gives a short introduction in the set-theoretical methods which will later be used to construct uncountable modules with prescribed properties in the universe V=L, i.e. the Diamond Principle.

Chapter 2 collects the model-theoretic generalizations of the original classification theorems for (countable) p-groups, Warfield modules and Warfield groups. Thus Section 2.1 contains the results of [GLLS], generalizing the original theorem for Warfield modules to $\mathbb{Z}_{(p)}$-modules with nice decomposition bases and Section 2.2 contains [JLLS] and receives a result for groups with partial decomposition bases.

Chapter 3 will provide a new tool to realize modules with a prescribed endomorphism ring in the universe V=L. The introduced new Step-Lemma will allow the construction of modules with additional control over the Ulm invariants of the achieved modules.

In Chapter 4 we will at last realize the desired class of partially isomorphic uncountable modules by giving a Realization Theorem and calculating the Ulm invariants of the realized modules. We receive a class of partially isomorphic modules which is (very) large and contains modules which differ in various regards and are clearly far from being isomorphic in the classical sense. Furthermore we will give an impression which sequences of Ulm-Kaplansky invariants can be realized by the achieved modules.

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Chapter 1

Preliminaries

1.1 Group- and module-theory

In our group- and module-theoretical setting we often refer to the **ring $\mathbb{Z}_{(p)}$ of integers localized at** p, for a prime p. Recall that $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : \mathrm{gcd(n, p) = 1}\}$. $\mathbb{Z}_p$ denotes the cyclic group of order p.
The classical theorem by Ulm [U] characterizes countable abelian p-groups, that is groups in which all elements have order a power of p, with the help of numerical invariants, the so-called Ulm-Kaplansky invariants. These make use of the subgroups (or $\mathbb{Z}_{(p)}$-submodules) $p^\alpha G$ of G, where $pG = \{px : x \in G\}, p^{\alpha+1}G = p(p^\alpha G)$ and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$, if α a limit ordinal. $p^\infty G = \bigcap_\alpha p^\alpha G$ is the maximal divisible subgroup (or $\mathbb{Z}_{(p)}$-submodule) of G, sometimes denoted by G_d. We then write $G = G_d \oplus G_r$, where G_r is **reduced** ($p^\infty G_r = 0$). G_d is also equal to $p^{l(G)}G$, where $l(G)$ is the **length** of G, the smallest ordinal τ such that $p^{\tau+1}G = p^\tau G$. Since $G = G_d \oplus G_r$, we are often able to restrict our considerations by regarding a reduced group $G = G_r$. The **(p-)height** of an element $x \in G$, $|x|$, is then defined by $|x| = \alpha$ if $x \in p^\alpha G \setminus p^{\alpha+1}G$ and $|x| = \infty$ if $x \in p^\infty G$. If we need to distinguish in which (sub-)group or (sub-)module the height is computed, we will indicate the relevant group or module as in $|x|_G$. Consindering the regulations $\infty > \infty > \alpha$ for each ordinal α and $|0| = \infty$, we always have

$$|x + y| = \min\{|x|, |y|\}, \text{ if } |x| \neq |y| \text{ and}$$

$$|x + y| \geq |x|, \text{ if } |x| = |y|.$$

Notice $|nx| \geq |x|$ for all $n \in \mathbb{Z}$, especially $|px| > |x|$.

We will make use of the fact that every abelian p-group is canonically interpretable as a $\mathbb{Z}_{(p)}$-module. Therefore, we will mostly consider $\mathbb{Z}_{(p)}$-modules or groups which are either abelian p-groups or, more generally, mixed groups induced by the additive structure of a $\mathbb{Z}_{(p)}$-module. Abelian groups of this type are called **p-local**.

To define our desired invariants we now consider the **(p-)socle** $p^\alpha G[p]$ of $p^\alpha G$, which contains all elements $x \in p^\alpha G$ with $px = 0$.

Definition 1.1.1. *For a group (or module) G, α an ordinal and p a prime consider the $\mathbb{Z}_p$-vector space $p^\alpha G[p]/p^{\alpha+1}G[p]$ and define*

$$u_p(\alpha, G) = \dim(p^\alpha G[p]/p^{\alpha+1}G[p])$$

the α-th **Ulm-Kaplansky invariant** *of G (or* **Ulm invariant** *for short) and*

$$u_p(\infty, G) = \dim(p^\infty G[p]).$$

Example 1.1.2. *Let $G = \mathbb{Z}_p^{(\alpha_1)} \oplus \mathbb{Z}_{p^2}^{(\alpha_2)} \oplus \mathbb{Z}_{p^3}^{(\alpha_3)}$. Then $pG \cong p\mathbb{Z}_{p^2}^{(\alpha_2)} \oplus p\mathbb{Z}_{p^3}^{(\alpha_3)} \cong \mathbb{Z}_p^{(\alpha_2)} \oplus \mathbb{Z}_{p^2}^{(\alpha_3)}$ and $p^2G \cong p^2\mathbb{Z}_{p^3}^{(\alpha_3)} \cong \mathbb{Z}_p^{(\alpha_3)}$. The socles are $pG[p] \cong \mathbb{Z}_p^{(\alpha_2)} \oplus p\mathbb{Z}_{p^2}^{(\alpha_3)} \cong \mathbb{Z}_p^{(\alpha_2)} \oplus \mathbb{Z}_p^{(\alpha_3)}$ and $p^2G[p] \cong \mathbb{Z}_p^{(\alpha_3)}$. The considered $\mathbb{Z}_p$-vector space to calculate $u_p(1, G)$ then is isomorphic to $(\mathbb{Z}_p^{(\alpha_2)} \oplus \mathbb{Z}_p^{(\alpha_3)})/\mathbb{Z}_p^{(\alpha_3)} \cong \mathbb{Z}_p^{(\alpha_2)}$ and consequently we have $u_p(1, G) = \alpha_2$. Similarly we receive $u_p(0, G) = \alpha_1, u_p(2, G) = \alpha_3$ and $u_p(\beta, G) = 0$ for all $\beta \geq 3$.*

We see, if G is a direct sum of cyclic groups, $u_p(\alpha, G)$ corresponds with the number of cyclic summands of order $p^{\alpha+1}(\alpha < \omega)$.
For reduced groups, which contain no divisible subgroups (and therefore $p^\infty G = 0$), Ulm's theorem then states the following

Theorem 1.1.3 (Ulm)**.** *For G and H countable, reduced p-groups the following are equivalent:*

(i) $G \cong H$;

(ii) $u_p(\alpha, G) = u_p(\alpha, H)$ for all ordinals α.

At the end of Section 1.1 we will sketch the proof of Theorem 1.1.3.
For uncountable (abelian) p-groups Ulm's theorem is false:

Example 1.1.4. *Let, for a fixed prime p, $G = t(\Pi_{i=1}^\infty \mathbb{Z}_{p^i})$, the torsion part of $\Pi_{i=1}^\infty \mathbb{Z}_{p^i}$, and $H = \bigoplus_{i=1}^\infty \mathbb{Z}_{p^i}$. Then G is uncountable and H is countable. Moreover, G is not a direct sum of cyclics, thus G and H are clearly not isomorphic, but the Ulm-Kaplansky invariants for G and H coincide: $u_p(\alpha, G) = u_p(\alpha, H) = 1$, for all $\alpha < \omega$, resp. $u_p(\alpha, G) = u_p(\alpha, H) = 0$, for all $\alpha \geq \omega$.*

A **Warfield module** is a direct summand of a **simply presented** $\mathbb{Z}_{(p)}$-module, that means it can be defined in terms of generators and relations such that the only relations are of the forms $px = 0$ or $px = y$. For example, the Prüfer-group or -module, $\mathbb{Z}(p^\infty)$ is simply presented, as it is defined to be generated by the elements $x_1, x_2, ...$ which fulfill $px_1 = 0, px_2 = x_1, ..., px_n = x_{n-1}, ...$. The group $\mathbb{Z}(p^\infty)$ is divisible. One may, for example, identify it with the multiplicative group of all complex p^n-th roots of unity. For the Prüfer group we have $u_p(\alpha, \mathbb{Z}(p^\infty)) = 0$ for all α, since $p^\alpha \mathbb{Z}(p^\infty) = \mathbb{Z}(p^\infty)$ for all α.

An equivalent definition for a Warfield module can be given with the help of the concept of a decomposition basis:

Definition 1.1.5. *A subset $X = \{x_i\}_{i \in I}$ of a $\mathbb{Z}_{(p)}$-module M is called a* **decomposition basis** *for M if*

(i) the elements x_i are independent;

(ii) the elements x_i all have infinite order;

(iii) $M/\langle X \rangle$ is torsion, where $\langle X \rangle$ denotes the $\mathbb{Z}_{(p)}$-submodule of M, generated by the elements of X, and

(iv) $\langle X \rangle = \bigoplus_{i \in I} \langle x_i \rangle$ is a **valuated coproduct** *in M,*

which means for $x = \sum_{i \in I} k_i x_i \in \langle X \rangle (k_i \in \mathbb{Z}_{(p)}), |x|_M = \min\{|k_i x_i|_M : i \in I\}$.

If only (i), (ii) and (iv) are fulfilled, X is called a **decomposition set**.
An easy example is the empty set, which is a decomposition basis for any torsion module. For $M = \bigoplus_{i \in I} M_i$, where each M_i has rank 1, we may collect elements $x_i \in M_i$ of infinite order (if existent). Then $\{x_i\}_{i \in I}$ forms a decomposition basis for M.
If X and X' are two decomposition bases of M and every element of X' is a nonzero multiple of an element of X, then X' is called a **subordinate** of X.
Further recall that a submodule N of M is called **nice** (in M) if

$$p^\alpha (M/N) = (p^\alpha M + N)/N \text{ for all ordinals } \alpha.$$

If a torsion module M has a family $\mathcal{C}$ of nice submodules which contains the zero group $\{0\}$, is closed under sums ($C_i \in \mathcal{C}, i \in I \Rightarrow \sum_{i \in I} C_i \in \mathcal{C}$) and fulfills that for $C \in \mathcal{C}$ and countable $X \subseteq M$ there exists $C' \in \mathcal{C}$ such that $C, X \subseteq C'$ and C'/C

is countably generated, then M is called **totally projective**. Hill proved that all totally projective modules are classified by their Ulm-Kaplansky invariants. Note also that a reduced p-group is totally projective if and only if it is simply presented.

An element $x \in M$ is called **proper with respect to** N if it has maximal height among all elements in the coset $x + N$. Then we have $|x + n| = \min\{|x|, |n|\}$ for all $n \in N$. The following can be found in [L]:

Lemma 1.1.6 (Loth)**.** *The following are equivalent*

(i) N is nice in M;

(ii) every coset of N in M contains an element which is proper with respect to N.

A decomposition basis X of M, where $\langle X \rangle$ is a nice submodule of M is called **nice decomposition basis of** M. A subordinate of a nice decomposition basis is again nice cf. [L].

We are now able to give a more practicable definition of Warfield modules.

Definition 1.1.7. *A $\mathbb{Z}_{(p)}$-module M is called* **Warfield module** *if it possesses a nice decomposition basis X such that the quotient $M/\langle X \rangle$ is simply presented.*

For M a $\mathbb{Z}_{(p)}$-module with submodule N and α an ordinal, put

$$N(\alpha) = p^\alpha M[p] \cap (N + p^{\alpha+1} M).$$

Considering $p^\alpha M[p], p^{\alpha+1} M[p]$ and $N(\alpha)$ as $\mathbb{Z}_p$-vector spaces, the α**-th Ulm-Kaplansky invariant of** M **relative to** N is defined by

$$u_p^N(\alpha, M) = \dim(p^\alpha M[p]/N(\alpha)).$$

If we let $u_p^{M,N}(\alpha) = \dim(N(\alpha)/p^{\alpha+1} M[p])$, we see that

$$\begin{aligned} \dim(p^\alpha M[p]/N(\alpha)) + \dim(N(\alpha)/p^{\alpha+1} M[p]) &= \\ \dim((p^\alpha M[p]/p^{\alpha+1} M[p])/(N(\alpha)/p^{\alpha+1} M[p]) + \dim(N(\alpha)/p^{\alpha+1} M[p]) &= \\ \dim(p^\alpha M[p]/p^{\alpha+1} M[p]) \end{aligned}$$

and therefore

$$u_p(\alpha, M) = u_p^N(\alpha, M) + u_p^{M,N}(\alpha). \tag{1.1}$$

To define further invariants for Warfield modules, we now need to consider sequences $\overline{\beta} = (\beta_i)_{i<\omega}$, where each β_i is an ordinal or the symbol ∞. An **Ulm sequence** $\overline{\beta} = (\beta_i)_{i<\omega}$ is ordered in the sense that $\beta_i < \beta_{i+1}$ for all i with the convention $\infty < \infty$. We put $p^k\overline{\beta} = (\beta_{i+k})_{i<\omega}$ and call two Ulm sequences $\overline{\beta}$ and $\overline{\gamma}$ **equivalent**, $\overline{\beta} \sim \overline{\gamma}$, if there exist $k, l < \omega$ with $p^k\overline{\beta} = p^l\overline{\gamma}$. This means that the sequences might have different starting values but are equal beyond a certain point. $[\overline{\beta}]$ denotes the equivalence class of $\overline{\beta}$. For an element $x \in M$, the **Ulm sequence of** x, $u(x)$ or $u_M(x)$ is given by $\{|p^i x|\}_{i<\omega}$. $u(x)$ has a **gap at** α, for an ordinal α, if $|p^i x| = \alpha$ and $|p^{i+1}x| > \alpha + 1$ for some $i < \omega$. If there is need to indicate which prime p is considered, we will also sometimes write $u_p(x)$ for an Ulm sequence of x.

Given a submodule N of M we now consider submodules of N, defined by

$$N(\overline{\beta}) = \{x \in N : |p^i x|_M \geq \beta_i \text{ for all } i < \omega\}$$

and

$$N(\overline{\beta}^*) = \langle x \in N(\overline{\beta}) : |p^i x|_M > \beta_i \text{ for infinitely many values of } i\rangle,$$

if $\beta_i \neq \infty$ for all $i < \omega$ and

$$N(\overline{\beta}^*) = tN(\overline{\beta}),$$

otherwise.
Note that $N(\overline{\beta})/N(\overline{\beta}^*)$ is a $\mathbb{Z}_p$-vector space in the case $\beta_i \neq \infty$ for all i, while $M(\overline{\beta})/M(\overline{\beta}^*)$ is a $\mathbb{Q}$-vector space in the case $\beta_i = \infty$ for some i.

Definition 1.1.8. *The* $\overline{\beta}$**-th Warfield invariant of** M *is defined by*

$$w_M(\overline{\beta}) = \dim(M(\overline{\beta})/M(\overline{\beta}^*)).$$

The calculation of the more general Warfield invariants uses the concept of the **direct limit** which considers for submodules N of M the sequence of modules $N(\overline{\beta})/N(\overline{\beta}^*)$, $N(p\overline{\beta})/N(p\overline{\beta}^*)$, $N(p^2\overline{\beta})/N(p^2\overline{\beta}^*)$, ..., where the quotients are related through monomorphisms $\Phi_{(i,j)}$ for all $i \leq j$. More precisely, we have

$$\Phi_{(i,j)} : N(p^i\overline{\beta})/N(p^i\overline{\beta}^*) \to N(p^j\overline{\beta})/N(p^j\overline{\beta}^*)$$
$$\text{with } \Phi_{(i,j)}(x + N(p^i\overline{\beta}^*)) = p^{j-i}x + N(p^j\overline{\beta}^*).$$

$\{N(p^i\overline{\beta})/N(p^i\overline{\beta}^*); \Phi_{(i,j)} : i, j \in \omega, i \leq j\}$ is a **direct system** and

$$
\begin{aligned}
W_N^M(\overline{\beta}) := \quad & \varinjlim_{i\in\omega} N(p^i\overline{\beta})/N(p^i\overline{\beta}^*) \\
& = (\textstyle\bigoplus_{i\in\omega} N(p^i\overline{\beta})/N(p^i\overline{\beta}^*))/\langle x + N(p^i\overline{\beta}^*) - (p^{j-i}x + N(p^j\overline{\beta}^*)) : i \leq j\rangle
\end{aligned}
$$

its **direct limit**. The dimension of $W_N^M(\overline{\beta})$ is the **$\overline{\beta}$-th Warfield invariant of** N, denoted by $w_N^M(\overline{\beta})$. One has $w_M^M(\overline{\beta}) = w_M(\overline{\beta})$. For a submodule N of M such that M/N is torsion, [HRW] showed $w_N^M(\overline{\beta}) = w_M(\overline{\beta})$. Thus, for any decomposition basis X of M:

$$
w_{\langle X\rangle}^M(\overline{\beta}) = w_M(\overline{\beta}).
$$

Then we see that $w_M(\overline{\beta})$ counts the elements in a decomposition basis which have an Ulm sequence equivalent to $\overline{\beta}$, since

$$
\langle X\rangle(\overline{\beta})/\langle X\rangle(\overline{\beta}^*) \cong \bigoplus_{x\in X} \langle x\rangle(\overline{\beta})/\langle x\rangle(\overline{\beta}^*),
$$

and the following small lemma holds:

Lemma 1.1.9. *If M is a Warfield module with decomposition basis X and $x \in X$, then $w_{\langle x\rangle}^M(\overline{\beta}) = 1$ iff $u_M(x) \sim \overline{\beta}$ and $w_{\langle x\rangle}^M(\overline{\beta}) = 0$ iff $u_M(x) \not\sim \overline{\beta}$.*

Proof. Evidently, $w_{\langle x\rangle}^M(\overline{\beta}) \in \{0, 1\}$.
Let us first assume $u_M(x) \sim \overline{\beta}$. Thus, there exist $k, l < \omega$ with $u_M(p^k x) = p^l\overline{\beta}$. If $\beta_i \neq \infty$ for all i, we have $\langle x\rangle(p^l\overline{\beta}) = \langle p^k x\rangle$ and $\langle x\rangle(p^l\overline{\beta}^*) = \langle p^{k+1}x\rangle$. Hence, $w_{\langle x\rangle}^M(\overline{\beta}) \geq \dim(\langle x\rangle(p^l\overline{\beta})/\langle x\rangle(p^l\overline{\beta}^*)) = \dim(\langle p^k x\rangle/\langle p^{k+1}x\rangle) = \dim(\mathbb{Z}_p) = 1$ and $w_{\langle x\rangle}^M(\overline{\beta}) = 1$ follows. Observe here that x is an element of a decomposition basis and thus has infinite order.
If $\beta_i = \infty$ for some i, we have $p^k x \in \langle x\rangle(p^l\overline{\beta})$ and $\langle x\rangle(p^l\overline{\beta}^*) = t\langle x\rangle(p^l\overline{\beta}) = 0$, since $\langle x\rangle$ is torsion-free. Hence, $w_{\langle x\rangle}^M(\overline{\beta}) \geq \operatorname{rank}(\langle x\rangle(p^l\overline{\beta})/\langle x\rangle(p^l\overline{\beta}^*) > 0$ and $w_{\langle x\rangle}^M(\overline{\beta}) = 1$ follows.
For the converse direction let us assume that $w_{\langle x\rangle}^M(\overline{\beta}) = 1$.
This implies $\langle x\rangle(p^l\overline{\beta})/\langle x\rangle(p^l\overline{\beta}) \neq 0$ for some $l < \omega$ and we can choose some element $y \in \langle x\rangle(p^l\overline{\beta}) \setminus \langle x\rangle(p^l\overline{\beta}^*)$. We have $|p^i y|_M = \beta_{l+1}$ for almost all $i < \omega$, thus $u_M(y) \sim p^l\overline{\beta}$ and $u_M(x) \sim u_M(y) \sim p^l\overline{\beta} \sim \overline{\beta}$. ∎

If M has a decomposition basis X, we then define

$$
X^{(\overline{\beta})} := \{x \in X : u_M(x) \sim \overline{\beta}\}.
$$

and have $w_M(\overline{\beta}) = |X^{(\overline{\beta})}|$. Obviously this number is independent from the choice of X.
The classification theorem by Warfield then states

Theorem 1.1.10 (Warfield)**.** *For two Warfield modules M and N the following are equivalent:*

(i) $M \cong N$*;*

(ii) *(a)* $u_p(\alpha, M) = u_p(\alpha, N)$ *for all ordinals* α*;*

(b) $u_p(\infty, M) = u_p(\infty, N)$*;*

(c) $w_M(\overline{\beta}) = w_N(\overline{\beta})$ *for all Ulm sequences* $\overline{\beta}$*.*

Since the model-theoretic generalizations of the classification theorems will deal with Ulm and Warfield invariants coinciding only up to a certain ordinal, it will be useful to provide some helpful definitions to handle initial segments of Ulm sequences.

Definition 1.1.11. *Let α be an ordinal or the symbol ∞ and $\overline{\beta}, \overline{\eta}$ two Ulm sequences.*

(i) The **α-initial sequence of** $\overline{\beta}$, $\mathrm{in}_\alpha(\overline{\beta})$, *is the sequence* $(\gamma_i)_{i<\omega}$ *where*

$$\gamma_i = \begin{cases} \beta_i & \text{if } \beta_i \leq \alpha, \\ \infty & \text{if } \beta_i > \alpha. \end{cases}$$

(ii) We call $\overline{\beta}$ *and* $\overline{\eta}$ **α-initially equivalent**, $\overline{\beta} \sim_\alpha \overline{\eta}$, *iff* $\mathrm{in}_\alpha(\overline{\beta}) \sim \mathrm{in}_\alpha(\overline{\eta})$.

Naturally we have $\mathrm{in}_\infty(\overline{\beta}) = \overline{\beta}$ and $\overline{\beta} \sim_\infty \overline{\eta}$ exactly if $\overline{\beta} \sim \overline{\eta}$. If, for example, $\overline{\beta} = (1, 2, 3, 4, 5, 6, 7, ...), \overline{\eta} = (1, 2, 3, 4, 5, 10, 15, 20, ...)$ then $\overline{\beta} \not\sim \overline{\eta}$ but $\overline{\beta} \sim_5 \overline{\eta}$. On the other hand, equivalent Ulm sequences are always α-initially equivalent for any ordinal α.

According to $X^{(\overline{\beta})}$ we then define

$$X_\alpha^{(\overline{\beta})} := \{x \in X : u_M(x) \sim_\alpha \overline{\beta}\}$$

for a decomposition basis X of M. If $\beta_i \leq \alpha$ for all $i < \omega$, then $\mathrm{in}_\alpha(\overline{\beta}) = \overline{\beta}$ and therefore $u_M(x) \sim_\alpha \overline{\beta}$ iff $\mathrm{in}_\alpha(u_M(x)) \sim \overline{\beta}$ for any $x \in X$, and this is the case exactly if $u_M(x) \sim \overline{\beta}$ since $\mathrm{in}_\alpha(u_M(x))$ and $u_M(x)$ are the same in all entries up to α. Would $u_M(x)$ have (one and therefore) infinitely many entries $> \alpha$, then so would $\mathrm{in}_\alpha(u_M(x))$ and consequently $u_M(x) \not\sim_\alpha \overline{\beta}$. It follows $X^{(\overline{\beta})} = X_\alpha^{(\overline{\beta})}$. If, on the other hand, $\beta_i > \alpha$ for some $i < \omega$, there are two possibilities regarding $u_M(x)$ for an element $x \in X$. Either $|p^i x|_M \leq \alpha$ for all $i < \omega$. Then $u_M(x) \not\sim_\alpha \overline{\beta}$, since $u_M(x)$ has no entry ∞, unlike $\overline{\beta}$. If, however, $|p^i x|_M > \alpha$ for some $i < \omega$, then $u_M(x) \sim_\alpha \overline{\beta}$, since both, $\mathrm{in}_\alpha(u_M(x))$ and $\mathrm{in}_\alpha(\overline{\beta})$ only have entries ∞ beyond a certain point and

certainly coincide from these points on. Moreover, they might have been equivalent to start with, which would imply that $u_M(x)$ has an entry $> \alpha$, too. Therefore in the case $\beta_i > \alpha$ for some $i < \omega$ we have $X_\alpha^{(\overline{\beta})} = X^{(\overline{\beta})} \cup \{x \in X \setminus X^{(\overline{\beta})} : |p^k x|_M > \alpha$ for some $k < \omega\}$, hence $X_\alpha^{(\overline{\beta})} = \{x \in X : p^k x \in p^{\alpha+1}M$ for some $k < \omega\}$.
It is possible that $X_\alpha^{(\overline{\beta})} = X_\alpha^{(\overline{\eta})}$, even if $\overline{\beta} \not\sim \overline{\eta}$. Namely we have $x \in X_\alpha^{(\overline{\beta})} \cap X_\alpha^{(\overline{\eta})}$ if $\overline{\beta}, \overline{\eta}$ and $u_M(x)$ all have entries $> \alpha$. Therefore we can say that $\sim_\alpha$ describes equivalence of sequences only up to α.
Nevertheless, the cardinality of the set $X_\alpha^{(\overline{\beta})}$, in case that X is a decomposition basis of a module M, can be identified as another invariant of M:

Definition 1.1.12. *Let α be an ordinal or the symbol ∞ and M a module. Let U be a complete set of representatives of distinct equivalence classes of Ulm sequences $\overline{\eta}$ which fulfill $w_M(\overline{\eta}) \neq 0$. Then*

$$w_M^\alpha(\overline{\beta}) = \sum_{\overline{\eta} \in U, \overline{\eta} \sim_\alpha \overline{\beta}} w_M(\overline{\eta}).$$

As indicated, $w_M^\alpha(\overline{\beta}) = |X_\alpha^{(\overline{\beta})}|$: In the case $\beta_i \leq \alpha$ for all $i < \omega$, there is no other Ulm sequence $\overline{\eta}$ which is α-initially equivalent to $\overline{\beta}$ but was not equivalent to $\overline{\beta}$ to start with. Therefore in this case $w_M^\alpha(\overline{\beta}) = w_M(\overline{\beta})$. In the case $\beta_i > \alpha$ for some $i < \omega$, we consider the set $\{x \in X \setminus X^{(\overline{\beta})} : |p^k x|_M > \alpha$ for some $k\}$ and see that it contains the elements $x \in X$ with $u_M(x) \not\sim \overline{\beta}$ but $u_M(x) \sim \overline{\eta}$ for some $\overline{\eta} \sim_\alpha \overline{\beta}$. All, $\overline{\beta}, u_M(x)$ and $\overline{\eta}$ must have entries $> \alpha$ in this case and therefore it is possible that later entries of $u_M(x)$ and $\overline{\eta}$ coincide, where those of $u_M(x)$ and $\overline{\beta}$ differ and $\overline{\eta}$ and $\overline{\beta}$ are equivalent up to α. Then

$$X_\alpha^{(\overline{\beta})} = X^{(\overline{\beta})} \cup \bigcup_{\overline{\eta} \sim_\alpha \overline{\beta}, \overline{\eta} \not\sim \overline{\beta}} X^{(\overline{\eta})}$$

which immediately leads to $w_M^\alpha(\overline{\beta}) = |X_\alpha^{(\overline{\beta})}|$.

Becoming even more general, Stanton [St] extended the definition of the $\overline{\beta}$-th Warfield invariants from $\mathbb{Z}_{(p)}$-modules to groups. To do so, sequences of Ulm sequences are needed. They are collected in a so-called **Ulm matrix**, a matrix $A = [a_{(p,i)}]_{(p,i) \in P \times \omega}$ in which each row is an Ulm sequence. (P denotes the set of all primes.) For an element of the group $x \in G$, $U(x)$ is the **Ulm matrix of** x, which has $u_p(x)$ as its p-th row. For $n < \omega$, nA has $a_{(p,i+|n|_p)}$ as its (p,i) entry, where $|n|_p$ denotes the p-height of n in $\mathbb{Z}$. We then have $nU(x) = U(nx)$. For $A = [a_{(p,i)}]$ and $B = [b_{(p,i)}]$, we set $A \geq B$ iff $a_{(p,i)} \geq b_{(p,i)}$ for all (p,i). We call A and B **compatible** and write

$A \sim B$, if there are $n, m < \omega$ such that $mB \geq A$ and $nA \geq B$. Compatible Ulm matrices form an equivalence class, called **compatibility class**. The compatibility class of a matrix A is denoted by $[A]$. If two Ulm matrices are compatible then their p-th rows coincide for almost all primes p.

A **decomposition basis** for a group is defined similar to Definition 1.1.5, where $\langle X \rangle$ denotes the generated subgroup and (iv) has to be fulfilled for all primes p ($|x|_p = \min\{|k_i x_i|_p : i \in I\}$). Also, a decomposition basis X of G is called **nice**, if $\langle X \rangle_p = \langle X \rangle \otimes \mathbb{Z}_{(p)}$ is a nice $\mathbb{Z}_{(p)}$-submodule of $G_p = G \otimes \mathbb{Z}_{(p)}$ for every prime p, that means each coset $x + \langle X \rangle_p$ contains an element of maximal p-height. A **Warfield group** G then possesses a nice decomposition basis X such that $G/\langle X \rangle$ is simply presented.

Definition 1.1.13. *For any abelian group G with decomposition basis X, p a prime, $\overline{\beta}$ an Ulm sequence and A an Ulm matrix set*

$$w_G(p, \overline{\beta}, A) = |\{x \in X : U(x) \sim A \text{ and } u_p(x) \sim \overline{\beta}\}|$$

the **(global) Warfield invariant of** *G.*

The classification theorem for Warfield groups then states

Theorem 1.1.14 (Hunter/Richman, Stanton)**.** *For two Warfield groups G and H the following are equivalent:*

(i) $G \cong H$;

(ii) *(a)* $u_p(\alpha, G_p) = u_p(\alpha, H_p)$ *for all primes p and ordinals α,*

(b) $u_p(\infty, G_p) = u_p(\infty, H_p)$ *for all primes p,*

(c) $w_G(p, \overline{\beta}, A) = w_H(p, \overline{\beta}, A)$ *for all primes p, Ulm sequences $\overline{\beta}$ and Ulm matrices A.*

Classical proofs regarding the classification of groups or modules, such as the theorems by Ulm and Warfield, often make use of **height-preserving** partial isomorphisms between subgroups or -modules M' of M and N' of N.

Definition 1.1.15. *An isomorphism $f : M' \to N'$ is called* **height-preserving** *partial isomorphism of M and N if*

$$|x|_M = |f(x)|_N$$

for all $x \in M'$.

Slightly extending the above definition, we want to be able to preserve heights below a certain ordinal α and equivalence of Ulm sequences up to $\sim_\alpha$ and follow [BE] with the following

Definition 1.1.16. *An isomorphism $f : M' \to N'$ is called* **α-height-preserving** *partial isomorphism of M and N for some ordinal α if for all $x \in M'$:*

(i) $|x|_M < \alpha \Rightarrow |f(x)|_N = |x|_M$;

(ii) $|x|_M \geq \alpha \Rightarrow |f(x)|_N \geq \alpha$.

An ∞-height-preserving isomorphism is a height-preserving isomorphism.

Certain sets I of (α-)height-preserving partial isomorphisms of groups or modules M and N possess the crucial

Back-and-forth property 1.1.17. *For any $f \in I$ and $a \in M$ (resp. $b \in N$) there is $g \in I$ such that g extends f and $a \in \mathrm{dom}(g)$ (resp. $b \in \mathrm{im}(g)$).*

This property allows the proof of two groups or modules being isomorphic by extending the domain as well as the range of isomorphisms by one element at a time, thus jumping 'back and forth' between the two groups (modules). Therefore the following definition is natural:

Definition 1.1.18. *If for two groups or modules M and N a set I with property 1.1.17 exists, then M and N are called* **partially isomorphic** *and we write $M \cong_p N$.*

Sketch of the proof of Theorem 1.1.3. The proof uses the hypothesis of countability and constructs the isomorphism between G and H step by step. Thus, at every stage in the proof, one has to worry only about finite subgroups of G and H. By numbering off all elements of G and all elements of H, and considering the n-th element of G in the (2n-1)-th step and the n-th element of H in the 2n-th step of the proof, one ensures to achieve an isomorphism between G and H and not just between parts of G and H. This is a method of argumentation which corresponds to the back-and-forth property mentioned above. Of course, the desired isomorphism between G and H has to be height-preserving, thus it is essential to begin the proof with a partial isomorphism which is also height-preserving.
One starts with finite subgroups S and T of G and H, resp. and a height-preserving isomorphism f between them, e.g. the trivial partial isomorphism $f : 0 \to 0$. Then

an element $x \in G$ is considered, with the following normalizing assumptions: x is proper with respect to S and $|px|$ is maximal among all those proper elements. Additionally we may assume that x is not in S, but px is in S. We set $|x| = \alpha$. All heights are computed in G and H, resp. We then have to find an element y in H which is proper with respect to T, has height α and fulfills $py = f(px)$.
In the first case we assume $|f(px)| = \alpha + 1$. Then any element $y \in p^\alpha H$ which fulfills $py = f(px)$ meets all requirements and can be chosen to extend f. The first case makes no use of the equality of the Ulm invariants.
The second case assumes $|f(px)| > \alpha + 1$. Then $px = px'$ with $x' \in p^{\alpha+1}G$. Thus the element $x - x'$ is in $p^\alpha G[p]$, $|x - x'| = \alpha$ and $x - x'$ is proper with respect to S. The existence of such an element, in combination with the equality of the Ulm invariants, ensures the existence of a certain mapping, which has properties that allow us to deduce that H contains an element y' which is proper with respect to T and which fulfills $py' = 0$ and $|y'| = \alpha$. (See e.g. [Kap] for the whole story.) By $|f(px)| > \alpha + 1$ we are able to write $f(px) = py''$ with $y'' \in p^{\alpha+1}H$. $y = y' + y''$ then provides the desired element to extend f.
An element of H is added to the range of f similarly by extending f^{-1}. ■

1.2 Model-theory

We follow the approach of Barwise and Eklof [BE] and define $\hat{g}$ for cardinal-valued functions g by

$$\hat{g}(x) = \begin{cases} g(x) & \text{if } g(x) < \aleph_0, \\ \infty & \text{if } g(x) \geq \aleph_0. \end{cases}$$

and extend this definition to

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } g(x) \leq \aleph_0, \\ \infty & \text{if } g(x) > \aleph_0. \end{cases}$$

Now, the model-theoretic framework of this work is given by an ordinary first order language L with identity, finitary relation and function symbols and constant symbols. We assume that L has a variable v_α for every ordinal α. Examples for atomic formulas in L are terms like "$x = a$" or "$A(x, a, z)$" where x, a, z are constant symbols or variables of L and A is a relation symbol of L. In order to define the language L_∞ (often denoted by $L_{\infty\omega}$, the first index indicating that conjunctions and disjunctions of arbitrary length may appear in a formula, the second index indicating that only finitely many quantifiers are allowed to appear <u>in a row</u>), which we will refer to mostly, we define for each ordinal α a collection L_α of formulas as follows: L_α is the smallest collection F of formulas which contains the atomic formulas and is closed under the following logical operations:

(L1) If $\varphi \in F$, then $\neg\varphi \in F$.

(L2) If $\Phi \subseteq F$, then $\bigwedge \Phi, \bigvee \Phi \in F$.

(L3) If $\varphi \in L_\beta$ for some $\beta < \alpha$ and v is a variable, then $\exists v\, \varphi, \forall v\, \varphi \in F$.

In (L2) $\bigwedge \Phi$ (resp. $\bigvee \Phi$) denotes the conjunction (resp. disjunction) of an arbitrary number of elements of the set Φ (possibly all). Notice that Φ can be of any cardinality. Following [BE], we let $L_\infty = \bigcup_\alpha L_\alpha$. The **quantifier rank** $\mathrm{qr}(\varphi)$ of a formula $\varphi \in L_\infty$ is defined to be the least ordinal α such that $\varphi \in L_\alpha$. As an example we will consider the statement $x \in p^\alpha G$ for a p-group G. Therefore recall that every ordinal α can be written as $\alpha = \omega\delta + n$ where δ is a unique ordinal and $n < \omega$. Barwise and Eklof [BE] pointed out that the statement can be expressed by a formula of quantifier rank δ ($\delta + 1$, resp.) if $n = 0$ ($n > 0$, resp.), as the statement is equivalent to the formula

$$\bigwedge_{\beta<\omega\delta} x \in p^\beta G, (n = 0) \text{ or}$$

$$\exists y(y \in p^{\omega\delta} G \wedge p^n y = x), (n > 0) \text{ resp.}$$

We can verify the quantifier rank of "$x \in p^\alpha G$" by induction. For $n = 0$ the statement $x \in p^\omega G$ is a conjunction of all the formulas $\exists y(y \in G \wedge p^i y = x)$, $0 \leq i < \omega$. Each of this formulas has quantifier rank 1, since "$y \in G \wedge p^i y = x$" has quantifier rank 0. According to (L2) the conjunction then has quantifier rank 1, too. Assuming now that $\mathrm{qr}(x \in p^{\omega\delta} G) = \delta$, the formula $x \in p^{\omega(\delta+1)} G$ is similarly a conjunction of all the formulas $\exists y(y \in p^{\omega\delta} G \wedge p^i y = x)$, $0 \leq i < \omega$. Each of these formulas now has quantifier rank $\delta + 1$, the same holds for the conjunction. As the formula $x \in p^{\omega\delta+n} G$ is one of the formulas $\exists y(y \in p^{\omega\delta} G \wedge p^i y = x)$, namely for $i = n$, it also has quantifier rank $\delta + 1$.

Note that the quantifier rank of a formula can be seen as a measure of its complexity because it gives us knowledge about the largest number of nested quantifiers which occur in the formula. The notion of a **sentence** is defined, as usual, as a formula containing no **free** variables i.e. containing no variables not **bound** by a quantifier.

Models for a language are understood to be sets of constants, which, in combination with the language-specific identity, functions, relations, etc. satisfy the axioms of the language. Models are denoted by $\mathfrak{A} = \langle A, ...\rangle$, where A is a set of constants. If $\varphi \in L_\infty$ is a formula with at most n variables, $a_1, ..., a_n \in A$ and $\varphi(a_1, ..., a_n)$ is true, we write $\mathfrak{A} \models \varphi[a_1, ..., a_n]$, and accordingly for a sentence φ which is true, $\mathfrak{A} \models \varphi$. For example, $\mathcal{N} = \langle \mathbb{N}, +, \cdot, ', 0\rangle$ is the standard model for the language of number theory which satisfies the following sentences:

(P1) $\forall x : x' \neq 0$

(P2) $\forall x \forall y : x' = y' \Rightarrow x = y$

(P3) $\forall x : x + 0 = x$

(P4) $\forall x \forall y : x + y' = (x + y)'$

(P5) $\forall x : x \cdot 0 = 0$

(P6) $\forall x \forall y : x \cdot y' = x \cdot y + x$

and the induction principle

(P7) $(\mathcal{N} \models \varphi(0) \wedge (\forall x : \mathcal{N} \models \varphi(x) \Rightarrow \mathcal{N} \models \varphi(x'))) \Rightarrow \forall x : \mathcal{N} \models \varphi(x)$.

Let α be an ordinal or the symbol ∞. Then two models $\mathfrak{A} = \langle A, ...\rangle$ and $\mathfrak{B} = \langle B, ...\rangle$ for L_∞ are called L_α**-equivalent**, and we write $\mathfrak{A} \equiv_\alpha \mathfrak{B}$, if for all sentences $\varphi \in L_\alpha$ we have

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathfrak{B} \models \varphi.$$

A basic tool to investigate isomorphism under a model-theoretic point of view is a theorem like the following by Karp [K] which appears in a modified way in most works dealing with this topic. This one links L_α-equivalence of models to the existence of a set of partial isomorphisms with the back-and-forth property:

Theorem 1.2.1 (Karp)**.** *Let $\mathfrak{A} = \langle A, \ldots\rangle$ and $\mathfrak{B} = \langle B, \ldots\rangle$ be models for L_∞ and δ an ordinal or the symbol ∞. Then the following are equivalent:*

(i) $\mathfrak{A} \equiv_\delta \mathfrak{B}$*;*

(ii) For each ordinal $\alpha \leq \delta$ there is a non-empty set I_α of isomorphisms on substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that

(a) if $\alpha \leq \beta$, then $I_\beta \subseteq I_\alpha$;

(b) if $\alpha + 1 \leq \delta$, $f \in I_{\alpha+1}$ and $a \in A$ ($b \in B$, resp.), then f extends to a map $f' \in I_\alpha$ such that $a \in \mathrm{dom}(f')$ ($b \in \mathrm{im}(f'), resp.$).

This leads to

Theorem 1.2.2 (Karp)**.** *Let G and H be some structures for the language L, then*

$$G \equiv_\infty H \Leftrightarrow G \cong_p H.$$

Investigating this relationship, Jon Barwise [B] 1973 proved the following generalization of the back and forth part of the original argument by Cantor (that any two linearly ordered sets without end points which are countable and dense are isomorphic):

Theorem 1.2.3 (Barwise)**.** *If M and N are countable or countably generated, then*

$$M \cong_p N \Leftrightarrow M \cong N.$$

This result, together with Theorem 1.2.2 then provides the relation between general isomorphism and model-theoretic isomorphism (infinitary equivalence) in the case of countable (or countably generated) groups, modules or structures, namely that

$$G \equiv_\infty H \Leftrightarrow G \cong H.$$

The link is the partial isomorphism which in the paper by Barwise [B] is considered "the strongest possible absolute notion of isomorphism". In Chapter 4 we will investigate how closely related uncountable structures are if they are partially isomorphic.

If $\mathfrak{A} \equiv_\delta \mathfrak{B}$ and there are sets of isomorphisms $I_\alpha (\alpha \leq \delta)$ as in Theorem 1.2.1 which consist entirely of δ-height-preserving isomorphisms, then we write

$$\mathfrak{A} \equiv_\delta^h \mathfrak{B}.$$

Since a wide range of modules can be described as direct sums of special modules, in order to study these it is helpful to know that L_α-equivalence is invariant under this construction:

Lemma 1.2.4 (Barwise-Eklof)**.** *The following hold:*

(i) If $\bigoplus_{i \in I} \mathfrak{A}_i$ denotes the direct sum of the models $\mathfrak{A}_i$, then $\mathfrak{A}_i \equiv_\alpha \mathfrak{B}_i$ for each $i \in I$ implies $\bigoplus_{i \in I} \mathfrak{A}_i \equiv_\alpha \bigoplus_{i \in I} \mathfrak{B}_i$.

(ii) If $\mathfrak{A}$ is a model for L_∞ and I and J are infinite index sets, the equality $\mathfrak{B}_i = \mathfrak{A} = \mathfrak{C}_j$ for each $i \in I$ and $j \in J$ implies $\bigoplus_{i \in I} \mathfrak{B}_i \equiv_\infty \bigoplus_{j \in J} \mathfrak{C}_j$.

We will now begin by investigating the quantifier rank of classifying sentences of our groups and modules, as we already did in [GLLS]. Remember $\alpha = \omega\delta + n$.
For $m < \omega$, the statements "rank$(p^\alpha G) \geq m$" and "$u(\alpha, G) \geq m$" can be expressed by sentences $\varphi_{\alpha,m}$ and $\psi_{\alpha,m}$ of quantifier rank $\delta + m$ and $\delta + m + 1$, resp., as the statements are equivalent to

$$\exists x_1 ... \exists x_m (\bigwedge_{i=1}^{m} x_i \in p^{\omega\delta} G \ \wedge \ p^n x_1, ..., p^n x_m \text{ are independent})$$

and

$$\exists x_1 ... \exists x_m (\bigwedge_{i=1}^{m} x_i \in p^{\omega\delta+n} G \ \wedge \ p x_i = 0 \ \wedge \ x_1, ..., x_m \text{ are}$$

$$\text{independent modulo } p^{\omega\delta+n+1} G)$$

Note that the expression "$x_1, ..., x_m$ are independent" is of quantifier rank zero since it is equivalent to an infinite chain of subjunctions, which itself is equivalent to an infinite chain of disjunctions and conjunctions of atomic formulas. Then "$x_1, ..., x_m$ are independent modulo $p^{\omega\delta+n+1}G$" is of quantifier rank $\delta + 1$. It is clear that these results carry over to $\mathbb{Z}_{(p)}$-modules. Similarly, we can express facts about Warfield invariants: let $\overline{\beta} = (\beta_i)_{i<\omega}$ be an Ulm sequence. First, assume that $\beta_i \neq \infty$ for all $i < \omega$ and write $\beta_i = \omega\delta_i + n_i$ where δ_i is an ordinal and $n_i < \omega$. Define

$$\delta_i' = \begin{cases} \delta_i & \text{if } n_i = 0 \\ \delta_i + 1 & \text{if } n_i > 0. \end{cases}$$

Then "$x \in M(\overline{\beta})$" can be expressed by a formula $\theta_{\overline{\beta}}(x)$ of quantifier rank $\xi = \sup\{\delta_i' : i < \omega\}$:

$$\theta_{\overline{\beta}}(x) = \bigwedge_{i<\omega} (p^i x \in p^{\beta_i} M).$$

The statement $x \in M(\overline{\beta}^*)$ means that x can be expressed as a linear combination of elements $x_i \in M(\overline{\beta})$ which satisfy $|p^j x_i| > \beta_j$ for infinitely many $j \in \omega$ and therefore is equivalent to $\exists x_1 \dots \exists x_k (x = \sum_{i=1}^{k} \lambda_i x_i$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{Z} \wedge \bigwedge_{i=1}^{k} (x_i \in M(\overline{\beta}) \wedge \ p^j x_i \in p^{\beta_j+1} M$ for infinitely many $j < \omega))$, for some $k < \omega$, hence it can be expressed by a formula $\theta_{\overline{\beta}^*}(x)$ of quantifier rank $\xi + \omega$.

Since the statement

$$w_M(\overline{\beta}) \geq m$$

is true if and only if $\exists x_1 ... \exists x_m$ $(x_1, \ldots, x_m \in M(\overline{\beta}) \ \wedge \ x_1, ..., x_m$ are independent modulo $M(\overline{\beta}^*))$, it can be expressed by a sentence $\theta_{\overline{\beta},m}$ of quantifier rank $\xi + \omega + m$.

Now suppose the module M has a decomposition basis X. If $\infty \neq \beta_i \leq \alpha$ for all $i < \omega$, then $w_M^\alpha(\overline{\beta}) = w_M(\overline{\beta})$ and otherwise

$$w_M^\alpha(\overline{\beta}) = |\{x \in X : p^k x \in p^{\alpha+1} M \text{ for some } k < \omega\}| = \operatorname{rank}(\bigoplus_{x \in X} (\langle x \rangle \cap p^{\alpha+1} M))$$

which coincides with the rank of $\langle X \rangle \cap p^{\alpha+1} M$ and therefore with the torsion-free rank of $p^{\alpha+1} M$. Therefore, by consulting $\theta_{\overline{\beta},m}$ and $\varphi_{\alpha,m}$, "$w_M^\alpha(\overline{\beta}) \geq m$" (where $m < \omega$) can be expressed by a sentence $\psi_{\alpha,\beta,m}$ whose quantifier rank is

$$\begin{array}{ll} \xi + \omega + m & \text{if } \infty \neq \beta_i \leq \alpha \text{ for all } i < \omega \\ \delta + m & \text{if } \beta_i > \alpha \text{ for some } i < \omega. \end{array}$$

Our observations yield one direction of some classifications, of which (i) was already formulated in [BE]:

Lemma 1.2.5. *Let M and N be modules. Suppose $M \equiv_\lambda N$ where λ is a limit ordinal. Then:*

(i) (a) $\hat{u}(\alpha, M) = \hat{u}(\alpha, N)$ if $\alpha < \omega\lambda$.
(b) If $l(M) < \omega\lambda$ and $l(N) < \omega\lambda$, then $\hat{u}(\infty, M) = \hat{u}(\infty, N)$.

(ii) If M and N have decomposition bases and if $\alpha < \omega\lambda$ where $\lambda = \omega\gamma$ and γ is a limit ordinal, then $\hat{w}_M^\alpha(\overline{\beta}) = \hat{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$.

Proof. Bear in mind that $\alpha = \omega\delta + n < \omega\lambda$ yields $\delta + k < \lambda$ for all $k < \omega$.
Hence (i)(a) follows since $\psi_{\alpha,m}$, for finite m is an element of L_λ, and with (L1) $\neg\psi_{\alpha,m}$ and also $u(\alpha, M) < m + 1 = \neg\psi_{\alpha,m+1}$, are in L_λ, too. Therefore $M \equiv_\lambda N$ yields $M \models \psi_{\alpha,m}, \neg\psi_{\alpha,m+1} \Leftrightarrow N \models \psi_{\alpha,m}, \neg\psi_{\alpha,m+1}$ which implies $u_p(\alpha, M) = u_p(\alpha, N)$. Since $m < \omega$ we achieve this equation only for the generalized Ulm-Kaplansky invariant $\hat{u}_p(\alpha, M)$.
For (b) observe that if $l(M) := \eta < \omega\lambda$ we have $x \in p^\infty M \Leftrightarrow x \in p^\eta M$ which is a formula of quantifier rank $< \lambda$ and therefore an element of L_λ. Then, "$u_p(\infty, M) \geq m$" is a formula similar to $\varphi_{\alpha,m}$ and thus in L_λ, too. The assertion then follows as in (a). In (ii) we write $\delta = \delta' + n'$ and have

$$\xi + \omega + m \leq \delta + \omega + m < \omega\delta' + \omega + \omega = \omega(\delta' + 2) < \omega\gamma = \lambda$$

and $\delta + m < \lambda$, hence for any $m < \omega$ the formula $\psi_{\alpha,\overline{\beta},m}$ is in L_λ and the assertion follows similar to (i). ∎

The converse direction will be contained in the main result of Section 2.1.

1.3 Set-theory – The Diamond

This section shall introduce the prediction principle known as Jensen's Diamond that holds e.g. in Gödel's universe (V=L).
Let us start with collecting the necessary definitions. We will follow [EM].

Definition 1.3.1. *For any limit ordinal α, a subset X of α is called* **unbounded** *or* **cofinal in** α*, if* $\sup(X) = \alpha$*, that means, for every $\beta < \alpha$ there exists some $\gamma \in X$ such that $\gamma > \beta$.*

Definition 1.3.2. *The* **cofinality** *of a limit ordinal α,* $\mathrm{cf}(\alpha)$*, is the least cardinal λ such that there exists a subset $X \subseteq \alpha$ of cardinality λ which is cofinal in α.*

Definition 1.3.3. *A cardinal λ is called* **regular***, iff* $\mathrm{cf}(\lambda) = \lambda$*. If λ is not regular, then it is called* **singular***.*

Every successor cardinal, $\aleph_{\alpha+1}$, is regular, since $\aleph_\alpha \cdot \aleph_\alpha = \max\{\aleph_\alpha, \aleph_\alpha\} = \aleph_\alpha$. If $\aleph_\alpha$ is a limit cardinal and $\alpha > 0$, then $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.

Definition 1.3.4. *A closed, unbounded subset of a limit ordinal will be called a* **cub***.*

For any limit γ, the set γ itself is always a cub in γ.
We want to give another example for a cub.

Example 1.3.5 (Eklof/Mekler)**.** *Suppose that λ is a regular uncountable cardinal. Let $f : \lambda^n \to \lambda$ for some integer $n \geq 1$. We set*

$$C = \{\sigma \in \lambda : f(x) \in \sigma \textit{ for each } x \in \sigma^n\}$$

Clearly, C is closed in λ. To see that it is also unbounded in λ, we have to find for each $\alpha \in \lambda$ a $\beta > \alpha$ such that $\beta \in C$. Therefore define a sequence $(\alpha_i)_{i<\omega}$ starting with $\alpha_0 = \alpha$. Now if α_m is defined, then the cardinality of α_m^n is less than λ, therefore there exists $\alpha_m < \alpha_{m+1} < \lambda$ with $f(x) \in \alpha_{m+1}$ for all $x \in \alpha_m^n$. This provides an increasing sequence $(\alpha_i)_{i<\omega}$ and if we let $\beta = \sup\{\alpha_i : i < \omega\}$, then $\beta < \lambda$ and $\beta \in C$.

Lemma 1.3.6 (Eklof/Mekler)**.** *Let λ be a limit ordinal of cofinality $> \aleph_0$. Then the intersection of less than* $\mathrm{cf}(\lambda)$ *cubs in λ is again a cub in λ.*

Proof. Let $(C_\nu)_{\nu<\mu}$ be a collection of cubs in λ, where $\mu < \mathrm{cf}(\lambda)$. The intersection $C' = \bigcap_{\nu<\mu} C_\nu$ is evidently closed. To show that it is a cub in λ we have to find for each $\alpha \in \lambda$ a $\beta \in C'$ with $\beta > \alpha$. Since $\mathrm{cf}(\lambda)$ is uncountable, μ can be assumed to be an infinite cardinal. Then, since $\mu \cdot \mu = \mu$, there exists a function $f : \mu \to \mu$ so that $f^{-1}[\delta]$ is cofinal in μ for all $\delta \in \mu$.
We define $(\alpha_\nu)_{\nu<\mu}$ by induction:

$$\alpha_\nu = \min\{x \in C_{f(\nu)} : x > \alpha + \sup\{\alpha_\tau : \tau < \nu\}\}$$

Since $C_{f(\nu)}$ is unbounded and $\nu < \mathrm{cf}(\lambda)$, this is possible. Then we set

$$\beta = sup\{\alpha_\nu : \nu < \mu\}.$$

Obviously $\beta > \alpha$. Since $\mu < \mathrm{cf}(\lambda)$, $\beta \in \lambda$ and also $\beta \in C_\delta$ for all $\delta < \mu$ by choice of f, since C_δ is closed and $\beta = \sup\{\alpha_\nu : f(\nu) = \delta\}$. Thus $\beta \in C'$, too. ∎

Definition 1.3.7. *A subset $E \subseteq \lambda$ is called* **stationary** *in λ iff $E \cap C \neq \emptyset$ for all cubs $C \subseteq \lambda$.*

We may introduce an equivalence relation $\sim$ on $\mathcal{P}(\lambda)$ by writing $X \sim Y$ for $X, Y \in \mathcal{P}(\lambda)$ iff $X \cap C = Y \cap C$ for some cub $C \in \lambda$. With respect to this equivalence relation a set $X \in \mathcal{P}(\lambda)$ is a cub if and only if $X \sim \lambda$, i.e. X is in a natural sense large, in terms of [EM] "relatively large". This is why cubs are also called **sets of measure 1**. Similarly, $X \in \mathcal{P}(\lambda)$ fails to be stationary if and only if $X \sim \emptyset$. Such sets are called **thin** in λ or **sets of measure 0**. Stationary sets are also called **sets of non-zero measure**.

Let λ be a regular, uncountable cardinal from now on. For any infinite regular cardinal $\rho < \lambda$, the set of all $\alpha \in \lambda$ with $\mathrm{cf}(\alpha) = \rho$ is a stationary subset of λ. Lemma 1.3.6 provides that the intersection of less than λ cubs is again a cub (in λ). Therefore cubs cannot be partitioned into cubs, but stationary sets can be partitioned into stationary sets. The following fact will become useful in Section 4.3:

Lemma 1.3.8 (Solovay)**.** *If λ is a regular uncountable cardinal, then any stationary subset of λ can be partitioned into λ (disjoint) stationary sets.*

For the non-trivial proof see [Jech], Theorem 85.

We are now ready to introduce the Diamond Principle and follow [R2] and [GT].

During his analysis of Gödel's constructible universe V=L, Jensen discovered an enumeration principle on the set $\mathcal{P}(\omega)$, namely

(#) there exists a sequence $\{W_\alpha \subseteq \mathcal{P}(\omega) : \alpha < \omega_1\}$, such that for every subset $X \subseteq \omega_1$, there exists $\omega < \alpha < \omega_1$ such that $X \cap \alpha = W_\alpha$

and gave it the name of **diamond**, $\diamondsuit$.
This statement arised from an equivalent formulation of Cantor's continuum hypothesis (CH, $2^{\aleph_0} = \aleph_1$) in the context of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

The Diamond Principle $\diamondsuit$ in the now most popular form results from (#) by considering stationary sets and can be used to 'predict' initial segments $X \cap \alpha$ ($\alpha < \lambda$) of a given subset $X \subseteq \lambda$, independent of the choice of X.

Definition 1.3.9. *[$\diamondsuit_\lambda E$] For E a stationary subset of λ we denote by $\diamondsuit_\lambda E$ the following axiom:*
($\diamondsuit_\lambda E$) There exist $W_\alpha \subseteq \alpha, (\alpha \in E)$, such that the set $\{\alpha \in E : W_\alpha = X \cap \alpha\}$ is stationary in λ for any subset $X \subseteq \lambda$.

In [Jen], Jensen proved that $\diamondsuit_\lambda E$ holds in V=L for all regular uncountable λ and all stationary $E \subseteq \lambda$. The paper also introduces the very first complicated combinatorial object constructed with the help of the Diamond Principle: a Souslin tree, that is a tree of height κ where every branch and every antichain has cardinality $< \kappa$ (for κ an infinite cardinal).
There are further versions of the Diamond Principle which are more useful for applications. One makes use of the concept of λ-filtrations:

Definition 1.3.10. *Let M be a set of cardinality $\leq \lambda$. A* **λ-filtration of** *M is an ordered sequence $\{M_\alpha : \alpha < \lambda\}$ of subsets of M, such that for all $\alpha, \beta \in \lambda$:*

(i) $|M_\alpha| < \lambda$,

(ii) *if* $\alpha < \beta$ *then* $M_\alpha \subseteq M_\beta$,

(iii) *for* $\alpha \in \lambda$ *a limit ordinal,* $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$,

(iv) $M = \bigcup_{\alpha<\lambda} M_\alpha$.

Theorem 1.3.11. *Suppose that $\diamondsuit_\lambda E$ holds. If M is a set of cardinality λ and $\{M_\alpha : \alpha < \lambda\}$ a λ-filtration of M, then there exist subsets $N_\alpha \subseteq M_\alpha$, $\alpha \in E$ such that for each $X \subseteq M$ the set $\{\alpha \in E : N_\alpha = X \cap M_\alpha\}$ is stationary in λ.*

The next version is especially useful when realizing endomorphism rings and is the version we will use in the remaining part of this work. It makes use of the so-called Jensen functions which are, corresponding to the initial segments W_α in Definition 1.3.9, predicted by the Diamond.

Theorem 1.3.12. *Suppose that $\diamondsuit_\lambda E$ holds. For any two λ-filtrations $M = \bigcup_{\alpha<\lambda} M_\alpha$ and $N = \bigcup_{\alpha<\lambda} N_\alpha$ there are Jensen functions*

$$g_\alpha : M_\alpha \to N_\alpha (\alpha \in E)$$

such that, for any function $g : M \to N$,

$$\{\alpha \in E : g_\alpha = g \restriction M_\alpha\} \text{ is stationary in } \lambda.$$

The statement $\diamondsuit_\lambda E$ (for regular uncountable λ and stationary $E \subseteq \lambda$) is independent from ZFC but consistent with it. It was known that V=L is sufficient for the validity of $\diamondsuit_\lambda E$. Recently, Shelah proved that the Generalized Continuum Hypothesis, $2^\lambda = \lambda^+$ (GCH), suffices to imply $\diamondsuit_{\lambda^+} E$ for all stationary $E \subseteq \{\delta < \lambda^+ : cf(\delta) \neq cf(\lambda)\}$. Rinot [R1] gave a (surprisingly) short proof for Shelah's observation. He also gave a survey of related prediction principles and the validity of Jensen's diamond for successor cardinals [R2].

Assuming only ZFC, results similar to those achieved with the Diamond Principle can be realized using e.g. (Shelah's) Black Box. There are several prediction principles known under this name, as is the case with the Diamond, but we will not discuss these techniques in detail. It can be said that results achieved using a Diamond Principle can also be proved in ZFC through use of a Black Box if one is willing to accept a slightly weaker version of the result. The general strategy of both, the Diamond Principle and the Black Box, is to predict unwanted objects (e.g. homomorphisms) that can be "killed" with the help of a Step-Lemma.

Chapter 2

Infinitary equivalence – extending the theorem by Ulm

Using infinitary equivalence and α-height preserving isomorphisms, Barwise and Eklof [BE] proved the following generalization of the classical Ulm theorem for p-groups of any cardinality:

Theorem 2.0.13 (Barwise/Eklof). *Let G and H be p-groups and let δ be an ordinal such that*

(i) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all* $\alpha < \omega\delta$,

(ii) $l(G) < \omega\delta \Leftrightarrow \hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$,

then $G \equiv_\delta H$.
If δ is a limit ordinal, the converse direction also holds.

If now G and H are countable reduced groups, $u_p(\alpha, G) = u_p(\alpha, H)$ for all α and thus $G \equiv_\infty H$. Thus Theorem 2.0.13 implies Thoerem 1.1.3 via Theorem 1.2.2 and Theorem 1.2.3.

2.1 $\mathbb{Z}_{(p)}$-modules with nice decomposition bases

We now consider the natural question of extending the Barwise-Eklof theorem 2.0.13 to Warfield modules. The results of this section have already been published in [GLLS]. There is no reasonable class of modules including the Warfield modules that is closed under $L_{\omega_\alpha\omega}$- equivalence. However, in analogy to [BE] we are able to completely classify L-equivalence classes of certain $\mathbb{Z}_{(p)}$-modules if we assume they posses a nice decomposition basis. Again, as in the Barwise-Eklof generalization, we need additional numerical invariants deduced from the original (α-th) Warfield invariants and obtain the classical lemma for countable Warfield modules as a corollary.

We will start with a fact about possible extensions of α-height-preserving isomorphisms.

Lemma 2.1.1. *Let S and T be submodules of modules M and N, respectively, and let $f : S \to T$ be an α-height-preserving isomorphism where α is an ordinal or the symbol ∞. Suppose $x \in M$ and $y \in N$ such that x has order p^r modulo S, y has order p^r modulo T and $f(p^r x) = p^r y$ for some positive integer r. If either*

(i) $r = 1$, $|x|_M = |y|_N$, x is proper with respect to S and y is proper with respect to T, or

(ii) $|x|_M \geq \alpha$ and $|y|_N \geq \alpha$,

then f extends to an α-height-preserving isomorphism

$$\langle S, x\rangle \to \langle T, y\rangle$$

by sending x onto y.

Proof. All heights in this proof are computed in M and N, respectively. It is clear that for $s \in S$ and $n \in \mathbb{Z}$, $s + nx \mapsto f(s) + ny$ defines an isomorphism $f' : \langle S, x\rangle \to \langle T, y\rangle$.
First, suppose condition (i) of the lemma holds. Assume $|s+x| < \alpha$. If $|s| < \alpha$, then $|f(s)| = |s|$ and therefore $\min\{|f(s)|, |x|\} = \min\{|s|, |x|\}$, and $\min\{|s|, |x|\} = |s+x|$ since x is proper with respect to S. If $|s| \geq \alpha$, then $|s + x| = \min\{|s|, |x|\} < \alpha$ implies $|s + x| = |x| < \alpha$. Also $|f(s)| \geq \alpha$ (since $|s| \geq \alpha$), hence $\min\{|f(s)|, |x|\} = \min\{|f(s)|, |s + x|\} = |x| = |s + x|$. In either case we have

$$|f(s) + y| = \min\{|f(s)|, |y|\} = \min\{|f(s)|, |x|\} = |s + x|.$$

Now assume $|s + x| \geq \alpha$. Since $|s + x| = \min\{|s|, |x|\}$ we have also $|x|, |s| \geq \alpha$ and thus $|f(s)| \geq \alpha$, too. This implies $|f(s) + y| = \min\{|f(s)|, |y|\} = \min\{|f(s), |x|\} \geq \alpha$. Therefore f' is α-height-preserving.
Now suppose condition (ii) holds. Then $|nx|, |ny| \geq \alpha$, too. If $|s + nx| < \alpha$, then $|s + nx| = \min\{|s|, |nx|\}$ implies $|s| < \alpha$ and therefore $|f(s)| < \alpha$. Then

$|f(s)| = \min\{|f(s)|, |ny|\} = |f(s) + ny|$ and if $|s + nx| \geq \alpha$ we have $|s| \geq \alpha$, too, which yields $|f(s)| \geq \alpha$ and hence $|f(s) + ny| \geq \alpha$. This completes the proof. ∎

Notice that in the proof above the condition "$r = 1$" in case (i) is necessary: for $x = 1 \in M = \mathbb{Z}/p^2\mathbb{Z}$ and $y = (1, p) \in N = \mathbb{Z}/p\mathbb{Z} \times Z/p^3\mathbb{Z}$ we have $|x|_M = |y|_N$ but $|px|_M \neq |py|_N$, hence $f : \{0\} \to \{0\}$ cannot be extended to a height-preserving isomorphism $\langle x \rangle \to \langle y \rangle$.

In the proof of his classification theorem (1.1.10), Warfield uses the following fact

Lemma 2.1.2 (Hunter/Richman/Walker [HRW]). *If X and Y are decomposition bases of modules M and N, respectively and $w_M(\overline{\beta}) = w_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, then there exist subordinates X' of X and Y' of Y such that there is a height-preserving isomorphism from $\langle X' \rangle$ to $\langle Y' \rangle$.*

In fact, also the converse direction holds. The proof of Lemma 2.1.2 provides a bijection $\phi : X \to Y$ which secures equivalent Ulm sequences $u(x)$ and $u(\phi(x))$. X' and Y' then can be chosen as the collection of certain p-power multiples of x and $\phi(x)$, respectively.
A consequence of this is that if $A \subseteq X'$ and $B \subseteq Y'$ are finite sets and $f : \langle A \rangle \to \langle B \rangle$ is a height-preserving isomorphism with $f(A) = B$, then for every $x \in X'$ (resp. $y \in Y'$) there is $y \in Y'$ (resp. $x \in X'$) such that f extends to a height-preserving isomorphism $\langle A, x \rangle \to \langle B, y \rangle$.
We can generalize this result to the following extension lemma which provides helpful α-height-preserving isomorphisms with a back-and-forth property in our case:

Lemma 2.1.3. *Let α be a fixed ordinal or the symbol ∞. If M and N are modules with decomposition bases X and Y, respectively, such that $\tilde{w}^\alpha_M(\overline{\beta}) = \tilde{w}^\alpha_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, then there exist subordinates X' of X and Y' of Y satisfying the following:*
If $A \subseteq X'$ and $B \subseteq Y'$ are countable sets and $f : \langle A \rangle \to \langle B \rangle$ is an α-height-preserving isomorphism with $f(A) = B$, then for every $x \in X'$ (resp. $y \in Y'$) there exists $y \in Y'$ (resp. $x \in X'$) such that f extends to

$$f' : \langle A, x \rangle \to \langle B, y \rangle$$

by sending $x \mapsto y$ and f' is α-height-preserving, too.

Proof. Since $X = \dot{\bigcup}_{\overline{\beta}} X^{(\overline{\beta})}_\alpha$, $Y = \dot{\bigcup}_{\overline{\beta}} Y^{(\overline{\beta})}_\alpha$ and each set $X^{(\overline{\beta})}_\alpha, Y^{(\overline{\beta})}_\alpha$ has cardinality $w^\alpha_M(\overline{\beta})$, respectively $w^\alpha_N(\overline{\beta})$, we will define the subordinates X' and Y' as disjoint

unions $X' = \dot{\bigcup}_{\overline{\beta}} X'^{(\overline{\beta})}_{\alpha}$ and $Y' = \dot{\bigcup}_{\overline{\beta}} Y'^{(\overline{\beta})}_{\alpha}$. We may therefore assume $X = X^{(\overline{\beta})}_{\alpha}$ and $Y = Y^{(\overline{\beta})}_{\alpha}$ for some fixed Ulm sequence $\overline{\beta}$.
Suppose there exists an $i < \omega$ such that $\beta_i > \alpha$. Then $X = \{x \in X : p^k x \in p^{\alpha+1}M$ for some $k < \omega\}$, so there exist subordinates $X' \subseteq p^{\alpha+1}M$ of X and $Y' \subseteq p^{\alpha+1}N$ of Y satisfying the required property since $|X'| = |Y'|$ or both X' and Y' are infinite sets.

Now assume that $\infty \neq \beta_i \leq \alpha$ for all $i < \omega$. Then $X = X^{(\overline{\beta})}$ and $Y = Y^{(\overline{\beta})}$. For some Ulm sequence $\overline{\gamma}$ let $X[\overline{\gamma}] = \{x \in X : u_M(x) = \overline{\beta}\}$.

Case 1: If $|X| = |Y| \leq \aleph_0$, then we will construct by induction subordinates X' and Y' and a countable (maybe finite) sequence of Ulm sequences

$$\overline{\beta}_1 < \overline{\beta}_2 < \ldots < \overline{\beta}_n < \ldots$$

which are all equivalent to $\overline{\beta}$ such that $X' = \dot{\bigcup}_n X'[\overline{\beta}_n]$ and $Y' = \dot{\bigcup}_n Y'[\overline{\beta}_n]$. Moreover, we will ensure that $|X'[\overline{\beta}_n]| = |Y'[\overline{\beta}_n]| = 1$. Assume that we have constructed X' and Y' as claimed. Then the unique element $x_n \in X'[\overline{\beta}_n]$ can only be mapped onto the unique element $y_n \in Y'[\overline{\beta}_n]$ by any α-height-preserving map. Thus, given an α-height-preserving isomorphism $f : \langle A \rangle \to \langle B \rangle$ such that f induces a bijection between A and B, and given $x \in X'$ (resp. $y \in Y'$) there is a unique $y \in Y'$ (resp. $x \in X'$) such that f can be extended to an α-height-preserving isomorphism by mapping x onto y (resp. y onto x).
Let $X = \{x_1, x_2, x_3, \ldots\}$ and $Y = \{y_1, y_2, y_3, \ldots\}$ be arbitrary enumerations of X and Y, respectively. Inductively we choose integers n_i and m_i such that

$$u_M(p^{n_i} x_i) = u_N(p^{m_i} y_i)$$

and $u_M(p^{n_i} x_i) < u_M(p^{n_{i+1}} x_{i+1})$ for all i. The desired subordinates are then given by $X' = \{p^{n_i} x_i : i = 1, 2, \ldots\}$ and $Y' = \{p^{m_i} y_i : i = 1, 2, \ldots\}$.
If $i = 1$ then there are n_1 and m_1 such that $u_M(p^{n_1} x_1) = u_N(p^{m_1} y_1)$ since the Ulm sequences of x_1 and y_1 are equivalent to β. Put $\beta_1 = u_M(p^{n_1} x_1)$.
Now assume that $n_1, m_1, n_2, m_2, \ldots, n_i, m_i$ and $\beta_1, \beta_2, \ldots, \beta_i$ are constructed as claimed. Choose k such that $p^k x_{i+1}$ and $p^k y_{i+1}$ have Ulm sequences strictly bigger than $u_M(p^{n_i} x_i) = u_N(p^{m_i} y_i)$. Now there are l and s such that $u_M(p^{k+l} x_{i+1}) = u_N(p^{k+s} y_{i+1})$. Put $n_{i+1} = k + l$ and $m_{i+1} = k + s$ and $\beta_{i+1} = u_M(p^{n_{i+1}} x_{i+1})$. This finishes Case 1.

Case 2: Suppose we are not in Case 1. Then without loss of generality, we may assume $|X| \geq |Y| > \aleph_0$ since by assumption $\omega_M(\overline{\beta}) = \omega_N(\overline{\beta})$. The strategy is to proceed as in Case 1 with minor changes. By induction we will choose subordinates X' and Y' and a countably infinite sequence of Ulm sequences

$$\overline{\beta}_1 < \overline{\beta}_2 < \ldots < \overline{\beta}_n < \ldots$$

which are all equivalent to $\overline{\beta}$ such that $X' = \dot{\bigcup}_n X'[\overline{\beta}_n]$ and $Y' = \dot{\bigcup}_n Y'[\overline{\beta}_n]$. Moreover, this time, we will ensure that $X'[\overline{\beta}_n]$ and $Y'[\overline{\beta}_n]$ are both uncountable.

Assume for the moment that we can do this. As in Case 1 any α-height-preserving isomorphism can map elements from $X'[\overline{\beta}_n]$ only to elements from $Y'[\overline{\beta}_n]$ and vice versa (for any n). Thus, given an α-height-preserving isomorphism $f : \langle A \rangle \to \langle B \rangle$ such that f induces a bijection between A and B, the uncountability of $X'[\overline{\beta}_n]$ and $Y'[\overline{\beta}_n]$ ensures that $X'[\overline{\beta}_n] \backslash A$ and $Y'[\overline{\beta}_n] \backslash B$ are still uncountable. Therefore, for any $x \in X' \setminus A$ (resp. $y \in Y' \setminus B$) there is some $y \in Y' \setminus B$ (resp. $x \in X' \setminus A$) such that f can be extended to an α-height-preserving isomorphism by mapping x onto y (resp. y onto x).

It remains to show that we can carry on the induction. First note that the set $\{p^n \overline{\beta} : n < \omega\}$ is countable. Hence (after replacing X and Y by suitable subordinates) we have

$$X = \dot{\bigcup}_{i \in I_X} X[\overline{\eta}_i] \text{ and } Y = \dot{\bigcup}_{i \in I_Y} Y[\overline{\mu}_i]$$

for some $I_X, I_Y \subseteq \omega$ and Ulm sequences $\overline{\eta}_i$, $\overline{\mu}_i$ equivalent to $\overline{\beta}$. Since X and Y are uncountable there must be $\overline{\eta}_k$ and $\overline{\mu}_l$ such that $X[\overline{\eta}_k]$ and $Y[\overline{\mu}_l]$ are uncountable as well. Therefore we can write $X[\overline{\eta}_k] = \dot{\bigcup}_{i<\omega} X_i$ where each set X_i is uncountable. Now let $i < \omega$. For every $i \in I_X$ there exist $n_i, m_i < \omega$ such that $p^{n_i}\overline{\eta}_i = p^{m_i}\overline{\eta}_k$, and we define

$$X_i^* = \{p^{n_i}x : x \in X[\overline{\eta}_i]\} \cup \{p^{m_i}x : x \in X_i\}$$

and replace X_k^* by the set $\{p^{m_k}x : x \in X_k\}$. If $i \notin I_X$ we let $m_i = 0$ and set $X_i^* = X_i$. Then it follows that $X^* = \dot{\bigcup}_{i<\omega} X_i^*$ is a subordinate of X such that each X_i^* is uncountable and $X_i^* = X_i^*[p^{m_i}\overline{\eta}_k]$. Similarly, we obtain a subordinate Y^* of Y so that $Y^* = \dot{\bigcup}_{i<\omega} Y_i^*$ where each Y_i^* is uncountable and $Y_i^* = Y_i^*[p^{r_i}\overline{\mu}_l]$ for some r_i. It is then straightforward to see, using similar arguments as in Case 1 that we may pass to subordinates $X' = \dot{\bigcup}_{n<\omega} X'[\overline{\beta}_n]$ and $Y' = \dot{\bigcup}_{n<\omega} Y'[\overline{\beta}_n]$ satisfying

$$\overline{\beta}_1 < \overline{\beta}_2 < \ldots < \overline{\beta}_n < \ldots .$$

This finishes the proof. ∎

In the above proof we see, that if $|X_\alpha^{(\overline{\beta})}| = |Y_\alpha^{(\overline{\beta})}|$ for all Ulm sequences $\overline{\beta}$, f' can be globally extended to an α-height-preserving isomorphism $\langle X' \rangle \to \langle Y' \rangle$. Moreover, the lemma can be generalized to adding countably infinitely many elements to the domain or range of f, thus receiving an α-height-preserving isomorphism

$$f' : \langle A, a_0, a_1, ... \rangle \to \langle B, b_0, b_1, ... \rangle.$$

Also, the property of f (and f') being α-height-preserving can be replaced by δ-height-preserving for any ordinal $\delta < \alpha$.
As a corollary we receive:

Corollary 2.1.4. *Let M and N be modules with decomposition bases X and Y, respectively, such that $\tilde{w}_M^\delta(\overline{\beta}) = \tilde{w}_N^\delta(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where δ is some fixed ordinal or the symbol ∞. Then there exist subordinates X' and Y' of X and Y such that $\langle X' \rangle \equiv_\delta^h \langle Y' \rangle$.*

Proof. Let X' and Y' be the subordinates of X and Y obtained in Lemma 2.1.3. For every ordinal α, let I_α be the set I of all δ-height-preserving isomorphisms

$$f : \langle A \rangle \to \langle B \rangle$$

where A and B are finite subsets of X' and Y' such that $f(A) = B$. Then Lemma 2.1.3 shows that for L_δ and the modules $\langle X' \rangle$ and $\langle Y' \rangle$, the sets I_α satisfy condition (ii) from Theorem 1.2.1, hence $\langle X' \rangle \equiv_\delta^h \langle Y' \rangle$. To see this, let $f : \langle A \rangle \to \langle B \rangle$ be a map in I such that A and B are finite subsets of X' and Y' and $f(A) = B$, and let $x \in \langle X' \rangle$. Then $x = \sum_{i=1}^{m} n_i x_i$ for some $x_i \in X'$ and $n_i \in \mathbb{Z}_{(p)}$. Thus, by Lemma 2.1.3, there is an extension $f' \in I$ of f which maps $\langle A, x_1, \ldots, x_m \rangle$ onto $\langle B, y_1, \ldots, y_m \rangle$ for some $y_i \in Y'$. Clearly, $x \in \mathrm{dom}(f')$. If $y \in \langle Y' \rangle$, then by symmetry f extends to a map $f^* \in I$ with $y \in \mathrm{im}(f^*)$. ∎

Since from now on we will use α-height-preserving isomorphisms with special properties, we will need a further

Definition 2.1.5. *For α an ordinal or the symbol ∞ and modules M and N with decomposition bases X, Y, resp., we set*

$$\mathrm{prs}_\alpha^{X,Y} = \{ f : E \to F :$$

(i) f is an α-height-preserving isomorphism,

(ii) E, F are finitely generated submodules of M, N, resp.,

(iii) *there exist generators* $x_1, ..., x_n$ *of* E, $y_1, ..., y_n$ *of* F, *resp. and a positive integer* $k < \omega$ *such that* $x_1, ..., x_k \in X \cup \{0\}, y_1, ..., y_k \in Y \cup \{0\}$,

(iv) $\forall i = 1, ..., n : f(x_i) = y_i$,

(v) *if* $k < n$, *then for* $i = k, ..., n-1$ *the submodules* $E_i = \langle x_1, ..., x_i \rangle$ *and* $F_i = \langle y_1, ..., y_i \rangle$ *fulfill*

(a) $|E_{i+1}/E_i| = |F_{i+1}/F_i| = p$,

(b) *if* $|x_{i+1}|_M < \alpha$ *or* $|y_{i+1}|_N < \alpha$, *then* x_{i+1} *is proper with respect to* $\langle X, E_i \rangle$ *and* y_{i+1} *is proper with respect to* $\langle Y, F_i \rangle\}$

Notice, that in (v)(b) the fact that $|x_{i+1}| < \alpha$ or $|y_{i+1}| < \alpha$ always implies both inequalities, since f is α-height-preserving and maps x_{i+1} onto y_{i+1} with the consequence that $|x_{i+1}|_M = |y_{i+1}|_N$.
It is clear that for two modules M and N with decomposition bases X, Y, resp.

$$\mathrm{prs}_\beta^{X,Y} \subseteq \mathrm{prs}_\alpha^{X,Y}, \text{ for ordinals } \alpha < \beta.$$

Also, each $\mathrm{prs}_\alpha^{X,Y}$ is non-empty since it contains the map $0 \mapsto 0$.
Now, Lemma 2.1.3 can be further extended to

Lemma 2.1.6. *Let* M *and* N *be modules with decomposition bases* X *and* Y, *resp., such that* $\tilde{w}_M^\alpha(\overline{\beta}) = \tilde{w}_N^\alpha(\overline{\beta})$ *for all Ulm sequences* $\overline{\beta}$ *where* α *is some fixed ordinal or the symbol* ∞. *Let* X' *and* Y' *be subordinates of* X *and* Y *as in Lemma2.1.3 and assume that* $f \in \mathrm{prs}_\alpha^{X',Y'}$. *Then for every* $x \in X'$ *(*$y \in Y'$*, resp.) there is* $y \in Y'$ *(*$x \in X'$*, resp.) such that* f *extends to a map*

$$f' : \langle E, x \rangle \to \langle F, y \rangle$$

with $f' \in \mathrm{prs}_\alpha^{X',Y'}$ *by sending* x *onto* y.

Proof. We prove the assertion by induction on $m = n - k$. The case $m = 0$ was shown in Lemma 2.1.3, so we assume that the claim is true for $m \geq 0$ and suppose

$$f : E_{k+m+1} = \langle x_1, \ldots, x_{k+m+1} \rangle \to F_{k+m+1} = \langle y_1, \ldots, y_{k+m+1} \rangle$$

is a map in $\mathrm{prs}_\alpha^{X',Y'}$. Let $x \in X'$ ($y \in Y'$, resp.). By induction hypothesis, $f \restriction_{E_{k+m}} : E_{k+m} \to F_{k+m}$ extends to a map

$$f^* : \langle E_{k+m}, x \rangle \to \langle F_{k+m}, y \rangle$$

in $\mathrm{prs}_\alpha^{X',Y'}$ with $f^*(x) = y \in Y'$ ($f^{*-1}(y) = x \in X'$, resp.). In case $|x_{k+m+1}| = |y_{k+m+1}| < \alpha$ the element x_{k+m+1} is proper with respect to $\langle E_{k+m}, x\rangle$ and has order p modulo $\langle E_{k+m}, x\rangle$, and the same is true for y_{k+m+1} and $\langle F_{k+m}, y\rangle$. With Lemma 2.1.1 the map f^* extends to an α-height-preserving isomorphism

$$f' : \langle E_{k+m}, x, x_{k+m+1}\rangle = \langle E_{k+m+1}, x\rangle \to \langle F_{k+m}, y, y_{k+m+1}\rangle = \langle F_{k+m+1}, y\rangle$$

by sending x_{k+m+1} onto y_{k+m+1}, hence f' extends f. It is clear that x_{i+1} has order p modulo $\langle E_i, x\rangle$ and y_{i+1} has order p modulo $\langle F_i, x\rangle$ for $i = k, \ldots, k+m$. Therefore, $f' \in \mathrm{prs}_\alpha^{X',Y'}$ and the induction is complete. ∎

Lemma 2.1.7. *Let M and N be modules with nice decomposition bases X and Y, respectively, such that $\tilde{w}_M^\alpha(\overline{\beta}) = \tilde{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞. Let X' and Y' be subordinates of X and Y as in Lemma 2.1.3 and assume that $f : E \to F$ is in $\mathrm{prs}_\alpha^{X',Y'}$. If $x \in M \setminus E$ and $px \in E$, then f extends to a map $f^* : E^* \to F^*$ in $\mathrm{prs}_\alpha^{X',Y'}$ for which there is an element $x^* \in M$ that is proper with respect to $\langle X', E^*\rangle$ and has order p modulo E^* such that $\langle E^*, x\rangle = \langle E^*, x^*\rangle$ and $\langle X', E^*\rangle = \langle X', E\rangle$.*

Proof. Since X' is a nice decomposition basis for M and finite extensions of nice submodules are nice, $\langle X', E\rangle$ is a nice submodule of M and contains therefore an element a such that $x^* = x + a$ is proper with respect to $\langle X', E\rangle$. There are elements $x_1^*, \ldots, x_s^* \in X'$ such that $E^* = \langle E, x_1^*, \ldots, x_s^*\rangle$ contains both a and px. By Lemma 2.1.6, f has an extension $f^* : E^* \to F^*$ in $\mathrm{prs}_\alpha^{X',Y'}$. Then $\langle E^*, x\rangle = \langle E^*, x^*\rangle$, $\langle X', E^*\rangle = \langle X', E\rangle$, and x^* is proper with respect to E^* and has order p modulo E^*. ∎

The next result establishes a connection between proper elements of a module and the relative Ulm-Kaplansky invariants:

Lemma 2.1.8. *Let N be a submodule of a module M and suppose $x \in M$ is proper with respect to N and has height $\beta \neq \infty$. Then:*

(i) If $|px| > \beta + 1$, there is an element $y \in p^{\beta+1}M$ such that $x - y \in p^\beta M[p]$ and $x - y \notin N + p^{\beta+1}M$.

(ii) If x has order p modulo N, then $u_p^N(\beta, M) = u_p^{\langle N,x\rangle}(\beta, M) + 1$ and $u_p^N(\alpha, M) = u_p^{\langle N,x\rangle}(\alpha, M)$ if $\alpha \neq \beta$.

Proof. (i): If $|px| > \beta+1$, then $px = py$ for some $y \in p^{\beta+1}M$, hence $x-y \in p^\beta M[p]$. For any $a \in N$ we have

$$|x - y + a| = \min\{|x + a|, |y|\} = |x + a| \leq \beta$$

therefore $x - y \notin N + p^{\beta+1}M$.

(ii): Let $N_1 = \langle N, x\rangle$. If the coset $x + N$ contains an element x' with $|x'| = \beta$ and $|px'| > \beta + 1$, we replace x by x'.

Case I: $|px| > \beta + 1$. By (i), there exists an element $y \in p^{\beta+1}M$ such that $x - y \in p^\beta M[p]$ and $x - y \notin N + p^{\beta+1}M$. Then

$$x - y \notin N(\beta) = p^\beta M[p] \cap (N + p^{\beta+1}M)$$

and therefore $N_1(\beta)/N(\beta) = \langle N, x - y\rangle(\beta)/N(\beta) \cong \mathbb{Z}_{(p)}$. But then

$$\dim(p^\beta M[p]/N(\beta)) = \dim(\frac{p^\beta M[p]/N(\beta)}{N_1(\beta)/N(\beta)}) + \dim(N_1(\beta)/N(\beta))$$

shows that $u_p^N(\beta, M) = u_p^{N_1}(\beta, M) + 1$. If $\alpha < \beta$ we have $x \in p^\beta M \subseteq p^{\alpha+1}M$ which implies $N_1(\alpha) = N(\alpha)$. Now assume that for some $\alpha > \beta$ there exists an element in $N_1(\alpha) \setminus N(\alpha)$. Then we can find elements $a \in N$ and $g \in p^{\alpha+1}M$ such that $a + kx + g \in p^\alpha M[p]$ for some positive integer $k < p$. But then there are integers m and n such that $mk + np = 1$, hence $ma + x - npx \in p^\alpha M$. Since x is proper with respect to N, this yields

$$\alpha \leq |ma + x - npx| \leq |x| = \beta,$$

a contradiction. It follows that $N_1(\alpha) = N(\alpha)$. Therefore $u_p^A(\alpha, M) = u_p^{A_1}(\alpha, M)$ if $\alpha \neq \beta$.

Case II: $|px| = \beta + 1$. Assume $N_1(\beta) \setminus N(\beta)$ is non-empty. As before, we can find a positive integer $k < p$ and elements $a \in N$ and $g \in p^{\beta+1}M$ such that $a + kx + g \in p^\beta M[p]$, so there are integers m and n satisfying $ma + x - npx + mg \in p^\beta M[p]$. But then $\beta \leq |ma + x| \leq |x|$, thus $ma + x$ has height β. Moreover,

$$|p(ma + x)| = |np^2x - mpg| > \beta + 1,$$

so we can replace x by $ma + x$ and are in Case I. If $\alpha < \beta$ or $\alpha > \beta$, we conclude that $N_1(\alpha) = N(\alpha)$ as in the previous case, hence we obtain $u_p^A(\alpha, M) = u_p^{A_1}(\alpha, M)$ if $\alpha \neq \beta$. ∎

Lemma 2.1.9. *Let M and N be modules with nice decomposition bases X and Y, resp., such that $\tilde{w}_M^\alpha(\overline{\beta}) = \tilde{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞, and let X' and Y' be subordinates of X and Y as in Lemma 2.1.3. Assume that $f : E \to F$ is a map in $\mathrm{prs}_\alpha^{X',Y'}$. Let $A = \langle X' \rangle$ and $B = \langle Y' \rangle$ and suppose γ is an ordinal $< \alpha$. Then there is $m < \omega$ such that*

$$u_p^A(\gamma, M) = u_p^{A+E}(\gamma, M) + m \text{ and } u_p^B(\gamma, N) = u_p^{B+F}(\gamma, N) + m.$$

In particular: If $\hat{u}_p^A(\gamma, M) = \hat{u}_p^B(\gamma, N)$, then $\hat{u}_p^{A+E}(\gamma, M) = \hat{u}_p^{B+F}(\gamma, N)$.

Proof. If $E \subseteq A$, then $F \subseteq B$ and there is nothing to show. Now assume that $E \not\subseteq A$. Then $F \not\subseteq B$ and we write $E = \langle x_1, \dots, x_k, x_{k+1}, \dots, x_n \rangle$ and $F = \langle y_1, \dots, y_k, y_{k+1}, \dots, y_n \rangle$ according to Definition 2.1.5. Let $A_i = \langle A, x_1, \dots, x_i \rangle$ and $B_i = \langle B, y_1, \dots, y_i \rangle$. Then every coset $x_{i+1} + A_i$ has an element x'_{i+1} of maximal height since A_i is nice in M, so there are $a_1, \dots, a_s \in X'$ such that

$$E_i = \langle x_1, \dots, x_i, a_1, \dots, a_s \rangle$$

contains $x_{i+1} - x'_{i+1}$ for all $i = k, \dots, n-1$. By Lemma 2.1.6, there are elements $b_1, \dots, b_s \in Y'$ such that f extends to a map

$$f' : \langle E, a_1, \dots, a_s \rangle \to \langle F, b_1, \dots, b_s \rangle$$

in $\mathrm{prs}_\alpha^{X',Y'}$ by sending each a_i onto b_i. Letting $F_i = \langle y_1, \dots, y_i, b_1, \dots, b_s \rangle$ we have

$$\langle E_i, x'_{i+1} \rangle = \langle E_i, x_{i+1} \rangle = E_{i+1} \text{ and } \langle F_i, f'(x'_{i+1}) \rangle = \langle F_i, y_{i+1} \rangle = F_{i+1}$$

and x'_{i+1} is proper with respect to E_i for all $i = k, \dots, n-1$.
Suppose $|x'_{i+1}|_M \leq \gamma$. Since $\gamma < \alpha$ and $f'(E_{i+1}) = F_{i+1}$, we have $|f'(x'_{i+1})|_N = |x'_{i+1}|_M$ and $f'(x'_{i+1})$ is proper with respect to F_i. If $|x'_{i+1}|_M > \gamma$, then $|f'(x'_{i+1})|_N > \gamma$ in which case $E_i + p^{\gamma+1}M = \langle E_i, x'_{i+1} \rangle + p^{\gamma+1}M$ and $F_i + p^{\gamma+1}N = \langle F_i, f'(x'_{i+1}) \rangle + p^{\gamma+1}N$, therefore

$$u_p^{E_i}(\gamma, M) = u_p^{\langle E_i, x'_{i+1} \rangle}(\gamma, M) \text{ and } u_p^{F_i}(\gamma, N) = u_p^{\langle F_i, f'(x'_{i+1}) \rangle}(\gamma, N).$$

Now we apply Lemma 2.1.8(ii) repeatedly and obtain $u_p^A(\gamma, M) = u_p^{A+E}(\gamma, M) + m$ and $u_p^B(\gamma, N) = u_p^{B+F}(\gamma, N) + m$ for some $0 \leq m \leq n - k$. ∎

Next we prove another generalization of a result by Barwise and Eklof. They showed for p-groups G and H([BE], Lemma A.3.2.):

Lemma 2.1.10 (Barwise/Eklof)**.** *If ν is an ordinal with $\omega(\nu+1) \leq l(G)$ and $\omega(\nu+1) \leq l(H)$ and $f : S \to T$, where S and T are finitely generated subgroups of G and H resp., is in $\mathrm{prs}^{G,H}_{\omega(\nu+1)}$ then the following holds:*
For each $a \in G$ with $|a|_G \geq \omega\nu$ there exists $b \in H$ and an extension f' of f with $f' : \langle S, a\rangle \to \langle T, b\rangle$ and $f' \in \mathrm{prs}^{G,H}_{\omega\nu}$.

which can be generalized to

Lemma 2.1.11. *Let M and N be modules with decomposition bases X and Y, respectively, and let ν be an ordinal such that $l(tN) \geq \omega(\nu+1)$. Let $f : E \to F$ be a map in $\mathrm{prs}^{X,Y}_{\omega(\nu+1)}$. If $x \in M \setminus E$ with $p^{r+1}x \in E$ for some $r < \omega$ and $|x| \geq \omega\nu$, then f extends to a map*

$$f' : \langle E, x\rangle \to \langle F, y\rangle$$

in $\mathrm{prs}^{X,Y}_{\omega\nu}$ by sending x onto y.

Proof. Let r be the smallest integer ≥ 0 such that $p^{r+1}x \in E$. Then $|p^{r+1}x| \geq \omega\nu + r + 1$, hence $|f(p^{r+1}x)| \geq \omega\nu + r + 1$, so we can write $f(p^{r+1}x) = p^{r+1}y_0$ for some $y_0 \in p^{\omega\nu}N$. If $p^r y_0 \notin F$ we let $y = y_0$. Now suppose $p^r y_0 \in F$ and let B be a basic subgroup of $p^{\omega\nu+r}(tN)$ (see [F] Vol. **I**, p. 139). It is clear that $B[p] \subseteq (p^{\omega\nu+r}N)[p]$. Assume the latter group is finite. Then $B[p]$ is finite, thus B is finite and we can write $p^{\omega\nu+r}(tN) = B \oplus D$ for some divisible group D (see [F] Theorem 27.5), hence $l(tN) < \omega(\nu+1)$, a contradiction. Consequently, $(p^{\omega\nu+r}N)[p]$ is an infinite group and therefore $p^{\omega\nu+r}N[p] \not\subseteq F[p]$. Then there is $y_1 \in p^{\omega\nu}N$ such that $p^r y_1 \notin F$ and $p^{r+1}y_1 = 0$. Letting $y = y_0 + y_1$ we obtain

$$|y| \geq \omega\nu,\ p^r y \notin F \text{ and } p^{r+1}y = f(p^{r+1}x).$$

By Lemma 2.1.1, $f \in \mathrm{prs}^{X,Y}_{\omega\nu}$ extends to an $\omega\nu$-height-preserving isomorphism

$$f' : \langle E, x\rangle \to \langle F, y\rangle$$

by sending x onto y. Since $f' \in \mathrm{prs}^{X,Y}_{\omega\nu}$, the proof is complete. ∎

The following result by Warfield will be needed (see [War, Lemma 5.1]).

Lemma 2.1.12 (Warfield)**.** *Let M be a module possessing a decomposition basis X. Then X has a subordinate X' such that for every ordinal α, $u_p^{M,\langle X'\rangle}(\alpha)$ is finite or $u_p(\alpha, M) = u_p^{\langle X'\rangle}(\alpha, M)$.*

According to [War], a subordinate fulfilling the properties stated in Lemma 2.1.12 is called **lower decomposition basis**. Note that $u_p^{M,\langle X'\rangle}(\alpha)$ corresponds to the dimension of $I_\alpha([X'])$ in Warfield's notation.
For the next lemma observe that equation (1.1) also holds for the $\hat{}$-generalized cardinals.

Lemma 2.1.13. *Let M and N be modules with nice decomposition bases X and Y, respectively, and let α be a limit ordinal or the symbol ∞. Suppose $\hat{u}_p(\delta, M) = \hat{u}_p(\delta, N)$ for all ordinals $\delta < \alpha$ and $\tilde{w}_M^\alpha(\overline{\beta}) = \tilde{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$. Then there exist subordinates X' and Y' of X and Y, respectively, such that every map $f : E \to F$ in $\mathrm{prs}_\alpha^{X',Y'}$ satisfies the following conditions:*

(i) If $x \in M \setminus E$ with $px \in E$ and $\infty \neq \sup\{|x + a| : a \in \langle X', E\rangle\} < \alpha$, then f extends to $f' \in \mathrm{prs}_\alpha^{X',Y'}$ such that $x \in \mathrm{dom}(f')$.

(ii) If $y \in N \setminus F$ with $py \in F$ and $\infty \neq \sup\{|y + b| : b \in \langle Y', F\rangle\} < \alpha$, then f extends to $f' \in \mathrm{prs}_\alpha^{X',Y'}$ such that $y \in \mathrm{im}(f')$.

Proof. First, we show that there exist subordinates X' and Y' of X and Y, respectively, such that $\langle X'\rangle \equiv_\alpha^h \langle Y'\rangle$ and $\hat{u}_p^{\langle X'\rangle}(\delta, M) = \hat{u}_p^{\langle Y'\rangle}(\delta, N)$ for all ordinals $\delta < \alpha$. Consider the equation

$$\hat{u}_p(\delta, M) = \hat{u}_p^{M,\langle X\rangle}(\delta) + \hat{u}_p^{\langle X\rangle}(\delta, M)$$

and note that $u_p^{M,\langle X\rangle}(\delta)$ is the number of elements in X whose Ulm sequences have a gap at δ ([War] p. 341). By Lemma 2.1.12, we can find subordinates X^* of X and Y^* of Y such that for every ordinal $\delta < \alpha$ the following is true: If both $u_p^{M,\langle X^*\rangle}(\delta, M)$ and $u_p^{N,\langle Y^*\rangle}(\delta)$ are infinite, then

$$\hat{u}_p^{\langle X^*\rangle}(\delta, M) = \hat{u}_p^{\langle Y^*\rangle}(\delta, N).$$

Notice that this statement remains true after replacing X^* and Y^* by subordinates X' and Y' as in Lemma 2.1.3 Then for every ordinal $\delta < \alpha$, X' and Y' have the same number of elements whose Ulm sequences have a gap at δ whenever $u_p^{M,\langle X'\rangle}(\delta)$ or $u_p^{N,\langle Y'\rangle}(\delta)$ is finite. Setting $A = \langle X'\rangle$ and $B = \langle Y'\rangle$, we obtain $A \equiv_\alpha^h B$ and

$$\hat{u}_p^A(\delta, M) = \hat{u}_p^B(\delta, N)$$

for all ordinals $\delta < \alpha$. The modules $A + E$ and $B + F$ are nice in M and N, respectively, hence in (i) and (ii) of the lemma, "sup" can be replaced by "max".

Let $f : E = \langle x_1, \ldots, x_n \rangle \to F = \langle y_1, \ldots, y_n \rangle$ be a map in $\mathrm{prs}_\alpha = \mathrm{prs}_\alpha^{X',Y'}$ and suppose x is an element of $M \setminus E$ with $px \in E$ and

$$\infty \neq \beta = \max\{|x + a| : a \in A + E\} < \alpha.$$

If $x \in A+E$, then Lemma 2.1.6 yields an extension $f' \in \mathrm{prs}_\alpha$ of f with $x \in \mathrm{dom}(f')$. Now suppose $x \notin A + E$. Then x has order p modulo $A + E$. By Lemma 2.1.7, f extends to a map $f^* : E^* \to F^*$ in prs_α for which there is an element $x^* \in M$ such that x^* is proper with respect to $A + E^*$ and has order p modulo E^*, $\langle E^*, x \rangle = \langle E^*, x^* \rangle$ and $A + E^* = A + E$. If possible, we choose $f^* \in \mathrm{prs}_\alpha$ and x^* so that $|px^*| > |x^*| + 1$.

To simplify notation we now write x for this element x^* and $f : E \to F$ for the function $f^* : E^* \to F^*$. Then $\infty \neq |x| = \beta < \alpha$ and $px \in p^{\beta+1}M$ which implies $f(px) \in p^{\beta+1}N$.

Case I: $|px| > \beta + 1$. By Lemma 2.1.8(i), there exists an element $x' \in p^{\beta+1}M$ such that $x - x' \in p^\beta M[p]$ and $x - x' \notin A + E + p^{\beta+1}M$, thus $\hat{u}_p^{A+E}(\beta, M) \neq 0$. Since $\beta < \alpha$, we can apply Lemma 2.1.9 and obtain $\hat{u}_p^{B+F}(\beta, N) \neq 0$. Then there exists an element $z \in p^\beta N[p]$ with $z \notin B + F + p^{\beta+1}N$, hence $|z| = \beta$ and $|z + h| \leq \beta$ for all $h \in B + F$. Since α is a limit ordinal or $\alpha = \infty$ we have $\beta + 2 < \alpha$, so there is an element $w \in p^{\beta+1}N$ such that $f(px) = pw$. Then

$$|w + z + h| = \min\{|w|, |z + h|\} \leq \beta = |w + z|$$

for all $h \in B + F$, hence $w + z$ is proper with respect to $B + F$. Notice that $w + z \notin B + F$, otherwise $|w| = |z - (w + z)| \leq \beta$ by the previous observation, contradicting the fact that $|w| \geq \beta + 1$. By Lemma 2.1.1 we can extend f to an α-height-preserving isomorphism

$$f' : \langle E, x \rangle \to \langle F, w + z \rangle$$

with $f'(x) = w + z$. Clearly, $f' \in \mathrm{prs}_\alpha$.

Case II: $|px| = \beta + 1$. Then $\beta + 1 \neq \infty$, therefore $f(px) = pw$ for some $w \in N$ with $|w| = \beta$. Suppose there exists an element $z \in B + F$ such that $|w + z| \geq \beta + 1$. Then $|z| = \beta$ and there are elements $y_{n+1}, \ldots, y_m \in Y'$ such that $z \in F' = \langle F, y_{n+1}, \ldots, y_m \rangle$. By Lemma 2.1.6, there are $x_{n+1}, \ldots, x_m \in X'$ so that f extends to a map

$$\overline{f} : E' = \langle E, x_{n+1}, \ldots, x_m \rangle \to F'$$

in prs_α. Letting $c = \overline{f}^{-1}(z)$ we have $|x+c| \geq \beta = \max\{|x+a| : a \in A+E\}$. Notice that $c \in A+E' = A+E$, therefore $x+c$ is proper with respect to $A+E'$. Since $x+c$ has order p modulo E' and $|p(w+z)| > \beta+1$ yields $|p(x+c)| > \beta+1$ we can replace x by $x+c$ and are in Case I. Therefore we may assume that $|w+z| \leq \beta = |w|$ for all $z \in B+F$, i.e., w is proper with respect to $B+F$. By Lemma 2.1.1, f extends to an α-height-preserving isomorphism

$$f' : \langle E, x\rangle \to \langle F, w\rangle$$

by mapping x to w. Again, $f' \in prs_\alpha$. The second assertion follows immediately, hence the proof is complete. ∎

Theorem 2.1.14. *Let M and N be reduced modules with nice decomposition bases. If $\hat{u}_p(\delta, M) = \hat{u}_p(\delta, N)$ for all ordinals δ and $\tilde{w}_M(\overline{\beta}) = \tilde{w}_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, then $M \equiv_\infty N$.*

Proof. Let $\mathrm{prs}_\infty^{X',Y'}$ be as in Lemma 2.1.13 and put $I_\delta = \mathrm{prs}_\infty^{X',Y'}$ for every ordinal δ. Since M and N are reduced, we can apply Lemma 2.1.13. Then Theorem 1.2.1 shows that $M \equiv_\infty N$. ∎

The following fact will be useful:

Lemma 2.1.15. *Let M be a module. If $l(tM) = \alpha$, then $l(M) \leq \alpha + \omega$.*

We are now able to prove our main result of this section:

Theorem 2.1.16. *Let M and N be modules with nice decomposition bases and let δ be an ordinal. Suppose*

(i) $\hat{u}_p(\alpha, M) = \hat{u}_p(\alpha, N)$ for all ordinals $\alpha < \omega\delta$;

(ii) if $l(tM) < \omega\delta$, then $\hat{u}_p(\infty, M) = \hat{u}_p(\infty, N)$;

(iii) $\tilde{w}_M^{\omega\nu}(\overline{\beta}) = \tilde{w}_N^{\omega\nu}(\overline{\beta})$ for all ordinals $\nu \leq \delta$ and all Ulm sequences $\overline{\beta}$.

Then $M \equiv_\delta N$. The converse holds if $\delta = \omega\gamma$ where γ is a limit ordinal and if $\tilde{w}_M^{\omega\delta}(\overline{\beta}) = \tilde{w}_N^{\omega\delta}(\overline{\beta})$ and $(w_M^{\omega\nu}(\overline{\beta}) \leq \aleph_0 \Leftrightarrow w_N^{\omega\nu}(\overline{\beta}) \leq \aleph_0)$ for all ordinals $\nu < \delta$ and all Ulm sequences $\overline{\beta}$.

Proof. Let M and N be modules with nice decomposition bases X and Y, respectively, satisfying (i)-(iii).

Case I: $l(tM) < \omega\delta$ (this implies $l(tN) < \omega\delta$ by (i)). We show that in this case $M \equiv_\infty N$. By Lemma 2.1.15, there is $\nu < \delta$ such that $l(M) \leq \omega(\nu+1)$, hence $w_M(\overline{\beta}) = w_M^{\omega(\nu+1)}(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, and a corresponding statement is true for N. Now write $\overline{\infty} = (\infty, \infty, \ldots)$,

$$X = X_\infty \dot\cup X_r \text{ and } Y = Y_\infty \dot\cup Y_r$$

with $X_\infty = \{x \in X : u(x) \sim \overline{\infty}\}$ and $Y_\infty = \{y \in Y : u(y) \sim \overline{\infty}\}$. Then

$$w_M(\overline{\infty}) = |X_\infty| \text{ and } w_N(\overline{\infty}) = |Y_\infty|,$$

so by our remarks on direct sums in Section 1.2 we obtain

$$p^\infty M \cong \bigoplus_{w_M(\overline{\infty})} \mathbb{Q} \oplus \bigoplus_{u_p(\infty,M)} \mathbb{Z}(p^\infty) \equiv_\infty \bigoplus_{w_N(\overline{\infty})} \mathbb{Q} \oplus \bigoplus_{u_p(\infty,N)} \mathbb{Z}(p^\infty) \cong p^\infty N.$$

By [F, Theorem 21.2], there are reduced modules M_r and N_r such that $M = p^\infty M \oplus M_r$, $N = p^\infty N \oplus N_r$, $\langle X_r \rangle \subseteq M_r$, and $\langle Y_r \rangle \subseteq N_r$. Then X_r and Y_r are nice decomposition bases for M_r and N_r, respectively. We have $w_{M_r}(\overline{\beta}) = 0 = w_{N_r}(\overline{\beta})$ for any Ulm sequence $\overline{\beta} \sim \overline{\infty}$, and

$$\hat{w}_{M_r}(\overline{\beta}) = \hat{w}_M(\overline{\beta}) = \hat{w}_N(\overline{\beta}) = \hat{w}_{N_r}(\overline{\beta})$$

whenever $\overline{\beta} \not\sim \overline{\infty}$. Since $\hat{u}_p(\alpha, M_r) = \hat{u}_p(\alpha, M) = \hat{u}_p(\alpha, N) = \hat{u}_p(\alpha, N_r)$ for all ordinals α, Theorem 2.1.14 yields $M_r \equiv_\infty N_r$. Therefore $M \equiv_\infty N$.

Case II: $l(tM) \geq \omega\delta$ (which implies $l(tN) \geq \omega\delta$). Let X' and Y' be subordinates of X and Y, respectively, as in Lemma 2.1.13 For any ordinal $\nu \leq \delta$ we define I_ν to be the set of all maps $f : E \to F$ in $\mathrm{prs}_{\omega\nu} = \mathrm{prs}_{\omega\nu}^{X',Y'}$. For $\nu + 1 \leq \delta$ let

$$f : E \to F$$

be a map in $I_{\nu+1}$ and suppose that x is an element of $M \setminus E$. We will extend f to a map $f' \in I_\nu$ with $x \in \mathrm{dom}(f')$.

Let $A = \langle X' \rangle$ and let r be the smallest integer ≥ 0 such that $p^{r+1}x \in A + E$. Then $\tilde{w}_M^{\omega(\nu+1)}(\overline{\beta}) = \tilde{w}_N^{\omega(\nu+1)}(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ because $\nu + 1 \leq \delta$. Therefore we can apply Lemma 2.1.6 to extend f to a map $f^* : E^* \to F^*$ in $\mathrm{prs}_{\omega(\nu+1)}$ such that $p^{r+1}x \in E^*$. To simplify notation, we write $f : E \to F$ for $f^* : E^* \to F^*$.

Suppose $|x| \geq \omega\nu$. Then by Lemma 2.1.11, f extends to some map $f' : \langle E, x\rangle \to \langle F, y\rangle$ in I_ν. Now assume $|x| < \omega\nu$.

<u>Case IIa:</u> Suppose that for all $m = 0, \ldots, r$ we have

$$\max\{|p^m x + z| : z \in \langle A + E, p^{m+1}x\rangle\} < \omega\nu.$$

Then we use Lemma 2.1.13 repeatedly to obtain an extension f' of f in $I_{\nu+1} \subseteq I_\nu$ with $x \in \mathrm{dom}(f')$.

<u>Case IIb:</u> Now assume that there exists $0 \leq m \leq r$ and an element $z \in \langle A + E, p^{m+1}x\rangle$ such that $|p^m x + z| \geq \omega\nu$. Let m be the smallest such integer. Using Lemma 2.1.6 again we extend f to a map

$$\overline{f} : \overline{E} \to \overline{F}$$

in $I_{\nu+1}$ with $z \in \langle \overline{E}, p^{m+1}x\rangle$. By Lemma 2.1.11, $\overline{f}$ extends to a map $\overline{f}' \in I_\nu$ with domain $\langle \overline{E}, p^m x + z\rangle$. Now write $z = e + \lambda p^{m+1}x$ where $e \in \overline{E}$ and $\lambda \in \mathbb{Z}_{(p)}$. Then $(1 + \lambda p)p^m x = p^m x + z - e \in \langle \overline{E}, p^m x + z\rangle$. Since $1 + \lambda p$ is a unit in the ring $\mathbb{Z}_{(p)}$, it follows that $\langle \overline{E}, p^m x\rangle = \langle \overline{E}, p^m x + z\rangle$. Finally, we use Lemma 2.1.13 repeatedly to extend $\overline{f}'$ to some map $f' \in I_\nu$ whose domain contains the elements $p^{m-1}x, \ldots, px, x$.

In view of Lemma 2.1.13(ii) the conditions of Theorem 1.2.1(ii) are satisfied and we conclude that $M \equiv_\delta N$. The last part of the theorem follows from Lemma 1.2.5. ∎

2.2 Groups with partial decomposition bases

The results in this section have already been published in [JLLS].

For a generalized classification theorem for groups, Jacoby [J1] introduced a generalization of the concept of a decomposition basis:

Definition 2.2.1. *Let R be an arbitrary principal ideal domain and G an R-module. A collection $\mathcal{C}$ of subsets of G is called* **partial decomposition basis for** G*, if*

(i) $\mathcal{C} \neq \emptyset$,

(ii) all elements in $\mathcal{C}$ are finite,

(iii) each element $X \in \mathcal{C}$ is a decomposition set and

(iv) for $X \in \mathcal{C}$ and $x \in G$ there is $Y \in \mathcal{C}$ with $X \subseteq Y$ and $rx \in \langle Y \rangle$ for some $0 \neq r \in R$.

If X is a decomposition basis for G, then the collection of all finite subsets of X is a partial decomposition basis for G. For $\mathbb{Z}_{(p)}$-modules, the following is a direct consequence:

Lemma 2.2.2. *Let G be a $\mathbb{Z}_{(p)}$-module with partial decomposition basis. Then G also possesses a partial decomposition basis $\mathcal{C}$ with*

(i) $X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$ and

(ii) $\{x_1, ..., x_n\} \in \mathcal{C}, a_1, ..., a_n \in \mathbb{Z}_{(p)} \setminus \{0\} \Rightarrow \{a_1x_1, ..., a_nx_n\} \in \mathcal{C}$.

Further investigating the idea of α-initially equivalent Ulm sequences as considered in Section 2.1 we will introduce an equivalent definition of equivalence related to sequences which start right from the beginning with coinciding entries.

Definition 2.2.3. *Let α be an ordinal.*

(i) Two Ulm sequences $\overline{\beta}$ and $\overline{\eta}$ are called **equal up to** α, *$\overline{\beta} =_\alpha \overline{\eta}$, if $\min\{\beta_i, \alpha\} = \min\{\eta_i, \alpha\}$ for all $i < \omega$.*

(ii) If $\overline{\beta} =_\alpha \overline{\eta}$, their equivalence classes $[\overline{\beta}]$ and $[\overline{\eta}]$ are called α-**equivalent**, *$[\overline{\beta}] \sim^\alpha [\overline{\eta}]$.*

(iii) If $[\overline{\beta}] \sim^\alpha [\overline{\eta}]$, any Ulm sequences $\overline{\beta}' \in [\overline{\beta}]$ and $\overline{\eta}' \in [\overline{\eta}]$ are called α-**equivalent**, *too, and we also write $\overline{\beta}' \sim^\alpha \overline{\eta}'$.*

We have $\overline{\beta} \sim^\alpha \overline{\eta}$ if and only if $[\overline{\beta}] \sim^\alpha [\overline{\eta}]$. We set

$$\hat{w}^\alpha_G(\overline{\beta}) = \min\{\sum_{\overline{\eta} \sim^\alpha \overline{\beta}} \hat{w}_G(\overline{\eta}), \omega\}$$

and can achieve a classification of $\mathbb{Z}_{(p)}$-modules with partial decomposition bases, which is the main result of [JL]:

Theorem 2.2.4 (Jacoby/Loth)**.** *Let G and H be $\mathbb{Z}_{(p)}$-modules with partial decomposition bases and δ an ordinal. If*

(i) $\hat{u}(\alpha, G) = \hat{u}(\alpha, H)$ for all $\alpha < \omega\delta$,

(ii) $l(tG) < \omega\delta$ implies $\hat{u}(\infty, G) = \hat{u}(\infty, H)$ and

(iii) $\hat{w}_G^{\omega(\nu+1)}(\overline{\beta}) = \hat{w}_H^{\omega(\nu+1)}(\overline{\beta})$ *for all Ulm sequences* $\overline{\beta}$ *and ordinals* $\nu < \delta$,

then $G \equiv_\delta H$.

For $A = [a_{(p,i)}]$ and $B = [b_{(p,i)}]$ two Ulm matrices and α an ordinal, we call A and B **equal up to** α and write $A =_\alpha B$, if $\min\{a_{(p,i)}, \alpha\} = \min\{b_{(p,i)}, \alpha\}$ for all primes p and $i < \omega$. We will define a further equivalence relation, this time for compatibility classes of Ulm matrices. We call two compatibility classes c and c' of Ulm matrices **α-compatible**, $c \sim_\alpha c'$, if there are $A \in c$ and $B \in c'$ such that $A =_\alpha B$. Any Ulm matrices $C \in c$ and $C' \in c'$ are then also called α-compatible and we write $C \sim_\alpha C'$, too. Notice that if any two Ulm matrices are α-compatible, their respective p-rows are equal up to α for almost all primes p.
For any group G with partial decomposition basis, ordinal α, prime p, Ulm sequence $\overline{\beta}$ and compatibility class c of Ulm matrices we set

$$\hat{w}_G^\alpha(p, \overline{\beta}, c) = \min\{\sum_{\overline{\eta} \sim^\alpha \overline{\beta}, c' \sim_\alpha c} \hat{w}_G(p, \overline{\eta}, c'), \omega\}.$$

For a finite decomposition set X of G and $\langle X \rangle^0 = \{x \in G : \exists 0 \neq r \in R : rx \in \langle X \rangle\}$ we then have

$$\hat{w}_{\langle X \rangle^0}^\alpha(p, \overline{\beta}, c) = \sum_{\overline{\eta} \sim^\alpha \overline{\beta}, c' \sim_\alpha c} |\{x \in X : U(x) \in c' \text{ and } u_p(x) \sim^\alpha \overline{\eta}\}|$$

$$= |\{x \in X : [U(x)] \sim_\alpha c \text{ and } u_p(x) \sim^\alpha \overline{\beta}\}|.$$

We will now establish some facts on the cardinals $\hat{w}_G^\alpha(p, \overline{\beta}, c)$, similar to the results in [J1].

Lemma 2.2.5. *Suppose G is a group with partial decomposition basis $\mathcal{C}$. If $X \in \mathcal{C}$, then we have: $\hat{w}_G^\alpha(p, \overline{\beta}, c) \geq \hat{w}_{\langle X \rangle^0}^\alpha(p, \overline{\beta}, c)$ for any ordinal α, compatibility class c of Ulm matrices, prime p and Ulm sequence $\overline{\beta}$.*

Proof. The set X is a decomposition basis of $\langle X \rangle^0$. Since $\hat{w}_G(p, \overline{\eta}, c') \geq \hat{w}_{\langle X \rangle^0}(p, \overline{\eta}, c')$ for all c', p, $\overline{\eta}$, we obtain $\hat{w}_G^\alpha(p, \overline{\beta}, c) \geq \hat{w}_{\langle X \rangle^0}^\alpha(p, \overline{\beta}, c)$. ■

We will need a result from [J1]:

Theorem 2.2.6 (Jacoby)**.** *Let G be a group with partial decomposition basis $\mathcal{C}$, p a prime, $\overline{\beta}$ an Ulm sequence and c a compatibility class of Ulm matrices such that $\hat{w}_G(p, \overline{\beta}, c) \geq n$. For any partial decomposition basis $\mathcal{D}$ for G and $Y \in \mathcal{D}$ there exists $Y' \in \mathcal{D}$ with $Y \subseteq Y'$ and Y' contains elements $y_1, \ldots, y_n$ such that $u_p(y_i) \sim \overline{\beta}$ and $U(y_i) \in c$ for all $i = 1, \ldots, n$.*

The next result shows that the cardinals $\hat{w}_G^\alpha(p,\overline{\beta},c)$ are independent of the choice of the partial decomposition basis:

Theorem 2.2.7. *Let G be a group with partial decomposition basis $\mathcal{C}$, α an ordinal and n a positive integer. If $\hat{w}_G^\alpha(p,\overline{\beta},c) \geq n$ and $Y \in \mathcal{C}$, then there exists $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' contains elements $y_1,\dots,y_n$ such that $[U(y_i)] \sim_\alpha c$ and $u_p(y_i) \sim^\alpha \overline{\beta}$ for all $i = 1,\dots,n$.*

Proof. Letting $E = \{(\overline{\eta},c') : \hat{w}_G(p,\overline{\eta},c') \neq 0\}$, we have

$$\hat{w}_G^\alpha(p,\overline{\beta},c) = \min\{\sum_{\overline{\eta}\sim^\alpha\overline{\beta},c'\sim_\alpha c,(\overline{\eta},c')\in E} \hat{w}_G(p,\overline{\eta},c'),\omega\}.$$

Let $(\overline{\eta},c') \in E$ and suppose $\hat{w}_G(p,\overline{\eta},c') \geq k$. By Theorem 2.2.6 there is $Y' \in \mathcal{C}$ such that $Y \subseteq Y'$ and Y' has k elements x with $U(x) \in c' \sim_\alpha c$ and $u_p(x) \sim^\alpha \overline{\eta} \sim^\alpha \overline{\beta}$. Repeat this for all elements in E until at least n such elements have been collected. ∎

Corollary 2.2.8. *Let G be a group with partial decomposition basis $\mathcal{C}$, p a prime, α an ordinal, $\overline{\beta}$ an Ulm sequence and c a compatibility class of Ulm matrices. Then $\hat{w}_G^\alpha(p,\overline{\beta},c)$ is the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1,\dots,x_n \in X$ such that $[U(x_i)] \sim_\alpha c$ and $u_p(x_i) \sim^\alpha \overline{\beta}$ for all $i = 1,\dots,n$. If no such n exists, $\hat{w}_G^\alpha(p,\overline{\beta},c) = \omega$.*

Proof. Suppose there is a largest integer n such that there is $X \in \mathcal{C}$ containing n elements x satisfying $[U(x)] \sim_\alpha c$ and $u_p(x) \sim^\alpha \overline{\beta}$. Then by Theorem 2.2.7, $\hat{w}_G^\alpha(p,\overline{\beta},c) \leq n$. On the other hand, $\hat{w}_G^\alpha(p,\overline{\beta},c) \geq \hat{w}_{\langle X\rangle^0}^\alpha(p,\overline{\beta},c) = n$ by Lemma 2.2.5 and Theorem 2.2.7. If no such n exists, $\hat{w}_G^\alpha(p,\overline{\beta},c) = \omega$ by Lemma 2.2.5. ∎

Corollary 2.2.9. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, p a prime, α an ordinal, $\overline{\beta}$ an Ulm sequence and c a compatibility class of Ulm matrices. Suppose $\hat{w}_G^\alpha(p,\overline{\beta},c) = \hat{w}_H^\alpha(p,\overline{\beta},c)$, $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and*

$$\hat{w}_{\langle X\rangle^0}^\alpha(p,\overline{\beta},c) > \hat{w}_{\langle Y\rangle^0}^\alpha(p,\overline{\beta},c).$$

Then there exists $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and there is $y \in Y' \setminus Y$ such that $[U(y)] \sim_\alpha c$ and $u_p(y) \sim^\alpha \overline{\beta}$.

Proof. Let $n = \hat{w}_{\langle Y\rangle^0}^\alpha(p,\overline{\beta},c)$. Then

$$\hat{w}_H^\alpha(p,\overline{\beta},c) = \hat{w}_G^\alpha(p,\overline{\beta},c) \geq \hat{w}_{\langle X\rangle^0}^\alpha(p,\overline{\beta},c) > \hat{w}_{\langle Y\rangle^0}^\alpha(p,\overline{\beta},c) = n.$$

By Theorem 2.2.7, there is $Y' \in \mathcal{D}$ such that $Y \subseteq Y'$ and Y' contains $n+1$ elements y such that $[U(y)] \sim_\alpha c$ and $u_p(y) \sim^\alpha \overline{\beta}$. Since Y contains only n such elements, one of them is in $Y' \setminus Y$. ∎

The following lemma will be needed:

Lemma 2.2.10 (Stanton)**.** *Let G be a group with decomposition basis X, p a prime and $x_1, x_2 \in X$ with α-compatible Ulm matrices. Then there are elements $y_1, y_2 \in \langle X \rangle$ such that $u_p(y_1) =_\alpha u_p(x_2)$, $u_p(y_2) =_\alpha u_p(x_1)$ and $u_q(y_1) =_\alpha u_q(x_1)$, $u_q(y_2) =_\alpha u_q(x_2)$ for all primes $q \neq p$. The set $Y = (X \setminus \{x_1, x_2\}) \cup \{y_1, y_2\}$ is a decomposition basis for G and $\langle X \rangle = \langle Y \rangle$.*

Stanton proved this for coinciding Ulm sequences, not just for sequences equal up to α. The proof applies just as well in this case.

The following fact was also proved in [J1]:

Lemma 2.2.11 (Jacoby)**.** *Let G be a group with (some) partial decomposition basis. Then G has a partial decomposition basis $\mathcal{C}$ such that*

(i) $X \in \mathcal{C}, X' \subseteq X \Rightarrow X' \in \mathcal{C}$;

(ii) $X \in \mathcal{C}, \langle X \rangle = \langle Y \rangle$, *$Y$ finite decomposition set* $\Rightarrow Y \in \mathcal{C}$;

(iii) $\{x_1, \ldots, x_n\} \in \mathcal{C}, a_1, \ldots, a_n \in \mathbb{Z} \Rightarrow \{a_1 x_1, \ldots, a_n x_n\} \in \mathcal{C}$.

The proof takes the union of a chain of partial decomposition bases for G that alternately satisfy conditions (i) and (iii) and condition (ii).

Lemma 2.2.12. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$ satisfying conditions (i) and (ii) of Lemma 2.2.11. Suppose α is an ordinal such that $\hat{w}^\alpha_G(p, \overline{\beta}, c) = \hat{w}^\alpha_H(p, \overline{\beta}, c)$ for every prime p, Ulm sequence $\overline{\beta}$ and compatibility class c of Ulm matrices. Assume $X \cup \{x\} \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that*

$$\hat{w}^\alpha_{\langle X \rangle^0}(p, \overline{\beta}, c) = \hat{w}^\alpha_{\langle Y \rangle^0}(p, \overline{\beta}, c)$$

for all p, $\overline{\beta}$ and c. Then there exists an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}^\alpha_{\langle X \cup \{x\} \rangle^0}(p, \overline{\beta}, c) = \hat{w}^\alpha_{\langle Y \cup \{y\} \rangle^0}(p, \overline{\beta}, c)$$

for all p, $\overline{\beta}$ and c. In fact, $U(x) \sim_\alpha U(y)$ and $u_p(x) \sim^\alpha u_p(y)$ for all primes p.

Proof. Suppose $x \in G\backslash X$ and let p_0 a prime, $\overline{\beta}_0$ an Ulm sequences with $u_{p_0}(x) \sim \overline{\beta}_0$ and c_0 be the compatibility class containing $U(x)$. By Corollary 2.2.8,

$$\hat{w}^{\alpha}_{\langle X\cup\{x\}\rangle^0}(p_0, \overline{\beta}_0, c_0) = \hat{w}^{\alpha}_{\langle X\rangle^0}(p_0, \overline{\beta}_0, c_0) + 1 > \hat{w}^{\alpha}_{\langle Y\rangle^0}(p_0, \overline{\beta}_0, c_0).$$

By Corollary 2.2.9 and condition (i) of Lemma 2.2.11 there is an element $z \in H \setminus Y$ satisfying

$$(*) \quad Y \cup \{z\} \in \mathcal{D},\ U(z) \sim_{\alpha} c_0 \text{ and } u_{p_0}(z) \sim^{\alpha} \overline{\beta}_0.$$

Then $U(x)$ and $U(z)$ are α-compatible and $u_p(x)$ and $u_p(z)$ are α-equivalent for all but finitely many primes p, say, $p_1, \dots, p_n$. We will show by induction on n that z can be replaced by an element $y \in H$ satisfying $(*)$ such that $u_p(x)$ and $u_p(y)$ are α-equivalent for all primes p. For $n = 0$ there is nothing to show, so assume the assertion is true for $n-1$ and let $\overline{\beta}_n = u_{p_n}(x)$. By our assumption, $u_{p_n}(z) \not\sim^{\alpha} \overline{\beta}_n$ and therefore

$$\hat{w}^{\alpha}_{\langle Y\cup\{z\}\rangle^0}(p_n, \overline{\beta}_n, c_0) = \hat{w}^{\alpha}_{\langle Y\rangle^0}(p_n, \overline{\beta}_n, c_0) < \hat{w}^{\alpha}_{\langle X\cup\{x\}\rangle^0}(p_n, \overline{\beta}_n, c_0).$$

By Corollary 2.2.9, there is $z' \in H$ such that $Y \cup \{z, z'\} \in \mathcal{D}$, $[U(z')] \sim_{\alpha} c_0$ and $u_{p_n}(z') \sim^{\alpha} \overline{\beta}_n$. Now we apply Lemma 2.2.10 to the group $M = \langle Y \cup \{z, z'\}\rangle^0$ with decomposition basis $Y \cup \{z, z'\}$, elements z, z' and prime p_n. Then there are elements $y, y' \in \langle Y \cup \{z, z'\}\rangle$ such that $Y \cup \{y, y'\}$ is a decomposition basis for M, $\langle Y, y, y'\rangle = \langle Y, z, z'\rangle$ and $u_{p_n}(y) =_{\alpha} u_{p_n}(z')$, $u_{p_n}(y') =_{\alpha} u_{p_n}(z)$ and $u_q(y) =_{\alpha} u_q(z)$, $u_q(y') =_{\alpha} u_q(z')$ whenever $q \neq p_n$. Notice that condition $(*)$ holds for y as $Y \cup \{y\} \in \mathcal{D}$ by conditions (i) and (ii) of Lemma 2.2.11, y and z have α-compatible Ulm matrices and $u_{p_0}(y) = u_{p_0}(z)$ up to α. To complete the induction, we need to verify that $u_p(x)$ and $u_p(y)$ are α-equivalent for all but $n-1$ primes p. Indeed, $u_q(y) = u_q(z) \sim^{\alpha} u_q(x)$ whenever $q \notin \{p_1, \dots, p_n\}$, and $u_{p_n}(y) = u_{p_n}(z') \sim^{\alpha} u_{p_n}(x)$. This completes the proof. ∎

We will need two further results. The first one is again from [J1]:

Lemma 2.2.13 (Jacoby)**.** *Let G be a group with decomposition basis X and S a finitely generated subgroup of G such that $S \cap \langle X\rangle = \langle S \cap X\rangle$. If $y \in X$ ($y \notin S$), then there is a positive integer n satisfying $|mny + s|_p = \min\{|mny|_p, |s|_p\}$ for all $m \in \mathbb{Z}$, $s \in S$ and primes p.*

The second one is Warfield's local-global lemma [War]:

Lemma 2.2.14 (Warfield)**.** *Let A and B be abelian groups, S and T subgroups such that A/S and B/T are torsion, and $f : S \to T$ a homomorphism. Suppose for every* *prime p, the induced map $f_p : S \otimes \mathbb{Z}_{(p)} \to T \otimes \mathbb{Z}_{(p)}$ extends to a homomorphism $g(p) : A \otimes \mathbb{Z}_{(p)} \to B \otimes \mathbb{Z}_{(p)}$. Then f extends to a homomorphism $g : A \to B$ such that $g_p = g(p)$ for all primes p. If each map $g(p)$ is injective (bijective), then g is injective (bijective).*

The main result of this section then is the following generalization of Theorem 2.2.4:

Theorem 2.2.15. *Let G and H be groups with partial decomposition bases $\mathcal{C}$ and $\mathcal{D}$, respectively, and let δ be an ordinal. Suppose*

(i) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and $\alpha < \omega\delta$;

(ii) if $l(tG_p) < \omega\delta$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$;

(iii) $\hat{w}_G^{\omega(\nu+1)}(p, \overline{\beta}, c) = \hat{w}_H^{\omega(\nu+1)}(p, \overline{\beta}, c)$ for every prime p, Ulm sequence $\overline{\beta}$, compatibility class c of Ulm matrices and $\nu < \delta$.

Then $G \equiv_\delta H$.

Proof. First, let $\mathcal{C}$ and $\mathcal{D}$ be the partial decomposition bases for G and H as provided by Lemma 2.2.11. For $\nu \leq \delta$ let I_ν be the set of all maps $f : S \to T$ with associated sets $X \in \mathcal{C}$, $Y \in \mathcal{D}$ such that $f(x) = Y$ which fulfill

(i) S and T are finitely generated subgroups of G and H, resp.;

(ii) f is an $\omega\nu$-height-preserving isomorphism;

(iii) $X \subseteq S \subseteq \langle X \rangle^0$ and $Y \subseteq T \subseteq \langle Y \rangle^0$;

(iv) for every $x \in X$, $U(x)$ and $U(f(x))$ are $\omega\nu$-compatible.

To prove that $G \equiv_\delta H$, we will show that the system $\{I_\nu : \nu \leq \delta\}$ satisfies condition (ii) of Karp's Theorem 1.2.1. Suppose $f \in I_{\nu+1}$ where $\nu < \delta$, say, $f : S \to T$ with associated $X \in \mathcal{C}$, $Y \in \mathcal{D}$, and let $x \in G \setminus S$. To find an extension $g \in I_\nu$ of f with $x \in \mathrm{dom}(g)$, we will show

(A) If x has a multiple in S, then there is such a map $g \in I_\nu$

and

(B) If $X \cup \{x\} \in \mathcal{C}$, then there is a map $g' \in I_{\nu+1}$ extending f such that $rx \in \mathrm{dom}(g')$ for some positive integer r.

Then repeated application of (B) followed by an application of (A) yields an extension $g \in I_\nu$ of f with $x \in \mathrm{dom}(g)$. To prove (A), suppose $rx \in S$ for some positive

integer r. In order to construct the map g, we will apply Warfield's Lemma 2.2.15 to the groups $A = \langle S, x\rangle$ and $B = T^0$, so let p be a prime and consider the induced map $f_p^* : S_p^* \to T_p^*$. Since the natural map $G \to G_p$ preserves p-heights, the modules G_p and H_p have induced partial decomposition bases. Let $\alpha < \omega\delta$, $\nu < \delta$ and $\overline{\beta}$ an Ulm sequence. Then $\hat{u}_p(\alpha, G_p) = \hat{u}_p(\alpha, H_p)$ by [F2, Part 2, Lemma 16].
Let C be a set of representatives of the $\omega(\nu+1)$-compatibility classes, one for each class. Since the natural map $G \to G_p$ preserves p-heights, repeated application of 2.2.6 yields $\hat{w}_{G_p}(\overline{\beta}) = \min\{\sum_c \hat{w}_G(p, \overline{\beta}, c), \omega\}$, for G a group with partial decomposition basis, p a prime and $\overline{\beta}$ an equivalence class of Ulm sequences. The summation is over all compatibility classes c for which $w_G(p, \overline{\beta}, c) \neq 0$. So

$$
\begin{aligned}
\hat{w}_{G_p}^{\omega(\nu+1)}(\overline{\beta}) &= \min\{\sum_{\overline{\eta}\sim^{\omega(\nu+1)}\overline{\beta}} \hat{w}_{G_p}(\overline{\eta}), \omega\} \\
&= \min\{\sum_{\overline{\eta}\sim^{\omega(\nu+1)}\overline{\beta}} \min\{\sum_c \hat{w}_G(p, \overline{\eta}, c), \omega\}, \omega\} \\
&= \min\{\sum_{\overline{\eta}\sim^{\omega(\nu+1)}\overline{\beta}} \sum_{c\in C} \sum_{c'\sim_{\omega(\nu+1)}c} \hat{w}_G(p, \overline{\eta}, c'), \omega\} \\
&= \min\{\sum_{c\in C}(\sum_{\overline{\eta}\sim^{\omega(\nu+1)}\overline{\beta}} \sum_{c'\sim_{\omega(\nu+1)}c} \hat{w}_G(p, \overline{\eta}, c')), \omega\} \\
&= \min\{\sum_{c\in C} \hat{w}_G^{\omega(\nu+1)}(p, \overline{\beta}, c), \omega\} \\
&= \min\{\sum_{c\in C} \hat{w}_H^{\omega(\nu+1)}(p, \overline{\beta}, c), \omega\} = \hat{w}_{H_p}^{\omega(\nu+1)}(\overline{\beta}).
\end{aligned}
$$

Now let $\mathcal{C}_{G_p}$ and $\mathcal{C}_{H_p}$ be the induced partial decomposition bases of G_p and H_p as in Theorem 2.2.2 and notice that the map f_p with associated sets $\{x \otimes 1 : x \in X\} \in \mathcal{C}_{G_p}$ and $\{y \otimes 1 : y \in Y\} \in \mathcal{C}_{H_p}$ can be extended to an $\omega\nu$-height-preserving isomorphism $g(p)$ with $x_p \in \mathrm{dom}(g(p))$. (See [JL] for more.) By Lemma 2.2.14 we have a homomorphism $g : A \to B$ where $g(x) = y$ for some $y \in B$ and $g_p = g(p)$ for all p. Each map $g(p) : A_p \to B_p$ is injective and $\omega\nu$-height-preserving, therefore $g : \langle S, x\rangle \to \langle T, y\rangle$ is an $\omega\nu$-height-preserving isomorphism. Then g with associated X and Y satisfies conditions (i)-(iv), hence $g \in I_\nu$.

To verify (B), assume that $X \cup \{x\} \in \mathcal{C}$. By condition (iv) we have

$$\hat{w}_{\langle X\rangle^0}^{\omega(\nu+1)}(p, \overline{\beta}, c) = \hat{w}_{\langle Y\rangle^0}^{\omega(\nu+1)}(p, \overline{\beta}, c)$$

for all primes p, Ulm sequences $\overline{\beta}$ and compatibility classes c of Ulm matrices. By

Lemma 2.2.12, there is an element $y \in H$ such that $Y \cup \{y\} \in \mathcal{D}$ and

$$\hat{w}^{\omega(\nu+1)}_{\langle X\cup\{x\}\rangle^0}(p,\overline{\beta},c) = \hat{w}^{\omega(\nu+1)}_{\langle Y\cup\{y\}\rangle^0}(p,\overline{\beta},c)$$

for all p, $\overline{\beta}$ and c where $U(x) \sim_{\omega(\nu+1)} U(y)$ and $u_p(x) \sim^{\omega(\nu+1)} u_p(y)$ for all primes p. Then there are positive integers k and l such that $u_p(kx) =_{\omega(\nu+1)} u_p(ly)$ for all primes p. Let $x' = kx$ and $y' = ly$. Now proceed as in the proof of the classification in $L_{\infty\omega}$ (see [J1, Theorem 14]): Letting $\tilde{X} = X \cup \{x'\}$, we have $S \cap \langle \tilde{X} \rangle = \langle S \cap \tilde{X} \rangle$, so we can apply Lemma 2.2.14 to the group $\langle S, x' \rangle^0$ with decomposition basis $\tilde{X}$ and the subgroup S, and similarly to $\langle T, y' \rangle^0$, $Y \cup \{y'\}$ and T. Then there is a positive integer n such that

$$|mnx' + s|_p = \min\{|mnx'|_p, |s|_p\} \text{ and } |mny' + t|_p = \min\{|mny'|_p, |t|_p\}$$

for all $m \in \mathbb{Z}$, $s \in S$, $t \in T$ and primes p. Finally, let $S' = \langle S, nx' \rangle$ and $T' = \langle T, ny' \rangle$. Then f extends to the map

$$g' : S' \to T'$$

by sending nx' onto ny'. It is clear that g' is $\omega(\nu+1)$-height-preserving. Let $X' = X \cup \{nx'\}$, $Y' = Y \cup \{ny'\}$. Then $X' \subseteq S' \subseteq \langle X' \rangle^0$ and $Y' \subseteq T' \subseteq \langle Y' \rangle^0$, therefore g' is a map in $I_{\nu+1}$ with associated sets $X' \in \mathcal{C}$, $Y' \in \mathcal{D}$ such that $nx' \in \mathrm{dom}(g')$.

Consequently, f extends to a map $g \in I_\nu$ with $x \in \mathrm{dom}(g)$, as desired. By symmetry, the conditions of Theorem 1.2.1(2) are satisfied and it follows that $G \equiv_\delta H$. ∎

Chapter 3

A new Step-Lemma

In order to construct classes of partially isomorphic modules of arbitrary cardinality in Gödel's universe, we will follow the classical approach that uses a Step-Lemma and the Diamond Principle. First, we will prove a new Step-Lemma for abelian p-groups G which are almost direct sums of cyclic groups and have a prescribed endomorphism ring $\mathrm{Small}\, G \oplus A$ where Small denotes the ideal of all small endomorphisms of G. The fact that $\mathrm{Small}\, G \oplus A$ is the endomorphism ring of G (resp. that such a p-group G exists) is equivalent to A being a ring satisfying the Pierce-condition [DG], i. e. A^+ is a p-adic completion of a free J_p-module, J_p being the ring of p-adic integers. (**Prüfer module**).

3.1 Characterizing small endomorphisms

Recall that for a p-group G the **torsion-completion** is defined as $\overline{G} = t\hat{G}$ with $\hat{G}$ being the p-adic completion of G. An element $x \in \overline{G}$ is a torsion-element which can be expressed in the form $x = \sum_{n<\omega} x_n$ such that $x_n \in G$ and $(x_n)_{n<\omega}$ is a sequence in G which converges to 0 in the p-adic topology. If G is a direct sum of cyclic p-groups, then we will write $G = \bigoplus_{n<\omega} G_n$ where G_n is homogeneous of type $\mathbb{Z}_{p^n}$, a direct sum of cyclic groups $\mathbb{Z}_{p^n}$, and we define $c(G) = \{n < \omega : G_n \neq 0\}$. Note that G_n is only unique up to isomorphism, but in the following we often fix G_n for each $n < \omega$. Let $I_n, n < \omega$ be disjunct index sets for $n \neq m$. Set $G_n := \bigoplus_{i \in I_n} (A/p^n A)e_i$. The A**-support** of $x_n \in G_n$ is the finite subset $[x_n] \subseteq I_n$ used for the non-trivial summands of x_n in this direct sum: $x_n = \sum_{i\in[x_n]} a_i e_i$ with $0 \neq a_i \in A/p^n A$. Moreover, $[x] = \bigcup_{n<\omega}[x_n]$ is the A**-support** of x. If $I \subseteq [x]$ then x_I is the element $x \restriction I = \sum_{n<\omega} x_n \restriction I$ with $x_n \restriction I = \sum_{i\in([x_n]\cap I)} a_i e_i \in \overline{G}$. Similarly, if $g \in \overline{G} = \overline{P} \oplus \overline{H}$, for some decomposition $P \oplus H$ of G, then we write for its unique components

$g \upharpoonright \overline{P} = g_P \in \overline{P}$ and $g \upharpoonright \overline{H} = g_H \in \overline{H}$.

An endomorphism $\varphi \in \operatorname{End} G$ is called **small**, if for any $n < \omega$ there is $k < \omega$ with $p^k G[p^n]\varphi = 0$. If G is p-separable, then $G \subseteq \overline{G}$ where $\overline{G}$ is the p-torsion completion of G. Clearly, each endomorphism φ then has a unique extension $\varphi : \overline{G} \longrightarrow \overline{G}$, which we may also denote by φ. We may characterize another class of endomorphisms, so-called **inessential** endomorphisms, by $\varphi \in \operatorname{Ines} G$ iff $\overline{G}\varphi \subseteq G$. $\operatorname{Small} G$ and $\operatorname{Ines} G$ are ideals of $\operatorname{End} G$.

We will characterize $\operatorname{Small} G$ in the case, that G is constructed as intended above, namely we will study generalizations of direct sums of cyclic p-groups which are separable p-groups and define a category of strongly-κ-direct-groups which is a generalization of the basic category of direct sums of cyclic p-groups.

Definition 3.1.1. *(i) A p-group G of size κ is a κ-direct sum of cyclics if every subset of size $< \kappa$ is contained in a pure subgroup U of G, which is a direct sum of cyclic groups. We will call G* κ**-direct-c**.

(ii) A p-group G of size κ is called **strongly-κ-direct** *if every subset of size $< \kappa$ is contained in a subgroup U of G which is a direct sum of cyclics such that the quotient G/U is κ-direct-c.*

A p-pure subgroup of a direct sum of cyclic p-groups is itself a direct sum of cyclics, as well as the union of such subgroups. Thus any κ-direct-c group G is the union of a continuous chain $G = \bigcup_{\alpha<\kappa} G_\alpha$ of p-pure subgroups G_α of size $|G_\alpha| < \kappa$ which are direct sums of cyclics, which we know from Definition 1.3.10 can be identified as a κ-filtration of G. For example let $\{g_\alpha : \alpha < \kappa\}$ be a list of all elements of G and let G_β be the p-pure subgroup of G generated by the elements g_α $(\alpha < \beta)$. The modules G_β then provide the continuous ascending chain of direct sums of cyclics. We let A be a ring with Pierce-condition and consider a p-group G which is at the same time an A-module such that $G = \bigcup_{\alpha<\kappa} G_\alpha$ is the union of a κ-filtration of p-pure A-modules G_α which are direct sums of cyclic A-modules of the form A/p^nA. We will also call G κ**-direct-c** in this case.

For particular p-groups, like those constructed in [CG] we have $\operatorname{Small} G = \operatorname{Ines} G$ and we will show that this holds in the category of κ-direct-c p-groups, too.

Theorem 3.1.2. *If κ is a regular (uncountable) cardinal and the p-torsion A-module G is κ-direct-c, then* $\operatorname{Small} G = \operatorname{Ines} G$.

Proof. Let $G = \bigcup_{\alpha<\kappa} G_\alpha$ be the union of a continuous chain of submodules G_α of size $|G_\alpha| < \kappa$ which are direct sums of cyclic (A/p^nA)-modules. G is κ-direct-c and p-torsion.

We will show that $\operatorname{Small} G \subseteq \operatorname{Ines} G$ and will first reduce the problem to a submodule G_α of G.

If $\varphi \in \operatorname{Small} G$ and $\overline{x} \in \overline{G}$, then we must show $\overline{x}\varphi \in G$ (hence $\overline{G}\varphi \subseteq G$). Let $\overline{x} = \sum_{n<\omega} x_n$ with $x_n \in G$. Then $\{x_n \mid n < \omega\}$ is countable and using that $\operatorname{cf}(\kappa) > \omega$ (κ is regular, uncountable) there is $\alpha < \kappa$ such that $\{x_n \mid n < \omega\} \subseteq G_\alpha$. By a back-and-forth argument ($\{x_n\varphi \mid n < \omega\}$ is countable, too) we can assume that $G_\alpha\varphi \subseteq G_\alpha$. We have $\overline{x} \in \overline{G}_\alpha$ and G_α is a direct sum of cyclics and $G_\alpha \subseteq_* G$. It now remains to show that $\overline{x}\varphi \in G_\alpha$.

Let $B \subseteq G_\alpha$ be a family of the A-cyclic generators of G_α. Thus $\overline{x} = \sum e_n a_n$ with $e_n \in B$, $a_n \in A$ and $e_n a_n$ a null-sequence in the p-adic topology. Since $\overline{x} \in \overline{G}_\alpha$ and G_α a p-group, there is $i < \omega$ with $\overline{x}p^i = 0$. Using that φ is small, we find (for i) some $k < \omega$ with $p^k G_\alpha[p^i]\varphi = 0$. The null-sequence provides $n_0 < \omega$ with $a_n \in p^k A$ for all $n \geq n_0$. Thus $\sum_{n\geq n_0} e_n a_n \in p^k G_\alpha[p^i]$ and it follows $\sum_{n\geq n_0} e_n a_n \varphi = 0$. As a consequence $\overline{x}\varphi = (\sum_{n<n_0} e_n a_n)\varphi \in G_\alpha$ and $\overline{G}_\alpha\varphi \subseteq G_\alpha$, resp. $\overline{G}\varphi \subseteq G_\alpha$ follows.

Next we will show the converse inclusion and again first reduce the problem to G_α. If $\varphi \in \operatorname{End} G \setminus \operatorname{Small} G$, then we must show that $\varphi \notin \operatorname{Ines} G$. By assumption we find $m < \omega$ such that $p^k G[p^m]\varphi \neq 0$ for all $k < \omega$. Thus there are elements $b_k \in p^k G[p^m]$ with $h_k = b_k\varphi \neq 0$ for all $k < \omega$. As above there is $\alpha < \kappa$ and a submodule G_α of G, such that $b_k \in G_\alpha$ for all $k < \omega$ and $G_\alpha\varphi \subseteq G_\alpha$. By the choice of G_α also $h_k \in G_\alpha$ holds for all $k < \omega$.

Now we can continue working in G_α which is a direct sum of cyclics and choose a basic set B with $G_\alpha = \bigoplus_{b\in B} bA$. Write $h_k = \sum_{b\in B} h_k^b b$ with respect to this decomposition. Thus $[h_k]$ is a non-empty, but finite subset of B. We want to select an infinite, strictly increasing subsequence $I = \{i_j \mid j < \omega\} \subseteq \omega$ such that $[h_{i_j}] \cap [h_{i_k}] = \emptyset$ whenever $j \neq k$. Moreover, b_{i_j} ($j < \omega$) will be a null-sequence, hence $\bar{b} = \sum_{j<\omega} b_{i_j} \in \overline{G}_\alpha$ is well-defined (with $\bar{b}p^m = 0$) and $\bar{b}\varphi = (\sum_{j<\omega} b_{i_j})\varphi = \sum_{j<\omega}(b_{i_j}\varphi) = \sum_{j<\omega} h_{i_j}$ has infinite support $[\bar{b}\varphi] = \bigcup_{j<\omega}[h_{i_j}]$. Thus, obviously $\bar{b}\varphi \in \overline{G}_\alpha \setminus G_\alpha$ (and $\bar{b}\varphi \notin G$ from $G_\alpha\varphi \subseteq G_\alpha$) will show that $\varphi \notin \operatorname{Ines} G$. We finally construct I by induction. Suppose that $\{i_j \mid j < r\}$ is constructed for some $r < \omega$, and let $s_r := \sum_{j<r} b_{i_j}\varphi \in G_\alpha$ which has finite support $[s_r] \subseteq B$. By a simple argument on direct sums of cyclic A-modules we can choose $i_r < \omega$ such that $i_r > i_{r-1}$ and $p^{i_r} G_\alpha \cap \bigoplus_{b\in[s_r]} bA = 0$. Now we select $b_{i_r} \in p^{i_r} G_\alpha[p^m]$ and the non-trivial element $h_{i_r} = b_{i_r}\varphi$ surely must also lie in $p^{i_r} G_\alpha[p^m]$. Hence $[s_r] \cap [h_{i_r}] = \emptyset$

by the above but $[h_{i_r}] \neq \emptyset$, and I is as desired. ■

3.2 The Step-Lemma

If G is as described before, a direct sum of cyclic p-groups, then G is unbounded if the set $c(G)$ is infinite. In the following G will also be an A-module over a ring A that satisfies the Pierce-condition and each $G_n = \bigoplus A/p^n A$ is a direct sum of cyclic A-modules which are again direct sums of cyclic p-groups $\mathbb{Z}_{p^n}$. Clearly, also in the case of A-modules, G is unbounded if the set $c(G)$ is infinite. We will shortly say in this case that the p-torsion group G is an **unbounded A-module** and will in the following very often assume that

$$(*) \quad G_n \cong (A/p^n A)^{(\kappa_n)} \text{ for some uncountable cardinal } \kappa_n > |A|$$

holds whenever $G_n \neq 0$.

Proposition 3.2.1. *Let A be a ring which satisfies the Pierce-condition and G an unbounded A-module with condition $(*)$, i.e. $G \cong \bigoplus (A/p^n A)^{(\kappa_n)}$. If $\varphi \in \operatorname{End} G \setminus A \oplus \operatorname{Ines} G$, then there exists a decomposition $G = P \oplus H$ with H,P both unbounded A-modules and $c(G) = c(H) = c(P)$ such that the following hold.*

$$\text{There is } n < \omega \text{ such that } \overline{P}[p^n](\varphi - a) \nsubseteq G \text{ for all } a \in A \tag{3.1}$$

and

$$|P| \leq |A| \text{ with } P\varphi \subseteq P. \tag{3.2}$$

Proof. Let $G = P_{(0)} \oplus H_{(0)}$ be a first decomposition of G such that $P_{(0)}, H_{(0)}$ are unbounded A-modules with $c(G) = c(H_{(0)}) = c(P_{(0)})$ and $|P_{(0)}| \leq |A|$ and let B be an A-basis of G with respect to this decomposition. To assure $P\varphi \subseteq P$, we 'close' $P_{(0)}$ under φ as follows: Say $B_0 \subseteq B$ is the basis of $P_{(0)}$. Then $|B_0| \leq |A|$ and $\langle B_0 \rangle = P_{(0)}$. We recursively choose for $n < \omega$ sets $B_n \subseteq B_{n+1} \subseteq B$ with $|B_n| \leq |A|$ and $\langle B_n, B_n\varphi \rangle \subseteq P_{(n+1)} := \langle B_{n+1} \rangle$ as A-modules. The fact that κ_n is chosen $> \aleph_0$ assures we can do so. Then $P := \bigcup_{n<\omega} P_{(n)} = \langle \bigcup_{n<\omega} B_n \rangle$ is still a direct summand of G (with respect to the basis B) and $|P| \leq |A|$. If now $x \in P$ then there exists n with $x \in P_{(n)}$ and accordingly $x\varphi \in P_{(n+1)} \subseteq P$ and the claim $P\varphi \subseteq P$ is guaranteed. Let $H := \langle B \setminus \bigcup_{n<\omega} B_n \rangle$ be the complement of P in G, $G = P \oplus H$. Since the basis B of G was fixed at the beginning of the proof, H exists. Thus $\overline{G} = \overline{P} \oplus \overline{H}$

and $\overline{P}, \overline{H}$ both look like $\overline{G}$ above, say $P = \bigoplus P_i, H = \bigoplus H_i$, where $c(P) = c(G)$ follows from $P_{(0)} \subseteq P$, while $c(H) = c(G)$ follows from $|P| \leq |A|$ and $(*)$.

If we are lucky, P also satisfies the claim (3.1). Then P is as required in the lemma and we can proceed. Otherwise there are $a_n \in A$ with $\overline{P}[p^n](\varphi - a_n) \subseteq G$ for all $n < \omega$. If $n \leq m$, then $\overline{P}[p^n] \subseteq \overline{P}[p^m]$, and therefore $\overline{P}[p^m](\varphi - a_m) \subseteq G$ implies $\overline{P}[p^n](\varphi - a_m) \subseteq G$ and in combination with $\overline{P}[p^n](\varphi - a_n) \subseteq G$ we get $\overline{P}[p^n]((\varphi - a_m) - (\varphi - a_n)) = \overline{P}[p^n](a_n - a_m) \subseteq G$. Now choose $t \in \overline{P}[p^n] \setminus P$ such that $t = \sum_{i \in c(P)} t_i$ with $0 \neq t_i \in P_i$ and (w.l.o.g.) $o(t_i) = p^n$ for all i (e.g. elements of the kind $\sum_{i \geq n, i \in c(P)} p^{i-n} e_i$). Then from $\overline{P}[p^n](a_n - a_m) \subseteq G$ follows $t(a_n - a_m) = (\sum_{i \in c(P)} t_i)(a_n - a_m) = \sum_{i \in c(P)} (t_i(a_n - a_m)) \in G$. Since each P_i is an A-module, $t_i(a_n - a_m) \in P_i$ for all i which implies $t_i(a_n - a_m) = 0$ for almost all $i \in c(P)$ (otherwise $t(a_n - a_m) \notin G$, since the element would possess an infinite support). Thus, using $\operatorname{Ann} t_i = p^n A$, we have $a_n - a_m \in p^n A$ which is equivalent to saying that $(a_n)_{n<\omega}$ is a Cauchy sequence. The ring A is complete (by the Pierce-condition), and we find $a_* \in A$ such that $a_n - a_* \in p^n A$ for all $n < \omega$. Thus $\varphi - a_n = (\varphi - a_*) + (a_* - a_n)$ where $a_* - a_n =: p^n a' \in p^n A$. With $\overline{P}[p^n](\varphi - a_n) = \overline{P}[p^n](\varphi - a_* + p^n a') = \overline{P}[p^n](\varphi - a_*) + \overline{P}[p^n](p^n a') = \overline{P}[p^n](\varphi - a_*) + 0$ it follows from above that

$$\overline{P}[p^n](\varphi - a_*) \subseteq G \text{ for all } n < \omega.$$

However $\varphi - a_* \notin \operatorname{Ines} G$ by assumption on φ, and so we find $g_* \in \overline{G}$ with $g_*(\varphi - a_*) \notin G$. We choose some $\bigcup_{n<\omega} B_n \subseteq B' \subseteq B$ with $|B'| \leq |A|$ and $g_* \in \langle \overline{B'} \rangle$. Then $P \sqsubset \langle B' \rangle \sqsubset G$ holds and 'closing' $\langle B' \rangle$ under φ we can find a new decomposition $G = P' \oplus H'$ with H', P' both unbounded A-modules such that $c(G) = c(H') = c(P'), |P'| \leq |A|$ and the following hold.

$$P \sqsubset P', P'\varphi \subseteq P', g_* \in \overline{P}' \text{ and } g_*(\varphi - a_*) \notin G. \tag{3.3}$$

If (3.1) does not hold for P', then by the argument above there are $b_n \in A$ with $\overline{P}'[p^n](\varphi - b_n) \subseteq G$ for all $n < \omega$ and the limit b_* of the p-adic sequence $(b_n)_{n<\omega}$ satisfies

$$\overline{P}'[p^n](\varphi - b_*) \subseteq G \text{ for all } n < \omega. \tag{3.4}$$

Restricting to $\overline{P}$ we have $\overline{P}[p^n](\varphi - b_*) \subseteq G$ as well as $\overline{P}[p^n](\varphi - a_*) \subseteq G$ for all $n < \omega$. It follows $\overline{P}[p^n](a_* - b_*) \subseteq G$ for all $n < \omega$. Since P is unbounded we can

choose for all $n < \omega$ some element c_n of the kind $c_n := \sum_{i \geq n, i \in c(P)} p^{i-n} e_i \in \overline{P}[p^n]$. Then c_n has infinite support and $c_n(a_* - b_*) \in G$ for all $n < \omega$ forces $a_* - b_* \in p^n A$ for all $n < \omega$ and thus $a_* - b_* \in \bigcap_{n<\omega} p^n A = 0$ and $a_* = b_*$.
Thus $\overline{P}'[p^n](\varphi - a_*) \subseteq G$ for all $n < \omega$ from (3.4). Applying this to $g_* \in \overline{P}'$ gives $g_*(\varphi - a_*) \in G$ in contradiction to (3.3). Thus (3.1) is shown. ∎

As a simple result similar to Proposition 3.2.1 we include

Corollary 3.2.1. *Let P be an unbounded A-module, $\varphi \in \operatorname{End} P$ and $a \in A \setminus p^k A$ for some $k > 0$. Then*

$$\overline{P}[p^k](p^k\varphi - a) \nsubseteq P$$

holds.

Proof. We have $\overline{P}[p^k](p^k\varphi - a) = \overline{P}[p^k]a \nsubseteq P$ as witnessed by $a \in A \setminus p^k A$ and the choice of some suitable test element of the kind $c = \sum_{i \geq k, i \in c(P)} p^{i-k} e_i \in \overline{P}[p^k]$. Keep in mind that $p^i e_i = 0$ since e_i is a basis element of the torsion module $G_i = (A/p^i A)^{(\kappa)}$. ∎

If G is unbounded with condition $(*)$ and we assume $G = P \oplus H$, with P, H unbounded A-modules as obtained by Proposition 3.2.1, then G, H and P are of the following form: $\bigoplus_{k<\omega} G_k$ (resp. H_k, P_k) where G_k (resp. H_k, P_k) $\cong (A/p^{i_k}A)^{(\kappa_k)}$ for suitable cardinals $\kappa_k > 0$. For G and H, $\kappa_k > |A|$, whereas $\kappa_k \leq |A|$ in the decomposition of P. We will collect all the exponents i_k and call $i_0 < i_1 < \ldots$ the **chain of** G. Obviously $c(G) = \{i_k : k < \omega\}$ holds and by $c(G) = c(H) = c(P)$ the chain of G is the same as that of H and P. Therefore in the following we will call $i_0 < i_1 < \ldots$ just a/the **chain**. We fix this chain and set

$$q_k := p^{i_k} \text{ and } d_k := i_{k+1} - i_k.$$

Then $q_k p^{d_k} = q_{k+1}$ and $d_k \geq 1$ for all $k < \omega$.

Definition 3.2.2. *Let G be an unbounded A-module as above.*

*(i) We call $(g_k)_{k<\omega} \subseteq \overline{G}$ a d_k-**chain** iff $g_k - p^{d_k} g_{k+1} \in G$ and $q_k g_k = 0$ for all $k < \omega$.*

*(ii) We call $(g_k)_{k<\omega} \subseteq \overline{G}$ a **basic-d_k-chain** iff $g_k - p^{d_k} g_{k+1} \in B$ and $\operatorname{Ann} g_k = q_k A$ for all $k < \omega$, where B is some basis of G ($G = \bigoplus_{k<\omega} G_k$).*

We will begin our construction with a first

Lemma 3.2.3. *Let G be an unbounded A-module as above. Then the following hold.*

(i) There exists a basic-d_k-chain $(g_k)_{k<\omega}$ of elements $g_k \in \overline{G} \setminus G$.

(ii) For every element $g \in \overline{G}$ and every $n \in \omega$ with $q_n g = 0$ there exists a d_k-chain $(g_k)_{k<\omega}$ with $g_n = g$.

Proof. i) We assume as above that each summand in $G = \bigoplus_{k<\omega} G_k$ is an A-module of the form: $G_k \cong (A/p^{i_k}A)^{(\kappa_k)}$. Now we choose for each $k < \omega$ an element $e_k \in G_k$ which belongs to a natural basis of G_k. Then we have $\operatorname{Ann} e_k = p^{i_k}A$. We set $g_k := \sum_{n\geq k} \frac{q_n}{q_k} e_n$. Then $g_k \in \overline{G}$ with $\operatorname{Ann} g_k = q_k A = p^{i_k}A$. We have $g_{k+1} = \sum_{n\geq k+1} \frac{q_n}{q_{k+1}} e_n$ and

$$p^{d_k} g_{k+1} = \frac{p^{i_{k+1}}}{p^{i_k}} g_{k+1} = \frac{q_{k+1}}{q_k} g_{k+1} = \sum_{n\geq k+1} \frac{q_n}{q_k} e_n = g_k - \frac{q_k}{q_k} e_k.$$

Therefore $g_k - p^{d_k} g_{k+1} = e_k$ and $(g_k)_{k<\omega}$ is a basic-d_k-chain. Notice that all g_k have infinite support, hence $g_k \in \overline{G} \setminus G$.

ii) Let $g \in \overline{G}$ and $n \in \omega$ with $q_n g = 0$ be given. We need to define a suitable d_k-chain $(g_k)_{k<\omega}$ with $g_n = g$. For this we set $g_n = g$ and extend recursively to $k < n$ by

$$g_k = p^{d_k} g_{k+1}$$

which will guarantee $q_k g_k = 0$ via $q_k p^{d_k} = q_{k+1}$. It remains to define g_k for $k > n$. For this aim let $\operatorname{Ann} g = p^m A$ and $g = \sum_{k<\omega} z_k$ be a representation of g in $\overline{G}$. Then, since $q_n g = 0$ and therefore $q_n \geq \operatorname{Ann} g$,

$$p^m \leq q_n, p^m z_k = 0 \text{ for all } k < \omega \tag{3.5}$$

and we let

$$z_k =: \frac{q_k}{p^m} z'_k \text{ with } z'_k \in G \text{ for all } k < \omega \text{ with } q_k \geq p^m \text{ (thus } \frac{q_k}{p^m} \geq 1). \tag{3.6}$$

Keep in mind that $(z_k)_{k<\omega}$ is a null-sequence in the p-adic topology and each z_k is divisible by q_k.

We have $\operatorname{Ann} z_k = p^m A$ and $\operatorname{Ann} z'_k = q_k A$. We now set for $k > n$

$$g_k := \sum_{q_l \geq p^m \frac{q_k}{q_n}} \frac{q_l}{p^m \frac{q_k}{q_n}} z'_l \in \overline{G}. \tag{3.7}$$

Thus with $q_k p^{d_k} = q_{k+1}$ and therefore $p^{d_k} \frac{q_l}{p^m \frac{q_{k+1}}{q_n}} = \frac{q_l}{p^m \frac{q_k}{q_n}}$ we have for $k \geq n$:

$$p^{d_k} g_{k+1} = \sum_{q_l \geq p^m \frac{q_{k+1}}{q_n}} \frac{q_l}{p^m \frac{q_k}{q_n}} z'_l, \text{ implying}$$

$$g_k - p^{d_k} g_{k+1} = \sum_{q_l \in [p^m \frac{q_k}{q_n}, p^m \frac{q_{k+1}}{q_n}[} \frac{q_l}{p^m \frac{q_k}{q_n}} z_l',$$

which is a finite sum of elements of G. Thus

$$g_k - p^{d_k} g_{k+1} \in G \text{ for } k > n$$

by (3.7) while for $k = n$ we have

$$p^{d_n} g_{n+1} = \sum_{q_l \geq p^m \frac{q_{n+1}}{q_n}} \frac{q_l}{p^m} z_l' = \sum_{q_l \geq p^m \frac{q_{n+1}}{q_n}} z_l$$

with (3.6). Then with $\frac{q_{n+1}}{q_n} = p^{d_n}$ we have

$$g_n - p^{d_n} g_{n+1} = \sum_{q_l \geq p^m} \frac{q_l}{p^m} z_l' - \sum_{q_l \geq p^m \frac{q_{n+1}}{q_n}} z_l = \sum_{q_l \in [p^m, p^{m+d_n}[} z_l,$$

a finite sum of elements in G and therefore $g_n - p^{d_n} g_{n+1} \in G$. Lastly see $p^m \frac{q_k}{q_n} \leq q_k$ from 3.5 and thus in $q_k g_k = \sum_{q_l \geq p^m \frac{q_k}{q_n}} q_k \frac{q_l}{p^m \frac{q_k}{q_n}} z_l'$, we have $\frac{q_k}{p^m \frac{q_n}{q_k}} = p^j$ for some $j > 0$. Then $g_k g_k = \sum_{q_l \geq p^m \frac{q_k}{q_n}} p^j q_l z_l'$ and since $q_l z_l' = 0$, also $p^j q_l z_l' = 0$ for all $l < \omega$ and we have $q_k g_k = 0$ for $k > n$. ∎

The elements g_k from Lemma 3.2.3(a) are so-called **branch-elements**. Notice that Definition 3.2.2 and Lemma 3.2.3 carry over to the modules P and H if $G = P \oplus H$ as constructed above.

Now we can realize a first extension:

Proposition 3.2.2. *Let $G = P \oplus H$ be a direct sum of unbounded A-modules with chain $i_0 < i_1 < \ldots$ and B be an A-module basis of G with respect to this decomposition. Then for any d_k-chain $(p_k)_{k<\omega} \subseteq \overline{P}$ and any basic-d_k-chain $(h_k)_{k<\omega} \subseteq \overline{H}$ constructed as in Lemma 3.2.3(a) the following hold for $G' := \langle G, (p_k + h_k)A : k < \omega \rangle$.*

(i) $(p_k + h_k)_{k<\omega} \subseteq \overline{G}$ *is a d_k-chain inside G'.*

(ii) $B' := B \setminus \{e_k : k < \omega\} \cup \{p_k + h_k : k < \omega\}$ *with $e_k = h_k - p^{d_k} h_{k+1} \in B \cap H$ is a basis of G' and G' is an unbounded A-module with chain $i_0 < i_1 < \ldots$.*

(iii) G' *is pure in $\overline{G}$.*

(iv) $G'/G \cong A(p^\infty)$.

Proof. i): Clearly $(p_k + h_k) - p^{d_k}(p_{k+1} + h_{k+1}) \in G'$ by definition of G'. Since $q_k p_k = 0$ and $q_k h_k = 0$ for all $k < \omega$ according to the properties of a (basic-)d_k-chain, also $q_k(p_k + h_k) = 0$ for all $k < \omega$ and a) holds.
ii): We show, that the elements in B' are independent. Keep in mind that $B = B_P \cup B_H$, where B_P and B_H are disjoint, since $G = P \oplus H$. Since $e_k \in B_H$ and furthermore $\langle e_k \rangle_{k<\omega} = \langle h_k \rangle_{k<\omega}$ according to our assumption and since $(h_k)_{k<\omega}$ is a basic-d_k-chain, we conclude that B is the disjoint union of B_P and $B_H \setminus \{e_k : k < \omega\} \cup \{h_k : k < \omega\}$, where $B_H \setminus \{e_k : k < \omega\}$ and $\{h_k : k < \omega\}$ are disjoint, too. Then we can consider B as the disjoint union of $B \setminus \{e_k : k < \omega\}$ and $\{h_k : k < \omega\}$. With this decomposition of B in mind we let $\sum_{e_l \in B\setminus\{e_k:k<\omega\}} a_l e_l + \sum_{k<\omega} a_k(p_k + h_k) = (\sum_{e_l \in B\setminus\{e_k:k<\omega\}} a_l e_l + \sum_{k<\omega} a_k p_k) + \sum_{k<\omega} a_k h_k = 0$. Since the support of the first sum belongs to a subset of $B \setminus \{e_k : k < \omega\}$ and the support of the last summand belongs to a subset of $\{e_k : k < \omega\}$ resp. $\{h_k : k < \omega\}$, the two sums have disjoint supports and we can conclude $\sum_{e_l \in B\setminus\{e_k:k<\omega\}} a_l e_l + \sum_{k<\omega} a_k p_k = 0$ and $\sum_{k<\omega} a_k h_k = 0$ which leads to $a_k h_k = 0$ for all $k < \omega$. Hence $a_k \in \mathrm{Ann}(h_k) = q_k A$ as $(h_k)_{k<\omega}$ is a basic-d_k-chain and therefore $a_k p_k = a'(q_k p_k) = 0$ as $(p_k)_{k<\omega}$ is a d_k-chain. Thus $a_k(p_k + h_k) = 0$ for all $k < \omega$, and it remains $\sum_{e_l \in B\setminus\{e_k:k<\omega\}} a_l e_l = 0$ which implies $a_l e_l = 0$ for all l, according to the properties for the original basis B. This proves independence. Furthermore, as $e_k = h_k - p^{d_k} h_{k+1} = (p_k + h_k) - p^{d_k}(p_{k+1} + h_{k+1}) - (p_k - p^{d_k} p_{k+1})$ with $p_k - p^{d_k} p_{k+1} \in P \subseteq \langle B \setminus \{e_k\} \rangle$ and $(p_k + h_k) - p^{d_k}(p_{k+1} + h_{k+1}) \in \langle \{h_k : k < \omega\} \rangle$, we have that B' generates all elements of B and hence is a generating system of G'. Therefore B' is a basis of G'. As a consequence we obtain that G' is a direct sum of cyclics and an unbounded A-module with chain $i_0 < i_1 < ...$ as $\mathrm{Ann}(p_k + h_k) = q_k A$.
iii): We know that G is pure in $\overline{G}$. We have to show, that the equation $px = g' \in G'$ is solvable in G' whenever it is solvable in $\overline{G}$. Consider $px = g' \in G'$. Then $px = \tilde{g} + \sum a_k(p_k + h_k)$, where $\tilde{g} \in G$ and $\sum a_k(p_k + h_k) = \sum a_k(p_k + (e_k + p^{d_k} h_{k+1}) + 0) = \sum a_k(p_k + (e_k + p^{d_k} h_{k+1}) + (p^{d_k} p_{k+1} - p^{d_k} p_{k+1})) = \sum p^{d_k} a_k(p_{k+1} + h_{k+1}) + (\sum a_k e_k + \sum a_k(p_k - p^{d_k} p_{k+1}))$, with the latter sum being an element of G according to the properties of the d_k-chain $(p_k)_{k<\omega}$ and since the e_k are basis-elements of B. We set $\tilde{g} + (\sum a_k e_k + \sum a_k(p_k - p^{d_k} p_{k+1})) =: \tilde{g}' \in G$ and have $px = \tilde{g}' + \sum p^{d_k} a_k(p_{k+1} + h_{k+1})$. Now the equation $py = \tilde{g}'$ is solvable in G since it is solvable in $\overline{G}$ and G is pure in $\overline{G}$. Therefore there exists some $g \in G$ with $pg = \tilde{g}'$ and since $d_k \geq 1$ we have $px = pg + \sum p^{d_k} a_k(p_{k+1} + h_{k+1}) = p(g + \sum p^{d_k - 1} a_k(p_{k+1} + h_{k+1}))$, where now $g + \sum p^{d_k-1} a_k(p_{k+1} + h_{k+1})$ is the required solution in G'.
iv): Since $G'/G = \langle A(p_k + h_k) + G : k < \omega \rangle$ and $p_k + h_k = p^{d_k}(p_{k+1} + h_{k+1})$ (mod G) we have G'/G is p-divisible of rank 1 and thus $G'/G \cong A(p^\infty)$. ∎

The elements $p_k + h_k$ from Proposition 3.2.2 are so-called **branch-like elements**.

For our step lemma we also need the following easy

Lemma 3.2.4. *If $(z_k)_{k<\omega} \subseteq \overline{G}$ is a d_k-chain (not necessarily basic), then $\frac{q_k}{q_n} z_k - z_n \in G$ for all $k \geq n$.*

Proof. We proof the lemma by induction on k.
For $k = n$ trivially $\frac{q_n}{q_n} z_n - z_n = 0 \in G$ holds.
For $k = n+1$, $\frac{q_{n+1}}{q_n} z_{n+1} - z_n = -(z_n - p^{d_n} z_{n+1}) \in G$, according to the definition of a d_k-chain.
Now assume

$$\frac{q_{k-1}}{q_n} z_{k-1} - z_n \in G \text{ for some } k > n. \tag{3.8}$$

Then we have $\frac{q_k}{q_n} z_k - z_n = \frac{q_{k-1}}{q_n} \frac{q_k}{q_{k-1}} z_k - z_n = \frac{q_{k-1}}{q_n} p^{d_{k-1}} z_k - z_n$. Since $(z_k)_{k<\omega}$ is a d_k-chain we have $p^{d_{k-1}} z_k - z_{k-1} \in G$. We set $p^{d_{k-1}} z_k - z_{k-1} =: g_{k-1} \in G$, thus $p^{d_{k-1}} z_k = g_{k-1} + z_{k-1}$. Substituting this we get $\frac{q_k}{q_n} z_k - z_n = \frac{q_{k-1}}{q_n}(g_{k-1} + z_{k-1}) - z_n = (\frac{q_{k-1}}{q_n} z_{k-1} - z_n) + \frac{q_{k-1}}{q_n} g_{k-1}$, where the first part is in G according to (3.8) and the rest is naturally in G. Thus the assumption follows. ∎

Next we want to prove a consequence of Proposition 3.2.1.

Corollary 3.2.5. *Let A be a ring which satisfies the Pierce-condition and G an unbounded A-module with basis B, chain $i_0 < i_1 < \ldots$ and condition $(*)$. If $\varphi \in \operatorname{End} G \backslash A \oplus \operatorname{Ines} G$, then there exists a decomposition $G = P \oplus H$ with respect to B as a direct sum of unbounded A-modules, a basic-d_k-chain $(h_k)_{k<\omega} \subseteq \overline{H}$ and a d_k-chain $(p_k)_{k<\omega} \subseteq \overline{P}$ such that for $G_1 := \langle G, h_k A : k < \omega \rangle$ and $G_2 := \langle G, (p_k + h_k) A : k < \omega \rangle$ the following hold.*

(a) G_i is an unbounded A-module with $G_i \subseteq_* \overline{G}$, $G_i/G \cong A(p^\infty)$ and chain $i_0 < i_1 < \ldots$ for $i = 1, 2$.

(b) There exists either some $k < \omega$ with $h_k \varphi \notin G_1$ or some $k < \omega$ with $(p_k + h_k)\varphi \notin G_2$.

Proof. Notice that $G_1 = \langle G, h_k A : k \geq n \rangle$ and $G_2 = \langle G, (p_k + h_k) A : k \geq n \rangle$ for any $n < \omega$ by the properties of the (basic-) d_k-chains $(h_k)_{k<\omega}$ and $(p_k)_{k<\omega}$: Observe, that recursively, for $h_k \in G_1$, with $h_{k-1} - p^{d_k} h_k \in G$ also $h_{k-1} \in G_1$ follows.
Let $\varphi \in \operatorname{End} G \backslash A \oplus \operatorname{Ines} G$ be given. Then, according to Proposition 3.2.1 there exists

some decomposition $G = P \oplus H$ with respect to B as a direct sum of unbounded A-modules P, H with chain $i_0 < i_1 < ...$ and some $m < \omega$ with

$$\overline{P}[p^m](\varphi - c) \nsubseteq G \text{ for all } c \in A. \tag{3.9}$$

First we try the basic-d_k-chain $(h_k)_{k<\omega}$ as constructed in Lemma 3.2.3(a). For this choice condition (a) for $i = 1$ holds trivially by Proposition 3.2.2. If we are lucky then there exists some $k < \omega$ with $h_k\varphi \notin G_1$, condition (b) holds and also (a) for $i = 2$ is true with the trivial choice $p_k = 0$ for all $k < \omega$.
We therefore assume the bad case

$$h_k\varphi \in G_1 \text{ for all } k < \omega. \tag{3.10}$$

We fix some $n < \omega$ with

$$p^m \leq q_n \tag{3.11}$$

and consider $G_1 = \langle G, h_kA : k \geq n\rangle$ with this n, now fixed. Then each element in G_1 is the sum of some $g \in G$ and a linear combination of elements $h_k, k \geq n$. Without loss of generality, by the properties of the (basic-)d_k-chain $(h_k)_{k<\omega}$, instead of a linear combination of elements h_k, it suffices to consider only one summand. So, by (3.10) there exist now some $r \geq n$ (note that $\frac{q_r}{q_n}$ exists), $g \in G$ and $a \in A$ with $h_n\varphi = g + ah_r$, resp.

$$h_n\varphi - ah_r \in G. \tag{3.12}$$

We now distinguish two cases.
<u>Case 1</u>: $a \notin \frac{q_r}{q_n}A$.
With (3.2) we have $\varphi \in \operatorname{End} P$ and applying Corollary 3.2.1 leads to $\overline{P}[\frac{q_r}{q_n}](\frac{q_r}{q_n}\varphi - a) \nsubseteq P$, which implies

$$\overline{P}[\frac{q_r}{q_n}](\frac{q_r}{q_n}\varphi - a) \nsubseteq G \tag{3.13}$$

as otherwise $\overline{P}[\frac{q_r}{q_n}](\frac{q_r}{q_n}\varphi - a) \subseteq \overline{P} \cap G = P$, a contradiction. Thus $\overline{P}[\frac{q_r}{q_n}](\frac{q_r}{q_n}\varphi - a) \subseteq \overline{P} \setminus G$.
With (3.13) we choose $z \in \overline{P}[\frac{q_r}{q_n}]$ with

$$z(\frac{q_r}{q_n}\varphi - a) \notin G. \tag{3.14}$$

Since $\frac{q_r}{q_n}z = 0$, also $q_rz = 0$ and applying Lemma 3.2.3(b) to $z \in \overline{P}$ there exists some d_k-chain $(p_k)_{k<\omega}$ with

$$p_r = z. \tag{3.15}$$

With this choice condition (a) for $i = 2$ holds trivially by Proposition 3.2.2 and it remains to check condition (b). We claim

$$(p_n + h_n)\varphi \notin G_2$$

for the n fixed in (3.11).
Towards a contradition we assume $(p_n + h_n)\varphi \in G_2$ and set $G_2 = \langle G, (p_k + h_k)A : k \geq r\rangle$ with the r as fixed in (3.12). With the same argument as above, there exist some $s \geq r \geq n$ and $a' \in A$ with

$$(p_n + h_n)\varphi - a'(p_s + h_s) \in G.$$

Combining (3.12) and (3.16) leads to

$$p_n\varphi + ah_r - a'(p_s + h_s) = (p_n\varphi - a'p_s) + (ah_r - a'h_s) \in G.$$

With Lemma 3.2.4 we can add the summand $(\frac{q_s}{q_r}h_s - h_r)a \in G$ and get

$$(p_n\varphi - a'p_s) + (\frac{q_s}{q_r}a - a')h_s \in G. \tag{3.16}$$

Now with (3.2) the first summand is an element of $\overline{P}$ while the second summand is an element of $\overline{H}$. Thus by disjointness of supports follows

$$p_n\varphi - a'p_s \in G \text{ and } (\frac{q_s}{q_r}a - a')h_s \in G.$$

Since h_s has infinite support (as secured by Lemma 3.2.3 (a)the basic-d_k-chain is $\in \overline{H} \setminus H$) and elements in G have finite support, we have $\frac{q_s}{q_r}a - a' \in \operatorname{Ann} h_s = q_sA$ by the properties of the basic-d_k-chain $(h_k)_{k<\omega}$ and also $q_sA \subseteq \operatorname{Ann} p_s$ from the properties of the d_k-chain $(p_k)_{k<\omega}$. Thus $(\frac{q_s}{q_r}a - a')p_s = 0$ which implies $\frac{q_s}{q_r}ap_s = a'p_s$ and from $p_n\varphi - a'p_s \in G$ follows

$$p_n\varphi - \frac{q_s}{q_r}ap_s \in G.$$

Applying Lemma 3.2.4, adding $(\frac{q_s}{q_r}p_s - p_r)a \in G$ implies $p_n\varphi - ap_r \in G$ and using $\frac{q_r}{q_n}p_r - p_n \in G$ (also with Lemma 3.2.4) leads to $p_r(\frac{q_r}{q_n}\varphi - a) \in G$. With (3.15) we have $z(\frac{q_r}{q_n}\varphi - a) \in G$ contradicting (3.14).
<u>Case 2</u>: $a \in \frac{q_r}{q_n}A$.
Hence there exists some $b \in A$ with

$$a = \frac{q_r}{q_n}b. \tag{3.17}$$

With (3.9) we choose $z \in \overline{P}[p^m]$ with

$$z(\varphi - b) \notin G \tag{3.18}$$

and as in (3.11) fix n with $p^m \leq q_n$. Then, since q_n is a p-power $\geq p^m$ and $z \in \overline{P}[p^m]$, $q_n z = 0$ and applying Lemma 3.2.3(b) there exists some d_k-chain $(p_k)_{k<\omega}$ with

$$p_n = z. \tag{3.19}$$

With this choice condition (a) for $i = 2$ holds trivially by Proposition 3.2.2 and it remains to check condition (b). We claim again

$$(p_n + h_n)\varphi \notin G_2.$$

Assuming $(p_n + h_n)\varphi \in G_2$ towards a contradiction we can follow the same arguments as in Case 1 up to the point

$$p_n\varphi - ap_r \in G.$$

Now with (3.17) we have $p_n\varphi - \frac{q_r}{q_n}bp_r \in G$ which implies $p_n\varphi - bp_n \in G$ by adding $(\frac{q_r}{q_n}p_r - p_n)b \in G$ with Lemma 3.2.4. Thus with (3.19) follows $z(\varphi - b) \in G$ contradicting (3.18). ∎

We can finally formulate the

Step-Lemma 3.2.6. *Let A be a ring which satisfies the Pierce-condition. Moreover, let G be an unbounded A-module and let $G = \bigcup_{l<\omega} G_l$ be a chain of unbounded A-modules such that $G_{l+1} = G_l \oplus H_l$ for suitable unbounded A-modules H_l with chain $i_0 < i_1 < \ldots$ and condition $(*)$ for all $l < \omega$. If $\varphi \in \operatorname{End} G \setminus A \oplus \operatorname{Ines} G$, then there exists an extension G' of G such that the following hold.*

(i) *G' is a pure A-submodule of $\overline{G}$.*

(ii) *G' is an unbounded A-module with chain $i_0 < i_1 < \ldots$.*

(iii) *$G_l \sqsubset G'$ for all $l < \omega$.*

(iv) *$G'/G \cong A(p^\infty)$.*

(v) *φ does not extend to an endomorphism of G'. More precisely there exists some d_k-chain $(g_k)_{k<\omega} \subseteq \overline{G}$ inside G' with $g_k\varphi \notin G'$ for some $k < \omega$.*

Proof. Let B_0 be a basis of G_0 and B_{l+1} be a basis of H_l for $l < \omega$. Then $C_l := \bigcup_{k \leq l} B_k$ is a basis of G_l and $B := \bigcup_{k<\omega} B_k$ is a basis of G.
As H_l is an unbounded A-module with condition $(*)$ for all $l < \omega$ it is possible to construct a basic-d_k-chain $(h_k)_{k<\omega}$ like in Lemma 3.2.3(a) by choosing suitable elements $e_k \in B_k$ and Corollary 3.2.5 applies to this particular choice of $(h_k)_{k<\omega}$ as indicated in the proof of this Corollary. According to Corollary 3.2.5 we now construct two A-modules G_1 and G_2 and can choose $G' \in \{G_1, G_2\}$ such that all the properties (i), (ii), (iv) and (v) hold. For the remaining property (iii) just observe that like in Proposition 3.2.2(a)

$$C_l' := B \setminus \{e_k : l \leq k < \omega\} \cup \{h_k : l \leq k < \omega\} \text{ is a basis of } G_1$$

and

$$C_l'' := B \setminus \{e_k : l \leq k < \omega\} \cup \{p_k + h_k : l \leq k < \omega\} \text{ is a basis of } G_2$$

for all $l < \omega$ and that $C_l \subseteq C_{l+1}', C_{l+1}''$ by our choice of $e_k \in B_k$. ∎

Chapter 4

Realizing classes of partially isomorphic modules

4.1 The Realization Theorem

Now, assuming the Diamond Principle $\diamondsuit_\lambda E$ for λ a regular uncountable cardinal and E a stationary, non-reflecting subset of λ we will realize modules M with $\mathrm{End}(M)$ of the desired structure $A \oplus \mathrm{Ines}(M)$.
First, we construct a filtration of modules, satisfying certain useful properties:

Lemma 4.1.1. *Let A be a ring with Pierce-condition and $i_0 < i_1 < \ldots$ a sequence of positive integers. If $\lambda > |A|^+$ and we assume $\diamondsuit_\lambda E$ for a non-reflecting, stationary subset E of λ consisting of limit ordinals of cofinality ω, then there exists a λ-filtration of modules M_ν, which fulfill*

(0) $|M_\nu| = |\nu| + |A|^+ = |M_{\nu+1} \setminus M_\nu|$

(i) *M_ν is an unbounded A-module with chain $i_0 < i_1 < \ldots$.*

(ii) *If $\rho \in \nu \setminus E$, then $M_\rho \sqsubset M_\nu$ with $M_\nu = M_\rho \oplus H_{\nu\rho}$, where $H_{\nu\rho}$ is also unbounded with chain $i_0 < i_1 < \ldots$.*

(iii) *If $\nu = \rho + 1$ with $\rho \in E$ and $g_\rho \in \mathrm{End}(M_\rho) \setminus A \oplus \mathrm{Ines}(M_\rho)$, then g_ρ does not extend to an endomorphism of M_ν*

for all $\nu \in \lambda$.

Proof. We will carry out the construction of M inductively, defining a module-structure on a set M of cardinality λ.

For each $\alpha < \lambda$ of cofinality ω we can choose a strictly increasing sequence $\alpha_n \in \alpha \setminus E$ with $\sup_{n<\omega}\alpha_n = \alpha$. Since E consists of limit ordinals (cofinal to ω), the α_n's may be chosen as successor ordinals. Also, since E is non-reflecting, for any $\alpha \in LORD$ of cofinality $> \omega$ we have $E \cap \alpha$ is not stationary and thus there is a cub $C \subseteq \alpha$ with $E \cap C = \emptyset$. Therefore we find a continuous sequence $(\alpha_{\nu'})_{\nu'<\mathrm{cf}(\alpha)}$ with $\sup_{\nu'<\mathrm{cf}(\alpha)}\alpha_{\nu'} = \alpha$ and $\alpha_{\nu'} \in \alpha \setminus E$.

Now we define an A-module structure on M_ν inductively. We begin with setting

$$M_0 := \bigoplus_{n<\omega} (A/p^{i_n}A)^{(|A|^+)}$$

and see $|M_0| = |A|^+$, according to (0). Also, we already recognize our chain $i_0 < i_1 < ....$ Since the filtration will be a continuous chain, the construction is now reduced to an inductive step passing from M_ν to $M_{\nu+1}$. In constructing M_ν we have to consider three possible cases for ν:

Case 1: $\nu \in LORD$. In this case

$$M_\nu := \bigcup_{\rho<\nu} M_\rho.$$

Since the modules M_ρ form an ascending chain, we have $|M_\nu| = \sup_\rho|M_\rho| = \sup_\rho|\rho| + |A|^+ = |\nu| + |A|^+$. For the cardinality of $|M_{\nu+1} \setminus M_\nu|$ compare the calculations in case 2. (0) holds.
As $\nu \in LORD$ we know, according to E non-reflecting, that there is a continuous chain $(\nu_\beta)_{\beta<\mathrm{cf}(\nu)}$ with $\sup_{\beta<\mathrm{cf}(\nu)}\nu_\beta = \nu$ and $\nu_\beta \in \nu \setminus E$. Then, since the modules M_β form an ascending chain, $M_\nu = \bigcup_{\beta<\nu} M_\beta$ can be also written as $\bigcup_{\beta<\mathrm{cf}(\nu)} M_{\nu_\beta}$. With (ii) and induction hypothesis for ordinals $< \nu$ we have $M_{\nu_\beta} = M'_{\nu_\beta} \oplus H_{\nu_\beta}$ for all $\beta < \mathrm{cf}(\nu)$. Thus, since M_ν is the union of modules M_{ν_β} with $\nu_\beta \notin E$, M_ν can ultimately be given as $M_0 \oplus \bigoplus_{\beta<\mathrm{cf}(\nu)} H_{\nu_\beta}$ and since all the summands are unbounded with chain $i_0 < i_1 < ...$ by (ii), M_ν is, too. Thus (i) holds.
For $\rho \in \nu \setminus E$ chose $\beta < \mathrm{cf}(\nu)$ with $\rho < \nu_\beta < \nu$. Then $M_\rho \sqsubset M_{\nu_\beta}$ by induction hypothesis and (ii) since $\rho \notin E$. Also we have $M_{\nu_\beta} \sqsubset M_\nu$ according to the filtration and sequence (ν_β) and thus $M_\rho \sqsubset M_\nu$ and (ii) holds.
(iii) is empty in this case.

Case 2: $\nu = \rho + 1$, where $\rho \notin E$. Then

$$M_\nu := M_\rho \oplus \bigoplus_{n<\omega} (A/p^{i_n}A)^{(|A|^+)},$$

where the latter is a module isomorphic to M_0. Thus M_ν is isomorphic to $M_\rho \oplus M_0$. We have $|M_\nu| = |M_\rho| \cdot |M_0| = (|\rho| + |A|^+) \cdot |A|^+ = \max\{|\rho| + |A|^+, |A|^+\} = |\rho| + |A|^+ = |\nu| + |A|^+$, since $|\rho| = |\nu|$, as $\nu = \rho + 1$. On the other hand $M_\nu \setminus M_\rho = \{x = x_\rho \oplus x_0 : x_\rho \in M_\rho, x_0 \in M_0, x_0 \neq 0\}$ (since $x_\rho \oplus x_0 = x_\rho \in M_\rho$) and thus $|M_\nu \setminus M_\rho| = |M_\rho| \cdot |M_0 \setminus \{0\}|$, where $|M_0 \setminus \{0\}| = |M_0| = |A|^+$. So $|M_\nu \setminus M_\rho| = \max\{|\rho| + |A|^+, |A|^+\}$, too, and the equality required in (0) holds.
By induction hypothesis, M_ρ is an unbounded A-module with chain $i_0 < i_1 < ...$ and M_0 was, too. So, again by collecting basis elements, also M_ν is an unbounded A-module with chain $i_0 < i_1 < ...$ and (i) holds.
Consider $\rho' < \nu, \rho' \notin E$. If $\rho' = \rho$, $M_{\rho'} = M_\rho \sqsubset M_\nu$ by construction of M_ν. If $\rho' < \rho$, (ii) is already fulfilled for M_ρ and thus $M_{\rho'} \sqsubset M_\rho$, which is a summand of M_ν by construction.
(iii) is again empty in case 2.

Case 3: $\nu = \rho + 1$, where $\rho \in E$. Then we check if g_ρ (the Jensen function) fulfills $g_\rho \notin \mathrm{End}(M_\rho) \setminus A \oplus \mathrm{Ines}(M_\rho)$. In this case we may proceed as in Case 2. By checking property (ii) however, the case $\rho' = \rho$ does not occur in this case.
If now $g_\rho \in \mathrm{End}(M_\rho) \setminus A \oplus \mathrm{Ines}(M_\rho)$, then we may use the Step-Lemma 3.2.6 from the proceeding chapter to realize a construction for a module which will be chosen as M_ν.
If M_ν is constructed with the help of the Step-Lemma 3.2.6, condition (iv) of the Step-Lemma states $M_\nu / M_\rho \cong A(p^\infty)$. Then $|\rho| + |A|^+ = |M_\rho| \leq |M_\nu| = |M_\nu / M_\rho| \cdot |M_\rho| = |A(p^\infty)| \cdot |M_\rho| = \max\{|A(p^\infty)|, |M_\rho|\}$. With $|A(p^\infty)| = |A| \leq |M_\rho|$ we have

$$|M_\nu| = |M_\rho| = |\rho| + |A|^+ = |\nu| + |A|^+.$$

On the other hand $|M_\nu \setminus M_\rho| = |(M_\nu / M_\rho) \setminus \{0\}| \cdot |M_\rho| = |M_\nu / M_\rho| \cdot |M_\rho| = |A(p^\infty)| \cdot |M_\rho| = |A| \cdot |M_\rho|$ as above, thus $|M_\nu \setminus M_\rho| = |\nu| + |A|^+$.
M_ν is unbounded with chain $i_0 < i_1 < ...$ by the properties guaranteed in the Step-Lemma.
We have $M_{\rho'} \sqsubset M_{\rho_l} \sqsubset M_\nu$ for $\rho' < \rho$ with $\rho' \notin E$ and $\rho' < \rho_l < \rho$. The first inclusion is guaranteed by (ii) and induction hypothesis, the second holds by (iii) of the Step-Lemma. Thus (ii) is verified in case 3.
(iii) holds with (v) of the Step-Lemma. This completes the proof of 4.1.1. ∎

After the construction in 4.1.1 we set

$$M = \bigcup_{\alpha < \lambda} M_\alpha$$

for the constructed λ-filtration $\{M_\alpha : \alpha < \lambda\}$. Then,

$$M_\alpha \text{ is unbounded with chain } i_0 < i_1 < ...$$

for all α and for $\beta < \alpha, \beta \notin E$

$$M_\beta \sqsubset M_\alpha, M_\alpha = M_\beta \oplus M_{\alpha\beta} \tag{4.1}$$

where $M_{\alpha\beta}$ is also unbounded with chain $i_0 < i_1 < ....$

Next we will prove a second

Lemma 4.1.2. *If M is constructed as in 4.1.1, M is strongly-λ-direct with $|M| = \lambda$.*

Proof. Let $S \subseteq M, |S| < \lambda$. The properties of the filtrations tell us that there is $\beta < \lambda, \beta \notin E$ with $S \subseteq M_\beta$. We already know M_β is an unbounded A-module. We have to show that M/M_β is λ-direct-c. For M/M_β we have

$$M/M_\beta = \bigcup_{\beta<\alpha<\lambda} M_\alpha/M_\beta$$

a further filtration. Each M_α/M_β (for $\beta \notin E$) is an unbounded A-module with 4.1 and thus M/M_β is λ-direct-c and it follows that M is strongly-λ-direct.

Since $|M| = |\lambda| + |A|^+ = \lambda$, the proof is complete. ■

Next we will ensure a prescribed structure of $\operatorname{End} M$ with the following

Lemma 4.1.3. *If M is as above, M has a prescribed endomorphism ring $A \oplus \operatorname{Ines} M$.*

Proof. We assume the opposite and let $\varphi \in \operatorname{End} M \setminus (A \oplus \operatorname{Ines} M)$. We set

$$C := \{\alpha < \lambda : \varphi \upharpoonright M_\alpha \in \operatorname{End} M_\alpha\}$$

By a familiar back-and-forth argument, C is a closed, unbounded set.

We will prove the following

$(\star)$ $\exists \beta < \lambda : \forall \alpha \in C, \beta < \alpha < \lambda : \varphi \upharpoonright M_\alpha \in \operatorname{End} M_\alpha \setminus (A \oplus \operatorname{Ines} M_\alpha)$

Again, we assume the opposite. Then, for every β there is an $\alpha_\beta \in C, \beta < \alpha_\beta < \lambda$ such that $\varphi \upharpoonright M_{\alpha_\beta} \notin \operatorname{End} M_{\alpha_\beta} \setminus (A \oplus \operatorname{Ines} M_{\alpha_\beta})$, which yields $\varphi \upharpoonright M_{\alpha_\beta} = a_{\alpha_\beta} + \varphi_{\alpha_\beta}$, where $a_{\alpha_\beta} \in A$ and $\varphi_{\alpha_\beta} \in \operatorname{Ines} M_{\alpha_\beta}$. Since $\{\alpha_\beta : \beta < \lambda\}$ is unbounded in λ, $|\{\alpha_\beta : \beta < \lambda\}| = \lambda > |A|$. Therefore we know there exists $a \in A$ such that $\{\alpha_\beta : a_{\alpha_\beta} = a\}$ is unbounded in λ since there are $|A| < \lambda$ many possibilities for a

and thus $|A|$ such sets, of which at least one has to be unbounded in order to secure that the set $\{\alpha_\beta : \beta < \lambda\}$ is unbounded in λ. But then we have

$$\varphi = \bigcup_{a_{\alpha_\beta}} \varphi_{\alpha_\beta} + a_{\alpha_\beta} = \bigcup_{a_{\alpha_\beta}} \varphi_{\alpha_\beta} + a,$$

since the $\varphi_{\alpha_\beta} + a_{\alpha_\beta}$ form an ascending chain of partial maps and we have

$$\varphi - a = \bigcup_{a_{\alpha_\beta}=a} \varphi_{\alpha_\beta},$$

which is an element of Ines M_{α_β}.
It remains to show that $\varphi - a \in \operatorname{Ines} M$. Let $g \in \overline{M}$, which means $g \in \overline{M}_{\alpha_\beta}$ for some $\alpha_\beta \in C$ with $\alpha_\beta = a$. This holds since C is unbounded and thus M is a limit and can be replaced by some M_{α_β} for α_β big enough. We have to show $g(\varphi - a) \in M$. This is guaranteed by

$$g(\varphi - a) = g\varphi_{\alpha_\beta} \in M$$

since $\varphi_{\alpha_\beta} \in \operatorname{Ines} M_{\alpha_\beta}, g \in \overline{M}_{\alpha_\beta}$, where α_β was big enough, so that $\varphi - a = \bigcup_{a_{\alpha_\beta}=a} \varphi_{\alpha_\beta}$. This proves $\varphi - a$ is in Ines M and thus $(\star)$ holds.
Now chose $\gamma \in C \cap \{\alpha < \lambda : \beta < \alpha\} \cap E' = C \cap (\beta, \lambda) \cap E'$ where $E' = \{\alpha \in E : g_\alpha = \varphi \restriction M_\alpha\}$ with g_α the Jensen function, is stationary. Since (β, λ) is a cub and C is too, so is $C \cap (\beta, \lambda)$.
We will now concentrate on the construction, passing from M_γ to $M_{\gamma+1}$. Since $\gamma \in E' \subseteq E$ this step is similar to case 3 in 4.1.1. We have

$$\gamma \in E' \Rightarrow g_\gamma = \varphi \restriction M_\gamma,$$

$$\gamma \in C \Rightarrow g_\gamma = \varphi \restriction M_\gamma \in \operatorname{End} M_\gamma \text{ and}$$

$$\gamma < \beta \Rightarrow g_\gamma = \varphi \restriction M_\gamma \in \operatorname{End} M_\gamma \setminus (A \oplus \operatorname{Ines} M_\gamma) \text{ with } (\star).$$

The Step-Lemma yields that in this case, $\varphi \restriction M\gamma$ does not lift to $M_{\gamma+1}$. We therefore know

$$\exists g \in M_{\gamma+1} : \varphi(g) \notin M_{\gamma+1}. \tag{4.2}$$

Furthermore the existence of a d_k-chain $(g_k)_{k<\omega}$ with $g_0 = g$ and $g_k \in M_{\gamma+1}$ for all $k < \omega$ (Corollary 3.2.5) is secured by the Step-Lemma and definition 3.2.2. We have $g_k - p^{d_k} g_{k+1} \in M_\gamma$. With Lemma 3.2.4 follows

$$\frac{q_k}{q_n} g_k - g_n \in M_\gamma, \text{ for all } k \geq n.$$

Thus also $\frac{q_k}{q_0} g_k - g_0 \in M_\gamma$.
On the other hand, obviously $\varphi(g), \varphi(g_k) \in M$ and there exists $\gamma + 1 < \delta$ with $\varphi(g), \varphi(g_k) \in M_\delta$. Consider $\varphi(g) + M_{\gamma+1} \in M_\delta / M_{\gamma+1}$. The d_k-chain tells us $g_k - p^{d_k} g_{k+1} \in M_\gamma$. We chose $g_n = g$. Lemma 3.2.4 then yields

$$\frac{q_k}{q_n} g_k - g_n = \frac{q_k}{q_n} g_k - g \in M_\gamma$$

which implies, since $\varphi \restriction M_\gamma \in \operatorname{End} M_\gamma$, that $\frac{q_k}{q_n}\varphi(g_k) - \varphi(g) \in M_\gamma \subseteq M_{\gamma+1}$. Thus $\frac{q_k}{q_n}(\varphi(g_k) + M_{\gamma+1}) - (\varphi(g) + M_{\gamma+1}) = 0$ and thus

$$\varphi(g) + M_{\gamma+1} = \frac{q_k}{q_n}(\varphi(g_k) + M_{\gamma+1})$$

where $\frac{q_k}{q_n}$ becomes an arbitrary high power of p for k chosen suitable. Thus $\varphi(g) + M_{\gamma+1} \in M_\delta / M_{\gamma+1}$ is p-divisible. But the Step-Lemma ((ii)) also yields $M_{\gamma+1} \sqsubset M_\delta$, since $\gamma \in E$ and thus $\gamma + 1 \notin E$, and therefore $M_\delta / M_{\gamma+1}$ is an unbounded A-module and thus contains no elements ($\neq 0$) divisible by p (is p-reduced). As a consequence $\varphi(g) + M_{\gamma+1} = 0$ which implies

$$\varphi(g) \in M_{\gamma+1},$$

a contradiction to 4.2.Thus φ has to be $\in A \oplus \operatorname{Ines} M$. ∎

Finally, 4.1.1, 4.1.2 and 4.1.3 yield the

Realization Theorem 4.1.4. *Let A be a ring with Pierce-condition and $i_0 < i_1 < ...$ a sequence of positive integers. If $\lambda > |A|^+$ and we assume $\diamondsuit_\lambda E$ for a non-reflecting, stationary subset E of λ consisting of limit ordinals of cofinality ω, then there exists a strongly-λ-direct A-module M of cardinality λ with* $\operatorname{End}(M) = A \oplus \operatorname{Ines}(M)$.

4.2 Calculating Ulm invariants

We consider an unbounded A-module

$$G = \bigoplus_{n<\omega} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$$

with chain $i_0 < i_1 < ...$ and $\kappa_i > |A| > \aleph_0$ for all i.
We will calculate it's Ulm invariants and prove the following

Lemma 4.2.1. *If G is an unbounded A-module as given above,*
$u_p(\beta, G) = \begin{cases} \kappa_l, & \text{if } \beta < \omega \text{ and there exists } l \text{ with } \beta + 1 = i_l, \\ 0, & \text{else.} \end{cases}$
And consequently
$\hat{u}_p(\beta, G) = \begin{cases} \infty, & \text{if } \beta < \omega \text{ and there exists } l \text{ with } \beta + 1 = i_l, \\ 0, & \text{else.} \end{cases}$

Proof. We consider $p^k G = \bigoplus_{n<\omega} p^k \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$. The multiplication with p^k leads to an annulment of all summands $\mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$ with $i_n \leq k$ and thus

$$p^k G = \bigoplus_{i_n > k} p^k \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$$

which is isomorphic to $\bigoplus_{i_n>k} \mathbb{Z}_{p^{i_n-k}}^{(\kappa_n)}$.
To consider the socle we will use the fact that for all elements $a \in \mathbb{Z}_{p^{i_n}}, o(a) = p^{i_n}$, thus the socle can be given by

$$p^k G[p] = \bigoplus_{i_n > k} p^{i_n - 1} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}$$

and

$$p^{k+1} G[p] = \bigoplus_{i_n > k+1} p^{i_n - 1} \mathbb{Z}_{p^{i_n}}^{(\kappa_n)}.$$

To calculate the Ulm invariant we have to consider the dimension of the vector space $p^k G[p]/p^{k+1}G[p]$. We see that in the case of the module G this vector space only exists iff there is i_l in the chain with $i_l = k + 1$. In this case

$$p^k G[p]/p^{k+1} G[p] \cong p^{k+1-1} \mathbb{Z}_{p^{k+1}}^{(\kappa_l)} = p^k \mathbb{Z}_{p^{k+1}}^{(\kappa_l)} \cong \mathbb{Z}_p^{(\kappa_l)}.$$

and we see

$$u_p(k, G) = \dim(p^k G[p]/p^{k+1} G[p]) = \kappa_l.$$

If $i_l = k + 1$ does not exist in the chain, we have $u_p(k, G) = 0$.
Keep in mind $p^\alpha G = 0$ for $\alpha \geq \omega$ since we have p-cyclic groups. Thus $u_p(\beta, G) = 0$ for $\beta \geq \omega$ anyway. ∎

We keep in mind that in our case the construction is chosen such that $\kappa_l > |A| > \aleph_0$.

Now we want to consider a module $G = \bigcup_{\alpha<\lambda} G_\alpha$ as realized with the help of the Step-Lemma and the Realization Theorem, where all G_α are unbounded A-modules and have the same chain $i_0 < i_1 < \ldots$. We will show how the Ulm invariants of G are related to those of the G_α and that the generalized Ulm invariants of all modules are the same.

Lemma 4.2.2. *Let $G = \bigcup_{\alpha<\lambda} G_\alpha$ with a λ-filtration as realized by the Step-Lemma and the Realization Theorem, then*

(i) $u_p(\beta, G) = \sup_{\alpha<\lambda} u_p(\beta, G_\alpha)$

(ii) $\hat{u}_p(\beta, G) = \hat{u}_p(\beta, G_\alpha) = \begin{cases} \infty, & \text{if } \beta < \omega \text{ and there exists } l \text{ with } \beta + 1 = i_l, \\ 0, & \text{else.} \end{cases}$

Proof. It is $p^kG/p^{k+1}G = (p^kG + p^{k+1}G)/p^{k+1}G = (\bigcup_{\alpha<\lambda} p^kG_\alpha + p^{k+1}G)/p^{k+1}G = \bigcup_{\alpha<\lambda}((p^kG_\alpha + p^{k+1}G)/p^{k+1}G)$. Then the isomorphism theorem by Noether tells us

$$\bigcup_{\alpha<\lambda}((p^kG_\alpha + p^{k+1}G)/p^{k+1}G) \cong \bigcup_{\alpha<\lambda}(p^kG_\alpha/(p^kG_\alpha \cap p^{k+1}G)$$

and thus

$$p^kG/p^{k+1}G \cong \bigcup_{\alpha<\lambda}(p^kG_\alpha/p^{k+1}G_\alpha)$$

and this holds for the socles, too. Since all the quotiens are $\mathbb{Z}_p$-vector spaces, the dimension of the union is the supremum of the dimension of the quotiens, which means the supremum of $u_p(k, G_\alpha), \alpha < \lambda$. With 4.2.1 the equation for the generalized Ulm invariants is immediate. ∎

In the next chapter we will now consider possible classes of partially isomorphic modules.

4.3 Classes of partially isomorphic modules

In the last chapter we saw that the generalized Ulm invariants of G and all G_α's coincide, that, in other words, all these modules are L_∞-equivalent (following Barwise-Eklof) or, with Karp, partially isomorphic.
For a given, infinite sequence of integers $< \omega$ (the given chain), we are therefore able to construct modules G_α, G with

$$\hat{u}_p(\beta, G) = \hat{u}_p(\beta, G_\alpha) = \begin{cases} \omega, & \text{for infinitely many } \beta < \omega, \\ 0, & \text{for } \beta \geq \omega. \end{cases}$$

Obviously, by choosing different chains there are at least $2^{\aleph_0}$ many different modules which provide the same generalized Ulm invariants and thus are partially isomorphic. See that $\mathbb{Z}_p^{(\kappa)} \cong_p \mathbb{Z}_p^{(\kappa')}$ but $\mathbb{Z}_p^{(\kappa)} \not\cong \mathbb{Z}_p^{(\kappa')}$ for any $\kappa \neq \kappa' > \aleph_0$.

However, the modules thus collected in the equivalence class of partially isomorphic modules are unbounded A-modules (in the sense of Section 3.1), where some have a prescribed endomorphism ring (G) and others don't (G_α).
With referring to chapter 9 of [GT] we will show how big the classes of partially isomorphic modules can become, regulated with help of the Diamond.
Obviously, by choosing λ in [GT] 9.1.19 accordingly, modules in the class can be constructed with arbitrary cardinality. Moreover, one can prove a corresponding

Theorem 4.3.1. *Assume $\diamondsuit_\lambda E$ for a non-reflecting stationary subset E of λ consisting of limit ordinals cofinal to ω. Moreover, let A be a ring with Pierce-condition and $|A^+| < \lambda$ and let $i_0 < i_1 < ..$ be a sequence of positive integers.*
Then there exist 2^λ pairwise non-isomorphic strongly-λ-direct A-modules M of cardinality λ with chain $i_0 < i_1 < ...$ and $\operatorname{End} M = A \oplus \operatorname{Ines} M$ which are partially isomorphic, i. e. L_∞-equivalent and have the same Ulm-Kaplansky invariants, also in the non-generalized case.

Proof. Theorem 9.1.17 of [GT] tells us, that if $\diamondsuit_\lambda E$ holds, there exists a decomposition $E = \bigcup_{\beta<\lambda} E_\beta$ such that $\diamondsuit_\lambda E_\beta$ holds for all $\beta < \lambda$. Every E_β is a stationary set.
Now, for each non-empty set $S \subseteq \lambda$, the set $E_S := \bigcup_{\beta \in S} E_\beta$ is stationary, too. For $S = \lambda$, we just have $E_S = E$. Obviously, there are $|\mathcal{P}(S)| = 2^\lambda$ many possibilities for S. We will, for each set S, construct a module M_S which fulfills the desired properties.
The construction of M_S will be similar to the construction of M in Section 4.1, with only two differences:

1. Instead of E and $\diamondsuit_\lambda E$ consider E_S and $\diamondsuit_\lambda E_S$.

2. By constructing M_ν, in the case $\nu = \rho + 1$ with $\rho \in E_S$ and the Jensen function $g_\rho \notin \operatorname{End} M_\rho \setminus A \oplus \operatorname{Ines} M_\rho$ we do not proceed as in the proof of the Realization Theorem and simply add a summand isomorphic to M_0, but will use a different construction instead.

In detail, in the case $\nu = \rho + 1, \rho \in E_S, g_\rho \notin \operatorname{End} M_\rho \setminus A \oplus \operatorname{Ines} M_\rho$ we construct $M_{\rho+1}$ from M_ρ with Proposition 3.2.2 by choosing an arbitrary d_k-chain $(p_k)_{k<\omega}$, as well as an arbitrary basic-d_k-chain $(h_k)_{k<\omega}$ as in Lemma 3.2.3 (a).
Then $|M_{\rho+1}| = |M_\rho| \cdot |M_{\rho+1}/M_\rho| = (|\rho| + |A|^+) \cdot |A(p^\infty)|$, where $|A(p^\infty)| < |A|$ and thus $|M_{\rho+1}| = |\rho| + |A|^+ = |\rho + 1| + |A|^+$.

The resulting M_S has all the properties as desired by the Realization Theorem and also the same Ulm invariants as a module realized with the Realization Theorem would have. Thus all M_{S_i} thus achieved are partially isomorphic. Moreover, one might prove a Step-Lemma similar to ours in Chapter 3, which will provide a module fulfilling all properties with exception of (v).
It remains to show that for $\emptyset \neq S_1, S_2 \in \lambda$, $S_1 \neq S_2$, $M_{S_1} \ncong M_{S_2}$. This will be verified with the help of the **Gamma invariant** $\Gamma(M_S)$. To define $\Gamma(M_S)$, at first we set

$$G_S := \{\nu < \lambda : M_S/M_\nu \text{ is not strongly-}\lambda\text{-direct}\} \subseteq \lambda \tag{4.3}$$

and consider the equivalence relation on $\mathcal{P}(\lambda)$ we already introduced in Section 1.3:

$$G_1 \sim G_2 :\Leftrightarrow \exists \text{ cub } C \subseteq \lambda : G_1 \cap C = G_2 \cap C. \tag{4.4}$$

See, that if $G_1 \sim G_2, G_2 \sim G_3$, $G_1 \cap C_1 = G_2 \cap C_1$ and $G_2 \cap C_2 = G_3 \cap C_2$ and therefore, since $C_1 \cap C_2$ is again a cub, $G_1 \cap (C_1 \cap C_2) = G_2 \cap (C_1 \cap C_2) = G_3 \cap (C_1 \cap C_2)$. Then

$$\Gamma(M_S) = [G_S]_\sim = G_S/\sim . \tag{4.5}$$

Since for $\alpha \in \lambda \setminus E_S : M/M_\alpha$ is strongly-λ-direct as secured by our construction and for $\alpha \in E_S : M_{\alpha+1}/M_\alpha \cong A(p^\infty)$ and $M_{\alpha+1}/M_\alpha \subseteq M/M_\alpha$ with M/M_α not strongly-λ-direct, we have

$$M_S/M_\nu \text{ is not strongly-}\lambda\text{-direct } \Leftrightarrow \nu \in E_S \tag{4.6}$$

and thus

$$\Gamma(M_S) = [E_S]_\sim. \tag{4.7}$$

But for $\emptyset \neq S_1, S_2$ and $S_1 \neq S_2$ we have $[E_{S_1}] \neq [E_{S_2}]$: Would $E_{S_1} \cap C = E_{S_2} \cap C$ for some cub C, then there would be $\alpha \in S_2, \alpha \notin S_1$ with $E_\alpha \cap C \neq \emptyset$ and, since $E_\alpha \subseteq E_{S_2}, E_\alpha \cap C \subseteq E_{S_2} \cap C = E_{S_1} \cap C$. If follows $E_{S_1} \cap E_\alpha \neq \emptyset$, which is a contradiction to $\alpha \notin S_1$. This completes the proof. ■

To further control the Ulm-Kaplansky invariants $u_p(\beta, G)$, resp. $u_p(\beta, G_\alpha)$ for $\beta < \omega$ but with no connection to the chain we now neglect the properties of the modules' endomorphism rings. Then, we can 'fill up' the 'realizable' sequence of Ulm-Kaplansky invariants κ_l as provided by Lemma 4.2.1 as in the following theorem:

Theorem 4.3.2. *Let A be a ring with Pierce-condition. For any sequence of Ulm-Kaplansky invariants*

$$u_p(\beta, G) = \begin{cases} \kappa_i, & \text{for countably many } \beta < \omega, \\ k_i, & \text{for finitely many (but different) } \beta < \omega, \\ 0, & \text{if } \beta \geq \omega, \end{cases}$$

where $\kappa_l > |A| > \aleph_0$ and k_i arbitrary, finite or infinite, and assuming $\diamondsuit_\lambda E$ for λ a regular uncountable cardinal and E a non-reflecting, stationary subset of λ consisting of limit ordinals cofinal to ω, there exist (at least) 2^λ pairwise non-isomorphic A-modules realizing this sequence of invariants which are all partially isomorphic and are unions of ascending chains of direct sums of cyclic p-groups. Assuming $\lambda > \kappa_i, k_i$ for all i and also $\lambda > |A^+|$, the modules all have cardinality λ.

Proof. At first, for a chosen chain $i_0 < i_1 < ...$, starting with the realizable sequence of Ulm-Kaplansky invariants as in Lemma 4.2.1, the modules are constructed as in Theorem 4.3.1. Then, for any $\beta < \omega$ where no l with $i_l = \beta + 1$ exists, we can secure $u_p(\beta, G)$ (resp. $u_p(\beta, G_\alpha)$)$= k$ (k as desired) by replacing G by $G \oplus \mathbb{Z}^{(k)}_{p^{\beta+1}}$ (resp. G_α by $G_\alpha \oplus \mathbb{Z}^{(k)}_{p^{\beta+1}}$). The replaced modules remain partially isomorphic (even isomorphic, if they were so to start with) and the modules G_α still provide a λ-filtration for G. By adding a suitably chosen sum $\bigoplus_{\beta_i} \mathbb{Z}^{(k_i)}_{p^{\beta_i+1}}$, finitely many values $u_p(\beta_i, G)$ (resp. $u_p(\beta_i, G_\alpha)$) can be prearranged as desired. ■

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