# EXISTENCE AND REGULARITY OF SOLUTION FOR A STOCHASTIC CAHN-HILLIARD/ALLEN-CAHN EQUATION WITH UNBOUNDED NOISE DIFFUSION

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Abstract. The Cahn-Hilliard/Allen-Cahn equation with noise is a simplified mean field model of stochastic microscopic dynamics associated with adsorption and desorption-spin flip mechanisms in the context of surface processes. For such an equation we consider a multiplicative space-time white noise with diffusion coefficient of linear growth. Applying technics from semigroup theory, we prove local existence and uniqueness in dimensions $d = 1, 2, 3$. Moreover, when the diffusion coefficient satisfies a sub-linear growth condition of order $\alpha$ bounded by $\frac{1}{3}$, which is the inverse of the polynomial order of the nonlinearity used, we prove for $d = 1$ global existence of solution. Path regularity of stochastic solution, depending on that of the initial condition, is obtained a.s. up to the explosion time. The path regularity is identical to that proved for the stochastic Cahn-Hilliard equation in the case of bounded noise diffusion. Our results are also valid for the stochastic Cahn-Hilliard equation with unbounded noise diffusion, for which previous results were established only in the framework of a bounded diffusion coefficient. As expected from the theory of parabolic operators in the sense of Petrovskĭı, the bi-Laplacian operator seems to be dominant in the combined model.

**Keywords:** Stochastic Cahn-Hilliard/Allen-Cahn equation, space-time white noise, convolution semigroup, Galerkin approximations, unbounded diffusion.

## 1. Introduction

### 1.1. The Stochastic equation.

We consider the Cahn-Hilliard/Allen-Cahn equation with multiplicative space-time noise:

$$
\left\{
\begin{array}{rcl}
u_t & = & -\varrho\Delta\Big(\Delta u - f(u)\Big) + \Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W} \quad \text{in} \quad \mathcal{D}\times[0,T), \\
u(x,0) & = & u_0(x) \quad \text{in} \quad \mathcal{D}, \\
\frac{\partial u}{\partial \nu} & = & \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{D}\times[0,T).
\end{array}
\right. \tag{1.1}
$$

Here, $\mathcal{D}$ is a rectangular domain in $\mathbb{R}^d$ with $d = 1, 2, 3$, $\varrho > 0$ is a "physical diffusion" constant, $f$ is a polynomial of degree 3 with a positive leading coefficient, such as $f = F'$ where $F(u) = (1 - u^2)^2$ is a double equal-well potential. The "noise diffusion" coefficient $\sigma(\cdot)$ is a Lipschitz function with sub-linear growth, $\dot{W}$ is a space-time white noise in the sense of Walsh [22], and $\nu$ is the outward normal vector. In addition, we assume that the initial condition $u_0$ is sufficiently integrable or regular, depending on the desired results on the solution. Obviously, when $\sigma := 1$, the noise in (1.1) becomes additive.

In this paper, as in [3], we will analyze the more general case of multiplicative noise. However, unlike [3], we consider a more general Lipschitz coefficient $\sigma$ with sub-linear growth such that

$$
|\sigma(u)| \le C(1 + |u|^\alpha), \tag{1.2}
$$

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for some $\alpha \in (0,1]$ and a positive constant $C$.

In the sequel, we will give sufficient conditions on the initial condition $u_0$ so that:

(1) a unique local maximal solution exists when $d = 1, 2, 3$, for $\alpha = 1$, that is when $\sigma$ satisfies the classical linear growth condition;
(2) when $\alpha < \frac{1}{3}$, i.e. when $\alpha$ is strictly smaller than the inverse of the polynomial order of the nonlinear function $f$, a global solution exists with Lipschitz path-regularity for $d = 1$.

The stochastic Cahn-Hilliard equation can be considered as a special case of our model. Therefore, when the function $\sigma$ satisfies the aforementioned sub-linear growth assumption, our method extends all the results of [3] on existence and uniqueness of a local maximal solution when $d = 1, 2, 3$, and on global existence and path-regularity, when $d = 1$, for the solution of the stochastic Cahn-Hilliard equation with a multiplicative noise; in reference [3] C. Cardon-Weber considered a bounded diffusion coefficient. It seems to us that there is a gap in the proof of global existence given in reference [3] on page 793. Indeed the various constraints imposed on the parameters $d$, $a$, $q$, $r$ and $\gamma'$ lead to a contradiction; however we did not disprove the statement of the corresponding Theorem 1.3. The argument we use in this paper to prove global existence is different from that in [3] and is based on the Gagliardo Nirenberg inequality. Using the factorization method for the stochastic term, we derive a path regularity similar to that obtained in [3]. The path regularity can also be obtained a.s. in dimensions $d = 2, 3$ for any time interval $[0, T]$ where $T$ is strictly smaller than the explosion time $T^*(\omega)$.

1.2. **The physical background.** Surface diffusion and adsorption/desorption consist the micromechanisms that are typically involved in surface processes or on cluster interface morphology. Chemical vapor deposition, catalysis, and epitaxial growth are surface processes involving transport and chemistry of precursors in a gas phase where the unconsumed reactants and radicals adsorb onto the surface of a substrate so that surface diffusion, or reaction and desorption back to the gas phase is observed. Such processes have been modelled by continuum-type reaction diffusion models where interactions between particles are neglected or treated phenomenologically, [18, 12]. Alternatively, a more precise microscocpic description is provided in statistical mechanics theories, [15]. For instance we can consider a combination of Arrhenius adsorption/desorption dynamics, Metropolis surface diffusion and simple unimolecular reaction; the corresponding mesoscopic equation is:

$$u_t - D\nabla \cdot \Big[\nabla u - \beta u(1-u)\nabla J * u\Big] - \Big[k_a p(1-u) - k_d u \exp\big(-\beta J * u\big)\Big] + k_r u = 0\,. \tag{1.3}$$

Here, $D$ is the diffusion constant, $k_r$, $k_d$ and $k_a$ denote respectively the reaction, desorption and adsorption constants while $p$ is the partial pressure of the gaseous species. The partial pressure $p$ is assumed to be a constant, although realistically it is given by the fluids equations in the gas phase. Furthermore, $J$ is the particle-particle interaction energy and $\beta$ is the inverse temperature.

Stochastic microscopic dynamics such as Glauber and Metropolis dynamics have been analyzed for adsorption/desorption-spin flip mechanisms in the context of surface processes; for more details we refer to the review article [20]. In addition, the Kawasaki and Metropolis stochastic dynamics models describe the diffusion of a particle on a surface, where sites cannot be occupied by more than one particle. Stochastic time-dependent Ginzburg-Landau type equations with additive Gaussian white noise source such as Cahn-Hilliard and Allen-Cahn appear as Model B and Model A respectively in the classical theory of phase transitions according to the universality classification of Hohenberg and Halperin [17]. A simplified mean field mathematical model, associated with the aforementioned mechanisms that describes surface diffusion, particle-particle interactions and as well as adsorption to and desorption from the surface, is a partial differential equation written as a combination of Cahn-Hilliard and Allen-Cahn type equations with noise. The Cahn-Hilliard operator is related to mass conservative phase separation and surface diffusion in the presence of interacting particles. On the other hand, the Allen-Cahn operator is related to adsorption and desorption and serves as a diffuse interface model for antiphase grain boundary coarsening.

At large space-time scales the random fluctuations are suppressed and a deterministic pattern emerges. Such a deterministic model has been analyzed by Katsoulakis and Karali in [19]. The so called mean field partial differential equation has the following form:

$$\begin{cases} u_t = -\varepsilon^2 \varrho \Delta \left( \Delta u - \dfrac{f(u)}{\varepsilon^2} \right) + \Delta u - \dfrac{f(u)}{\varepsilon^2}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

where $f(u) = F'$ for $F = (1-u^2)^2/4$ a double-well potential with wells $\pm 1$, $\varrho > 0$ is the diffusion constant and $0 < \varepsilon \ll 1$ is a small parameter. In [19], the authors rigorously derived the macroscopic cluster evolution laws and transport structure as a motion by mean curvature depending on surface tension to observe that due to multiple mechanisms an effective mobility speeds up the cluster evolution.

**Remark 1.1.** *The stochastic equation analyzed in this work is a simplified mean field model for interacting particle systems used in statistical mechanics. These systems are Markov processes set on a lattice corresponding to a solid surface. A typical example is the Ising-type systems defined on a multi-dimensional lattice; see* [14]. *Assuming that the particle-particle interactions are attractive, then the resulting system's Hamiltonian is nonnegative (attractive potential). Hence, the diffusion constant $\varrho$ of the SPDE* (1.1) *is considered positive, as in* [19].

**Remark 1.2.** *Ginzburg-Landau type operators are usually supplemented by Neumann or periodic boundary conditions. In order to obtain an initial and boundary value problem we consider the SPDE* (1.1) *with the standard homogeneous Neumann boundary conditions on $u$ and $\Delta u$. These conditions are frequently used for the deterministic or stochastic Cahn-Hilliard equation; see e.g.* [11, 7, 3].

1.3. **Main results.** As a first step for a rigorous mathematical analysis of the stochastic model, in Section 2, we will prove the existence and uniqueness of a local maximal solution to (1.1) when the initial condition $u_0$ belongs to $L^q(\mathcal{D})$ for $q \in [3, \infty)$ if $d = 1, 2$ and $q \in [6, \infty)$ if $d = 3$. Section 4 describes some possible general assumptions on the domain $\mathcal{D}$ which would lead to the same result obtained in dimensions $2, 3$, and presents the stochastic Cahn-Hilliard equation as a special case of a Cahn-Hilliard/Allen-Cahn stochastic model. Note that the approach used in this paper to solve this nonlinear SPDE with a polynomial growth is similar to that developed by J.B. Walsh [22] and I. Gyöngy [16] for the stochastic heat equation and related SPDEs. Unlike these references, the smoothing effect of the bi-Laplace operator enables us to deal with a stochastic perturbation driven by a space-time white noise in dimension 1 up to 3.

The existence-uniqueness proof is similar to that of Cardon-Weber in [3], and relies on upper estimates of the fundamental solution, Galerkin approximations and the application of a cut-off function. However, the fact that the diffusion coefficient $\sigma$ is unbounded requires to multiply $\sigma$ by the cut-off function in order to estimate properly the stochastic integral, and then to use *a priori* estimates for the remaining part.

With our method we prove existence of a unique local maximal solution under the requirement that $\sigma$ satisfies the classical linear growth condition: $|\sigma(u)| \le C(1 + |u|)$ for some positive constant $C$. Furthermore, if $\sigma$ satisfies a sub-linear growth condition $|\sigma(u)| \le C(1 + |u|^\alpha)$ with $0 < \alpha < \frac{1}{3}$, we prove global existence and path regularity for $d = 1$. Here, we point out that the supremum of $\alpha$ coincides with the inverse of the polynomial order of the nonlinearity $f$. Our argument does not extend in dimensions $d = 2, 3$ even if we know that the $L^2$ norm of the local maximal solution remains bounded on any given time interval, and if we have not proved that explosion of the $L^q$ norm takes place in finite time.

The upper estimates on the Green's function stated in sections 2 and 3 obviously show that all the results in [3] can be extended to our framework if $\sigma$ is bounded. Note that in many papers dealing with some multiplicative random perturbation of a PDE with polynomial non-linearity, when the diffusion coefficient $\sigma$ is defined "point wise" and the driving noise is a space time white noise or a gaussian noise which is white in space and colored in time, then $\sigma$ is bounded. This is the case for the Burgers equation in [16] and subsequent papers, in [8], [3] and [4] for the 2 and 3D Cahn-Hilliard equation and related parabolic equations. Therefore, one of the main contributions of this paper is to deal with some unbounded noise coefficient $\sigma$ for the stochastic Cahn-Hilliard equation and Cahn-Hilliard/Allen-Cahn equations. Certain attempts to go beyond the fact that $\sigma$ is bounded were done by S. Cerrai in [5] for the stochastic Allen-Cahn equation (see also the work of M. Kuntze and J. van Neerven, [21], for a more general framework). In these papers, the authors first proved the existence of a global solution when $\sigma$ is sub-linear. Then using dissipativity, they extended this result to the linear growth assumption on $\sigma$ as in the classical case of SDEs. Note that we could not apply the technique introduced by [5] for the stochastic Allen-Cahn equation to go from sub-linear to linear growth; this is due to the fact that in our model, in contrast to the Allen-Cahn equation, the Laplace operator is applied to the nonlinearity. However, we believe that global solutions with an analogous path regularity could exist in higher dimensions for smoother noise in space; for example the formal derivative of a Fourier series of Brownian motions. Moreover, the nonlinear function $f$ could be defined as the derivative of a general double equal well potential of higher polynomial order.

Our method based on the factorization method for the deterministic and random forcing terms, yields for $d = 1$ the same regularity as that proven in [3], where $\sigma$ is bounded.

As usual we denote by $C$ a generic constant and by $C(s)$ a constant depending on some parameter $s$. For $p \in [1, \infty]$, the $L^p(\mathcal{D})$-norm is denoted by $\| \cdot \|_p$. Finally, given real numbers $a$ and $b$ we let $a \vee b$ (resp. $a \wedge b$) denote the maximum (resp. the minimum) of $a$ and $b$.

## 2. The corresponding evolution equation

2.1. **Preliminaries.** For simplicity and to ease notation, without restriction of generality, we will assume that the "physical diffusion" constant $\varrho$ is equal to 1 and that $\mathcal{D}$ is the unitary cube. Extension to more general domains will be addressed in the next section.

In order to give a mathematical meaning to the stochastic PDE (1.1) we integrate in time and space and use the initial and boundary conditions (see e.g. [22]). For a strict definition of solution, we say that $u$ is a weak (analytic) solution of the equation (1.1) if it satisfies the following weak formulation:

$$\begin{aligned} &\int_{\mathcal{D}} \Big( u(x,t) - u_0(x) \Big) \phi(x)\, dx = \\ &\int_0^t \int_{\mathcal{D}} \Big( - \Delta^2 \phi(x) u(x,s) + \Delta\phi(x) [f(u(x,s)) + u(x,s)] - \phi(x) f(u(x,s)) \Big)\, dxds \\ &+ \int_0^t \int_{\mathcal{D}} \phi(x) \sigma(u(x,s))\, W(dx,ds), \end{aligned} \tag{2.1}$$

for all $\phi \in \mathcal{C}^4(\mathcal{D})$ with $\frac{\partial \phi}{\partial \nu} = \frac{\partial \Delta \phi}{\partial \nu} = 0$ on $\partial \mathcal{D}$. Note that this $u$ stands as a probabilistic 'strong solution' since we keep the given space-time white noise and do not only deal with the distribution of the processes.

The random measure $W(dx, ds)$ is the $d$-dimensional space-time white noise, that is induced by the one-dimensional $(d+1)$-parameter (with $d$ space variables and one time variable) Wiener process $W$ defined as $W := \big\{ W(x,t) : \ t \in [0,T], \ x \in \mathcal{D} \big\}$. For every $t \geq 0$ we let $\mathcal{F}_t :=$

$\sigma\big(W(x,s):\ s\le t,\ x\in\mathcal{D}\big)$ denote the filtration generated by $W$, cf. [22, 3, 2]. Furthermore, we assume that the coefficient $\sigma:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and satisfies the following growth condition for some $\alpha\in(0,1]$ and $C>0$:

$$|\sigma(x)|\le C(1+|x|^\alpha),\ \forall x\in\mathbb{R}.$$

2.2. **Estimates for the Green's function.** Let $\Delta$ denote the Laplace operator; we shall use the Green's function for the operator $\mathcal{T}:=-\Delta^2+\Delta$ on $\mathcal{D}$ with the homogeneous Neumann conditions, that is the fundamental solution to $\partial_t u-\mathcal{T}u=0$ on $\mathcal{D}$ with the boundary conditions $\frac{\partial u}{\partial\nu}=\frac{\partial\Delta u}{\partial\nu}=0$ on $\partial\mathcal{D}\times[0,T)$. Let $k=(k_i,\ i=1,\cdots,d)$ denote a multi-index with non-negative integer components $k_i$ and let $\|k\|^2:=\sum_i k_i^2$. We set $\epsilon_0(x):=\frac{1}{\sqrt{\pi}}$, and for any positive integer $j$ we define $\epsilon_j(x):=\sqrt{\frac{2}{\pi}}\cos(jx)$. Finally for $k=(k_i)\in\mathbb{N}^d$ and $x\in\mathcal{D}$ let $\epsilon_k(x):=\prod_i\epsilon_{k_i}(x_i)$. Then $(\epsilon_k,\ k\in\mathbb{N}^d)$ is an orthonormal basis of $L^2(\mathcal{D})$ consisting on eigenfunctions of $\mathcal{T}$ corresponding to the eigenvalues $-\lambda_k^2-\lambda_k$ where $\lambda_k=\|k\|^2$. Of course, $\epsilon_0$ is related to the null eigenvalue.

Let $S(t):=e^{(-\Delta^2+\Delta)t}$ be the semi-group generated by the operator $\mathcal{T}$; if $u:=\sum_k(u,\epsilon_k)\,\epsilon_k$ then

$$\mathcal{T}u=\sum_k-(\lambda_k^2+\lambda_k)(u,\epsilon_k)_{L^2(\mathcal{D})}\,\epsilon_k,$$

and (see e.g. [7, 3]) the convolution semigroup is defined by

$$S(t)U(x):=\sum_k e^{-(\lambda_k^2+\lambda_k)t}(U,\epsilon_k)_{L^2(\mathcal{D})}\epsilon_k(x),$$

for any $U$ in $L^2(\mathcal{D})$ with the associated Green's function given by

$$G(x,y,t)=\sum_k e^{-(\lambda_k^2+\lambda_k)t}\,\epsilon_k(x)\,\epsilon_k(y), \tag{2.2}$$

for $t>0$, $x,y\in\mathcal{D}$. Using the Definition 1.3 of [10], we deduce that $\mathcal{T}=-\Delta^2+\Delta$ is uniformly strongly parabolic in the sense of Petrovskiĭ. Thus, as proved in [9], the following upper estimates of the Green function $G$ and its various derivatives hold true. Notice that they are similar to those of the Green's function used in [3] for the operator $-\Delta^2$.

**Lemma 2.1.** *Let $G$ be the Green's function defined by* (2.2). *Then there exist positive constants $c_1$ and $c_2$ such that for any $t\in(0,T]$, any $x,y\in\mathcal{D}$ and any multi-index $k=(k_i,\ i=1,\cdots,d)$ with $|k|=\sum_{i=1}^d k_i\in\{1,2\}$, the next inequalities are satisfied:*

$$|G(x,y,t)|\ \le\ c_1\,t^{-\frac{d}{4}}\exp\Big(-c_2\,|x-y|^{\frac43}\,t^{-\frac13}\Big), \tag{2.3}$$

$$|\partial_x^k G(x,y,t)|\ \le\ c_1\,t^{-\frac{d+|k|}{4}}\exp\Big(-c_2\,|x-y|^{\frac43}\,t^{-\frac13}\Big), \tag{2.4}$$

$$|\partial_t G(x,y,t)|\ \le\ c_1\,t^{-\frac{d+4}{4}}\exp\Big(-c_2\,|x-y|^{\frac43}\,t^{-\frac13}\Big). \tag{2.5}$$

Furthermore, given any $c>0$ there exists a positive constant $C(c)$ such that

$$\int_{\mathbb{R}^d}\exp\big(-c|x|^{\frac43}t^{-\frac13}\big)dx=C(c)t^{\frac{d}{4}}. \tag{2.6}$$

Let us also define for $x\in\mathcal{D}$ and $t>s\ge0$

$$h(x,t,s):=-c_2\,|x|^{\frac43}\,(t-s)^{-\frac13}. \tag{2.7}$$

The following lemma gathers several estimates for integrals of space (respectively time) increments of $G$. Note once more that the results are the same as those of Lemma 1.8 in [3] and are deduced from the explicit formulation (2.2) of $G$ by using similar arguments.

**Lemma 2.2.** *Let $G$ be the Green's function defined by* (2.2)*. Given positive constants $\gamma, \gamma'$ with $\gamma < (4-d)$, $\gamma \le 2$ and $\gamma' < 1 - \frac{d}{4}$, there exists a constant $C > 0$ such that for any $t > s \ge 0$ and any $x, y \in \mathcal{D}$ the next estimates hold true:*

$$\int_0^t \int_{\mathcal{D}} |G(x,z,t-r) - G(y,z,t-r)|^2 \, dzdr \le C|x-y|^\gamma, \tag{2.8}$$

$$\int_0^s \int_{\mathcal{D}} |G(x,z,t-r) - G(x,z,s-r)|^2 \, dzdr \le C|t-s|^{\gamma'}, \tag{2.9}$$

$$\int_s^t \int_{\mathcal{D}} |G(x,z,t-r)|^2 \, dzdr \le C|t-s|^{\gamma'}. \tag{2.10}$$

2.3. **Integral representation.** Using the Green's function, we can present the solution of equation (2.1) in an integral form for any $x \in \mathcal{D}$ and $t \in [0,T]$, that is the following mild solution:

$$\begin{aligned} u(x,t) = & \int_{\mathcal{D}} u_0(y) G(x,y,t) \, dy \\ & + \int_0^t \int_{\mathcal{D}} \big[\Delta G(x,y,t-s) - G(x,y,t-s)\big] f(u(y,s)) \, dyds \\ & + \int_0^t \int_{\mathcal{D}} G(x,y,t-s) \sigma(u(y,s)) \, W(dy,ds). \end{aligned} \tag{2.11}$$

Application of the inequality (2.6) and Hölder's inequality lead to the following bound for the term involving the initial condition.

**Lemma 2.3.** *Let $G(x,y,t)$ be the Green's function defined by* (2.2)*. For every $1 \le q < \infty$ and $T > 0$ there exists a constant $C := C(T,q)$ such that*

$$\sup_{t \in [0,T]} \|G_t u_0\|_q \le C \|u_0\|_q, \tag{2.12}$$

*where $G_0 = Id$ and $G_t u_0$ is defined for $t > 0$ by*

$$G_t u_0(x) := \int_{\mathcal{D}} u_0(y) G(x,y,t) \, dy. \tag{2.13}$$

2.4. **Well posedness of the truncated equation.** In order to prove the existence of the solution $u$ to (2.11), as a first step we consider an appropriated cut-off SPDE, cf. [3]. Let $\chi_n \in C^1(\mathbb{R}, \mathbb{R}^+)$ be a cut-off function satisfying $|\chi_n| \le 1$, $|\chi_n'| \le 2$ for any $n > 0$ and

$$\chi_n(x) = \begin{cases} 1 & \text{if } \ |x| \le n, \\ 0 & \text{if } \ |x| \ge n+1. \end{cases}$$

For fixed $n > 0$, $x \in \mathcal{D}$, $t \in [0,T]$ and $q \in [3, +\infty)$, we consider the following cut-off SPDE:

$$\begin{aligned} u_n(x,t) = & \int_{\mathcal{D}} u_0(y) G(x,y,t) \, dy \\ & + \int_0^t \int_{\mathcal{D}} \big[\Delta G(x,y,t-s) - G(x,y,t-s)\big] \, \chi_n(\|u_n(\cdot,s)\|_q) \, f(u_n(y,s)) \, dyds \\ & + \int_0^t \int_{\mathcal{D}} G(x,y,t-s) \, \chi_n(\|u_n(\cdot,s)\|_q) \, \sigma(u_n(y,s)) \, W(dy,ds). \end{aligned} \tag{2.14}$$

In this section we suppose that $\sigma$ satisfies (1.2) with $\alpha \in (0,1]$, and that the following condition $(\mathbf{C}_\alpha)$ holds:

**Condition $(\mathbf{C}_\alpha)$** *One of the following properties (i) or (ii) is satisfied:*

*(i) $d = 1,2$ and $q \in [3, +\infty)$, or $d = 3$ and $q \in [6, +\infty)$,*

*(ii)* $d = 3$ *and* $q \in \big(3 \vee [6(1-\alpha)], 6\big)$.

We show the existence and uniqueness of the solution to the SPDE (2.14) in the set $\mathcal{H}_T$ defined by

$$\mathcal{H}_T := \Big\{u(\cdot,t) \in L^q(\mathcal{D}) \text{ for } t \in [0,T]: \ u \text{ is } (\mathcal{F}_t)\text{-adapted and } \|u\|_{\mathcal{H}_T} < \infty\Big\},$$

where

$$\|u\|_{\mathcal{H}_T} := \sup_{t\in[0,T]} \Big(E\|u(\cdot,t)\|_q^\beta\Big)^{\frac{1}{\beta}}, \tag{2.15}$$

for $\beta \in (\frac{q}{\alpha}, \infty)$ if Condition $(\mathbf{C}_\alpha)$**(i)** holds, or for $\beta \in (\frac{q}{\alpha}, \frac{6q}{(6-q)})$ if Condition $(\mathbf{C}_\alpha)$**(ii)** holds.

**Remark 2.4.** *In order to present our results in a more general framework we consider the growth condition* (1.2) *with* $\alpha \in (0,1]$*; the upper bound of* $\alpha$ *will be restricted in the sequel.*

**Remark 2.5.** *Note that if* $d = 3$*, the inequality* $6(1-\alpha) < q < 6$ *implies that the interval* $(\frac{q}{\alpha}, \frac{6q}{(6-q)})$ *is not empty.*

**Theorem 2.6.** *Let* $\sigma$ *be globally Lipschitz and satisfy the assumption* (1.2) *with* $\alpha \in (0,1]$*, let* $u_0 \in L^q(\mathcal{D})$ *and let Condition* $(\mathbf{C}_\alpha)$ *hold. Furthermore, let* $\beta \in (\frac{q}{\alpha}, +\infty)$ *if Condition* $(\mathbf{C}_\alpha)$**(i)** *is satisfied (resp.* $\beta \in (\frac{q}{\alpha}, \frac{6q}{6-q})$ *if Condition* $(\mathbf{C}_\alpha)$**(ii)** *is satisfied). Then the SPDE* (2.14) *admits a unique solution* $u_n$ *in every time interval* $[0,T]$ *and* $u_n \in \mathcal{H}_T$*. Moreover, if for some stopping time* $\tau$ *a local process* $\big(\tilde{U}(.,t), t \in [0,\tau)\big)$ *is a local solution to* (2.14)*, then the processes* $u_n(.,t)_{|[0,\tau)\times\Omega}$ *and* $\tilde{U}_{|[0,\tau)\times\Omega}$ *are equivalent.*

*Proof.* We define the operators $\mathcal{M}_n$ and $\mathcal{L}_n$ on $\mathcal{H}_T$ by

$$\mathcal{M}_n(u)(x,t) := \int_0^t\int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)]\chi_n(\|u(\cdot,s)\|_q) f(u(y,s))\,dyds, \tag{2.16}$$

$$\mathcal{L}_n(u)(x,t) := \int_0^t\int_{\mathcal{D}} G(x,y,t-s)\,\chi_n(\|u(\cdot,s)\|_q)\,\sigma(u(y,s))\,W(dy,ds), \tag{2.17}$$

with $u \in \mathcal{H}_T$. Then obviously (2.14) is written as

$$u_n(x,t) = \int_{\mathcal{D}} u_0(y)G(x,y,t)\,dy + \mathcal{M}_n(u_n)(x,t) + \mathcal{L}_n(u_n)(x,t). \tag{2.18}$$

We claim that if $T > 0$ is sufficiently small, then the operator $\mathcal{M}_n + \mathcal{L}_n$ is a contraction mapping from $\mathcal{H}_T$ to $\mathcal{H}_T$.

First we consider the mapping $\mathcal{M}_n$. For an arbitrary function $u \in \mathcal{H}_T$, by Minkowski's inequality, (2.3) and (2.4) we have

$$\begin{aligned}\|\mathcal{M}_n(u)(\cdot,t)\|_q \le c_1 \int_0^t &(t-s)^{-\frac{d+2}{4}} \\ &\times \Big\{ \int_{\mathcal{D}} \Big| \int_{\mathcal{D}} \exp\Big(-c_2\frac{|x-y|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}\Big)\chi_n(\|u(\cdot,s)\|_q) f(u(y,s))\,dy\Big|^q dx\Big\}^{\frac{1}{q}}\,ds.\end{aligned}$$

By using Young's inequality with exponents $\rho$ and $r$ in $[1,\infty)$ such that $\frac{1}{\rho} + \frac{1}{r} = \frac{1}{q} + 1$, we obtain for $h(x,t,s) := -c_2\frac{|x|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}$ defined by (2.7)

$$\begin{aligned}\|\mathcal{M}_n(u)(\cdot,t)\|_q &\le c_1 \int_0^t (t-s)^{-\frac{d+2}{4}}\|\exp(h(\cdot,t,s))\|_r \Big\|\chi_n(\|u(\cdot,s)\|_q) f(u(.,s))\Big\|_\rho\,ds \\ &\le C \int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r}} \Big\|\chi_n(\|u(\cdot,s)\|_q) f(u(.,s))\Big\|_\rho\,ds,\end{aligned} \tag{2.19}$$

where the last inequality follows from (2.6). We now choose $\rho = \frac{q}{3} \geq 1$ since $q \in [3,\infty)$ and $r \in [1,\infty)$ satisfying $\frac{1}{\rho} + \frac{1}{r} = \frac{1}{q} + 1$ . The function $f$ is a polynomial of degree 3, so, for $n \geq 1$ we have

$$\Big\| \chi_n(\|u(\cdot,s)\|_q) f(u(.,s)) \Big\|_{\frac{q}{3}} \leq C n^3. \tag{2.20}$$

Since $q > d$ we deduce that $-\frac{d+2}{4} + \frac{d}{4r} > -1$; hence the above inequalities yield

$$\|\mathcal{M}_n(u)\|_{\mathcal{H}_T} = \sup_{t\in[0,T]} E\Big( \|\mathcal{M}_n(u(\cdot,t))\|_q^\beta \Big)^{\frac{1}{\beta}} \leq C\, n^3\, T^{-\frac{d}{4}+\frac{d}{4r}+\frac{1}{2}}. \tag{2.21}$$

Therefore, $\mathcal{M}_n$ is a mapping from $\mathcal{H}_T$ to $\mathcal{H}_T$.

Moreover, for arbitrary $u$ and $v$ in $\mathcal{H}_T$ such that $\|u(\cdot,s)\|_q \leq \|v(\cdot,s)\|_q$, we shall prove that for $q \in [3,\infty)$ and $\rho = \frac{q}{3}$ the next inequality holds true:

$$\Big\| \chi_n(\|u(\cdot,s)\|_q) f(u(\cdot,s)) - \chi_n(\|v(\cdot,s)\|_q) f(v(\cdot,s)) \Big\|_\rho \leq C\, n^3 \|u(\cdot,s) - v(\cdot,s)\|_q. \tag{2.22}$$

Indeed, we have

$$\begin{aligned} &\Big\| \chi_n(\|u(\cdot,s)\|_q) f(u(\cdot,s)) - \chi_n(\|v(\cdot,s)\|_q) f(v(\cdot,s)) \Big\|_\rho \\ &\quad \leq \Big\| \big[\chi_n(\|u(\cdot,s)\|_q) - \chi_n(\|v(\cdot,s)\|_q)\big]\, f(u(\cdot,s)) \Big\|_\rho + \Big\| \chi_n(\|v(\cdot,s)\|_q)\, \big[f(u(\cdot,s)) - f(v(\cdot,s))\big] \Big\|_\rho . \end{aligned}$$

Note that $\|v(\cdot,s)\|_q \geq \|u(\cdot,s)\|_q$ and

$$\chi_n(\|u(\cdot,s)\|_q) - \chi_n(\|v(\cdot,s)\|_q) = 0 \;\text{ if }\; \|u(\cdot,s)\|_q \geq n+1.$$

Hence, for $n \geq 1$ and $\rho = \frac{q}{3}$ we obtain the existence of $C > 0$ such that for all $n \geq 1$

$$\begin{aligned} \Big\| \big[\chi_n(\|u(\cdot,s)\|_q) - \chi_n(\|v(\cdot,s)\|_q)\big]\, f(u(\cdot,s)) \Big\|_\rho &\leq C\Big(1 + (n+1)^3\Big) \Big| \|u(\cdot,s)\|_q - \|v(\cdot,s)\|_q \Big| \\ &\leq C\, n^3\, \|u(\cdot,s) - v(\cdot,s)\|_q. \end{aligned}$$

Using again the inequality $\|v(\cdot,s)\|_q \geq \|u(\cdot,s)\|_q$, then for any $n \geq 1$ we deduce the existence of $C > 0$ such that

$$\begin{aligned} &\Big\| \chi_n(\|v(\cdot,s)\|_q)\, \big[f(u(\cdot,s)) - f(v(\cdot,s))\big] \Big\|_\rho \\ &\qquad \leq C \chi_n(\|v(\cdot,s)\|_q) \Big(1 + \|v(\cdot,s)\|_q^2 + \|u(\cdot,s)\|_q^2\Big) \|u(\cdot,s) - v(\cdot,s)\|_q \\ &\qquad \leq C n^2\, \|u(\cdot,s) - v(\cdot,s)\|_q, \end{aligned}$$

holds for any $n \geq 1$. Thus, (2.22) is valid.

Inequality (2.22) and an argument similar to that used for proving (2.19) yield

$$\begin{aligned} &\|\mathcal{M}_n(u)(\cdot,t) - \mathcal{M}_n(v)(\cdot,t)\|_q \\ &\quad \leq \int_0^t |t-s|^{-\frac{d+2}{4}+\frac{d}{4r}} \Big\| \chi_n(\|u(\cdot,s)\|_q) f(u(\cdot,s)) - \chi_n(\|v(\cdot,s)\|_q) f(v(\cdot,s)) \Big\|_\rho\, ds \\ &\quad \leq C\, n^3 \int_0^t |t-s|^{-\frac{d+2}{4}+\frac{d}{4r}} \|u(\cdot,s) - v(\cdot,s)\|_q\, ds. \end{aligned} \tag{2.23}$$

Therefore, by inequality (2.23) and Hölder's inequality, since $\beta \in [q,\infty)$, we deduce

$$\begin{aligned}\|\mathcal{M}_n(u)-\mathcal{M}_n(v)\|_{\mathcal{H}_T} &\le C\,n^3 \sup_{t\in[0,T]}\Big\{E\Big|\int_0^t |t-s|^{-\frac{d+2}{4}+\frac{d}{4r}}\|u(\cdot,s)-v(\cdot,s)\|_q\,ds\Big|^\beta\Big\}^{1/\beta}\\
&\le C\,n^3 \sup_{t\in[0,T]}\Big\{\Big(\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r}}ds\Big)^{\beta-1}\int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r}}E\|u(.,s)-v(.,s)\|_q^\beta ds\Big\}^{\frac{1}{\beta}}\\
&\le C\,n^3\,T^{(-\frac{d+2}{4}+\frac{d}{4r}+1)}\sup_{t\in[0,T]}\Big(E\|u(\cdot,t)-v(\cdot,t)\|_q^\beta\Big)^{\frac{1}{\beta}}\\
&\le C\,n^3\,T^{-\frac{d+2}{4}+\frac{d}{4r}+1}\|u-v\|_{\mathcal{H}_T}.\end{aligned}\tag{2.24}$$

Obviously, by (2.21) and (2.24) it follows that for fixed $n\ge 1$ and $T>0$, the map $\mathcal{M}_n$ is Lipschitz from $\mathcal{H}_T$ to $\mathcal{H}_T$.

For the mapping $\mathcal{L}_n$ defined in terms of a stochastic integral, at first notice that since $\alpha \in (0,1]$, the inequality $\beta > \frac{q}{\alpha}$ yields $\beta\in(q,\infty)$. Thus, the Hölder, Burkholder and Minkowski inequalities, and the growth condition (1.2) on $\sigma$ yield

$$\begin{aligned}E\|\mathcal{L}_n(u(\cdot,t))\|_q^\beta &\le C\int_{\mathcal{D}} E|\mathcal{L}_n(u(x,t)|^\beta\,dx\\
&\le C\int_{\mathcal{D}} E\Big|\int_0^t\int_{\mathcal{D}}\big|G(x,y,t-s)\,\chi_n(\|u(\cdot,s)\|_q)\,\sigma(u(y,s))|^2\,dyds\Big|^{\beta/2}dx\\
&\le C\Big(E\int_0^t\Big\|\int_{\mathcal{D}}G^2(\cdot,y,t-s)\,\chi_n(\|u(\cdot,s)\|_q)\,\big[1+|u(y,s)|^{2\alpha}\big]\,dy\Big\|_{\beta/2}ds\Big)^{\beta/2}.\end{aligned}$$

Since $\beta\in(\frac{q}{\alpha},\infty)$, we have $\frac{2\alpha}{q}>\frac{2}{\beta}$ and we may choose $\bar r\in(1,\infty)$ such that $\frac{2\alpha}{q}+\frac{1}{\bar r}=\frac{2}{\beta}+1$.

Let once more $h(x,t,s)$ be defined by (2.7); Young's inequality and (2.3) imply

$$\begin{aligned}E\|\mathcal{L}_n(u(\cdot,t)\|_q^\beta &\le C\Big(E\int_0^t (t-s)^{-\frac{d}{2}}\Big\|\exp(h(\cdot,t,s))\Big\|_{\bar r}\chi_n(\|u(\cdot,s)\|_q)\,\big\|[1+|u(\cdot,s)|^{2\alpha}]\big\|_{\frac{q}{2\alpha}}ds\Big)^{\beta/2}\\
&\le C\Big(E\int_0^t (t-s)^{-\frac{d}{2}+\frac{d}{4\bar r}}(1+n^{2\alpha})ds\Big)^{\beta/2}.\end{aligned}$$

Note that the inequalities $d<4$, $q\ge 3$, $\alpha\in(0,1]$ and $\beta>\frac{q}{\alpha}$ yield $-\frac{d}{2}+\frac{d}{4\bar r}>-1$. Hence, for any $u\in\mathcal{H}_T$ we obtain the existence of $C>0$ such that

$$\|\mathcal{L}_n(u)\|_{\mathcal{H}_T}\le C(1+n^\alpha)T^{\frac{1}{2}[-\frac{d}{2}+\frac{d}{4\bar r}+1]},\tag{2.25}$$

holds for every $n\ge 1$. Therefore, $\mathcal{L}_n$ is also a mapping from $\mathcal{H}_T$ to $\mathcal{H}_T$.

Recall that $\sigma$ is Lipschitz. Therefore, an argument similar to that used to prove (2.22) with $q$ instead of $\rho$ shows that for $u,v\in\mathcal{H}_T$, we have

$$\|\delta(u,v,\cdot,s)\|_q\le C(1+n^\alpha)\|u(\cdot,s)-v(\cdot,s)\|_q,\tag{2.26}$$

for

$$\delta(u,v,y,s):=\chi_n(\|u(\cdot,s)\|_q)\sigma(u(y,s))-\chi_n(\|v(\cdot,s)\|_q)\sigma(v(y,s)).$$

Recall that $\alpha \in (0,1]$ and $\beta > \frac{q}{\alpha}$, so that $\beta > q$; thus the Hölder, Burkholder-Davies-Gundy and Minkowski inequalities together with (2.3) yield for $u$, $v$ in $\mathcal{H}_T$

$$\begin{aligned} E\|\mathcal{L}_n(u)(\cdot,s)-\mathcal{L}_n(v)(\cdot,s)\|_q^\beta &\le C\int_{\mathcal{D}} E|\mathcal{L}_n(u)(\cdot,s)-\mathcal{L}_n(v)(\cdot,s)|^\beta\,dx \\ &\le CE\int_{\mathcal{D}}\Big|\int_0^t\int_{\mathcal{D}} G^2(x,y,t-s)\delta^2(u,v,y,s)dyds\Big|^{\beta/2}dx \\ &\le CE\Big|\int_0^t (t-s)^{-\frac{d}{2}}\Big\|\exp(h(\cdot,t,s))*\delta^2(u,v,\cdot,s)\Big\|_{\beta/2}ds\Big|^{\beta/2}. \end{aligned}$$

The inequality $\beta > q$ implies the existence of $r_2 \in (1,\infty)$ such that $\frac{2}{\beta}+1=\frac{2}{q}+\frac{1}{r_2}$. Using once more the assumptions on $q$, $\alpha$ and $\beta$ in Condition ($\mathrm{C}_\alpha$), in particular the assumption $\beta(6-q) < 6q$ and $q \in [3,6)$ for $d=3$, we deduce $-\frac{d}{2}+\frac{d}{4r_2} > -1$. Thus Young's inequality and (2.26) imply

$$\begin{aligned} E\|\mathcal{L}_n(u)(\cdot,s)-\mathcal{L}_n(v)(\cdot,s)\|_q^\beta &\le CE\Big|\int_0^t (t-s)^{-\frac{d}{2}+\frac{d}{4r_2}}\|\delta(u,v,\cdot,s)\|_{q/2}ds\Big|^{\beta/2} \\ &\le C(1+n^{\alpha\beta})T^{(-\frac{d}{2}+\frac{d}{4r_2}+1)\frac{\beta}{2}}\sup_{t\in[0,T]} E\|u(\cdot,s)-v(\cdot,s)\|_q^\beta, \end{aligned}$$

and therefore,

$$\|\mathcal{L}_n(u)-\mathcal{L}_n(v)\|_{\mathcal{H}_T} \le C\,(1+n^\alpha)\,T^{-\frac{d}{4}+\frac{d}{8r_2}+\frac{1}{2}}\,\|u-v\|_{\mathcal{H}_T}. \tag{2.27}$$

So, for fixed $n$ and $T>0$, the map $\mathcal{L}_n$ is also a Lipschitz mapping from $\mathcal{H}_T$ to $\mathcal{H}_T$.

The upper estimates (2.24) and (2.27) imply that the mapping $\mathcal{M}_n+\mathcal{L}_n$ is Lipschitz from $\mathcal{H}_T$ to $\mathcal{H}_T$ with the Lipschitz constant bounded by

$$C(n,T) := C\Big[n^3T^{-\frac{d+2}{4}+\frac{d}{4r}+1}+Cn^\alpha T^{-\frac{d}{4}+\frac{d}{8r_2}+\frac{1}{2}}\Big].$$

For fixed $n \ge 1$, there exists $T_0(n)$ sufficiently small (which does not depend on $u_0$) such that $C(n,T) < 1$ for $T \le T_0(n)$, so that $\mathcal{M}_n+\mathcal{L}_n$ is a contraction mapping from the space $\mathcal{H}_T$ into itself. Thus for $T \le T_0(n)$, the map $\mathcal{M}_n+\mathcal{L}_n$ has a unique fixed point in the set $\Big\{u \in \mathcal{H}_T : \; u(\cdot,0)=u_0\Big\}$. This implies that in $[0,T]$, for $T \le T_0(n)$, there exists a unique solution $u_n$ for the SPDE (2.14).

If $T > T_0(n)$, let $\bar u_0(x) = u_n(x,T_0(n))$ and $\bar W(t,x) = W(T_0(n)+t,x)$; then $\dot{\bar W}$ is a space-time white noise related to the filtration $(\mathcal{F}_{T_0(n)+t}, t\ge 0)$ independent of $\mathcal{F}_{T_0(n)}$. A similar argument proves the existence and uniqueness of the solution $\bar u_n$ to an equation similar to (2.14) with $u_0$ and $W$ replaced by $\bar u_0$ and $\bar W$ respectively. Hence, (2.14) has a unique solution $u_n$ on the interval $[0,2T_0(n)]$, defined by $u_n(x,t) := \bar u_n(x,t-T_0(n))$ for $t \in [T_0(n),2T_0(n)]$. Since there exists $N \ge 1$ such that $NT_0(n) \ge T$ an easy induction argument proves existence and uniqueness of the solution to (2.14) on any given time interval $[0,T]$.

Finally, to prove the last assertion, let $\tau$ be a stopping time and let $T_0(n) > 0$ be defined as above. Then the fixed point theorem proves that for $\tau_1 = \tau \wedge T_0(n)$, the processes $u_n(.,t)|_{[0,\tau_1)\times\Omega}$ and $\tilde{U(.,t)}|_{[0,\tau_1)\times\Omega}$ are equivalent. We then deduce by induction that for every positive integer $k$ and $\tau_k = \tau \wedge (kT_0(n))$, the processes $\tilde U|_{[0,\tau_k)\times\Omega}$ and $u_n(.,t)|_{[0,\tau_k)\times\Omega}$ are equivalent. Since the sequence $\tau_k$ increases to $\tau$, the proof is complete. □

2.5. **Local maximal solutions of the stochastic equation.** Our aim is to prove the existence of a local maximal solution to equation (2.11). For every positive integer $n$, let $T_n$ be the stopping time defined by

$$T_n = \inf\big\{t \ge 0 \;:\; \|u_n(.,t)\|_q \ge n\big\} \wedge n.$$

Then replacing in (2.14) the deterministic time $t$ by the random one $t \wedge T_n$, and using the fact that for $s \le t \wedge T_n$ we have $\chi_n(\|u_n(.,s)\|_q) = 1$, using the local property of stochastic integrals, we deduce that the process $(u_n(.,t \wedge T_n), t \ge 0)$ is a solution of the equation:

$$\begin{aligned} u_n(x, t \wedge T_n) = & \int_{\mathcal{D}} u_0(y) G(x,y,t)\, dy \\ & + \int_0^{t\wedge T_n} \int_{\mathcal{D}} \big[\Delta G(x,y,t\wedge T_n - s) - G(x,y,t\wedge T_n - s)\big] f(u_n(y,s))\, dyds \\ & + \int_0^{t\wedge T_n} \int_{\mathcal{D}} G(x,y,t\wedge T_n - s) \sigma(u_n(y,s))\, W(dy,ds). \end{aligned}$$

Therefore, the process $(u_n(.,t), t < T_n)$ is a solution to (2.11).

Let us fix two positive integers $n, k$ with $n < k$; then the processes $u_n(.,t)|_{[0,T_n\wedge T_k]}$ and $u_k(.,t)|_{[0,T_n\wedge T_k]}$ are both solutions of equation (2.14); the uniqueness stated in Theorem 2.6 proves that they are indistinguishable. Let $T^* := \limsup_n T_n$; then for any $\epsilon > 0$ the processes $u_n(.,t)$ and $u_k(.,t)$ agree on the time interval $[0, (T^* - \epsilon) \vee 0]$ for any $n$ and $k$ such that $T_n \ge (T^* - \epsilon) \vee 0$ and $T_k \ge (T^* - \epsilon) \vee 0$. This proves the existence of a solution $u(.,t)$ to the equation (2.11) on the time interval $[0, T^*)$; this solution is defined by the limit value of the truncated processes along a subsequence (which depends on $\omega$). Furthermore, for almost every $\omega$, there exists a strictly increasing sequence $n_i(\omega)$ of integers such that $T_{n_i(\omega)}(\omega)$ converges to $T^*(\omega)$ and for $i$ large enough $u(., T_{n_i(\omega)})(\omega) = u_{n_i(\omega)}(., T_{n_i(\omega)}(\omega))$, so that

$$\sup\{\|u(.,t)(\omega)\|_q : t \le T^*(\omega)\} \ge \sup\{\|u(.,t)(\omega)\|_q : t \le T_{n_i(\omega)}(\omega)\} = n_i.$$

Hence, $\sup\{\|u(.,t)\|_q : t < T^*\} = \infty$ a.s. and $u$ is a maximal solution to (2.11) on $[0, T^*)$.

In the sequel, we will give sufficient conditions such that $T^* = \infty$ a.s., and thus, to obtain that the equation (2.11) admits a unique global solution. This requires to first prove some bounds of the stochastic integral.

2.6. **Bound for the stochastic integral.** Our aim will be to prove moment estimates for the (space-time) uniform norm for $\mathcal{L}_n(u_n)$ which will be needed later.

This norm is defined as follows:

$$\|\mathcal{L}_n(u_n)\|_{L^\infty} := \sup_{t\in[0,T]} \sup_{x\in\mathcal{D}} |\mathcal{L}_n(u_n)(x,t)|.$$

**Lemma 2.7.** *Let $\sigma$ satisfy Condition (1.2) with $\alpha \in (0,1]$, let Condition $(\mathbf{C}_\alpha)$ hold, and let $u_n$ be the solution to the SPDE (2.14). Furthermore, suppose that $q \ge \tilde{q} > \frac{2\alpha d}{4-d}$. Then for any $p \in [1,\infty)$ there exists a positive constant $C_p(T)$ such that for every $n \ge 1$, we have:*

$$E\Big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^{2p}\Big) \le C_p(T) \Big[n^{2\alpha p} \wedge \sup_{t\in[0,T]} E\big(\|u_n(.,t)\|_{\tilde{q}}^{2\alpha p}\big)\Big]. \tag{2.28}$$

*Proof.* Since $d < 4$, $\alpha \in (0,1]$ and $q \in [3,\infty)$, we have $q > 2\alpha$ and may choose $\tilde{q} \le q$ with $\tilde{q} > (2\alpha) \vee \frac{2\alpha d}{4-d}$. For $t \in [0,T]$, using the Burkholder-Davis-Gundy inequality, (2.3), the growth condition (1.2) on $\sigma$ and Hölder's inequality with conjugate exponents $\frac{\tilde{q}}{2\alpha}$ and $\frac{\tilde{q}}{\tilde{q}-2\alpha}$, we obtain for any $t \in [0,T]$ and $x \in \mathcal{D}$

$$\begin{aligned}
E|\mathcal{L}_n(u_n)(x,t)|^{2p} &\leq C_p E\Big| \int_0^t \int_{\mathcal{D}} (t-s)^{-\frac{d}{2}} \exp(h(x-y,t,s))\chi_n(\|u_n(\cdot,s)\|_q)[1+|u_n(y,s)|^{2\alpha}]dyds\Big|^p \\
&\leq C_p E\Big| \int_0^t (t-s)^{-\frac{d}{2}} \Big\| \exp(h(\cdot,t,s))\Big\|_{\frac{\tilde q}{\tilde q-2\alpha}} \chi_n(\|u_n(\cdot,s)\|_q)[1+\|u_n(y,s)\|_{\tilde q}]^{2\alpha} ds\Big|^p \\
&\leq C_p \Big| \int_0^t (t-s)^{-\frac{d}{2}+\frac{d(\tilde q-2\alpha)}{4\tilde q}} [1+E\big(\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(.,s)\|_q]^{2\alpha}\big) ds\Big|^p \\
&\leq C_p(T)\Big[1+\sup_{t\in[0,T]} E\big(\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(.,t)\|_{\tilde q}^{2\alpha p}\big)\Big],
\end{aligned}$$

where as above we let $h(x,t,s)$ be defined by (1.2). The last inequality is valid provided that $-\frac{d}{4}\Big(1+\frac{2\alpha}{\tilde q}\Big) > -1$, which holds true since $q \geq \tilde q > \frac{2\alpha d}{4-d} \vee (2\alpha)$.

Similar computations using (2.3), (2.4) and the Taylor formula imply that for $x, \xi \in \mathcal{D}$ and $t \in [0,T]$, we have for $\lambda \in (0,1)$, $\tilde q \leq q$, $p \in [1,\infty)$ and $n \geq 1$:

$$\begin{aligned}
E|\mathcal{L}_n(u_n)(x,t) - \mathcal{L}_n(u_n)(\xi,t)|^{2p} &\leq C_p E\Big| \int_0^t \int_{\mathcal{D}} |G(x,y,t-s)-G(\xi,y,t-s)|^2 \chi_n(\|u_n(\cdot,s)\|_q) \\
&\quad \times \big[1+|u_n(y,s)|^{2\alpha}\big]\, dy\, ds\Big|^p \\
\leq& C_p|x-\xi|^{2\lambda p} E\Big| \int_0^t (t-s)^{-\frac{(d+1)\lambda}{2}} (t-s)^{-\frac{d}{2}(1-\lambda)} \chi_n(\|u_n(\cdot,s)\|_q) \\
&\quad \times \|\exp(h(\cdot,t,s))\|_{\frac{\tilde q}{\tilde q-2\alpha}} [1+\|u_n(\cdot,s)\|_{\tilde q}]^{2\alpha} ds\Big|^p \\
\leq& C_{p,q}|x-\xi|^{2\lambda p} \Big|\int_0^t (t-s)^{-\frac{d+\lambda}{2}+\frac{d(\tilde q-2\alpha)}{4\tilde q}} ds\Big|^p \Big[1+\sup_{s\in[0,T]} E\big(\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(.,s)\|_{\tilde q}^{2p\alpha}\big)\Big]
\end{aligned}$$

$$\leq C_{p,q}(T)|x-\xi|^{2\lambda p}\Big[1+\sup_{s\in[0,T]} E\big(\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(.,s)\|_{\tilde q}^{2p\alpha}\big)\Big], \tag{2.29}$$

provided that $-\frac{d+\lambda}{2} + \frac{d(\tilde q-2\alpha)}{4\tilde q} > -1$, which holds true if $0 \leq \lambda < \Big(2-\frac{d}{2}\Big) \wedge 1$ and $\tilde q > \frac{2\alpha d}{4-d-2\lambda}$. Hence, for $q \geq \tilde q > \frac{2\alpha d}{4-d}$ one can find $\lambda \in (0,1)$ small enough to fulfill this constraint.

Using again the Taylor formula, (2.3) and (2.5), we obtain, for $0 \leq t' \leq t \leq T$ and $\mu \in [0,1]$

$$\begin{aligned}
E|\mathcal{L}_n(u_n)(x,t) - \mathcal{L}_n(u_n)(x,t')|^{2p} &\leq C_p E\Big| \int_0^t \int_{\mathcal{D}} \big[|G(x,y,t-s)|^{2(1-\mu)} + |G(x,y,t'-s)|^{2(1-\mu)}\big] \\
&\quad \times |G(x,y,t-s)-G(x,y,t'-s)|^{2\mu} \chi_n(\|u_n(\cdot,s)\|_q)[1+|u_n(y,s)|^{2\alpha}]dyds\Big|^p \\
\leq& |t-t'|^{2\mu p} E\Big| \int_0^t (t-s)^{-2\mu(\frac{d}{4}+1)-(1-\mu)\frac{d}{2}} \|\exp(h(\cdot,t,s))\|_{\frac{\tilde q}{\tilde q-2\alpha}} \\
&\quad \times \big(1+\|u_n(\cdot,s)\|_{\tilde q}^{2\alpha}\big)\chi_n(\|u_n(\cdot,s)\|_q) ds\Big|^p
\end{aligned}$$

$$\leq C_{p,q}(T)|t-t'|^{2\mu p}\Big[1+\sup_{s\in[0,T]} E\big(\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(.,s)\|_{\tilde q}^{2p\alpha}\big)\Big], \tag{2.30}$$

where the last inequality holds true if $-\frac{d}{2} - 2\mu + \frac{d}{4}\Big(1-\frac{2\alpha}{\tilde q}\Big) > -1$; this is similar to the previous requirement used to prove (2.29) replacing $\frac{\lambda}{2}$ by $2\mu$. Thus, since $q \geq \tilde q > \frac{2\alpha d}{4-d}$, we may find $\mu \in (0,1)$ which satisfies this constraint, and such that (2.30) holds for any $p \in [1,+\infty)$.

The upper estimates (2.29), (2.30) yield the existence of some positive constants $\lambda$ and $\mu$, and, given $p \in [1,\infty)$, of some positive constant $C_p(T)$ (independent of $n$) such that for $x, x' \in \mathcal{D}$ and $t, t' \in [0,T]$, we have for every $n \geq 1$

$$E|\mathcal{L}_n(u_n)(x,t) - \mathcal{L}_n(u_n)(x',t')|^{2p} \leq C_p(T)\big[|x-\xi|^{2\lambda p} + |t-t'|^{2\mu p}\big] \times \Big[1 + \sup_{s\in[0,T]} E\big(\chi_n(\|u_n(\cdot,s)\|_q \|u_n(.,s)\|_{\tilde{q}}^{2p\alpha}\big)\Big].$$

Therefore, the Garsia-Rodemich-Rumsey Lemma yields the upper estimate (2.28). Indeed, for every $n \geq 1$ we have $\|\chi_n\|_\infty \leq 1$ and $\tilde{q} \leq q$; hence, Hölder's inequality gives

$$\chi_n(\|u_n(\cdot,s)\|_q)\|u_n(\cdot,s)\|_{\tilde{q}}^{\lambda} \leq n^\lambda \wedge C \sup_{s\in[0,T]} \|u_n(\cdot,s)\|_{\tilde{q}}^{\lambda},$$

for every $s \in [0,T]$, and $\lambda > 0$.

□

## 3. Investigation of global existence and uniqueness of solution

In this section, we investigate the existence and uniqueness of a global solution for the stochastic evolution equation (2.1). This requires some stronger integrability assumption on the initial condition $u_0$, which should belong to $L^4(\mathcal{D})$. More precisely, we suppose that the following condition $(\tilde{\mathbf{C}}_\alpha)$ is satisfied.

**Condition $(\tilde{\mathbf{C}}_\alpha)$** *One of the following properties is satisfied:*
*(i)* $d = 1,2$ *and* $q \in [3,\infty)$.
*(ii)* $d = 3$ *and* $q \geq 4$ *is such that* $q \in \big(6(1-\alpha) \vee (6\alpha), \infty\big)$.

Note that if Condition $(\tilde{\mathbf{C}}_\alpha)$(ii) is satisfied for $d = 3$, we have $\|\cdot\|_4 \leq C\|\cdot\|_q$ for some positive constant $C$, while this upper estimate fails in dimensions $d = 1,2$.

The proof is decomposed in two steps. We at first upper estimate the $L^4(\mathcal{D})$ norm of $(u_n)$ by that of $\mathcal{L}_n(u_n)$ and some constant defined in terms of the basis of eigenfunctions $(\epsilon_k)$ presented in subsection 2.2. This is achieved by using the Galerkin approximation. Then for $\alpha \in (0, \frac{1}{3})$, when $d = 1$, we derive global existence of solution.

### 3.1. Galerkin approximation and $L^4(\mathcal{D})$ estimate of $u_n$.

For any $n \geq 1$, let us define

$$v_n := u_n - \mathcal{L}_n(u_n).$$

Then, formally, $v_n$ satisfies the following equation:

$$\begin{aligned} &\partial_t v_n + [\Delta^2 - \Delta]v_n - (\Delta - Id)\Big(\chi_n(\|v_n + \mathcal{L}_n(u_n)\|_q) f(v_n + \mathcal{L}_n(u_n)\Big) = 0 \text{ in } \mathcal{D}\times[0,T), \\ &v_n(x,0) = u_0(x) \ \text{ in } \ \mathcal{D}, \\ &\frac{\partial v_n}{\partial \nu} = \frac{\partial \Delta v_n}{\partial \nu} = 0 \ \text{ on } \ \partial\mathcal{D}\times[0,T). \end{aligned} \tag{3.1}$$

For a strict definition of solution, we say that $v_n$ is a weak solution of the above equation (3.1) if for all $\phi \in \mathcal{C}^4(\mathcal{D})$ with $\frac{\partial\phi}{\partial\nu} = \frac{\partial\Delta\phi}{\partial\nu} = 0$ on $\partial\mathcal{D}$, we have:

$$\int_{\mathcal{D}} \Big(v_n(x,t) - u_0(x)\Big)\phi(x)\,dx = \int_0^t \int_{\mathcal{D}} \Big\{\big[-\Delta^2 + \Delta\big]\phi(x)\,v_n(x,s) + \big[\Delta\phi(x) - \phi(x)\big]\chi_n(\|v_n + \mathcal{L}_n(u_n)\|_q) f(v_n + \mathcal{L}_n(u_n))\Big\}\,dxds.$$

Using the Green's function $G$ defined by (2.2), we deduce the integral form of this equation:

$$v_n(x,t) = \int_{\mathcal{D}} u_0(y)G(x,y,t)\,dy + \int_0^t \int_{\mathcal{D}} \big[\Delta G(x,y,t-s) - G(x,y,t-s)\big]$$
$$\times\, \chi_n(\|v_n + \mathcal{L}_n(u_n)\|_q) f\big(v_n(y,s) + \mathcal{L}_n(u_n)(y,s)\big)\,dyds. \tag{3.2}$$

We will use the Galerkin method to prove the existence of the solution $v_n$ for the equation (3.1). Let us denote by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ the eigenvalues of Neumann Laplacian operator inducing $\{w_i\}_{i=0}^\infty$ as an orthonormal basis of $L^2(\mathcal{D})$ of eigenfunctions, i.e., $(w_i, w_j)_{L^2(\mathcal{D})} = \delta_{ij}$ and

$$-\lambda_i w_i = \Delta w_i \ \text{ in } \mathcal{D}, \quad \frac{\partial w_i}{\partial \nu} = 0 \ \text{ on } \partial\mathcal{D} \ \text{ for } i = 0,1,2,\cdots. \tag{3.3}$$

Let $P_m$ denote the orthogonal projection from $L^2(\mathcal{D})$ onto $\text{span}\{w_0, w_1, \cdots, w_m\}$. For every $m = 0,1,2,\cdots$ we consider the function $v_n^m$

$$v_n^m(x,t) = \sum_{i=0}^m \rho_i^m(t) w_i(x),$$

defined by the Galerkin ansatz, where

$$\begin{cases} \frac{\partial}{\partial t} v_n^m + \big(\Delta^2 - \Delta\big) v_n^m - \big(\Delta - Id\big)\Big[\chi_n(\|v_n^m + \mathcal{L}_n(u_n)\|_q)\, P_m\big(f(v_n^m + \mathcal{L}_n(u_n))\big)\Big] = 0, \\ v_m^n(x,0) = P_m(u_0) \ \text{ in } \mathcal{D}, \quad \frac{\partial v_n^m}{\partial \nu} = \frac{\partial \Delta v_n^m}{\partial \nu} = 0 \ \text{ on } \partial\mathcal{D}. \end{cases} \tag{3.4}$$

This yields an initial value problem of ODE satisfied by $\rho_i^m(t)$ for $i = 0,1,\cdots,m$. By standard arguments of ODE, this initial value problem has a local solution. We will show that a global solution exists.

Multiplying by $v_n^m$ both sides of (3.4) and integrating in space, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_n^m(\cdot,t)\|_2^2 + \|\Delta v_n^m(.,t)\|_2^2 + \|\nabla v_n^m(\cdot,t)\|_2^2 \\ &\quad = \chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} f\big(v_n^m(x,t) + \mathcal{L}_n(u_n)(x,t)\big)\big[\Delta v_n^m(x,t) - v_n^m(x,t)\big]\,dx \\ &\quad = \sum_{i=1}^3 T_i(t), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} T_1(t) &= \chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} \big[f\big(v_n^m(x,t) + \mathcal{L}_n(u_n)(x,t)\big) - f\big(v_n^m(x,t)\big)\big] \\ &\qquad \times \big[\Delta v_n^m(x,t) - v_n^m(x,t)\big]dx, \\ T_2(t) &= \chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} f\big(v_n^m(x,t)\big)\Delta v_n^m(x,t)dx, \\ T_3(t) &= -\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} f\big(v_n^m(x,t)\big) v_n^m(x,t)dx. \end{aligned}$$

Since $f$ is a polynomial of degree 3, then we have for $x, y \in \mathbb{R}$:

$$|f(x+y) - f(x)| \le c|y|(1 + x^2 + y^2).$$

Thus, by Cauchy-Schwarz and Young inequalities we obtain for any $\varepsilon > 0$

$$\begin{aligned} T_1(t) \le & C\chi_n(\|v_n^m(\cdot) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} |\mathcal{L}_n(u_n)(x,t)| \big[1 + |v_n^m(x,t)|^2 + |\mathcal{L}_n(u_n)(x,t)|^2\big] \\ & \times \big[|\Delta v_n^m(x,t)| + |v_n^m(x,t)|\big] dx \\ \le & C\, \chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \|\mathcal{L}_n(u_n)(\cdot,t)\|_\infty \big[1 + \|v_n^m(\cdot,t)^2\|_2 + \|\mathcal{L}_n(u_n)(\cdot,t)^2\|_2\big] \\ & \times \big[\|\Delta v_n^m(\cdot,t)\|_2 + \|v_n^m(\cdot,t)\|_2\big] \\ \le & \varepsilon \big[\|\Delta v_n^m(\cdot,t)\|_2^2 + \|v_n^m(\cdot,t)\|_2^2\big] t \\ & + \frac{C}{\varepsilon}\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q)\|\mathcal{L}_n(u_n)(\cdot,t)\|_\infty^2 \big[1 + \|v_n^m(\cdot,t)\|_4^4 + \|\mathcal{L}_n(u_n)(\cdot,t)\|_4^4\big]. \end{aligned} \tag{3.6}$$

Observe that $f(x) = ax^3 + g(x)$, where $a > 0$, and $g$ is a polynomial of degree 2. Hence, it follows that $f'(x) = 3ax^2 + 2bx + c$ for some real constants $b$, $c$, and $f'(x) \ge 2ax^2 - \tilde{c}$ for some non-negative constant $\tilde{c}$. So, an integration by parts yields for any $\varepsilon > 0$

$$\begin{aligned} T_2(t) = & -\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} f'(v_n^m(x,t)) |\nabla v_n^m(x,t)|^2 dx \\ \le & C\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} \big[ -2a|v_n^m(x,t)|^2 |\nabla v_n^m(x,t)|^2 + \tilde{c}|\nabla v_n^m(x,t)|^2\big] dx \\ \le & \; - C\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \int_{\mathcal{D}} v_n^m(x,t) \Delta v_n^m(x,t) dx \\ \le & \varepsilon \|\Delta v_n^m(\cdot,t)\|_2^2 + \frac{C}{\varepsilon}\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q)\|v_n^m(\cdot,t)\|_2^2. \end{aligned} \tag{3.7}$$

Finally, since $xf(x) \ge \frac{7}{8}ax^4 - \tilde{C}$ with $a$, $\tilde{C} > 0$, we obtain

$$T_3(t) \le \chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(\cdot,t)\|_q) \Big[ \int_{\mathcal{D}} -\frac{7}{8} a |v_n^m(x,t)|^4 dx + \tilde{C}|\mathcal{D}|\Big]. \tag{3.8}$$

The above upper estimates of $T_i(t)$, $i = 1, 2, 3$, imply that for $\varepsilon > 0$ small enough,

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\|v_n^m(\cdot,t)\|_2^2 + \frac{1}{2}\|\Delta v_n^m(\cdot,t)\|_2^2 + \|\nabla v_n^m(\cdot,t)\|_2^2 \le & C\chi_n(\|v_n^m(\cdot,t) + \mathcal{L}_n(u_n)(.,t)\|_q) \\ \times \Big(\|\mathcal{L}_n(u_n(\cdot,t))\|_\infty^2 \big[1 + \|v_n^m(\cdot,t)\|_4^4 + \|\mathcal{L}_n(u_n(\cdot,t))\|_4^4\big] & + \|v_n^m(\cdot,t)\|_2^2 + 1\Big). \end{aligned} \tag{3.9}$$

Since

$$\|v_n^m(\cdot,0)\|_2 = \|P_m u_0\|_2 \le \|u_0\|_2,$$

integrating $(3.9)$ from 0 to $t \in (0,T]$, and using Hölder's and Young's inequalities, we finally obtain that for some positive constants $C, C_0$:

$$\begin{aligned} \|v_n^m(\cdot,t)\|_2^2 + \int_0^t \|\Delta v_n^m(\cdot,s)\|_2^2\, ds \le & \|u_0\|_2^2 + CT\Big(1 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^6\Big) \\ & + C_0\Big(1 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big) \int_0^t \chi_n(\|v_n^m(\cdot,s) + \mathcal{L}_n(u_n)(\cdot,s)\|_q)\big(\|v_n^m(\cdot,s)\|_4^4 + 1\big)\, ds. \end{aligned} \tag{3.10}$$

Note that the $H^2(\mathcal{D})$-norm is equivalent to $\Big( \int_{\mathcal{D}} \big(|\Delta u(x)|^2 + |u(x)|^2\big)\, dx\Big)^{\frac{1}{2}}$ under the boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0$ on $\partial\mathcal{D}$.

Let $q \in [3,4)$ in dimensions $d = 1, 2$; then the Gagliardo-Nirenberg inequality implies that for any function $U \in \text{Dom }(\Delta)$ we have:

$$\|U\|_4 \le C\|\Delta U\|_2^a\, \|U\|_q^{1-a} + C\|U\|_q,$$

for some positive constant $C$ and $a\in(0,1)$ such that $\frac14=\big(\frac12-\frac2d\big)a+\frac{1-a}{q}$, that is $a=\frac{4-q}{2[q(4-d)+2d]}$. Then for $d=1,2$ we have $4a<2$; for any $\epsilon>0$, we deduce that for some posiitve constant $C(a)$

$$\|U\|_4^4\le \epsilon\|\Delta U\|_2^2+C(a)\epsilon^{-\frac{2a}{1-2a}}\,\|U\|_q^{\frac{4(1-a)}{1-2a}}+C\|U\|_q.$$

Hence, for any $\epsilon>0$ we have for the constant $C_0$ appearing in (3.10):

$$\begin{aligned}
C_0\Big(1+&\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)\int_0^t\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\big(\|v_n^m(\cdot,s)\|_4^4+1\big)\,ds\\
\le& C_0\epsilon\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)\int_0^t\|\Delta v_n^m(.,s)\|_2^2ds\\
&+C_0C(a)\epsilon^{-\frac{2a}{1-2a}}\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)\int_0^t\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\,\|v_n^m(\cdot,s)\|_q^{\frac{4(1-a)}{1-2a}}\,ds\\
&+C_0C\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)\int_0^t\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\|v_n^m(\cdot,s)\|_q^4\,ds. \qquad (3.11)
\end{aligned}$$

If the cut-off function $\chi_n$ applied to the $\|.\|_q$ norm of some function $U$ is not zero, the we can deduce that

$$\|U\|_4\le\bar C(n+1)\le Cn.$$

Recall that $|\chi_n|\le1$; hence, the triangular inequality implies that

$$\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\|v_n^m(\cdot,s)\|_q\le C\big(\|\mathcal{L}_n(u_n)\|_{L^\infty}+n\big).$$

Choose $\epsilon^{-1}=2C_0\epsilon\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)$; plugging the inequality (3.11) in (3.10) and using the above upper estimate of $\|v_n^m(\cdot,s)\|_q$, we deduce:

$$\begin{aligned}
\|v_n^m(\cdot,t)\|_2^2+\frac12\int_0^t\|\Delta v_n^m(\cdot,s)\|_2^2\,ds\le&\|u_0\|_2^2+CT\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^6\Big)\\
&+C(q,d,T)\Big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big)^{1+\frac{2a}{1-2a}}\Big(n^{\frac{4(1-a)}{1-2a}}+\|\mathcal{L}_n(u_n)\|_{L^\infty}^{\frac{4(1-a)}{1-2a}}\Big). \qquad (3.12)
\end{aligned}$$

Suppose now that $q\ge4$; a similar simpler argument based on the triangular inequality and properties of the cut-off function $\chi_n$ yields

$$\begin{aligned}
\int_0^t\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\big(\|v_n^m(\cdot,s)\|_4^4+1\big)ds\le& CT\big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^4\big)\\
&+C\int_0^t\chi_n(\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q)\|v_n^m(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_4^4\,ds\\
\le& CT\big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^4+n^4\big). \qquad (3.13)
\end{aligned}$$

The upper estimates (3.12) or (3.10) and (3.13) imply that

$$\sup_{t\in[0,T]}\|v_n^m(\cdot,t)\|_2^2\le\|u_0\|_2^2+C\big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^6\big)+CTn^{N_1}\big(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^{N_2}\big),$$

and

$$\begin{aligned}
\int_0^T[\|v_n^m(\cdot,t)\|_2^2+\|\Delta v_n^m(\cdot,t)\|_2^2]dt\le&(T+1)\|u_0\|^2+C(T^2+1)(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^6)\\
&+C(T^2+1)n^{N_1}(1+\|\mathcal{L}_n(u_n)\|_{L^\infty}^{N_2}), \qquad (3.14)
\end{aligned}$$

for some exponents $N_1$ and $N_2$ which do not depend on $m$.

Note that the right-hand sides of both inequalities in (3.14) depend on $n$ but are independent of the index $m$. Thus, a standard weak compactness argument proves that for fixed $n$, as

$m\to\infty$, a subsequence of $(v_n^m, m\geq 1)$ converges weakly in $L^2(0,T;H^2(\mathcal{D}))$ to a solution $v_n$ of (3.1) with homogeneous Neumann boundary conditions.

Let $(\epsilon_k)$ denote the orthonormal basis defined in Section 2.2 and set

$$B(u_0) := \frac{1}{2}\Big\| \sum_{k\in\mathbb{N}^d} [\lambda_k+1]^{-\frac12}(u_0,\epsilon_k)_{L^2(\mathcal{D})}\epsilon_k\Big\|_2^2 = \frac12 \sum_{k\in\mathbb{N}^d}[\lambda_k+1]^{-1}(u_0,\epsilon_k)^2_{L^2(\mathcal{D})}. \tag{3.15}$$

Note that if $\mathcal{D}$ is the unitary cube, we have $B(u_0)\leq \frac12\|u_0\|_2^2$. The following lemma provides estimates of the $L^4(\mathcal{D})$-norm of $u_n$ properly localized in terms of that of $\mathcal{L}_n(u_n)$.

**Lemma 3.1.** *Let $\sigma$ be Lipschitz and satisfy the sub-linearity condition* (1.2) *with $\alpha\in(0,1]$, and let $u_0\in L^q(\mathcal{D})$ where $q$ satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$. Let $u_n$ be the solution to the SPDE* (2.14) *and $B(u_0)$ be defined by* (3.15)*. Then, there exists a constant $C:=C(t,\mathcal{D})$ independent of the index $n$ satisfying almost surely*

$$\int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\|u_n(\cdot,s)\|_4^4 ds \leq C\Big\{1+B(u_0)+\int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\|\mathcal{L}_n(u_n)(\cdot,s)\|_4^4 ds\Big\}. \tag{3.16}$$

*Proof.* Using the orthonormal basis $(\epsilon_k)$ defined at the beginning of Section 2.2, we write $v_n\in L^2(\mathcal{D})$ as

$$v_n(x,t) = \sum_{k\in\mathbb{N}^d}\rho_k(t)\epsilon_k(x).$$

To ease notation, for $x\in\mathcal{D}$ and $s\in[0,T]$ we set

$$Q(x,s) := \chi_n(\|v_n(\cdot,s)+\mathcal{L}_n(u_n)(\cdot,s)\|_q) f(u_n(x,s)).$$

Then the equation (3.1) is written as follows

$$\partial_t v_n + (-\Delta+Id)(-\Delta)v_n + (-\Delta+Id)Q = 0, \tag{3.17}$$

with the boundary conditions $v_n(x,0)=u_0(x)$ in $\mathcal{D}$, and $\frac{\partial v_n}{\partial\nu} = \frac{\partial \Delta v_n}{\partial \nu} = 0$ on $\partial\mathcal{D}\times[0,T)$.

We set $A=-\Delta+Id$, apply $A^{-1}$ to (3.17) and take the $L^2$-inner product with $v_n(\cdot,t)$. The $L^2$-orthogonality of the eigenfunctions $\epsilon_k$ of $\Delta$ gives

$$\sum_{k\in\mathbb{N}^d}[\lambda_k+1]^{-1}\rho_k(t)\partial_t\rho_k(t) + \sum_{k\in\mathbb{N}^d}\lambda_k\rho_k(t)^2 + \big(Q(\cdot,t),v_n(\cdot,t)\big)=0.$$

Integrating this identity from 0 to $t$ yields

$$\int_0^t \big(Q(\cdot,s),v_n(\cdot,s)\big)ds = \sum_{k\in\mathbb{N}^d}\frac12[\lambda_k+1]^{-1}\rho_k^2(0) - \sum_{k\in\mathbb{N}^d}\Big[[\lambda_k+1]^{-1}\rho_k^2(t)+\int_0^t\lambda_k\rho_k^2(s)ds\Big].$$

Since $\lambda_k\geq 0$ for all $k$, we obtain

$$\int_0^t \big(Q(\cdot,s),v_n(\cdot,s)\big)_{L^2}\, ds \leq \frac12\sum_{k\in\mathbb{N}^d}[\lambda_k+1]^{-1}\rho_k^2(0) = B(u_0). \tag{3.18}$$

Furthermore, $f$ is a polynomial of degree 3; therefore, $f(u_n)\geq \frac45 u_n^4 - c$ for some non negative constant $c$. This yields

$$\int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\|u_n(\cdot,s)\|_4^4 ds \leq C\Big\{1+\int_0^t\int_{\mathcal{D}}\chi_n(\|u_n(\cdot,s)\|_q)f(u_n(\cdot,s))u_n(\cdot,s)dxds\Big\}.$$

Since $Q=\chi_n(\|v_n+\mathcal{L}_n(u_n)\|_q)f(u_n)$ and $u_n=\mathcal{L}_n(u_n)+v_n$, using (3.18) in the previous identity we obtain

$$\int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\|u_n(\cdot,s)\|_4^4\, ds \leq C\Big\{1+\int_0^t\int_{\mathcal{D}}\chi_n(\|u_n(\cdot,s)\|_q)f(u_n(\cdot,s))\mathcal{L}_n(u_n(\cdot,s))\,dxds$$
$$+B(u_0)\Big\}. \tag{3.19}$$

Using once more the fact that $f(u_n)$ is a third degree polynomial, Young's inequality implies that for any $\epsilon > 0$ and $s \in [0,T]$,

$$\begin{aligned}\int_{\mathcal{D}} \chi_n(\|u_n(\cdot,s)\|_q) f(u_n(x,s)) \mathcal{L}_n(u_n(x,s))\, dx &\le \epsilon \int_{\mathcal{D}} \chi_n(\|u_n(\cdot,s)\|_q) |f(u_n(.,s))|^{4/3} dx \\ &\quad + \frac{C}{\epsilon} \int_{\mathcal{D}} \chi_n(\|u_n(\cdot,s)\|_q) |\mathcal{L}_n(u_n(x,s))|^4\, dx \\ &\le C\epsilon \chi_n(\|u_n(\cdot,s)\|_q) \|u_n(\cdot,s)\|_4^4 + C + \frac{C}{\epsilon} \chi_n(\|u_n(\cdot,s)\|_q) \|\mathcal{L}_n(u_n(\cdot,s))\|_4^4.\end{aligned}$$

Consequently, plugging this upper estimate in (3.19) and choosing $\epsilon$ small enough, we complete the proof of (3.16). □

3.2. **Some $L^2$-estimates of $u_n$.** The Sobolev embedding theorem implies that for $d = 1,2,3$, $H^2(\mathcal{D}) \subset L^4(\mathcal{D})$. Hence, computations similar to that used to prove (3.10) using analogues of (3.6)-(3.8) with the weak $H^2(\mathcal{D})$-limit $v_n$ of $v_n^m$ taken instead of $v_n^m$, show that for any $\epsilon > 0$ we have

$$\begin{aligned}\frac{1}{2}\|v_n(\cdot,t)\|_2^2 + \int_0^t \|\Delta v_n(\cdot,s)\|_2^2\, ds &\le \frac{1}{2}\|u_0\|_2^2 + C\int_0^t \tilde{T}_1(s) ds + \epsilon \int_0^t \|\Delta v_n(\cdot,s)\|_2^2 ds \\ &\quad + C(1+\epsilon^{-1})\Big[T + \int_0^t \chi_n^2(\|v_n(\cdot,s) + \mathcal{L}_n(u_n)(\cdot,s))\|_q) \|v_n(\cdot,s)\|_4^4 ds\Big],\end{aligned}$$

where the Cauchy-Schwarz and Young inequalities yield for any $\epsilon > 0$

$$\tilde{T}_1(s) \le \epsilon\big[\|\Delta v_n(\cdot,s)\|_2^2 + \|v_n(\cdot,s)\|_2^2\big] + \frac{C}{\epsilon}\chi_n(\|v_n(\cdot,s) + \mathcal{L}_n(u_n)(\cdot,s))\|_q) \bar{T}_1(s),$$

for $\bar{T}_1(s)$ defined by

$$\begin{aligned}\bar{T}_1(s) &:= \int_{\mathcal{D}} |\mathcal{L}_n(u_n(x,s))|^2 \big[1 + |v_n(x,s)|^4 + |\mathcal{L}_n(u_n(x,s))|^4\big] dx \\ &\le C\big[1 + \|\mathcal{L}_n(u_n(\cdot,s))\|_\infty^6 + \|\mathcal{L}_n(u_n(\cdot,s))\|_\infty^2 \|v_n(\cdot,s)\|_4^4\big].\end{aligned}$$

Recall that $u_n = v_n + \mathcal{L}_n(u_n)$. Choosing $\epsilon$ small enough, using the Gronwall Lemma and Lemma 3.1, we deduce that for $t \in [0,T]$ there exists a positive constant $C(T)$ such that

$$\begin{aligned}\|v_n(\cdot,t)\|_2^2 + \int_0^t \|\Delta v_n(\cdot,s)\|_2^2 ds &\le C(T)\Big(\|u_0\|_2^2 + 1 \\ &\quad + \int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\big[1 + \|\mathcal{L}_n(u_n(\cdot,s))\|_6^6 + \big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^2 + 1\big)\ \|v_n(\cdot,s)\|_4^4\big] ds\Big) \\ &\le C(T)\Big[\|u_0\|_2^2 + 1 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^6 \\ &\quad + \big(1 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^2\big) \int_0^t \chi_n(\|u_n(\cdot,s)\|_q)\big(\|u_n(\cdot,s)\|_4^4 + \|\mathcal{L}_n(u_n(\cdot,s))\|_4^4\big)\ ds\Big] \\ &\le C(T)\Big[1 + \|u_0\|_2^2 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^6 + (1 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^2) B(u_0)\Big] \\ &\le C(T)\Big[1 + \|u_0\|_2^2 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^6 + \|u_0\|_2^2 \|\mathcal{L}_n(u_n)\|_{L^\infty}^2\Big] \\ &\le C(T)\Big[1 + \|u_0\|_2^4 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^6\Big], \qquad (3.20)\end{aligned}$$

holds for every $n \ge 1$, where the last inequality is a consequence of (3.16), of Young's inequality and of the upper bound $B(u_0) \le C\|u\|_0^2$.

The above upper estimate and bounds of moments of the stochastic integral proved in Section 2.6 yield the following lemma.

**Lemma 3.2.** *Let $\sigma$ be Lipschitz and satisfy the sub-linear growth condition* (1.2) *with* $\alpha \in \big(0,1] \cap \big(0, \frac{4-d}{d}\big)$*. Let $u_0 \in L^q(\mathcal{D})$ where $q$ satisfies the condition ($\tilde{C}_\alpha$); then, for every $p \in [2,\infty)$ there exists a positive constant $C_p(T)$ such that for every $n \geq 1$ we have the estimate*

$$E\Big(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_2^p\Big) \leq C_p(T)\Big[1 + \|u_0\|_2^\lambda + (1+\|u_0\|_2^p) E\Big(\sup_{t\in[0,T]} \|u_n(.,t)\|_2^{3\alpha p}\Big)\Big]. \tag{3.21}$$

*Proof.* Since $u_n = v_n + \mathcal{L}_n(u_n)$, the inequality (3.20) and the Cauchy Schwarz inequality imply that for any $t \in [0,T]$,

$$\begin{aligned}\|u_n(\cdot,t)\|_2 &\leq \|v_n(\cdot,t)\|_2 + C\|\mathcal{L}_n(u_n)(\cdot,t)\|_{L^\infty} \\ &\leq C(T)\big[1 + \|u_0\|_2 + \|\mathcal{L}_n(u_n)\|_{L^\infty}^3 + \|u_0\|_2 \|\mathcal{L}_n(u_n)\|_{L^\infty}\big].\end{aligned}$$

Note that the assumption $\alpha < \frac{4-d}{d}$ implies that for $\tilde{q} := 2$, we have $q \geq 3 \geq \tilde{q} > \frac{2\alpha d}{4-d}$. Thus, using the upper estimate of moments of the $\|\mathcal{L}_n(u_n)\|_{\tilde{q}}$ proved in (2.28), we deduce that for any $p \in [2,\infty)$, we have

$$\begin{aligned}E\big(\|u_n(\cdot,t)\|_2^p\big) &\leq C_p(T)\Big[1 + \|u_0\|_2^p + E\big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^3\big) + \|u_0\|_2^p E\big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^p\big)\Big] \\ &\leq C_p(T)\big[1 + \|u_0\|_2^p\big]\Big[1 + E\Big(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_2^{3\alpha p}\Big)\Big].\end{aligned}$$

This completes the proof. ☐

This lemma yields an upper bound of the $L^2$-norm of $u_n$ which does not depend on $n$.

**Lemma 3.3.** *Let $\sigma$ be Lipschitz and satisfy the sub-linear growth condition* (1.2) *with $\alpha \in \big(0, \frac{1}{3}\big)$. Let $u_0 \in L^q(\mathcal{D})$ where $q$ satisfies the condition ($\tilde{C}_\alpha$); then, for every $p \in [2,\infty)$ there exists a positive constant $C_p(T)$ such that for every $n \geq 1$ we have*

$$E\Big(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_2^p\Big) \leq C_p(T)\big[1 + \|u_0\|_2^{\frac{p}{1-3\alpha}}\big]. \tag{3.22}$$

*Proof.* First note that for $d = 1,2,3$, we have $\frac{1}{3} \leq \frac{4-d}{d}$ so that Lemma 3.2 can be used to deduce that by Hölder's inequality,

$$E\Big(1 + \sup_{t\in[0,T]} \|u_n(.,t)\|_2^p\Big) \leq C_p(T)\Big[1 + \|u_0\|_2^p\big] + \big[1 + \|u_0\|_2^p\big]\Big\{E\Big(1 + \sup_{t\in[0,T]} \|u_n(.,t)\|_2^p\Big)\Big\}^{3\alpha}\Big].$$

Dividing by $\Big\{E\Big(1 + \sup_{t\in[0,T]} \|u_n(.,t)\|_2^p\Big)\Big\}^{3\alpha}$ we deduce that

$$E\Big(1 + \sup_{t\in[0,T]} \|u_n(.,t)\|_2^p\Big)^{1-3\alpha} \leq 2C_p(T)\big[1 + \|u_0\|_2^p\big],$$

which concludes the proof. ☐

3.3. **Existence of a global solution for $d = 1$.** We now suppose that $d = 1$; using the Gagliardo-Niremberg inequality and results from the previous section, we will derive the existence of a global solution in $L^q$ for $q \in [3,\infty)$. We first prove an $L^q$ estimate for $\mathcal{M}_n(u_n)(\cdot,t)$ uniformly in $t \in [0,T]$.

**Lemma 3.4.** *Let $d = 1$ and $\sigma$ be Lipschitz and satisfy the sub-linearity condition* (1.2) *with $\alpha \in (0, \frac{1}{3})$, $u_0 \in L^q(\mathcal{D})$, where $q \in [3,\infty)$. Let $u_n$ be the solution of the SPDE* (2.14) *and let $\beta \in [2,\infty)$. Then for $\mathcal{M}_n(u_n)$ defined by* (2.16) *there exists a positive constant $C := C(T, \|u_0\|_2)$ such that for and every $n \geq 1$:*

$$E\Big(\sup_{0\leq t\leq T} \|\mathcal{M}_n(u_n)(\cdot,t)\|_q^\beta\Big) \leq C. \tag{3.23}$$

*Proof.* Computations similar to those used when proving (2.19), yield for $\frac{1}{\rho}+\frac{1}{r}=\frac{1}{q}+1$,

$$\|\mathcal{M}_n(u_n)(\cdot,t)\|_q \le C \int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{4r}} \Big\|\chi_n(\|u_n(\cdot,s)\|_q) f(u_n(.,s))\Big\|_\rho \, ds.$$

Since $f$ is a third degree polynomial, choosing $\rho=\frac{q}{3}$ as before, we get

$$\Big\|\chi_n(\|u_n(\cdot,s)\|_q) f(u_n(.,s))\Big\|_{\frac{q}{3}} \le C\Big(1+\|u_n(\cdot,s)\|_q^3\Big).$$

Thus, for $\rho=\frac{q}{3}$ and $r$ such that $\frac{2}{q}+\frac{1}{r}=1$, we obtain for any $t\in[0,T]$:

$$\|\mathcal{M}_n(u_n)(\cdot,t)\|_q \le C\int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{4r}}\Big(1+\|u_n(\cdot,s)\|_q^3\Big)\, ds.$$

Let $\gamma\in(1,\infty)$ be such that $(-\frac{3}{4}+\frac{1}{4r})\gamma>-1$, and let $\gamma'$ be the conjugate exponent of $\gamma$. Then $\gamma'>\frac{2q}{q-1}$ and Hölder's inequality yields the existence of a positive constant $C$ such that

$$\|\mathcal{M}_n(u_n)(\cdot,t)\|_q \le C(T)\Big\{1+\Big(\int_0^t \|u_n(\cdot,s)\|_q^{3\gamma'}\, ds\Big)^{\frac{1}{\gamma'}}\Big\}. \tag{3.24}$$

Let $\tilde a=\frac{q-2}{4q}$; then the Gagliardo-Nirenberg inequality (see [1]) implies the existence of a constant $C$ such that for any function $\phi\in H^2(\mathcal{D})$, we have

$$\|\phi\|_q \le C\|D^2\phi\|_2^{\tilde a}\,\|\phi\|_2^{1-\tilde a}+C\|\phi\|_2.$$

The Dirichlet boundary conditions we have imposed imply that the norms $\|\phi\|_{H^2}$ and $\big(\|\phi\|_2+\|\Delta\phi\|_2\big)$ are equivalent; therefore, we obtain

$$\|\phi\|_q \le C\|\Delta\phi\|_2^{\tilde a}\,\|\phi\|_2^{1-\tilde a}+C\|\phi\|_2. \tag{3.25}$$

Note that since $3(q-2)<4(q-1)$, we can choose the conjugate exponents $\gamma$ and $\gamma'$ such that $3\gamma'\tilde a<2$. Thus, the identity $u_n=\mathcal{L}_n(u_n)+v_n$, (3.25) and (3.20) imply that for any $t\in[0,T]$, we have

$$\begin{aligned}
\Big(\int_0^t \|u_n(\cdot,s)\|_q^{3\gamma'}ds\Big)^{1/\gamma'} &\le C\Big(\int_0^t (\|\mathcal{L}_n(u_n)\|_{L^\infty}^{3\gamma'}+\|v_n(\cdot,s)\|_q^{3\gamma'})ds\Big)^{1/\gamma'}\\
&\le C(T)\Big[\|\mathcal{L}_n(u_n)\|_{L^\infty}^3+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^3\Big]\\
&\quad + C\sup_{s\in[0,T]}\|v_n(\cdot,s)\|_2^{3(1-\tilde a)}\Big(\int_0^t\|\Delta v_n(\cdot,s)\|_2^{3\tilde a\gamma'}ds\Big)^{1/\gamma'}\\
&\le C(T)\Big[\|\mathcal{L}_n(u_n)\|_{L^\infty}^3+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^3\Big]\\
&\quad + C\Big[\|\mathcal{L}_n(u_n)\|_{L^\infty}^{3(1-\tilde a)}+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^{3(1-\tilde a)}\Big]\Big(\int_0^t\|\Delta v_n(\cdot,s)\|_2^2ds\Big)^{3\tilde a/2}\\
&\le C(T)\Big[\|\mathcal{L}_n(u_n)\|_{L^\infty}^3+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^3\Big]\\
&\quad + C(T)\Big(1+\|u_0\|_2^{6\tilde a}+\|\mathcal{L}_n(u_n)(\cdot,s)\|_{L^\infty}^{9\tilde a}\Big)\Big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^{3(1-\tilde a)}+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^{3(1-\tilde a)}\Big)\\
&\le C(T)(1+\|u_0\|_2^{6\tilde a})\Big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^3+\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^3\Big)\\
&\quad + C(T)\|\mathcal{L}_n(u_n)\|_{L^\infty}^{6\tilde a+3}+C(T)\|\mathcal{L}_n(u_n)\|_{L^\infty}^{9\tilde a}\sup_{s\in[0,T]}\|u_n(\cdot,s)\|_2^{3(1-\tilde a)}.
\end{aligned} \tag{3.26}$$

Thus, choosing the conjugate exponents $\gamma$ and $\gamma'$ such that $\gamma' > \frac{2q}{q-1}$ and $3\gamma'\tilde{a} < 2$, plugging the upper estimate (3.26) into (3.24), using Hölder's inequality, (3.22) and (2.28) we deduce that for any $\beta \in [1,\infty)$ and $\alpha \in (0,\frac{1}{3})$, we have

$$E\Big(\sup_{t\in[0,T]} \|\mathcal{M}_n(u_n)(.,t)\|_q^\beta\Big) \leq C\Big[1 + E\big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^{(6\tilde{a}+3)\beta}\big) + E\Big(\sup_{s\in[0,T]} \|u_n(.,s)\|_2^{3\beta}\Big) + \Big\{E\Big(\|\mathcal{L}_n(u_n)\|_{L^\infty}^{9\beta}\Big)\Big\}^{\tilde{a}}\Big\{E\Big(\sup_{s\in[0,T]} \|u_n(.,s)\|_2^{3\beta}\Big)\Big\}^{1-\tilde{a}}\Big],$$

for some constant $C$ depending on $T, \|u_0\|_2$ and $\beta$. Since $2 > \frac{2\alpha}{3}$ for any $\alpha \in (0,1]$, we deduce from (2.28) applied with $\tilde{q} = 2$ that

$$E\big(\|\mathcal{L}_n(u_n)\|_\infty^p\big) \leq C_p(T) E\Big(\sup_{t\in[0,T]} \|u_n(.,t)\|_2^{\alpha p}\Big),$$

for every $p \in [2,\infty)$. The inequality (3.22) concludes the proof. □

**Remark 3.5.** *Note that the above arguments highly depend on the fact that $d = 1$. Indeed, for $d = 2,3$ the choice of $\gamma$ to ensure time integrability gives an exponent $\gamma'$ too large to be compatible with that of the Laplace operator coming from the Gagliardo-Niremberg inequality. Dimensions $d = 2,3$ would require a more regular noise, such as a noise white in time but colored in space.*

By using Lemmas 2.7 and 3.4 we can complete the proof of existence of a global solution of (2.11) for $d = 1$ and $\alpha \in \big(0,\frac{1}{3}\big)$.

Indeed, for every integer $n \geq 1$ let us define the stopping time $T_n$ as follows:

$$T_n := \inf\Big\{t \geq 0 : \ \|u_n(\cdot,t)\|_q \geq n\Big\}. \tag{3.27}$$

Then for every integer $n \geq 1$, the process $u(\cdot,t) = u_n(\cdot,t)$ is a solution of (2.11) on the interval $[0, T_n \wedge T]$. Assuming that $\alpha \in (0,\frac{1}{3})$, we will show that $\lim_{n\to\infty} T_n = \infty$ a.s., which will enable us to solve (2.11) on $[0,T]$ a.s. for any fixed $T$.

**Theorem 3.6.** *Let $d = 1$, and suppose that $\sigma$ is globally Lipschitz and satisfies the sub-linearity condition* (1.2) *with $\alpha \in (0,\frac{1}{3})$. Let $u_0 \in L^q(\mathcal{D})$ where $q$ satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$. Then for any $T > 0$ there exists a unique solution $u$ to the SPDE* (2.11) *in the time interval $[0,T]$ (or equivalently if $T_n$ is defined by* (3.27)*, $T_n \to \infty$ a.s. as $n \to \infty$); this solution belongs to $L^\infty\big([0,T]; L^q(\mathcal{D})\big)$ a.s. Furthermore, given any $\beta \in [2,\infty)$ we have*

$$E\Big(1_{\{T\leq T_n\}} \sup_{t\leq T} \|u(.,t)\|_q^\beta\Big) \leq C,$$

*for some constant $C$ depending on $T, \beta, q$ and $\|u_0\|_2$.*

*Proof.* The sequence $T_n$ is clearly non decreasing. Fix $T > 0$; by the definition of $T_n$, on the set $\{T_n < T\}$ we have for any $\beta \in [2,\infty)$

$$\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_q^\beta \geq n^\beta.$$

Thus, the Chebyshev inequality, (2.12), (2.18), (2.28) and (3.23) yield the existence of a constant $C$ depending on $T$, $\|u_0\|_q$, $\|u_0\|_2$ and $\beta$ such that for every $n \geq 1$ the next inequality holds true

$$P(T_n < T) \leq n^{-\beta} E\Big(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_q^\beta\Big) \leq C n^{-\beta}. \tag{3.28}$$

Since $\beta \in [2,\infty)$, the Borel-Cantelli Lemma implies that $P(\limsup_{n\to\infty}\{T_n < T\}) = 0$, that is $\lim_{n\to\infty} T_n \geq T$ a.s. Since $T$ is arbitrary, this yields $T_n \to \infty$ a.s. as $n \to \infty$. The uniqueness of the solution to (2.14) implies that a process $u$ can be uniquely defined setting $u(\cdot,t) = u_n(\cdot,t)$ on

$[0, T_n]$. Since $T_n \to \infty$ a.s., we conclude that for any fixed $T > 0$, equation (2.11) has a unique solution and the upper estimate of moments of the $q$-norm of the solution follows from (2.12), (2.18), (2.28) and (3.23). □

**Remark 3.7.** *In a future work we aim to treat the problem in its full generality, investigating also equal-well potentials of higher polynomial order $4+k$ for $k > 0$, resulting to a higher order nonlinearity (of order $3+k$), together with the restriction on the growth $\alpha$. Based on this paper's results we conjecture that global solutions exist for $\alpha \in \left[0, \frac{1}{3+k}\right)$ in dimension 1 and that a larger dimension $d$ would require a driving noise more regular in the space variable.*

## 4. Generalization

The stochastic local existence proof for the Cahn-Hilliard/Allen-Cahn equation with noise, could easily be modified to hold for domains with more general geometry, cf. in [2] the proposed eigenvalue-formulae-free approach for the stochastic Cahn-Hilliard equation.

Furthermore, all our results proven so far, are also valid for the more general model

$$\left\{\begin{array}{rcl} u_t &=& -\varrho\Delta\Big(\Delta u - f(u)\Big) + \tilde{q}\Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W} \quad \text{in} \quad \mathcal{D}\times[0,T), \\ u(x,0) &=& u_0(x) \quad \text{in} \quad \mathcal{D}, \\ \frac{\partial u}{\partial \nu} &=& \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{D}\times[0,T), \end{array}\right. \tag{4.1}$$

for some constants $\varrho > 0$ and $\tilde{q} \geq 0$. The proof is very similar with the following simple modifications:

(1) We have to replace the Green's function $G$ defined by (2.2) by the following $\varrho, \tilde{q}$-dependent one

$$G^{\varrho,\tilde{q}}(x,y,t) := \sum_{k\in\mathbb{N}^d} e^{(-\varrho\lambda_k^2+\tilde{q}\lambda_k)t}\epsilon_k(x)\epsilon_k(y).$$

All the estimates used on $G$ also hold for $G^{\varrho,\tilde{q}}$, since the operator $-\varrho\Delta^2+\tilde{q}\Delta$ is parabolic in the sense of Petrovksĭı.

(2) The estimate (3.16) also holds for $\varrho > 0$ and $\tilde{q} > 0$ when $B(u_0)$ is defined as

$$B(u_0) := \frac{1}{2}\Big\| \sum_{k\in\mathbb{N}^d} [\varrho\lambda_k + \tilde{q}]^{-\frac{1}{2}}(u_0, \epsilon_k)_{L^2(\mathcal{D})}\epsilon_k \Big\|_2^2.$$

Since $\varrho > 0$ and $\lambda_k \geq 0$, one may also invert $\varrho\lambda_k + \tilde{q}$ if $\tilde{q} > 0$ for any $k \in \mathbb{N}^d$.

If $\tilde{q} = 0$ (for $\varrho = 1$ we get the Cahn-Hilliard equation) then

$$B(u_0) := \frac{1}{2}\Big\| \sum_{k\in\mathbb{N}^{*d}} [\varrho\lambda_k]^{-\frac{1}{2}}(u_0, \epsilon_k)_{L^2(\mathcal{D})}\epsilon_k \Big\|_2^2.$$

and $\varrho\lambda_k$, for any $k \in \mathbb{N}^{*d}$, is invertible.

While the stochastic Cahn-Hilliard equation is a special case for our analysis (with $\rho = 1$ and $\tilde{q} = 0$), this is not true for the stochastic Allen-Cahn equation. In our model the assumption that $\varrho > 0$ is crucial; indeed, since the fourth order operator is still acting, the operator $-\rho\Delta^2 + \tilde{q}\Delta$ is also parabolic in the sense of Petrovskĭı. Thus, the higher order differential operator is dominating and all the upper estimates of the Green's function and their derivatives stated in Section 2.2 remain valid.

## 5. Path regularity

In this section, we investigate the path regularity for the global solution to stochastic solution of (1.1) when $d = 1$ under certain regularity assumptions for the initial condition $u_0$. Our arguments will also provide a.s. regularity of the maximal solution of the solution to (1.1), that is when restricted to the time interval $[0, T^*)$ defined in section 2.5.

More precisely, we prove that when the coefficient $\sigma$ has an appropriate sub-linear growth, the paths of the solution to equation (1.1) have a.s. a Hölder regularity depending on that of the initial condition. The path regularity proven here is the same as that obtained for the stochastic Cahn-Hilliard equation obtained in [3], where the coefficient $\sigma$ was supposed to be bounded. We follow the main lines of the proof presented in [3]; nevertheless some modifications are needed. Indeed, the factorization method is used both for the deterministic and stochastic integrals.

In this section we suppose that the assumptions of Theorem 3.6 are satisfied. Let us recall that the integral form of the solution $u$ given by (2.11) can be decomposed as follows:

$$u(t,x) = G_t u_0(x) + \mathcal{I}(x,t) + \mathcal{J}(x,t), \tag{5.1}$$

where $G_t u_0$ is defined by (2.13), and

$$\mathcal{I}(x,t) = \int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)] f(u(y,s))\, dyds,$$

$$\mathcal{J}(x,t) = \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\sigma(u(y,s))\, W(dy,ds). \tag{5.2}$$

Let us study the regularity of each term in the decomposition (5.1) of $u$.

The series decomposition of $G$ given in (2.2) is similar to that in [3]; hence an argument similar to the proof of Lemma 2.1 of [3] if $u_0$ is continuous, and to the first part of Lemma 2.2 of [3] if $u_0$ is $\delta$-Hölder continuous, yields the following regularity result for $G_\cdot u_0(\cdot)$.

**Lemma 5.1.** *If $u_0$ is continuous, then the function $G_t u_0$ is continuous. If $u_0$ belongs to $C^\delta(\mathcal{D})$ for $0 < \delta < 1$, then the function $(x,t) \to G_t u_0(x)$ is $\delta$-Hölder continuous in the space variable $x$ and $\frac{\delta}{4}$-Hölder continuous in the time variable $t$.*

Let us now consider the drift term $\mathcal{I}(x,t)$ and use the factorization method (see e.g. [8] or [3]).

We remark that, as proved in Theorem 3.6, if $u_0$ is bounded, then $u$ belongs a.s. to $L^\infty(0,T; L^q(\mathcal{D}))$ for any $q < \infty$ large enough.

The definition of the Green's function yields

$$\Delta G(x,y,t) = \int_{\mathcal{D}} G(x,y,t-s)\Delta G(z,y,s)\, dz, \tag{5.3}$$

and

$$G(x,y,t) = \int_{\mathcal{D}} G(x,y,t-s) G(z,y,s)\, dz. \tag{5.4}$$

For some $a \in (0,1)$ define the operators $\mathcal{F}$ and $\mathcal{H}$ on $L^\infty(0,T; L^q(\mathcal{D}))$ as follows:

$$\mathcal{F}(v)(t,x) := \int_0^t \int_{\mathcal{D}} G(x,z,t-s)(t-s)^{-a} v(z,s)\, dzds,$$

$$\mathcal{H}(v)(z,s) := \int_0^s \int_{\mathcal{D}} \big[\Delta G(z,y,s-s') - G(z,y,s-s')\big](s-s')^{a-1} f(v(y,s'))\, dyds'.$$

Therefore, using relations (5.3) and (5.4) we deduce that

$$\mathcal{I}(x,t) = c_a \mathcal{F}(\mathcal{H}(u))(x,t),$$

where $c_a := \pi^{-1}\sin(\pi a)$ obviously depends only on $a$.

First we claim that, for $q$ satisfying condition $(\tilde{\mathbf{C}}_\alpha)$, the operator $\mathcal{H}$ maps $L^\infty(0,T;L^q(\mathcal{D}))$ into itself. Indeed, the estimates on the Green's function in Lemma 2.1 and arguments similar to those used in Section 2.4 to prove (2.19) with $\rho = \frac{q}{3}$ (based on the Minkowski and Young inequalities) prove that if $v \in L^\infty(0,T;L^q(\mathcal{D}))$ then

$$\|\mathcal{H}(v)(\cdot,t)\|_q \le \int_0^t (t-s)^{-1+a-\frac12-\frac{d}{2q}} \big(1+\|v(\cdot,s)\|_q^3\big)\, ds.$$

For the boundedness of the above integral we need that $-1+a-\frac12-\frac{d}{2q} > -1$. Since $q > d$, this inequality holds for some $a \in \big(\frac12+\frac{d}{2q},1\big)$. Then, an argument similar to that used in [3] proves that if $v \in L^\infty([0,T];L^q(\mathcal{D}))$ then $\mathcal{F}(v)$ belongs to $\mathcal{C}^{\lambda,\mu}(\mathcal{D}\times[0,T])$ for any $\lambda < 1$ and $\mu < \frac12$. Indeed, the upper estimates of the Green's function from Lemma 2.2 are the same as that for the Green's function of the Cahn-Hilliard equation which only involves the fourth order derivatives.

Considering the stochastic integral $\mathcal{J}$ defined in (5.2), we observe that the fact that $\sigma$ is not bounded any more does not allow us to use the related argument from the proof of Lemma 2.2 in [3] stated on page 797. Instead, we also use the factorization method for the stochastic integral. Let $T^*$ be defined by section 2.5; recall that given any $n \ge 1$, for

$$T_n = \inf\{t \ge 0 : \|u_n(.,t)\|_q \ge n\} \wedge T^*,$$

we have $1_{\{T<T_n\}}u(.,t) = 1_{\{T<T_n\}}u_n(.,t)$, where $u_n$ is the solution to (2.14). The local property of stochastic integrals implies that for any $n$ and $t \in [0,T]$:

$$1_{\{T<T_n\}}\mathcal{J}(x,t) = 1_{\{T<T_n\}} \int_0^t \int_{\mathcal{D}} G(t-s,x,y) 1_{\{s\le T_n\}} \sigma(u_n(y,s)) W(dy,ds).$$

The process $u_n$ is adapted and since $u_0$ is bounded Condition $(\tilde{\mathbf{C}}_\alpha)$ holds true for $q$ large enough. Furthermore, if $\gamma > 0$ and $\beta \in (1,\infty)$ are such that $\beta\gamma \in [2,\infty)$, using the inequalities (2.28) and (3.2) we have $E\big|\int_0^T \|u_n(.,t)\|_q^\gamma dt\big|^\beta \le C(n,T)$. Fix $a \in (0,1)$, let $\mathcal{K}(u_n)$ be defined as follows:

$$\mathcal{K}(u_n)(x,t) = \int_0^t \int_{\mathcal{D}} G(x,y,t-s)(t-s)^{a-1} 1_{\{s<T_n\}} \sigma(u_n(y,s)) W(dy,ds).$$

We first check that this stochastic integral makes sense for fixed $t \in [0,T]$ and $x \in \mathcal{D}$, and that a.s. $\mathcal{K}(u_n) \in L^\infty(0,T;L^q(\mathcal{D}))$, so that $1_{\{T\le T_n\}}\mathcal{J}(x,t) = 1_{\{T\le T_n\}} c_a \mathcal{F}(\mathcal{K}(u_n))(x,t)$.

Indeed, for fixed $t \in [0,T]$, $x \in \mathcal{D}$ and $p \in [1,\infty)$, the Burkholder inequality yields

$$\begin{aligned} E|\mathcal{K}(u_n)(x,t)|^{2p} \le & E\Big| \int_0^t \int_{\mathcal{D}} G^2(x,y,t-s)(t-s)^{2(a-1)} 1_{\{s\le T_n\}} \sigma^2(u_n(s,y)) dy ds\Big|^p \\ \le & C(n) \Big| \int_0^t (t-s)^{-\frac{d}{2}+2(a-1)+\frac{d}{4}} ds\Big|^p \end{aligned}$$

Let $a \in (\frac12+\frac{d}{8},1)$; then we have $-\frac{d}{2}+2(a-1)+\frac{d}{4} > -1$, which yields

$$E|\mathcal{K}(u_n)(x,t)|^{2p} < \infty, \ \ \forall p \in [1,\infty). \tag{5.5}$$

Let us now prove moment upper estimates of increments of $\mathcal{K}(u_n)$; this together with (5.5) will imply by Garsia's Lemma that

$$E\big(\|\mathcal{K}(u_n)\|^{2\rho}_{L^\infty(\mathcal{D}\times[0,T])}\big) < \infty.$$

Arguments similar to those used in the proof of (2.29) prove that for $\tilde\lambda \in (0,1)$, $\tilde q \in (2\alpha, q)$ and $n \geq 1$, we have for $t \in [0,T]$, $x, \xi \in \mathcal{D}$:

$$\begin{aligned} E\big|\mathcal{K}(u_n)(x,t) - \mathcal{K}(u_n)(\xi,t)\big|^{2p} &\leq C_p|x-\xi|^{2\tilde\lambda p} \\ &\quad\times \Big|\int_0^t (t-s)^{-\frac{d+1}{2}\tilde\lambda - \frac{d}{2}(1-\tilde\lambda)+2(a-1)} 1_{\{s\leq T_n\}} \|\exp(h(.,t,s)\|_{\frac{\tilde q}{\tilde q - 2\alpha}} \big[1 + \|u_n(.,s)\|_{\tilde q}^{2\alpha}\big] ds\Big|^p \\ &\leq C_p(n)|x-\xi|^{2\tilde\lambda p}\Big|\int_0^t (t-s)^{-\frac{d+\tilde\lambda}{2}+2(a-1)+\frac{d(\tilde q-2\alpha)}{4\tilde q}} ds\Big|^p \\ &\leq C_p(n,T)|x-\xi|^{2\tilde\lambda p} \end{aligned}$$

for some finite constant $C_p(n,T)$, provided that the time integrability constraint $-\frac{d+\tilde\lambda}{2} + 2(a-1) + \frac{d(\tilde q - 2\alpha)}{4\tilde q} > -1$ holds true. Recall that for $d = 1,2$ we have $q \in [2,\infty)$ while for $d = 3$ we have required $q > 6\alpha$; in both cases we deduce that $\alpha < q\big(\frac{2}{d} - \frac12\big)$. Thus, given $\bar\lambda \in \big(0, 2 - \frac{d}{2} - \frac{d\alpha}{q}\big) \cap (0,1)$ one can find $\tilde q \in (2\alpha, q)$ and $a \in \big(\frac12 + \frac{d}{8}, 1\big)$ such that the time integrability is fulfilled. Furthermore, if we may choose $q$ as large as we want (because the initial condition is bounded) and $\bar\lambda \in \big(0, \big[2 - \frac{d}{2}\big] \wedge 1\big)$, then the choice of $\tilde q < q$ and $a < 1$ ensuring the time integrability is still possible.

Let $0 \leq t' \leq t \leq T$, $x \in \mathcal{D}$ and $\tilde\mu \in \big(0, \frac12 - \frac{d}{8}\big)$; arguments similar to that proving (2.30) imply

$$\begin{aligned} E\big|\mathcal{K}(u_n)(x,t) - \mathcal{K}(u_n)(x,t')\big|^{2p} &\leq C_p|t-t'|^{2\tilde\mu p} \\ &\quad\times \Big|\int_0^t (t-s)^{-2(\frac{d}{4}+1)\tilde\mu - \frac{d}{2}(1-\tilde\mu)+2(a-1)} 1_{\{s\leq T_n\}} \|\exp(h(.,t,s)\|_{\frac{\tilde q}{\tilde q - 2\alpha}} \big[1 + \|u_n(.,s)\|_{\tilde q}^{2\alpha}\big] ds\Big|^p \\ &\leq C_p(n)|t-t'|^{2\tilde\mu p}\Big|\int_0^t (t-s)^{-\frac{d}{2}-2\tilde\mu+2(a-1)+\frac{d(\tilde q-2\alpha)}{4\tilde q}} ds\Big|^p \\ &\leq C_p(n,T)|t-t'|^{2\tilde\mu p}, \end{aligned}$$

for some finite constant $C_p(n,T)$, provided that $-\frac{d}{4} - 2\tilde\mu - 2(1-a) - \frac{\alpha d}{2\tilde q} > -1$. Once more, given $\tilde\mu \in \big(0, \frac12 - \frac{d}{8} - \frac{\alpha d}{4q}\big)$, we can find $\tilde q \in (2\alpha, q)$ and $a \in \big(\frac12 + \frac{d}{8}, 1\big)$ such that this inequality holds true. Furthermore, if we may choose $q$ as large as we want (because the initial condition is bounded) and $\bar\mu \in \big(0, \frac12 - \frac{d}{8}\big)$, then the choice of $\tilde q < q$ and $a < 1$ ensuring the time integrability is still possible.

Hence, given a bounded initial condition $u_0$, $\bar\lambda \in \big(0, \big[2 - \frac{d}{2}\big] \wedge 1\big)$ and $\bar\mu \in \big(0, \frac12 - \frac{d}{8}\big)$, for every $n \geq 1$ and $p \in [1,\infty)$, we can find some positive constant $C_p(n,T)$ such that

$$E|\mathcal{K}(u_n)(x,t) - \mathcal{K}(u_n)(\xi,t')|^{2p} \leq C_p(n,T)\big(|\xi - x|^{2\tilde\lambda p} + |t-t'|^{2\tilde\mu p}\big),$$

for $0 \leq t' \leq t \leq T$ and $x, \xi \in \mathcal{D}$.

The Garsia-Rodemich-Rumsey lemma implies that

$$E\big(\|\mathcal{K}(u_n)\|^{2p}_{L^\infty(\mathcal{D}\times[0,T])}\big) < \infty, \quad \forall p \geq 1,$$

and

$$E\big(\|\mathcal{K}(u_n)\|_q^{2p}\big) \leq E\big(\|\mathcal{K}(u_n)\|^{2p}_{L^\infty(\mathcal{D}\times[0,T])}\big) < \infty, \quad \forall p \geq 1.$$

This gives on one hand the stated time and space Hölder regularity, and on the other hand the previous space-time Hölder moments estimates of $\mathcal{K}(u_n) \in L^\infty(0,T,L^q(\mathcal{D}))$ a.s.

Since $\mathcal{F}$ maps $L^\infty(0,T;L^q(\mathcal{D}))$ into $\mathcal{C}^{\lambda,\mu}(\mathcal{D}\times[0,T])$ for $\lambda < \big[2 - \frac{d}{2}\big] \wedge 1$ and $\mu < \frac12 - \frac{d}{8}$, and since $\mathcal{J}(x,t) = c_a \mathcal{F}(\mathcal{K}(u_n))(x,t)$ on the set $\{T < T_n\}$, we deduce that $\mathcal{J} \in \mathcal{C}^{\lambda,\mu}(\mathcal{D}\times[0,T])$ a.s. on the set $\{T < T_n\}$.

Finally, Theorem 3.6 implies that as $n \to \infty$ the sets $\{T < T_n\}$ increase to $\Omega$ when $d = 1$ and to $t < T^*$ for $d = 2, 3$; this proves that a.s. $\mathcal{J} \in \mathcal{C}^{\lambda,\mu}(\mathcal{D} \times [0, T])$ for $\lambda < \left[2 - \frac{d}{2}\right] \wedge 1$ and $\mu < \frac{1}{2} - \frac{d}{8}$.

As a consequence (cf. [3]), we obtain the following regularity of the trajectories for $d = 1$.

**Theorem 5.2.** *Let $d = 1$, $\sigma$ be globally Lipschitz and satisfy the sub-linearity condition* (1.2) *with $\alpha \in (0, \frac{1}{3})$, and let $u_0 \in L^\infty(\mathcal{D})$. Then we have:*

*(i) If $u_0$ is continuous, then the global solution of* (2.11) *has almost surely continuous trajectories.*

*(ii) If $u_0$ is $\beta$-Hölder continuous for $0 < \beta < 1$, then the trajectories of the global solution to* (2.11) *are almost surely $\beta \wedge \left(2 - \frac{d}{2}\right)$-continuous in space and $\frac{\beta}{4} \wedge \left(\frac{1}{2} - \frac{d}{8}\right)$-continuous in time.*

A similar result holds for the maximal solution of (2.11) in dimension $d = 2, 3$.

**Theorem 5.3.** *Let $d = 2, 3$, $\sigma$ be globally Lipschitz and let $u_0 \in L^\infty(\mathcal{D})$. Let $T^*$ denote the stopping time introcuded in section 2.5; then we have:*

*(i) If $u_0$ is continuous, then the local maximal solution of* (2.11) *has almost surely continuous trajectories on $\mathcal{D} \times [0, T^*)$.*

*(ii) If $u_0$ is $\beta$-Hölder continuous for $0 < \beta < 1$, then the trajectories of the global solution to* (2.11) *are almost surely $\beta \wedge \left(2 - \frac{d}{2}\right)$-continuous in space and $\frac{\beta}{4} \wedge \left(\frac{1}{2} - \frac{d}{8}\right)$-continuous in time on $\mathcal{D} \times [0, T^*)$.*

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