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Mixed-type functional differential equations: A numerical approach (Extended version)

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Abstract

We consider forward-backward (mixed) functional equations of the general form

$$x'(t) = a(t)x(t) + b(t)x(t-1) + c(t)x(t+1).$$

We give some initial insights into the analytical properties of the equation and its solutions(s) and we describe some of the particular challenges presented. We give an existence and uniqueness theorem for certain linear autonomous equations. We introduce numerical schemes based on linear θ-methods and investigate their effectiveness in producing approximate solutions on $(0, k-1]$where the boundary conditions are given on the intervals $[-1, 0]$ and $(k-1, k]$, $k \in \mathbb{N}$. The paper concludes with some illustrative examples.

1 Introduction and background information

Functional differential equations with both delayed and advanced arguments are generally referred to as mixed (type) functional differential equations (MFDEs) or forward-backward equations. The study of such equations was motivated by optimal control problems [12] but applications also arise in other areas (see, for example, nerve conduction theory [2] economic dynamics [13], travelling waves in a spatial lattice [1]). In many example applications the independent variable is likely to be spatial rather than temporal.

The analysis of MFDEs presents a significant challenge and it is appropriate for us to spend some time in this paper gathering together various insights and results. Our aim is to provide a reasonably firm foundation on which to

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build the numerical scheme that we shall discuss later. To be precise, we shall investigate solutions to equations of the specific form:

$$u'(t) = au(t) + bu(t-1) + cu(t+1). \tag{1}$$

One might reasonably expect to gain some insights into these problems from the existing literature on retarded differential equations of the form

$$u'(t) = au(t) + bu(t-1) \tag{2}$$

and this will be our starting point in this paper. In Section 5 we adopt the approach by Iakovleva and Vanegas [7]. We consider a more general equation than in [7] and prove existence and uniqueness of solutions to equation (3) under a stated assumption. We introduce the aims of our numerical investigation and our methodology in Section 6 and in Section 7 we include numerical examples to support our work. We summarise our results in Section 8 and indicate the potential for further research and investigation.

2 MFDEs as boundary value problems

We are accustomed to needing to specify initial or boundary conditions for a differential equation in order to be able to arrive at a unique solution. Therefore a natural starting point is to consider the appropriate form of conditions that will be needed for a MFDE of the form (3). One can apply insights from the study of retarded differential equations to realise that the conditions will be required in terms of function values over one or more intervals. Alternatively, application of Laplace transforms to (3) leads to a formulation for the Laplace transform of the solution in terms of a *boundary function* ϕ whose values need to be given over the intervals $[-1, 0]$ and $(k-1, k]$ if a solution is sought over the interval $[0, k-1)$.

Unfortunately, as we shall see later, the specification of such a boundary function is not sufficient to ensure that a solution can be found. It turns out that the underlying problem is both ill-posed (see section 3) and highly unstable (the characteristic equation has solutions with positive real part). We shall discuss this further in the next sections. In Section 4 we determine solutions to equation (3) of the form $x(t) = e^{\lambda t}$.

$$x'(t) = ax(t) + bx(t-1) + cx(t+1) \tag{3}$$

3 Working with MFDEs: Insight, problems and some solutions

In general, the Cauchy problem is not well posed for both positive and negative times. This fact is cited by Rustichini [12], in 1989, as the "single most relevant difficulty encountered when dealing with MFDEs". Evidence supporting this statment is readily found in the existing literature. Consequences of the problem being ill-posed include the following:

- Linear equations with constant coefficients may not have a solution for a given initial condition. For example, equation (3) with $a = 0, b = 1, c = 1$ and initial condition $x_0 \equiv 1$ has no everywhere continuous solution [6,12].
- 'a variation of constants formula analagous to that for ODEs and DDEs does not seem possible' [13].
- 'translation along solutions does not generate strongly continuous semigroups' [10].
- Krisztin [8] states that, since an upper bound to the real parts of the characteristic equation does not exist, then 'we cannot expect, in general, asymtotic expansions of the form $x(t) = \Sigma_{j=0}^{\ell} p_j(t)e^{z_j t} + O(e^{\gamma t})$ for $t \to \infty$' for all solutions to the linear autonomous FDE.
- 'does not generate a semi-flow on any space that contains all its eigenfunctions' [6].

This problem of ill-posedness has been *avoided* or *overcome* by several authors. Mallet-Paret and Verduyn Lunel decompose solutions as sums of 'forward' and 'backward' solutions [10]. Rustichini [13] restricts the action of a linear operator of mixed type to functions which are periodic. This enables the operator to be identified with one of the delay type. Härterich et al discuss finding functions $\phi(t)$ for which a solution exists on either $\mathbb{R}^+$ or $\mathbb{R}^-$. However, they comment on the difficulty of determining whether or not the set of eigenfunctions is complete. Verduyn Lunel [14] gives conditions which guarantee completeness of the set of eigenfunctions for autonomous FDEs. This links with our work on small solutions [4,5,11] but this is not our concern here. Mallet-Paret establishes an existence theory for a class of MFDEs of mixed type using a linear Fredholm theory and the implicit funtion theory [9]. Collard et al. [3] develop and implement a numerical procedure 'to solve for the short run dynamics of a neoclassical growth model with a simple time-to-build lag'. (They use a Runge-Kutta type of algorithm combined with a shooting method). Abell et al. have developed a general purpose code, COLMTFDE, for MFDEs. We refer the reader to [1] for further details. Progress has been made since the statement in 1989 by Rustichini. However, in 2005 Abell et al [1] still refer to the 'lack of analytical techniques and numerical solvers for differential equations with both forward and backward delays'.

4 The characteristic quasi-polynomial

Many of the interesting features and challenges of MFDEs become apparent when one considers the roots of the characteristic quasi-polynomial, which takes the form $\lambda = a + be^{-\lambda} + ce^{\lambda}$. With $\lambda = x + iy$ and $x \neq \frac{1}{2}\ln\left|\frac{b}{c}\right|$, we obtain

$$\cos y = \frac{(x-a)}{(be^{-x} + ce^{x})}, \tag{4}$$

and

$$\sin y = \frac{y}{(-be^{-x} + ce^{x})}, \tag{5}$$

leading to

$$y = \pm \left(ce^{x} - be^{-x}\right)\sqrt{1 - \frac{(x-a)^2}{(be^{-x} + ce^{x})^2}}. \tag{6}$$

In Figure 1 we give an illustrative example of the graph of y against x given by (4) and (6). In Figure 2 we provide an illustrative example of the graph for $\cos(y)$ against x. The characteristic values λ will be given by the values of x, y corresponding to points where the graphs given by (4) and (6) intersect. We can see that there are infinitely many characteristic values with positive x coefficient and infinitely many with negative x coefficient. The characteristic roots do not have any cluster points. In the special cases where either $b = 0$ or $c = 0$, either the left hand or right hand branch of one of the graphs is absent which accords with our previous experience for retarded differential equations.

The implication of these characteristic values is significant. There are eigenfunctions with arbitrarily large exponential growth rates and eigenfunctions with arbitrarily large exponential decay rates. This means that, whether we project forward in time from the left hand end of the interval or backwards in time from the right hand end, the solution we obtain will be arbitrarily highly unstable with respect to small changes in the boundary function. This explains the possible motivation for decomposing the solution into forward and backward components.

For a given a, b, c we can obtain values $x = x_1$, find y_1 such that $\lambda = x_1 + iy_1$ and hence give a solution to the equation of the form $x(t) = e^{x_1 t}(\cos y_1 t + i \sin y_1 t)$. In Figures 3 and 4 we illustrate typical solutions, $x(t)$, for $x_1 < 0$ and $x_1 > 0$ respectively. In each case the same equation is used for all four graphs but the time period over which the solution is given varies. We observe that, as expected, the solution converges for $x_1 < 0$ but not for $x_1 > 0$.

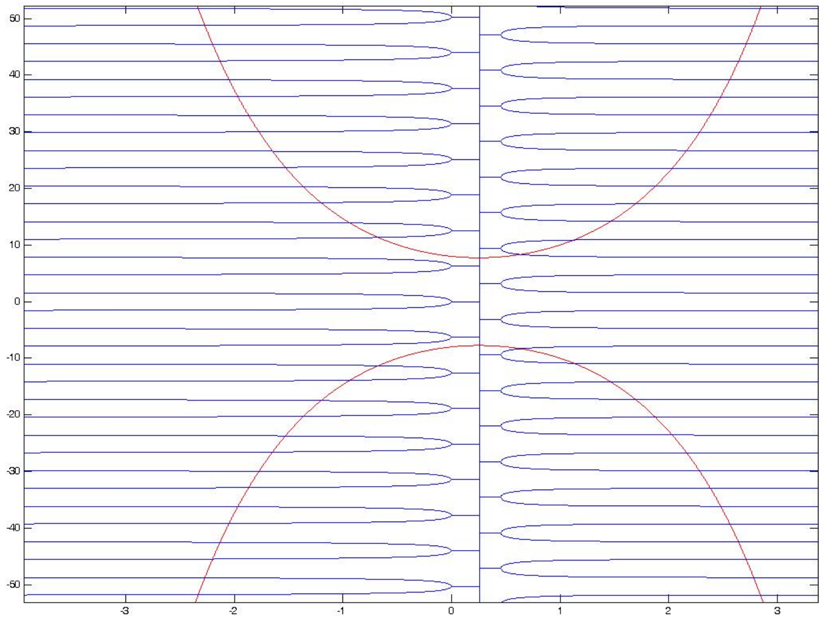

Fig. 1. y against x for equation $x'(t) = 2x(t) - 5x(t-1) + 3x(t+1)$;

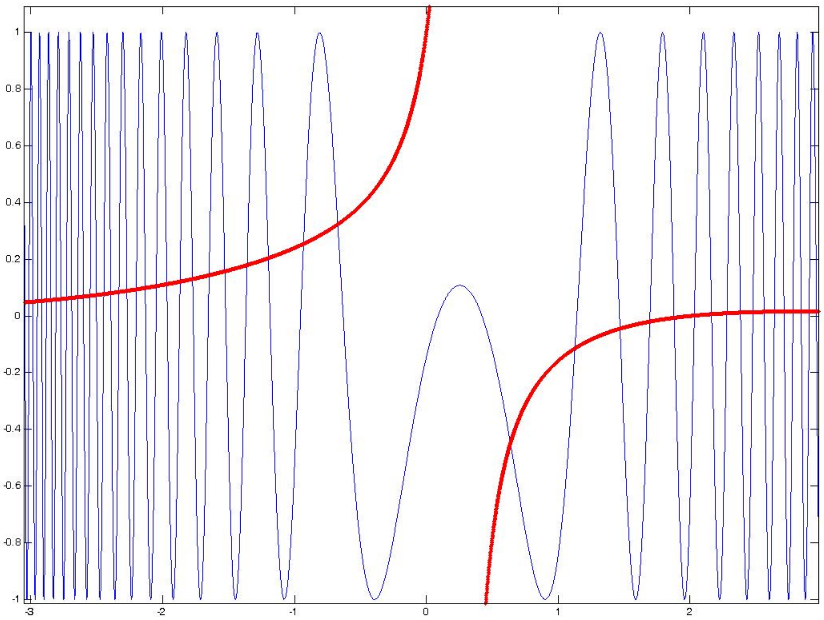

Fig. 2. $\cos(y)$ against x for equation $x'(t) = 2x(t) - 5x(t-1) + 3x(t+1)$;

5 Existence and uniqueness of solutions

Comments referring to the limited knowledge of existence and uniqueness theory for MFDEs are readily found in the relatively small (but growing)

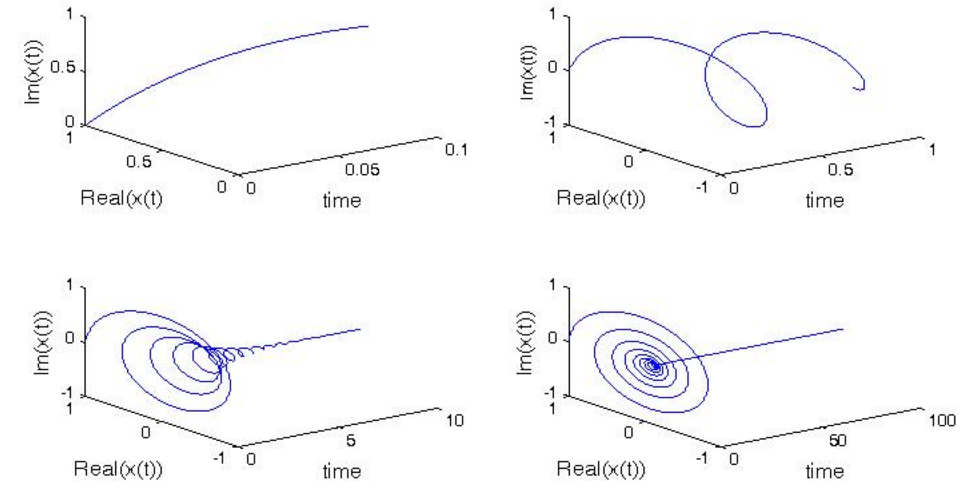

Fig. 3. Equation $x'(t) = 2x(t) - 5x(t-1) + 3x(t+1)$; $x_1 = -0.6723$

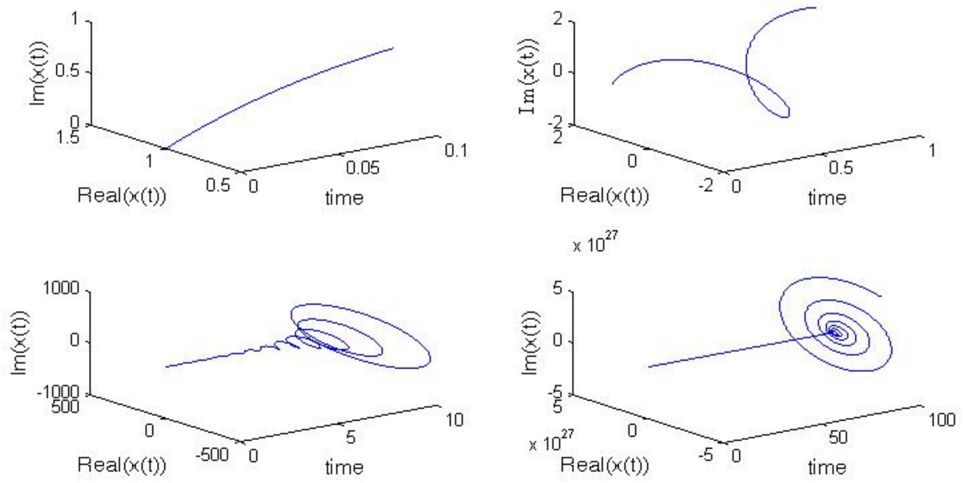

Fig. 4. Equation $x'(t) = 2x(t) - 5x(t-1) + 3x(t+1)$; $x_1 = 0.6379$

amount of literature available on this type of functional equation (see [1,9]). Prior to discussing the uniqueness and existence of solutions we first establish what is meant by a solution of a MFDE. Definition 5.1 is given by Mallet-Paret and Verduyn Lunel. [see [8,12] for alternative definitions]

Definition 5.1 *[10] A solution of equation (3) on an interval* $[t_1, t_2] \subseteq \mathbb{R}$ *is a continuous function* $x : [t_1 - 1, t_2 + 1] \to \mathbb{C}$ *which is absolutely continuous on* $[t_1, t_2]$ *and which satisfies equation (3) for almost every* $t \in [t_1, t_2]$.

Remark 5.1 *Using* $x(t) = e^{at}y(t)$ *we can transform (3) into an equation of the form* $y'(t) = \alpha y(t+1) + \beta y(t-1)$. *We are thus able to focus our attention* *on equations which do not include an instantaneous term.*

We follow the methodology adopted in [7] and extend the solution to the right. We consider equation (7) [equation (3) transformed to remove the instantaneous term]

$$x'(t) = \alpha x(t+1) + \beta x(t-1), \tag{7}$$

with the initial conditions $x(t) = \phi(t)$, $t \in [-1, 1]$,
with $\phi(t) = \begin{cases} \phi_1(t) \text{ for } t \in [-1, 0], \\ \phi_2(t) \text{ for } t \in (0, 1]. \end{cases}$
We rewrite (7) in the form

$$x(t) = ax'(t-1) + bx(t-2) \text{ where } a = \frac{1}{\alpha}, \qquad b = -\frac{\beta}{\alpha}. \tag{8}$$

Table 1 gives the solution using the method of steps for $x(t)$ for $t \in (1, 5)$.

Interval for t	Solution $x(t)$
(1,2)	$a\phi_2'(t-1) + b\phi_1(t-2)$
(2,3)	$a^2\phi_2''(t-2) + b\phi_2(t-2) + ab\phi_1'(t-3)$
(3,4)	$a^3\phi_2'''(t-3) + 2ab\phi_2'(t-3) + a^2b\phi_1''(t-4) + b^2\phi_1(t-4)$
(4,5)	$a^4\phi_2^{(4)}(t-4) + 3a^2b\phi_2''(t-4) + b^2\phi_2(t-4) + a^3b\phi_1'''(t-5)$ $+2ab^2\phi_1'(t-5)$

Table 1
Solution, $x(t)$, to $x(t) = ax'(t-1) + bx(t-2)$ for $t \in (1, 5)$, with $x(t) = \phi(t), t \in [-1, 1]$, $\phi(t) = \phi_1(t)$ for $t \in [-1, 0)$, $\phi(t) = \phi_2(t)$ for $t \in [0, 1]$.

Continuing in this way we find that for $t \in (2\ell - 1, 2\ell)$ the solution $x(t)$ is given by

$$x(t) = \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k}a^{2k}b^{\ell-k}\phi^{(2k)}(t-2\ell) + \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k-1}\phi^{(2k+1)}(t-(2\ell-1)) \tag{9}$$

and for $t \in (2\ell, 2\ell + 1)$ by

$$x(t) = \Sigma_{k=0}^{\ell}\delta_{\ell,2k}a^{2k}b^{\ell-k}\phi^{(2k)}(t-2\ell) + \Sigma_{k=0}^{\ell-1}\delta_{\ell,2k+1}a^{2k+1}b^{\ell-k}\phi^{(2k+1)}(t-(2\ell+1)), \tag{10}$$

where $\gamma_{v,w}$ and $\delta_{v,w}, v, w \in \mathbb{N}$, are defined for $\ell \geq 1$ and $v \leq 2\ell - 1$ as follows:

$$\gamma_{\ell,2k} = \Sigma_{i=0}^{k}\gamma_{\ell-k-1+i,2i} + \Sigma_{i=0}^{k-1}\delta_{\ell-k-1+i,2i+1} \tag{11}$$
$$\gamma_{\ell,2k+1} = \Sigma_{i=0}^{k}\gamma_{\ell-k-1+i,2i} + \Sigma_{i=0}^{k}\delta_{\ell-k-1+i,2i+1} \tag{12}$$
$$\delta_{\ell,2k} = \gamma_{\ell+1,2k} \tag{13}$$
$$\delta_{\ell,2k+1} = \gamma_{\ell,2k+1}. \tag{14}$$

We can show that $\gamma_{\ell,0} = 1$, $\gamma_{\ell,2\ell-1} = 1$, $\gamma_{\ell,2\ell-2} = 1$, $\delta_{\ell,0} = 1, \delta_{\ell,2\ell-1} = 1$, $\delta_{\ell,2\ell} = 1$, and establish the following relationships

$$\gamma_{p,2k-1} + \gamma_{p,2k} = \gamma_{p+1,2k} \tag{15}$$
$$\gamma_{p+1,2k} + \gamma_{p,2k+1} = \gamma_{p+1,2k+1} \tag{16}$$

We note that $\gamma_{v,w} = \delta_{v,w} = 0$ for $v < 0$ and that $\gamma_{\ell,w} = \delta_{\ell,w} = 0$ for $j > 2\ell - 1$. We prove the validity of expressions (9) and (10) by induction. The cases $\ell = 1, \ell = 2$ are already proven (see Table 1). Introduce $\tilde{t} = t - 2p$. We assume that (9) and (10) hold for $\ell = p$. Hence, on $(2p+1, 2p+2)$, using (8) gives

$$\begin{aligned}
&x\ (t) = \\
&\Sigma_{k=0}^{p}\delta_{p,2k}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) + \Sigma_{k=0}^{p-1}\delta_{p,2k+1}a^{2k+2}b^{p-k}\phi^{(2k+2)}(\tilde{t}-2) \\
&\quad +\Sigma_{k=0}^{p-1}\left\{\gamma_{p,2k}a^{2k}b^{p+1-k}\phi^{(2k)}(\tilde{t}-2) + \gamma_{p,2k+1}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1)\right\}, \\
&= \Sigma_{k=0}^{p}\gamma_{p+1,2k}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) + \Sigma_{k=0}^{p-1}\gamma_{p,2k+1}a^{2k+2}b^{p-k}\phi^{(2k+2)}(\tilde{t}-2) \\
&\quad +\Sigma_{k=0}^{p-1}\gamma_{p,2k+1}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) + \Sigma_{k=0}^{p-1}\gamma_{p,2k}a^{2k}b^{p+1-k}\phi^{(2k)}(\tilde{t}-2) \\
&= \Sigma_{k=0}^{p}[\gamma_{p+1,2k} + \gamma_{p,2k+1}]a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) - \gamma_{p,2p+1}a^{2p+1}\phi^{(2k+1)}(\tilde{t}-1) \\
&\quad +\Sigma_{k=0}^{p-1}\gamma_{p,2k+1}a^{2k+2}b^{p-k}\phi^{(2k+2)}(\tilde{t}-2) + \Sigma_{k=0}^{p-1}\gamma_{p,2k}a^{2k}b^{p+1-k}\phi^{(2k)}(\tilde{t}-2).
\end{aligned} \tag{17}$$

Using $\gamma_{p,2p+1} = 0$, equations (13) and (14) and introducing $s = k+1$ leads to

$$\begin{aligned}
&x\ (t) = \\
&\Sigma_{k=0}^{p}\gamma_{p+1,2k+1}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) + \Sigma_{s=1}^{p}\gamma_{p,2s-1}a^{2s}b^{p-s+1}\phi^{(2s)}(\tilde{t}-2) \\
&\quad +\Sigma_{k=0}^{p-1}\gamma_{p,2k}a^{2k}b^{p+1-k}\phi^{(2k)}(\tilde{t}-2) \\
&= \Sigma_{k=0}^{p}\gamma_{p+1,2k+1}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) - \gamma_{p,-1}b^{p-1}\phi(\tilde{t}-2) \\
&\quad +\gamma_{p,2p}a^{2p}b\phi^{(2p)}(\tilde{t}-2) + \Sigma_{k=0}^{p}[\gamma_{p,2k-1} + \gamma_{p,2k}]a^{2k}b^{p-k+1}\phi^{(2k)}(\tilde{t}-2) \\
&= \Sigma_{k=0}^{p}\left\{\gamma_{p+1,2k+1}a^{2k+1}b^{p-k}\phi^{(2k+1)}(\tilde{t}-1) + \gamma_{p+1,2k}a^{2k}b^{p-k+1}\phi^{(2k)}(\tilde{t}-2)\right\}.
\end{aligned} \tag{18}$$

(18) is equivalent to (9) with $\ell = p+1$. The expression (10), for $t \in (2p+2, 2p+3)$, is proved similarly.

Remark 5.2 *The solution can be extended to the left in a similar manner by rewriting the equation in the form* $x(t) = ax'(t+1) + bx(t+2)$.

Theorem 5.1 *Define* $x(t)$ *as in equations (9)and (10).*
For $m = 1, 2, ..., k$, $x(m^+) = x(m^-)$ *and* $x'(m^+) = x'(m^-)$
if and only if $\phi^{(n+1)}(0) = \alpha\phi^{(n)}(1) + \beta\phi^{(n)}(-1)$ *for* $n = 0, 1, ..., k$.

Proof
We observe that $x(1^-) = \phi(1^-) = \phi(1)$, $\phi'(0^+) = \phi(0)$.
We first consider $k = 1$.
Using $x(t) = a\phi'(t-1) + b\phi(t-2)$ gives $x(1^+) = a\phi'(0^+) + b\phi(-1^+)$.
Since $x(1^+) = x(1^-)$ if and only if $\phi(1) = a\phi'(0^+) + b\phi(-1^+)$ we obtain the condition $\phi'(0) = \frac{1}{a}\phi(1) - \frac{b}{a}\phi(-1^+) = \alpha\phi(1) + \beta\phi(-1)$.
Using the derivative of (8) we obtain $x'(1^+) = a\phi''(0^+) + b\phi'(-1^+)$.
Since $x'(1^-) = \phi'(1^-)$ we see that $x'(1^+) = x'(1^-)$ if and only if $\phi'(1^-) = a\phi''(0^+) + b\phi'(-1^+)$ requiring that $\phi''(0) = \alpha\phi'(1) + \beta\phi'(-1)$. Hence the theroem is true for $k = 1$. We now consider $k = 2$.

$$\begin{aligned} x(2+) &= \Sigma_{k=0}^{1}\gamma_{2,2k}a^{2k}b^{1-k}\phi^{(2k)}(0^+) + \Sigma_{k=0}^{0}\gamma_{1,2k+1}a^{2k+1}b^{1-k}\phi^{(2k+1)}(-1^+) \\ &= b\phi(0^+) + a^2\phi''(0^+) + ab\phi'(-1^+) \end{aligned} \tag{19}$$

$$\begin{aligned} x(2^-) &= \Sigma_{k=0}^{0}\gamma_{1,2k}a^{2k}b^{1-k}\phi^{(2k)}(0^-) + \Sigma_{k=0}^{0}\gamma_{1,2k+1}a^{2k+1}b^{-k}\phi^{(2k+1)}(1^-)) \\ &= b\phi(0^-) + a\phi'(1^-) \end{aligned} \tag{20}$$

Using (19) and (20) we see that $x(2^+) = x(2^-) \Leftrightarrow \phi''(0) = \alpha\phi'(1) + \beta\phi'(-1)$.
We then use the first derivatives of equations (9) and (10) to show that

$$\begin{aligned} x'(2^+) &= \Sigma_{k=0}^{1}\delta_{1,2k}a^{2k}b^{1-k}\phi^{(2k+1)}(0^+) + \Sigma_{k=0}^{0}\delta_{1,2k+1}a^{2k+1}b^{1-k}\phi^{(2k+2)}(-1^+) \\ &= b\phi(0^+) + a^2\phi'''(0^+) + ab\phi''(-1^+) \end{aligned} \tag{21}$$

$$\begin{aligned} x'(2^-) &= \Sigma_{k=0}^{0}\left\{\gamma_{1,2k}a^{2k}b^{1-k}\phi^{(2k+1)}(0^-) + \gamma_{1,2k+1}a^{2k+1}b^{-k}\phi^{(2k+2)}(1^-)\right\} \\ &= b\phi(0^-) + a\phi''(1^-). \end{aligned} \tag{22}$$

Using (21) and (22) we can see that $x'(2^+) = x'(2^-) \Leftrightarrow \phi'''(0) = \alpha\phi''(1) + \beta\phi''(-1)$.
Hence the theorem is proved for $k = 2$.

We now assume that the Theorem 5.1 is true for $k = 2\ell - 1$. Using equation (8) we see that $x(2\ell^+) = x(2\ell^-) \Leftrightarrow ax'((2\ell-1)^+) + bx((2\ell-2)^+) = ax'((2\ell-1)^-) + bx((2\ell-2)^-)$ and observe that this is true by the inductive assumption.

$x'(2\ell^+) = x'(2\ell^-)$

$$\begin{aligned} &\Leftrightarrow \Sigma_{k=0}^{\ell}\delta_{\ell,2k}a^{2k}b^{\ell-k}\phi^{(2k+1)}(0^+) + \Sigma_{k=0}^{\ell-1}\delta_{\ell,2k+1}a^{2k+1}b^{\ell-k}\phi^{(2k+2)}(-1^+) \\ &= \Sigma_{k=0}^{\ell-1}\left\{\gamma_{\ell,2k}a^{2k}b^{\ell-k}\phi^{(2k+1)}(0^-) + \gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k-1}\phi^{(2k+2)}(1^-)\right\} \end{aligned} \tag{23}$$

$$\Leftrightarrow \Sigma_{k=0}^{\ell}\gamma_{\ell+1,2k}a^{2k}b^{\ell-k}\phi^{(2k+1)}(0^+) + \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k}\phi^{(2k+2)}(-1^+)$$
$$= \Sigma_{k=0}^{\ell-1}\left\{\gamma_{\ell,2k}a^{2k}b^{\ell-k}\phi^{(2k+1)}(0^-) + \gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k-1}\phi^{(2k+2)}(1^-)\right\} \quad (24)$$

$$\Leftrightarrow \gamma_{\ell+1,2\ell}a^{2\ell}\phi^{(2\ell+1)}(0^+) + \Sigma_{k=0}^{\ell-1}[\gamma_{\ell+1,2k} - \gamma_{\ell,2k}]a^{2k}b^{\ell-k}\phi^{(2k+1)}(0)$$
$$= \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k-1}[\phi^{(2k+2)}(1^-) - b\phi^{(2k+2)}(-1^+)] \quad (25)$$

$$\Leftrightarrow \gamma_{\ell+1,2\ell}a^{2\ell}\phi^{(2\ell+1)}(0^+) + \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k-1}a^{2k}b^{\ell-k}\left[\frac{1}{a}\phi^{(2k)}(1) - \frac{b}{a}\phi^{(2k)}(-1)\right]$$
$$= \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k+1}a^{2k+1}b^{\ell-k-1}[\phi^{(2k+2)}(1^-) - b\phi^{(2k+2)}(-1^+)] \quad (26)$$

$$\Leftrightarrow \gamma_{\ell+1,2\ell}a^{2\ell}\phi^{(2\ell+1)}(0^+) + \Sigma_{k=0}^{\ell-1}\gamma_{\ell,2k-1}a^{2k-1}b^{\ell-k}\left[\phi^{(2k)}(1) - b\phi^{(2k)}(-1)\right]$$
$$= \Sigma_{r=1}^{\ell}\gamma_{\ell,2r+1}a^{2r-1}b^{\ell-r}[\phi^{(2r)}(1^-) - b\phi^{(2r)}(-1^+)] \text{ using } r = k+1 \quad (27)$$

$$\Leftrightarrow \gamma_{\ell+1,2\ell}a^{2\ell}\phi^{(2\ell+1)}(0^+) + a^{-1}b^{\ell}\gamma_{\ell,-1}[\phi(1) - b\phi(-1)]$$
$$= \gamma_{\ell,2\ell-1}a^{2\ell-1}[\phi^{(2l)}(1^-) - b\phi^{(2\ell)}(-1^+)]. \quad (28)$$

Hence, $a\phi^{(2\ell+1)}(0^+) = \phi^{(2\ell)}(1^-) - b\phi^{(2\ell)}(-1^+)$, leading to $\phi^{(2\ell+1)}(0) = \alpha\phi^{(2\ell)}(1) + \beta\phi^{(2\ell)}(-1)$. Hence, if the theorem is true for $k = 2\ell-1$ then it is true for $k = 2\ell$. In a similar way we can prove that if the theorem is true for $k = 2\ell$ then it is true for $k = 2\ell+1$.
True for $k = 1 \Rightarrow$ true for $k = 2 \Rightarrow$ true for $k = 3$ and so on. Hence the theorem is proved.

Theorem 5.2 *The solution to equation (7) with $\phi \in \mathbb{C}^{\infty}_{[-1,1]}$ exists and is differentiable if and only if $\phi^{(n+1)}(0) = \alpha\phi^{(n)}(1) + \beta\phi^{(n)}(-1)$ for $n = 0, 1, 2, ...$.*

Proof
Since $\phi \in \mathbb{C}^{\infty}_{[-1,1]}$, for each interval $(m, m+1)$ the function $x(t)$ exists and is infinitely many times differentiable. To establish the continuity of $x(t)$ and the existence of derivatives at the end points m and $m+1$ (and hence the existence of $x(t)$ at such points) we need to prove the equalities

$$x^{(i)}(m^+) = x^{(i)}(m^-), \text{ for } i = 0, 1, 2; \qquad m = 1, 2, ... \quad (29)$$

where $x^{(i)}(m^+) := \lim_{\epsilon\to 0,\epsilon>0} x^{(i)}(m+\epsilon)$, $x^{(i)}(m^-) := \lim_{\epsilon\to 0,\epsilon>0} x^{(i)}(m-\epsilon)$. Theorem 5.1 comlpletes the proof.

Theorem 5.3 *Let $\phi \in \mathbb{C}^{\infty}_{[-1,1]}$. If a solution $x(t)$ of equation (7) exists and is differentiable then the solution is unique.*

Proof On the open intervals the solution coincides with one of equations (9) and (10), hence is uniquely defined. In the integer points the solution is obtained uniquely due to the continuity of $x(t)$.

Remark 5.3 *Discontinuities in the solution may exist at the integer points if no conditions, such as those in Theorem 5.1, are imposed on ϕ.*

6 Aims and methodology

Having established in Section 5 that solutions to equation (7) exist and are unique we now consider the case when $\phi_2(t)$ is unknown but $x(t)$ is known to equal $f(t)$ for $t \in (k-1, k]$. We investigate the potential of numerical methods to find a *correct* solution on $(0, 1]$ and then to use this solution, along with the given $\phi_1(t)$ on $[-1, 0]$, to extend the solution beyond $t = 1$.

We introduce $y_{n+N} = (\, x_{n+N}\; x_{n+N-1} \;\ldots\ldots\; x_{n+1}\; x_n \;\ldots\ldots\; x_{n-N}\,)^T$ and the matrices $M_1, M_2, M_3, M_4, B, C, D$ and F where $M_1 \in \mathbb{R}^{1\times 2N}$, $M_2 \in \mathbb{R}^{1\times 1}$, $M_3 \in \mathbb{R}^{2N\times 2N}$, $M_4 \in \mathbb{R}^{2N\times 1}$, $B \in \mathbb{R}^{N\times(N+1)}$, $C \in \mathbb{R}^{(N+1)\times 1}, D \in \mathbb{R}^{N\times N}$ and $F \in \mathbb{R}^{N\times 1}$.
Discretization of (3) using a θ-method, with $h = \frac{1}{N}$, leads to an equation of the form

$$y_{n+N} = Ay_{n+N-1} \text{ with } A = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{30}$$

where M_1 takes the form $\left(-\frac{(1-\theta)}{\theta}\; 0 \ldots 0\; \frac{(1-h\theta a)}{h\theta c}\; \frac{-[1+h(1-\theta)a]}{h\theta c}\; 0 \ldots 0\; \frac{-b}{c} \right)$,
M_2 equals $\left(\frac{-b(1-\theta)}{\theta c}\right)$, M_3 is the identity matrix and $M_4 = (0.....0)^T$.
Since $y_{n+N} = Ay_{n+N-1} = A^2 y_{n+N-2} = A^3 y_{n+N-3} = ...$, we find that $y_{n+N} = A^N y_n$.
Similarly, $y_{n+2N} = A^{2N} y_n, y_{n+3N} = A^{3N} y_n, ..., y_{n+kN} = A^{kN} y_n$.

We find that $y_{kN} = A^{(k-1)N} y_N$ which leads to an equation of the form

$$\begin{pmatrix} F \\ * \end{pmatrix} = \begin{pmatrix} D & B \\ * & * \end{pmatrix} \begin{pmatrix} X \\ C \end{pmatrix}$$

where X, $X \in \mathbb{R}^{N\times 1}$, is an unknown matrix. We do not need to concern ourselves with matrix entries marked $*$. We observe that for a particular problem F, D, B and C will be known. Hence we are able to find X, a set of solution points on $(0, 1]$. We compute X for different values of k and with different step lengths, h. We compare the computed X with the X_E, the *expected* X by calculating $\frac{1}{h}\times$ the 2-Norm of $(X - X_E)$, thus giving the mean squared error. If we then consider e, the mean squared error (mse), to be proportional to $(h^2)^p = h^{2p}$ and find the regression line of $\log e$ on $\log h$, which has gradient $2p$, we obtain an estimate of p. We find that $p = 0.9306$ in the case of the Forward Euler method and $= 1.0506$ for the Backward Euler method. However, since $h_i = 2h_{i+1}$, $p_i = \frac{1}{2}\left(\frac{\log e_{i-1} - \log e_i}{\log h_{i-1} - \log h_i}\right) = \frac{1}{2\log 2}\log\left(\frac{e_{i-1}}{e_i}\right)$ gives the gradient of the line joining (h_{i-1}, e_{i-1}) to (h_i, e_i), enabling a comparison of estimates of p as the step size decreases.

7 Numerical examples

In each example we use equations with a known exact solution and use the conditions that $x(t)$ is equal to this function on $[-1, 0]$ and $(k - 1, k]$. We discretize the equation and, following the method outlined in Section 6, obtain a solution on $(0, 1]$. We estimate the 2-norm of the error by finding $\mathcal{E} = h \times \|(X - X_E\|_2$. The values of $\mathcal{E}^2$ are tabulated for the examples that follow and enable us to estimate the order of convergence, p, as $h \to 0$ of X to X_E.

Example 7.1 *We consider* $x'(t) = \frac{1}{2}x(t+1) - \frac{1}{2}x(t-1)$ *on* $(0, 1)$ *given* $\phi_1(t) = t^2, t \in [-1, 0]$, *and* $f(t) = t^2, t \in (3, 4]$.

In Table 2 we present values of $\mathcal{E}^2$ and estimates for the order p for the forward and backward Euler methods and for the trapezium rule. We observe that $p_i \to 1$ as i increases (and h decreases) for both Euler methods. This result is consistent with these methods being $O(h)$. The trapezium rule, being $O(h^2)$, gives zero error (as expected with $f(t)$ and $\phi(t)$ as quadratic functions).

We now consider equation (31), which has as an exact solution $x(t) = e^{\alpha t}$, with $\phi_1(t)$ and $f(t)$ both equal to $e^{\alpha t}$. We present two examples , each time using the trapezium rule to discretize the equation.

$$x'(t) = (\alpha - be^{-\alpha} - ce^{\alpha})x(t) + bx(t-1) + cx(t+1) \tag{31}$$

Example 7.2 *We consider equation (31) with* $\alpha = 1$ *and with* $k = 2, 3$ *and* 4. *In Table 3 we present the values of* $\mathcal{E}^2$ *and estimates for the order* p. *Results are as expected (with an order* 2 *observed for the trapezium rule) until the method becomes unreliable when the combination of small* h *and larger* k *leads to a nearly singular matrix. The values* x *in the table correspond to those obtained where Matlab reports that the results will be unreliable.*

Remark 7.1 *We have seen that the results in our tables are limited, for small step lengths* h, *by the fact that the matrices in our computations may become*

		Mean squared error (mse) and p_i				
	Step length	Forward Euler method		Backward Euler method		Trapezium rule
i	h_i	mse	p_i	mse	p_i	mse
1	0.5	4.508574		1.909368		0
2	0.25	1.721040	0.6947	0.398952	1.1294	0
3	0.125	0.528169	0.8521	0.086295	1.1044	0
4	0.0625	0.145308	0.9309	0.019675	1.0665	0
5	0.03125	0.038016	0.9672	0.004670	1.0374	0
6	0.015625	0.009716	0.9841	0.001136	1.0197	0
7	0.0078125	0.002455	0.9923	0.000280	1.0102	0
8	0.00390625	0.000617	0.9962	0.000069	1.0104	0

Table 2
Equation: $x'(t) = \frac{1}{2}x(t+1) - \frac{1}{2}x(t-1)$ given $\phi_1(t) = t^2, t \in [-1, 0]$, and $f(t) = t^2, t \in (3, 4]$.
Mean squared error for the solution on $(0, 1]$ and estimates of p.

close to singular. This is a limitation of our existing method (which is designed to provide a prototype algorithm) and could be avoided by the use of more sophisticated techniques from numerical linear algebra for the solution of the underlying systems of equations. This is beyond the scope of the present paper.

Finally, we consider the approximation of the solution over the subsequent interval $(1, k-1]$. In Figures 5 and 6 we illustrate the trajectories of the solution on $(0, k]$ for $k = 2, 3, 4, 5$. In Figures 5 and 6 we use step lengths $h = \frac{1}{50}$ and $\frac{1}{128}$ respectively. We can see the effect of the near singular matrices reflected in the trajectories because in these cases the original solution given over $[k-1, k]$ is not recovered accurately. As we remarked earlier, this particular problem could be overcome by the use of improved solution methods for the linear systems.

Example 7.3 *We consider equation (31) for different values of α. In Table 4 we present the mean squared errors and estimates of p for three values of α. Again, we observe that the numerical method achieves its order providing that the matrix involved in the computation is not singular.*

Remark 7.2 *We have seen that the results in our tables are limited, for small step lengths h, by the fact that the matrices in our computations may become close to singular. This is a limitation of our existing method (which is designed to provide a prototype algorithm) and could be avoided by the use of more sophisticated techniques from numerical linear algebra for the solution of the underlying systems of equations. This is beyond the scope of the present paper.*

		Mean squared error (mse) and p_i to 4 significant figures					
	Step length	k=2		k=3		k=4	
i	h_i	mse	p_i	mse	p_i	mse	p_i
1	0.5	6.024×10^{-4}		2.998×10^{-3}		8.069×10^{-3}	
2	0.25	2.449×10^{-5}	2.310	1.234×10^{-4}	2.301	3.332×10^{-4}	2.299
3	0.125	1.193×10^{-6}	2.180	6.068×10^{-6}	2.173	1.644×10^{-5}	2.171
4	0.0625	6.516×10^{-8}	2.097	3.332×10^{-7}	2.093	9.047×10^{-7}	2.092
5	0.03125	3.796×10^{-9}	2.051	1.947×10^{-8}	2.049	5.293×10^{-8}	2.048
6	0.015625	2.288×10^{-10}	2.026	1.176×10^{-9}	2.025	3.200×10^{-9}	2.024
7	0.0078125	1.404×10^{-11}	2.013	7.222×10^{-11}	2.013	3.288×10^{-11}	3.302
8	0.00390625	8.7×10^{-13}	2.006	4.48×10^{-12}	2.005	6.339×10^{-8}	-5.456

Table 3
(Trapezium Rule) Equation: $x'(t) = (1-0.5e^{-1}-0.5e^{1})x(t)+0.5x(t-1)+0.5x(t+1)$ given $\phi_1(t) = e^t, t \in [-1, 0]$, and $f(t) = e^t, t \in (k-1, k]$:
Mean squared error for solution on $(0, 1]$ and estimates of p.

Finally, we consider the approximation of the solution over the subsequent interval $(1, k-1]$*. In Figures 5 and 6 we illustrate the trajectories of the solution on* $(0, k]$ *for* $k = 2, 3, 4, 5$*. In Figures 5 and 6 we use step lengths* $h = \frac{1}{50}$ *and* $\frac{1}{128}$ *respectively. We can see the effect of the near singular matrices reflected in the trajectories because in these cases the original solution given over* $[k-1, k]$ *is not recovered accurately. As we remarked earlier, this particular problem could be overcome by the use of improved solution methods for the linear systems. n Figures 5 and 6 we illustrate the trajectories of the solution on* $(0, k]$ *for* $k = 2, 3, 4, 5$.

Remark 7.3 *We have also applied our approach to equations where* $\phi_1(t)$ *and* $f(t)$ *are not exact solutions of the MFDE or when the assumptions of Theorem 5.2 are not met . In this case the method produces a solution on* $(0, 1]$ *but,as expected, it is not continuous or differentiable.*

		Mean squared error and p_i (to 4 significant figures)					
	Step length	$\alpha = -0.5$		$\alpha = 0.6$		$\alpha = 3$	
i	h_i	mse	p_i	mse	p_i	mse	p_i
1	0.5	6.226×10^{-7}		1.291×10^{-4}		4.966	
2	0.25	3.146×10^{-8}	2.153	5.627×10^{-6}	2.260	2.202×10^{-1}	2.248
3	0.125	1.772×10^{-9}	2.075	2.863×10^{-7}	2.148	1.051×10^{-2}	2.195
4	0.0625	1.052×10^{-10}	2.037	1.602×10^{-8}	2.080	5.589×10^{-4}	2.117
5	0.03125	6.410×10^{-12}	2.018	9.454×10^{-10}	2.041	3.199×10^{-5}	2.063
6	0.015625	4.000×10^{-13}	2.001	5.741×10^{-11}	2.021	1.910×10^{-6}	2.033
7	0.0078125	2.100×10^{-13}	4.648	1.080×10^{-11}	1.205	1.166×10^{-7}	2.017
8	0.00390625	2.766×10^{-10}	-5.182	5.868×10^{-9}	-4.543	6.863×10^{-9}	2.043

Table 4
(Trapezium Rule) Equation: $x'(t) = (\alpha - be^{-\alpha} - ce^{\alpha})x(t) + bx(t-1) + cx(t+1)$ given $\phi_1(t) = e^{\alpha t}, t \in [-1, 0]$, and $f(t) = e^{\alpha t}, t \in (3, 4]$.
Mean squared error for the solution on $(0, 1]$ and estimates of p.

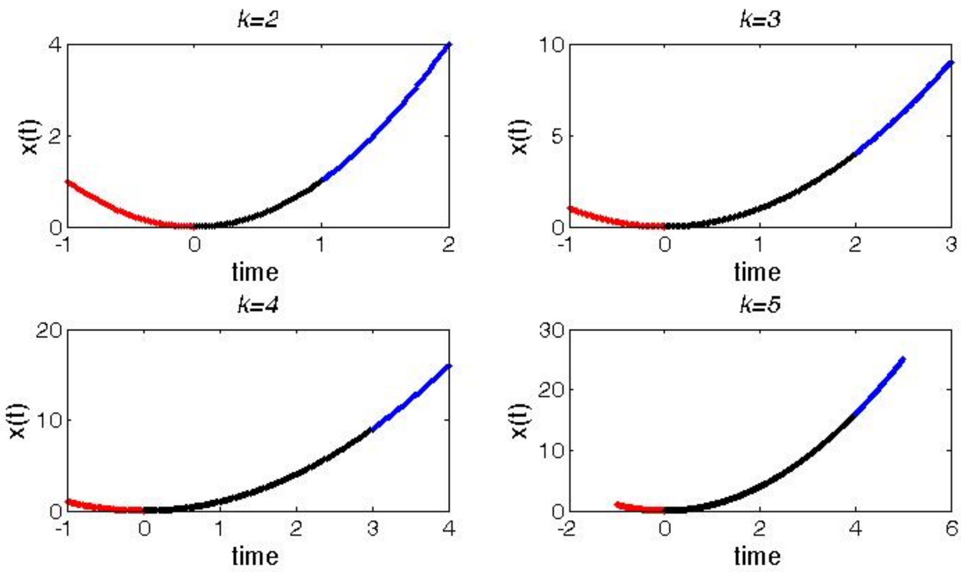

Fig. 5. See example 7.3: Trajectory of the solution found on (0, k] for k=2, 3, 4, 5; N=50

8 Conclusions and limitations of the approach

In this paper we have

(1) established the uniqueness and existence of the solutions to an autonomous

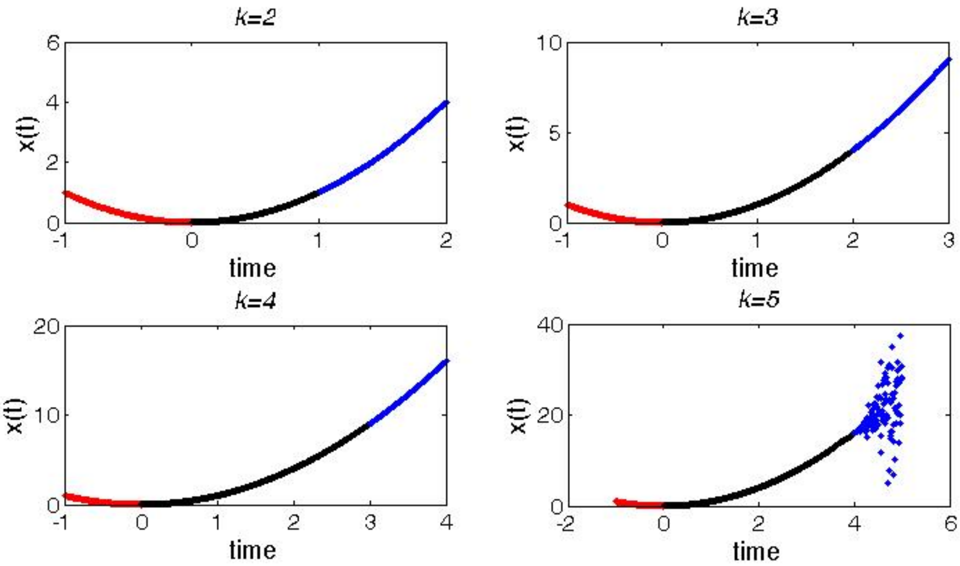

Fig. 6. See example 7.3: Trajectory of the solution found on (0, k] for k=2, 3, 4, 5; N=128

MFDE under a stated assumption.

(2) developed amd demonstrated a numerical approach to solving an autonomous MFDE given the solution on $[-1, 0]$ and $(k-1, k]$, $k \in \mathbb{N}$.

(3) demonstrated that our approach produces numerical results that are consistent with the order of the methods being used.

(4) indicated a problem with our approach, relating to the near singularity of a matrix, when the dimension increases.

The results of the paper are very satisfactory in providing a reliable method for finding solutions to MFDEs where the boundary conditions are given. However, we need to return to the question of whether or not a solution to such a problem exists! As we remarked at the beginning of the paper, the boundary value problem of MFDEs is not well-understood, but it is known that such a problem often has no solution. As we saw when we considered the initial value problem, there are also quite stringent conditions required on the initial function in order to ensure that the solution has certain properties, such as being continuously differentiable, for example.

Given *any set of boundary conditions and any MFDE of the form we considered here*, the proposed numerical method will provide a solution. If the original problem had a continuously differentiable solution, then the solution found by the method in this paper is likely (assuming any problems with near singular matrices are circumvented) to be a good approximation to the true solution. However, if the original problem does not have a solution, then the numerical scheme will nevertheless provide a solution, but this time to a perturbation of the original problem! Caveat emptor!

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