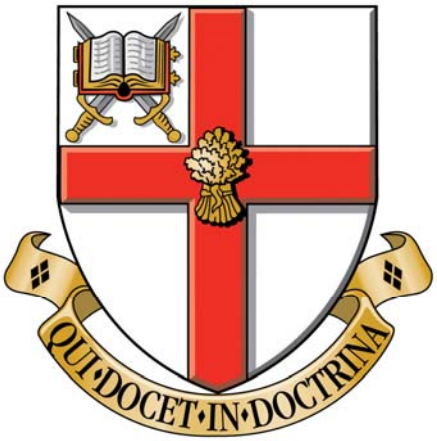

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AN INVERSE PROBLEM FOR DELAY DIFFERENTIAL EQUATIONS – ANALYSIS VIA INTEGRAL EQUATIONS

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An Inverse Problem for Delay Differential Equations – Analysis via Integral Equations

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Abstract

We address the problem of determining the initial function $\varphi(t)$ (for $t \in [-\tau, 0]$) given the solution $y(t) \equiv y(\varphi; t)$ of the linear delay differential equation

$$y'(t) - A(t)y(t) - B(t)y(t-\tau) = f(t) \quad (t \in [0, T]),$$

for which

$$y(t) = \varphi(t) \quad (t \in [-\tau, 0]).$$

The function $\varphi(t)$ is approximated by the function $\varphi_\star(t)$ that minimizes a certain parameter-dependent quadratic functional. The optimal function $\varphi_\star(t)$ is shown to satisfy a Fredholm integral equation, and the rôle of a regularization parameter is transparent from the form of this equation. (There is a related integral equation for $\varphi(t)$.) The convergence properties of an iterative method for finding $\varphi_\star(t)$, using an iteration that is based upon the delay equation for $y(t)$ and a corresponding adjoint equation, are established by considering an iteration for the solution of the Fredholm integral equation.

Keywords: Delay differential equations, initial function, adjoint equations, identification problem, data assimilation, fundamental matrices, regularization parameter.

1 The nature of the problem.

Studies have been undertaken, in the context of the mathematical modelling of biological data (see Remark 1.3), of the problem of determining a parametrized

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retarded differential equation, along with the corresponding initial function, such that the solution is a good fit to an observed function. This type of problem has been addressed by others (*e.g.* [7, 20]). Our practical approach to answering the question relies upon the numerical solution of differential equations with deviating arguments, and the minimization of an objective function appearing in this paper; we discuss numerical experiments elsewhere. However, some interesting analysis involving integral equations arises in arriving at a theory for our technique; part of this material is presented below.

Here, we describe a method for determining an initial function, given the solution to a linear delay differential equation that it generates. This is shown – using the adjoint equations with deviating arguments – to be equivalent to the solution of a Fredholm integral equation (see §1.3). The integral equation can be of a type which is generally recognized as ill-posed, and the effect of regularization parameters is established. Our results here extend those for "data assimilation problems" in ordinary and partial differential equations (*cf.* [16] and related works). Extensions to nonlinear delay differential equations (where the initial function need not be unique) are discussed by the authors elsewhere.

Since inverse problems are frequently ill-posed, the connection with integral equations of the first kind (and their regularization to integral equations of the second kind) is, though interesting, unsurprising. Moreover, the rôle of adjoint equations in variation-of-constants formulae can be found in the literature. Nevertheless, our results appear to be new and the theory provides insight into a practical method, insight which does not appear to be readily available by a different approach.

1.1 Analysis.

Consider an n-dimensional system of linear delay differential equations (DDEs) with *time-dependent* coefficients, of the form

$$\frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t-\tau) = f(t), \quad \text{for } t \in [0,T], \tag{1a}$$

subject to

$$y(t) = \varphi(t) \quad \text{for } t \in [-\tau, 0]. \tag{1b}$$

Here

$$y(t),\ f(t),\ \varphi(t)\ \in \mathbb{R}^{n\times 1}, \quad A(t),\ B(t)\ \in \mathbb{R}^{n\times n},$$

τ is a prescribed positive constant (the "lag"); $\varphi(t)$ is the "initial function", $f(t)$ the "inhomogeneous term". We find it convenient to suppose that $T \geq \tau$. The solution of (1) is dependent on the function φ in (1b), and when $y(t) = \varphi(t)$ (for $t \in [-\tau, 0]$) we can therefore write

$$y(t) \equiv y(\varphi; t) \quad (\text{for } t \in [-\tau, T]). \tag{2}$$

Remark 1.1 *If $B(t)$ does not vanish for $t \in [0,T]$ then the functions $y(\varphi_1;t)$ and $y(\varphi_2;t)$ differ when the functions φ_1 and φ_2 differ, and φ is uniquely determined if $y(\varphi;t)$ is given on $[0,\tau]$.*

The problem addressed here involves the determination (given prescribed $\tau > 0$, $A(t)$, $B(t)$, $f(t)$) of an initial function $\varphi(t)$, chosen from some class $\mathcal{F}$ of functions defined on $[-\tau,0]$, such that the solution $y(\varphi;t)$ of the given retarded equation is a "best" approximation to an observed function. Throughout, we suppose $\mathcal{F} \subseteq PC[-\tau,0])$, so that the initial function is required to be piecewise continuous with finite jumps at points of discontinuity. We emphasize the case where

$$\varphi \in \mathcal{F}, \qquad \text{where } \mathcal{F} := \big\{\varphi \in C[-\tau,0) \ \text{ with bounded } \varphi(0)\big\}; \tag{3a}$$

$$A,\ B,\ f \in C[0,T]. \tag{3b}$$

We determine an estimation of the initial function $\varphi(t)$ by minimization of a quadratic functional (given below) involving the deviation of $y(\varphi;t)$ from the observed function.

1.2 Details of the identification problem.

Suppose $\alpha, \beta, \gamma \geq 0$ and $y(\varphi;0) = \varphi(0))$ and $\widehat{\varphi} = \widehat{\varphi}(t)$ and $\widehat{y} = \widehat{y}(t)$ are given functions. We introduce the functional

$$S_\alpha^{\beta,\gamma}(\varphi) := \tfrac{\alpha}{2} \int_{-\tau}^{0} \big\|\varphi(t) - \widehat{\varphi}(t)\big\|_2^2 dt + \tfrac{\beta}{2}\big\|\varphi(0) - \widehat{\varphi}(0)\big\|_2^2 + \tag{4}$$

$$\tfrac{\gamma}{2}\big\|y(\varphi;0) - \widehat{y}(0)\big\|_2^2 + \tfrac{1}{2}\int_0^T \big\|y(\varphi;t) - \widehat{y}(t)\big\|_2^2 dt$$

where $y(\varphi;t)$ satisfies (1). We intend the value γ (introduced to provide some flexibility in the formulation) to assume the value 0 or 1.

We can now formulate the *identification problem* (or *data assimilation problem*) as follows:

Definition 1.1 *Let $\mathcal{F} \subseteq PC[-\tau,0]$ be a smoothness class of bounded functions on $[-\tau,0]$. Then the corresponding data assimilation problem for the identification of φ is formulated as follows.* Find a function $\varphi_\star \in \mathcal{F}$, such that $y(\varphi_\star;t)$ satisfies (1) and

$$S_\alpha^{\beta,\gamma}(\varphi_\star) = \min_{\varphi\in\mathcal{F}} S_\alpha^{\beta,\gamma}(\varphi), \tag{5}$$

where $S_\alpha^{\beta,\gamma}(\varphi)$ is defined by (4), in which $y(\varphi;t)$ satisfies (1).

This formulation embodies parameters $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, which (when positive) are "regularization parameters" (see [9], for example). This applies,

in particular, to α. Thus, if we introduce an abstract operator $\mathcal{A}$ such that $y(\varphi;t) = \mathcal{A}\varphi(t)$ we have

$$S_\alpha^{0,0}(\varphi) = \frac{\alpha}{2}\int_{-\tau}^{0} \left\|\varphi(t) - \widehat{\varphi}(t)\right\|_2^2 dt + \frac{1}{2}\int_0^T \left\|\mathcal{A}\varphi - \widehat{y}(t)\right\|_2^2 dt,$$

(which is of the form associated with Tikhonov regularization [19] for recovery of $\varphi(t)$ from $\mathcal{A}\varphi(t) = \widehat{y}(t)$). Clearly, $\varphi_\star$ depends on these parameters α, β, γ.

We consider an idealized situation where the functions $\widehat{y}(t)$ and $\widehat{\varphi}(t)$ are supposed to be unambiguously defined, but in practice $\widehat{y}(t)$ is usually defined by *a priori* observational data which is subject to noise. The choice of $\widehat{\varphi}(t)$ is determined by modelling considerations.

1.3 Our results in brief.

We show below that the optimal initial function $\varphi_\star$ satisfies a coupled set of delay equations (see (10)), involving "adjoint equations", and we give an iterative technique for obtaining successive approximations φ_n to $\varphi_\star$. We show that the function $\varphi_\star$ identified by our chosen formulation is associated with the solution of a Fredholm integral equation, and the iteration we propose is related to an iterative solution of the integral equation. Our discussion establishes a connection with a regularization method due to Lavrent'ev [12].

Theorem 1.1 *For appropriate $\mathfrak{g}_\alpha^{\beta,\gamma}(t)$, $K_{\beta,\gamma}(t,s)$ and $\gamma_0(\beta,\gamma)$, the function $\varphi_\star(t)$ solving the data assimilation problem satisfies equations of the form*

$$\alpha\varphi_\star(t) + \int_{-\tau}^{0} K_{\beta,\gamma}(t,s)\varphi_\star(s)ds = \mathfrak{g}_\alpha^{\beta,\gamma}(t), \quad \textit{for } t \in [-\tau, 0), \tag{6}$$

$$\varphi_\star(0) = \gamma_0(\beta,\gamma).$$

In (6), the kernel $K_{\beta,\gamma}(t,s)$ is self-adjoint ($K_{\beta,\gamma}(t,s) = K_{\beta,\gamma}^{\mathsf{T}}(s,t)$) and positive-definite.

Remark 1.2 *If $\alpha > 0$, then equation (6) is a Fredholm equation of the second kind, and if $\alpha = 0$ it is a Fredholm equation of the first kind. The positive-definiteness of the kernel implies that the equation of the second kind ($\alpha > 0$) is uniquely solvable. However, Fredholm equations of the first kind (the case $\alpha = 0$) with well-behaved kernels are ill-posed. For this reason, the introduction of $\alpha > 0$ is said to regularize the problem.*

The above theorem is derived through the use of an adjoint differential equation with a deviating argument. (The link with a variation of parameters formula is exploited.) We shall obtain properties of the kernel $K_{\beta,\gamma}(t,\sigma)$ using the fundamental matrix solution for a delay differential equation. The iteration that yields successive φ_n is related to the integral equation in (6) and we investigate the convergence of that iteration by using properties of the kernel.

Remark 1.3 *Note that we concentrate on the identification of the initial function $\varphi(t)$, it being assumed that the DDE is known. For discussions of parameter estimation for DDEs, in particular in the context of cell dynamics, see Baker, Bocharov & Paul [3], Baker, Bocharov, Paul, & Rihan [4], and [5], and the references therein. Ordinary differential equations (ODEs) and parabolic partial differential equations (PPDEs) also arise in modeling of cell populations. Similar (so-called — see [1, 2, 15, 16, 17]) data assimilation problems have been discussed for ODEs, see [15], and PPDEs, see [1, 2, 16]; the present work fills a gap in the discussion of DDEs.*

2 The optimization problem.

In order to consider the minimum of the functional (4), we analyze $S_\alpha^{\beta,\gamma}(\varphi+\varepsilon\psi)$. If $\varphi_\star \in \mathcal{F}$ provides a minimum of the functional $S_\alpha^{\beta,\gamma}(\varphi)$, we have $S_\alpha^{\beta,\gamma}(\varphi_\star) \leq S_\alpha^{\beta,\gamma}(\varphi_\star+\varepsilon\psi)$, where ε is a real parameter and ψ is an arbitrary function in the linear space $\mathcal{F}$. We note $S_\alpha^{\beta,\gamma}(\varphi) \geq 0$, and $S_\alpha^{\beta,\gamma}(\varphi+\varepsilon\psi)$ is a quadratic in ε. To write down $S_\alpha^{\beta,\gamma}(\varphi_\star+\varepsilon\psi)$ we need an expression for $y(\varphi_\star+\varepsilon\psi;t)$, and we have it in the following result.

Let us write $Ly(t) := y'(t) - A(t)y(t) - B(t)y(t-\tau)$ (for $t \in [0,T]$), and $My(t) = y(t)$ (for $t \in [-\tau,0]$). By virtue of the linearity of L and M,

$$y(\varphi_\star+\varepsilon\psi;t) = y(\varphi_\star;t) + \varepsilon z(\psi;t) \tag{7}$$

where $z(t) \equiv z(\psi;t)$ satisfies $Lz(t) = 0 \quad (\text{for } t \in [0,T])$ and $Mz(t) = \psi(t) \quad (\text{for } t \in [-\tau,0])$, that is,

$$\frac{dz(\psi;t)}{dt} - A(t)z(\psi;t) - B(t)z(\psi;t-\tau) = \quad 0, \quad \text{for } t \in [0,T], \tag{8a}$$

$$z(\psi;t) = \psi(t), \quad \text{for } t \in [-\tau,0), \text{ and } z(\psi;0) = \psi(0). \tag{8b}$$

The condition $z(\psi;t) = \psi(t)$, for $t \in [-\tau,0]$ is expressed as $z(\psi;t) = \psi(t)$, for $t \in [-\tau,0)$, and $z(\psi;0) = \psi(0)$ in order to *emphasize* the possibility that $\psi(0-)$ may not equal $\psi(0)$. The function $z(\psi;t)$ vanishes if and only if $\psi(t)$ vanishes.

We can write

$$S_\alpha^{\beta,\gamma}(\varphi+\varepsilon\psi) = S_\alpha^{\beta,\gamma}(\varphi) + \varepsilon P_\alpha^{\beta,\gamma}(\varphi,\psi) + \varepsilon^2 Q_\alpha^{\beta,\gamma}(\psi), \tag{9a}$$

where

$$P_\alpha^{\beta,\gamma}(\varphi,\psi) = \alpha \int_{-\tau}^{0} \{\varphi(t) - \widehat{\varphi}(t)\}^{\mathsf{T}} \psi(t)dt + \tag{9b}$$

$$\int_0^T \{y(\varphi;t) - \widehat{y}(t)\}^{\mathsf{T}} z(\psi;t)dt + \beta\{\varphi(0) - \widehat{\varphi}(0)\}^{\mathsf{T}} \psi(0) + \gamma\{y(\varphi;0) - \widehat{y}(0)\}^{\mathsf{T}} z(\psi;0);$$

$$Q_\alpha^{\beta,\gamma}(\psi) = \frac{\alpha}{2}\int_{-\tau}^{0} \|\psi(t)\|_2^2 dt + \frac{1}{2}\int_0^T \|z(\psi;t)\|_2^2 dt + \frac{\beta}{2}\|\psi(0)\|_2^2 + \frac{\gamma}{2}\|z(\psi;0)\|_2^2.$$

We observe that in the latter expressions we have $z(\psi;0) = \psi(0)$. We obtain the following result.

Lemma 2.1 *If* $J_\alpha^{\beta,\gamma}(\varphi,\psi) \equiv \frac{\alpha}{2}\int_{-\tau}^{0}\{\varphi(t)-\widehat{\varphi}(t)\}^{\mathsf{T}}\{\psi(t)-\widehat{\varphi}(t)\}dt +$
$$\frac{1}{2}\int_0^T \{y(\varphi;t)-\widehat{y}(t)\}^{\mathsf{T}}\{y(\psi;t)-\widehat{y}(t)\}dt + \frac{\beta}{2}\{\varphi(0)-\widehat{\varphi}(0)\}^{\mathsf{T}}\{\psi(0)-\widehat{\varphi}(0)\} +$$
$$\frac{\gamma}{2}\{y(\varphi;0)-\widehat{y}(0)\}^{\mathsf{T}}\{y(\psi;0)-\widehat{y}(0)\};$$
then $S_\alpha^{\beta,\gamma}(\varphi) = J_\alpha^{\beta,\gamma}(\varphi,\varphi)$, *and* $P_\alpha^{\beta,\gamma}(\varphi,\psi) = J_\alpha^{\beta,\gamma}(\varphi,\psi) + J_\alpha^{\beta,\gamma}(\psi,\varphi) = 2J_\alpha^{\beta,\gamma}(\varphi,\psi)$. *Finally,* $Q_\alpha^{\beta,\gamma}(\psi) = \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\{J_\alpha^{\beta,\gamma}(\varphi+\varepsilon\psi,\varphi+\varepsilon\psi)\}$, *and* $Q_\alpha^{\beta,\gamma}(\psi) \geq 0$ *and* $Q_\alpha^{\beta,\gamma}(\psi) = 0$ *if and only if* $\psi = 0$ *on* $[-\tau,0]$.

We thus have the following result.

Theorem 2.1 *A function* $\varphi_\star(t)$ *defined on* $[-\tau,0]$ *minimizes* $S_\alpha^{\beta,\gamma}(\varphi)$ *in (4) for* $\varphi \in \mathcal{F}$ *if and only if* $P_\alpha^{\beta,\gamma}(\varphi_\star,\psi)$ *in (9b) vanishes for all* $\psi \in \mathcal{F}$, *where* $z = z(\psi;t)$ *satisfies (7).*

We also note the following result (the proof of which is provided in Appendix A).

Lemma 2.2 *The bilinear form* $\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\psi)$ *is symmetric and positive semidefinite (definite when* $\alpha > 0$*) on* $\mathcal{F}$.

2.1 A method for finding the optimal $\varphi_\star$.

We propose a method for finding the initial function $\varphi_\star \in \mathcal{F}$ which minimizes $S_\alpha^{\beta,\gamma}(\varphi)$ on $\mathcal{F}$. This method comprises the solution of a set of coupled equations, written in the form

$$\frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t-\tau) = f(t), \quad \text{for } t \in [0,T], \tag{10a}$$

$$y(t) = \varphi_\star(t) \quad \text{for } t \in [-\tau,0), \quad y(0) = \varphi_\star(0), \tag{10b}$$

$$-\frac{dx^{\mathsf{T}}(t)}{dt} - x^{\mathsf{T}}(t)A(t) - x^{\mathsf{T}}(t+\tau)B(t+\tau) = [y(\varphi_\star;t) - \widehat{y}(t)]^{\mathsf{T}}, \tag{10c}$$

for $t \in [0,T]$,

$$x^{\mathsf{T}}(t) = 0, \quad \text{for } t \in [T, T+\tau], \tag{10d}$$

$$\alpha(\varphi_\star(t) - \widehat{\varphi}(t)) + [B(t+\tau)]^{\mathsf{T}} x(t+\tau) = 0, \ \text{ for } t \in [-\tau,0), \tag{10e}$$

$$x(0) + \beta\{\varphi_\star(0) - \widehat{\varphi}(0)\} + \gamma\{\varphi_\star(0) - \widehat{y}(0)\} = 0. \tag{10f}$$

Remark 2.1 *If equation (10e) were to hold for $t \in [-\tau, 0]$ then the equation for $t = 0$ would have the form $\alpha\varphi_\star(0) + [B(\tau)]^{\mathsf{T}} x(\tau) = \alpha\widehat{\varphi}(0)$. Equation (10f) defines $\varphi_\star(0)$ if $\beta + \gamma \neq 0$.*

In the process of determining $\varphi_\star(t)$ we also determine the corresponding $y(\varphi_\star; t)$ and $x^{\mathsf{T}}(\varphi_\star; t)$. We shall show that the solution of (10) does provide a minimum.

The method proposed for the solution of (10) is based upon the following iteration:

Definition 2.1 (An iteration for finding the function $\varphi_\star(t)$)

$$\frac{dy_n(t)}{dt} - A(t)y_n(t) - B(t)y_n(t-\tau) = f(t), \quad \textit{for } t \in [0, T], \tag{11a}$$

$$y_n(t) = \varphi_n(t) \quad \textit{for } t \in [-\tau, 0), \quad y_n(0) = \varphi_n(0), \tag{11b}$$

$$-\frac{dx_n^{\mathsf{T}}(t)}{dt} - x_n^{\mathsf{T}}(t)A(t) - x_n^{\mathsf{T}}(t+\tau)B(t+\tau) = [y_n(\varphi_n; t) - \widehat{\mathsf{y}}(t)]^{\mathsf{T}}, \tag{11c}$$

for $t \in [0, T]$,

$$x_n^{\mathsf{T}}(t) = 0, \quad \textit{for } t \in [T, T+\tau], \tag{11d}$$

$$\varphi_{n+1}(t) = \varphi_n(t) + \delta_n\big(\alpha(\varphi_n(t) - \widehat{\varphi}(t)) + [B(t+\tau)]^{\mathsf{T}} x_n(t+\tau)\big), \tag{11e}$$

for $t \in [-\tau, 0)$ and

$$\varphi_{n+1}(0) = \varphi_n(0) + \delta'_n\big\{(\beta+\gamma)\varphi_n(0) + x_n(0) - \beta\widehat{\varphi}(0) - \gamma\widehat{\mathsf{y}}(0)\big\}, \tag{11f}$$

for $n = 0, 1, 2, \ldots$ and $\{\delta_j\}$, $\{\delta'_j\}$ are appropriately chosen scalars.

What is proposed is to find the solution of (10) using the following procedure:

- *Choose a starting approximation for the initial function ($\varphi_0(s)$, $s \in [-\tau, 0]$).*
- *For $n = 0, 2, \ldots, N$, where the choice of N yields appropriate accuracy:*
  1. *Obtain the solution $y_n = y_n(t)$ of the original problem (11a) for $\varphi = \varphi_n(s)$.*
  2. *Obtain the solution $x_n^{\mathsf{T}} = x_n^{\mathsf{T}}(t)$ of the adjoint problem (11c) with known right-hand side $y_n(t) - \widehat{\mathsf{y}}(t)$ when $t \in [0, T]$ with the condition $x_n^{\mathsf{T}}(s) = 0$, $s \in [T, T+\tau]$.*
  3. *Find the next iterate $\varphi_{n+1}(s)$ using equations (11e) and (11f).*

We shall establish that if we allow N to tend to infinity, this iteration converges to give $\varphi_\star$, for a feasible range of values of α and δ. Our convergence result (see Theorem 4.2) follows very simply from the fact that the solution of (10) satisfies an integral equation of the form

$$\alpha\varphi_\star(t) + \int_{-\tau}^{0} K_{\beta,\gamma}(t, s)\varphi_\star(s)ds = \mathfrak{g}_\alpha^{\beta,\gamma}(t), \tag{12}$$

for $t \in [-\tau, 0)$, in which $K_{\beta,\gamma}(t, s)$ and $\mathfrak{g}_\alpha^{\beta,\gamma}(t)$ are given in detail in (23). The iteration above is related to an iterative method for solving the integral equation (see (36), below).

3 The underlying theory.

The theoretical discussion of our method depends upon aspects of the theory of DDEs and of integral equations.

3.1 A rôle for the adjoint equation.

We shall obtain an equivalent formulation of the problem (5), based upon adjoint equations. This provides an alternative characterization to that of Theorem 2.1. The purpose of this approach is to derive (10) and (11) in order to solve the "data assimilation problem" (5). The results presented here arise, in effect, due to the relation between the fundamental solution, certain adjoint problems, and variation of parameters formulae [6].

Lemma 3.1 *Let $y = y(\varphi; t)$ be a solution of the problem* (1) *and let $z = z(\psi; t)$ be a solution of the homogeneous problem* (8)*. Then the first variation of the functional $S_\alpha^{\beta,\gamma}(\varphi)$ can be represented in the form* $P_\alpha^{\beta,\gamma}(\varphi,\psi) = \int_{-\tau}^{0} \Big\{\alpha[\varphi(t) - \widehat{\varphi}(t)] + x^{\mathsf{T}}(t+\tau)B(t+\tau)\Big\}^{\mathsf{T}} \psi(t)dt +$

$$\Big(x^{\mathsf{T}}(0) + \beta[\varphi(0) - \widehat{\varphi}(0)]^{\mathsf{T}} + \gamma[y(\varphi;0) - \widehat{\mathsf{y}}(0)]^{\mathsf{T}}\Big)\psi(0), \tag{13}$$

where $x^{\mathsf{T}}(t) \in \mathbb{R}^{1\times n}$ is the solution ($x^{\mathsf{T}}(t) \equiv x^{\mathsf{T}}(\varphi; t)$) of the problem

$$-\frac{dx^{\mathsf{T}}(t)}{dt} - x^{\mathsf{T}}(t)A(t) - x^{\mathsf{T}}(t+\tau)B(t+\tau) = [y(\varphi;t) - \widehat{\mathsf{y}}(t)]^{\mathsf{T}}, \tag{14a}$$

for $t \in [0, T]$,

$$x^{\mathsf{T}}(t) = 0, \quad \text{for } t \in [T, T+\tau]. \tag{14b}$$

Equations (14) correspond to an adjoint problem, with a special forcing term $y(\varphi; t) - \widehat{\mathsf{y}}(t)$.

Remark 3.1 *If required, the derivative in (14a) is interpreted as the left-hand derivative. The derivative of the function $x^{\mathsf{T}}(t)$ satisfying (14) inherits from $y(\varphi; t) - \widehat{\mathsf{y}}(t)$ any jump discontinuities at points in $[0, T]$; if $y(\varphi; t) - \widehat{\mathsf{y}}(t)$ is continuous (in particular if $y(t)$ satisfies (1) where $f \in C[0, T]$ and $\varphi \in C[-\tau, 0]$) then $\frac{d}{dt}x^{\mathsf{T}}(t)$ is continuous on $[0, T]$.*

Proof. We have

$$P_\alpha^{\beta,\gamma}(\varphi,\psi) = \alpha \int_{-\tau}^{0} [\varphi(t) - \widehat{\varphi}(t)]^{\mathsf{T}} \psi(t)dt +$$

$$\beta[\varphi(0)-\widehat{\varphi}(0)]^{\mathsf{T}}\psi(0)+\gamma[y(\varphi;0)-\widehat{y}(0)]^{\mathsf{T}}z(\psi;0)$$
$$\underbrace{-\int_0^T \frac{dx^{\mathsf{T}}(t)}{dt}z(\psi;t)dt}_{\text{term }(i)}\underbrace{-\int_0^T x^{\mathsf{T}}(t)A(t)z(\psi;t)dt}_{\text{term }(ii)} \tag{15}$$
$$\underbrace{-\int_0^T x^{\mathsf{T}}(t+\tau)B(t+\tau)z(\psi;t)dt\,.}_{\text{term }(iii)}$$

Using integration by parts, for (i) in (15) and writing (iii) as

$$\int_\tau^{T+\tau} x^{\mathsf{T}}(s)B(s)z(\psi;s-\tau)ds=\int_0^T x^{\mathsf{T}}(s)B(s)z(\psi;s-\tau)ds$$
$$-\int_{-\tau}^0 x^{\mathsf{T}}(s+\tau)B(s+\tau)\psi(s)ds,$$

we obtain $P_\alpha^{\beta,\gamma}(\varphi,\psi)=\alpha\int_{-\tau}^0[\varphi(t)-\widehat{\varphi}(t)]^{\mathsf{T}}\psi(t)dt+x^{\mathsf{T}}(0)z(\psi;0)+\beta[\varphi(0)-\widehat{\varphi}(0)]^{\mathsf{T}}\psi(0)+$
$\gamma[y(\varphi;0)-\widehat{y}(0)]^{\mathsf{T}}z(\psi;0)+\int_{-\tau}^0 x^{\mathsf{T}}(t+\tau)B(t+\tau)\psi(t)dt+$
$\int_0^T x^{\mathsf{T}}(t)\left(\frac{dz(\psi;t)}{dt}-A(t)z(\psi;t)-B(t)z(\psi,t-\tau)\right)dt$. Then $z(\psi;t)$ satisfies the homogeneous equation (8). For $P_\alpha^{\beta,\gamma}(\varphi,\psi)$ we thus obtain the expression in (13) and our lemma follows.

¿From (9a), since $y(\psi;0)=\varphi(0)$, we have the following theorem.

Theorem 3.1 *A function $\varphi_\star(t)$ defined on $[-\tau,0]$ minimizes $S_\alpha^{\beta,\gamma}(\varphi)$ for $\varphi\in\mathcal{F}$ where $\mathcal{F}=C[-\tau,0)\cap\{\varphi\big|\varphi(0)$ is bounded$\}$ if, where x^{T} satisfies (7),*

$$\int_{-\tau}^0\{\alpha[\varphi(t)-\widehat{\varphi}(t)]+x^{\mathsf{T}}(t+\tau)B(t+\tau)\}^{\mathsf{T}}\psi(t)dt=0 \tag{16a}$$

for all $\psi\in C[-\tau,0)$, and

$$\{x(0)+\beta[\varphi(0)-\widehat{\varphi}(0)]+\gamma[\varphi(0)-\widehat{y}(0)]\}^{\mathsf{T}}\psi(0)=0. \tag{16b}$$

If we consider the problem of minimizing over a subset $\mathcal{F}_n$ of $\mathcal{F}$, we obtain analogous results with $\mathcal{F}$ replaced in $\mathcal{F}_n$.

3.2 A rôle for fundamental solutions.

We introduce (see [10, pp. 359-363], [11, pp. 148-150]) the fundamental solution for a linear delay differential equation of the form

$$\frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t-\tau) = f(t), \quad \text{for } t \in [0,T], \tag{17a}$$

with initial condition

$$y(t) = \varphi(t) \quad \text{for } t \in [-\tau, 0]. \tag{17b}$$

Lemma 3.2 *Let $Y(s,t)$ be a solution of the equation*

$$\frac{\partial Y(s,t)}{\partial s} + Y(s,t)A(s) + Y(s+\tau,t)B(s+\tau) = 0, \quad \textit{for } s < t, \tag{18a}$$

which satisfies

$$Y(s,t) = 0, \ \textit{for } \ t < s, \quad Y(t,t) = I \tag{18b}$$

Then the solution of the system (17) is given by

$$y(t) = Y(0,t)\varphi(0) + \int_{-\tau}^{0} Y(s+\tau,t)B(s+\tau)\varphi(s)ds + \int_{0}^{t} Y(s,t)f(s)ds. \tag{19}$$

For related results, see [6]. One may see that the essential structure in (19) is that

$$y(\varphi;t) \equiv y(t) = Y(0,t)\varphi(0) + \mathrm{F}\varphi(t) + \mathrm{V}f(t) \tag{20a}$$

where F is a Fredholm integral operator defined on $\mathcal{F}$ and V is a Volterra integral operator defined on $C[0,T]$. Hence, also, the solution of the homogeneous problem with initial function ψ is

$$z(\psi;t) \equiv z(t) = Y(0,t)\psi(0) + \mathrm{F}\psi(\mathrm{t}). \tag{20b}$$

Lemma 3.3 *The solution $x^{\mathsf{T}}(t)$ of the adjoint problem*

$$-\frac{dx^{\mathsf{T}}(t)}{dt} - x^{\mathsf{T}}(t)A(t) - x^{\mathsf{T}}(t+\tau)B(t+\tau) = g^{\mathsf{T}}(t), \ \textit{for } t \in [0,T], \tag{21a}$$

$$x^{\mathsf{T}}(t) = v^{\mathsf{T}}(t), \quad \textit{for } t \in [T, T+\tau]. \tag{21b}$$

is expressible as

$$x^{\mathsf{T}}(t) = v^{\mathsf{T}}(T)Y(t,T) + \int_{T}^{T+\tau} v^{\mathsf{T}}(s)B(s)Y(t,s-\tau)ds + \int_{t}^{T} g^{\mathsf{T}}(s)Y(t,s)ds,$$

It follows that the solution $x^{\mathsf{T}}(t)$ of (14) is given by

$$x^{\mathsf{T}}(t) = \int_{t}^{T} \left[y(\varphi;s) - \widehat{\mathrm{y}}(s)\right]^{\mathsf{T}} Y(t,s)ds, \tag{22}$$

4 An integral equation for the optimal initial function $\varphi_\star$.

In the next pages, we shall establish the following theorem.

Theorem 4.1 *For $t \in [-\tau, 0)$,*

$$\alpha\varphi_\star(t) + \int_{-\tau}^{0} K_{\beta,\gamma}(t,s)\varphi_\star(s)ds = \mathfrak{g}_\alpha^{\beta,\gamma}(t), \tag{23}$$

in which

$$K_{\beta,\gamma}(t,s) = \int\limits_{\xi=t+\tau}^{T} \Phi^{\mathsf{T}}(t+\tau,\xi)\Phi(s+\tau,\xi)d\xi + \tag{24a}$$

$$- \int\limits_{\xi=t+\tau}^{T}\int\limits_{\mu=s+\tau}^{T} \Phi^{\mathsf{T}}(t+\tau,\xi)Y(0,\xi)\mathcal{D}_{\beta,\gamma}^{-1}[Y(0,\mu)]^{\mathsf{T}}\Phi(s+\tau,\mu)d\mu d\xi$$

and

$$\mathfrak{g}_\alpha^{\beta,\gamma}(t) = \alpha\widehat{\varphi}(t) - \int\limits_{t+\tau}^{T} \Phi^{\mathsf{T}}(t+\tau,s)Y(0,s)\mathcal{D}_{\beta,\gamma}^{-1}F_0(\widehat{\varphi}(0),\widehat{\mathrm{y}},f)ds + \tag{24b}$$

$$- \int\limits_{t+\tau}^{T} \Phi^{\mathsf{T}}(t+\tau,s)\left(\int\limits_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathrm{y}}(s)\right)ds$$

where

$$\mathcal{D}_{\beta,\gamma} := (\beta+\gamma)I + \int\limits_0^T [Y(0,s)]^{\mathsf{T}}Y(0,s)ds \tag{24c}$$

is a symmetric and positive-definite constant matrix. Here $\Phi(t+\tau,s) = Y(t+\tau,s)B(t+\tau)$. Additionally, $\varphi_\star(0)$ satisfies a relation

$$\mathcal{D}_{\beta,\gamma}\varphi_\star(0) = \beta\widehat{\varphi}(0) + \gamma\widehat{\mathrm{y}}(0) + \int\limits_{-\tau}^{0}\int\limits_{s+\tau}^{T} [Y(0,\xi)]^{\mathsf{T}}\Phi(s+\tau,\xi)\varphi(s)d\xi ds -$$

$$\int\limits_0^T [Y(0,s)]^{\mathsf{T}}\left(\int\limits_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathrm{y}}(s)\right)ds, \tag{25}$$

Remark 4.1 *It is easy to show that $\mathcal{D}_{\beta,\gamma}$ is symmetric positive-definite (and therefore has an inverse). Consider $\mathcal{J}(t,s) = \int_0^T [Y(t,\xi)]^{\mathsf{T}} Y(s,\xi)d\xi$; we have $\mathcal{J}(t,s) = \mathcal{J}^{\mathsf{T}}(s,t)$, and therefore $D_{\beta,\gamma} \equiv (\beta+\gamma)I + \mathcal{J}(0,0) = D_{\beta,\gamma}^{\mathsf{T}}$. By definition, $\int_0^T u^{\mathsf{T}}(t) \int_0^T [Y(0,s)]^{\mathsf{T}} Y(0,s)dsu(t)dt =$*

$$\int_0^T\int_0^T u^{\mathsf{T}}(t)[Y(0,s)]^{\mathsf{T}} Y(0,s)u(t)dsdt = \int_0^T \|v(t)\|^2 dt \geq 0 \tag{26}$$

for some $v(t)$, and $\beta \geq 0$, $\gamma \geq 0$ so $\left(D_{\beta,\gamma}u, u\right) \geq 0$. Thus the positive semi-definiteness of $D_{\beta,\gamma}$ is obtained. Since $Y(0,s)$ is a solution of the equation (18a) we can conclude from (26) that $(D_{\beta,\gamma}u, u) = 0$ implies $u \equiv 0$. The existence of $\mathcal{D}_{\beta,\gamma}^{-1}$ follows.

Corollary 4.1 *When $\alpha = 0$, we have*

$$\int_{-\tau}^0 K_{\beta,\gamma}(t,s)\varphi_\star(s)ds = \mathfrak{g}_0^{\beta,\gamma}(t) \;\; \textit{for} \;\; t \in [-\tau, 0), \tag{27}$$

where $K_{\beta,\gamma}(t,s)$ and $\mathfrak{g}_\alpha^{\beta,\gamma}(t)$ are defined by (24).

Lemma 4.1 *The kernel $K_{\beta,\gamma}(t,s)$ defined by (24a) is self-adjoint and positive semi-definite.*

The proof is provided in Appendix B.

The equation (23) is obtainable from (27) by applying Lavrent'ev's method ([12, 18], [21, p. 89]) – which is sometimes called the "method of singular perturbation" [13, 18] – to (27). The function φ satisfies

$$\int_{-\tau}^0 K_{0,0}(t,s)\varphi(s)ds = \mathfrak{g}_0^{0,0}(t), \;\; \text{for } t \in [-\tau, 0). \tag{28}$$

4.1 Proof of Theorem 4.1.

In this section we shall establish that the initial function which satisfies equations (16) also satisfies the integral equation (23).

According to §3.2 we may write the solution of the adjoint problem (14) in the form

$$x^{\mathsf{T}}(t) = \int_t^T \left[y(\varphi;s) - \widehat{y}(s)\right]^{\mathsf{T}} Y(t,s)ds. \tag{29}$$

Using (19), we can write (29) as

$$x^{\mathsf{T}}(t) = \int_t^T \left[Y(0,s)\varphi(0) + \int_{-\tau}^0 Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi \right]^{\mathsf{T}} Y(t,s)ds +$$

$$\int_t^T \left[\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathsf{y}}(s) \right]^{\mathsf{T}} Y(t,s)ds$$

(for $0 \le t \le T$), and therefore

$$x^{\mathsf{T}}(t+\tau) = \int_{t+\tau}^T \left[\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathsf{y}}(s) \right]^{\mathsf{T}} Y(t+\tau,s)ds +$$

$$+ \int_{t+\tau}^T \left[Y(0,s)\varphi(0) + \int_{-\tau}^0 Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi \right]^{\mathsf{T}} Y(t+\tau,s)ds. \qquad (30)$$

By virtue of (10e), $\varphi(t)$ satisfies $\alpha\varphi(t) + \left[x^{\mathsf{T}}(t+\tau)B(t+\tau)\right]^{\mathsf{T}} = \alpha\widehat{\varphi}(t)$ for $t \in [-\tau, 0)$. Thus, using (30), we have for $t \in [-\tau, 0)$

$$\alpha\varphi(t) + \left[\int_{t+\tau}^T \left[Y(0,s)\varphi(0) + \int_{-\tau}^0 Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi \right]^{\mathsf{T}} \times \right.$$

$$\left. \times Y(t+\tau,s)dsB(t+\tau) \right]^{\mathsf{T}} = \alpha\widehat{\varphi}(t) -$$

$$\left[\int_T^{t+\tau} \left[\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathsf{y}}(s) \right]^{\mathsf{T}} Y(t+\tau,s)dsB(t+\tau) \right]^{\mathsf{T}}. \qquad (31)$$

¿From the expression for $x^{\mathsf{T}}(t)$, we have

$$x^{\mathsf{T}}(0) = \int_0^T \left[\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathsf{y}}(s) \right]^{\mathsf{T}} Y(0,s)ds +$$

$$+ \int_0^T \left[Y(0,s)\varphi(0) + \int_{-\tau}^0 Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi \right]^{\mathsf{T}} Y(0,s)ds. \qquad (32)$$

Therefore, we can write $\varphi(0)$ as (see (10f)) $(\gamma+\beta)\varphi(0)+\left[x^{\mathsf{T}}(0)\right]^{\mathsf{T}} = \beta\widehat{\varphi}(0)+\gamma\widehat{\mathrm{y}}(0)$. Here, using (32), we can write $(\beta+\gamma)\varphi(0)+$

$$\left[\int_0^T \left[Y(0,s)\varphi(0) + \int_{-\tau}^0 Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi\right]^{\mathsf{T}} Y(0,s)ds\right]^{\mathsf{T}} =$$

$$\beta\widehat{\varphi}(0) + \gamma\widehat{\mathrm{y}}(0) - \left[\int_0^T \left[\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathrm{y}}(s)\right]^{\mathsf{T}} Y(0,s)ds\right]^{\mathsf{T}}. \tag{33}$$

The latter equation allows us to eliminate $\varphi_\star(0)$ from (31) to obtain an integral equation for $\varphi_\star$. Taking the transposes we can write (33) in the form

$$(\gamma+\beta)\varphi(0) + \int_0^T [Y(0,s)]^{\mathsf{T}} Y(0,s)\varphi(0)ds =$$

$$-\int_0^T\int_{-\tau}^0 [Y(0,s)]^{\mathsf{T}} Y(\xi+\tau,s)B(\xi+\tau)\varphi(\xi)d\xi ds + F_0(\widehat{\varphi}(0),\widehat{\mathrm{y}},f),$$

where $F_0(\widehat{\varphi}(0),\widehat{\mathrm{y}},f) = \beta\widehat{\varphi}(0) + \beta\widehat{\mathrm{y}}(0) -$

$$\int_0^T [Y(0,s)]^{\mathsf{T}} \left(\int_0^s Y(\xi,s)f(\xi)d\xi - \widehat{\mathrm{y}}(s)\right) ds.$$

We may therefore write

$$\mathcal{D}_{\beta,\gamma}\varphi(0) = -\int_0^T\int_{-\tau}^0 [Y(0,s)]^{\mathsf{T}} \Phi(\xi+\tau,s)\varphi(\xi)d\xi ds + F_0(\widehat{\varphi}(0),\widehat{\mathrm{y}},f). \tag{34}$$

Using the equation (34), we may obtain from (31) the result

$$\alpha\varphi(t) + \int_{t+\tau}^T\int_{-\tau}^0 \Phi^{\mathsf{T}}(t+\tau,s)\Phi(\xi+\tau,s)\varphi(\xi)d\xi ds + \tag{35}$$

$$-\int_{t+\tau}^{T}\int_{0}^{T}\Big\{\int_{-\tau}^{0}\Phi^{\mathsf{T}}(t+\tau,s)Y(0,s)\mathcal{D}_{\beta,\gamma}^{-1}[Y(0,\mu)]^{\mathsf{T}}\,\Phi(\xi+\tau,\mu)\varphi(\xi)d\xi\Big\}d\mu ds =$$

$$\alpha\widehat{\varphi}(t)-\int_{t+\tau}^{T}\Phi^{\mathsf{T}}(t+\tau,s)Y(0,s)\mathcal{D}_{\beta,\gamma}^{-1}F_0(\widehat{\varphi}(0),\widehat{\mathrm{y}},f)ds+$$

$$-\int_{t+\tau}^{T}\Phi^{\mathsf{T}}(t+\tau,s)\left(\int_{0}^{s}Y(\xi,s)f(\xi)d\xi-\widehat{\mathrm{y}}(s)\right)ds.$$

Remark 4.2 *From Lemma 3.2 we have $Y(\xi+\tau,\mu)=0$ for $\mu<\xi+\tau$. We may therefore change the lower limit in the third term in equation (35) to $\mu=\xi+\tau$.*

We have derived (35) which is the required integral equation formulation and Theorem 4.1 is now established.

Remark 4.3 *The solution $\varphi_\star$ of (23) is unique in $L_2[-\tau,0]$ and hence in $\mathcal{F}$. Using a similar method to that above, we can show that in a finite dimensional subspace $\mathcal{F}_n$ of $\mathcal{F}$ the integral equation has the form (23), where all functions are treated as functions from $\mathcal{F}_n$.*

4.2 A convergence result.

We shall consider the convergence of the iteration described in §2.1 by studying the iteration

$$\frac{\varphi_{n+1}(t)-\varphi_n(t)}{\delta_n}=\mathfrak{g}_\alpha^{\beta,\gamma}(t)-\Big(\alpha\varphi_n(t)+\int_{-\tau}^{0}K_{\beta,\gamma}(t,s)\varphi_n(s)ds\Big). \tag{36}$$

This iteration is based upon the integral equation (23). It is, in a certain sense, linked to the "method of sequential approximations" [12, p. 272]. In (36), $K_{\beta,\gamma}(t,s)$ has been shown to be symmetric and positive-definite; the corresponding integral operator on $L_2[-\tau,0]$ is bounded, self-adjoint, and positive-definite. We state the following result.

Lemma 4.2 *The iteration (36) is equivalent to the iteration (11) described in §2.1. (For a given φ_0, the two sequences $\{\varphi_n\}$ are identical.)*

Proof. From (11e), the functions defined by the iteration (11) satisfy the relation

$$\frac{\varphi_{n+1}(t)-\varphi_n(t)}{\delta_n}=\alpha(\varphi_n(t)-\widehat{\varphi}(t))+[B(t+\tau)]^{\mathsf{T}}x_n(t+\tau)\quad\text{for } t\in[-\tau,0)$$

and we have shown in §4.1 that

$$\alpha(\varphi_n(t)-\widehat{\varphi}(t))+[B(t+\tau)]^{\mathsf{T}}x_n(t+\tau)=\alpha\varphi_n(t)+\int_{-\tau}^{0}K_{\beta,\gamma}(t,s)\varphi_n(s)ds-\mathfrak{g}_\alpha^{\beta,\gamma}(t),$$

so the result is immediate.

Theorem 4.2 (Convergence) *Suppose $\rho(K_{\beta,\gamma})$ is the spectral radius of the integral operator $K_{\beta,\gamma}$ on $L_2[-\tau,0]$ having the kernel $K_{\beta,\gamma}(t,s)$. Then, a sufficient condition for the iteration (11) in Definition 2.1 to converge in the mean-square norm is*

$$\delta_n \leq \frac{2}{\max(\alpha, \rho(K_{\beta,\gamma}))}, \quad \textit{for all } n. \tag{37}$$

Proof. We shall write $\mathfrak{L}_\alpha^{\beta,\gamma}\varphi(t) = \alpha\varphi(t) + \int_{-\tau}^{0} K_{\beta,\gamma}(t,s)\varphi(s)ds$ and the operator $\mathfrak{L}_\alpha^{\beta,\gamma}$ on $L_2[-\tau,0]$ inherits self-adjointness and (with $\alpha > 0$) positive-definiteness from the corresponding properties of the integral operator $K_{\beta,\gamma}$. For a sequence $\{\delta_n\}$ with $\delta_n > 0$ for all n, we can write the iteration process (36) in the form

$$\frac{\varphi_{n+1}(t) - \varphi_n(t)}{\delta_n} = \mathfrak{g}_\alpha^{\beta,\gamma}(t) - \mathfrak{L}_\alpha^{\beta,\gamma}\varphi_n(t). \tag{38}$$

Let $\varphi_\star$ be the solution of the equation $\mathfrak{L}_\alpha^{\beta,\gamma}\varphi_\star(t) = \mathfrak{g}_\alpha^{\beta,\gamma}(t)$ and let us define $\varepsilon_{n+1} = \varphi_{n+1} - \varphi_\star$. Then, according to (38), we have the relation $\varepsilon_{n+1} = (I - \delta_n\mathfrak{L}_\alpha^{\beta,\gamma})\varepsilon_n$, and

$$\varepsilon_{n+1} = \prod_{i=0}^{n}(I - \delta_n\mathfrak{L}_\alpha^{\beta,\gamma})\varepsilon_0. \tag{39}$$

The iteration (38) converges in the mean-square norm if $\|\varepsilon_n\|_2 \to 0$ as $n \to \infty$. ¿From (39) we have

$$\|\varepsilon_{n+1}\|_2 \leq \Big\| \prod_{i=0}^{n}(I - \delta_n\mathfrak{L}_\alpha^{\beta,\gamma}) \Big\|_2 \|\varepsilon_0\|_2 \leq \prod_{i=0}^{n} \Big\|(I - \delta_n\mathfrak{L}_\alpha^{\beta,\gamma})\Big\|_2 \|\varepsilon_0\|_2.$$

Thus, a sufficient condition for convergence of this iteration is

$$\|I - \delta_n\mathfrak{L}_\alpha^{\beta,\gamma}\|_2 \leq \vartheta < 1 \quad \text{for all } n. \tag{40}$$

Given the properties of $\mathfrak{L}_\alpha^{\beta,\gamma}$ on $L_2[-\tau,0]$, we have $\|\mathfrak{L}_\alpha^{\beta,\gamma}\|_2 = \max_r \kappa_r$ (the spectral radius $\rho(\mathfrak{L}_\alpha^{\beta,\gamma})$), where $\{\kappa_r\}_{r\geq 0}$ are the positive eigenvalues of $\mathfrak{L}_\alpha^{\beta,\gamma}$. Indeed, $\kappa_r = \alpha + \varkappa_r$, where $\{\varkappa_r\}_{r\geq 0}$ are the positive eigenvalues of $K_{\beta,\gamma}$. Then condition (40) becomes

$$\max_r |1 - \delta_n\alpha - \delta_n\varkappa_r| < 1.$$

We have $1 - \delta_n\alpha - \delta_n\varkappa_r \in [1 - \delta_n\alpha - \delta_n\rho(K_{\beta,\gamma}), 1 - \delta_n\alpha) \subseteq (-1, 1)$ provided $1 - \delta_n\alpha - \delta_n\rho(K_{\beta,\gamma}) > -1$ and Theorem 4.2 established.

Remark 4.4 *In general, the explicit form of the kernel $K_{\beta,\gamma}(t,s)$ is unknown and we cannot implement the iteration process for the integral equation itself; we use the iterative process (11).*

Appendices

A Proof of Lemma 2.2

¿From (9b) we have

$$P_\alpha^{\beta,\gamma}(\varphi,\psi)=\alpha\int_{-\tau}^{0}\{\varphi(t)-\widehat{\varphi}(t)\}^{\mathsf{T}}\psi(t)dt+\int_{0}^{T}\{y(\varphi,f;t)-\widehat{y}(t)\}^{\mathsf{T}}z(\psi;t)dt+$$

$$\beta\{\varphi(0)-\widehat{\varphi}(0)\}^{\mathsf{T}}\psi(0)+\gamma\{y(\varphi;0)-\widehat{y}(0)\}^{\mathsf{T}}z(\psi;0).$$

Clearly,

$$P_\alpha^{0,0}(\varphi,\psi)=\alpha\int_{-\tau}^{0}\{\varphi(t)-\widehat{\varphi}(t)\}^{\mathsf{T}}\psi(t)dt+\int_{0}^{T}\{y(\varphi,f;t)-\widehat{y}(t)\}^{\mathsf{T}}z(\psi;t)dt=$$

$$=\alpha\int_{-\tau}^{0}\varphi^{\mathsf{T}}(t)\psi(t)dt+\int_{0}^{T}y^{\mathsf{T}}(\varphi,f;t)z(\psi;t)dt$$

$$-\Big\{\alpha\int_{-\tau}^{0}\widehat{\varphi}^{\mathsf{T}}(t)\psi(t)dt+\int_{0}^{T}\widehat{y}(t)\}^{\mathsf{T}}z(\psi;t)dt\Big\}.$$

Using the expressions for $y(t)$ and $z(t)$ associated with (20),

$$\int_{0}^{T}\{y(\varphi,f;t)-\widehat{y}(t)\}^{\mathsf{T}}z(\psi;t)dt=$$

$$\Big\{\int_{0}^{T}\{Y(0,t)\varphi(0)+\mathrm{F}\varphi(t)+\mathrm{V}f(t)-\widehat{y}(t)\}^{\mathsf{T}}Y(0,t)dt\Big\}\psi(0)+$$

$$\int_{0}^{T}\Big\{Y(0,t)\varphi(0)+\mathrm{F}\varphi(t)+\mathrm{V}f(t)-\widehat{y}(t)\Big\}^{\mathsf{T}}\mathrm{F}\psi(t)dt.$$

Let us write $P_\alpha^{\beta,\gamma}(\varphi,\psi)$ in the form

$$P_\alpha^{\beta,\gamma}(\varphi,\psi)=\overbrace{\underbrace{{}^0P_\alpha^{0,0}(\varphi,\psi)}_{\text{terms in }\varphi}+\underbrace{{}^1P_\alpha^{0,0}(\widehat{\varphi},\psi)}_{\text{terms in }\widehat{\varphi}}+\underbrace{{}^2P_0^{0,0}(\{\mathrm{V}f-\widehat{y}\},\psi)}_{\text{terms in } f\ \&\ \widehat{y}}}^{\text{integrals with }\psi(t)\text{ in the integrand}}$$
$$+\underbrace{{}^0\nabla P_0^{0,0}(\varphi)+{}^1\nabla P_0^{0,0}(\{\mathrm{V}f-\widehat{y}\})}_{\text{integrals involving }\psi(0)}\qquad(41)$$

$$+ \overbrace{\underbrace{{}^0\Delta P_0^{0,0}}_{\text{independent of } \beta,\ \gamma} + \underbrace{{}^1\Delta P_0^{\beta,\gamma}}_{\text{involving } \beta \text{ and } \gamma} + \overbrace{{}^2\Delta P_0^{\beta,\gamma}(\widehat{\varphi},\widehat{y})}^{\text{involving only } \psi(0)}}^{\text{discrete terms: involving } \varphi(0) \text{ and } \psi(0);}.$$

Let us now consider the bilinear form

$$\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\psi) = {}^0P_\alpha^{0,0}(\varphi,\psi) + \Big({}^0\nabla P_0^{0,0}(\varphi) + {}^0\Delta P_0^{0,0} + {}^1\Delta P_0^{\beta,\gamma}\Big)\psi(0) \tag{42}$$

If we return to the detailed expression for ${}^0P_\alpha^{0,0}(\varphi,\psi)$, ${}^0\nabla P_0^{0,0}(\varphi)$, ${}^0\Delta P_0^{0,0}$ and ${}^1\Delta P_0^{\beta,\gamma}$ we obtain the following expression for the bilinear form $\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\psi) =$

$$\begin{aligned}
&\alpha\int_{-\tau}^{0}\varphi^{\mathsf{T}}(s)\psi(s)ds + \int_0^T\int_{-\tau}^0\int_{-\tau}^0\varphi^{\mathsf{T}}(s)\Phi^{\mathsf{T}}(s+\tau,t)\Phi(\mu+\tau,t)\psi(\mu)d\mu dsdt+\\
&\qquad\int_0^T\int_{-\tau}^0\varphi^{\mathsf{T}}(s)\Phi^{\mathsf{T}}(s+\tau,t)Y(0,t)\psi(0)dsdt+\\
&\qquad\int_0^T\int_{-\tau}^0\varphi^{\mathsf{T}}(0)[Y(0,t)]^{\mathsf{T}}\Phi(s+\tau,t)\psi(s)dsdt+\\
&\varphi^{\mathsf{T}}(0)(\beta+\gamma)\psi(0) + \int_0^T\varphi^{\mathsf{T}}(0)[Y(0,t)]^{\mathsf{T}}Y(0,t)\psi(0)dt.
\end{aligned} \tag{43}$$

It is easy to see from (43) that $\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\psi) = \mathcal{P}_\alpha^{\beta,\gamma}(\psi,\varphi)$. Now consider $\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\varphi)$. We can write

$$\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\varphi) = \alpha\int_{-\tau}^0\varphi^{\mathsf{T}}(s)\varphi(s)ds\ + (\beta+\gamma)\varphi^{\mathsf{T}}(0)\varphi(0)+$$
$$\int_0^T\Big[Y(0,t)\varphi(0)+\int_{-\tau}^0\Phi(s+\tau,t)\varphi(s)ds\Big]^{\mathsf{T}}\Big[Y(0,t)\varphi(0)+\int_{-\tau}^0\Phi(s+\tau,t)\varphi(s)ds\Big]dt.$$

Thus $\mathcal{P}_\alpha^{\beta,\gamma}(\varphi,\varphi) > 0$; positive definiteness is established.

B Proof of Lemma 4.1

Since the bilinear form (42) is positive semidefinite for all $\varphi(0)$ we can pick some particular value of the function φ at $t = 0$, namely, let

$$\varphi(0) = -\mathcal{D}_{\beta,\gamma}^{-1}\int_{-\tau}^0\int_{s+\tau}^T[Y(0,\xi)]^{\mathsf{T}}Y(s+\tau,\xi)B(s+\tau)\varphi(s)d\xi ds,$$

and we define the bilinear form (43) after substitution of $\varphi(0)$ as $\widehat{\mathcal{P}}_{\alpha}^{\beta,\gamma}(\varphi,\varphi)$.

If, for notational convenience, we define

$$\bar{\varphi}(\mu,t) := \Phi(\mu+\tau,t)\varphi(\mu)$$

then we have

$$\widehat{\mathcal{P}}_{\alpha}^{0,0}(\varphi,\varphi) = \alpha\int\limits_{-\tau}^{0}\varphi^{\mathsf{T}}(s)\varphi(s)ds + \int\limits_{0}^{T}\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\bar{\varphi}^{\mathsf{T}}(s,t)\bar{\varphi}(\mu,t)d\mu ds dt -$$

$$\int\limits_{0}^{T}\int\limits_{-\tau}^{0}\bar{\varphi}^{\mathsf{T}}(s,t)Y(0,t)\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{\mu+\tau}^{T}[Y(0,\xi)]^{\mathsf{T}}\bar{\varphi}(\mu,\xi)d\xi d\mu ds dt +$$

$$\Bigg\{ -\int\limits_{0}^{T}\int\limits_{-\tau}^{0}\Big[\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{\mu+\tau}^{T}[Y(0,\xi)]^{\mathsf{T}}\bar{\varphi}(\mu,\xi)d\xi d\mu\Big]^{\mathsf{T}}[Y(0,t)]^{\mathsf{T}}\bar{\varphi}(s,t)dsdt +$$

$$\Big[\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{t+\tau}^{T}[Y(0,\mu)]^{\mathsf{T}}\bar{\varphi}(t,\mu)d\mu dt\Big]^{\mathsf{T}}\Big((\beta+\gamma)I + \int\limits_{0}^{T}[Y(0,t)]^{\mathsf{T}}Y(0,t)dt\Big)\times$$

$$\times\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{s+\tau}^{T}[Y(0,\xi)]^{\mathsf{T}}\bar{\varphi}(s,\xi)d\xi ds \Bigg\}$$

Let us consider the last two terms in the expression for $\widehat{\mathcal{P}}_{\alpha}^{\beta,\gamma}(\varphi,\varphi)$ (within the braces). Taking into account that $\mathcal{D}_{\beta,\gamma} = (\beta+\gamma)I + \int\limits_{0}^{T}[Y(0,t)]^{\mathsf{T}}Y(0,t)dt$ we can write

$$-\int\limits_{0}^{T}\int\limits_{-\tau}^{0}\Big\{\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{\mu+\tau}^{T}[Y(0,\xi)]^{\mathsf{T}}\bar{\varphi}(\mu,\xi)d\xi d\mu\Big\}^{\mathsf{T}}[Y(0,t)]^{\mathsf{T}}\bar{\varphi}(s,t)dsdt +$$

$$\Big\{\mathcal{D}_{\beta,\gamma}^{-1}\int\limits_{-\tau}^{0}\int\limits_{t+\tau}^{T}[Y(0,\mu)]^{\mathsf{T}}\bar{\varphi}(t,\mu)d\mu dt\Big\}^{\mathsf{T}}\int\limits_{-\tau}^{0}\int\limits_{s+\tau}^{T}[Y(0,\xi)]^{\mathsf{T}}\bar{\varphi}(s,\xi)d\xi ds = 0.$$

Hence, we have

$$0 \le \widehat{\mathcal{P}}_{\alpha}^{\beta,\gamma}(\varphi,\varphi) = \alpha(\varphi,\varphi) + (K_{\beta,\gamma}\varphi,\varphi), \tag{44}$$

where $K_{\beta,\gamma}$ is the integral operator with kernel

$$K_{\beta,\gamma}(t,s) = \int\limits_{\xi=t+\tau}^{T}\Phi^{\mathsf{T}}(t+\tau,\xi)\Phi(s+\tau,\xi)d\xi + \tag{45}$$

$$- \int_{\xi=t+\tau}^{T} \int_{\mu=s+\tau}^{T} \Phi^{\mathsf{T}}(t+\tau,\xi) Y(0,\xi) \mathcal{D}_{\beta,\gamma}^{-1} [Y(0,\mu)]^{\mathsf{T}} \Phi(s+\tau,\mu) d\mu d\xi$$

Since $\mathcal{D}_{\beta,\gamma}^{-1}$ is symmetric we can show, from (44) and (45), that $\widehat{\mathcal{P}}_{\alpha}^{\beta,\gamma}(\varphi,\psi)$ is symmetric and Lemma 4.1 is established.

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