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Numerical Methods for Space-Fractional Partial Differential Equations

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Abstract

In this dissertation, we consider numerical methods for solving space-fractional PDEs. We first consider finite difference method then we consider finite element methods for solving space-fractional PDEs. The error estimates are obtained. Finally we consider the matrix transform technique (MTT) for solving space-fractional PDEs which include finite difference and finite element method. Numerical examples are given.

Key words

- finite difference method
- finite element method
- Backward Euler method
- Matrix transform technique
- Space-fractional partial differential equations
- Error estimate

This work is original and has not been previously submitted for any academic purpose.

Signed: ....

Date: ....

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Contents

Chapter 1

Introduction

The importance of fractional calculus is noticeable in the mathematical research in recant years. Because the use of derivative of fractional order in mathematical models has become increasingly in demand. Many mathematical models with fractional derivative have been successfully applied in finance [12][29][31], biology [37], physics [25][2][26][39], chemistry and biochemistry [4] and hydrology [16][15][3]. These models are effective for description of the memory and inherent properties of different substances [28]. The great use of fractional order derivative include specific properties of the process are reformulating the integer order of the diffussion, advection diffusion and Fokker-plank equations. The fractional PDEs can be arranged into two kinds :space-fractional partial differential equations and time-fractional partial differential equations. Evaluation of the analytic solutions of the fractional order partial differential equations are difficult beacuse it obtained Green functions [22]. The evolution of numerical techniques to solve the fractional order problem is increasing in a rapid way. Many authors have discussed the numerical methods for solving the space-fractional derivatives [28]. Liu et al.[15] transform the space fractional Fokker-Plank equation into a system a differential equations. In 2004, Meerchaet and Tadjeran [24] obtained the numerical solution of space FADE in one dimension and [23] presented a shifted Grünwald estimation for the two-sided space FPDE. Also in [11],[13],[24],[14],[23],[36],[15],[20] and [7],[9],[40] authors are discussed the finite difference method and finite element method, respectively for solving space-fractional partial differential equations.

In this dissertaion, we consider the following space-fractional partial differential equations for $\frac{1}{2} < \alpha < 1$

$$u_t - \frac{d^{2\alpha}u}{d|x|^{2\alpha}} = f, \qquad 0 < x < 1, \quad t > 0 \tag{1.1}$$

subject to the boundary and initial conditions are given by

$$u(t,0) = u(t,1) = 0 \tag{1.2}$$

$$u(0,x) = u_0(x) = 0 \tag{1.3}$$

Here,

$$-\frac{d^{2\alpha}u(x)}{d|x|^{2\alpha}} = \frac{1}{2\ cos\Big(\pi\alpha\Big)}\Big[{}_{0}^{R}D_{x}^{2\alpha}u(x) + {}_{x}^{R}D_{1}^{2\alpha}u(x)\Big]$$

and ${}_{0}^{R}D_{x}^{\beta}u(t,x)$ denotes the left Riemann- Liouville fractional derivative with respect to x defined by

$$ {}_{0}^{R}D_{x}^{\beta}u(t,x) = \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_0^x (x-y)^{1-\beta}\ u(y)dy, 1<\beta<2 \quad (1.4)$$

and ${}_{x}^{R}D_{1}^{\beta}u(t,x)$ denotes the right Riemann- Liouville fractional derivative with respect to x defined by

$$ {}_{x}^{R}D_{1}^{\beta}u(t,x) = \frac{1}{\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_0^x (y-x)^{1-\beta}\ u(y)\ dy, 1<\beta<2. \quad (1.5)$$

Where Γ denotes the gamma function.

There are different numerical methods to solve the equations (1.1)-(1.3).In this dissertation we will discuss finite difference and element methods and then we will also discuss about matrix transform technique (MTT) for space-fractional PDEs. In this dissertation include

- Introduction of fractional calculus which contains definitions and relavent functions.

- Finite difference and element methods to solve space-fractional partial differential equations with matrix transform technique.

  In chapter 2 we consider some basic functions, definitions and relations of fractional derivatives.

  In chapter 3 we consider the finite difference method to solve the space-fractional PDEs.

  In chapter 4 we discuss the finite element method to solve elliptic, parabolic and space-fractional partial differential equations. We also discusse some useful functional spaces and their properties.

  In chapter 5 we discuss matrix transform technique for solving space-fractional PDEs which include finite difference and finite element methods.

  In chapter 6 we we summarize the dissertation and mention some future work in this topic.

Chapter 2

Fractional calculus

2.1 Basic functions and transforms

We will discuss about some functions which play an important role in the definition of fractional operators. In this section we outline important definitions and lemmas used throghout the remaining chapters of this papers. First include here information on the gamma functions, beta functions, the mittag-leffler functions, laplace transform, fouriar transform.

Gamma function: Occurs in the definition of fractional differential and integral operators. The basic function of fractional calculus namely the Euler Gamma function is important for generalization of factorial function which only accept positive integer. For $\alpha > 0$ the Gamma function in the integral form as [28]

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{2.1}$$

Properties of the Gamma function:

1.The Gamma function reduction formula:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha\,!, \quad Re\ \alpha > 0$$

Example:

$$\Gamma(1) = 0\,!$$

2. Ratio of Gamma functions:

$$\frac{\Gamma(z-\alpha)}{\Gamma(z+1)} = \frac{\Gamma(z-\alpha) - z\Gamma(z-\alpha-1)}{\Gamma(z+1)}$$
$$= \frac{\Gamma(-\alpha) - z\Gamma(z-\alpha-1)}{\Gamma(-\alpha-1)\Gamma(z+1)}$$

Beta function: The Beta function is defined by

$$B(\alpha,\beta) = \int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1}d\tau = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)} \tag{2.2}$$

The beta-function can be express in terms of gamma-functions [17]

$$B(z,w) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}$$

The Mittag-Leffler function: The Mittag-Leffler function is an entire function defined by the series

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)} \quad \alpha > 0 \tag{2.3}$$

Example: Let $\alpha = 1$ the (2.3) becomes

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^t$$

For the two parameters the Mittag-Leffler function is given by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)} \quad \alpha > 0,\ \beta > 0 \tag{2.4}$$

Example: Let $\alpha = \beta = 2$ the (2.4) becomes

$$E_{2,2}(-t^2) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(2k+2)} = \frac{sint}{t}.$$

Laplace transform:[17] Let a function $f(x)$ is defined for $0 < x < \infty$, then the Laplace transform $F(s)$ is defined as

$$\mathcal{L}f(t)(s) = F(s) = \int_0^\infty e^{-st}f(t)dt \tag{2.5}$$

at least for those s for which the integral converges. Let $f(x)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $c \in \mathcal{R}$ and $x > 0$

$$sup\frac{|f(x)|}{e^{cx}} < \infty$$

In this case the Laplace transform (2.5) exists for all $s > c$ [6].

Inverse Laplace transform: The Inverse Laplace Transform of $F(s)$ is defined as

$$\mathcal{L}f(t)(s) = F(s) = \int_0^{\infty} e^{-st} f(t)dt \tag{2.6}$$

Fourier transform: [28] Let $f(t)$ is a continuous function of a real variable $t \in (-\infty, +\infty)$. Then the Fourier transform of $f(t)$ is defined by

$$\mathcal{F}\{f(\omega)\} = F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)dt \tag{2.7}$$

Inverse Fourier transform: The inverse Fourier transform is defined as

$$\mathcal{F}^{-1}\{f(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} F(\omega)d\omega \tag{2.8}$$

Convolution of Fourier transform: Let f and g be two functions. The convolution of f and g is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)du, \quad t \in \mathcal{R}$$

Now , the Fourie transform of above becomes

$$\mathcal{F}\{(f * g)(\omega)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\} = F(\omega)G(\omega)$$

Fourier transformation of Riemann-Liouville integral:[28],[9] Let $\alpha > 0$, then the Fourier Transfor of the Riemann-Liouville integral with a lower terminal of $-\infty$ is defined as

$$\mathcal{F}\{{}^{R}_{-\infty}D_t^{-\alpha} f(t)\} = (i\omega)^{-\alpha} F(\omega)$$

2.2 Definitions of fractional derivatives

There are several definitions given to fractional calculus. In this section we will give some important definitions of fractional both the integral and derivative.

Grüwald-Letnikov fractional integral:

Let $0 < \alpha < 1$, the Grüwald-Letnikov fractional integral ([28] pp. 47) defined by

$${}^{G}_{0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \quad , \tag{2.9}$$

Here ${}^{G}_{0}D_t^{-\alpha} f(t)$ is the Grüwald-Letnikov fractional integral of order α with respect to t, where α is non-integer.

Riemann-Liouville fractional integral: The Riemann-Liouville fractional integral of order $0 < \alpha < 1$, defined by

$$ {}_0^R D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{2.10} $$

Reimann-Liouville fractional derivative: The Riemann-Liouville fractional derivative ([28]pp.55) is defined with $\alpha > 0$ and $n-1 < \alpha \le n$ as

$$ {}_0^R D_t^{\alpha} f(t) = D^n [D_t^{\alpha-n} f(t)] = D^n \frac{1}{\Gamma(n-\alpha)} \int_\alpha^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \tag{2.11} $$

Caputo fractional integral: For $\alpha > 0$ the Caputo fractional derivative ([28]pp.79) is defined with $n-1 < \alpha < n$ and $\alpha = n$ are, respectively , given by

$$ {}_0^R D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} [D^n f(\tau)] d\tau \tag{2.12} $$

$$ {}_0^R D_t^{n} f(t) = \frac{d^n}{dt^n} f(t) \tag{2.13} $$

Riesz fractional operator:[36][30] For $n-1 < \alpha < n$ on an open interval $0 < x < 1$ is defined as

$$ \frac{d^\alpha}{d|x|^\alpha} u(t,x) = -C_\alpha ({}_0^R D_x^\alpha u(t,x) + {}_x^R D_1^\alpha u(t,x)) \tag{2.14} $$

where $C_\alpha = \frac{1}{2cos\left(\frac{\pi\alpha}{2}\right)}, \alpha \neq 1$ and

$$ {}_0^R D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}} d\xi $$

$$ {}_x^R D_1^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{u(\xi,t)}{(\xi-x)^{\alpha+1-n}} d\xi $$

2.3 Relations between the different fractional derivatives

Lemma 2.3.1. *[36][30] For a function $u(x)$ defined on the infinite domain $[-\infty < x < \infty]$, the following equality holds:*

$$ -\frac{d^\alpha u}{d|x|^\alpha} = (-\Delta)^{\alpha/2} u = \frac{1}{2\ cos\left(\frac{\pi\alpha}{2}\right)} \left[{}_{-\infty}^{R} D_x^\alpha u + {}_x^R D_\infty^\alpha u \right] $$

Proof: Fractional power of the Laplace operator defined as

$$-(-\Delta)^{\alpha/2}u(x) = -\mathcal{F}^{-1}\Big(|x|^{\alpha}\mathcal{F}u(x)\Big)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and inverse Fourier transform respectively.

By the definition of Fourier transform and inverse Fourier transform, we get

$$\begin{aligned}
-(-\Delta)^{\alpha/2}u(x) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}|\xi|^{\alpha}\int_{-\infty}^{\infty} e^{i\eta\xi}u(\eta)\,d\eta\,d\xi \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}|\xi|^{\alpha}\Big[u(\eta)\frac{e^{i\eta\xi}}{i\xi}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(\eta)\frac{e^{i\eta\xi}}{i\xi}d\eta\Big]d\xi \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} u'(\eta)\Big[i\int_{-\infty}^{\infty} e^{-i\xi(\eta-x)}\frac{|\xi|^{\alpha}}{\xi}d\xi\Big]d\eta \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} u'(\eta)i\Big[\int_{0}^{\infty} e^{-i\xi(\eta-x)}\xi^{\alpha-1}d\xi - \int_{0}^{\infty} e^{-i\xi(x-\eta)}\xi^{\alpha-1}d\xi\Big]d\eta \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} u'(\eta)i\Big[\frac{\Gamma(\alpha)}{[i(x-\eta)]^{\alpha}} - \frac{\Gamma(\alpha)}{[i(\eta-x)]^{\alpha}}\Big]d\eta \\
&\qquad\qquad\qquad [\text{By Gamma function for } 0<\alpha<1] \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} u'(\eta)\frac{sign(x-\eta)\Gamma\alpha\Gamma(1-\alpha)}{|(x-\eta)|^{\alpha}\Gamma(1-\alpha)}[i^{\alpha-1}+(-i)^{\alpha-1}]d\eta
\end{aligned}$$

Using $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{sin\pi\alpha}$ and $i^{\alpha-1}+(-i)^{\alpha-1} = 2sin\frac{\pi\alpha}{2}$, we obtain

$$-(-\Delta)^{\alpha/2}u(x) = \frac{-1}{2cos\frac{\pi\alpha}{2}}\Big[\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{u'(\eta)}{(x-\eta)} - \frac{1}{\Gamma(1-\alpha)}\int_{x}^{\infty}\frac{u'(\eta)}{(\eta-x)}\Big]$$

Note that, for $0<\alpha<1$, the Riemann -Liouville fractional derivative in [a,x] is given by

$$\begin{aligned}
{}_0^R D_x^{\alpha}u &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}(x-\eta)^{-\alpha}\,u(\eta)d\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\Big[u(a)\frac{(x-a)^{1-\alpha}}{1-\alpha} + \int_{a}^{x} u'(\eta)\frac{(x-\eta)^{1-\alpha}}{1-\alpha}d\eta\Big] \\
&= \frac{1}{\Gamma(1-\alpha)}\Big[u(a)(x-a)^{-\alpha} + \int_{a}^{x} u'(\eta)(x-\eta)^{-\alpha}d\eta\Big] \\
&= \frac{u(a)(x-a)^{-\alpha}}{\Gamma(1-\alpha)} + \int_{a}^{x}\frac{u'(\eta)(x-\eta)^{-\alpha}}{\Gamma(1-\alpha)}d\eta
\end{aligned}$$

Therefore, $\lim_{a\to-\infty}({}_0^R D_x^\alpha u) = {}_{-\infty}^{R}D_x^\alpha u = \int_{-\infty}^{x} \frac{u'(\eta)(x-\eta)^{-\alpha}}{\Gamma(1-\alpha)} d\eta$

Similarly, we can write ${}_x^R D_\infty^\alpha u = \int_x^\infty \frac{u'(\eta)(\eta-x)^{-\alpha}}{\Gamma(1-\alpha)} d\eta$

Finally, we get

$$-\frac{d^\alpha u}{d|x|^\alpha} = (-\Delta)^{\alpha/2} u = \frac{1}{2\,cos\left(\frac{\pi\alpha}{2}\right)}\left[{}_{-\infty}^{R}D_x^\alpha u + {}_x^R D_\infty^\alpha u\right] \quad \text{for } 0 < \alpha < 1$$

where

$$ {}_{-\infty}^{R}D_x^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{u(\eta)}{(x-\eta)^\alpha} d\eta$$

$$ {}_x^R D_\infty^\alpha u = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{u(\eta)}{(\eta-x)^\alpha} d\eta$$

Further, when $n-1 < \alpha < n$ then

$$-\frac{d^\alpha u}{d|x|^\alpha} = (-\Delta)^{\alpha/2} u = \frac{1}{2\,cos\left(\frac{\pi\alpha}{2}\right)}\left[{}_{-\infty}^{R}D_x^\alpha u + {}_x^R D_\infty^\alpha u\right]$$

where

$$ {}_{-\infty}^{R}D_x^\alpha u = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} \frac{u(\eta)}{(x-\eta)^{\alpha+1-n}} d\eta$$

$$ {}_x^R D_\infty^\alpha u = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty \frac{u(\eta)}{(\eta-x)^{\alpha+1-n}} d\eta$$

Lemma 2.3.2. *By lemma 2.3.1, we have with* $1 < \alpha < 2$ *the Riemann-Liouville fractional derivatives on [0,1] are respectively express as*

$$ {}_0^R D_x^\alpha u = \frac{u(0)(x-0)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{u'(0)(x-0)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u''(y)}{(x-y)^{\alpha-1}} dy \tag{2.15}$$

$$ {}_x^R D_1^\alpha u = \frac{u(1)(1-x)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{u'(1)(1-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_x^1 \frac{u''(y)}{(y-x)^{\alpha-1}} dy \tag{2.16}$$

Proof: We first prove (2.15). By definition [36]

$$
\begin{aligned}
{}_0^R D_x^\alpha u &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x (x-y)^{1-\alpha}\; u(y)dy \\
&= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[u(y)\frac{(x-y)^{2-\alpha}}{2-\alpha}\Big|_0^x + \int_0^x u'(y)\frac{-(x-y)^{2-\alpha}}{2-\alpha}dy\Big] \\
&= \frac{1}{\Gamma(3-\alpha)}u(0)\frac{d^2}{dx^2}(x^{2-\alpha}) + \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[\int_0^x \frac{(x-y)^{2-\alpha}}{2-\alpha}u'(y)dy\Big] \\
&= \frac{1}{\Gamma(3-\alpha)}u(0)\frac{d^2}{dx^2}(x^{2-\alpha}) + \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[u'(y)\Big(\frac{-(x-y)^{3-\alpha}}{(2-\alpha)(3-\alpha)}\Big)\Big|_0^x + \\
&\qquad \textstyle\int_0^x u''(y)\frac{(x-y)^{3-\alpha}}{(2-\alpha)(3-\alpha)}dy\Big] \\
&= \frac{1}{\Gamma(3-\alpha)}u(0)\frac{d^2}{dx^2}(x^{2-\alpha}) + \frac{1}{\Gamma(2-\alpha)}u'(0)\frac{d^2}{dx^2}\Big(\frac{(x)^{3-\alpha}}{(2-\alpha)(3-\alpha)}\Big) + \\
&\qquad \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x u''(y)\frac{(x-y)^{3-\alpha}}{(2-\alpha)(3-\alpha)}dy \\
&= \frac{1}{\Gamma(3-\alpha)}u(0)\frac{d}{dx}\{(2-\alpha)(x^{1-\alpha})\} + \frac{1}{\Gamma(2-\alpha)}u'(0)\frac{d}{dx}\Big(\frac{(3-\alpha)(x)^{2-\alpha}}{(2-\alpha)(3-\alpha)}\Big) + \\
&\qquad \frac{1}{\Gamma(2-\alpha)}\frac{d}{dx}\{\int_0^x \frac{(3-\alpha)(x-y)^{2-\alpha}}{(2-\alpha)(3-\alpha)}u''(y)dy\} \\
&= \frac{u(0)(x^{-\alpha})}{\Gamma(1-\alpha)} + \frac{u'(0)(x^{1-\alpha})}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-y)^{1-\alpha}u''(y)dy
\end{aligned}
$$

Similarly, we will prove (2.16). By definition

$$
\begin{aligned}
{}_x^R D_1^\alpha u &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_x^1 (y-x)^{1-\alpha}\; u(y)dy \\
&= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[u(y)\frac{(y-x)^{2-\alpha}}{2-\alpha}\Big|_x^1 - \int_x^1 u'(y)\frac{(y-x)^{2-\alpha}}{2-\alpha}dy\Big] \\
&= \frac{1}{\Gamma(3-\alpha)}u(1)\frac{d^2}{dx^2}((1-x)^{2-\alpha}) - \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[\int_x^1 u'(y)\frac{(y-x)^{2-\alpha}}{2-\alpha}u'(y)dy\Big] \\
&= \frac{1}{\Gamma(3-\alpha)}u(1)\frac{d^2}{dx^2}(1-x)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\Big[u'(y)\Big(\frac{(y-x)^{3-\alpha}}{(2-\alpha)(3-\alpha)}\Big)\Big|_x^1 - \\
&\qquad \int_x^1 u''(y)\frac{(y-x)^{3-\alpha}}{(2-\alpha)(3-\alpha)}dy\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(3-\alpha)} u(1) \frac{d^2}{dx^2}(1-x)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2}\Big[u'(1) \frac{(1-x)^{3-\alpha}}{(2-\alpha)(3-\alpha)} - \\
&\qquad\qquad \int_x^1 u''(y) \frac{(y-x)^{3-\alpha}}{(2-\alpha)(3-\alpha)} dy \Big] \\
&= \frac{1}{\Gamma(3-\alpha)} u(1) \frac{d}{dx}\{-(2-\alpha)(1-x)^{1-\alpha}\} + \frac{1}{\Gamma(2-\alpha)} \frac{d}{dx} u'(1) \frac{(3-\alpha)(1-x)^{2-\alpha}}{(2-\alpha)(3-\alpha)} \\
&\qquad - \frac{1}{\Gamma(2-\alpha)} \frac{d}{dx} \int_x^1 u''(y) \frac{-(3-\alpha)(y-x)^{2-\alpha}}{(2-\alpha)(3-\alpha)} dy \Big] \\
&= \frac{u(1)(1-x)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{u'(1)(1-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_x^1 \frac{u''(y)}{(y-x)^{\alpha-1}} dy
\end{aligned}
$$

Chapter 3

Finite difference method for solving space-fractional partial differential equations

3.1 Finite difference approximations

One of the simplest and of the oldest methods to solve differential equations is the finite difference method. L.Euler(1707-1783) ca. 1908 and C.Runge(1856-1927) ca.1908 shall probably be credited for formulating finite difference method in one dimension of space and extended one dimension to two dimensions, respectively. The development of finite difference techniques in numerical application introduced in the early 1950s and development reproduce by the appearence of computers that offered a easy system to working with complex problems of science and technology.

The principle of this method is replacing the region over which the independent variables in the PDE are defined by a finite mesh points at which the dependent variable is approximated. The partial derivative in the PDE at each mesh point are approximated from neighbouring values by Taylor's theorem.

Let $u(x)$ have the n derivatives over the interval [a,b], then for $a < x_0 < x_0 + h < b$ the Taylor series gives

$$u(x_0 + h) = u(x_0) + h\, u_x(x_0) + h^2 \frac{u_{xx}(x_0)}{2!} + \cdots + h^{n-1} \frac{u_{(n-1)}(x_0)}{(n-1)!} + O(h^n)$$

$$\Rightarrow u(x_0 + h) = u(x_0) + h\, u_x(x_0) + O(h^2)$$

[Truncating after first derivative term]

$$\Rightarrow u_x(x_0) = \frac{u(x_0 + h) - u(x_0)}{h} + O(h)$$

Neglecting the $O(h)$ term gives

$$u_x(x_0) = \frac{u(x_0 + h) - u(x_0)}{h}$$

which is called a first order finite difference approximation to $u_x(x_0)$.

Example: Let us recall the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 \leq x \leq 1 \tag{3.1}$$

with the initial and boundary conditions :

$$u(x, 0) = u_0(x), \tag{3.2}$$

$$u(0, t) = u(1, t) = 0, \tag{3.3}$$

Let $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$ be a partition of [0,1] and h=1/4 be the step size.

At $x = x_j$, j=1,2,3, we have

$$\left.\frac{\partial u}{\partial t}\right|_{x=x_j} - \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_j} = 0 \tag{3.4}$$

By Taylor formula, we have

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_j} = \frac{u(x_{j-1}, t) - 2u(x_j, t) + (x_{j+1}, t)}{h^2} + O(h^2)$$

Thus we get

$$\left.\frac{\partial u}{\partial t}\right|_{x=x_j} - \frac{u(x_{j-1}, t) - 2u(x_j, t) + (x_{j+1}, t)}{h^2} + O(h^2) = 0$$

Denote $U_j(t) \approx u(x_j, t)$ the approximate solution of $u(x_j, t)$. Then we get

$$\frac{d}{dt}U_j(t) - \frac{U_{j-1}(t) - 2U_j(t) + U_{j+1}(t)}{h^2} = 0$$

Denote $U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix}$

We get

$$\frac{d}{dt}\begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix} = 0$$

$$Denote,\ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and we can write

$$\frac{d}{dt}U(t) + AU(t) = 0 \tag{3.5}$$

$$U(0) = U_0 \tag{3.6}$$

Here (3.4) is an ordinary differential system which can be solved by MATLAB.

The exact solution of (3.4) is $U(t) = E(t)U_0$, where $E(t) = e^{-tA}$

3.2 Finite difference method for space-fractional partial differential equations

In this section, we provide a numerical solution for solving the following space-fractional partial differential equation with $1 < \alpha < 2$,

$$u_t - \frac{d^\alpha u}{d|x|^\alpha} = 0, \qquad 0 < x < 1, \quad t > 0 \tag{3.7}$$

$$u(t,0) = u(t,1) = 0 \tag{3.8}$$

$$u(0,x) = u_0(x) = 0 \tag{3.9}$$

where,

$$-\frac{d^\alpha u}{d|x|^\alpha} = (-\Delta)^{\alpha/2}u = \frac{1}{2\ cos\left(\frac{\pi\alpha}{2}\right)}\left[{}_0^R D_x^\alpha u + {}_x^R D_1^\alpha u\right]$$

and

$${}_0^R D_x^\alpha u(t,x) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x (x-y)^{1-\alpha}\ u(y)\ dy, 1 < \alpha < 2 \tag{3.10}$$

$${}_x^R D_1^\alpha u(t,x) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x (y-x)^{1-\alpha}\ u(y)\ dy,\ 1 < \alpha < 2. \tag{3.11}$$

Let, $0 = x_0 < x_1 < x_2 \cdots < x_l < x_{N-1} < x_N = 1$ be a partition. Since we know that $u(t,0) = u(t,1) = 0$, we only need to find the numerical solutions at $x_l, l = 1,2,3,\dots N-1$.

At $x = x_l,\ l = 1,2,\dots N-1$, we have by (2.15) and (2.16)

$$\frac{\partial u}{\partial t}\Big|_{x=x_l} + \frac{1}{2\,cos\left(\frac{\pi\alpha}{2}\right)}\Big[\frac{u(0)(x_l)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{u'(0)(x_l)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^{x_l} \xi^{1-\alpha}u''(x_l-\xi)d\xi +$$

$$\frac{u(1)(1-x_l)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{u'(1)(1-x_l)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^{1-x_l} \xi^{1-\alpha}u''(x_l+\xi)d\xi\Big] = 0$$

$$. \tag{3.12}$$

Note that, [21]

$$\frac{1}{\Gamma(2-\alpha)}\int_{x_0}^{x_l} \xi^{1-\alpha}u''(x_l-\xi)d\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\int_{x_j}^{x_{j+1}} \xi^{1-\alpha}u''(x_l-\xi)d\xi$$
$$\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\int_{x_j}^{x_{j+1}} \xi^{1-\alpha}u''(x_l-x_j)d\xi$$
$$\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\frac{u(x_l-x_{j+1})-2u(x_l-x_j)+u(x_l-x_{j-1})}{h^2}\int_{x_j}^{x_j+1} \xi^{1-\alpha}d\xi$$
$$= \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{l-1}\Big[u(x_l-x_{j+1})-2u(x_l-x_j)+u(x_l-x_{j-1})\Big][(j+1)^{2-\alpha}-j^{2-\alpha}]$$

Since,

$$\int_{x_j}^{x_j+1} \xi^{1-\alpha}d\xi = \frac{1}{(2-\alpha)}\,\xi^{2-\alpha}\Big|_{\xi=x_j}^{\xi=x_{j+1}}$$
$$= \frac{1}{(2-\alpha)}[(x_{j+1})^{2-\alpha}-(x_j)^{2-\alpha}] = \frac{h^{2-\alpha}}{(2-\alpha)}[(j+1)^{2-\alpha}-j^{2-\alpha}]$$

Again, for $l = 1, 2, \ldots N-1$ we have

$$\frac{1}{\Gamma(2-\alpha)}\int_0^{1-x_l} \xi^{1-\alpha}u''(x_l+\xi)d\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\int_{x_0}^{x_N-x_l} \xi^{1-\alpha}u''(x_l+\xi)d\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\Big[\int_{x_0}^{x_1} + \int_{x_1}^{x_2} \cdots + \int_{x_N-x_{l+1}}^{x_N-x_l}\Big]\,\xi^{1-\alpha}\,u''(x_l+\xi)d\xi$$
$$\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{N-l-1}\frac{u(x_l+x_{j-1})-2u(x_l+x_j)+u(x_l+x_{j+1})}{h^2}\int_{x_j}^{x_j+1} \xi^{1-\alpha}d\xi$$

$$= \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{N-l-1} \Big[u(x_l + x_{j-1}) - 2u(x_l + x_j) + u(x_l + x_{j+1})\Big] [(j+1)^{2-\alpha} - j^{2-\alpha}]$$

Denote $u(x_l, t) \approx U_l(t),\ l = 1, 2 \ldots N-1$, (3.12) can be approximated by

$$\frac{d}{dt}U_l(t) + \frac{1}{2\ cos\left(\frac{\pi\alpha}{2}\right)} \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \Big[\frac{(1-\alpha)(2-\alpha)U_0(t)}{l^{\alpha}} + \frac{(2-\alpha)}{l^{\alpha-1}}(U_1(t) - U_0(t)) +$$
$$\sum_{j=0}^{l-1} \Big(U_{l-j+1} - 2U_{l-j} + U_{l-j-1}\Big)[(j+1)^{2-\alpha} - j^{2-\alpha}]\Big] + \frac{1}{2\ cos\left(\frac{\pi\alpha}{2}\right)} \frac{h^{-\alpha}}{\Gamma(3-\alpha)}$$
$$\Big[\frac{(1-\alpha)(2-\alpha)U_N(t)}{(N-l)^{\alpha}} + \frac{(2-\alpha)}{(N-l)^{\alpha-1}}(U_N(t) - U_{N-1}(t)) +$$
$$\sum_{j=0}^{N-l-1} \Big(U_{l+j-1} - 2U_{l+j} + U_{l+j+1}\Big)[(j+1)^{2-\alpha} - j^{2-\alpha}]\Big] = 0$$
$$. \tag{3.13}$$

We can express this equation in matrix form

$$\frac{d}{dt}U(t) + AU(t) = 0 \tag{3.14}$$

where

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \\ . \\ . \\ U_{N-1}(t) \end{bmatrix}$$

,

$$U_0(t) \approx u(0,t) = 0,\ U_N(t) \approx u(1,t) = 0$$

And A is a $(n-1) \times (n-1)$.

For example, when n=6, we have with $c_j = [(j+1)^{2-\alpha} - j^{2-\alpha}],\ j = 1, 2, 3, 4, 5.$

$$A = \frac{1}{2\ cos\left(\frac{\pi\alpha}{2}\right)} \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \begin{bmatrix} -2c_1 & c_1 & 0 & 0 & 0 \\ -2c_2 + c_1 & c_2 - 2c_1 & c_1 & 0 & 0 \\ -2c_3 + c_2 & c_3 - 2c_2 + c_1 & c_2 - 2c_1 & c_1 & 0 \\ -2c_4 + c_3 & c_4 - 2c_3 + c_2 & c_3 - 2c_2 + c_1 & c_2 - 2c_1 & c_1 \\ -2c_5 + c_4 & c_5 - 2c_4 + c_3 & c_4 - 2c_3 + c_2 & c_3 - 2c_2 + c_1 & c_2 - 2c_1 \end{bmatrix}$$

$$+\frac{1}{2\,cos\left(\frac{\pi\alpha}{2}\right)}\frac{h^{-\alpha}}{\Gamma(3-\alpha)}\begin{bmatrix} c_2-2c_1 & c_3-2c_2+c_1 & c_4-2c_3+c_2 & c_5-2c_4+c_3 & -2c_5+c_4 \\ c_1 & c_2-2c_1 & c_3-2c_2+c_1 & c_4-2c_3+c_2 & -2c_4+c_3 \\ 0 & c_1 & c_2-2c_1 & c_3-2c_2+c_1 & -2c_3+c_2 \\ 0 & 0 & c_1 & c_2-2c_1 & -2c_2+c_1 \\ 0 & 0 & 0 & c_1 & -2c_1 \end{bmatrix}$$

$$+\frac{1}{2\,cos\left(\frac{\pi\alpha}{2}\right)}\frac{h^{-\alpha}}{\Gamma(3-\alpha)}\begin{bmatrix} \frac{2-\alpha}{1^{\alpha-1}} & 0 & 0 & 0 & -\frac{2-\alpha}{(N-1)^{\alpha-1}} \\ \frac{2-\alpha}{2^{\alpha-1}} & 0 & 0 & 0 & -\frac{2-\alpha}{(N-2)^{\alpha-1}} \\ \frac{2-\alpha}{3^{\alpha-1}} & 0 & 0 & 0 & -\frac{2-\alpha}{(N-3)^{\alpha-1}} \\ \frac{2-\alpha}{4^{\alpha-1}} & 0 & 0 & 0 & -\frac{2-\alpha}{(N-4)^{\alpha-1}} \\ \frac{2-\alpha}{5^{\alpha-1}} & 0 & 0 & 0 & -\frac{2-\alpha}{(N-5)^{\alpha-1}} \end{bmatrix}$$

The system (3.13) can be solved by using MATLAB.

The stability and error estimates can be found from the reference [36].

Chapter 4

Finite element method for solving space-fractional partial differential equations

The German Mathematician Ricard Courant (1888-1972) shall probably be credited for formulating the finite element method [Courant, 1943].The great use of FEM are in almost every field of engineering analysis. The FEM method is known as one of the most powerful and versatile method for solving boundary value problems. The principle of FEM method is nearly similar to FDM method. Both methods consider a partition of the domain into a number of small pieces. In the FEM, the domain is subdivided into a partition or mesh or a collection of geometrically simple elements and the approximation space is composed of piecewise polynomial functions on each element of the partition. The main idea of the FEM is to replace the Hilbert space V in which the variational formulation is posed by a finite dimensional subspace. Ervin and Roop [9] [10] employed the finite element method to find the variational solution of fractional advection equation, in where the fractional derivative depends on the space, related to the nonlocal operator.

4.1 Finite element method for solving elliptic equation

Consider the elliptic problem

$$-u'' = f \qquad 0 < x < 1 \tag{4.1}$$

$$u(0) = u(1) = 0 \tag{4.2}$$

Definition 4.1.1. *(Classical solutions)*

We say that equation (4.1) has the classical solution if there exists $u \in C^2[0,1] \cap C_0[0,1]$such that equation (4.1) holds.Here $u \in C^2[0,1]$ and $C_0[0,1]$ denote the some continuous function spaces defined by

C[0,1]={ u:u(x) is continuous on [0,1] }

$C^2[0,1] = \{u : u'(x), u''(x)$ are continuous on [0,1] }

$C_0[0,1] =$ { u:u(x) is continuous on [0,1] and u(0)=u(1)=0}

In application, the equation (4.1) has no classical solution for any f.

Therefore we need to find weak solution of equation (4.1).To introduce weak solution of equation (4.1), we need to introduce some sobolev spaces or integrable function spaces.

Definition 4.1.2. *(Sobolev spaces) We define,*

$H = L_2(0,1) = \{u : \int_0^1 u^2(x)dx < \infty\}$

$H^1(0,1) = \{u : u, u' \in L_2(0,1)\}$

$H^2(0,1) = \{u : u, u', u'' \in L_2(0,1)\}$

$H_0^1(0,1) = \{u : u \in H_0^1(0,1)$ and $u(0) = u(1) = 0\}$

$L_\infty(0,1) = \{u : |u(x)|$ is bounded almost everywhere$\}$

Theorem 4.1.1. *The space $H = L_2(0,1)$ is a Hilbert space with respect to the inner product $(u,v)_{L_2(0,1)} = \int_0^1 u(x)v(x)dx$. The induced norm is $||u||_{L_2(0,1)} = \Big[\int_0^1 u^2(x)dx\Big]^{\frac{1}{2}}$.*

Theorem 4.1.2. *The spaces $H^r(0,1), r = 1,2$ are Hilbert spaces with respect to the inner product $(u,v)_{H^1(0,1)} = \int_0^1 u(x)v(x)dx + \int_0^1 u'(x)v'(x)dx$ and $(u,v)_{H^2(0,1)} = \int_0^1 u(x)v(x)dx + \int_0^1 u'(x)v'(x)dx + \int_0^1 u''(x)v''(x)dx = (u,v)_{L_2(0,1)} + (u',v')_{L_2(0,1)} + (u'',v'')_{L_2(0,1)}$ respectively.*

The induced norms are $||u||^2_{H^r(0,1)} = \sum_{l=0}^{r} ||u^l(x)||^2_{L_2}, r = 1,2.$

There are some sobolev embedding theorems which show the relations between continuous function spaces and Sobolev spaces.

Theorem 4.1.3. *(Sobolev embedding theorem) Assume that $u \in H^1(0,1)$ then $u \in L_\infty(0,1)$.*

Proof: By using embedding inequality, we have

$$||u||_{L_\infty} \leq C||u||_{H^1} \tag{4.3}$$

Thus if $u \in H^1(0,1)$, we get $u \in L_\infty(0,1)$.

Theorem 4.1.4. *(Poincare inequality)[5] Assume that $u \in H^1(0,1)$ then*

$$||u||_{L_2(0,1)} \leq C||u'||_{L_2(0,1)} \tag{4.4}$$

Proof: Note that, $u(x) = u(0) + \int_0^x u'(y)dy$.

Since $u \in H_0^1(0,1)$, we have $u(0) = 0$ then

$$||u||^2_{L_2(0,1)} = \int_0^1 |u(x)|^2 dx = \int_0^1 |\int_0^x u'(y)dy|^2 dx$$

Using Cauchy-Schwartz inequality i.e. $(f,g) \leq ||f||_{L_2}||g||_{L_2}$

$$or, \int_0^1 f(x)g(x)dx \leq \Big(\int_0^1 f^2(x)dx\Big)^{\frac{1}{2}}\Big(\int_0^1 g^2(x)dx\Big)^{\frac{1}{2}}$$

We get,

$$||u||^2_{L_2(0,1)} \leq \int_0^1 \Big(\int_0^x dy\Big)\int_0^x |u'(y)|^2 dy\ dx$$

$$\leq \int_0^1 \Big(\int_0^1 dy\Big)\int_0^1 |u'(y)|^2 dydx$$

$$\leq \int_0^1 |u'(y)|^2 dy$$

$$\leq ||u'||^2_{L_2(0,1)}$$

Which implies

$$||u||_{L_2(0,1)} \leq C||u'||_{L_2(0,1)}.$$

Now we come to the weak solution of (1.1). Assume that u is a classical solution of (4.1). That is , u satisfies

$$-u'' = f \qquad 0 < x < 1 \tag{4.5}$$
$$u(0) = u(1) = 0$$

Multiplying a smooth test function $v \in C_0^\infty(0,1)$.

Here $C_0^\infty(0,1) = \{u(x) : u \in C^\infty(0,1)$ and u has compact support on (0,1) i.e.

$$supp(u) \subset (0,1) \text{and } supp(u) = \{x : u(x) \neq 0\}\}$$

Of course, if $v \in C_0^\infty(0,1)$, then $u(0) = u(1) = 0$.

Thus equation (4.5) becomes

$$-u'' \cdot v = f \cdot v, \qquad v \in C_0^\infty(0,1)$$

Integrating on (0,1), we get

$$\int_0^1 (-u'' \cdot v)dx = \int_0^1 f \cdot v \, dx, \qquad v \in C_0^\infty(0,1)$$

Integrating by parts, noting that $v(0) = v(1) = 0$, we get

$$\int_0^1 u'v'dx = \int_0^1 fv \, dx, \qquad v \in C_0^\infty(0,1) \tag{4.6}$$

Now we see that to get (4.4) from (4.5), we only need $v \in H_0^1(0,1)$

Definition 4.1.3. *(Weak solution) We say that $u \in H_0^1(0,1)$ is a weak solution of equation (4.1) if $u \in H_0^1(0,1)$ satisfies*

$$\int_0^1 u'v'dx = \int_0^1 fv \, dx, \qquad v \in H_0^1(0,1) \tag{4.7}$$

The equation (4.7) is also called variational form of (4.1).

Definition 4.1.4. *(Strong solution) We say that u is a strong solution of (4.1) if $u \in H^2(0,1) \cap H_0^1(0,1)$ such that (4.1) hold.*

Theorem 4.1.5. *Assume that u is strong solution of (4.1). Then u is also a weak solution of equation (4.7). On the other hand, assume that u is a weak solution of equation (4.7) and $u \in H^2(0,1)$, then u is also a strong solution of (4.1).*

We have the following Lax-Milgram lemma [33].

Theorem 4.1.6. *(Lax-Milgram Lemma)*

Let H be a Hilbert space with norm $||.||_H$ and inner product $(\cdot,\cdot)_H$. Let V be Hilbert space with norm $||\cdot||_V$ and inner product $(\cdot,\cdot)_V$. Let $V \subset H$ be a subspace of H. Let $a(u,v) : V \times V \to \mathbb{R}$ be a bilinear form on $V \times V$. Let $F : V \to \mathbb{R}$ be a linear fractional on V.

Assume that

1) $a(\cdot,\cdot)$ is bounded on $V \times V$

i.e. $|a(u,v)| \leq C_1||u||_V \cdot ||v||_V$.

2) $a(\cdot,\cdot)$ is coercivity on $V \times V$

i.e. $|a(u,v)| \geq C_2||v||_V^2, \;\; C_2 > 0$.

3) $F : V \to \mathbb{R}$ is bounded on V

i.e.$|F(v)| \leq C_3||v||_V$.

Then there exist a unique solution $u \in V$ such that $a(u,v) = F(v) \; \forall v \in V$ and $||u||_v V \leq C||F||_{L(V,\mathcal{R})}$.

Existence and uniqueness of the weak solution of (4.7)**:** Assume that $f \in L_2(0,1)$. There exists a unique solution $u \in H^1(0,1)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H^1(0,1) \tag{4.8}$$

Here $a(u,v) = \displaystyle\int_0^1 u'v'dx, \; (f,v) = \int_0^1 fvdx$.

Then (4.7) can be written into , find $u \in H_0^1(0,1)$, such that, with $F(v) = (f,v)$

$$a(u,v) = F(v), \quad \forall v \in H^1(0,1) \tag{4.9}$$

We will use Lax-Milgram Lemma to prove the existence and uniqueness of (4.9). Choose $V = H^1(0,1)$, with norm $|| \cdot ||_{H^1}$ and inner product $(u,v)_{H^1} = (u,v)_{L^2} + (u',v')_{L^2}, \; v \in H^1(0,1)$.

Choose $H = L^2(0,1)$ with norm $|| \cdot ||_{L_2}$ and inner product $(u,v)_{L_2} = \displaystyle\int_0^1 v^2(x)dx$.

Let us check the conditions in Lax-Milgram lemma

1) a$(\cdot,\cdot)$ is boundedness of $a(\cdot,\cdot)$ on $V \times V, \;\; V = H^1(0,1)$. In fact we have $|a(u,v)| = |(u',v')| \leq ||u'||_{L_2} \cdot ||v'||_{L_2}$ [By Cauchy-Schwartz inequality]

$$\leq ||u||_{H^1} \cdot ||v||_{H^1}$$

2) Coercivity of a$(\cdot,\cdot)$ on $V \times V$. In fact $a(u,v) = (u',v') = ||v'||^2_{L_2}$.
By poincare inequality, if $v \in H^1_0(0,1)$, we have $||v||_{L_2} \leq C_0||v'||_{L_2}$ for some $C_0 > 0$.

$||v||^2_{H^1} = ||v||^2_{L_2} + ||v'||^2_{L_2} \leq C_0^2||v'||^2_{L_2} + ||v'||^2_{L_2} \leq (C_0^2+1)||v'||^2_{L_2}$

i.e, $||v'||^2_{L_2} \geq \frac{1}{C_0^2+1}||v||^2_{H^1} = C_2||v||^2_{H^1}$

3) $F : V \to \mathbb{R}$ is bounded on V. In fact, we have

$$|F(v)| = |(f,v)| \leq ||f||_{L_2}||v||_{L_2} \leq ||f||_{L_2}||v||_{H^1}. \tag{4.10}$$

Thus, by Lax-Milgram lemma (4.9) has an unique solution $u \in H^1(0,1)$.

Further, by (4.10) we have $||F||_{L(V,\mathbb{R})} \leq ||f||_{L_2}$

Regularity: Let $f \in L_2(0,1)$. Assume that $u \in H^1(0,1)$ is the weak solution of equation (4.8) then $u \in H^2(0,1)$. In other words, if u is the weak solution of (4.8) i.e.$u \in H^1(0,1)$ satisfies

$$a(u,v) = f(u,v), \quad \forall u \in H^1(0,1)$$

then we can expect u has more regularity $u \in H^2(0,1)$.

Based on the above theorems, we know that if $f \in L_2(0,1)$, then the weak solution of (4.8) actually is a strong solution. Therefore we are now confident to work on the weak solution.

Now, we use finite element method to solve the variational form of (4.7) i.e. find $u \in H^1_0(0,1)$ such that

$$or, \quad (u',v') = (f,v) \tag{4.11}$$

To do this we need to introduce the finite element basis functions. Let $0 = x_0 < x_1 < x_2 \cdot < x_M = 1$ be a partition of [0,1].

Then define some finite element basis functions

$$\phi_1(x) = \begin{cases} \dfrac{x-x_1}{x_0-x_1}, & \text{if } x_0 < x < x_1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} \dfrac{x - x_1}{x_2 - x_1}, & \text{if } x_1 < x < x_2 \\ \dfrac{x - x_3}{x_2 - x_3}, & \text{if } x_2 < x < x_3 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{i-1} < x < x_j \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & \text{if } x_j < x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

where, $j = 1, 2, \dots M - 1$

Let us define a finite element space
$S_h = \{u_h(x) : U_h$is piecewise linear and continuous functions on [0,1] and
$u_h(0) = u_h(1) = 0\}$
Any element in S_h can be written into the linear combination of the basis function $\phi_j, \quad j = 1, 2, 3, \dots M - 1$. i.e. $U_h(x) = \sum_{j=1}^{M-1} \alpha_j \phi_j(x)$

Here we don't need to consider $\phi_0(x), \phi_M$ since we consider the homogeneous boundary conditions. If we consider non-homogeneous boundary condition, we then need to include basis functions $\phi_0(x), \phi_M(x)$.

The finite element method of (4.11) is to find $u_h \in S_h$ such that

$$(u_h', \chi') = (f, \chi), \qquad \forall \chi \in S_h \tag{4.12}$$

To find the solution of (4.12), let the solution of (4.12) have the form

$$u_h(x) = \sum_{j=1}^{M-1} \alpha_j \phi_j(x) \tag{4.13}$$

Substituting (4.13) into (4.12) and choosing $\chi = \phi_l(x)$, we get

$$\sum_{j=1}^{M-1} (\phi_j', \phi_l') \alpha_j = (f, \phi_l), \quad l = 1, 2, \dots M - 1 \tag{4.14}$$

In matrix form, we have

$$\vec{S} * \vec{\alpha} = \vec{F}$$

$$\text{Where, } \vec{S} = \begin{bmatrix} (\phi'_1, \phi'_1) & (\phi'_1, \phi'_2) & (\phi'_1, \phi'_3) & ..... & ... & (\phi'_1, \phi'_{M-1}) \\ (\phi'_2, \phi'_1) & (\phi'_2, \phi'_2) & (\phi'_2, \phi'_3) & .... & ... & (\phi'_2, \phi'_{M-1}) \\ .... & ... & .... & .... & ... & .... \\ .... & ... & .... & .... & ... & .... \\ .... & ... & .... & .... & ... & .... \\ (\phi'_3, \phi'_1) & (\phi'_3, \phi'_2) & (\phi'_3, \phi'_3) & ..... & ... & (\phi'_{M-1}, \phi'_{M-1}) \end{bmatrix}$$

$$\textit{and } \ \vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ . \\ \alpha_{M-1} \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ (f, \phi_3) \\ ...... \\ ..... \\ (f, \phi_{M-1}) \end{bmatrix}.$$

Here $\vec{S}$ is a positive definite matrix, hence $\vec{S}^{-1}$ exists. We can get

$$\vec{\alpha} = \vec{S}^{-1} * \vec{F}$$

Therefore we get the approximate finite element solution

$$u_h(x) = \sum_{j=1}^{M-1} \alpha_j \phi_j(x)$$

where $u_h(x_1) = \alpha_1, \ u_h(x_2) = \alpha_2, u_h(x_{M-1}) = \alpha_{M-1}$.

Error estimate:

Now, we will consider the error estimates of the finite element method for solving elliptic problem.

We have the following error estimates [18].

Theorem 4.1.7. *Let u_h and u be the solutions of (4.12) and (4.11) respectively. Then we have*

$$||u_h - u||_{L_2} \leq Ch^2 ||u||_{H^2}, \ \ ||u'_h - u'||_{L_2} \leq Ch||u||_{H^2}$$

where h is the step size.

To prove this theorem, we need the error estimates for the interpolation function on S_h. Let $0 = x_0 < x_1 < x_2 \cdot < x_M$ be a partition. Let V be a smooth function, say $v \in H^2(0, 1)$. Then we can define the Lagrange piecewise linear interpolation polynomial $I_h v(x)$ such that $I_h v(x) = v(x_j), \ \ j = 1, 2, \ldots M$.

Lemma 4.1.1. *(Interpolation error) Let $v \in H^2(0,1) \cap H^1(0,1)$.*
Let $I_h : H^2(0,1) \cap H^1(0,1) \to S_h$ be the interpolation operator. We have

$$||I_h v - v||_{L_2} + h||\nabla(I_h v - v)||_{L_2} \leq Ch^2||v||_{H^2} \tag{4.15}$$

Here ∇ denote the derivative i.e. $\nabla v = v'$ in 1-d case.

We also need the following finite element orthogonality i.e.

Lemma 4.1.2. *(Finite element orthogonality) Let u_h and u be the solutions of (4.12) and (4.11) respectively. Then we have*

$$(\nabla(u_h - u), \nabla\chi) = 0, \quad \forall \chi \in S_h \tag{4.16}$$

Proof: By (4.11) we have

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1 \tag{4.17}$$

By (4.12) we have

$$(\nabla u_h, \nabla\chi) = (f, \chi), \quad \forall \chi \in S_h \tag{4.18}$$

Since $S_h \subset H_0^1$ is a subspace of H_0^1, (4.17) implies that

$$(\nabla u, \nabla\chi) = (f, \chi), \quad \forall \chi \in S_h \tag{4.19}$$

Thus we have, by (4.18) and (4.19),

$$(\nabla(u_h - u), \nabla\chi) = 0, \quad \forall \chi \in S_h \tag{4.20}$$

To prove an L_2-norm error estimate of the finite element approximation to elliptic problem, we also need the following so called the elliptic regularity inequality.

Lemma 4.1.3. *(Elliptic regularity) Let $f \in L_2(0,1)$. Let $u \in H^2 \bigcap H_0^1$ be the solution of (4.1). Then we have the following elliptic regularity estimate*

$$||u||_{H^2} \leq C||u''||_{L^2} \leq C||f||_{L^2}$$

Note that here

$$||u||_{H^2}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2 + ||u''||_{L^2}^2$$

Remark: In general, we have

$$||u''||_{L^2} \leq ||u||_{H^2}, \ \forall u \in H^2$$

However, if u is the solution of (4.1), we can get

$$||u||_{H^2} \leq C||u''||_{L^2}, \ \forall u \in H^2 \subset H_0^1$$

Now we are ready to prove theorem 4.1.7.

Proof of theorem 4.1 : We will prove this theorem step by step.

Step 1 : We first prove

$$||\nabla(u_h - u)|| \leq Ch||u||_{H^2}$$

Note that $\quad ||\nabla(u_h - u)||^2 = (\nabla(u_h - u), \nabla(u_h - u))$

$$= (\nabla(u_h - u), \nabla(u_h - \chi + \chi - u))$$

$$= (\nabla(u_h - u), \nabla(u_h - \chi)) + (\nabla(u_h - u), \nabla(\chi - u))$$

$$= (\nabla(u_h - u), \nabla(\chi - u))$$

(finite element orthogonality)

$$\leq ||\nabla(u_h - u)||.||\nabla(\chi - u)||, \quad \forall \chi \in S_h$$

Hence, we get

$$||\nabla(u_h - u)|| \leq ||\nabla(\chi - u)||, \quad \forall \chi \in S_h$$

$$i.e. \;\; ||\nabla(u_h - u)|| \leq \inf_{\chi \in S_h} ||\nabla(\chi - u)||$$

By the interpolation error, we have

$$\inf_{\chi \in S_h} ||\nabla(\chi - u)|| \leq Ch||u||_{H^2}$$

Hence, we get

$$||\nabla(u_h - u)|| \leq Ch||u||_{H^2}$$

Next we prove the L_2 -norm of error estimate.

Step 2: We now prove

$$||(u - u_h)|| \leq Ch^2||u||_{H^2}$$

Let $g \in L_2$, consider

$$-\psi'' = g, \;\; 0 < x < 1$$

$$\psi(0) = \psi(1) = 0$$

We have, by elliptic regularity,

$$||\psi||_{H^2} \leq C||\psi||_{L_2}$$

Hence, we have

$$(u_h - u, g) = (u_h - u, -\psi'') = (\nabla(u_h - u), \nabla\psi)$$

$$= (\nabla(u_h - u), \nabla(\psi - \chi)) \text{ [Orthogonality]}$$

$$\leq ||(\nabla(u_h - u)||_{L_2} ||\nabla(\psi - \chi)||_{L_2}$$

$$\leq ||(\nabla(u_h - u)||_{L_2} \inf_{\chi \in S_h} ||\nabla(\psi - \chi)||_{L_2}$$

Note that, by step 1,

$$||\nabla(u_h - u)|| \leq Ch||u||_{H^2}$$

and

$$\inf_{\chi \in S_h} ||\nabla(\psi - \chi)||_{L_2} \leq ||\nabla(\psi - I_h\psi)||_{L_2} \leq Ch||g||_{H^2} \leq Ch||\psi||_{L_2}$$

Thus we get

$$(u_h - u, g) \leq (Ch||u||_{H^2}).(Ch||g||_{L_2})$$

$$= Ch^2||u||_{H^2}||\psi||_{L_2}$$

Choose $g = u_h - u$. we get

$$||u_h - u||_{L_2}^2 \leq Ch^2||u||_{H^2}||u_h - u||_{L_2}$$

which implies that

$$||u_h - u||_{L_2} \leq Ch^2||u||_{H^2}$$

4.2 Finite element method for solving parabolic equation

Consider the parabolic problem

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) \tag{4.21}$$

with the initial condition :

$$u(0,x) = u_0(x), \qquad 0 \leq x \leq 1 \tag{4.22}$$

and subject to the boundary conditions :

$$u(t,0) = u(t,1) = 0, \qquad 0 \leq t \leq T \tag{4.23}$$

We will describe step by step to solve equation (4.21)- (4.23) by finite element method as follows :

Step 1: Variational formulation in an infinite dimensional space, $V = H_0^1$.

Consider the following Hilbert space:

$$H_0^1 = \{v(x) : (0,1) \to \mathbb{R} : \int_0^1 (v^2(x) + (v'(x))^2)dx < \infty, v(0) = v(1) = 0\}$$

Now, multiplying equation (4.21) by test function $v(x) \in H_0^1$ and integrate by parts on the limit [0,1] to get

$$\int_0^1 \frac{\partial u(x,t)}{\partial t} v(x)dx - \int_0^1 \frac{\partial^2 u(x,t)}{\partial x^2} v(x)dx = \int_0^1 f(x,t)v(x)dx$$

$$or, \int_0^1 \frac{\partial u(x,t)}{\partial t} v(x)dx - v(x)\frac{\partial u(x,t)}{\partial x}\Big|_0^1 + \int_0^1 \frac{\partial u(x,t)}{\partial x} v'(x)dx = \int_0^1 f(x,t)v(x)dx$$

$$or, \int_0^1 \frac{\partial u(x,t)}{\partial t} v(x)dx + \int_0^1 u'(x,t)v'(x)dx = \int_0^1 f(x,t)v(x)dx$$

Thus the variational form is to find $u(\cdot,t) \in H_0^1$ such that [34]

$$\int_0^1 \frac{\partial u(x,t)}{\partial t} v(x)dx + \int_0^1 u'(x,t)v'(x)dx = \int_0^1 f(x,t)v(x)dx, \quad v \in H_0^1$$

Denote the inner product $(u,v) := \int u(x,t)v(x)dx$ for any fixed time t.

We can write our variational form as

$$(\frac{\partial u}{\partial t}, v) + (u'_x, v'_x) = (f, v), \qquad \forall v \in H_0^1 \tag{4.24}$$

By well known Lax-Milgram theorem, we can say equation (4.24) has a unique solution.

Step 2: Variational formulation in the finite dimensional space, S_h.

First divide the interval [0,1] into $0 = x_0 < x_1 < x_2 \cdots < x_M = 1$.

Then define some linear finite element basis functions $\phi_1, \phi_2, \cdot\phi_{M-1}$ where $\phi_i \in S_h$ for $i = 1, 2, \ldots M-1$. defined by

$$\phi_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

More precisely for $j = 1, 2, \dots M-1$ can be express as

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{j-1} < x < x_j \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & \text{if } x_j < x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

where $\phi_j(x)$ is a linear continuous function on the mesh $[x_0, x_m]$.
For the grid parameter h define the finite element space S_h by the basis function as

$$S_h = \{u_h : u_h = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_{M-1}\phi_{M-1}\}$$

$$= \{u_h : u_h \text{ is a piecewise continuous linear function, } u_h(0) = u_h(1) = 0\}$$

The function u_h is a finite element solution of equation (4.21).
The finite element method is to find $u_h(\cdot, t) = u_h(t) \in S_h$ i.e. for fixed t such that

$$(\frac{\partial u_h}{\partial t}, \chi) + (u_h', \chi') = (f, \chi), \quad \forall\, \chi \in S_h \in H_0^1 \tag{4.25}$$

where $'$ is for derivative with respect to space variable x.

Step 3: Discretization with respect to time [1]:

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition of [0,T]. We will discretize from time $t_0, t_1, \dots t_n$.

When time $t = t_0$, we get $u_h(0) = P_h u_0$, where $P_h = L_2 \to S_h\ is\ L_2$ projection operator.

Let, $P_h u_0 = \sum_{j=1}^{M-1} \alpha_j \phi_j$.

By definition, $(P_h u_0, \phi_l) = (u_0, \phi_l), l = 1, 2, 3 \dots M-1$

$$\Rightarrow (\textstyle\sum_{j=1}^{M-1} \alpha_j \phi_j, \phi_l) = (u_0, \phi_l)$$

$$\Rightarrow \textstyle\sum_{j=1}^{M-1} \alpha_j (\phi_j, \phi_l) = (u_0, \phi_l)$$

$$\Rightarrow \vec{\alpha} * \text{ mass} = \text{right hand vector}$$

At time t_1 the equation (4.24) becomes

$$\Big(\frac{\partial u_h}{\partial t}, \chi\Big)\Big|_{t=t_1} + (u_h', \chi')\Big|_{t=t_1} = (f, \chi)\Big|_{t=t_1} \tag{4.26}$$

By the backward Euler method, we know

$$\Big(\frac{\partial u_h}{\partial t}\Big)\Big|_{t=t_1} \approx \frac{u_h(t_1) - u_h(t_0)}{\Delta t}$$

So, equation (4.26) can be written as

$$\Big(\frac{u_h(t_1) - u_h(t_0)}{\Delta t}, \chi\Big) + (u_h'(t_1), \chi') = (f, \chi), \quad \chi \in S_h$$

$$or, (u_h(t_1), \chi) + \Delta t(u_h'(t_1), \chi') = \Delta t(f, \chi) + (u_h(t_0), \chi), \quad \chi \in S_h$$

Let, $u_h(t_1) = \sum_{j=1}^{M-1} \alpha_j(t_1)\phi_j, \quad \phi_j$ is basis function.

$$\Big(\sum_{j=1}^{M-1} \alpha_j(t_1)\phi_j, \chi\Big) + \Delta t\Big(\sum_{j=1}^{M-1} \alpha_j(t_1)\phi_j', \chi'\Big) = \Delta t(f, \chi) + (u_h(t_0), \chi), \quad \chi \in S_h$$

Choose $\chi = \phi_l, \quad l = 1, 2, \ldots M - 1$ and we obtain:

$$\Big(\sum_{j=1}^{3} \alpha_j(t_1)\phi_j, \phi_l\Big) + \Delta t\Big(\sum_{j=1}^{3} \alpha_j(t_1)\phi_j', \phi_l'\Big) = \Delta t(f, \phi_l) + (u_h(t_0), \phi_l), \quad \chi \in S_h$$

In matrix form, e.g. M=4, we get

$$\Big[(\phi_j, \phi_l)\Big] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \Delta t\Big[(\phi_j', \phi_l')\Big] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \Delta t \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ (f, \phi_3) \end{bmatrix} + \begin{bmatrix} (u_h(t_0), \phi_1) \\ (u_h(t_0), \phi_2) \\ (u_h(t_0), \phi_3) \end{bmatrix}$$

$$or, (mass + \Delta t * stiffness)\ \vec{\alpha}(t_1) = \Delta t\ \vec{F} + initial\ \ vector$$

$$or, \vec{\alpha}(t_1) = (mass + \Delta t * stiffness)^{-1}\ \vec{\alpha}(t_1)[\Delta t\ \vec{F} + initial\ \ vector]$$

Here, stiffness= $\begin{bmatrix} (\phi_1', \phi_1') & (\phi_1', \phi_2') & (\phi_1', \phi_3') \\ (\phi_2', \phi_1') & (\phi_2', \phi_2') & (\phi_2', \phi_3') \\ (\phi_3', \phi_1') & (\phi_3', \phi_2') & (\phi_3', \phi_3') \end{bmatrix}$

$$, \qquad mass = \begin{bmatrix} (\phi_1, \phi_1) & (\phi_1, \phi_2) & (\phi_1, \phi_3) \\ (\phi_2, \phi_1) & (\phi_2, \phi_2) & (\phi_2, \phi_3) \\ (\phi_3, \phi_1) & (\phi_3, \phi_2) & (\phi_3, \phi_3) \end{bmatrix},$$

$$and \;\; \vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ (f, \phi_3) \end{bmatrix}, \quad Initial\ vector = \begin{bmatrix} (u_h(t_0), \phi_1) \\ (u_h(t_0), \phi_2) \\ (u_h(t_0), \phi_3) \end{bmatrix}$$

Step 4: Construction of co-efficients:

We will construct stiffness matrix, mass matrix, right vector and initial vector.

Construction of stiffness matrix: In general form, the local stiffness matrix on element $e_i = [x_i, x_{i+1}]$ is

local stiffness matrix= $\begin{bmatrix} \int_{x_i}^{x_{i+1}} \phi_i'(x)\phi_i'(x)dx & \int_{x_i}^{x_{i+1}} \phi_i'(x)\phi_{i+1}'(x)dx \\ \int_{x_i}^{x_{i+1}} \phi_{i+1}'(x)\phi_i'(x)dx & \int_{x_i}^{x_{i+1}} \phi_{i+1}'(x)\phi_{i+1}'(x)dx \end{bmatrix}$

By refferencing method we can redefine our basis function $\phi_i(x)\ as\ \hat{\phi}_i(y)$

$$where, \;\; [x_i, x_{i+1}] \Rightarrow [0, 1]$$
$$x_i \to 0, \quad x_{i+1} \to 1$$

$$y = \frac{x - x_i}{x_{i+1} - x_i}, \qquad x = x_i + hy$$

$$\phi_i(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} = \frac{x_i + hy - x_{i+1}}{x_i - x_{i+1}} = 1 - y = \hat{\phi}_i(y)$$

$$and \qquad \phi_{i+1}(x) = \frac{x - x_i}{x_i - x_{i+1}} = \frac{x_i + hy - x_i}{x_i - x_{i+1}} = y = \hat{\phi}_{i+1}(y)$$

Therefore, $\phi_i'(x) = \dfrac{d}{dx}\phi_i(x) = \dfrac{d}{dy}(\hat{\phi}_i(y))\dfrac{dy}{dx} = \dfrac{-1}{h}$ and

$$\phi_{i+1}'(x) = \frac{d}{dx}\phi_{i+1}(x) = \frac{d}{dy}(\hat{\phi}_{i+1}(y))\frac{dy}{dx} = \frac{1}{h}$$

Now we will estimate elements of local stiffness matrix.

$$\int_{x_i}^{x_{i+1}} \phi_i'(x)\phi_i'(x)dx = \int_0^1 (-)\frac{1}{h}(-)\frac{1}{h}\ h\ dy = \frac{1}{h},$$

$$\int_{x_i}^{x_{i+1}} \phi_i'(x)\phi_{i+1}'(x)dx = \frac{-1}{h},$$

$$\int_{x_i}^{x_{i+1}} \phi_{i+1}'(x)\phi_{i+1}'(x)dx = \frac{-1}{h}$$

Local stiffness matrix becomes

$$\begin{bmatrix} \frac{1}{h} & \frac{-1}{h} \\ \frac{-1}{h} & \frac{-1}{h} \end{bmatrix}$$

Denote global stiffness matrix is

$$\begin{bmatrix} \int_0^1 \phi_1'(x)\phi_1'(x)dx & \int_0^1 \phi_2'(x)\phi_1'(x)dx & \int_0^1 \phi_3'(x)\phi_1'(x)dx \\ \int_0^1 \phi_1'(x)\phi_2'(x)dx & \int_0^1 \phi_2'(x)\phi_2'(x)dx & \int_0^1 \phi_3'(x)\phi_2'(x)dx \\ \int_0^1 \phi_1'(x)\phi_3'(x)dx & \int_0^1 \phi_2'(x)\phi_3'(x)dx & \int_0^1 \phi_3'(x)\phi_3'(x)dx \end{bmatrix}$$

On element $e_0 = [x_0, x_1]$ the global stiffness matrix is

$$\begin{bmatrix} \int_{x_0}^{x_1} \phi_1'(x)\phi_1'(x)dx & \int_{x_0}^{x_1} \phi_2'(x)\phi_1'(x)dx & \int_{x_0}^{x_1} \phi_3'(x)\phi_1'(x)dx \\ \int_{x_0}^{x_1} \phi_1'(x)\phi_2'(x)dx & \int_{x_0}^{x_1} \phi_2'(x)\phi_2'(x)dx & \int_{x_0}^{x_1} \phi_3'(x)\phi_2'(x)dx \\ \int_{x_0}^{x_1} \phi_1'(x)\phi_3'(x)dx & \int_{x_0}^{x_1} \phi_2'(x)\phi_3'(x)dx & \int_{x_0}^{x_1} \phi_3'(x)\phi_3'(x)dx \end{bmatrix} = \begin{bmatrix} \frac{1}{h} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On element $e_1 = [x_1, x_2]$ the global stiffness matrix is

$$\begin{bmatrix} \int_{x_1}^{x_2} \phi_1'(x)\phi_1'(x)dx & \int_{x_1}^{x_2} \phi_2'(x)\phi_1'(x)dx & \int_{x_1}^{x_2} \phi_3'(x)\phi_1'(x)dx \\ \int_{x_1}^{x_2} \phi_1'(x)\phi_2'(x)dx & \int_{x_1}^{x_2} \phi_2'(x)\phi_2'(x)dx & \int_{x_1}^{x_2} \phi_3'(x)\phi_2'(x)dx \\ \int_{x_1}^{x_2} \phi_1'(x)\phi_3'(x)dx & \int_{x_1}^{x_2} \phi_2'(x)\phi_3'(x)dx & \int_{x_1}^{x_2} \phi_3'(x)\phi_3'(x)dx \end{bmatrix} = \begin{bmatrix} \frac{1}{h} & \frac{-1}{h} & 0 \\ \frac{-1}{h} & \frac{1}{h} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

similarly, on element $e_2 = [x_2, x_3]$ the global stiffness matrix is

$$\begin{bmatrix} \int_{x_2}^{x_3} \phi_1'(x)\phi_1'(x)dx & \int_{x_2}^{x_3} \phi_2'(x)\phi_1'(x)dx & \int_{x_2}^{x_3} \phi_3'(x)\phi_1'(x)dx \\ \int_{x_2}^{x_3} \phi_1'(x)\phi_2'(x)dx & \int_{x_2}^{x_3} \phi_2'(x)\phi_2'(x)dx & \int_{x_2}^{x_3} \phi_3'(x)\phi_2'(x)dx \\ \int_{x_2}^{x_3} \phi_1'(x)\phi_3'(x)dx & \int_{x_2}^{x_3} \phi_2'(x)\phi_3'(x)dx & \int_{x_2}^{x_3} \phi_3'(x)\phi_3'(x)dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{h} & \frac{-1}{h} \\ 0 & \frac{-1}{h} & \frac{1}{h} \end{bmatrix}$$

On element $e_3 = [x_3, x_4]$ the global stiffness matrix is

$$\begin{bmatrix} \int_{x_3}^{x_4} \phi_1'(x)\phi_1'(x)dx & \int_{x_3}^{x_4} \phi_2'(x)\phi_1'(x)dx & \int_{x_3}^{x_4} \phi_3'(x)\phi_1'(x)dx \\ \int_{x_3}^{x_4} \phi_1'(x)\phi_2'(x)dx & \int_{x_3}^{x_4} \phi_2'(x)\phi_2'(x)dx & \int_{x_3}^{x_4} \phi_3'(x)\phi_2'(x)dx \\ \int_{x_3}^{x_4} \phi_1'(x)\phi_3'(x)dx & \int_{x_3}^{x_4} \phi_2'(x)\phi_3'(x)dx & \int_{x_3}^{x_4} \phi_3'(x)\phi_3'(x)dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{h} \end{bmatrix}$$

Now, we will assemble all elements matrix to get our required global stiffness matrix, which is

$$\begin{bmatrix} \frac{2}{h} & \frac{-1}{h} & 0 \\ \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} \\ 0 & \frac{-1}{h} & \frac{2}{h} \end{bmatrix}$$

Construction of mass matrix: In general form, the local mass matrix on element $e_i = [x_i, x_{i+1}]$ is

$$\text{local mass matrix}= \begin{bmatrix} \int_{x_i}^{x_{i+1}} \phi_i(x)\phi_i(x)dx & \int_{x_i}^{x_{i+1}} \phi_i(x)\phi_{i+1}(x)dx \\ \int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\phi_i(x)dx & \int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\phi_{i+1}(x)dx \end{bmatrix}$$

In construction of stiffness matrix we discussed about refferencing $\phi_i(x)$ *to* $\hat{\phi}_i(y)$. So, we can estimate all elements of mass matrix by refference method.

$$\int_{x_i}^{x_{i+1}} \phi_i(x)\phi_i(x)dx = \int_0^1 (1-y)(1-y)h\,dy = \frac{h}{3}$$
$$\int_{x_i}^{x_{i+1}} \phi_i(x)\phi_{i+1}(x)dx = \frac{h}{6},$$
$$\int_{x_i}^{x_{i+1}} \phi_{i+1}(x)\phi_{i+1}(x)dx = \frac{h}{3}$$

Local mass matrix becomes

$$\begin{bmatrix} \frac{h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{h}{3} \end{bmatrix}$$

Denote global mass matrix is

$$\begin{bmatrix} \int_0^1 \phi_1(x)\phi_1(x)dx & \int_0^1 \phi_2(x)\phi_1(x)dx & \int_0^1 \phi_3(x)\phi_1(x)dx \\ \int_0^1 \phi_1(x)\phi_2(x)dx & \int_0^1 \phi_2(x)\phi_2(x)dx & \int_0^1 \phi_3(x)\phi_2(x)dx \\ \int_0^1 \phi_1(x)\phi_3(x)dx & \int_0^1 \phi_2(x)\phi_3(x)dx & \int_0^1 \phi_3(x)\phi_3(x)dx \end{bmatrix}$$

On element $e_0 = [x_0, x_1]$ the global mass matrix is

$$\begin{bmatrix} \int_{x_0}^{x_1} \phi_1(x)\phi_1(x)dx & \int_{x_0}^{x_1} \phi_2(x)\phi_1(x)dx & \int_{x_0}^{x_1} \phi_3(x)\phi_1(x)dx \\ \int_{x_0}^{x_1} \phi_1(x)\phi_2(x)dx & \int_{x_0}^{x_1} \phi_2(x)\phi_2(x)dx & \int_{x_0}^{x_1} \phi_3(x)\phi_2(x)dx \\ \int_{x_0}^{x_1} \phi_1(x)\phi_3(x)dx & \int_{x_0}^{x_1} \phi_2(x)\phi_3(x)dx & \int_{x_0}^{x_1} \phi_3(x)\phi_3(x)dx \end{bmatrix} = \begin{bmatrix} \frac{h}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On element $e_1 = [x_1, x_2]$ the global mass matrix is

$$\begin{bmatrix} \int_{x_1}^{x_2} \phi_1(x)\phi_1(x)dx & \int_{x_1}^{x_2} \phi_2(x)\phi_1(x)dx & \int_{x_1}^{x_2} \phi_3(x)\phi_1(x)dx \\ \int_{x_1}^{x_2} \phi_1(x)\phi_2(x)dx & \int_{x_1}^{x_2} \phi_2(x)\phi_2(x)dx & \int_{x_1}^{x_2} \phi_3(x)\phi_2(x)dx \\ \int_{x_1}^{x_2} \phi_1(x)\phi_3(x)dx & \int_{x_1}^{x_2} \phi_2(x)\phi_3(x)dx & \int_{x_1}^{x_2} \phi_3(x)\phi_3(x)dx \end{bmatrix} = \begin{bmatrix} \frac{h}{3} & \frac{h}{6} & 0 \\ \frac{h}{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

similarly, on element $e_2 = [x_2, x_3]$ the global mass matrix is

$$\begin{bmatrix} \int_{x_2}^{x_3} \phi_1(x)\phi_1(x)dx & \int_{x_2}^{x_3} \phi_2(x)\phi_1(x)dx & \int_{x_2}^{x_3} \phi_3(x)\phi_1(x)dx \\ \int_{x_2}^{x_3} \phi_1(x)\phi_2(x)dx & \int_{x_2}^{x_3} \phi_2(x)\phi_2(x)dx & \int_{x_2}^{x_3} \phi_3(x)\phi_2(x)dx \\ \int_{x_2}^{x_3} \phi_1(x)\phi_3(x)dx & \int_{x_2}^{x_3} \phi_2(x)\phi_3(x)dx & \int_{x_2}^{x_3} \phi_3(x)\phi_3(x)dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h}{3} & \frac{h}{6} \\ 0 & \frac{h}{6} & \frac{h}{3} \end{bmatrix}$$

On element $e_3 = [x_3, x_4]$ the global mass matrix is

$$\begin{bmatrix} \int_{x_3}^{x_4} \phi_1(x)\phi_1(x)dx & \int_{x_3}^{x_4} \phi_2(x)\phi_1(x)dx & \int_{x_3}^{x_4} \phi_3(x)\phi_1(x)dx \\ \int_{x_3}^{x_4} \phi_1(x)\phi_2(x)dx & \int_{x_3}^{x_4} \phi_2(x)\phi_2(x)dx & \int_{x_3}^{x_4} \phi_3(x)\phi_2(x)dx \\ \int_{x_3}^{x_4} \phi_1(x)\phi_3(x)dx & \int_{x_3}^{x_4} \phi_2(x)\phi_3(x)dx & \int_{x_3}^{x_4} \phi_3(x)\phi_3(x)dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{h}{3} \end{bmatrix}$$

Now, assembling all elements matrix to get our required global mass matrix, which is

$$\begin{bmatrix} \frac{2h}{3} & \frac{h}{6} & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} \\ 0 & \frac{h}{6} & \frac{2h}{3} \end{bmatrix}$$

Construction of right vector: The right hand vector is denoted by

$$\vec{F} = \begin{bmatrix} \int_0^1 f(x)\ \phi_1(x)\ dx \\ \int_0^1 f(x)\ \phi_2(x)\ dx \\ \int_0^1 f(x)\ \phi_3(x)\ dx \end{bmatrix}$$

We will estimate all elements by midpoint method to get the right hand vector.

By midpoint method, we can define

$$x_{mid} = \frac{x_i + x_{i+1}}{2}$$

So, we get

$$\int_0^1 f(x)\ \phi_1(x)dx = \Big[\int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4}\Big] f(x)\ \phi_1(x)dx$$

$$= \Big[\int_{x_0}^{x_1} + \int_{x_1}^{x_2}\Big] f(x)\ \phi_1(x)dx$$

$$\approx \Big[f(x_{mid}\phi_1(x_{mid}) \int_{x_0}^{x_1}\ dx + f(x_{mid}\phi_1(x_{mid})\ \int_{x_1}^{x_2}\ dx\Big]$$

$$= \frac{h}{2} f(x_{mid}) + \frac{h}{2} f(x_{mid})$$

$$= h\ f(x_{mid})$$

$$Similarly,\ \int_0^1\ f(x)\ \phi_2(x)\ dx = h\ f(x_{mid})$$

$$and\ \ \int_0^1\ f(x)\ \phi_3(x)\ dx = h\ f(x_{mid}).$$

Construction of initial vector : Denote the initial vector is

$$\Big(u_h(t_0),\ \phi_l\Big) = \begin{bmatrix} (u_h(t_0), \phi_1) \\ (u_h(t_0), \phi_2) \\ (u_h(t_0), \phi_3) \end{bmatrix}$$

Let $u_h(t_0) = \sum_{j=1}^{3} \alpha_j\ \phi_j$,which implies

$$(u_h(t_0),\ \phi_l) = \Big(\sum_{j=1}^{3} \alpha_j\ \phi_j,\ \phi_l\Big) = \sum_{j=1}^{3} \alpha_j(t_0)\ (\phi_j,\ \phi_l)$$

In matrix form

$$\begin{bmatrix} (u_h(t_0), \phi_1) \\ (u_h(t_0), \phi_2) \\ (u_h(t_0), \phi_3) \end{bmatrix} = \Big[(\phi_j,\ \phi_l)\Big] \begin{bmatrix} \alpha_1(t_0) \\ \alpha_2(t_0) \\ \alpha_3(t_0) \end{bmatrix}$$

$$or, \Big(u_h(t_0), \phi_l\Big)_{l=1}^{3} = mass *\ \vec{\alpha}(t_0)$$

Error estimate : First we rewrite equation (4.23) and (4.24) for

$$u_h(t) = \sum_{j=1}^{M-1} \alpha_j(t)\phi_j(t), \quad u_h(t) \in S_h$$

as

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \;\; \forall \, v \in H_0^1 \tag{4.27}$$

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \;\; \forall \, \chi \in S_h \tag{4.28}$$

Now we have the error estimates [34].

Theorem 4.2.1. *Let u_h and u be the solutions of (4.28) and (4.27), respectively. Then*

$$||u_h(t) - u(t)||_{L_2} \le ||u_{h,0} - u_0||_{L_2} + Ch^2\Big(||u_0||_{H^2} + \int_0^t ||u_t(s)||_{H^2} ds\Big) \tag{4.29}$$

To prove this error estimates we need to introduce so called elliptic or Ritz projection R_h on S_h.

Ritz projection: Let $v \in H_0^1$, we defined $R_h : H_0^1 \to S_h$ by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall \chi \in S_h, \; v \in H_0^1 \tag{4.30}$$

It is easy to see that $R_h : H_0^1 \to S_h$ is well define. In fact, let $v \in H_0^1$. Assume that,

$$R_h v = \sum_{j=M-1}^{5} \alpha_j \phi_j$$

Then we have by (4.30), with $\chi = \phi_l$,

$$\sum_{j=1}^{M-1} \alpha_j (\nabla \phi_j, \nabla \phi_l) = (\nabla v, \nabla \phi_l), \quad l = 1, 2, \ldots M-1.$$

or, in the matrix form

$$\text{stiffness} * \vec{\alpha} = \vec{F}$$

where stiffness and $\vec{\alpha}$ are defined and

$$\vec{F} = \begin{bmatrix} (\nabla v, \nabla \phi_1) \\ (\nabla v, \nabla \phi_2) \\ (\nabla v, \nabla \phi_3) \\ \dots \\ (\nabla v, \nabla \phi_{M-1}) \end{bmatrix}$$

It is easy to check that stiffness matrix is positive definite, therefore stiffness^{-1} exists. We therefore obtain

$$\vec{\alpha} = \text{stiffness}^{-1} * \vec{F}$$

Remark: The Ritz projection $R_h v \in S_h$ is the finite element approximation of v, where v is the solution of the elliptic problem

$$(\nabla v, \nabla \chi) = (f, \chi), \quad \chi \in S_h,$$

for some $f \in L_2$. Hence we have

Lemma 4.2.1. *Ritz projections: Let $v \in H^2 \cap H_0^1$. Let $R_h : H_0^1 \to S_h$ be the Ritz projections. We have*

$$||R_h v - v||_{L_2} + h||\nabla(R_h v - v)||_{L_2} \leq Ch^2 ||v||_{H^2}, \quad v \in H^2 \cap H_0^1$$

Proof of Theorem 4.2.1: Step 1: We write

$$u_h(t) - u(t) = \theta(t) + \rho(t), \text{ where } \theta(t) = u_h(t) - R_h u(t), \ \rho(t) = R_h u(t) - u(t)$$

By Lemma 4.2.1, we have

$$||\rho(t)||_{L_2} = ||R_h u(t) - u(t)||_{L_2} \leq Ch^2 ||u(t)||_{H^2}$$

Note that

$$u(t) = u(0) + \int_0^t u_t(s) ds$$

We get

$$||u(t)||_{H^2} \leq ||u_0||_{H^2} + \int_0^t ||u_t(s)||_{H^2} \, ds$$

Hence

$$||\rho(t)||_{L_2} \leq Ch^2 \Big(||u_0||_{H^2} + \int_0^t ||u_t(s)||_{H^2} \, ds \Big)$$

Step 2: Estimate $\theta(t) = u_h(t) - R_h u(t)$.

$\theta(t)$ satisfies the equations

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = (u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) - (R_h u_t, \chi) - (\nabla R_h u_t, \nabla\chi)$$

$$= (f, \chi) - (R_h u_t, \chi) - (\nabla u, \nabla\chi)$$

$$= (u_t - R_h u_t, \chi)$$

$$= (-\rho_t, \chi)$$

Hence θ satisfies

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = -(\rho_t, \chi), \ \forall\chi \in S_h$$

Choose, $\chi = \theta(t)$, we get

$$(\theta_t, \theta) + (\nabla\theta, \nabla\theta) = -(\rho_t, \theta)$$

$$\frac{1}{2}\frac{d}{dt}||\theta||_{L_2}^2 + ||\nabla\theta||_{L_2}^2 = -(\rho_t, \theta) \leq ||\rho(t)||_{L_2}.||\theta||_{L_2}$$

Note that $||\nabla\theta||_{L_2}^2 > 0$, we get

$$\frac{1}{2}\frac{d}{dt}||\theta||_{L_2}^2 \leq ||\rho(t)||_{L_2}.||\theta||_{L_2}$$

$$\Rightarrow ||\theta||_{L_2}\frac{d}{dt}||\theta||_{L_2} \leq |\rho_t||_{L_2}.||\theta||_{L_2}$$

$$\Rightarrow \frac{d}{dt}||\theta||_{L_2} \leq |\rho_t||_{L_2}$$

By integration on [0,t]

$$\Rightarrow ||\theta(t)||_{L_2} - ||\theta(0)||_{L_2} \leq \int_0^t |\rho_t(s)||_{L_2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} - ||\theta(0)||_{L_2} \leq \int_0^t Ch^2|u_t(s)||_{L_2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||\theta(0)||_{L_2} + \int_0^t Ch^2|u_t(s)||_{H^2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - R_h u(0)||_{L_2} + \int_0^t Ch^2|u_t(s)||_{H^2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - u(0)||_{L_2} + ||u(0) - R_h u(0)||_{L_2} + \int_0^t Ch^2 |u_t(s)||_{H^2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - u(0)||_{L_2} + Ch^2 ||u(0)||_{H^2} + \int_0^t Ch^2 |u_t(s)||_{H^2} ds$$

Thus, combining step 1 and step 2, we get

$$||u_h(t) - u(t)||_{L_2} \leq ||\rho(t)||_{L_2} + ||\theta(t)|_{L_2} \leq ||u_{h,0} - u_0||_{L_2} + Ch^2 \Big(||u_0||_{H^2} + \int_0^t ||u_t(s)||_{H^2} ds \Big)$$

4.3 Finite element method for solving space-fractional PDEs

In this section, we will consider finite element method to solve the space-fractional PDEs. Consider, with $\frac{1}{2} < \alpha < 1$

$$u_t - \frac{d^{2\alpha} u}{d|x|^{2\alpha}} = f, \qquad 0 < x < 1, \quad t > 0 \tag{4.31}$$

$$u(t,0) = u(t,1) = 0 \tag{4.32}$$

$$u(0,x) = u_0(x) = 0 \tag{4.33}$$

where,

$$-\frac{d^{2\alpha} u}{d|x|^{2\alpha}} = (-\Delta)^{\alpha} u = \frac{1}{2\, cos(\alpha\pi)} \Big[{}_0^R D_x^{2\alpha} u + {}_x^R D_1^{2\alpha} u \Big]$$

Here, $\Delta = \dfrac{\partial^2}{\partial {x_1}^2} + \dfrac{\partial^2}{\partial {x_2}^2} + \dfrac{\partial^2}{\partial {x_3}^2}$ with respect to the x denotes the Laplacian operator, ${}_0^R D_x^{\alpha} u(t,x)$ denotes the left Riemann- Liouville fractional derivative with respect to x defined by ${}_0^R D_x^{\alpha} u(t,x) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{d^2}{dx^2} \displaystyle\int_0^x (x-y)^{1-\alpha}\ u(y)\ dy, 1 < \alpha < 2$ and ${}_x^R D_1^{\alpha} u(t,x)$ denotes the right Riemann- Liouville fractional derivative with respect to x defined by ${}_x^R D_1^{\alpha} u(t,x) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{d^2}{dx^2} \displaystyle\int_0^x (y-x)^{1-\alpha}\ u(y)\ dy,\ 1 < \alpha < 2.$ Where Γ denotes the gamma function.

Preparation : Before solving equation (4.31)-(4.33) we will recall some useful definitions and properties. There are two ways to define fractional Sobolev space on $\mathbb{R}^1$ and then confine them on bounded domain.[9] Another way is to define the fractional sobolev spaces on bounded domain by the Riemann -Liouville fractional derivatives directly. This way enables us to deduce some useful results for bounded domain [28].

Definition 4.3.1. *We define* $C_0^\infty(\Omega) = \{u \in C^\infty(\Omega),$ *u has compact support in* Ω *i.e.* $supp(u) = \{x : u(x) \neq 0\} \subset \Omega\}$. $C_0^\infty(\Omega)$ *denotes the space of infinitely differentiable compactly supported functions.*

Definition 4.3.2. *Let* $\alpha > 0$, *we define*

$$\begin{aligned} H_0^\alpha(\Omega) =& \{\textit{the restriction on } \Omega \textit{ of the closure of } C_0^\infty(\Omega) \textit{ in } H^\alpha(\Re)\} \\ &= \{u \in L_2(\Omega) : \tilde{u}\Big|_\Omega = u, \text{ there exist } u_n \in C_0^\infty(\Omega) \subset C_0^\infty(\Re) \\ &\text{such that } ||u_n - \tilde{u}||_{H^\alpha(\Re)} \longrightarrow 0, \tilde{u} \in H^\alpha(\Re). \end{aligned}$$

Since $u_n \in C_0^\infty(\Omega)$ and $||u_n - \tilde{u}||_{H^\alpha(\Re)} \longrightarrow 0$, we see that $\tilde{u} = 0$ *on* $x \notin \Omega$.

Definition 4.3.3. *For any* $\alpha \geq 0$, *we define*

${}^lH^\alpha(0,1) = v : ||v||_{{}^lH^\alpha(0,1)} < \infty$ with
$||v||_{{}^lH^\alpha(0,1)} = \Big(||v||^2_{L_2(0,1)} + |v|^2_{{}^lH^\alpha(0,1)}\big)^{\frac{1}{2}}$ and $|v|_{{}^lH^\alpha(0,1)} = ||{}_0^RD_x^\alpha v||_{L_2(0,1)}$

Definition 4.3.4. *Let* $\alpha \geq 0$, ${}^lH_0^\alpha(0,1)$ *denotes the closure of* $C_0^\infty(0,1)$ *with respect* to the norm $||v||_{{}^lH^\alpha(0,1)}$ i.e.
${}^lH^\alpha(0,1) = \{v : \textit{there exists } \phi_n \in C_0^\infty(0,1) \textit{such that } ||\phi_n - v||_{{}^lH^\alpha(0,1)} \longrightarrow 0\}$.
Similarly, we can define ${}^rH^\alpha(0,1)$ *and* ${}^rH_0^\alpha(0,1)$

Definition 4.3.5. *Let* $\alpha > 0,\ \alpha \neq n + \frac{1}{2},\ n \geq 0,\ n \in Z^+$.

We define, ${}^cH^\alpha(0,1) = \{v : ||v||_{{}^cH^\alpha(0,1)} < \infty\}$
with ${}^cH^\alpha(0,1) = \Big(||v||^2_{L_2(0,1)} + |v|^2_{{}^cH^\alpha(0,1)}\Big)^{\frac{1}{2}}$ *and* $|v|_{{}^cH^\alpha(0,1)} = \Big|\Big({}_0^RD_x^\alpha\, v, {}_x^RD_1^\alpha\, v\Big)\Big|^{\frac{1}{2}}$
Here we have

$$\Big({}_0^RD_x^\alpha\, v, {}_x^RD_1^\alpha\, v\Big) = cos(\pi\alpha)||{}_0^RD_x^\alpha||^2, \quad \alpha \neq n + \frac{1}{2}$$

Definition 6: Let $\alpha \geq 0$, ${}^cH_0^\alpha(0,1)$ denotes the closure of $C_0^\infty(0,1)$ with respect to the norm $||v||_{{}^cH^\alpha(0,1)}$

Definition 4.3.6. *Let* $\alpha \geq 0$,*we define*
$H^\alpha(0,1) = \{v \in L^2(0,1),\ \textit{there exists } \tilde{v} \in H^\alpha(\Re),\ \textit{such that } \tilde{v}|_{(0,1)} = v\}$

with norm $||v||_{H^\alpha(0,1)} = inf||\tilde{v}||_{H^\alpha(\Re)}$.In particular, if $v \in C_0^\infty(0,1)$, then $||v||_{H^\alpha(0,1)} = ||\tilde{v}||_{H^\alpha(\Re)}$,

$$\tilde{v} = \begin{cases} v(x), & \text{for } x \in (0,1) \\ 0, & \text{for } x \notin (0,1) \end{cases}$$

Definition 4.3.7. *Let* $\alpha > 0,\ \alpha \neq n + \frac{1}{2},\ n \geq 0,\ n \in \mathbb{Z}^+$.

We define, ${}^cH^\alpha(\mathcal{D})$ *to be the closure of* $C_0^\infty(\mathcal{D})$ *with respect to the norm* $||v||_{{}^cH^\alpha(\mathcal{D})}$, *where*

$$||v||^2_{{}^cH^\alpha(\mathcal{D})} := ||v||^2_(L_2(\mathcal{D}))$$

Property 4.31: [28] If $0 < p < 1, 0 < q < 1, u(0) = 0, t > 0$ then

$${}_0D_x^{p+q}u = {}_0D_x^p{}_0D_x^q u = {}_0D_x^q{}_0D_x^p u.$$

Lemma 4.3.1. *[35]Let* $0 < \alpha < 1$ *if* $u \in H^\alpha(0,1), v \in C_0^\alpha(0,1)$ *then*

$$\Big({}_0^RD_x^\alpha u(x), v(x)\Big) = \Big(u(x), {}_x^RD_1^\alpha v(x)\Big)$$

Proof: By using integration by parts , we get

$$\begin{aligned}
\Big({}_0^RD_x^\alpha u(x), v(x)\Big) &= \int_0^1 \Big({}_0^RD_x^\alpha u(x)\Big) v(x)dx \\
&= \int_0^1 \Big(\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\tau)^{-\alpha}u(\tau)d\tau\Big)\ v'(x)dx \\
&= \frac{v(x)}{\Gamma(1-\alpha)}\int_0^x (x-\tau)^{-\alpha}u(\tau)d\tau\Big|_0^1 - \int_0^1 \Big(\frac{1}{\Gamma 1-\alpha}\int_0^x (x-\tau)^{-\alpha}u(\tau)d\tau\Big)\ v(x)dx \\
&= \Big(v \in C_0^\alpha(0,1), v(0) = v(1) = 0\Big) \\
&= \frac{-1}{\Gamma(1-\alpha)}\int_0^1 \Big(\int_0^x (x-\tau)^{-\alpha}u(\tau)d\tau\Big)\ v'(x)dx \\
&= \frac{-1}{\Gamma(1-\alpha)}\int_0^1 \Big(\int_\tau^1 (x-\tau)^{-\alpha}u(\tau)\ v(x)dx\ d\tau\Big)
\end{aligned}$$

[exchange the integration variables]

$$= \int_0^1 \frac{-1}{\Gamma(1-\alpha)}\Big(\int_\tau^1 \tau(x-\tau)^{-\alpha}\ v(x)dx\Big)u(\tau)d\tau$$

Claim that, ${}_\tau^RD_1^\alpha u(\tau) = \dfrac{-1}{\Gamma(1-\alpha)}\displaystyle\int_\tau^1 \tau(x-\tau)^{-\alpha}\ v'(x)dx$

Infact,

$$\begin{aligned}
{}_\tau^RD_1^\alpha u(\tau) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_\tau^1 \tau(x-\tau)^{-\alpha}\ v(x)dx \\
&= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{d\tau}\Big[v(x)\frac{(x-\tau)^{-\alpha+1}}{-\alpha+1}\Big|_\tau^1 - \int_\tau^1 v'(x)\frac{(x-\tau)^{-\alpha+1}}{-\alpha+1}dx\Big] \\
&= (v(x) = 0)
\end{aligned}$$

$$= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_\tau^1 v'(x) \frac{(x-\tau)^{-\alpha+1}}{-\alpha+1} dx \Big]$$
$$= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \Big[\int_\tau^1 v'(x)(x-\tau)^{-\alpha} dx + v'(x) \frac{(x-\tau)^{-\alpha+1}}{-\alpha+1} \Big|_\tau^1 \Big]$$
$$= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \Big[\int_\tau^1 v'(x)(x-\tau)^{-\alpha} dx$$

Thus we get,
$\Big({}_0^R D_x^\alpha u(x), v(x)\Big) = \Big(u(x), {}_x^R D_1^\alpha u(x)\Big)$

Lemma 4.3.2. *[19]Let* $0 < \alpha < 1$ *if* $u, v \in C_0^\alpha(0,1)$

then
$$\Big({}_0^R D_x^{2\alpha} u(x), v(x)\Big) = \Big({}_0^R D_x^\alpha u(x), {}_x^R D_1^\alpha v(x)\Big) \tag{4.34}$$
$$\Big({}_x^R D_1^{2\alpha} u(x), v(x)\Big) = \Big({}_x^R D_1^\alpha u(x), {}_0^R D_x^\alpha v(x)\Big) \tag{4.35}$$

Proof : Since $v \in H_0^\alpha$, by definition there exist $v_n \in C_0^\alpha(0,1)$ such that
$$||v_n - v||_{H_0^\alpha} \to 0 \text{ as } n \to \alpha$$

By property 4.31 and for all $u \in H_0^\alpha$ we get
$$\Big({}_0^R D_x^{2\alpha} u(x), v_n\Big) = \Big({}_0^R D_x^\alpha u(x), {}_0 D_x^\alpha u(x) v_n\Big), \quad v_n \in C_0^\alpha(0,1)$$

By lemma 4.3.1, we have
$$\Big({}_0^R D_x^\alpha {}_0^R D_x^\alpha u(x), v_n\Big) = \Big({}_0^R D_x^\alpha u(x), {}_x^R D_1^\alpha v_n\Big)$$

Thus we get
$$\Big({}_0^R D_x^{2\alpha} u(x), v_n\Big) = \Big({}_0^R D_x^\alpha u(x), {}_x^R D_1^\alpha v_n\Big)$$

Note that,
$$\lim_{n\to\infty} \Big({}_0^R D_x^\alpha u(x), v_n\Big) = \Big({}_0^R D_x^\alpha u(x), v\Big)$$

Since
$$\Big| \Big({}_0^R D_x^{2\alpha} u(x), v_n\Big) - \Big({}_0^R D_x^{2\alpha} u(x), v\Big) \Big|$$
$$= \Big| \Big({}_0^R D_x^{2\alpha} u(x), v_n - v\Big) \Big|$$
$$\leq ||{}_0^R D_x^{2\alpha} u(x)||_{L_2} . ||v_n - v||_{L_2} \to 0$$

Note that,

$$\lim_{n\to\infty}\left({}_0^RD_x^\alpha u(x), {}_x^RD_1^\alpha v_n\right) = \left({}_0^RD_x^\alpha u(x), {}_x^RD_1^\alpha v\right)$$

Since

$$\left|\left({}_0^RD_x^{2\alpha} u(x), {}_x^RD_1^\alpha v_n\right) - \left({}_0^RD_x^{2\alpha} u(x), {}_x^RD_1^\alpha v\right)\right|$$

$$= \left|\left({}_0^RD_x^{2\alpha} u(x), {}_x^RD_1^\alpha (v_n - v)\right)\right|$$

$$\leq ||{}_0^RD_x^{2\alpha} u(x)||_{L_2}.||v_n - v||_{H^\alpha} \to 0$$

Thus we get

$$\left({}_0^RD_x^{2\alpha} u(x), v(x)\right) = \left({}_0^RD_x^\alpha u(x), {}_x^RD_1^\alpha v(x)\right), \quad u, v \in H_0^\alpha(0,1)$$

Similarly, we can prove

$$\left({}_x^RD_1^{2\alpha} u(x), v(x)\right) = \left({}_x^RD_1^\alpha u(x), {}_0^RD_x^\alpha v(x)\right), \quad u, v \in H_0^\alpha(0,1)$$

Lemma 4.3.3. *Poincare inequality: For $u \in H_0^\alpha(\mathcal{D})$, $\frac{1}{2} < \alpha < 1$, we have*

$$||u||_{L^2(\mathcal{D})} \leq C|u|_{H_0^\alpha(\mathcal{D})}$$

and for $0 < s < u, s \neq n - \frac{1}{2}, n \in \mathcal{N}$,

$$|u|_{H_0^s(\mathcal{D})} \leq C|u|_{H_0^u(\mathcal{D})}$$

Lemma 4.3.4. *Let $\sigma > 0$, $\sigma \neq n - \frac{1}{2}$, $n \in \mathcal{Z}$. The spaces ${}^lH_0^\sigma(\mathcal{D}), {}^rH_0^\sigma(\mathcal{D}), {}^cH_0^\sigma(\mathcal{D}) and H_0^\sigma(\mathcal{D})$ are equal in the sense that all the norms $||\cdot||_{{}^lH_0^\sigma(\mathcal{D})}, ||\cdot||_{{}^rH_0^\sigma(\mathcal{D})}, ||\cdot||_{{}^cH_0^\sigma(\mathcal{D})}, ||\cdot||_{H_0^\sigma(\mathcal{D})}$ are equivalent.*

Finite element method : In this section we will consider how to solve the one dimensional space fractional partial differential equation by finite element method.

Consider the space fractional partial differential equation with the Reimann -Liouville type:

$$u_t + \frac{1}{2cos\ \alpha\pi}\left[{}_0^RD_x^{2\alpha} u + {}_x^RD_1^{2\alpha} u\right] = f, \qquad 0 < x < 1, \quad t > 0 \tag{4.36}$$

$$u(t,0) = u(t,1) = 0 \tag{4.37}$$

$$u(0,x) = u_0(x) = 0 \tag{4.38}$$

The variational form of (4.36) is to find $u \in H_0^\alpha(0,1)$

$$(u_t, v) + \frac{1}{2cos\ \alpha\pi}\left[\left({}_0^R D_x^{2\alpha} u, v\right) + \left({}_x^R D_1^{2\alpha} u, v\right)\right] = (f, v),\ v \in H_0^\alpha(0,1) \tag{4.39}$$

By lemma 4.3.2 we can write equation (4.39) as

$$(u_t, v) + \frac{1}{2cos\ \alpha\pi}\left[\left({}_0^R D_x^{\alpha} u, {}_x^R D_1^{\alpha} v\right) + \left({}_x^R D_1^{\alpha} u, {}_0^R D_x^{\alpha} v\right)\right] = (f, v), v \in H_0^\alpha(0,1)$$

The finite element method is to find a solution $u_h \in S_h$ such that

$$(u_{h,t}, \chi) + \frac{1}{2cos\ \alpha\pi}\left[\left({}_0^R D_x^{\alpha} u_h, {}_x^R D_1^{\alpha} \chi\right) + \left({}_x^R D_1^{\alpha} u_h, {}_0^R D_x^{\alpha} \chi\right)\right] = (f, \chi), \chi \in S_h$$

Where S_h denote the set of piecewise linear functions on [0,1] i.e.
$S_h = \{v(x) : v(x)$ is a piecewise continuous function on[0,1] and $v(0) = v(1) = 0\}$

Let $u_h(t) = \sum_{j=1}^{m-1} \alpha_j(t)\phi_j$ and choose $\chi = \phi_i$

We get,

$$\sum_{j=1}^{m-1}(\phi_j, \phi_i)\alpha_j'(t) + \frac{1}{2cos\ \alpha\pi}\sum_{j=1}^{m-1}\left[\left({}_0^R D_x^{\alpha}\phi_j, {}_x^R D_1^{\alpha}\phi_i\right) + \left({}_x^R D_1^{\alpha}\phi_j, {}_0^R D_x^{\alpha}\phi_i\right)\right]\alpha_i(t) = (f, \phi_i)$$

In matrix form

$$(\phi_j, \phi_i)_{i,j=1}^{m-1}\begin{bmatrix}\alpha_1'(t)\\ \alpha_2'(t)\\ .....\\ .....\\ \alpha_{m-1}'(t)\end{bmatrix} + \frac{1}{2cos\ \alpha\pi}\left[\left({}_0^R D_x^{\alpha}\phi_j, {}_x^R D_1^{\alpha}\phi_i\right)_{i,j=1}^{m-1}\right.$$

$$\left. + \left({}_x^R D_1^{\alpha}\phi_j, {}_0^R D_x^{\alpha}\phi_i\right)_{i,j=1}^{m-1}\right]\begin{bmatrix}\alpha_1(t)\\ \alpha_2(t)\\ .....\\ .....\\ \alpha_{m-1}(t)\end{bmatrix} = \begin{bmatrix}(f,\phi_1)\\ (f,\phi_2)\\ .....\\ .....\\ (f,\phi_{m-1})\end{bmatrix}$$

Denote, M is the mass matrix

$$M = (\phi_j, \phi_i)_{i,j=1}^{m-1}$$

and S_l and S_r are stiffness matrix such as

$$S_l = \left({}_0^R D_x^{\alpha}\phi_j, {}_x^R D_1^{\alpha}\phi_i\right)_{i,j=1}^{m-1}$$

$$S_r = \left({}_x^R D_1^\alpha \phi_j, {}_0^R D_x^\alpha \phi_i \right)_{i,j=1}^{m-1}$$

F is the vector values at the right hand side

$$F = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ ..... \\ ..... \\ (f, \phi_{m-1}) \end{bmatrix}$$

and

$$\alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ ..... \\ ..... \\ \alpha_{m-1}(t) \end{bmatrix}$$

Thus we obtain matrix equation

$$\begin{aligned} M\alpha'(t) &= \frac{1}{2cos\ \alpha\pi}(S_l + S_r)\alpha(t) + F \\ or,\ \alpha'(t) &= M^{-1}\frac{1}{2cos\ \alpha\pi}(S_l + S_r)\alpha(t) + M^{-1}F \\ &\text{and } \alpha(0) = \alpha_0 \end{aligned} \tag{4.40}$$

Construction of matrices : For construct the matrices let us consider the nodes $x_0 < x_1 < .... < x_n = 1$. Moreover, from the stiffness matrices, one can see that $S_l = S_r^T$.Here the inner product of the piecewise linear function ϕ_j is by [35]

$${}_0^R D_x^\alpha \phi_i(x) = \begin{cases} 0, & 0 < x < x_{i-1} \\ \dfrac{1}{h\Gamma(2-\alpha)}(x - x_{i-1})^{1-\alpha}, & x_{i-1} < x < x_i \\ \dfrac{1}{h\Gamma(2-\alpha)}[(x - x_{i-1})^{1-\alpha} - 2(x - x_i)^{1-\alpha}], & x_i < x < x_{i+1} \\ \dfrac{1}{h\Gamma(2-\alpha)}[(x - x_{i-1})^{1-\alpha} - 2(x - x_i)^{1-\alpha} + (x - x_{i+1})^{1-\alpha}], & x_{i+1} < x < 1 \end{cases}$$

$${}_x^R D_1^\alpha \phi_i(x) = \begin{cases} \dfrac{1}{h\Gamma(2-\alpha)}[(x_{i+1} - x)^{1-\alpha} - 2(x_i - x)^{1-\alpha} + (x_{i-1} - x)^{1-\alpha}], & 0 < x < x_{i-1} \\ \dfrac{1}{h\Gamma(2-\alpha)}[(x_{i+1} - x)^{1-\alpha} - 2(x_i - x)^{1-\alpha}], & x_{i-1} < x < x_i \\ \dfrac{1}{h\Gamma(2-\alpha)}(x_{i+1} - x)^{1-\alpha}, & x_i < x < x_{i+1} \\ 0, & x_{i+1} < x < 1 \end{cases}$$

Remark:
1. ${}^R_0D^\alpha_x\phi_i(x)$ has compact support on $[x_{i-1},1]$, i.e. ${}^R_0D^\alpha_x\phi_i(x)=0$ for $0<x<x_{i-1}$.

2. ${}^R_xD^\alpha_1\phi_i(x)$ has compact support on $[0,x_{i+1}]$, i.e. ${}^R_xD^\alpha_1\phi_i(x)=0$ for $x_{i+1}<x<1$.

Note that,

$$\Big({}^R_0D^\alpha_x\phi_l(x),{}^R_xD^\alpha_1\phi_m(x)\Big)=\Big[\int_{x_0}^{x_1}+\int_{x_1}^{x_2}+...+\int_{x_{i-1}}^{x_i}+....+\int_{x_{n-1}}^{x_n}\Big]{}^R_0D^\alpha_x\phi_l(x),{}^R_xD^\alpha_1\phi_m(x)\,dx$$

We will consider the integral element by element i.e.we first consider the integral on $[x_0,x_1]$, then $[x_1,x_2]$, then $[x_2,x_3]$.Finally we assemble them to obtain our matrix S_l and S_r.On the element $[x_{i-1},x_i]$, when $l\geq i+1$ or $m\geq i-2$ we get

$${}^R_0D^\alpha_x\phi_l(x)=0\ \ or,\ \ {}^R_xD^\alpha_1\phi_m(x)=0.$$

Hence in this case $\Big({}^R_0D^\alpha_x\phi_l(x),{}^R_xD^\alpha_1\phi_m(x)\Big)=0.$

Therefore, the element $[x_{i-1},x_i]$ will give contribution to the following elements

$$\begin{bmatrix}
\Big({}^R_0D^\alpha_x\phi_0,{}^R_xD^\alpha_1\phi_{i-1}\Big) & ... & ... & \Big({}^R_0D^\alpha_x\phi_{i-1},{}^R_xD^\alpha_1\phi_{i-1}\Big) & \Big({}^R_0D^\alpha_x\phi_i,{}^R_xD^\alpha_1\phi_{i-1}\Big) & \\
\Big({}^R_0D^\alpha_x\phi_0,{}^R_xD^\alpha_1\phi_i\Big) & .... & ... & \Big({}^R_0D^\alpha_x\phi_{i-1},{}^R_xD^\alpha_1\phi_i\Big) & \Big({}^R_0D^\alpha_x\phi_i,{}^R_xD^\alpha_1\phi_i\Big) & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
.... & .... & .... & .... & ..... & .... \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\Big({}^R_0D^\alpha_x\phi_0,{}^R_xD^\alpha_1\phi_m\Big) & ... & ... & \Big({}^R_0D^\alpha_x\phi_{i-1},{}^R_xD^\alpha_1\phi_m\Big) & \Big({}^R_0D^\alpha_x\phi_i,{}^R_xD^\alpha_1\phi_m\Big) &
\end{bmatrix}$$

Now let us find the contribution on $[x_{i-1},x_i]$ on (i-1)th row, we have

$$\begin{aligned}
\Big({}^R_0D^\alpha_x\phi_i,{}^R_xD^\alpha_1\phi_{i-1}\Big) &= \int_{x_{i-1}}^{x_i}\Big({}^R_0D^\alpha_x\phi_i\Big).\Big({}^R_xD^\alpha_1\phi_{i-1}\Big)dx\\
&= \int_{x_{i-1}}^{x_i}\frac{1}{h\Gamma(2-\alpha)}(x-x_{i-1})^{1-\alpha}\frac{1}{h\Gamma(2-\alpha)}(x_i-x)^{1-\alpha}dx\\
&= \Big[\frac{1}{h\Gamma(2-\alpha)}\Big]^2\int_{x_{i-1}}^{x_i}(x-x_{i-1})^{1-\alpha}(x_i-x)^{1-\alpha}dx\\
&= \Big(t=\frac{(x-x_{i-1})}{(x_i-x_{i-1})},\ (x-x_{i-1})=th,\ (x_i-x)=(1-t)h\Big)
\end{aligned}$$

$$= \Big[\frac{1}{h\Gamma(2-\alpha)}\Big]^2 \int_0^1 h^{2-2\alpha}\, t^{1-\alpha}(1-t)^{1-\alpha} h\; dt$$

$$= \frac{h^{1-2\alpha}}{(\Gamma(2-\alpha))^2} \int_0^1 t^{1-\alpha}(1-t)^{1-\alpha} dt$$

$$= L \int_0^1 t^{1-\alpha}(1-t)^{1-\alpha} dt, \text{ where } L = \frac{h^{1-2\alpha}}{(\Gamma(2-\alpha))^2}$$

Similarly, we have

$$\Big({}_0^R D_x^\alpha \phi_{i-1}, {}_x^R D_1^\alpha \phi_{i-1}\Big) = \int_{x_{i-1}}^{x_i} \Big({}_0^R D_x^\alpha \phi_{i-1}\Big).\Big({}_x^R D_1^\alpha \phi_{i-1}\Big) dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{1}{(h\Gamma(2-\alpha))^2}[(x-x_{i-2})^{1-\alpha} - 2(x-x_i)^{1-\alpha}](x_i - x)^{1-\alpha} dx$$

$$= L \int_0^1 [(1+t)^{1-\alpha} - 2t^{1-\alpha}](1-t)^{1-\alpha} dt, \text{ where } L = \frac{h^{1-2\alpha}}{(\Gamma(2-\alpha))^2}$$

$$\Big({}_0^R D_x^\alpha \phi_{i-2}, {}_x^R D_1^\alpha \phi_{i-1}\Big) = \int_{x_{i-1}}^{x_i} \Big({}_0^R D_x^\alpha \phi_{i-2}\Big).\Big({}_x^R D_1^\alpha \phi_{i-1}\Big) dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{1}{(h\Gamma(2-\alpha))^2}[(x-x_{i-3})^{1-\alpha} - 2(x-x_{i-2})^{1-\alpha} +$$

$$(x-x_{i-1})^{1-\alpha}](x_i - x)^{1-\alpha} dx$$

$$= L \int_0^1 [(2+t)^{1-\alpha} - 2(1+t)^{1-\alpha} + t^{1-\alpha}](1-t)^{1-\alpha} dt,$$

where $L = \frac{h^{1-2\alpha}}{(\Gamma(2-\alpha))^2}$

$$\Big({}_0^R D_x^\alpha \phi_{i-3}, {}_x^R D_1^\alpha \phi_{i-1}\Big) = \int_{x_{i-3}}^{x_i} \Big({}_0^R D_x^\alpha \phi_{i-1}\Big).\Big({}_x^R D_1^\alpha \phi_{i-1}\Big) dx$$

$$= L \int_0^1 [(3+t)^{1-\alpha} - 2(2+t)^{1-\alpha} + (1+t)^{1-\alpha}](1-t)^{1-\alpha} dt,$$

where $L = \frac{h^{1-2\alpha}}{(\Gamma(2-\alpha))^2}$

$$\Big({}_0^R D_x^\alpha \phi_0, {}_x^R D_1^\alpha \phi_{i-1}\Big) = \int_{x_0}^{x_i} \Big({}_0^R D_x^\alpha \phi_{i-1}\Big).\Big({}_x^R D_1^\alpha \phi_{i-1}\Big) dx$$

$$= L \int_0^1 [(i+t)^{1-\alpha} - 2(i-1+t)^{1-\alpha} + (i-2+t)^{1-\alpha}](1-t)^{1-\alpha} dt,$$

Now let us find the contribution on $[x_{i-1}, x_i]$ on ith row, we have

$$\left({}_0^RD_x^\alpha\phi_i, {}_x^RD_1^\alpha\phi_i\right) = \int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_i\right).\left({}_x^RD_1^\alpha\phi_i\right)dx$$

$$= \int_{x_{i-1}}^{x_i}\frac{1}{(h\Gamma(2-\alpha))^2}(x-x_{i-1})^{1-\alpha}[(x_{i+1}-x)^{1-\alpha}-2(x_i-x)^{1-\alpha}]dx$$

$$= L\int_0^1 t^{1-\alpha}\,[(2-t)^{1-\alpha}-2(1-t)^{1-\alpha}]dt,$$

$$\left({}_0^RD_x^\alpha\phi_{i-1}, {}_x^RD_1^\alpha\phi_i\right) = \int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_{i-1}\right).\left({}_x^RD_1^\alpha\phi_i\right)dx$$

$$= \int_{x_{i-1}}^{x_i}\frac{1}{(h\Gamma(2-\alpha))^2}[(x-x_{i-2})^{1-\alpha}-2(x-x_{i-1})^{1-\alpha}]$$

$$[(x_{i+1}-x)^{1-\alpha}-2(x_i-x)^{1-\alpha}]dx$$

$$= L\int_0^1 [(1+t)^{1-\alpha}-2t^{1-\alpha}]\,[(2-t)^{1-\alpha}-2(1-t)^{1-\alpha}]dt,$$

$$\left({}_0^RD_x^\alpha\phi_{i-2}, {}_x^RD_1^\alpha\phi_i\right) = \int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_{i-2}\right).\left({}_x^RD_1^\alpha\phi_i\right)dx$$

$$= \int_{x_{i-1}}^{x_i}\frac{1}{(h\Gamma(2-\alpha))^2}[(x-x_{i-3})^{1-\alpha}-2(x-x_{i-2})^{1-\alpha}+$$

$$(x-x_{i-1})^{1-\alpha}][(x_{i+1}-x)^{1-\alpha}-2(x_i-x)^{1-\alpha}]dx$$

$$= L\int_0^1 [(2+t)^{1-\alpha}-2(1+t)^{1-\alpha}+t^{1-\alpha}]\,[(2-t)^{1-\alpha}-2(1-t)^{1-\alpha}]dt$$

$$\left({}_0^RD_x^\alpha\phi_0, {}_x^RD_1^\alpha\phi_i\right) = \int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_0\right).\left({}_x^RD_1^\alpha\phi_i\right)dx$$

$$= L\int_0^1 [(i+t)^{1-\alpha}-2(i-1+t)^{1-\alpha}+(i-2+t)^{1-\alpha}]$$

$$[(2-t)^{1-\alpha}-2(1-t)^{1-\alpha}]dt$$

Now let us find the contribution on $[x_{i-1}, x_i]$ on (i+1)th row, we have

$$\left({}_0^RD_x^\alpha\phi_i, {}_x^RD_1^\alpha\phi_{i+1}\right) = \int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_i\right).\left({}_x^RD_1^\alpha\phi_{i+1}\right)dx$$

$$= L\int_0^1 t^{1-\alpha}\,[(3-t)^{1-\alpha}-2(2-t)^{1-\alpha}+(1-t)^{1-\alpha}]dt,$$

$$\left({}_0^RD_x^\alpha\phi_{i-1},{}_x^RD_1^\alpha\phi_{i+1}\right)=\int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_{i-1}\right).\left({}_x^RD_1^\alpha\phi_{i+1}\right)dx$$

$$= L\int_0^1 [(t+1)^{1-\alpha}-2t^{1-\alpha}]\,[(3-t)^{1-\alpha}-2(2-t)^{1-\alpha}+(1-t)^{1-\alpha}]dt,$$

$$\left({}_0^RD_x^\alpha\phi_0,{}_x^RD_1^\alpha\phi_{i+1}\right)=\int_{x_{i-1}}^{x_i}\left({}_0^RD_x^\alpha\phi_0\right).\left({}_x^RD_1^\alpha\phi_{i+1}\right)dx$$

$$= L\int_0^1 [(t+i)^{1-\alpha}-2(t+i-1)^{1-\alpha}-(t+i-2)^{1-\alpha}]$$

$$[(3-t)^{1-\alpha}-2(2-t)^{1-\alpha}+(1-t)^{1-\alpha}]dt,$$

Now let us find the contribution on $[x_{i-1},x_i]$ on nth row, we have
$\left({}_0^RD_x^\alpha\phi_i,{}_x^RD_1^\alpha\phi_n\right)=\left({}_0^RD_x^\alpha\phi_i,{}_x^RD_1^\alpha\phi_{i+n-1}\right)$

$$= L\int_0^1 [t^{1-\alpha}]\,[(n+2-i-t)^{1-\alpha}-2(n+1-i-t)^{1-\alpha}+(n-i-t)^{1-\alpha}]dt$$

$$\left({}_0^RD_x^\alpha\phi_0,{}_x^RD_1^\alpha\phi_n\right)=\left({}_0^RD_x^\alpha\phi_0,{}_x^RD_1^\alpha\phi_{i+n-1}\right)$$

$$= L\int_0^1 [(t+i)^{1-\alpha}-2(t+i-1)^{1-\alpha}-(t+i-2)^{1-\alpha}]$$

$$[(n+2-i-t)^{1-\alpha}-2(n+1-i-t)^{1-\alpha}+(n-i-t)^{1-\alpha}]\,dt,$$

Error estimate : We can rewrite the variational form of equation (4.31) for $v\in H_0^\alpha(\mathcal{D})$.

$$(u_t,v)+\frac{1}{2cos\ \alpha\pi}\left[\left({}_0^RD_x^\alpha u,{}_x^RD_1^\alpha v\right)+\left({}_x^RD_1^\alpha u,{}_0^RD_x^\alpha v\right)\right]=(f,v),\quad v\in H_0^\alpha(\mathcal{D})\quad(4.41)$$

Denote, $B_\alpha(u,v)=\frac{1}{2cos\ \alpha\pi}\left[\left({}_0^RD_x^\alpha u,{}_x^RD_1^\alpha v\right)+\left({}_x^RD_1^\alpha u,{}_0^RD_x^\alpha v\right)\right]$
Then the finite element method of (4.39) to find $u_h(t)\in S_h$ such that

$$(u_{h,t},\chi)+B_\alpha(u_h,\chi)=(f,\chi),\quad\forall\chi\in S_h\quad(4.42)$$
$$u_h(0)=u_{h,0}$$

Assume that $u_h(t) \in S_h$ has the form $u_h(t) = \sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x)$,

In this case we get the matrix equation (4.40) as

$$or, \ \vec{\alpha'}(t) + \vec{A}_\alpha * \vec{\alpha}(t) = \vec{F}(t) \tag{4.43}$$

where $\vec{A}_\alpha = mass^{-1} *$ stiff-alpha

Denoted that, M is the mass matrix

$$M = (\phi_j, \phi_i)_{i,j=1}^{M-1}$$

and stiff-alpha matrix $=[B_\alpha(\phi_j, \phi_l)]_{l,j=1}^{M-1}$

F is the vector values at the right hand side

$$F = \begin{bmatrix} (f(t), \phi_1) \\ (f(t), \phi_2) \\ \ldots\ldots. \\ \ldots\ldots. \\ (f(t), \phi_{M-1}) \end{bmatrix}$$

and

$$\alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \ldots.. \\ \ldots.. \\ \alpha_{M-1}(t) \end{bmatrix}$$

Now we have the following error estimates.

Theorem 4.3.1. *Let u_h and u be the solutions of (4.41) and (4.39), respectively. Then*

$$||u_h(t) - u(t)||_{L_2} \leq ||u_{h,0} - u_0||_{L_2} + Ch^{2\alpha}\Big(||u_0||_{H^{2\alpha}} + \int_0^t ||u_t(s)||_{H^{2\alpha}} ds\Big) \ \ \frac{1}{2} < \alpha < 1 \tag{4.44}$$

To prove this error estimates we need to introduce so called elliptic or Ritz projection R_h on S_h.

Ritz projection: Let $v \in H_0^\alpha$, $\frac{1}{2} < \alpha < 1$, we define $R_{\alpha h} : H_0^\alpha \to S_h$ by

$$B_\alpha(R_{\alpha h} v, \chi) = B_\alpha(v, \chi), \ \ \forall \chi \in S_h, \ v \in H_0^\alpha \tag{4.45}$$

It is easy to see that $R_{\alpha h}: H_0^\alpha \to S_h$ is well defined. In fact, let $v \in H_0^\alpha$. Assume that $R_{\alpha h} v = \sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x)$

Thus we have by (4.45), with $\chi = \phi_l$,

$\sum_{j=1}^{M-1} \alpha_j [B_\alpha(\phi_j, \phi_l)] = B_\alpha(v, \phi_l), \quad \phi_l \in S_h, \ l = 1, 2, \dots M-1.$

or, in the matrix form,

$$\text{stiff-alpha} * \vec{\alpha} = \vec{F} \tag{4.46}$$

where stiff-alpha and $\vec{\alpha}$ are defined as above. It is easy to check that stiffness alpha is positive definite. In fact, we have,

$$\begin{aligned}
\vec{\alpha}^T * \text{stiff-alpha} * \vec{\alpha} &= B_\alpha\Big(\sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x) \sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x)\Big) \\
&\geq C||{}_0^R D_x^\alpha \phi_i(\sum_{j=1}^{M-1} \alpha_j \phi_j)||^2 \geq 0
\end{aligned}$$

where we use the fact that the norms in ${}^l H_0^\alpha(\mathcal{D})$ and ${}^c H_0^\alpha(\mathcal{D})$ are equivalent. Further, by using the poincare inequality, Lemma 4.3.4, assume that

$B_\alpha\Big(\sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x) \sum_{j=1}^{M-1} \alpha_j(t)\phi_j(x)\Big) = 0,$

we have

$$||\sum_{j=1}^{M-1} \alpha_j \phi_j||_{L_2}^2 \leq ||{}_0^R D_x^\alpha\Big(\sum_{j=1}^{M-1} \alpha_j \phi_j\Big)||_{L_2}^2 \leq C\Big({}_0^R D_x^\alpha\Big(\sum_{j=1}^{M-1} \alpha_j \phi_j\Big), {}_x^R D_1^\alpha\Big(\sum_{j=1}^{M-1} \alpha_j \phi_j\Big)\Big) = 0$$

which implies that $\sum_{j=1}^{M-1} \alpha_j \phi_j = 0$ and therefore $\alpha_j = 0, \ j = 1, 2, \dots M-1$.

Hence stiffness alpha is positive definite.Thus we get the unique solution of (4.46).

Lemma 4.3.5. *(Interpolation error)Let $v \in H^{2\alpha}(0,1) \cap H_0^\alpha(0,1)$. Let $I_h : H^{2\alpha}(0,1) \cap H_0^\alpha(0,1) \to S_h$ be the interpolation operator. We have*

$$||I_h v - v||_{L_2} \leq Ch^r ||v||_{H^r}, \ r \leq 2\alpha$$

$$||I_h v - v||_{H_0^\alpha} \leq Ch^{r-\alpha} ||v||_{H^r}, \ r \leq 2\alpha$$

Lemma 4.3.6. *(Ritz projection) Let $v \in H^{2\alpha} \cap H_0^\alpha$. Let $R_{\alpha h} : H_0^\alpha \to S_h$ be the Ritz projection, we have*

$$||R_{\alpha h}v - v||_{L_2} + h^\alpha ||R_{\alpha h}v - v||_{H^\alpha} \leq Ch^r ||v||_{H^r}, \ \ \alpha \leq r \leq 2\alpha \tag{4.47}$$

Note that $R_{\alpha h}v$ is the finite element solution of the elliptic problem with exact solution v.

Proof : We first prove

$$||R_h v - v||_{H^\alpha} \leq Ch^{r-\alpha} ||v||_{H^r}, \ \ \alpha \leq r \leq 2\alpha \tag{4.48}$$

By definition, we have $B_\alpha(v - R_{\alpha h}v, \chi) = 0, \ \ \forall \chi \in S_h$.

Thus $B_\alpha(v - R_{\alpha h}v, v - R_{\alpha h}v) = B_\alpha(v - R_{\alpha h}v, v - \chi + \chi - R_{\alpha h}v)$

$$= B_\alpha(v - R_{\alpha h}v, v - \chi)$$

$$= ||v - R_{\alpha h}v||_{H_0^\alpha(\mathcal{D})} ||v - \chi||_{H_0^\alpha(\mathcal{D})}$$

Note that $B_\alpha(v - R_{\alpha h}v, v - R_{\alpha h}v) \geq ||v - R_{\alpha h}v||^2_{H_0^\alpha(\mathcal{D})}$

Thus we get $||v - R_{\alpha h}v||^2_{H_0^\alpha(\mathcal{D})} \leq ||v - \chi||_{H_0^\alpha(\mathcal{D})}, \ \ \chi \in H_0^\alpha(\mathcal{D})$

i.e. $||v - R_{\alpha h}v||^2_{H_0^\alpha(\mathcal{D})} \leq \inf_{\chi \in S_h} ||v - \chi||_{H_0^\alpha(\mathcal{D})} \leq ||v - I_h v||^2_{H_0^\alpha(\mathcal{D})}$

By using interpolation error, Lemma 4.3.5 we get

$||v - R_{\alpha h}v||^2_{H_0^\alpha(\mathcal{D})} \leq Ch^{r-\alpha} ||v||_{H^r}, \ r \leq 2\alpha$

We next prove the L_2 error estimate (4.47). i.e.

$$||R_{\alpha h}v - v||_{L_2} \leq Ch^r ||v||_{H^r}, \ \ \alpha \leq r \leq 2\alpha$$

Let $\phi \in L_2$. Consider the elliptic problem

$$(-\Delta)^\alpha \psi = \frac{-1}{2\, cos(\alpha\pi)} \left[{}_0^R D_x^{2\alpha} \psi(x) + {}_x^R D_1^{2\alpha} \psi(x) \right] = \phi, \ 0 < x < 1$$

$$\psi(0) = \psi(1) = 0$$

We first show the following regularity result:

$$||\psi||_{H^{2\alpha}((D))} \leq C||\phi||_{L_2(\mathcal{D})}$$

Infact, we have $C_\alpha = \dfrac{1}{2\ cos(\alpha\pi)}$

$$\begin{aligned}
||\phi||^2_{L_2(\mathcal{D})} &= ((-\Delta)^\alpha\psi, (-\Delta)^\alpha\psi) \\
&= [\frac{1}{2\ cos(\alpha\pi)}]^2\Big[{}_0^RD_x^{2\alpha}\psi(x) + {}_x^RD_1^{2\alpha}\psi(x), {}_0^RD_x^{2\alpha}\psi(x) + {}_x^RD_1^{2\alpha}\psi(x)\Big] \\
&= C_\alpha^2\Big[({}_0^RD_x^{2\alpha}\psi(x), {}_0^RD_x^{2\alpha}\psi(x)) + 2({}_x^RD_1^{2\alpha}\psi(x), {}_0^RD_x^{2\alpha}\psi(x)) + \\
&\quad ({}_x^RD_1^{2\alpha}\psi(x), {}_x^RD_1^{2\alpha}\psi(x))\Big] \\
&\geq C||\psi||^2_{H^{2\alpha}(\mathcal{D})}
\end{aligned}$$

where in the last inequality we use the facts that the norms in $H_0^{2\alpha}(\mathcal{D}), {}^lH_0^{2\alpha}(\mathcal{D}), {}^rH_0^{2\alpha}(\mathcal{D})$ and ${}^cH_0^{2\alpha}(\mathcal{D})$ are equivalent.

Hence we have,

$$\begin{aligned}
(R_{\alpha h}v - v, \phi) &= (R_{\alpha h}v - v, (-\Delta)^\alpha\psi) \\
&= B_\alpha(R_{\alpha h}v - v, \psi) \\
&= B_\alpha(R_{\alpha h}v - v, \psi - \chi) \text{ (Orthogonality)} \\
&\leq ||R_{\alpha h}v - v||_{H_0^\alpha(\mathcal{D})}||\psi - \chi||_{H_0^\alpha(\mathcal{D})},\ \forall\ \chi \in S_h
\end{aligned}$$

Therefore $(R_{\alpha h}v - v, \phi) \leq ||R_{\alpha h}v - v||_{H_0^\alpha(\mathcal{D})}||\psi - I_h\psi||_{H_0^\alpha(\mathcal{D})}$

where I_h is the interpolation operator $I_h : H_0^\alpha \to S_h$.

By (4.48), we get

$$||R_{\alpha h}v - v||_{H_0^\alpha(\mathcal{D})} \leq Ch^\alpha||v||_{H^{2\alpha}}.$$

By interpolation error and regularity, we get, we get $r = 2\alpha$,

$$||\psi - I_h\psi||_{H_0^\alpha(\mathcal{D})} \leq Ch^{r-\alpha}||\psi||_{H^r} \leq Ch^\alpha||\phi||_{L_2}$$

Thus

$$\begin{aligned}
(R_{\alpha h}v - v, \phi) &\leq (Ch^\alpha||v||_{H^{2\alpha}})(Ch^\alpha||\phi||_{L_2}) \\
&\leq Ch^{2\alpha}||v||_{H^{2\alpha}}||\phi||_{L_2}
\end{aligned}$$

Choose $\phi = R_{\alpha h}v - v$, we get

$$||(R_{\alpha h}v - v||_{L_2} \leq Ch^{2\alpha}||v||_{H^{2\alpha}}, \;\; \frac{1}{2} < \alpha \leq 1.$$

Combining these estimates complete the proof of Lemma 4.3.6.
Now we are ready to prove Theorem 4.3.1.

Proof of theorem 4.3.1: We write

$$u_h(t) - u(t) = \theta(t) + \rho(t), \text{ where } \theta(t) = u_h(t) - R_{\alpha h}u(t), \; \rho(t) = R_{\alpha h}u(t) - u(t)$$

By lemma 4.3.4, we have

$$||\rho(t)||_{L_2} \leq Ch^r||u(t)||_{H^r}, \; \alpha \leq r \leq 2\alpha, \; 0 < \alpha < 1$$

Note that

$$u(t) = u(0) + \int_0^t u_t(s)ds$$

We get

$$||u(t)||_{H^r} \leq ||u_0||_{H^r} + \int_0^t ||u_t(s)||_{H^r} \, ds$$

Hence

$$||\rho(t)||_{L_2} \leq Ch^r\Big(||u_0||_{H^r} + \int_0^t ||u_t(s)||_{H^r} \, ds\Big)$$

We next consider $\theta(t) = u_h(t) - R_{\alpha h}u(t)$.

$\theta(t)$ satisfy the equations

$$(\theta_t, \chi) + B_\alpha(\theta, \chi) = (u_{h,t}, \chi) + B_\alpha(u_h, \chi) - (R_h u_t, \chi) - B_\alpha(u, \chi)$$

$$= (f, \chi) - (R_{\alpha h}u_t, \chi) - B_\alpha(u, \chi)$$

$$= (u_t - R_h u_t, \chi)$$

$$= (-\rho_t, \chi)$$

Choose, $\chi = \theta(t)$, we get

$$(\theta_t, \theta) + B_\alpha(\theta, \theta) = -(\rho_t, \theta)$$

$$\frac{1}{2}\frac{d}{dt}||\theta||^2_{L_2} + ||\theta||^2_{H_\alpha} \leq -(\rho_t, \theta) \leq ||\rho(t)||_{L_2}.||\theta||_{L_2}$$

Note that $||\theta||_{H^\alpha} > 0$, we get

$$\frac{1}{2}\frac{d}{dt}||\theta||_{L_2}^2 \leq ||\rho(t)||_{L_2}.||\theta||_{L_2}$$

$$\Rightarrow ||\theta||_{L_2}\frac{d}{dt}||\theta||_{L_2} \leq ||\rho_t||_{L_2}.||\theta||_{L_2}$$

$$\Rightarrow \frac{d}{dt}||\theta||_{L_2} \leq ||\rho_t||_{L_2}$$

By integration on [0,t]

$$||\theta(t)||_{L_2} \leq ||\theta(0)||_{L_2} + \int_0^t ||\rho_t(s)||_{L_2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||\theta(0)||_{L_2} + \int_0^t Ch^r ||u_t(s)||_{L_2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||\theta(0)||_{L_2} + \int_0^t Ch^r ||u_t(s)||_{H^2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - R_{\alpha h}u(0)||_{L_2} + \int_0^t Ch^r ||u_t(s)||_{H^2} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - u(0)||_{L_2} + ||u(0) - R_{\alpha h}u(0)||_{L_2} + \int_0^t Ch^2 ||u_t(s)||_{H^r} ds$$

$$\Rightarrow ||\theta(t)||_{L_2} \leq ||u_h(0) - u(0)||_{L_2} + Ch^r||u(0)||_{H^r} + \int_0^t Ch^r ||u_t(s)||_{H^r} ds$$

Thus,

$$||u_h(t) - u(t)||_{L_2} \leq ||\rho(t)||_{L_2} + ||\theta(t)|_{L_2} \leq ||u_{h,0} - u_0||_{L_2} + Ch^r\Big(||u_0||_{H^r} + \int_0^t ||u_t(s)||_{H^r} ds\Big)$$

Chapter 5

Matrix transform technique (MTT) for solving space-fractional partial differential equations

In this chapter we will use Matrix Transform Technique (MTT) for solving the space-fractional equation. The MTT for solving space-fractional equation proceed by noting that numerical discritizations of the standard equation. A is the matrix representation of the laplacian obtained via a chosen discretisation method, such as the finite difference, finite volume, finite element method.Here we include finite differnce method and finite element method.

Laplace operator: Let $A = -\Delta,\ \mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. Assume that A has orthonormal eigenfunction e_j corresponding to eigenvalues $\lambda_j,\ j = 1, 2, 3\ldots$ i.e. $Ae_j = \lambda_j e_j,\ j = 1, 2, 3\ldots$ Then we have with $1 < \alpha < 2$

$$A^{\frac{\alpha}{2}} e_j = \lambda_j^{\frac{\alpha}{2}} e_j$$

5.1 Finite differnce method

Let us consider the example of space-fractional partial diffrential equations to apply matrix transform technique.

$$\frac{\partial u}{\partial t} + A^{\frac{\alpha}{2}} u = 0 \tag{5.1}$$

$$u(0) = u_0$$

where $A = -\frac{d^2}{dx^2},\ \mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$.

Solution:[13]Assume that the solution of (5.1) has the form

$$u(x,t) = \sum_{n=1}^{+\infty} C_n(t) e_j \tag{5.2}$$

Substituting (5.2) into (5.1), we get

$$\sum_{n=1}^{+\infty} C_n'(t)e_n + \sum_{n=1}^{+\infty} C_n(t)\lambda_n^{\frac{\alpha}{2}} e_n = 0$$

$$C_n(0) = (u_0, e_n),$$

i.e.

$$C_n'(t) + C_n(t)\lambda_n^{\frac{\alpha}{2}} = 0$$

$$C_n(0) = (u_0, e_n)$$

Thus

$$C_n(t) = C_n(0)e^{-\lambda_n^{\frac{\alpha}{2}}} t$$

Thus the exact solution of (5.1) is

$$u(x,t) = \sum_{n=1}^{+\infty} C_n(0)e^{-\lambda_n^{\frac{\alpha}{2}} t} e_n(x)$$

When $C_n(0) = \int_0^1 u_0(x)\sqrt{2}sin(n\pi x)dx = \sqrt{2}\int_0^1 u_0(x)sin(n\pi x)dx$
or,

$$u(x,t) = \sum_{n=1}^{+\infty} b_n sin(n\pi x)e^{-(n^2\pi^2)^{\frac{\alpha}{2}} t}$$

where

$$b_0 = 2\int_0^1 u_0(x)sin(n\pi x)dx$$

To find the approximate solution. We denote $\mathcal{Z}_n = \mathcal{L}(e_1, e_2, e_3, \ldots e_n)$.

The approximate solution can be defined by $u_N(x,t) \in \mathcal{Z}_n$, where

$$u_N(x,t) = \sum_{n=1}^{N} C_n(0)e^{-\lambda_n^{\frac{\alpha}{2}} t} e_n(x)$$

Matrix representation of the finite difference method:[32][13] We use spectral method to solve the equation (5.1). But in application, we don't know the eigenvalues and eigenfunctions of the operator A. We have to use the finite difference method to solve (5.1).

Consider the eigen value problem

$$-u''(x) = \lambda u(x), \quad 0 < x < 1 \tag{5.3}$$
$$u(0) = u(1) = 0$$

Here $\lambda = \lambda_j = j^2\pi^2,\ u(x) = e_j(x) = \sqrt{2}sinj\pi x.$

Let $0 = x_0 < x_1 \cdots < x_{N-1} = x_N = 1$, be a partition of [0,1].

At $x = x_j,\ j = 1, 2, \ldots N-1$. We have

$$u''(x)|_{x=x_j} = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + O(h^2)$$

Let $U_j \approx u(x_j),\ j = 1, 2, \ldots N-1$ be the approximate solution of $u(x_j)$,

then we have

$$\vec{A}\vec{U} \approx \lambda\vec{U} \tag{5.4}$$

with $\vec{U} = \begin{bmatrix} U_1 \\ U_2 \\ .. \\ .. \\ U_{N-1} \end{bmatrix}$, $\vec{A} = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & .. \\ -1 & 2 & -1 & .. \\ .. & .. & .. & .. \\ .. & .. & .. & -1 \\ .. & .. & -1 & 2 \end{bmatrix}$

Note that $\vec{A}$ has eigenvalues

$$U_j = \left[4sin^2\left(\frac{j\pi}{2N}\right)\right]N^2 = 4N^2 sin^2\left(\frac{j\pi}{2N}\right)$$

It is easy to see that $U_j \to \lambda_j = j^2\pi^2$.
Since

$$U_j = 4N^2 sin^2\left(\frac{j\pi}{2N}\right) = \frac{sin^2\left(\frac{j\pi}{2N}\right)}{\left(\frac{j\pi}{2N}\right)}.(j\pi)^2 \to (j\pi)^2 = \lambda_j \ as\ N \to \infty$$

Thus U_j is the approximation of $\lambda_j,\ j = 1, 2, \ldots N$ In otherword, $\vec{A}$ is an approximation of A. We hope that $\vec{A}^{\frac{\alpha}{2}}$ is a good approximation of $A^{\frac{\alpha}{2}}$.

5.2 Finite element method

Let us consider the elliptic equation [32],

$$-\Delta u = f \qquad 0 < x < 1 \tag{5.5}$$

$$u(0) = u(1) = 0$$

The abstract form is

$$Au = f \tag{5.6}$$

where $A = -\Delta, \mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$.

The finite element method is to find $u_h \in S_h$, such that

$$(\nabla u_h, \nabla \chi) = (f, \chi), \quad \forall \chi \in S_h$$

Let $u_h = \sum_{j=1}^{N} \alpha_j \phi_j$ and $\chi = \phi_l,\ l = 1, 2, \ldots N$

We get

$$\sum_{j=1}^{N} (\nabla \phi_j, \nabla \phi_l) \alpha_j = (f, \phi_l), \quad l = 1, 2, 3, \ldots N$$

The matrix form is

$$\vec{S}.\vec{\alpha} = \vec{f} \tag{5.7}$$

Here $\vec{S} = (\nabla \phi_j, \nabla \phi_l)$ is called the stiffness matrix, $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \cdots\cdot \\ \alpha_N \end{bmatrix}$, $\quad \vec{f} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ (f, \phi_3) \\ \cdots \\ \cdots\cdot \\ (f, \phi_N) \end{bmatrix}$

Multiplying $\vec{M}^{-1}$ in both sides. We get

$$(\vec{M}^{-1}\vec{S})\vec{\alpha} = \vec{M}^{-1}\vec{f}$$

Denote $\vec{A_h} = \vec{M}^{-1}\vec{S}$, we get

$$\vec{A_h}\vec{\alpha} = \vec{M}^{-1}\vec{f} \tag{5.8}$$

We will prove that $A_h : \mathbb{R} \to \mathbb{R}$ is a good approximation of $A : \mathcal{D}(A) \to H$

Remark: In (5.8), we have

$$\vec{S}.\vec{\alpha} = \vec{f}$$

But $\vec{S} : \mathbb{R}^N \to \mathbb{R}^N$ is not a good approximation of A. Instead

$$\vec{A_h} = \vec{M}^{-1}\vec{S}$$

is good approximation in same sense.

Now we consider the fractional equation

$$A^{\frac{\alpha}{2}}u = f \tag{5.9}$$

We define the approximate solution of (5.9) by

$$\vec{A}_N^{\frac{\alpha}{2}}\vec{\alpha} = \vec{M}^{-1}\vec{f} \tag{5.10}$$

By (5.10), we have

$$\vec{\alpha} = \left(\vec{A}_N^{\frac{\alpha}{2}}\right)^{-1}\vec{M}^{-1}\vec{f} = \left(\vec{A}_N^{\frac{\alpha}{2}}\right)^{-1}P_N f$$

Here we use the following definitions and lemmas:

Definition 5.2.1. *We define the operator* $P_N : H \to \mathbb{R}^N$ *by*

$$P_N f = \vec{M}^{-1}\vec{f} = \vec{M}^{-1}\begin{bmatrix}(f,\phi_1)\\(f,\phi_2)\\....\\(f,\phi_N)\end{bmatrix}$$

Definition 5.2.2. *We define the operator* $E_N : \mathbb{R}^N \to S_h \subset H$ *by*

$$E_N\vec{\alpha} = \sum_{i=1}^{N}\alpha_i\phi_i(x) \in S_h$$

Lemma 5.2.1. *1.* $P_N E_N \vec{\alpha} = \vec{\alpha}$

2.$P_h f = E_N P_N f$ where $P_h : H \to S_h$ is the L_2 projection defined by

$$(P_h f, \chi) = (f, \chi),\ \forall\chi \in S_h$$

3.$||P_N f|| \leq C||f||$

Lemma 5.2.2. $||E_N\vec{\alpha}|| \leq C||\vec{\alpha}||$

Proof:

$$||E_N\vec{\alpha}||^2 = ||\sum_{i=1}^{N}\alpha_i\phi_i(x)||^2$$

$$= \Big\langle \sum_{i=1}^{N}\alpha_i\phi_i(x), \sum_{i=1}^{N}\alpha_i\phi_i(x)\Big\rangle$$

$$= \vec{\alpha}^T\vec{M}\vec{\alpha} = \Big\langle \vec{M}\vec{\alpha}, \vec{\alpha}\Big\rangle \leq ||\vec{M}\vec{\alpha}||.||\vec{\alpha}||$$

$$\leq ||\vec{M}||.||\vec{\alpha}||^2$$

Note that $||\vec{M}|| = \sup_j \lambda_j h \leq Ch^2 \leq C.$

Thus

$$||E_N\vec{\alpha}|| \leq C||\vec{\alpha}||$$

Definition 5.2.3. *[32][4] A non-negative Stieltjes function is a function*

$f : (0,\infty) \to [0,\infty)$ *which can be written in the form*

$$f(\lambda) = z + \int_0^\infty \frac{1}{\lambda + t}\mu dt$$

where $z \geq 0,\ z = \lim_{\lambda\to\infty} f(\lambda)$

are non-negative constant and μ *is a measure on* $[0,\infty)$*such that* $\int_0^\infty \frac{1}{1+t}\mu dt < \infty$*. We denote the family of all Stieltjes functions by S.*

Example : $f(\lambda) = \lambda^{\alpha-1},\ 0 < \alpha < 1$ is a Stieltjes function.

Proof:[32] Choose $\mu\ dt = \dfrac{sin\alpha\pi}{\pi}t^{\alpha-1}dt$

$$z = \lim_{\lambda\to\infty}\lambda^{\alpha-1} = 0$$

Then we have

$$\lambda^{\alpha-1} = \int_0^\infty \frac{1}{\lambda+t}\frac{sin\alpha\pi}{\pi}t^{\alpha-1}dt,\ \lambda > 0$$

which follows from

$$\int_0^\infty \frac{1}{\lambda+t}t^{\alpha-1}dt$$

$$= \Big(\frac{1}{\lambda+t} = \int_0^\infty e^{-\lambda s}e^{-ts}\ ds\Big)$$

$$= \int_0^\infty e^{-\lambda s}\Big(\int_0^\infty t^{\alpha-1}e^{-ts}\,dt\Big)ds$$

$$= \int_0^\infty e^{-\lambda s}s^{-\alpha}\Gamma(\alpha)ds$$

$$= \lambda^{\alpha-1}\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin\alpha\pi}$$

Further, we note that

$$\int_0^\infty \frac{1}{1+t}\frac{sin\alpha\pi}{\pi}t^{\alpha-1}dt = \frac{sin\alpha\pi}{\pi}\int_0^\infty \frac{1}{1+t}t^{\alpha-1}dt = \frac{sin\alpha\pi}{\pi} < \infty$$

Denote $g(t) = t^{-\frac{\alpha}{2}}$, $1 < \alpha < 2$. The solution of (5.9) can be written into

$$u = g(A)f \tag{5.11}$$

The solution of (5.10) can be written into

$$u_N = E_N\vec{\alpha} = E_N\Big(\vec{A}_N^{\frac{\alpha}{2}}\Big)^{-1}P_Nf = E_N\; g(\vec{A_N})P_Nf \tag{5.12}$$

Now we have the following theorem.

Theorem 5.2.1. *[32] Let u and u_N be the solutions of (5.11) and (5.12) respectively. Then we have*

$$||u - u_N|| \leq Ch^2||Af||$$

Proof:

Step 1: Since $g(t) = t^{-\frac{\alpha}{2}}$ is a Stieltjes function . So, we have

$$g(A)f = \int_0^\infty R(\lambda; A)fd\mu(\lambda)$$

Where $R(\lambda; A) = (\lambda I + A)^{-1}$, $\int_0^\infty \frac{1}{1+t}d\mu(t) < \infty$.

Thus we get

$$u - u_N = g(A)f - E_N\; g(\vec{A_N})P_Nf$$

$$= \int_0^\infty \Big[R(\lambda; A)f - E_N\; R(\lambda; A_N)P_Nf\Big]d\mu(\lambda)$$

Step 2: For this step first we recall Resolvent equality lemma.

Lemma 5.2.3. *(Resolvent equality)*

$$R(\lambda; A) - R(\mu; A) = (\mu - \lambda)R(\lambda; A)R(\mu; A)$$

where

$$R(\lambda; A) = (\lambda I - A)^{-1}, \ R(\mu; A) = (\mu I - A)^{-1}.$$

Proof of Resolvent equality: We can write

$$\mu - \lambda = \mu - \lambda$$

$$\Rightarrow (\mu I - A) - (\lambda I - A) = \mu - \lambda$$

Divide by $(\lambda I - A)$ in both sides

$$(\lambda I - A)^{-1}(\mu I - A) - I = (\lambda I - A)^{-1}(\mu - \lambda)$$

$$\Rightarrow [(\lambda I - A)^{-1} - (\mu I - A)^{-1}](\mu I - A) = (\lambda I - A)^{-1}(\mu - \lambda)$$

Divide by $(\mu I - A)$ in both sides

$$[(\lambda I - A)^{-1} - (\mu I - A)^{-1}] = (\lambda I - A)^{-1}(\mu - \lambda)(\mu I - A)^{-1}$$

$$\Rightarrow R(\lambda; A) - R(\mu; A) = (\mu - \lambda)R(\lambda; A)R(\mu; A)$$

So that , from Resolvent equality we get

$$R(\lambda; A) - E_N \ R(\lambda; A_N)P_N$$

$$= \Big(I + (\lambda_0 - \lambda)E_N \ R(\lambda; A_N)P_N\Big)\Big(R(\lambda_0; A) - E_N \ R(\lambda_0; A_N)P_N\Big)\Big((A + \lambda_0 I)R(\lambda; A)\Big)$$

where $\lambda, \ \lambda_0 \in [0, \infty)$.

Step 3: Note that E_N, P_N are bounded and

$$||R(\lambda; A_N)|| \leq \frac{K_N}{\lambda + \omega_N}. \ \forall \lambda + \omega_N > 0,$$

$$||R(\lambda; A)|| \leq \frac{K_N}{\lambda + \omega}. \ \forall \lambda + \omega > 0$$

We get

$$||R(\lambda;A)f - E_N\ R(\lambda;A_N)P_N f||$$
$$\leq \Big(1+(\lambda_0-\lambda)\frac{K_N}{\lambda+\omega_N}C\Big)||(R(\lambda_0;A)-E_N\ R(\lambda_0;A_N)P_N)R(\lambda;A)(A+\lambda_0 I)f||$$

Note that, with $\lambda_0 = 0$

$$\Big(R(\lambda_0;A)-E_N\ R(\lambda_0;A_N)P_N\Big)\omega||$$

$$= ||A^{-1}\omega - E_N\ A_N^{-1}P_N\omega||$$

$$\leq Ch^2||\omega||$$

Thus we get

$$||R(\lambda;A)f - E_N\ R(\lambda;A_N)P_N f||$$
$$\leq \Big(1+(\lambda_0-\lambda)\frac{K_N}{\lambda+\omega_N C}\Big)Ch^2||R(\lambda;A)||.||Af||$$

$$\leq \Big(1+(\lambda_0-\lambda)\frac{K_N}{\lambda+\omega_N}C\Big)Ch^2\frac{1}{\lambda+\omega}.||Af||$$

Hence

$$||u-u_N|| \leq ||\int_0^\infty \Big[R(\lambda;A)f - E_N\ R(\lambda;A_N)P_N f\Big]d\mu(t)||$$
$$\leq \Big(1+(\lambda_0-\lambda)\frac{K_N}{\lambda+\omega_N}C\Big)Ch^2\Big[\int_0^\infty \frac{1}{\lambda+\omega}d\mu(\lambda)\Big]||Af||$$
$$\leq Ch^2||Af||$$

Since $\int_0^\infty \frac{1}{\lambda+\omega}d\mu(\lambda) < \infty$

Thus,

$$||u-u_N|| \leq Ch^2||Af||$$

5.3 Numerical example

5.3.1 Example:Finite difference method

Solve the given space-fractional PDEs from[32]

$$(-\Delta)^{\frac{\alpha}{2}}u = f, \quad 1 < \alpha < 2,\ 0 < x < 1 \tag{5.13}$$

$$u(0) = u(1) = 0$$

by using finite difference method.

Here $f(x) = \pi^{\alpha} sin\pi x,\ Au = -\frac{d^2}{dx^2}u = \Delta u$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$.

Solution: We will solve this equation step by step.

Step 1: We are solving

$$(-\Delta)u = 0, \quad 0 < x < 1$$
$$u(0) = u(1) = 0$$

Let $0 = x_0 < x_1 < \dots x_M = 1$, be a partition of [0,1].

At $x = x_j,\ j = 1, 2, 3, 4, 5 \dots, M-1$. we get

$$-\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f(x_j)$$

or,

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & .. & .. & 0 \\ -1 & 2 & .. & .. & 0 \\ .. & .. & .. & ... & .. \\ .. & .. & .. & 2 & -1 \\ 0 & 0 & .. & -1 & 2 \end{bmatrix}_{(M-1)\times(M-1)} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ ... \\ U_{M-1} \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ .... \\ f(x_{M-1}) \end{bmatrix}$$

We get

$$\vec{A_h} * \vec{\alpha} = \vec{F_h}$$

where

$$\vec{A_h} = \begin{bmatrix} 2 & -1 & .. & 0 & 0 \\ -1 & 2 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & 2 & -1 \\ 0 & 0 & .. & -1 & 2 \end{bmatrix}_{(M-1)\times(M-1)},$$

and

$$\vec{\alpha} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ ... \\ U_{M-1} \end{bmatrix}, \quad \vec{F}_h = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ .. \\ f(x_{M-1}) \end{bmatrix}$$

Step 2: The finite difference solution of (5.13) is defined by

$$\vec{A}_h^{\frac{\alpha}{2}} * \vec{\alpha} = \vec{F}_h$$

where

$$\vec{\alpha} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ .. \\ U_{M-1} \end{bmatrix}$$

is the solution of (5.13) .

Step 3: Matlab program will calculate $\vec{\alpha}$.

Figure 5.1

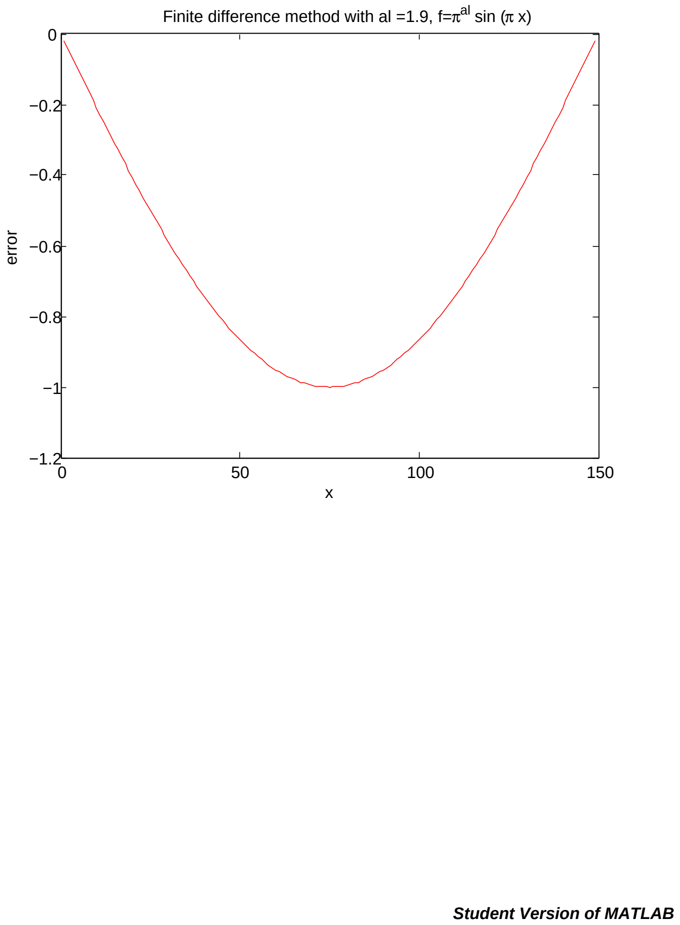

Figure 5.1: Error in Finite Difference Method for example 5.3.1

5.3.2 Example:Finite element method

Solve the given space-fractional PDEs from[32]

$$(-\Delta)^{\frac{\alpha}{2}}u = f, \quad 1 < \alpha < 2,\ 0 < x < 1 \tag{5.14}$$
$$u(0) = u(1) = 0$$

by using finite element method.

Here $f(x) = \pi^{\alpha} sin\pi x,\ Au = -\frac{d^2}{dx^2}u = \Delta u.$

Solution: We will solve this equation step by step.

Step 1: We are solving

$$(-\Delta)u = 0, \quad 0 < x < 1 \tag{5.15}$$
$$u(0) = u(1) = 0$$

Let $0 = x_0 < x_1 < \dots x_M = 1,$ be a partition of [0,1] i.e step size $h = \frac{1}{M}$.

Define the finite element basis functions $\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \dots \phi_{M-1}(x)$.

Let $U_h(t) = \sum_{j=1}^{M-1} \alpha_j \phi_j(x)$ be the finite element solution of 5.15 .

Then we have

$$(U_h', \chi') = (f, \chi), \quad \forall \chi \in S_h$$

or,

$$\sum_{j=1}^{M-1} \alpha_j (\phi_j', \chi') = (f, \chi), \quad \forall \chi \in S_h$$

Choose $\chi = \phi_l, \quad l = 1, 2, 3, 4, 5 \dots, M-1$.We get

$$\sum_{j=1}^{M-1} \alpha_j (\phi_j', \phi_l') = (f, \phi_l), \quad l = 1, 2, 3, 4, 5, \dots, M-1.$$

$$\text{or, stiffness} * \vec{\alpha} = \vec{f}$$

Here $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ ... \\ \alpha_{M-1} \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ (f,\phi_3) \\ ..... \\ (f,\phi_{M-1}) \end{bmatrix},$

$$stiffness = \begin{bmatrix} (\phi_1',\phi_1') & .... & (\phi_1',\phi_{M-1}') \\ (\phi_2',\phi_1') & .... & (\phi_2',\phi_{M-1}') \\ ..... & .... & .... \\ ..... & .... & .... \\ (\phi_{M-1}',\phi_1') & .... & (\phi_{M-1}',\phi_{M-1}') \end{bmatrix} = \frac{1}{h} \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & .. & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & ... & 0 & 0 \\ 0 & \frac{1}{6} & ... & \frac{1}{6} & 0 \\ .. & .. & ... & .... & ... \\ 0 & 0 & .. & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

Denote, $mass = \begin{bmatrix} (\phi_1,\phi_1) & .... & (\phi_1,\phi_{M-1}) \\ (\phi_2,\phi_1) & .... & (\phi_2,\phi_{M-1}) \\ ..... & .... & .... \\ ..... & .... & .... \\ (\phi_{M-1},\phi_1) & .... & (\phi_{M-1},\phi_{M-1}) \end{bmatrix} = h \begin{bmatrix} 2 & -1 & .. & 0 & 0 \\ -1 & 2 & .. & 0 & 0 \\ 0 & -1 & .. & -1 & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & .. & -1 & 2 \end{bmatrix}$

Thus we get

$$mass^{-1} * stiffness * \vec{\alpha} = mass^{-1} * \vec{f}$$

$$or, \quad \vec{A}_h * \vec{\alpha} = \vec{F}_h$$

When we get $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ ..... \\ \alpha_{M-1} \end{bmatrix}$, we get the finite element solution $U_h(x) = \sum_{j=1}^{5} \alpha_j \phi_j'(x)$.

Step 2: The finite element solution of (5.14) is defined by $U_h(x) = \sum_{j=1}^{M-1} \alpha_j \phi_j'(x)$.

$$\vec{A}_h^{\frac{\alpha}{2}} * \vec{\alpha} = \vec{F}_h$$

Step 3: Matlab program will calculate $\vec{\alpha}$.

Figure 5.2

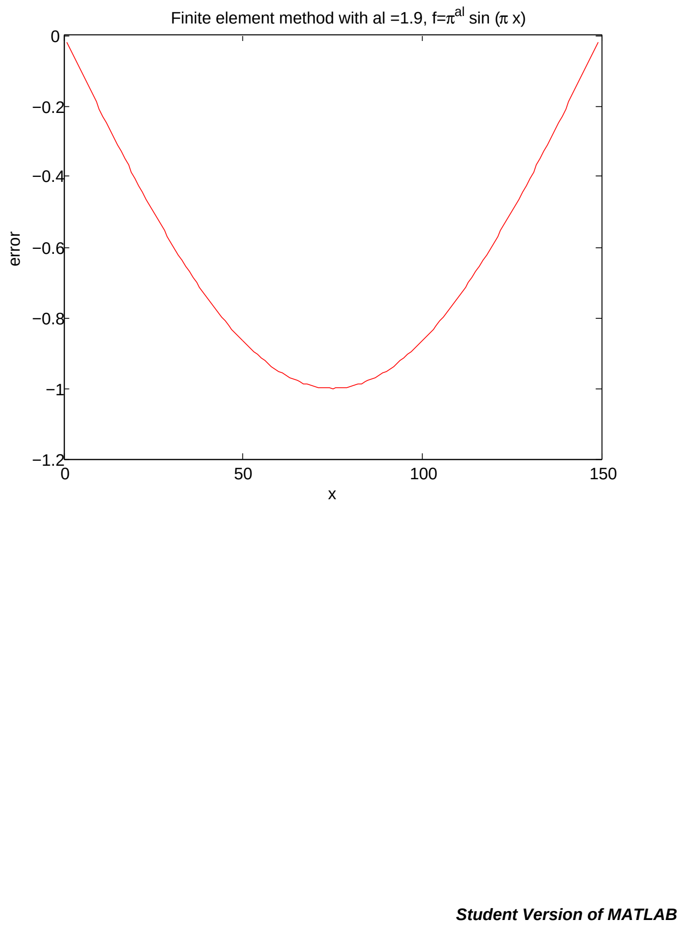

Figure 5.2: Error in Finite Element Method for example5.3.2

Chapter 6

Conclusion and future work

In this dissertation, we consider numerical methods for space-fractional PDEs and obtain the error estimates for the finite element method. In the future work, we will consider numerical methods for linear space-fractional PDEs in 2-dimensional case. We can also consider nonlinear space-fractional PDEs.

Chapter 7

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Chapter 8

Appendix

MATLAB program for equation (5.3) of example 5.3.1

```
%finite difference method
%Reference: Simpson, Turner and Illic, "A general matrix transfer technique for
%the numerical solution of space-fractional PDEs".
% A^{al/2}u=f, 0<x<1
%u(0)=u(1)=0;
%where A^{al/2}=(-\Delta)^{al/2}, 1<al<2,
%let us consider, f(x)=pi^al* sin(pi*x);
% exact solution \sum_{j=1}^{infty} lambda_{j}^{-al/2} {f, e_{j}) e_{j};
%By finite difference the approximate solution is
%A_h*alpha=F_h
%for the nodes 0=x_0<x_1<x_2<x_3<x_4<x_5<x_6=1, i.e h=1/6
% where A_h=-(1/h^2)*[-2 1 0 0 0; 1 -2 1 0 0; 0 0 1 -2 1; 0 0 0 -1 -2,
%F_h=[f(x_1); f(x_2); f(x_3);f(x_4); f(x_5)]

function [] =my_fractional_pde()
clear
al =default('al= is the fractional order, (default is al=1.9)',1.9);

h=1/150; % step size
x=[0:h:1]; % x-coordinate

n=size(x,2); % the number of the nodes

A_h= (-2)*diag(ones(n-2,1),0)+diag(ones(n-3,1),1)+diag(ones(n-3,1),-1);
A_h=-(1/h^2)*A_h;
A_h_al=expm((al/2)-logm(A_h));
xinnodes =x(2:n-1);
F_h=pi^(al)*sin (pi.*xinnodes');
```

```
U1=A_h_al\F_h;
U=[0;U1;0];

exact=0;
for i=1:10
y=quad(@(t)((pi^(al)*sin(pi*t)).*(sqrt(2)*sin(i*pi.*t))),0,1,1e-10);
exact=exact+(i*pi)^(-al)*y*sqrt(2)*sin(i*pi*x');
end
error=U-exact;
error=error(2:end-1);
figure(1)
plot(error,'r')
title('Finite difference method with al =1.9, f=\pi^{al} sin (\pi x)')
xlabel ('x')
ylabel('error')
function reply = default(query,value)
global BATCH FID
if exist('BATCH') & BATCH==1,
replycell=textscan(FID,'%f%*[^\n]',1);
reply=deal(replycell{:});
disp(query)
disp(reply)
else
reply=input([query,' : ']);
if isempty(reply), reply=value; end
end
return
```

MATLAB program for equation (5.4) of example 5.3.2

```
%finite element method
%Reference: Simpson, Turner and Illic, "A general matrix transfer technique for
%the numerical solution of space-fractional PDEs".
% A^{al/2}u=f, 0<x<1
%u(0)=u(1)=0;
%where A^{al/2}=(-\Delta)^{al/2}, 1<al<2,
%let us consider, f(x)=pi^al* sin(pi*x);
% exact solution \sum_{j=1}^{infty} lambda_{j}^{-al/2} {f, e_{j}) e_{j};
%By finite element the approximate solution is
%A_h*alpha=F_h
%where A_h=mass^{-1}*stiffness, F_h=mass^{-1}*F
% and for the finite element basis function (varphi_j),F=(f, varphi_j).
% Denote that,
```

```
% mass=h*[2/3 1/6 0 0; 1/6 2/3 1/6 0; 0 1/6 2/3 1/6;0 0 1/6 2/3]
%stiffness=1/h[2 -1 0 0; -1 2 -1 0; 0-1 2 -1; 0 0 -1 2]


function [] =my_fractional_pdeelement()
clear
al =default('al= is the fractional order, (default is al=1.9)',1.9);

h=1/150; % step size
x=[0:h:1]; % x-coordinate

n=size(x,2); % the number of the nodes

nodes1=1:n-1;
nodes2=2:n;

for i=1:n-1;
nodes(i,1)=nodes1(i);
nodes(i,2)=nodes2(i);
end


A=zeros(n,n);
b=zeros(n,1);
mass=zeros(n,n);
stiffness=zeros(n,n);

for e1=1:n-1
[localmass,localstiffness,db]=elementcontributions_elliptic(x,nodes,e1,al);
nn=nodes(e1,:);
mass(nn,nn)=mass(nn,nn)+localmass;
stiffness(nn,nn)=stiffness(nn,nn)+localstiffness;
b(nn)=b(nn)+db;
end

A=inv(mass)*stiffness;
b=inv(mass)*b;


innodes=2:n-1;
A1=A(innodes,innodes);
A1_al_2= expm((al/2)-logm(A1));
b1=b(innodes);
```

```
U1=A1_al_2\b1;
U=[0;U1;0];
%exact sol
exact=0;
for i=1:10
y=quad(@(t)((pi^(al)*sin(pi*t)).*(sqrt(2)*sin(i*pi.*t))),0,1,1e-10);
exact=exact+(i*pi)^(-al)*y*sqrt(2)*sin(i*pi*x');
end



error=U-exact;
error=error(2:end-1);
figure(1)
plot(error,'r')
title('Finite element method with al =1.9, f=\pi^{al} sin (\pi x)')
xlabel ('x')
ylabel('error')
function [localmass,localstiffness,db]=elementcontributions_elliptic(x,nodes,e1,al)
n1=nodes(e1,1);
n2=nodes(e1,2);
x1=x(n1);
x2=x(n2);
length=x2-x1;
mid_point=(x2+x1)/2;
f=(pi)^(al)*sin(pi*mid_point);

%f=mid_point
localmass=[1/3*length 1/6*length;1/6*length 1/3*length];
localstiffness=[1/length -1/length;-1/length 1/length];
db=f*length*[1/2;1/2];


function reply = default(query,value)
global BATCH FID
if exist('BATCH') & BATCH==1,
replycell=textscan(FID,'%f%*[^\n]',1);
reply=deal(replycell{:});
disp(query)
disp(reply)
else
reply=input([query,' : ']);
if isempty(reply), reply=value; end
```

```
end
return
```