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A nonpolynomial collocation method for fractional terminal value problems

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Abstract

In this paper we propose a non-polynomial collocation method for solving a class of terminal (or boundary) value problems for differential equations of fractional order α, $0 < \alpha < 1$. The approach used is based on the equivalence between a problem of this type and a Fredholm integral equation of a particular form. Taking into account the asymptotic behaviour of the solution of this problem, we propose a non-polynomial collocation method on a uniform mesh. We study the order of convergence of the proposed algorithm and a result on optimal order of convergence is obtained. In order to illustrate the theoretical results and the performance of the method we present several numerical examples.

Keywords: fractional calculus, Caputo derivative, boundary value problem, nonpolynomial collocation method.
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1. Introduction

There is rapidly increasing interest in the study of fractional differential equations because recent investigations in science and engineering have indicated that the dynamics of many systems can be described more accurately by using differential equations of non-integer order.

The theory for initial value problems is well established, but much less is known concerning terminal (or boundary) value problems. In this work we are concerned with the solution of boundary fractional differential equations of order α, $0 < \alpha < 1$, of the form

$$\begin{aligned} D_*^\alpha y(t) &= f(t, y(t)), \quad t \in [0, a], && (1.1)\\ y(a) &= y_a, && (1.2) \end{aligned}$$

where f is a suitably behaved function and D_*^α denotes the Caputo differential operator of order $\alpha \notin \mathbb{N}$ ([5]), defined by

$$D_*^\alpha y(t) := D^\alpha(y - T_{k-1}[y])(t), \quad k-1 < \alpha \leq k, \; k \in \mathbb{N},$$

where $T_{k-1}[y]$ is the Taylor polynomial of degree $k-1$ for y, centered at 0, and D^α is the Riemann-Liouville derivative of order α ([19]):

$$D^\alpha y(t) := D^k \left(J^{k-\alpha} y\right)(t),$$

where D^k is the classical integer order derivative and $J^{k-\alpha}$ is the Riemann-Liouville integral operator, given by

$$J^{k-\alpha} y(t) := \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} y(s) ds.$$

If $a = 0$, the problem (1.1)-(1.2) reduces to an initial value problem, that has been investigated by several authors. We refer, for example, to the work of Diethelm and Ford ([10]) where the authors, by considering the equivalence of the problem with a Volterra integral equation, established sufficient conditions for the existence and uniqueness of the solution, and investigated the sensitivity of the solution to changes in the parameters of the problem, namely, in the initial value, in the order of the derivative and in the right-hand side function f. Concerning the numerical solution of (1.1)-(1.2) with $a = 0$ we refer, for instance, to the articles [6], [8], [12], [13], [15], [17] among many others.

The case where $a > 0$ has not been discussed in much detail. That is, the case where the initial condition is not prescribed at the basis point of the fractional differential operator, which in this paper is assumed to be zero. This kind of problem, called a terminal (or boundary) value problem, arises naturally in the simulation of processes that are observed (i.e. measured) at a later point, some time after the process has started. In other words, one does not know the values of the quantities of interest at $t = 0$. Recently, in [14], Ford and Morgado considered the case where $a > 0$ and they proved that the boundary value problem (1.1)-(1.2) is equivalent to the following Fredholm integral equation:

$$y(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_0^a \left((t-s)^{\alpha-1} \chi_{[0,t]}(s) - (a-s)^{\alpha-1} \right) f(s, y(s)) ds, \quad (1.3)$$

where $\chi_{[0,t]}$ is the indicator function of the interval $[0, t]$. Furthermore, using the reformulation of (1.1)-(1.2) as a Fredholm integral equation, existence and uniqueness results were established.

Lemma 1.1. *If the function f is continuous and satisfies a Lipschitz condition with Lipschitz constant $L > 0$ with respect to its second argument, and if $\frac{2La^\alpha}{\Gamma(\alpha+1)} < 1$, then the boundary value problem (1.1)-(1.2) is equivalent to the integral equation (1.3). Furthermore the boundary (or terminal) value problem (1.1)-(1.2) has a unique solution on $[0, a]$.*

As pointed out in [14], after proving the existence of $y(0)$, existence and uniqueness results for $t > a$ are inherited from the corresponding initial value problem theory. In order to approximate the solution of (1.1)-(1.2), the authors of this paper previously proposed a simple shooting method. We considered the initial value problem

$$D_*^\alpha y(t) = f(t, y(t)), \quad t \in (0, T] \quad (1.4)$$
$$y(0) = y_0, \quad (1.5)$$

and for a certain value of y_0, we determined its numerical solution using standard initial value problem solvers. Then we used an iterative scheme to find the appropriate y_0, for which the solution of the initial value problem passes throught the point (a, y_a).

In this paper, we follow a different approach. We develop a nonpolynomial collocation method to approximate the solution of (1.3), which is equivalent to approximating the solution to the terminal value problem (1.1)-(1.2). A similar approach was used in [15] in the case where in (1.1)-(1.2) we have $a = 0$. Although the extension to the case $a > 0$ may at first seem straightforward, a closer look shows that this is not the case, either analytically or numerically. As remarked in (1.3), in the case $a > 0$ problem (1.1)-(1.2) is equivalent to a Fredholm equation, while in the case $a = 0$, we have an equivalence with a Volterra integral equation.

A well known feature of fractional differential equations is that, in general, we cannot expect the solution of (1.1) to be smooth, even if the right-hand side is smooth, a fact that makes the construction of numerical methods with a reasonable order of convergence difficult. A result concerning the smoothness of the solution of (1.4)-(1.5) can be found, for example in [7] and establishes the following:

Lemma 1.2. *Assume that the solution of (1.4)-(1.5) exists and is unique on $[0, T]$, for a certain $T > 0$. If $\alpha = \frac{p}{q}$, where $p \geq 1$ and $q \geq 2$ are two relatively prime integers and if the right-hand side function f can be written in the form $f(t, y(t)) = \overline{f}(t^{1/q}, y(t))$, where $\overline{f}$ is analytic in a neighborhood of $(0, y_0)$, then the unique solution of the problem (1.4)-(1.5) can be represented in terms of powers of $t^{1/q}$:*

$$y(t) = \sum_{i=0}^{\infty} a_i t^{i/q}, t \in [0, r), \tag{1.6}$$

for some $0 < r < T$, where the coefficients a_i are constants.

Therefore, if the solution of (1.1)-(1.2) exists and is unique, problem (1.1)-(1.2) coincides with a unique initial value $y_0 = y(0)$

$$y_0 = y_a - \frac{1}{\Gamma(\alpha)} \int_0^a (a - s)^{\alpha - 1} f(s, y(s)) ds,$$

and from Lemma 1.2 it follows that the solution can be written as a sum $y = y_1 + y_2$ where, for a fixed integer m, $y_1 \in C^m([0, T])$ and y_2 is the nonsmooth part of the solution. This result will be the starting point to the construction of the numerical method presented here. The paper is organised as follows: in section 2 we describe the numerical scheme to approximate the solution of (1.1)-(1.2). In section 3 we present the error analysis in the linear case. Finally, in section 4 we consider several terminal value problems for which the analytical solution is known, and we present for each one of them the numerical results we obtained which illustrate the performance of the proposed method, ending with some conclusions.

2. A numerical method based on a nonpolynomial approximation

In this section we approximate the solution of (1.3) and consequently the solution of the terminal value problem (1.1)-(1.2). In order to do that we take Lemma 1.2 into account and follow the approach in [2], also used in [4], and [15]. In [15], we used this approach to approximate the solution of (1.1)-(1.2), and in that paper we mentioned that the initial value problem solver we obtained could be used to approximate the solution of fractional terminal value problems of the form (1.1)-(1.2), on the basis of a shooting method: first we consider the initial value problem (1.4)-(1.5) for some arbitrary value(s) of y_0. Next, we determine its numerical solution using the initial value problem solver. Then we use an iterative scheme (the bisection method, for example) to find the appropriate y_0, for which the solution of the initial value problem passes through the point (a, y_a). Here, in order to avoid the shooting method, we derive a similar nonpolynomial collocation method for terminal value problems. The main difference between the method presented in [15] and the method presented here is that while in the first one we had to discretise a Volterra integral equation, in this one we are dealing with a Fredholm integral equation.

For $t > 0$, $m \in \mathbb{N}$ and $0 < \alpha < 1$ let us define a finite dimensional space V_m^α, with dimension $\ell = \#V_m^\alpha$, of nonpolynomial functions by

$$V_m^\alpha = span\left\{t^{i+j\alpha} : \quad i, j \in \mathbb{N}_0,\ i + j\alpha < m\right\}. \tag{2.1}$$

Introducing the index set

$$W_{\alpha,m} = \{i + j\alpha : \quad i, j \in \mathbb{N}_0,\ i + j\alpha < m\} = \{\nu_k : k = 1, 2, ..., \ell\},$$

we can write V_m^α as

$$V_m^\alpha = span\left\{t^{\nu_k},\ k = 1, ..., \ell\right\}. \tag{2.2}$$

Denoting by $\mathcal{P}_m$ the space of polynomials of degree $\leq m - 1$, clearly $\mathcal{P}_m \subset V_m^\alpha$. Moreover, the space V_m^α contains the polynomial and nonpolynomial functions t^{ν_j}, $j = 1, \ldots, \ell$, which may reflect the nonregularity of the solution of (1.3) (see 1.6).

Consider a partition of the interval $[0, a]$ into N subintervals of equal size $h = a/N$

$$\Delta_N = \{t_i = i\,h, \quad i = 0, 1, ..., N\}\,. \tag{2.3}$$

In each one of the subintervals $\sigma_0 = [0, t_1]$, $\sigma_i = (t_i, t_{i+1}]$, $i = 1, \ldots, N-1$, we define ℓ collocation points $t_{ik} = t_i + c_k h$, $k = 1, 2, \ldots, \ell$, where $c_1,\ c_2, ..., c_\ell$ are some fixed collocation parameters which do not depend on i and N and satisfy

$$0 \leq c_1 < c_2 < ... < c_\ell \leq 1, \tag{2.4}$$

and the set

$$V_m^\alpha(\Delta_N) = \{v :\ v|_{\sigma_i} \in V_m^\alpha,\ i = 0, 1, \ldots, N-1\}\,.$$

In what follows, for two given Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of bounded linear operators $A : E \to F$ with the norm $\|A\|_{\mathcal{L}(E,F)} = \sup\{\|Az\|_F :\ z \in E,\ \|z\|_E \leq 1\}$.

In order to define the interpolation operator

$$P_N \equiv P_{N,\ell} : C([0, a]) \longrightarrow V_m^\alpha(\Delta_N)$$

on the interval $[0, a]$, we define the Lagrange piecewise polynomial basis functions by setting, for $j = 0, 1, ..., N-1$, $k = 1, 2, ..., \ell$,

$$L_{jk}(t) = \begin{cases} \displaystyle\sum_{p=1}^{\ell} \beta_{kp}^j t^{\nu_p}, & t \in \sigma_j \\ \\ 0, & \text{otherwise} \end{cases}, \tag{2.5}$$

where the ℓ coefficients $\{\beta_{kp}^j\}_{p=1,\ldots,\ell}$ may be determined by solving, for each $j = 0, 1, ..., N-1$ and $k = 1, \ldots, \ell$, the linear systems of equations

$$L_{jk}(t_{j\gamma}) = \delta_{k\gamma}, \quad \gamma = 1, \ldots, \ell \quad (\ell = \#V_m^\alpha). \tag{2.6}$$

The linear system of equations (2.6) has a unique solution as long as the interpolation points, $t_{j\gamma}$, are all distinct (cf. [4]).

The operator P_N is then defined by

$$(P_N g)(t) = \sum_{i=0}^{N-1} \sum_{j=1}^{\ell} g(t_{ij}) L_{ij}(t), \quad t \in I = [0,a]. \tag{2.7}$$

It is obvious that $(P_N g)(t_{ij}) = g(t_{ij}),\ i = 0,1,...,N-1,\ j = 1,...,\ell$ and the interpolation operators P_N are uniformly bounded,

$$\|P_N\|_{\mathcal{L}(C([0,a]),L^\infty([0,a]))} \leq c, \quad \text{(cf. [4])}. \tag{2.8}$$

On the other hand, from [4] we have the following global convergence result for the interpolation operator P_N.

Theorem 2.1. *Let P_N be the interpolation operator defined in (2.7) associated with the partition Δ_N, $m \in \mathbb{N}$ and $0 < \alpha < 1$. Suppose that g has a decomposition $g = u + v$, where $u \in C^m$ and $v \in V_m^\alpha$. Then, there exists a positive constant c independent of N such that for all such functions g*

$$\|g - P_N g\|_\infty \leq c\, N^{-m}.$$

In this work, we seek a $u \in V_m^\alpha(\Delta_N)$ that satisfies (1.3) at the collocation points, that is:

$$\begin{aligned} u(t_{ik}) &= y(a) + \frac{1}{\Gamma(\alpha)} \int_0^a \left((t_{ik}-s)^{\alpha-1}\chi_{[0,t_{ik}]}(s) - (a-s)^{\alpha-1}\right) f(s,u(s))ds \\ &= y(a) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left((t_{ik}-s)^{\alpha-1}\chi_{[0,t_{ik}]}(s) - (a-s)^{\alpha-1}\right) f(s,u(s))ds, \\ & \qquad i = 0,1,...,N-1,\ k = 1,2,...,\ell. \end{aligned} \tag{2.9}$$

As an approximation of $g(s) = f(s,u(s))$ we consider

$$f(s,u(s)) \approx (P_N g)(s), \quad s \in [0,a], \tag{2.10}$$

meaning that

$$f(s,u(s)) \approx \sum_{k=1}^{\ell} L_{jk}(s) f(t_{jk}, u(t_{jk})), \quad s \in \sigma_j. \tag{2.11}$$

Substituting (2.5) and (2.11) in (2.9), and denoting by y_{ik} the approximated value of $y(t_{ik})$, we obtain the following discretisation of (1.3):

$$\begin{aligned} y_{ik} &= y_a + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \sum_{k=1}^{\ell} f(t_{jk}, y_{jk}) \times \\ &\sum_{p=0}^{l-1} \int_{t_j}^{t_{j+1}} \left((t_{ik}-s)^{\alpha-1}\chi_{[0,t_{ik}]}(s) - (a-s)^{\alpha-1}\right) s^{\nu_p} ds \beta_{kp}^{j} \\ &= y_a + \sum_{j=0}^{N-1} \sum_{k=1}^{\ell} f(t_{jk}, y_{jk}) \sum_{p=0}^{l-1} w_{jk}^{i,p} \beta_{kp}^{j}, \qquad (2.12) \end{aligned}$$

$i = 0, 1, \ldots, N-1,\ k = 1, \ldots, \ell$, where the weights are given by

$$\Gamma(\alpha) w_{jk}^{i,p} = \begin{cases} \displaystyle\int_{t_j}^{t_{j+1}} \left((t_{ik}-s)^{\alpha-1} - (a-s)^{\alpha-1}\right) s^{\nu_p} ds, & 0 \le j < i \\ \displaystyle\int_{t_i}^{t_{ik}} \left((t_{ik}-s)^{\alpha-1} - (a-s)^{\alpha-1}\right) s^{\nu_p} ds - \int_{t_{ik}}^{t_{i+1}} (a-s)^{\alpha-1} s^{\nu_p} ds, & j = i \\ \displaystyle -\int_{t_j}^{t_{j+1}} (a-s)^{\alpha-1} s^{\nu_p} ds, & i < j < N \end{cases}.$$

If the fractional differential equation (1.1) is linear, the system of equations (2.12) is linear; if not, in order to solve the nonlinear system (2.12), we use Newton's method with the vector $[y(a),\ y(a), \cdots, y(a)]^T$ ($N \times \ell$ components) as initial approximation.

Having determined the y_{ik}, $i = 0, 1, \ldots, N-1,\ k = 1, \ldots, \ell$, the solution of (1.1)-(1.2) is given by

$$y(t) = \sum_{k=1}^{\ell} L_{ik}(s) y_{ik}, \quad t \in \sigma_i,\ i = 0, 1, \ldots, N-1.$$

3. Convergence analysis

In this section we prove the convergence of the proposed algorithm applied to the boundary value problem (linear case):

$$\begin{aligned} D_*^{\alpha} y(t) &= \beta y(t) + g(t) = f(t, y(t)), \quad t \in [0, a], && (3.1) \\ y(a) &= y_a, && (3.2) \end{aligned}$$

where g is a continuous function and $\beta \in \mathbb{R}$.
In this case we can rewrite equation (1.3) as

$$y(t) = (T_\alpha y)(t) + \overline{g}(t), \quad t \in [0, a], \tag{3.3}$$

where T_α is the integral operator given by

$$(T_\alpha y)(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^a \left((t-s)^{\alpha-1} \chi_{[0,t]}(s) - (a-s)^{\alpha-1} \right) y(s) ds \tag{3.4}$$

and

$$\overline{g}(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_0^a \left((t-s)^{\alpha-1} \chi_{[0,t]}(s) - (a-s)^{\alpha-1} \right) g(s) ds. \tag{3.5}$$

Therefore, the method described in the previous section for problem (3.1)-(3.2) consists of obtaining an approximate solution $u_N \in V_m^\alpha(\Delta_N)$ such that

$$u_N(t_{ik}) = (T_\alpha u_N)(t_{ik}) + \overline{g}(t_{ik}), \quad i = 0, 1, ..., N-1, \; k = 1, 2, ..., \ell. \tag{3.6}$$

Conditions (3.6) have an operator equation representation in the form:

$$u_N(t) = (P_N T_\alpha u_N)(t) + (P_N \overline{g})(t), \tag{3.7}$$

where P_N is the interpolation operator defined by (2.7).
In order to obtain u_N we must solve a linear system of equations with respect to the coefficients $u_N(t_{jk}) = u_{jk}$:

$$u_{jk} = \sum_{i=0}^{N-1} \sum_{\mu=1}^{\ell} u_{i\mu} (T_\alpha L_{i\mu})(t_{jk}) + \overline{g}(t_{jk}), \quad j = 0, 1, ..., N-1, \; k = 1, 2, ..., \ell \tag{3.8}$$

where $L_{i\mu}$ are the Lagrange functions defined by (2.5). After solving (3.8) an approximation to the solution of (3.2) will be given by

$$y(s) \approx u_N(s) = \sum_{j=0}^{N-1} \sum_{k=1}^{\ell} u_{jk} L_{jk}(s), \quad s \in [0, a].$$

In the next theorem we prove the convergence of the proposed algorithm and prove the convergence order for arbitrary collocation parameters $c_1, ..., c_\ell$ satisfying (2.4).
Note that taking Lemma 1.2 into account, the solution of (3.1)-(3.2) may be written in the form $y = y_1 + y_2$ where, for a fixed integer m, $y_1 \in C^m([0, T])$ and $y_2 \in V_m^\alpha$ whenever $g(t) = \tilde{g}\left(t^{\frac{1}{q}}\right)$, and $\tilde{g}$ is analytical in the neighborhood of $t = 0$.

Theorem 3.1. *Let y be the unique solution of the boundary value problem (3.1)-(3.2), which may be written as a sum $y = y_1 + y_2$ where, for a fixed integer m, $y_1 \in C^m([0,T])$ and $y_2 \in V_m^\alpha$. Then, there exists an integer $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ equation (3.6) has a unique solution $u_N \in V_m^\alpha(\Delta_N)$ and*

$$\|y - u_N\|_\infty = \mathcal{O}(N^{-m}). \tag{3.9}$$

Proof.

Let $u_N(s) = (P_N y)(s) = \sum_{j=0}^{N-1} \sum_{k=1}^{\ell} y(t_{jk}) L_{jk}(s)$. From (3.3) and (3.7) we have

$$\begin{aligned}
(I - P_N T_\alpha)(y - u_N) &= y - u_N - P_N T_\alpha y + P_N T_\alpha u_N + P_N \overline{g} - P_N \overline{g} \\
&= y - u_N - P_N (T_\alpha y + \overline{g}) + P_N T_\alpha u_N + P_N \overline{g} \\
&= y - u_N - P_N y + u_N \qquad (3.10)\\
&= y - P_N y
\end{aligned}$$

For $0 < \alpha < 1$, the operator T_α, from $L^\infty((0,a))$ into $C([0,a])$, is obviously linear and is compact (since it is an integral operator with a weakly singular kernel). Moreover, equation $z = T_\alpha z$ has a unique solution in $L^\infty((0,a))$, the trivial solution $z = 0$. Therefore, there exists an inverse operator

$$(I - T_\alpha)^{-1} \in \mathcal{L}\left(L^\infty((0,a)), L^\infty((0,a))\right). \tag{3.11}$$

On the other hand, since the interpolation operators P_N, $N \in \mathbb{N}$, are uniformly bounded (see (2.8)), then

$$\|y - P_N y\|_\infty \longrightarrow 0 \quad \text{as} \quad N \to \infty. \tag{3.12}$$

From (3.11), (3.12) and using a standard argument (see e.g. [1], [3]), we conclude that there exists an integer N_0 such that for $N \geq N_0$ the operators $I - P_N T_\alpha$ are invertible in $L^\infty(0,a)$ and there exists a positive constant C, that does not depend on N, such that

$$\|(I - P_N T_\alpha)^{-1}\|_{\mathcal{L}(L^\infty((0,a)), L^\infty((0,a)))} \leq C, \quad N \geq N_0. \tag{3.13}$$

Using the above result on (3.10) we obtain

$$y - u_N = (I - P_N T_\alpha)^{-1} (y - P_N y), \quad N \geq N_0, \tag{3.14}$$

and consequently,

$$\|y - u_N\|_\infty \le C\|y - P_N y\|_\infty, \quad N \ge N_0. \tag{3.15}$$

From Theorem 2.1 it follows that $\|y - P_N y\|_\infty \to 0$ (for every y satisfying the conditions of the function g in that theorem) as $N \to \infty$ and

$$\|y - P_N y\|_\infty = \mathcal{O}(N^{-m}).$$

Using these results on (3.15) we prove the convergence, $\|y - u_N\|_\infty \to 0$, as $N \to \infty$, and (3.9).

□

4. Numerical examples

In this section we illustrate the theoretical results we have obtained and the performance of the proposed nonpolynomial collocation method, (NPCM), presented in Section 2. In order to do this we consider several examples. We compare the numerical results obtained by the proposed method with the numerical results obtained with the shooting algorithm based on the secant method where the initial value problem solver used is the nonpolynomial collocation method proposed in [15].
All the numerical experiments have been coded in *Mathematica* and run on a personal computer with processor Intel(R) Core(TM) i5-3230M, CPU 2.60 GHz under operating system Microsoft Windows 8.

Throughtout this section, ε_N and $\widehat{\varepsilon}_N$ represent the errors at the collocation and discretisation points, respectively, and p the experimental order of convergence:

$$\varepsilon_N = \max_{j=0,1,\dots,N-1} \max_{k=1,\dots,\ell} |y(t_{jk}) - u_N(t_{jk})|, \tag{4.1}$$

$$\widehat{\varepsilon}_N = \max_{j=0,1,\dots,N} |y(t_j) - y_N(t_j)| \tag{4.2}$$

$$p = \log\left(\frac{\varepsilon_N}{\varepsilon_{2N}}\right) / \log(2). \tag{4.3}$$

First we consider a simple boundary value problem (BVP):

$$\begin{cases} D_*^{1/2} y(t) = \frac{1}{2} y, \quad t \in [0,1] \\ y_a = y(1) = E_{1/2}(0.5) = 1.9523604891825572237875 30..., \quad a = 1 \end{cases}, \tag{4.4}$$

where $E_{1/2}$ denotes the Mittag-Leffler function, $E_{1/2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)}$, $z > 0$, and the exact solution is $y(t) = E_{1/2}(0.5\sqrt{t})$.

In Tables 1 and 2 we present the numerical results obtained by the application of the nonpolynomial collocation method for different values of N. From Tables 1 and 2 we can see that the maximum of the errors, using the NPCM on the space $V_m^{1/2}(\Delta_N)$, at the mesh points, converges to zero with order m which is in agreement with the result provided by Theorem 3.1.

Table 1: *Maximum of the errors and experimental order of convergence for the NPCM, applied to the BVP (4.4) on the spaces $V_1^{1/2}(\Delta_N)$ and $V_2^{1/2}(\Delta_N)$, with collocation parameters $c_1 = 0.5,\ c_2 = 1$ and $c_1 = 0.25,\ c_2 = 0.5,\ c_3 = 0.75$ and $c_4 = 1$, respectively.*

	NPCM on the space $V_1^{1/2}(\Delta_N)$				**NPCM on the space $V_2^{1/2}(\Delta_N)$**			
N	$\widehat{\varepsilon}_N$	p	ε_N	p	$\widehat{\varepsilon}_N$	p	ε_N	p
10	$2.2513 \cdot 10^{-2}$	–	$2.1556 \cdot 10^{-4}$	–	$1.3317 \cdot 10^{-4}$	–	$7.8782 \cdot 10^{-7}$	–
20	$1.0473 \cdot 10^{-2}$	1.10	$7.9273 \cdot 10^{-5}$	1.43	$3.0172 \cdot 10^{-5}$	2.14	$1.3021 \cdot 10^{-7}$	2.60
40	$4.9805 \cdot 10^{-3}$	1.07	$2.9419 \cdot 10^{-5}$	1.44	$7.0415 \cdot 10^{-6}$	2.10	$2.1931 \cdot 10^{-8}$	2.57
80	$2.4042 \cdot 10^{-3}$	1.05	$1.0878 \cdot 10^{-5}$	1.45	$1.6775 \cdot 10^{-6}$	2.07	$3.7446 \cdot 10^{-9}$	2.55
160	$1.1727 \cdot 10^{-3}$	1.04	$3.9937 \cdot 10^{-6}$	1.45	$4.0538 \cdot 10^{-7}$	2.05	$6.4573 \cdot 10^{-10}$	2.54
320	$5.7622 \cdot 10^{-4}$	1.03	$1.4555 \cdot 10^{-6}$	1.46	$9.8954 \cdot 10^{-8}$	2.03	$1.1215 \cdot 10^{-10}$	2.53

Table 2: *Maximum of the errors and experimental order of convergence for the NPCM , applied to the BVP (4.4) on the space $V_3^{1/2}(\Delta_N)$ with collocation parameters $c_1 = 0.165,\ c_2 = 0.33,\ c_3 = 0.495,\ c_4 = 0.66,\ c_5 = 0.825$ and $c_6 = 1$.*

	NPCM on the space $V_3^{1/2}(\Delta_N)$			
N	$\widehat{\varepsilon}_N$	p	ε_N	p
10	$4.6753 \cdot 10^{-7}$	–	$1.5895 \cdot 10^{-9}$	–
20	$5.1999 \cdot 10^{-8}$	3.17	$1.2765 \cdot 10^{-10}$	3.64
40	$5.9899 \cdot 10^{-9}$	3.12	$1.0544 \cdot 10^{-11}$	3.60
80	$7.0652 \cdot 10^{-10}$	3.08	$8.8751 \cdot 10^{-13}$	3.57
160	$8.2977 \cdot 10^{-11}$	3.09	$7.4607 \cdot 10^{-14}$	3.57

The numerical results obtained by the shooting algorithm are similar to the ones obtained by the proposed method. However, by looking at the relative CPU times in Figure 1 we can see that the computational cost (measured by the running time) of the shooting algorithm is considerably higher.

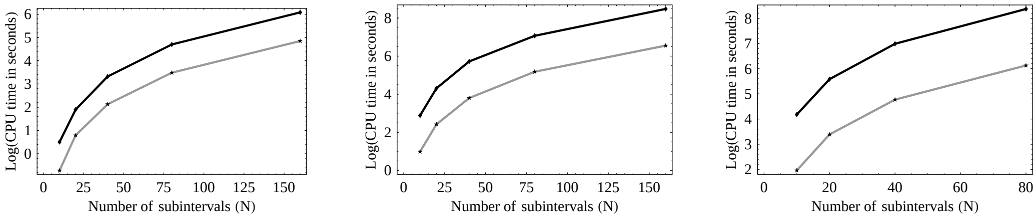

Figure 1: Example 4.4: The computational cost of the nonpolynomial collocation method (grey line) and shooting method (black line) on the spaces $V_m^\alpha(\Delta_N)$, $m=1$ (left), $m=2$ (center) and $m=3$ (right).

Now we consider two linear terminal value problems whose analytical solutions are not smooth:

$$\begin{cases} D_*^{1/2}y(t) = -\frac{1}{4}t^{3/2} + \frac{3\sqrt{\pi}}{4}t + \frac{1}{4}y, \quad t \in [0,a], \\ y_a = \frac{1}{2\sqrt{2}}, \quad a = 1/2. \end{cases} \tag{4.5}$$

$$\begin{cases} D_*^{1/2}y(t) = t, \quad t \in [0,a], \\ y_a = 1 + \frac{4}{3\sqrt{\pi}}, \quad a = 1. \end{cases} \tag{4.6}$$

The exact solution of the terminal value problems (4.5), (4.6) are $y(t) = t^{3/2}$ and $y(t) = \frac{4}{3\sqrt{\pi}}t^{3/2} + 1$, respectively.

The maximum of the absolute errors using the NPCM on the space $V_1^{1/2}(\Delta_N)$ are presented in Table 3.

In Table 4 we compare the results obtained by the NPCM and the shooting method, at $t=0$ for examples (4.5) and (4.6). The shooting method seems to converge with a higher order than the NPCM. However, by looking at the relative CPU times in Table 5 we observe, once again, that the computational cost of the shooting method is considerably higher.

Next we consider another example

$$\begin{cases} D_*^{1/2}y(t) = \frac{2}{\Gamma(5/2)}t^{3/2} - \frac{1}{\Gamma(3/2)}t^{1/2} + t^2 - t - y, \quad t \in [0,a], \\ y_a = 0, \quad a = 1, \end{cases} \tag{4.7}$$

Table 3: *Maximum of the errors, ε_N, and experimental order of convergence for the NPCM and shooting method on the space $V_1^{1/2}(\Delta_N)$ with collocation parameters $c_1 = 0.5,\ c_2 = 1$ applied to the BVPs (4.5) and (4.6).*

	BVP (4.5)				BVP (4.6)			
	NPCM		Shooting method		NPCM		Shooting method	
N	ε_N	p	ε_N	p	ε_N	p	ε_N	p
10	$7.52 \cdot 10^{-4}$	-	$1.56 \cdot 10^{-2}$	-	$1.57 \cdot 10^{-3}$	-	$3.31 \cdot 10^{-2}$	-
20	$2.90 \cdot 10^{-4}$	1.38	$5.53 \cdot 10^{-3}$	1.49	$6.07 \cdot 10^{-4}$	1.37	$1.17 \cdot 10^{-2}$	1.49
40	$1.10 \cdot 10^{-4}$	1.40	$1.96 \cdot 10^{-3}$	1.50	$2.31 \cdot 10^{-4}$	1.39	$4.17 \cdot 10^{-3}$	1.49
80	$4.09 \cdot 10^{-5}$	1.43	$6.95 \cdot 10^{-4}$	1.50	$8.65 \cdot 10^{-5}$	1.42	$1.48 \cdot 10^{-3}$	1.5
160	$1.51 \cdot 10^{-5}$	1.44	$2.46 \cdot 10^{-4}$	1.50	$3.20 \cdot 10^{-5}$	1.44	$5.24 \cdot 10^{-4}$	1.5

Table 4: *Values for $y(0)$ for the BVPs (4.5) and (4.6), and obtained by the NPCM on the space $V_1^{1/2}(\Delta_N)$ with collocation parameters $c_1 = 0.5,\ c_2 = 1$ and shooting method (with non polynomial collocation method on the space $V_1^{1/2}(\Delta_N)$ and collocation parameters $c_1 = 0.5,\ c_2 = 1$ to solve the IVP).*

	BVP (4.5)		BVP (4.6)	
N	NPC method	shooting method	NPC method	shooting method
10	$-0.155754 \cdot 10^{-1}$	$0.389817 \cdot 10^{-3}$	0.966934	1.000766
20	$-0.552759 \cdot 10^{-2}$	$0.110050 \cdot 10^{-3}$	0.988257	1.000217
40	$-0.196098 \cdot 10^{-2}$	$0.30524 \cdot 10^{-4}$	0.995832	1.000060
80	$-0.695319 \cdot 10^{-3}$	$0.835930 \cdot 10^{-5}$	0.998521	1.000016
160	$-0.246415 \cdot 10^{-3}$	$0.226749 \cdot 10^{-5}$	0.999476	1.000004
p	1.5	1.9	1.5	1.9

Table 5: *The CPU running-time, in seconds, for the NPCM and shooting method on the space $V_1^{1/2}(\Delta_N)$, applied to the BVPs (4.5) and (4.6).*

	Example 4.6		Example 4.4	
N	NPC method	Shooting method	NPC method	Shooting method
10	0.5	1.828	0.485	1.735
20	1.985	7.922	1.969	7.265
40	8.296	30.39	8.375	29.594
80	33.844	114.563	32.609	115.984
160	140.281	463.015	131.094	453.172

whose analytical solution is known and is given by $y(t) = t^2 - t$. In this case the solution is smooth but the fractional derivative $D^{1/2}y$ is nonsmooth.

For this example, the maximum of the errors and the experimental orders of convergence are presented, for different values of the stepsize h, in Table 6. From Table 6, we observe that the values obtained by the two methods are the same, in this example. For the approximation on the nonpolynomial space $V_1^{1/2}$ the convergence orders increase and is approximately 1. On the other hand, for the approximation on the nonpolynomial space $V_2^{1/2}$ a convergence order 2 is obtained for all values of the step size.

Table 6: *Maximum of the errors and experimental orders of convergence for the NPCM and shooting method, applied to the BVP (4.7), on the spaces $V_1^{1/2}(\Delta_N)$ and $V_2^{1/2}(\Delta_N)$ with collocation parameters $c_1 = 0.5,\ c_2 = 1$ and $c_1 = 0.25,\ c_2 = 0.5,\ c_3 = 0.75,\ c_4 = 1$, respectively.*

	NPC method				Shooting algorithm			
	$m=1$		$m=2$		$m=1$		$m=2$	
N	$\widehat{\varepsilon}_N$	p	$\widehat{\varepsilon}_N$	p	$\widehat{\varepsilon}_N$	p	$\widehat{\varepsilon}_N$	p
10	$5.17 \cdot 10^{-2}$	-	$3.06 \cdot 10^{-3}$	-	$5.17 \cdot 10^{-2}$	-	$3.06 \cdot 10^{-3}$	-
20	$3.06 \cdot 10^{-2}$	0.76	$7.65 \cdot 10^{-4}$	2	$3.06 \cdot 10^{-2}$	0.76	$7.65 \cdot 10^{-4}$	2
40	$1.65 \cdot 10^{-2}$	0.89	$1.91 \cdot 10^{-4}$	2	$1.65 \cdot 10^{-2}$	0.89	$1.91 \cdot 10^{-4}$	2
80	$8.54 \cdot 10^{-3}$	0.95	$4.78 \cdot 10^{-5}$	2	$8.54 \cdot 10^{-3}$	0.95	$4.78 \cdot 10^{-5}$	2
160	$4.34 \cdot 10^{-3}$	0.98	$1.20 \cdot 10^{-5}$	2	$4.34 \cdot 10^{-3}$	0.98	$1.20 \cdot 10^{-5}$	2

Finally, we consider a nonlinear example:

$$\begin{aligned} D_*^\alpha y(t) &= \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) \\ &\quad + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3 - (y(t))^{3/2}, \quad t \in [0,1], \\ y(1) &= 1. \end{aligned} \tag{4.8}$$

The exact solution of this terminal value problem is $y(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^\alpha$, meaning that the solution $y(t)$ can be written as $y(t) = u(t) + v(t)$ with $u(t) = \frac{9}{4}t^\alpha \in V_m^\alpha$ and $v(t) = t^8 - 3t^{4+\alpha/2} \in C^m([0,1]),\ m = 1$.

We consider the NPCM on the space $V_1^\alpha(\Delta_N)$ applied to the problem (4.8), for several values of α. For $\alpha = 1/2$ and $\alpha = 2/3$ we have $\ell = 2$ and in these cases the following collocation parameters have been used $c_1 = 0.5$ and

$c_2 = 1.0$. For $\alpha = 1/4$ and $\alpha = 1/3$ we have, $\ell = 4$ and $\ell = 3$, respectively, and then in these cases we have set $c_1 = 0.25,\ c_2 = 0.5,\ c_4 = 0.75,\ c_3 = 1$, for $\alpha = 1/4$ and $c_1 = 0.3,\ c_2 = 0.6,\ c_3 = 1$, for $\alpha = 1/3$.

In Table 7 some results of numerical experiments for different values of the stepsize h are presented. We also compare these with the numerical results obtained by the shooting method.

Table 7: *Maximum of the errors and experimental order of convergence for the NPCM and shooting method on the space $V_1^\alpha(\Delta_N)$, applied to the BVP (4.8).*

N	$\alpha = 1/4$		$\alpha = 1/3$		$\alpha = 1/2$		$\alpha = 2/3$	
	ε_N	p	ε_N	p	ε_N	p	ε_N	p
	NPCM							
10	$8.69 \cdot 10^{-5}$	-	$6.88 \cdot 10^{-4}$	-	$5.08 \cdot 10^{-3}$	-	$5.25 \cdot 10^{-3}$	-
20	$6.24 \cdot 10^{-6}$	3.80	$1.05 \cdot 10^{-4}$	2.71	$1.16 \cdot 10^{-3}$	2.13	$2.04 \cdot 10^{-3}$	1.37
40	$4.39 \cdot 10^{-7}$	3.83	$1.55 \cdot 10^{-5}$	2.76	$3.40 \cdot 10^{-4}$	1.77	$6.97 \cdot 10^{-4}$	1.55
80	$3.03 \cdot 10^{-8}$	3.86	$2.21 \cdot 10^{-6}$	2.81	$1.22 \cdot 10^{-4}$	1.48	$2.08 \cdot 10^{-4}$	1.74
160	$2.04 \cdot 10^{-9}$	3.89	$3.08 \cdot 10^{-7}$	2.84	$3.82 \cdot 10^{-5}$	1.68	$5.81 \cdot 10^{-5}$	1.84
	Shooting method							
10	$1.67 \cdot 10^{-2}$	-	$3.23 \cdot 10^{-3}$	-	$1.18 \cdot 10^{-3}$	-	$5.21 \cdot 10^{-3}$	-
20	$9.69 \cdot 10^{-4}$	4.10	$2.69 \cdot 10^{-4}$	3.58	$8.50 \cdot 10^{-4}$	-	$2.21 \cdot 10^{-3}$	1.24
40	$5.63 \cdot 10^{-5}$	4.10	$2.67 \cdot 10^{-5}$	3.33	$3.83 \cdot 10^{-4}$	1.15	$7.26 \cdot 10^{-4}$	1.61
80	$3.27 \cdot 10^{-6}$	4.11	$3.03 \cdot 10^{-6}$	3.14	$1.32 \cdot 10^{-4}$	1.54	$2.13 \cdot 10^{-4}$	1.77
160	$1.89 \cdot 10^{-7}$	4.11	$3.71 \cdot 10^{-7}$	3.03	$4.00 \cdot 10^{-5}$	1.72	$5.87 \cdot 10^{-5}$	1.85

Finally in Table 8 we list the approximations of $y(t)$ at $t = 0$, for several values of α, obtained by the NPCM on the space $V_1^\alpha(\Delta_N)$.

Table 8: *Comparison with the exact solution $y(t)$ of the BVP (4.8), for several values of α, at $t = 0$ (NPCM on the space $V_1^\alpha(\Delta_N)$).*

N	Approximations of $y(0)$				
	$\alpha = 1/4$	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 2/3$	$\alpha = 9/10$
5	$2.8443 \cdot 10^{-1}$	$-4.5403 \cdot 10^{-2}$	$-7.7533 \cdot 10^{-4}$	$5.6169 \cdot 10^{-3}$	$3.9180 \cdot 10^{-2}$
10	$1.6625 \cdot 10^{-2}$	$-3.2275 \cdot 10^{-3}$	$6.4858 \cdot 10^{-4}$	$5.2140 \cdot 10^{-3}$	$1.4152 \cdot 10^{-2}$
20	$9.6863 \cdot 10^{-4}$	$-2.6901 \cdot 10^{-4}$	$8.5029 \cdot 10^{-4}$	$2.2136 \cdot 10^{-3}$	$4.1093 \cdot 10^{-3}$
40	$5.6318 \cdot 10^{-5}$	$-2.6677 \cdot 10^{-5}$	$3.8329 \cdot 10^{-4}$	$7.2568 \cdot 10^{-4}$	$1.1073 \cdot 10^{-3}$
80	$3.2678 \cdot 10^{-6}$	$-3.0257 \cdot 10^{-6}$	$1.3159 \cdot 10^{-4}$	$2.1260 \cdot 10^{-4}$	$2.8863 \cdot 10^{-4}$
160	$1.8926 \cdot 10^{-7}$	$-3.7056 \cdot 10^{-7}$	$3.9975 \cdot 10^{-5}$	$5.8671 \cdot 10^{-5}$	$7.3927 \cdot 10^{-5}$

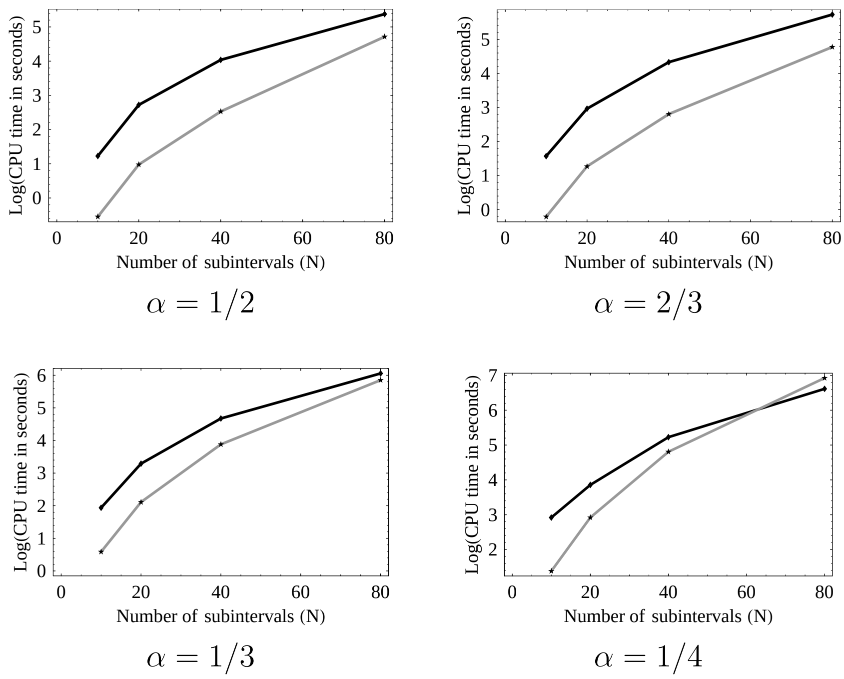

Figure 2: Example 4.8: The computational cost of the nonpolynomial collocation method (grey line) and shooting method (black line) on the sapace $V_1^{\alpha}(\Delta_N)$, for several values of α.

Remark 4.1.

In order to compute the coefficients β_{kp}^{j}, $p = 0, \ldots, \ell-1$, $k = 1, \ldots, \ell$, $j = 0, \ldots, N-1$, that define the nonpolynomial approximation $u \in V_m^{\alpha}(\Delta_N)$ of the boundary fractional differential equation of order α (1.1)-(1.2), we must solve the systems (2.6). One possible drawback of the proposed method is that for high values of ℓ, the systems (2.6) can be strongly ill-conditioned. In order to solve the $\ell \times \ell$ linear systems we use the function LinearSolve[] of the software *Mathematica*.
Let $Cond_N^{\ell} = \max\limits_{j=0,1,\ldots,N-1,\, k=1,2,\ldots,\ell} cond(A_{j,k})$, with $cond(A) = \|A\|_\infty \|A^{-1}\|_\infty$ and $A_{j,k}$ be the $\ell \times \ell$ matrices associated with the linear systems (2.6) and $Res_N^{\ell} = \max\limits_{j=0,1,\ldots,N-1,\, k=1,2,\ldots,\ell} \|A_{j,k}\boldsymbol{\beta}_{j,k} - \boldsymbol{\delta}_k\|_\infty$, where $\boldsymbol{\beta}_{j,k}$ is the solution computed and $\boldsymbol{\delta}_k$ the independent vector associated to the systems of equations (2.6). In Tables 9 and 10 we list $Cond_N^{\ell}$, $\ell = \#V_m^{\alpha}(\Delta_N)$, for $m = 1,\ 2$, $a = 1$

and several values of N and α. In each case we observe that the values of Res_N^ℓ are approximately zero.

Table 9: *The maximum of the condition numbers associated with the matrices of the systems (2.6) and the maximum of the residues Res_N^ℓ, for several values of N and α ($\ell = \#V_1^\alpha(\Delta_N)$)*

	$V_1^\alpha(\Delta_N)$							
	$\alpha = 1/4;\ \ell = 4$		$\alpha = 1/3;\ \ell = 3$		$\alpha = 1/2;\ \ell = 2$		$\alpha = 2/3;\ \ell = 2$	
N	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ
10	$6 \cdot 10^7$	10^{-94}	$8 \cdot 10^4$	10^{-96}	$2 \cdot 10^2$	10^{-99}	$1 \cdot 10^2$	10^{-99}
20	$5 \cdot 10^8$	10^{-93}	$3 \cdot 10^5$	10^{-96}	$3 \cdot 10^2$	10^{-98}	$2 \cdot 10^2$	10^{-98}
40	$4 \cdot 10^9$	10^{-92}	$1 \cdot 10^6$	10^{-95}	$6 \cdot 10^2$	10^{-98}	$5 \cdot 10^2$	10^{-98}
80	$3 \cdot 10^{10}$	10^{-91}	$6 \cdot 10^6$	10^{-94}	$1 \cdot 10^3$	10^{-98}	$1 \cdot 10^3$	10^{-98}
160	$3 \cdot 10^{11}$	10^{-90}	$2 \cdot 10^7$	10^{-94}	$3 \cdot 10^3$	10^{-98}	$2 \cdot 10^3$	10^{-98}
320	$2 \cdot 10^{12}$	10^{-89}	$9 \cdot 10^7$	10^{-93}	$5 \cdot 10^3$	10^{-97}	$4 \cdot 10^3$	10^{-97}
640	$2 \cdot 10^{13}$	10^{-88}	$4 \cdot 10^8$	10^{-93}	$1 \cdot 10^4$	10^{-97}	$8 \cdot 10^3$	10^{-97}
1280	$1 \cdot 10^{14}$	10^{-86}	$1 \cdot 10^9$	10^{-92}	$2 \cdot 10^4$	10^{-97}	$2 \cdot 10^4$	10^{-97}

Table 10: *The maximum of the condition numbers associated with the matrices of the systems (2.6) and the maximum of the residues Res_N^ℓ, for several values of N and α ($\ell = \#V_2^\alpha(\Delta_N)$)*

	$V_2^\alpha(\Delta_N)$							
	$\alpha = 1/4;\ \ell = 8$		$\alpha = 1/3;\ \ell = 6$		$\alpha = 1/2;\ \ell = 4$		$\alpha = 2/3;\ \ell = 5$	
N	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ	$Cond_N^\ell$	Res_N^ℓ
10	$5 \cdot 10^{18}$	10^{-84}	$3 \cdot 10^{12}$	10^{-90}	$8 \cdot 10^6$	10^{-94}	$6 \cdot 10^9$	10^{-92}
20	$7 \cdot 10^{20}$	10^{-82}	$9 \cdot 10^{13}$	10^{-88}	$6 \cdot 10^7$	10^{-93}	$1 \cdot 10^{11}$	10^{-91}
40	$9 \cdot 10^{22}$	10^{-80}	$3 \cdot 10^{15}$	10^{-86}	$5 \cdot 10^8$	10^{-93}	$2 \cdot 10^{12}$	10^{-89}
80	$1 \cdot 10^{25}$	10^{-78}	$1 \cdot 10^{17}$	10^{-85}	$4 \cdot 10^9$	10^{-92}	$3 \cdot 10^{13}$	10^{-88}
160	$2 \cdot 10^{27}$	10^{-76}	$3 \cdot 10^{18}$	10^{-83}	$3 \cdot 10^{10}$	10^{-91}	$4 \cdot 10^{14}$	10^{-87}
320	$2 \cdot 10^{29}$	10^{-74}	$1 \cdot 10^{20}$	10^{-82}	$3 \cdot 10^{11}$	10^{-90}	$7 \cdot 10^{15}$	10^{-86}
640	$3 \cdot 10^{31}$	10^{-71}	$3 \cdot 10^{21}$	10^{-80}	$2 \cdot 10^{12}$	10^{-89}	$1 \cdot 10^{17}$	10^{-85}
1280	$3 \cdot 10^{33}$	10^{-69}	$1 \cdot 10^{23}$	10^{-79}	$2 \cdot 10^{13}$	10^{-88}	$2 \cdot 10^{18}$	10^{-82}

5. Conclusions

In this paper we have presented a nonpolynomial collocation method for solving fractional terminal value problems. As previously established, these problems are equivalent to Fredholm integral equations. Therefore, we have developed our previous approach for initial value problems (equivalent to Volterra integral equations), and have discretised the integral equation directly. As expected, the convergence order of the numerical scheme is also optimal as it was for initial value problems. Some numerical examples are considered in order to illustrate the performance of the method. The solution can also be determined by using a shooting method. Although the numerical results obtained by the two approaches are similar (in terms of convergence order and errors), the direct discretisation of the Fredholm equation presents an advantage against shooting when we compare the computational time, which in most cases, reduces significantly when using the nonpolynomial collocation method proposed here.

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