A STUDY OF SOME $M^{[X]}/G/1$ TYPE QUEUES WITH RANDOM BREAKDOWNS AND BERNOULLI SCHEDULE SERVER VACATIONS BASED ON A SINGLE VACATION POLICY

A thesis submitted for the degree of
Doctor of Philosophy

by
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August 2008

Abstract

Queueing systems arise in modelling of many practical applications related to computer sciences, telecommunication networks, manufacturing and production, human computer interaction, and so on. The classical queueing system, even vacation queues or queues subject to breakdown, might not be sufficiently realistic.

The purpose of this research is to extend the work done on vacation queues and on unreliable queues by studying queueing systems which take into consideration both phenomena. We study the behavior of a batch arrival queueing system with a single server, where the system is subject to random breakdowns which require a repair process, and on the other hand, the server is allowed to take a vacation after finishing a service. The breakdowns are assumed to occur while serving a customer, and when the system breaks down, it enters a repair process immediately while the customer whose service is interrupted comes back to the head of the queue waiting for the service to resume. Server vacations are assumed to follow a Bernoulli schedule under single vacation policy.

We consider the above assumptions for different queueing models: queues with generalized service time, queues with two-stages of heterogeneous service, queues with a second optional service, and queues with two types of service. For all the models mentioned above, it is assumed that the service times, vacation times, and repair times all have general arbitrary distributions.

Applying the supplementary variable technique, we obtain probability generating functions of queue size at a random epoch for different states of the system, and some performance measures such as the mean queue length, mean waiting time in the queue, proportion of server's idle time, and the utilization factor. The results obtained in this research, show the effect of vacation and breakdown parameters upon main performance measures of interest. These effects are also illustrated using some numerical examples and graphs.

Dedication

This thesis is dedicated in loving memory to my beloved father
WHO
used to pray for me, everyday, for three years
used to ask me, everyday, how much left to finish my study
BUT
his late illness didn't leave him for another three weeks,
to celebrate my success

I deeply feel downhearted that
I will never be able to tell him
"Father, I have finished."

May Allah bless his soul

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Acknowledgment

All Praise is to Almighty Allah who gave me the strength, perseverance and patience to carry out this work successfully. I thank Allah for the guidance, compassion, and mercy which He has bestowed upon me throughout my entire life and in particular while working on this thesis.

I am truly grateful to my privileged supervisor Professor Kailash C. Madan for his guidance, patience and support over the years. His wide knowledge and encouragement were invaluable for me. I greatly appreciate your help and your willingness to answer my questions whenever I knock your door. I would like to thank you for introducing me to the world of Queueing Theory, a field that is very fascinating and challenging.

My appreciation and sincere thanks are extended to my privileged supervisor Professor Ken Darby-Dowman whose guidance was exceptional as he supported and encouraged me throughout this process. Thank you for providing me with your generous time and valuable feedback whenever I needed your help.

I wish to express my gratitude to the Ministry of Education, Kingdom of Bahrain, for providing me with the scholarship. I am especially grateful for the support provided by his Excellency Dr. Majid Bin Ali Al-Noaimi, the Minister of Education.

I am truly fortunate to have a loving family who always support me. My parents, my husband and my sister: it has been because of your warm love and continued support and prayers that I could realise my dreams. Thank you for your patience, understanding and encouragement during the difficult times. I really believe that this thesis is a result of priceless sacrifices from you. For this, I love you.

Declaration

From the research carried out during the course of this PhD study and connected research, the following papers have been accepted for publication:

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. (2009) 'Bernoulli schedule vacation queues with batch arrivals and random system breakdowns having general repair time distribution', *International Journal of Operational Research*, 4 (3-4). Accepted in June 2007.

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. (2009) 'Batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time distribution', *International Journal of Information and Management Sciences*, 20 (1). Accepted in March 2008.

Also, the following papers have been sent for publication:

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'A repairable $M^X/G/1$ queue with Generalised Bernoulli server vacations', *Journal of Applied Statistical Science*, by Nova Science Publishers, USA.

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'A single server queue with two-stage heterogeneous service, Bernoulli server vacations and random breakdowns having general repair time distribution', *Computers & Mathematics with Applications*. Published by Elsevier, The Netherlands.

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'A two-phase batch arrival queue with Bernoulli schedule vacations having general vacation times and random breakdowns having general repair times', *Stochastic Analysis and Applications*. Published by Taylor & Francis, USA.

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'Batch Arrival Vacation Queue with Second Optional Service and Random System Breakdowns', *Journal of Statistical Theory and Practice.* Published by University of North Carolina at Greensboro, USA.

Maraghi, F. A., Madan, K. C. & Darby-Dowman, K. 'Batch arrival Bernoulli schedule vacation queue with two types of general heterogeneous service and random system breakdowns', *Journal of Probability and Statistical Science.* Published by Susan Rivers' Cultural Institute, Hsinchu, Taiwan, ROC.

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Notations and Abbreviations

m	Number of customers in the system.
n	Number of customers in the queue.
λ	The average rate of customers entering the queue system.
μ	The average rate of serving customers per server.
ρ	A measure of traffic congestion for single server system which is defined as $\rho \equiv \lambda/\mu$
$P_m(t)$	Transient state probability of having exactly m customers in the system at time t.
P_m	Steady state probability of having exactly m customers in the system.
$N(t)$	Random Variable representing the total number of customers in the system at time t.
$N_q(t)$	Random Variable representing the total number of customers in the queue at time t.
$N_s(t)$	Random Variable representing the total number of customers in service at time t.
T	Random variable representing the time a customer spends in the system.
T_q	Random variable representing the time a customer spends waiting in the queue prior to entering service.
S	Random variable representing the service time with mean service time $E[S] = 1/\mu$
L	The mean number of customers in the system, i.e. $L = E[N]$
L_q	The mean number of customers in the queue, i.e. $L_q = E[N_q]$
W	The mean waiting time in the system, i.e. $W = E[T]$
W_q	The mean waiting time in the queue, i.e. $W_q = E[T_q]$
$P_n(x,t)$	Probability that, at time t, the server is providing a service and there are n ($n \geq 0$) customers in the queue excluding the one being served and the elapsed service time for this customer is x.
$P_n(t)$	$P_n(t) = \int_0^\infty P_n(x,t)dx$ denotes the probability that, at time t, the server

	is providing a service and there are n ($n \geq 0$) customers in the queue excluding the one being served irrespective of the value of x.
$V_n(x,t)$	Probability that at time t, the server is on vacation with elapsed vacation time x and there are n ($n \geq 0$) customers waiting in the queue for service.
$V_n(t)$	$V_n(t)=\int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.
$R_n(x,t)$	Probability that at time t, the system is inactive due to system breakdown and the system is under repair with elapsed repair time x, while there are n ($n \geq 0$) customers in the queue waiting for service.
$R_n(t)$	$R_n(t)=\int_0^\infty R_n(x,t)dx$ denotes the probability that at time t, the system is inactive due to system breakdown and the system is under repair irrespective of the value of x, while there are n ($n \geq 0$) customers in the queue waiting for service.
$Q(t)$	Probability that at time t, there are no customers in the system and the server is idle but available in the system.
$P_n^{(j)}(x,t)$	Probability that, at time t, the server is providing the j^{th} stage of service, $j = 1, 2$, and there are n ($n \geq 0$) customers in the queue excluding the one in j^{th} stage of service, and the elapsed service time for this customer is x.
$P_n^{(j)}(t)$	$P_n^{(j)}(t)=\int_0^\infty P_n^{(j)}(x,t)dx$ denotes the probability that at time t, the server is providing the j^{th} stage of service, $j = 1, 2$, and there are n ($n \geq 0$) customers in the queue excluding the one in j^{th} stage of service irrespective of the value of x.
$P_n^{(e)}(x,t)$	Probability that, at time t, the server is providing the first essential service and there are n ($n \geq 0$) customers in the queue excluding the one being provided the first essential service, and the elapsed service time for this customer is x.
$P_n^{(e)}(t)$	$P_n^{(e)}(t)=\int_0^\infty P_n^{(e)}(x,t)dx$ denotes the probability that, at time t, the

server is providing the first essential service and there are n $(n \geq 0)$ customers in the queue excluding the one being provided the first essential service irrespective of the value of x.

$P_n^{(o)}(x,t)$	Probability that, at time t, the server is providing the second optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being provided the second optional service, and the elapsed service time for this customer is x.
$P_n^{(o)}(t)$	$P_n^{(o)}(t) = \int_0^\infty P_n^{(o)}(x,t)dx$ denotes the probability that, at time t, the server is providing the second optional service and there are n $(n \geq 0)$ customers in the queue excluding the one being provided the second optional service irrespective of the value of x.
$P_n^{(\kappa j)}(x,t)$	Probability that, at time t, the server is providing the j^{th} kind of service, $j = 1, 2$, and there are n $(n \geq 0)$ customers in the queue excluding the one being provided the j^{th} kind of service, and the elapsed service time for this customer is x.
$P_n^{(\kappa j)}(t)$	$P_n^{(\kappa j)}(t) = \int_0^\infty P_n^{(\kappa j)}(x,t)dx$ denotes the probability that, at time t, the server is providing the j^{th} kind of service, $j = 1, 2$, and there are n $(n \geq 0)$ customers in the queue excluding the one being provided the j^{th} kind of service irrespective of the value of x.

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Chapter 1

Preliminaries

1.1 Introduction

Everyone has experienced the annoyance of having to wait in queues. Unfortunately, this phenomenon continues to be common in congested, urbanized societies. For example, in the United States, it has been estimated that Americans spend 37,000,000,000 hours per year waiting in queues. If this time could be spent productively instead, it would amount to nearly 20 million person-years of useful work each year! (Hillier & Lieberman, 2005)

We, as customers, do not generally like to wait in queues, and the managers of the establishments at which we wait also do not like us to wait, since it may cost them business, but this is the situation whenever the current demand for a service exceeds the current capacity to provide that service. Decisions regarding the amount of capacity to provide must be made frequently in industry and elsewhere. However, these are difficult decisions since it is often impossible to accurately predict when customers will arrive requiring a service and/or how much time will be needed to provide the required service. Adding more servers in the system is costly; on the other hand, fewer servers would cause the waiting queue to become excessively long. Therefore, the goal is to achieve an economical balance between the cost and the waiting line length. Queueing theory itself does not directly find this balance. However, it does contribute vital information required for such a decision by predicting various characteristics of the waiting queue such as the average waiting time, the average number of customers in the queue, etc. (Gross & Harris, 1998; Hillier & Lieberman, 2005).

Willig (2004) described a queueing system as a service centre and a population of customers, that may enter the service centre at various points of time in order to get service. In many cases, the service centre can only serve a limited number of customers at a time. If a new customer arrives and there is no free server, the customer enters a waiting line and waits until the service facility becomes available. Figure 1.1 shows the elements of a simple queueing model.

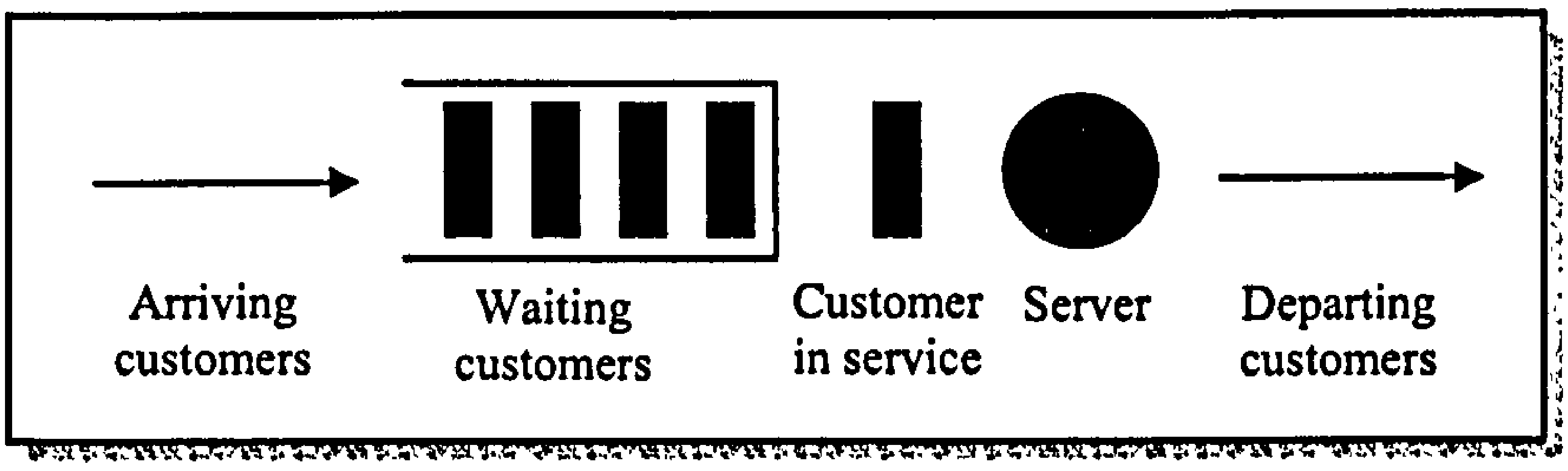

Figure 1.1 *The classical queueing model*

The term customer is used in a general sense and doesn't imply necessarily a human customer. For example, a customer could be a ball bearing waiting to be polished, an airplane waiting in line to take off, or a computer command waiting to be performed (Gross & Harris, 1998).

It is not necessary that there actually be a physical waiting line in front of the service provider. In other words, the members of the queue may be scattered throughout an area waiting for the server to come to them, e.g. machines waiting to be repaired (Hillier & Lieberman, 2005).

There are many valuable applications of the theory, most of which have been documented in the literature of probability, operations research, management science, and industrial engineering. The following are some examples in which queues occur as described by Adan and Resing (2001), Willig (2004) and Daigle (2005).

- Supermarkets: Customers wait at the checkouts, and queues become longer at peak-hours. The number of available checkouts is an important decision.
- Traffic lights: Traffic lights should be regulated in a way such that the waiting times are managed.

- Computers: A computer with a single processor runs a set of tasks. Each task requires some computation time. The task completion times are of interest, and if there are other shared resources like disks, printers, etc., then jobs waiting for these resources often have to be queued.
- Dental clinic: When there is no appointments system, patients arrive and queue to see the doctor for treatment.
- Call centres: In some companies and institutes, questions by phone are handled by a call centre. The call centre has a team structure where each operator helps a customer. A customer calling for help might wait in a queue till an operator becomes available.
- Post Office: In a post office there may be specialized counters for stamps, packages, financial transactions, etc. Counters with the same specialisation could have one common queue or separate queues.
- Data communication: As data traverses the Internet, it is multiplexed onto and demultiplexed from data communications lines at a number of switches, the interconnection of which forms an end-to-end path. Queues form at many points along the path.

1.2 Historical Background

Queueing theory history goes back nearly 100 years when A. K. Erlang, the Danish engineer who worked for the Copenhagen Telephone Exchange, published "The Theory of Probabilities and Telephone Conversations" in 1909. In later works, he observed that a telephone system could be modeled as Poisson input which represents the random arrival of calls, exponential or constant holding (service) times, and single or multiple channels. His papers written during the following 20 years contain some of the most important concepts and techniques, such as the notion of statistical equilibrium, the method of writing down balance equations, and optimization of a queueing system.

In 1927 Molina published his paper "Application of the Theory of Probability to Telephone Trunking Problems", and after a year Fry published "Probability and its Engineering Uses".

In the early 1930's, Pollaczek investigated queueing systems during finite time intervals. At that time, several theoreticians became interested in such problems and developed general models which could be used in more complex situations, such as Kolmogorov and Khintchine in Russia, Crommelin in France, and Palm in Sweden.

Queueing theory as an identifiable body of literature was essentially defined by the fundamental research of the 1950's and 1960's. In 1966, Saaty stated: "in the past seven years the literature on queueing theory has increased by half of its amount for the previous fifty years". Kovalenko (1974) presented a survey of mathematical research in queueing theory in the period from 1964 to 1970.

The queue with Poisson arrival, exponential service, and single server is one of the earliest systems to be analysed. Bailey used generating functions for the differential equations governing the system in 1954, Lederman and Reuter used spectral theory in their solution in 1956, while Laplace transformations were used later. Kendall initiated a probabilistic approach to the analysis in 1951 and 1953.

In the 1960's several authors investigated the use of approximation in the analysis of queueing systems. Kingman initiated the analysis under heavy traffic in 1965, Newell suggested fluid approximation in 1968, and Iglehart investigated diffusion approximation and weak convergence in 1970.

Computer technology inspired the field of queueing theory. The first article on queueing networks was by Jackson in 1957, while complex queueing network problems have been investigated extensively since the early 1970's. In 1975, Neuts developed an analysis technique for the transform method to multivariables. Some special queueing models of the 1950's and 1960's have found broader applicability in the context of computers and communication systems such as Polling models, Vacation models, and Retrial models.

Since the 1970's, with the advent of new processes in manufacturing, the application of queueing theory results and the development of new techniques have occurred at a phenomenal rate. Dallery and Gershwin wrote about manufacturing flow line systems. In 1992 Bitran and Dasu reported open queueing network models of manufacturing systems.

Clark gave the first theoretical treatment of the estimation problem in 1957, while the first paper on estimating parameters in a non-Markovian system is by Goyal and Harris in 1972. Hillier's paper in 1963 on economic models for industrial waiting line problems is the first paper to introduce standard optimization techniques to queueing problems. Since then, operations researchers trained in mathematical optimization techniques explored their use in a large number of queueing systems. Bäuerle considered an optimal control problem in a queueing network in 2002. (Bhat, 1969; 2008; Gross & Harris, 1998)

1.3 Characteristics of a Queueing System

In describing any queueing system, some characteristics should be specified in order to understand the nature of the system. Tanner (1995), Gross and Harris (1998), Adan and Resing (2001), and others discussed such characteristics, and they mentioned the following:

1.3.1 Arrival Pattern of Customers

In usual queueing situations, the process of arrivals is stochastic, and it is thus necessary to know the probability distribution describing the times between successive customers arrivals (interarrival times). The rate of arrivals or the average number of arrivals per minute can be defined. A common assumption is that arrivals are at random. Customers may arrive one by one or in batches, and if customers arrive in batches, it is necessary to know the probability distribution describing the size of batches. Langaris and Moutzoukis (1995) and Drezner (1999) among many others have investigated queueing systems with batch arrivals.

The population of customers may be finite or infinite. It is often assumed that the population is infinite even when the actual size is some relatively large finite number. The manner in which the pattern of arrivals changes with time should be considered. An arrival pattern could be:

- Stationary, i.e., the probability distribution describing the input process is time-independent.
- Nonstationary, i.e., the probability distribution describing the input process depends on time.

1.3.2 The Behavior of Customers

It is necessary to know the reaction of a customer upon entering the system. A customer may decide to enter the system no matter how long is the queue, or, on the other hand, a customer may decide not to enter the system if the queue is too long. Customers who decided to enter the system may be patient and willing to wait (for a long time), or may be impatient and leave after a while. Customers' impatience in queues was analysed by Altman and Yechiali (2006). In case there are two or more parallel waiting lines, customers may switch from one to another if they think they will get served faster by so doing.

1.3.3 Service Discipline

The service discipline is the manner in which customers are selected for service when a queue has formed. Some possibilities for the order in which customers enter service are:

- FIFO: First In First Out. Also called 'FCFS' which refers to First Come First Served. This is the most common discipline that can be observed in everyday life.
- LIFO: Last In First Out. Also called 'LCFS' which refers to Last Come First Served. This is applicable to many inventory systems when it is easier to reach the nearest item, which is the last in.
- SIRO: Serve In Random Order.
- RR (Round Robin): If the servicing of a customer is not completed at the end of a time slice of specified length, the customer is preempted and returned to the queue, which is served according to FCFS. This action is repeated until the customer service is completed.
- PS (Processor Sharing): This strategy corresponds to round robin with

infinitesimally small time slices. It is as if all customers are served simultaneously and the service time is increased correspondingly.

- IS (Infinite Server): There is an ample number of servers so that no queue ever forms.
- Priorities: Where customers are given priorities upon entering the system, the ones with higher priorities are served first, regardless of their time of arrival to the system. A priority discipline could be preemptive or nonpreemptive. In the preemptive case the customer with the highest priority is allowed to enter service immediately even if the server is busy with a customer with lower priority; that is the lower-priority customer in service is preempted, its service stopped, to be resumed after the higher priority customer is served. While in the nonpreemptive case the highest priority customer goes to the head of the queue but can not get into service until the current service is completed.

1.3.4 System Capacity

In some queueing systems there is waiting room capacity, so that when the number of customers in the queue reaches a certain limit, no further customers are allowed to enter until a service of a customer completes. These are called finite queueing systems, in which there is a finite limit to the maximum system size. Other queueing systems have no capacity limitation. Finch (1958) investigated the effect of the size of waiting room on a simple queue.

1.3.5 Number of Service Channels

As mentioned earlier, adding service channels to the system helps in reducing the waiting time of customers. A number of service channels mean many parallel service stations which can serve customers simultaneously, and it is generally assumed that the service mechanisms of the parallel channels operate independently of each other.

A multi-channel system could be fed with a single queue or each channel could have a separate queue. Figure 1.2 illustrates these two different types. Adan, Boxma, and Resing (2001) explored queueing models with multiple waiting lines. An example of the first type could be airplanes waiting at the holding point for an empty runway to take off, while petrol stations could fit the second type.

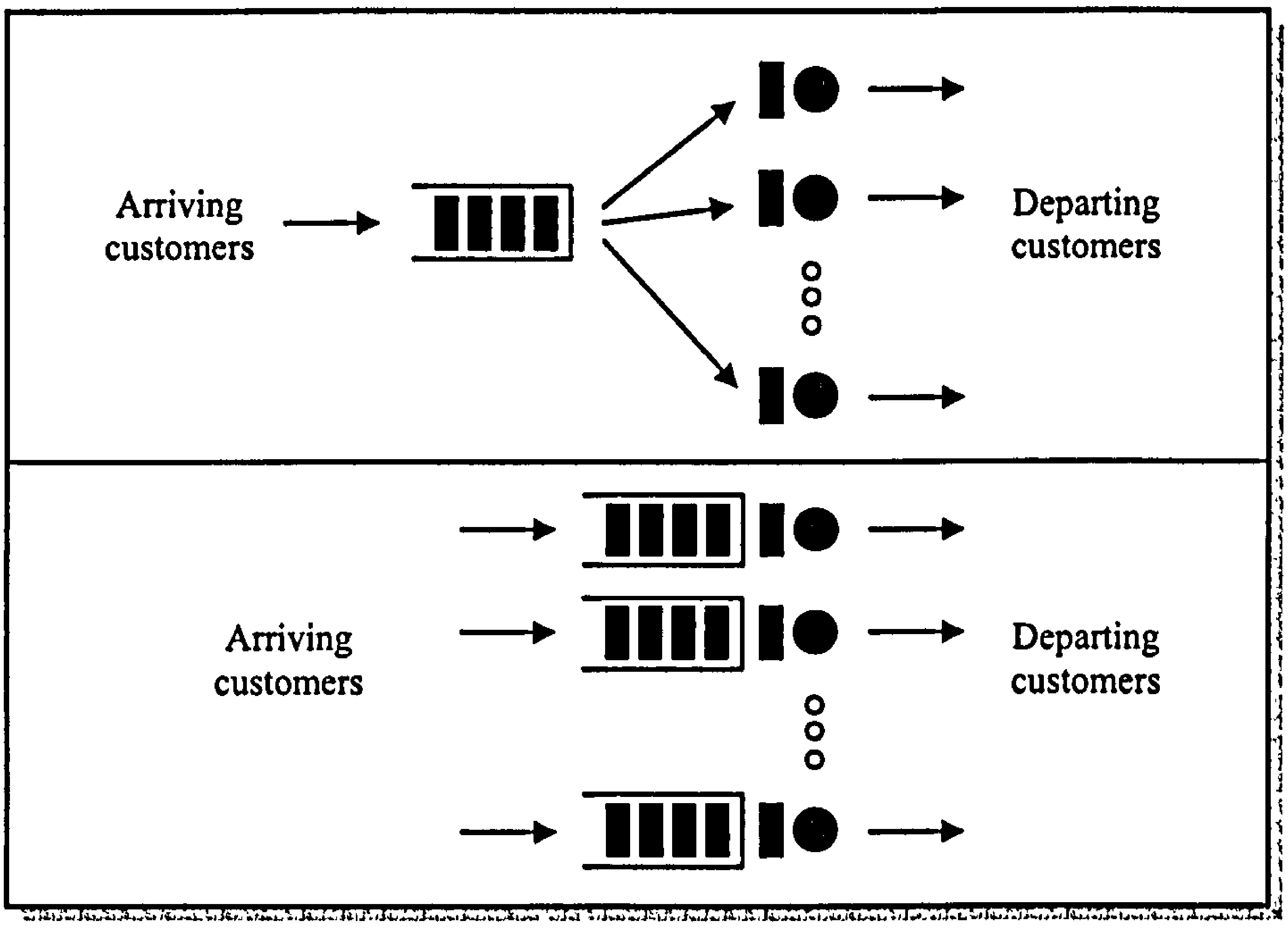

Figure 1.2 *Multi-channel queueing systems*

1.3.6 Service Mechanism

Many important aspects of service mechanism should be considered. Most importantly, a probability distribution is needed to describe service times. Service may also be single or batches. There are many situations where customers may be served simultaneously by the same server, such as people boarding a train or a computer with parallel processor. Service rate may depend on the length of the waiting line. A Server may work faster if the line is building up. The situation which service depends on the number of customers waiting is referred as state-dependent service.

Even if the service rate is high, it is very likely that some customers will be delayed waiting for the service. In general, customers arrive and depart in irregular intervals; hence the queue length will assume no definitive pattern unless arrivals and service are deterministic.

Service, like arrivals, can be stationary or non-stationary with respect to time. Usually it is assumed that the service times for customers are independent and do not depend upon the arrival process.

1.3.7 Stages of Service

A queueing system may have only a single stage as in the supermarket example, or it may have several stages, in which the customer enters a queue waits for service, gets served, departs the service station to enter a new queue for another service, and so on. An example of a multistage queueing system would be a causeway between to countries where each traveler must proceed through several stages; such as paying fee; visas; customs; and so on. Some multistage queueing systems allow recycling or feedback. Recycling could be seen in manufacturing processes where items that do not meet quality standards are sent back for reprocessing. A queueing system with feed back is illustrated in Figure 1.3.

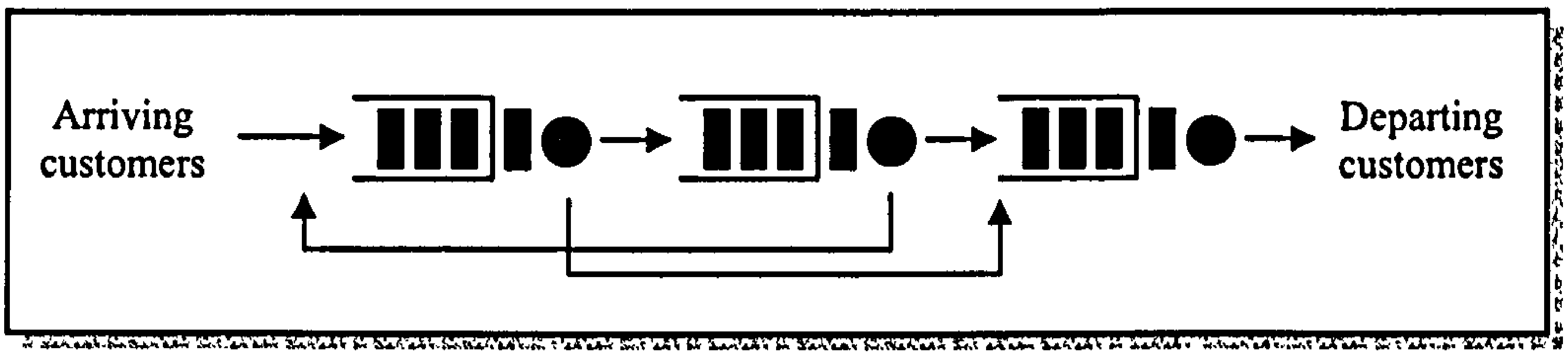

Figure 1.3 *A multistage queueing system with feedback*

1.4 Queue Notation

Describing queueing systems in short, Kendall's (1953) notation is widely used to demonstrate elementary queueing systems. Kendall's notation as described in Gelenbe and Pujolle (1998), Gross and Harris (1998), and Taha (2007) takes the following form:

$$\boxed{A/B/C/K/M/Z}$$

Where:

A: indicates the probability distribution of inter-arrival times

B: indicates the probability distribution of service time

C: Number of parallel service channels

K: The restriction on system capacity
M: The size of population of customers
Z: is the queue discipline

Some standard symbols for these characteristics are presented in Table 1.1. For example:

- *M*/*D*/1/∞/∞/FCFS: indicates a queueing process with exponential interarrival times, Deterministic service times, a single server, no restriction on the maximum number of customers allowed in the system, the population of customers is infinite and first come-first served queue discipline.
- *G*/E_k/2/10/100/SIRO: indicates a queueing system with general interarrival time distribution, an Erlang-*k* service distribution, two servers, a maximum of ten customers are allowed in the system at any one time, only one hundred users can occupy this queue and the service discipline is random.

Table 1.1 *Queueing notation A/B/C/K/M/Z*

Characteristic	Symbol	Explanation
Interarrival-time distribution (*A*) Service-time distribution (*B*)	*M*	Exponential (Markovian)
	D	Deterministic
	E_k	Erlang distribution with *k* phases
	H_k	Mixture of *k* exponentials
	C_k	Cox distribution with *k* phases
	PH	Phase type
	GI	General Independent distribution
	G	General
Number of parallel servers (*C*)	1, 2, ..., ∞	
System capacity (*K*)	1, 2, ..., ∞	
Population of users (*M*)	1, 2, ..., ∞	
Queue discipline (Z)	FIFO	First In, first Out
	LIFO	Last In, first Out
	SIRO	Serve In Random Order
	PR	Priority
	RR	Round Robin
	IS	Infinite Server
	GD	General Discipline

In many situations the last three elements of Kendall's notation are omitted if the system capacity is not limited to a certain number (*K* = ∞), the population of customers is infinite ($M = \infty$), and the service discipline is first come, first served (Z = FCFS). Thus $M/G/1$ is a queueing system with exponential inter-arrival times, general service time distribution, single server, infinite waiting rooms, infinite population, and first come first served discipline.

The current research deals with $M^{[X]}/G/1$ queueing system, that is Poisson arrivals (exponential interarrival times), general service time distribution, single server, no system capacity, infinite population of customers and first come – first served queue discipline. The superscript $^{[X]}$ indicates that the customers arrive to the system in batches of variable size.

1.5 Performance Measures

There are many parameters which are of importance when measuring the system effectiveness. These parameters could be classified into two categories and mentioned by Bose (2002):

1. Parameters of interest for a customer arriving to the queue:
   - Probability distribution of the queueing time.
   - Probability distribution of the total time spent by a customer in the system.
   - Probability distribution of the number of customers in the queue.
   - Probability distribution of the number of customers in the System.
   - Blocking probability for the finite capacity queues.
   - Probability that the customer has to wait for service.
2. Parameters of interest for the service provider:
   - Probability distribution for the service utilization.
   - Probability distribution for the buffer utilization.
   - Total revenue obtained or total revenue lost.
   - Customer satisfaction.

Usually mean performance measures are of interest in studying queueing systems such as mean waiting time in the queue or in the system, mean number of customers in the queue or in the system, mean utilization of system facility, etc.

Generally the queueing analyst has one of the following two tasks (Gross & Harris, 1998):

- Determine the values of appropriate measures of effectiveness for a given process. This could be done by relating waiting delays, queue lengths and such to the given properties of the arrival pattern and the service procedure.
- Design an optimal system to reach a balance between customer waiting time and the idle time of servers.

1.6 Server Vacations

In many real world queueing systems, server(s) may become unavailable for a random period of time at the service completion instant when there are no customers waiting in the queue, or even if there are customers. The random periods in which the server is absent are called a server vacation (Zhang & Tian, 2003b). Vacation periods could be deterministic, exponential, hyper-exponential, general, etc.

Queueing systems with vacations could provide (LaMaire, 1992):

- Exhaustive service – as the server cannot go for vacation until all the customers presently in the system have been served.
- Gated service – where the server only serves those customers that it finds in the system when it first starts service following its vacation. It then leaves for vacation again.
- Limited service – where the maximum number of customers that can be served before the server goes on a vacation has a fixed limit. Thus the server serves until either the limit is reached or all the customers eligible for service have been served.
- Limited with limit variation – where the maximum number of customers that can be served before the server goes on a vacation has a varying limit.

The vacation model itself may be of different types as Dshalalow (1998) and Bose (2002) stated:

- Multiple vacation model where a server, on returning from a vacation, goes for another vacation if it finds the system still empty. In this case the server resumes normal service if it finds one or more customers waiting after returning from a vacation.
- Single vacation model where the server goes for only one vacation after a service completion. Even if the queue is empty when it returns from the vacation, it stays at the system waiting for a customer to arrive.
- *N*-Policy vacation model where the server can go on multiple vacations and resumes service only when it finds *N* or more customers waiting when it returns from a vacation.

Doshi (1986), Alfa (2003) and Fiems and Bruneel (2002) listed some other queueing systems which could be considered as vacation queues including:

- Polling systems: Where many queues are served by only one server who attends to only one queue at a time, so it is considered as being away on vacation for customers in queues which are not receiving service. Polling systems are very common in computer systems where a processor has to attend to several queues of jobs. Also a road intersection controlled by traffic signals is an example.
- Priority queues: While the higher priority customer is receiving service, the lower priority customers consider the server as being away in vacation.
- Breakdowns and repairs: When the system breaks down and is being repaired, the system is considered to be on vacation
- Scheduled maintenance periods can also be considered as vacation periods.

1.7 Random Breakdowns

Some systems might suddenly break down such as computer systems, communication systems, networks, and many others. For example, consider a machine that always needs some maintenance, especially it needs some oil from time to time. If the machine runs out of oil, it will break down and a repairman must be called, who fills a new portion of oil into the machine.

However, due to several circumstances the machine uses (randomly) varying rate of oil, so the next time when the machine runs out of oil, will be random.

Whenever a service channel suddenly fails, then this service channel either enters a repair process immediately, or waits in a queue for the repairman. In both cases, the customer whose service is interrupted comes back to the head of the queue or might even leave the system. The service to customers will be provided once the problem is covered. The repair time could be deterministic, exponential, hyper-exponential, general, etc.

1.8 Some Definitions and Notations

In this section, some concepts and notations related to queueing theory are defined and presented.

1.8.1 Laplace and Laplace-Stieltjes Transforms

The *Laplace* transform of the probability density function $f(x)$ for $x > 0$ is defined by the following integral:

$$L\{f(x)\} = \tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx$$

Theorem: If X_1 and X_2 are two independent random variables with Laplace transformations $\tilde{f}_1(s)$ and $\tilde{f}_2(s)$, respectively, and $Y = X_1 + X_2$ has probability density function $f(y)$, then

$$\tilde{f}(s) = \tilde{f}_1(s) + \tilde{f}_2(s)$$

Where $\tilde{f}(s)$ is the Laplace transformation of $f(y)$.

The *Laplace-Stieltjes* transform of a function $f(x)$ for $x > 0$ is defined by the following integral:

$$\bar{f}(s) = \int_0^\infty e^{-sx} df(x)$$

The Laplace-Stieltjes transform shares many properties with the Laplace transform.

1.8.2 Probability Generating Function

The idea of probability generating function is a useful tool in the analysis of queueing systems. If N is a discrete random variable which can assume the values $n = 0, 1, 2, \ldots$ with probability p_n, then the probability generating function is defined as

$$P(z) = E[z^N] = \sum_{n=0}^{\infty} p_n z^n, \quad |z| \leq 1$$

Of course

$$P(1) = \sum_{n=0}^{\infty} p_n = 1$$

If p_n represents the probability that there are n customers in the queue, then the mean number of customers in the queue could be found using the probability generating function as follows (Bunday, 1996):

$$\sum_{n=0}^{\infty} np_n = \frac{d}{dz} P_q(z)\bigg|_{z=1} \tag{1.1}$$

1.8.3 Stochastic Process

A stochastic process is the mathematical abstraction of an empirical process whose development is governed by probabilistic laws such as a Poisson process. A stochastic process is a family of random variables, $\{X(t),\ t \in T\}$, defined over some index set or parameter space T. The set T is sometimes also called the time range, and $X(t)$ denotes the state of the process at time t. The process is classified as discrete-parameter or continuous-parameter depending upon the nature of the time range as follows (Gross & Harris, 1998):

- The stochastic process is said to be a discrete-parameter process defined on the index set T if T is a countable sequence, for example, $T = \{0, \pm 1, \pm 2, \ldots\}$ or $T = \{0, 1, 2, \ldots\}$.
- The stochastic process is said to be a continuous-parameter process defined on the index set T if T is an interval or an algebraic combination of intervals, for example, $T = \{t : -\infty < t < \infty\}$ or $T = \{t : 0 < t < \infty\}$.

In general, a stochastic process may be put into one of four broad categories determined by how time is measured and by how the states of the process are classified. Time can be discrete or continuous, and the states can be discrete or continuous. Examples for the four categories could be:

- Discrete time – discrete states process: The number of computers waiting for service when time is measured in days.
- Discrete time – continuous states process: The distance traveled by a truck driver each day.
- Continuous time – discrete states process: Number of calls arriving at a switchboard during the day.
- Continuous time – continuous states process: atmospheric temperature which changes continuously in time.

Depending on the nature of the process, the states may be numerical or non-numerical quantities. In queueing systems, the states are often taken to be the number of customers waiting for service.

1.8.4 Markov Process

A discrete-parameter stochastic process or a continuous-parameter stochastic process is said to be a Markov process if each outcome is linked to the one immediately preceding it. Mathematically, this could be stated in the following definition.

Definition:

For times $n = 0, 1, 2, \ldots$ let $\{X(n)\}$ denote a stochastic process and let $\{S(n)\}$ denote any collection of states of the process. The process is said to satisfy the Markov property if

$$P[X(n+1) = s(n+1) \mid X(n) = s(n), X(n-1) = s(n-1), \ldots, X(0) = s(0)]$$
$$P[X(n+1) = s(n+1) \mid X(n) = s(n)]$$

for $n = 0, 1, 2, \ldots$ (Higgins & Keller-McNulty, 1995).

For example, many board games have the Markov property. That is, the next position of a token on the board depends only on the present position and the roll of the dice.

1.8.5 Time-Independent Solution (Steady State)

When the behavior of the queueing system becomes independent of time, a steady state solution is said to prevail. When the queueing system has reached a steady-state condition, the state probabilities $P_m(t)$ become constants independent of time but yet the process is not deterministic. Mathematically, reaching a steady state needs the following condition to be satisfied

$$\lim_{t\to\infty} P_m(t) = P_m$$

so that

$$\lim_{t\to\infty}\left\{\frac{dP_m(t)}{dt}\right\} = 0$$

In some cases, the steady state is never reached. This happens for systems having arrival rate greater than service rate. In this case, the queue becomes larger and larger as time goes, the queue will become beyond control, and hence the steady state is never reached.

1.8.6 Supplementary Variable Technique

There are different techniques for analysing queueing systems with fairly general assumptions, such as the imbedded Markov chain, matrix-geometric method, and supplementary variable technique. In this research we use the supplementary variable technique in which one or more extra variables, called supplementary variables, are introduced in the definitions of the states of the system, so that the process with the new definition becomes Markovian. This technique was introduced by Cox (1955). For the $M/G/1$ system we need one supplementary variable, being the time since the last departure. While in the current research, we assume that the server takes vacations and breakdowns may occur at random which require repair process; additional supplementary variables are introduced being the elapsed vacation time and the elapsed repair time. The supplementary variable method is illustrated for the classic $M/G/1$ system in section 1.10 of this chapter.

Compared with the imbedded Markov chain approach, it is more straightforward to obtain the steady-state probabilities at an arbitrary instant and practically interesting performance measures via the supplementary variable method (Niu & Takahashi, 1999). It was shown by Choi, Hwang and Han (1998) that this method is simple and elegant. Gupta and Sikdar (2006) gave more advantages of supplementary variable techniques. They used this technique to develop the relations between the queue length distributions when the server is busy/vacation at arbitrary and departure epochs. They justified the use of this technique over other methods by that one can obtain several other results by using simple algebraic manipulation of transform equations such as mean length of idle period. Also, the supplementary variable method has the advantage over the imbedded Markov chain method that here we can study the system in continuous time instead of at discrete time points (Kashyap & Chaudhry, 1988). As expected by Choi, Hwang and Han (1998), the supplementary variable method is widely used in the analysis of variants of the $M/G/1$ queue. Several authors used this technique in their analyses for queueing system involving general distributions (Frey & Takahashi, 1999; Lee & Jeon, 1999; Madan, 2000a; 2000b; 2001; Wang, Cao & Li, 2001; Ke, 2003a; Niu, Shu & Takahashi, 2003; Arumuganathan & Jeyakumar, 2005; Madan & Choudhury, 2005; Kumar & Arumuganathan, 2008).

1.8.7 Some General Results

There are some general results and relationships for $G/G/1$ and $G/G/c$ queues which are useful in the study of queueing theory.

The standard mathematical "little o" notation will be used. Thus, $o(\Delta t)$ represents any function of Δt which goes to zero faster than Δt itself so that

$$\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$$

For example, in any calculation, if $(\Delta t)^2$ appeared, it could be replaced by $o(\Delta t)$ since

$$\lim_{\Delta t\to 0}\frac{(\Delta t)^2}{\Delta t}=\lim_{\Delta t\to 0}\Delta t=0$$

This notation will be very useful for summarizing negligible terms which do not enter into the final results.

Comparing arrivals rate with the service rate yields the following three cases:

- $\rho > 1$ $(\lambda > c\mu)$: The average number of arrivals into the system exceeds the maximum average service rate of the system, and it is expected that, as time goes on, the queue to get bigger and bigger, unless at some point customers were not allowed to enter the system. Thus, there is no steady state when $\rho > 1$, since the queue size never settles down (assuming customers are not prevented from entering the system).
- $\rho = 1$ $(\lambda = c\mu)$: When the arrivals rate exactly equals to the maximum average service rate of the system, the steady state does not exist unless arrivals and service are deterministic and perfectly scheduled, since randomness will prevent the queue from ever emptying out and allowing the server to catch up, thus causing the queue to grow without bound.
- $\rho < 1$ $(\lambda < c\mu)$: When the average arrival rate is strictly less than the maximum average service rate, a steady state solution exists since the queue size will be under control. Therefore, knowing the average arrival rate and the average service rate helps in finding the minimum number of parallel servers which guarantee a steady state solution by calculating the smallest c satisfying $\lambda/c\mu < 1$.

Finding the probability distribution for the total number of customers in the system is important in solving queueing models. This probability distribution is made up of those waiting in the queue, $N_q(t)$, plus those in the service, $N_s(t)$. That is, $N = N_q + N_s$. Let $P_m(t) = \Pr\{N(t) = m\}$, and $P_m = \Pr\{N = m\}$ in the steady state. Considering c-server queues in steady state, two expected value measures of major interest are the mean number of customers in the system,

$$L = \mathrm{E}[N] = \sum_{m=0}^{\infty} m p_m ,$$

and the mean number of customers in the queue,

$$L_q = \mathrm{E}[N_q] = \sum_{m=c+1}^{\infty} (m-c) p_m$$

1.8.8 Little's Law

One of the most powerful relationships in queueing theory was developed by John D. C. Little in the early 1960s. Little's law is a general results holding for *G/G/*1 queues; it also applies to other service disciplines than FIFO. It establishes a relationship between the average number of customers in the system, the mean arrival rate, and the mean system response time (that is the time between entering and leaving the system after finishing the service) in the steady state.

Little's law states that "the average number of customers in a system (over some interval) is equal to their average arrival rate, multiplied by their average time in the system". That is

$$L = \lambda W \tag{1.2}$$

Similarly "the average number of customers in the queue (over some interval) is equal to their arrival rate, multiplied by their average time spent in the queue". That is

$$L_q = \lambda W_q \tag{1.3}$$

Thus, in view of Little's formulae, it is necessary to find only one of the four expected value measures $E[N]$, $E[N_q]$, $E[T]$, or $E[T_q]$; which are L, L_q, W, and W_q, respectively, and the fact that $E[T] = E[T_q] + E[S]$, that is

$$W = W_q + E[S] \tag{1.4}$$

Or the fact that $E[N] = E[N_q] + \rho$, that is

$$L = L_q + \rho \tag{1.5}$$

Rosenkrantz (1992) presented a stochastic integral approach to Little's theorem. A distributional form of Little's law was studied by Takine (2001).

1.8.9 Related Distributions

1.8.9.1 Bernoulli Distribution

The Bernoulli distribution is a discrete probability distribution, which takes value 1 with probability p and value 0 with probability $1-p$.

A random variable N has a Bernoulli distribution, if and only if its probability distribution is given by

$$P(N=n)=p^n(1-p)^{1-n} \qquad \text{for } n=0,\,1$$

The mean and the variance of the Bernoulli distribution are given by

$$E(N)=p \quad \text{and} \quad \sigma^2(N)=p(1-p)$$

And its probability generating function is given by

$$P(z)=pz+(1-p)$$

Many Queueing theorists considered the Bernoulli distribution in their researches (Choi & Park, 1990; Wortman, Disney & Kiessler, 1991; Weststrate & Van der Mei, 1994; Feng, Kowada & Adachi, 1998; Lee *et al.*, 1999; Atencia *et al.*, 2006).

1.8.9.2 Poisson Distribution

The Poisson distribution is a discrete probability distribution which expresses the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate $\lambda > 0$, and are independent of the time since the last event.

A random variable N has a Poisson distribution, if and only if its probability distribution is given by

$$P(N=n)=\frac{\lambda^n e^{-\lambda}}{n!} \qquad \text{for } n=0,\,1,\,2,\,\ldots$$

where λ is the expected number of arrivals during a given interval. The mean and the variance of the Poisson distribution are given by

$$E(N)=\lambda \quad \text{and} \quad \sigma^2(N)=\lambda$$

And its probability generating function is given by

$$P(z)=e^{-\lambda(1-z)}$$

Many authors have investigated the applications of Poisson distribution in queueing systems (Kendall, 1951; Prabhu, 1960; Shanbhag, 1966; Berenshtein, Vainshtein & Kreinin, 1989). The Poisson distribution is widely used in queueing theory. In fact, all studies concern with queues having '*M*' letter for the first or second elements in Kendall's notations, deal with Poisson arrivals or Poisson distribution for service completions, respectively.

1.8.9.3 Exponential Distribution

The Exponential distribution is a continuous probability distribution which is often used to model the time between events that happen at a constant average rate, such as arrivals in queueing theory.

A random variable X has an exponential distribution, if and only if its probability density is given by

$$f(x) = \begin{cases} \mu e^{-\mu x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean and the variance of the exponential distribution are given by

$$E(X) = \frac{1}{\mu} \quad \text{and} \quad \sigma^2(X) = \frac{1}{\mu^2}$$

An important property of the exponential distribution is that it is memoryless. This means that if a random variable X is exponentially distributed its conditional probability obeys

$$P(X > s + x \mid X > x) = P(X > s), \quad \text{for all} \, s, x \geq 0$$

This says that the conditional probability that we need to wait, for example, more than another 10 seconds before the first arrival, given that the first arrival has not yet happened after 30 seconds, is no different from the initial probability that we need to wait more than 10 seconds for the first arrival. That is,

$$P(X > 40 \mid X > 30) = P(X > 10)$$

The most common stochastic queueing models assume that customers arrive at random which implies that the arrivals follow a Poisson distribution, and this leads to an exponentially distributed inter-arrival times. Similarly, the service completions follows a Poisson distribution when it is considered to be random, and this leads to exponentially distributed service times. For derivation of these results, see Gross & Harris (1998).

1.8.9.4 Deterministic Distribution

The letter D in the first (second) element of Kendall's notation indicates that the interarrival time (service time) is constant. In the case of constant interarrival times, also called regular arrivals or regular input, when each interarrival time is a, say, where $0 < a < \infty$, we have the input distribution

$$A(u) = \begin{cases} 0 & u < a \\ 1 & u \geq a \end{cases}$$

And the density function

$$a(u) = \delta(u - a)$$

where $\delta(\cdot)$ is the Dirac delta function. When service times are constant, we have similar expressions for the service time distribution (Kashyap & Chaudhry, 1988).

Jansson (1966), Pack (1977), Servi (1986), Pinotsi and Zazanis (2005), and Van-Leeuqaarden (2005) are among many others who have studied queueing models with deterministic interarrival times. Other authors including Stadje (1998), Ahn, Leeb and Jeona (2000), Brun and Garcia (2000), Koba (2000), Ahn and Jeon (2002), Nakagawa (2002), Jelenković, Mandelbaum & Momćilović (2004) and Kentaro *et al.* (2007) investigated queues with deterministic service time distribution. The deterministic distribution could be used also for vacation times as studied by Madan (2001).

1.8.9.5 Erlang Distribution (E_k)

The Erlang distribution was developed by A. K. Erlang to examine the number of telephone calls which might be made at the same time to the operators of the switching stations (See page 3 for Erlang's contribution to the theory of queues). The Erlang distribution is a continuous distribution, which has a positive value for all real numbers greater than zero, and is given by two parameters: the shape k, which is an integer, and the rate μ, which is a real.

A random variable X has an Erlang-k distribution if X is the sum of k independent random variables X_1, X_2, ...X_k having a common exponential distribution with parameter μ. The probability density of an Erlang distribution is given by

$$f(x;k,\mu) = \frac{\mu^k x^{k-1} e^{-\mu x}}{(k-1)!} \quad \text{for } x > 0$$

When the shape parameter k equals 1, the distribution simplifies to the exponential distribution.

The Erlang distribution is a special case of the Gamma distribution where the shape parameter k is an integer. In the Gamma distribution, this parameter is a real.

The mean and the variance of the Erlang-k distribution are given by

$$E(X)=\frac{k}{\mu} \quad \text{and} \quad \sigma^2(X)=\frac{k}{\mu^2}$$

Hillier and Boling (1967), Ackere and Ninios (1993), Adan and Wessels (1996), and Bertsimas and Mourtzinou (1999) among several others have studied queueing models with Erlang distribution.

1.9 The *M*/*M*/1 Queueing System

The *M*/*M*/1 systems can be described as follows: arrivals come at rate $\lambda > 0$, and hence the interarrival times are identically distributed and have exponential distribution with parameter λ (mean $1/\lambda$). The service rate is $\mu > 0$, and hence the service times are also identically distributed and have exponential distribution with parameter μ (mean $1/\mu$). There is only one server and customers are served in order of arrival. The waiting line and population of customers are infinite. Accordingly, we have the following probabilities for arrivals and service:

$$\Pr\{\text{arrival occurs between } t \text{ and } t+\Delta t\} = \lambda\Delta t + o(\Delta t)$$

$$\Pr\{\text{more than one arrival between } t \text{ and } t+\Delta t\} = o(\Delta t)$$

$$\Pr\{\text{no arrivals between } t \text{ and } t+\Delta t\} = 1 - \lambda\Delta t + o(\Delta t)$$

$$\Pr\{\text{one service completion between } t \text{ and } t+\Delta t\} = \mu\Delta t + o(\Delta t)$$

$$\Pr\{\text{more than one service completion between } t \text{ and } t+\Delta t\} = o(\Delta t)$$

$$\Pr\{\text{no service completion between } t \text{ and } t+\Delta t\} = 1-\mu\Delta t + o(\Delta t)$$

The aim is to calculate $P^{(m)}(t)$, the probability of m arrivals up to time t. To do so, we start with calculating the probability of the system state at time $t + \Delta t$ as follows

$$P^{(m)}(t+\Delta t) = P^{(m)}(t)(1-\lambda\Delta t)(1-\mu\Delta t) + P^{(m-1)}(t)(\lambda\Delta t)(1-\mu\Delta t) + P^{(m+1)}(t)(1-\lambda\Delta t)(\mu\Delta t), \quad m \geq 1 \tag{1.6}$$

$$P^{(0)}(t+\Delta t) = P^{(0)}(t)(1-\lambda\Delta t) + P^{(1)}(t)(1-\lambda\Delta t)(\mu\Delta t) \tag{1.7}$$

Simplifying (1.6) and (1.7) and ignoring terms with $(\Delta t)^2$ and higher order terms, we get

$$P^{(m)}(t+\Delta t) = P^{(m)}(t)(1-\lambda\Delta t-\mu\Delta t) + P^{(m-1)}(t)\lambda\Delta t + P^{(m+1)}(t)\mu\Delta t, \quad m \geq 1 \tag{1.8}$$

$$P^{(0)}(t+\Delta t) = P^{(0)}(t)(1-\lambda\Delta t) + P^{(1)}(t)\mu\Delta t \tag{1.9}$$

Taking the limits as $\Delta t \to 0$, we get the following differential equations

$$\frac{dP^{(m)}(t)}{dt} = -(\lambda+\mu)P^{(m)}(t) + \lambda P^{(m-1)}(t) + \mu P^{(m+1)}(t), \quad m \geq 1 \tag{1.10}$$

$$\frac{dP^{(0)}(t)}{dt} = -\lambda P^{(0)}(t) + \mu P^{(1)}(t) \tag{1.11}$$

The steady state solution for these differential equations can be obtained according to the following conditions

$$\frac{dP^{(m)}(t)}{dt} = 0 \quad \text{and} \quad \lim_{t\to\infty} P^{(m)}(t) = P^{(m)} \quad \text{for all } m \geq 0,$$

Replacing λ/μ by ρ, with $\rho < 1$ for stability, and subject to the condition $\sum_{m=0}^{\infty} P^{(m)}(t) = 1$ for all t, we get the system state probabilities to be

$$P^{(m)} = \rho^m(1-\rho), \qquad m \geq 0 \tag{1.12}$$

Knowing the steady state (equilibrium) probabilities $P^{(m)}$, various mean performance measures can be computed as follows:

a) Probability of finding the system empty on arrival

The customer will get served immediately on arrival if the system is empty, the probability of this is

$$P^{(0)} = 1-\rho$$

b) Server utilization

The server is busy whenever there is at least one customer in the system, i.e. the system state being zero. Hence, the utilization of the server will be

$$1-P^{(0)} = \rho$$

c) Mean number in the system

$$L = \sum_{m=0}^{\infty} mP^{(m)} = \sum_{m=0}^{\infty} m\rho^m(1-\rho) = (1-\rho)\sum_{m=0}^{\infty} m\rho^m = \frac{\rho}{1-\rho}$$

d) Mean number in queue

$$L_q = \sum_{m=1}^{\infty}(m-1)P^{(m)} = \frac{\rho}{1-\rho} - (1-P^{(0)}) = \frac{\rho^2}{1-\rho}$$

e) Average waiting time in system

W can be computed using equation (1.2)

$$W = \frac{L}{\lambda} = \frac{1}{\mu(1-\rho)}$$

f) Average waiting time in the queue

W_q can be computed using equation (1.3)

$$W_q = \frac{L_q}{\lambda} = \frac{\rho}{\mu(1-\rho)}$$

1.10 The *M*/*G*/1 Queueing System

The *M*/*G*/1 model assumes that the queueing system has a single server and a Poisson input process (exponential interarrival times) with a fixed mean arrival rate λ. As usual, it is assumed that the customers have independent service times with the same probability distribution. However, no restrictions are imposed on the specific service-time distribution.

Let $\mu(x)\mathrm{d}x$ be the conditional probability density of service completion during the interval $(x, x + \mathrm{d}x]$, given that the elapsed time is x, so that

$$\mu(x) = \frac{g(x)}{1-G(x)} \tag{1.13}$$

Where $G(s)$ and $g(s)$ are the distribution function and the density function of the service time, respectively. Accordingly, we have

$$G(s) = 1 - e^{-\int_0^s \mu(x)dx} \tag{1.14}$$

$$g(s) = \mu(s)e^{-\int_0^s \mu(x)dx} \tag{1.15}$$

For the steady state, we consider the limiting probability density

$$P_n(x) = \lim_{t\to\infty} P_n(x,t)$$

and the limiting probability

$$P_n = \lim_{t\to\infty} P_n(t) = \lim_{t\to\infty} \int_0^\infty P_n(x,t)dx$$

$$Q = \lim_{t\to\infty} Q(t)$$

Arguing as in section 1.9, we have the following transition equations

$$P_n(x+\Delta x) = P_n(x)\left[1-(\lambda+\mu(x))\Delta x\right] + P_{n-1}(x)\lambda\Delta x, \qquad n \ge 1$$

$$P_0(x+\Delta x) = P_0(x)\left[1-(\lambda+\mu(x))\Delta x\right]$$

$$Q = Q(1-\lambda\Delta x) + \int_0^\infty \mu(x)\Delta x \cdot P_0(x)dx$$

Hence, the steady state equations governing the *M/G/*1 system are

$$\frac{d}{dx}P_n(x) + (\lambda+\mu(x))P_n(x) = \lambda P_{n-1}(x), \qquad n \ge 1 \tag{1.16}$$

$$\frac{d}{dx}P_0(x) + (\lambda+\mu(x))P_0(x) = 0 \tag{1.17}$$

$$\lambda Q = \int_0^\infty P_0(x)\mu(x)dx \tag{1.18}$$

These equations are to be solved with the following boundary conditions

$$P_n(0) = \int_0^\infty P_{n+1}(x)\mu(x)dx, \quad n \ge 1 \tag{1.19}$$

$$P_0(0) = \int_0^\infty P_1(x)\mu(x)dx + \lambda Q \tag{1.20}$$

and the normalization condition

$$Q + \sum_{n=0}^{\infty}\int_0^\infty P_n(x)dx = 1 \tag{1.21}$$

Define the generating functions

$$P_q(x,z) = \sum_{n=0}^{\infty} z^n P_n(x), \quad P_q(z) = \sum_{n=0}^{\infty} z^n P_n, \tag{1.22}$$

Now, multiplying equation (1.16) by z^n, summing over n from 1 to ∞, adding to (1.17), and using the generating functions defined in (1.22), we get

$$\frac{d}{dx}P_q(x,z) + (\lambda - \lambda z + \mu(x))P_q(x,z) = 0 \tag{1.23}$$

Similarly we get

$$zP_q(0,z) = \int_0^\infty P_q(x,z)\mu(x)dx + \lambda(z-1)Q \tag{1.24}$$

Whose solution is

$$P_q(x,z) = P_q(0,z)e^{-(\lambda-\lambda z)x - \int_0^x \mu(t)dt} \tag{1.25}$$

Integrating equation (1.25) by parts with respect to x yields

$$P_q(z) = P_q(0,z)\left[\frac{1-\bar{G}[\lambda-\lambda z]}{\lambda-\lambda z}\right] \tag{1.26}$$

where $\bar{G}[\lambda-\lambda z] = \int_0^\infty e^{-(\lambda-\lambda z)x} \cdot dG(x)$ is the Laplace-Stieltjes transform of the service time. Now, multiplying both sides of equation (1.25) by $\mu(x)$ and integrating the resulting equation over x, we get

$$\int_0^\infty P_q(x,z)\mu(x)dx = P_q(0,z)\bar{G}[\lambda-\lambda z] \tag{1.27}$$

Using equation (1.27) in equation (1.24) we get

$$P_q(0,z) = \frac{\lambda(z-1)Q}{z-\bar{G}[\lambda-\lambda z]} \tag{1.28}$$

Substituting for $P_q(0,z)$ in equation (1.26) and finding Q using the normalization condition (1.21), we get

$$P_q(z) = \frac{(1-\bar{G}[\lambda-\lambda z])(1-\lambda E(S))}{\bar{G}[\lambda-\lambda z]-z} \tag{1.29}$$

where $E(S)$ is the mean service time. Equation (1.29) gives the probability generating function of the number of customers in the queue. Using this equation, equation (1.1) and Little's laws, various mean performance measures can be computed as follows:

a) **Probability of finding the system empty on arrival**

The customer will get served immediately on arrival if the server is idle. The probability of this is

$$Q = 1 - \rho = 1 - \lambda E(S)$$

b) **Server utilization**

The server is busy whenever there is at least one customer in the system

$$\rho = 1 - Q = \lambda E(S)$$

c) **Mean number in queue**

Let us write $P_q(z)$ given in (1.29) as $P_q(z)=N(z)/D(z)$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (1.29), respectively, then from equation (1.1) we have

$$L_q = \frac{d}{dz} P_q(z)\Big|_{z=1} = \frac{D(1)N'(1) - N(1)D'(1)}{(D(1))^2}$$

This is of 0/0 form since $N(1) = D(1) = 0$. Then we use L'Hopital's rule twice, we get

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z) = \lim_{z \to 1} \left\{ \frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2} \right\}$$
$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \qquad (1.30)$$

Using this result we get

$$L_q = \frac{\lambda^2 E(S^2)}{2(1-\rho)}$$

d) Mean number in the system

Using equation (1.5) we have

$$L = \rho + \frac{\lambda^2 E(S^2)}{2(1-\rho)}$$

e) Average waiting time in the queue

Using equation (1.3) we have

$$W_q = \frac{\lambda E(S^2)}{2(1-\rho)}$$

f) Average waiting time in system

Using equation (1.2) or (1.4) we get

$$W = \frac{\rho}{\lambda} + \frac{\lambda E(S^2)}{2(1-\rho)}$$

On the other hand, the probability generating function of the number of customers in the system at a random epoch $P(z)$ can be found using equation (1.29) and the equation given by Kashyap and Chaudhry (1988)

$$P(z) = zP_q(z) + Q \qquad (1.31)$$

hence, we get

$$P(z) = \frac{\bar{G}[\lambda - \lambda z](1-z)(1-\lambda E(S))}{\bar{G}[\lambda - \lambda z] - z} \qquad (1.32)$$

The above results obtained for $M/G/1$ can be applied for queueing systems with any given distribution for the service time by making the appropriate substitutions for $\bar{G}[\lambda-\lambda z]$, $E(S)$ and $E(S^2)$. Accordingly, the equations, solutions, and performance measures for the queueing systems *M*/*M*/1, *M*/*D*/1 and $M/E_k/1$ can be obtained as special cases of the results discussed in this section.

The *M*/*G*/1 queueing system has been studied extensively due to its wide applicability. Apart from the theoretical interest, it has been successfully applied in operations research and management sciences problems, in particular, for production planning. It has also become a tool for the performance prediction of complex computer and telecommunication systems (Kumar, Arivudainambi & Vijayakumar, 2002). Various aspects of *M*/*G*/1 queueing models have been studied by Levy and Yechiali (1975), Heyman (1977), Scholl and Kleinrock (1983), Ott (1984), Schassberger (1984), Willmot (1988), Madan (1994; 2000a), Nelson (1995), Harrison and Pitel (1996), Li and Zhu (1996), Xi (1996), Hur and Paik (1999), Bischof (2001), Choudhury (2003b; 2005; 2006), Eddins (2004), Kella, Zwart and Boxma (2005) and Taha (2007) among several others.

1.11 The $M^{[X]}/G/1$ Queueing System

Queueing systems may have arrivals coming in groups or the service is rendered in groups. The size of the groups will be regarded as a random variable given by a probability distribution. In this section, we introduce a single server queueing system with Poisson arrivals and general service time distribution in which customers arrive to the system in batches of variable size. The queueing systems described in earlier sections, with a batch size being always equal to one, become particular cases of the model discussed here.

The $M^{[X]}/G/1$ queue assumes similar assumptions underlying *M*/*G*/1 queues which were explained in section 1.10. Further, we let $\lambda c_i dt$ (i = 1, 2, 3, ...) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. Accordingly, the following equations will govern the system at steady-state.

$$\frac{d}{dx}P_n(x)+(\lambda+\mu(x))P_n(x)=\lambda\sum_{i=1}^{n-1}c_iP_{n-i}(x), \qquad n\geq 1 \tag{1.33}$$

$$\frac{d}{dx}P_0(x)+(\lambda+\mu(x))P_0(x)=0 \tag{1.34}$$

$$\lambda Q=\int_0^\infty P_0(x)\mu(x)dx \tag{1.35}$$

$$P_n(0)=\int_0^\infty P_{n+1}(x)\mu(x)dx+\lambda c_{n+1}Q, \quad n\geq 0 \tag{1.36}$$

Define the probability generating function for the batch size as follows

$$C(z)=\sum_{i=1}^{\infty}z^ic_i \tag{1.37}$$

From equations (1.33) – (1.37) we get

$$\frac{d}{dx}P_q(x,z)+\left(\lambda-\lambda C(z)+\mu(x)\right)P_q(x,z)=0 \tag{1.38}$$

$$zP_q(0,z)=\int_0^\infty P_q(x,z)\mu(x)dx+\lambda\left(C(z)-1\right)Q \tag{1.39}$$

Solving this system we get the following probability generating function for the number of customers in the queue at a random epoch

$$P_q(z)=\frac{\left(1-\bar{G}[\lambda-\lambda C(z)]\right)\left(1-\lambda E(I)E(S)\right)}{\bar{G}[\lambda-\lambda C(z)]-z} \tag{1.40}$$

where $\bar{G}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot dG(x)$ is the Laplace-Stieltjes transform of the service time and *E*(*I*) is the mean batch size of the arriving customers. Similar to *M*/*G*/1 queues; using equation (1.31), we can find the probability generating function of the number of customers in the system at a random epoch

$$P(z)=\frac{\bar{G}[\lambda-\lambda C(z)](1-z)\left(1-\lambda E(I)E(S)\right)}{\bar{G}[\lambda-\lambda C(z)]-z} \tag{1.41}$$

The following performance measures for $M^{[X]}/G/1$ queues are obtained using equation (1.40) and Little's laws, knowing that *E*(*I*(*I*-1)) is the second factorial moment of the batch size of arriving customers.

a) Probability of finding the system empty on arrival

The customer will get served immediately on arrival if the server is idle. The probability of this is

$$Q = 1 - \rho = 1 - \lambda E(I)E(S)$$

b) Server utilization

The server is busy whenever there is at least one customer in the system, i.e.

$$\rho = 1 - Q = \lambda E(I)E(S)$$

c) Mean number in queue

$$L_q = \frac{\lambda E(S)E(I(I-1)) + (\lambda E(I))^2 E(S^2)}{2(1-\rho)}$$

d) Mean number in the system

$$L = \rho + \frac{\lambda E(S)E(I(I-1)) + (\lambda E(I))^2 E(S^2)}{2(1-\rho)}$$

e) Average waiting time in the queue

$$W_q = \frac{E(S)E(I(I-1)) + \lambda (E(I))^2 E(S^2)}{2(1-\rho)}$$

f) Average waiting time in system

$$W = \frac{\rho}{\lambda} + \frac{E(S)E(I(I-1)) + \lambda (E(I))^2 E(S^2)}{2(1-\rho)}$$

In the current research, we consider an $M^{[X]}/G/1$ queueing system where the arrival occurs according to a marked Poisson process in batches of variable size. We attempt to generalise the results obtained for the classical $M^{[X]}/G/1$ queue. There is extensive literature on the $M^{[X]}/G/1$ queues, which has been studied in various forms by numerous authors including Lee and Srinivasan (1989), Lucantoni (1991), Lee, Lee and Chae (1994), Lucantoni, Choudhury and Whitt (1994), Choudhury (2000; 2003a; 2007), Choi *et al.* (2001), Fakinos and Economou (2001), Ke (2001; 2007b), Lee, Baek and Jeon (2005) and Madan and Al-Rawwash (2005), among several others.

1.12 Literature Review and the Current Work

In classical queueing theory, it is generally assumed that the server(s) are always available providing service for customers. However, this is not always the case in real life situations. A more realistic situation observed in real life examples is that the server may be leaving for a vacation, under repair, stopped for maintenance, attending to other queues, or simply forced to stop serving.

Vacation queues are a very important class of queues in real life. During the past two decades, there have been considerable contributions on queueing systems with server vacations, which have been successfully used in various applied problems including telecommunication engineering, computer networks, flexible manufacturing and production systems.

The well known stochastic decomposition result is one of the most significant results of the research on vacation models which was discovered by Levy and Yechiali (1975). It states that the waiting time in the queue for *M/GI/*1 models with vacations is distributed as the sum of two independent components – one distributed as the waiting time in the queue without vacations and the other as the equilibrium residual time in a vacation. An intuitive and simple explanation for this result is given by Fuhrmann (1984). This result was further studied by Fuhrmann and Cooper (1985) for *M/G/*1 queues with generalised vacations. Doshi (1985) extended the decomposition result for the *GI/G/*1 queue with single and multiple vacation models through different techniques, while Miyazawa (1994) provided a unified treatment of the stochastic decomposition results for *M/G/*1 and *GI/G/*1 queueing systems in more general settings. Tian, Li and Cao (1999) investigated the conditional stochastic decomposition in *M/M/c* queue with server vacation. Choudhury and Borthakur (2000) presented the stochastic decomposition results of batch arrival Poisson queue with a grand vacation process. Recently, Chang and Takine (2005) applied the same result to bulk queues with generalised vacations.

Among the different vacation models, Zhang and Tian (2003a), Madan and Abu Al-Rub (2004) and Madan and Al-Rawwash (2005) have considered queues with single vacation policy, while Zhang and Tian (2001), Tian and Zhang (2002), Kumar and Madheswari (2005), Banik, Gupta, and Pathak (2007), Choudhury, Tadj, and Paul (2007) and Xu *et al.* (2007) studied queues with multiple vacation policy. In other vacation queues, the server is turned on when *N* or more customers are present, and off only when the system is empty. After the server is turned off, the service will not operate until at least *N* customers are present in the system. This is called an *N*-Policy as introduced earlier in this chapter, and it is the model studied by Igaki (1992), Ke (2003a) and Choudhury and Madan (2005). Arumuganathan and Jeyakumar (2005) considered bulk queues with multiple vacations, setup times with *N*-policy and closedown times. Ke (2001; 2003b) conducted studies on queueing systems with server startup and two vacation types.

The vacation time could be exponentially distributed as assumed by Tian, Zhang and Cao (1989), Madan, Abu-Dayyeh and Saleh (2002), Ke (2003a), and Kumar and Mahdheswari (2005); deterministic as assumed by Madan (2001); or has general distribution as considered by Chatterjee and Mukherjee (1990), Madan (1991), Wortman, Disney and Kiessler (1991), Borthakur and Choudhury (1997), Chae, Lee, and Ahn (2001), Chang *et al.* (2002) and Madan and Choudhury (2005).

Shin and Pearce (1998) considered batch arrival queues in which the lengths of vacation times depend on the number of customers in the system at the beginning of a vacation. Finite capacity queueing systems with server vacation have been investigated by Jacob and Madhusoodanan (1987), Loris-Teghem (1988), Blondia (1989) and Niu and Takahashi (1999). Vacation queues with batch arrivals have been studied by Niu, Shu, and Takahashi (2003), Hur and Ahn (2005), Madan and Al-Rawwash (2005) and Ke (2007a; 2007b).

In queueing systems, assuming that the server is available on a permanent basis is apparently practically unrealistic. As discussed above, the server might leave for scheduled vacations. Another reason for the server being unavailable all the time is that the system may well be subjected to lengthy and unpredictable breakdowns while serving a customer. For instance, in manufacturing systems the machine may breakdown due to machine or job related problems; in computer systems, besides the scheduled backups, the machine may be subjected to unpredictable failures. In such systems server breakdowns result in a period of unavailable time until it is repaired. Understanding the behavior of an unreliable server, and the effect of machine breakdowns and repairs in these systems, is important from both the queueing and reliability points of view (Li, Shi & Chao, 1997).

In recent years, queues with server failure or breakdowns have emerged as one of the important areas of queueing theory. Avi-Itzhak and Naor (1963) presented five interesting models of queueing problems with service station subject to breakdown. Numerous researchers including Federgruen and So (1990), Jayawardene and Kella (1996), Takine and Sengupta (1997), Núñez-Queija (2000), Wang, Chiang, and Ke (2003), Vinck and Bruneel (2006) and Ke (2007a) studied queueing systems subject to breakdowns. Joseph and Manoharan (1997) obtained the transient distribution as well as the steady state distribution of a repairable system having different failure modes. Aissani and Artalejo (1998), Kulkarni and Choi (1990), Wang, Cao, and Li (2001) and Sherman (2006) studied retrial queues subject to breakdowns. A queue with two servers and random breakdowns was studied by Madan, Abu-Deyyed, and Gharaibeh (2003a). Gray, Wang and Scott (2004) studied a queueing model with multiple type of breakdowns in which each type of breakdown requires a finite random number of stages of repair. Matis, Feldman and Curry (2008) used an $M/M/1$ model with server failures as the basis for an approximation to more general systems with nonexponential failure times.

Almost all the work on queueing systems mentioned above have a common characteristic; they either consider queues with server vacations or queues subject to breakdowns. Some authors considered system breakdowns as the server being in vacation as mentioned in section 1.6. In our work, we consider batch arrival queueing systems with server vacations *and* random breakdowns, where vacation and breakdown are considered as two different concepts, since breakdowns may occur randomly while vacations are scheduled. The vacations are based on a single vacation policy and Bernoulli schedule; that is, after a service completion, the server may go for a vacation with probability p $(0 \leq p \leq 1)$ or may continue to serve the next customer, if any, with probability $1-p$. When a vacation period ends, the server comes back to the system irrespective of the presence of any customer in the system. Breakdowns are assumed to occur at random while serving a customer, and when the system fails, a repair process starts immediately to overcome the problem. In this case, the customer who was in service when the system failed goes back to the head of the queue waiting for the service to resume.

The present study was motivated by its various applications in real life situations. Our results can be applied to many systems such as a flexible manufacturing system as a tool of performance evaluation. In general, the parts to be processed arrive to a workstation in batches. A workstation may be in charge of several jobs at once. So the vacation in our models may correspond to the time duration that it is working on other secondary jobs such as maintenance work or simply taking a rest. Furthermore, a workstation may suddenly stop performing jobs due to any problems and thus a repair process starts. Server vacations and breakdowns have significant effects on system performance. We wish to understand such an effect on measures of system performance such as queue length and mean waiting time.

To the best of the researcher's knowledge, very little work considers both vacations and breakdowns in the system, and these published papers rely on different assumptions for the queueing systems than those considered in the current study.

Li, Shi and Chao (1997) studied the reliability analysis of $M/G/1$ queueing systems with server breakdowns and Bernoulli vacations. They assumed single arrivals and exponential vacation times, while in the current work we assume batch arrivals of variable size and generally distributed vacation times.

Ke (2007a) analysed the system characteristics of batch arrival queues under vacation policies with server breakdowns and startup/closedown times. In his model, Ke assumed that when all the customers are served in the system exhaustively, the server shuts downs by a closedown time and after shutdown, the server takes vacation. This is different from a Bernoulli schedule server vacation which is assumed in the current work.

Wang and Li (2008) studied a repairable $M/G/1$ retrial queue with Bernoulli vacation and a two-phase service. Although retrial queues belong to different class of queueing systems than the systems considered in the current research, yet there are some other differences in the assumptions underlying their model and the current model. They have assumed single arrivals rather than batch arrivals. Moreover, they have assumed no waiting space in front of the server; therefore if an arriving customer finds the server idle, he may obtain service immediately, and otherwise he either leaves the system or joins a retrial queue. This differs from the current research assumption of infinite waiting space.

1.13 Research Objectives

As discussed in the above section, the literature on queueing theory lacks studies conducted in depth on queues with server vacations and random breakdowns. Thus, the aim of this research is to extend both the classical $M^{[X]}/G/1$ queue with server vacations and the classical $M^{[X]}/G/1$ queue with random breakdowns by looking at the effect of server vacations *and* random breakdowns on queue size distribution and performance measures of some batch arrival queueing systems with single server and generalised service time. Server vacations are assumed to follow a Bernoulli schedule under single vacation policy, and breakdowns are assumed to occur at random while serving a customer. When the system fails, it enters a repair process immediately, and the customer whose service is interrupted comes back to the head of the queue.

Therefore, this research is conducted with the following objectives:

1. To determine the time-dependent behavior and steady-state behavior of batch arrival queueing systems with Bernoulli schedule vacation and random breakdown, where service times follow an arbitrary general distribution, while vacation times and repair times follow exponential distributions.
2. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule vacation and random breakdown, where service times, vacation times and repair times all are assumed to have general (arbitrary) distributions.
3. To determine the steady-state behavior of batch arrival queueing systems with Bernoulli schedule vacation and random breakdown, in which the customer undergoes two stages of heterogeneous service, where the times

of both stages of service, vacation times and repair times are all assumed to have general (arbitrary) distributions.

4. To determine the steady-state behavior of batch arrival queueing system with Bernoulli schedule vacation and random breakdown, in which the server provides a compulsory service to arriving customers as well as an optional service, where the times of both services, vacation times and repair times are all assumed to have general (arbitrary) distributions.
5. To determine the steady-state behavior of batch arrival queueing system with Bernoulli schedule vacation and random breakdown, in which an arriving customer chooses one of the two kinds of heterogeneous service provided by the server, where the times of both kinds of service, vacation times and repair times are all assumed to have general (arbitrary) distributions.

1.14 Research Methods

The Research aims and objectives discussed in the previous section could be achieved by obtaining probability generating functions for the queue size distribution at a random epoch. Thus, time dependent and steady state probability generating functions have been obtained to achieve the first objective, while to achieve objectives 2 to 5, steady state probability generating functions have been obtained.

Since the queueing systems under consideration have different states, for each queueing model, the following have been obtained:

1. Probability generating function for the number of customers in the queue while the server is providing a service.
2. Probability generating function for the number of customers in the queue while the server is on vacation.
3. Probability generating function for the number of customers in the queue while the system is under repair.
4. Probability generating function for the number of customers in the queue irrespective of the state of the system.

To understand the behavior of the queueing systems under consideration, we have also obtained some system performance measures such as mean queue length, mean waiting time in the queue, the proportion of time that the server is idle, and the utilization factor. Numerical results and some graphs have been given to demonstrate the behavior of the queues in an efficient way.

Among different methods in analysing such queueing systems, the supplementary variable method has been implemented to find the necessary probability generating functions. This method is preferred over the other methods due to its advantages outlined in section 1.8.6 of this chapter.

1.15 Outline of Forthcoming Chapters

This work consists of seven chapters. The content of forthcoming chapters is briefly outlined below.

In chapter 2, we analyse a batch arrival queueing system with a single server in which the server takes Bernoulli schedule vacations and the system is subject to breakdowns. Using the supplementary variable method, we obtain a time-dependent solution for the queue size distribution at a random epoch, and then by taking the limit as t goes to infinity, we obtain a steady state solution as well. As a starting point, in this chapter we consider the exponential distribution for vacation times and repair times, while the service time has a general distribution. For this queueing model, the mean queue size, the mean waiting time in the queue, the proportion of time that the server is idle and the utilization factor are obtained as some performance measures of the queueing system. As particular cases, we consider a queueing system with (1) k-Erlang service time, (2) exponential service time, (3) deterministic service time, (4) no server vacations, (5) no system breakdowns, and (6) the classical $M^{[X]}/G/1$ queue. These cases are of interest as some other authors' work can be classified into the above particular cases. A numerical illustration and three dimensional graphs are presented for the special cases with k-Erlang, exponential and deterministic service time distributions.

The queueing system considered in chapter 2 is generalised in chapter 3 by considering general arbitrary distributions for both the vacation times and repair times instead of the special case of the exponential distribution. For this model, steady state solutions are obtained and some performance measures of the queueing system. As particular cases, we consider (1) a queue with exponential vacation time, (2) exponential repair time, and (3) exponential vacation time and repair time in which the problem reduces to the steady state part of chapter 2.

In chapter 4, the queueing model is extended to a system with two-stage heterogeneous service. In this chapter a batch arrival queueing system with a two-stage heterogeneous service, single server, Bernoulli schedule vacations and random breakdowns is analysed. Each stage of service time has a different arbitrary distribution. Also vacation times and repair times are all generally distributed. The assumptions underlying arrivals, vacations and breakdown are similar to those assumed in chapters 2 and 3, while the service is provided in two compulsory stages by the same server. Again, for this model, steady state solutions and some performance measures are obtained. The following particular cases are discussed: (1) exponential vacation time and repair time, (2) No server vacations, (3) no system breakdowns with exponential vacations, and (4) No server vacations and no system breakdowns. For the first special case, we give some numerical results and graphs.

A queueing system with second optional service is studied in chapter 5, where the server provides an essential service to arriving customers and a second optional service, both having different general distributions for service time. As the aim of this research requires, the server takes vacation and the system is subjected to breakdowns while serving the customers in the first essential service. Vacation times and repair times are generally distributed. We obtain steady state results for this batch arrival queue along with some performance measures. As special cases we assume (1) no customer requires the second optional service, and (2) the system never fails. A numerical illustration is given along with some graphs for the special case of exponential service times, vacation times and repair times.

In chapter 6, a batch arrival queueing system with two kinds of general heterogeneous service is analysed. As in the previous chapters, the server may take Bernoulli vacations and the system may breakdown at random. In this model, an arriving customer has the option to choose one of the two types of service provided by the server. On the contrary to the assumption in chapter 5, here we assume that the system may breakdown while serving a customer in either kind of service. Steady state results are obtained and performance measures of the queueing model are derived. If no customer chooses the second service, the model reduces to the one considered in chapter 3. This is the first special case analysed. The second special case assumes no breakdowns may occur. As in the previous chapters, numerical and graphical illustrations are given.

In chapter 7, we conclude the work carried out in this research by summarising the important results, outlining contributions of the research, reflecting upon certain resulting concerns and views, and recommending further research studies.

* * *

Chapter 2

Time Dependent and Steady State Solutions for an $M^{[X]}/G/1$ Queue with Bernoulli Schedule Exponential Server Vacations, Random Breakdowns and Exponential Repair Times

2.1 Introduction

In this chapter we study a batch arrival queueing system in which the service facility suffers random breakdowns from time to time, and the server has the option to take a vacation after any service completion. When the server breaks down, it immediately enters a repair process of random length. The durations of the server vacation are of random length as well.

Many queueing researchers assume exponential distribution for interarrival times, service time, setup times, retrial times, vacation times or repair times. In this chapter we assume exponential distributions for both vacation times and repair times, while we assume a general distribution for service time.

Tian, Zhang and Cao (1989) analysed the $GI/M/1$ queue with exhaustive service and multiple exponential vacations. Kumar and Madheswari (2005) studied a queue with two heterogeneous servers and multiple vacations where interarrival times, service times and vacation times are all assumed to be exponentially distributed. By using matrix geometric method, the stationary queue length distribution and mean system size have been obtained. Zhang (2005) also assumed exponential distributions for interarrival times, service times and vacation times.

When a queueing system is subject to breakdowns, repair times could be assumed exponentially distributed according to Madan, Abu-Dayyed and Gharaibeh (2003b). They studied two models of a single bulk queueing system with random breakdowns in which the repair times are assumed to be exponential in the first model and deterministic in the second model. Wang, Chiang and Ke (2003) analysed the cost of the unloader queueing system with a single unloader subject to breakdowns in which repair times were exponentially distributed.

In the above mentioned work, authors worked out only steady state solutions. In the current chapter time dependent solution as well as steady state solution is obtained. Time dependent solutions are useful to monitor the behavior of the queueing system over time. Takagi (1990) studied the time dependent analysis of $M/G/1$ vacation models with exhaustive service. Garg and Kumari (1998) studied the time dependent solution for a bulk queue. Garg (2003) derived the time dependent solution of a single channel queueing system with service in batches of variable size where interarrival times and service times are exponentially distributed. Madan and Abu-AlRub (2004) provided time dependent equations governing a single server queue with optional server vacations based on exhaustive service, but the solutions to those equations were not obtained. In another work by Madan (2000a), the time dependent probability generating functions were obtained for an $M/G/1$ queue with second optional service where the service times of the optional service are exponential. Also Madan, Abu-Deyyeh and Gharaibeh (2003a) obtained time dependent results for a queueing system with two parallel servers subject to random breakdowns where both the service time and repair time are assumed to be exponential.

All the above mentioned papers, considered a queueing system with either server vacations or random breakdowns. In reality, it is not uncommon to find queueing systems in which the server is allowed to take vacations and the system is subject to breakdowns. That's why we consider such systems in the current work. We further assume that the customers arrive to the service station in batches of variable size, but are served one by one. It is useful to understand how breakdowns and vacations affect the performance measures of a queueing system. Hence, we obtain time dependent results and steady state results in terms of the queue size distribution as well as the probabilities for various states of the system.

The rest of the chapter is organised as follows: section 2.2 gives the assumptions underlying the queueing system considered. Equations governing the system are formulated in section 2.3 while the time dependent solutions to those equations are given in section 2.4. In section 2.5 the steady state results of the system are found. Mean queue size and mean waiting time for a customer are obtained in section 2.6. Some special cases of interest have been discussed in section 2.7. To demonstrate how the assumption of vacations and breakdowns affect the performance measures of the system, some numerical tables and graphs are given in section 2.8.

2.2 The Mathematical Model

The mathematical model of this chapter can be characterised by the following assumptions:

a) Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$ (i = 1, 2, 3, ...) to be the first order probability that a batch of i customers arrives at the system during a short interval of time (t, $t + dt$], where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

b) There is a single server who provides one by one service to arriving customers on a "first come, first served" basis and the service time follows a general (arbitrary) distribution with distribution function $G(s)$ and density function $g(s)$. Let $\mu(x)dx$ be the conditional probability density of service completion during the interval (x, $x + dx$], given that the elapsed time is x, so that

$$\mu(x) = \frac{g(x)}{1 - G(x)} \tag{2.1}$$

and, therefore

$$g(s) = \mu(s) e^{-\int_0^s \mu(x)dx} \tag{2.2}$$

c) The system may break down at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue.

d) Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

e) As soon as a service is completed, the server may take a vacation with probability p, or may stay in the system with probability $1\text{-}p$, $0 \leq p \leq 1$.

f) The duration of vacations follows exponential distribution with rate $\gamma > 0$ and hence mean vacation time $1/\gamma$.

g) Various stochastic processes involved in the system are independent of each other.

2.3 Equations Governing the System

According to the assumptions mentioned above, the system has the following set of differential-difference equations

$$\frac{\partial}{\partial x}P_n(x,t)+\frac{\partial}{\partial t}P_n(x,t)+(\lambda+\mu(x)+\alpha)P_n(x,t)=\lambda\sum_{i=1}^{n-1}c_iP_{n-i}(x,t), \quad n\geq 1 \tag{2.3}$$

$$\frac{\partial}{\partial x}P_0(x,t)+\frac{\partial}{\partial t}P_0(x,t)+(\lambda+\mu(x)+\alpha)P_0(x,t)=0 \tag{2.4}$$

$$\frac{d}{dt}V_n(t)+(\lambda+\gamma)V_n(t)=\lambda\sum_{i=1}^{n-1}c_iV_{n-i}(t)+p\int_0^\infty P_n(x,t)\mu(x)dx, \quad n\geq 1 \tag{2.5}$$

$$\frac{d}{dt}V_0(t)+(\lambda+\gamma)V_0(t)=p\int_0^\infty P_0(x,t)\mu(x)dx \tag{2.6}$$

$$\frac{d}{dt}R_n(t)+(\lambda+\beta)R_n(t)=\lambda\sum_{i=1}^{n-1}c_iR_{n-i}(t)+\alpha\int_0^\infty P_{n-1}(x,t)dx, \quad n\geq 1 \tag{2.7}$$

$$\frac{d}{dt}R_0(t)+(\lambda+\beta)R_0(t)=0 \tag{2.8}$$

$$\frac{d}{dt}Q(t)+\lambda Q(t)=\gamma V_0(t)+\beta R_0(t)+(1-p)\int_0^\infty P_0(x,t)\mu(x)dx \tag{2.9}$$

The above equations are to be solved subject to the following boundary conditions

$$P_n(0,t)=(1-p)\int_0^\infty P_{n+1}(x,t)\mu(x)dx+\gamma V_{n+1}(t)+\beta R_{n+1}(t)+\lambda c_{n+1}Q(t),\ n\geq 0 \tag{2.10}$$

We assume that initially there is no customer in the system and the server is idle, so that the initial conditions are

$$P_n(0)=0, \qquad V_n(0)=0, \qquad R_n(0)=0, \qquad Q(0)=1 \qquad \text{for } n \geq 0 \tag{2.11}$$

2.4 Time Dependent Solution

To find the solution, first we define the probability generating functions

$$P_q(x,z,t)=\sum_{n=0}^{\infty} z^n P_n(x,t),$$

$$V_q(z,t)=\sum_{n=0}^{\infty} z^n V_n(t),$$

$$R_q(z,t)=\sum_{n=0}^{\infty} z^n R_n(t),$$

$$C(z)=\sum_{i=1}^{\infty} z^i c_i \tag{2.12}$$

Next, we take Laplace transform of equations (2.3) – (2.10) using the initial conditions given in (2.11), we get

$$\frac{\partial}{\partial x}\tilde{P}_n(x,s)+\left(s+\lambda+\mu(x)+\alpha\right)\tilde{P}_n(x,s)=\lambda\sum_{i=1}^{n-1} c_i \tilde{P}_{n-i}(x,s), \qquad n \geq 1 \tag{2.13}$$

$$\frac{\partial}{\partial x}\tilde{P}_0(x,s)+\left(s+\lambda+\mu(x)+\alpha\right)\tilde{P}_0(x,s)=0 \tag{2.14}$$

$$\left(s+\lambda+\gamma\right)\tilde{V}_n(s)=\lambda\sum_{i=1}^{n-1} c_i \tilde{V}_{n-i}(s)+p\int_0^{\infty}\tilde{P}_n(x,s)\mu(x)dx, \qquad n \geq 1 \tag{2.15}$$

$$\left(s+\lambda+\gamma\right)\tilde{V}_0(s)=p\int_0^{\infty}\tilde{P}_0(x,s)\mu(x)dx \tag{2.16}$$

$$\left(s+\lambda+\beta\right)\tilde{R}_n(s)=\lambda\sum_{i=1}^{n-1} c_i \tilde{R}_{n-i}(s)+\alpha\int_0^{\infty}\tilde{P}_{n-1}(x,s)dx, \qquad n \geq 1 \tag{2.17}$$

$$\left(s+\lambda+\beta\right)\tilde{R}_0(s)=0 \tag{2.18}$$

$$\left(s+\lambda\right)\tilde{Q}(s)=1+(1-p)\int_0^{\infty}\tilde{P}_0(x,s)\mu(x)dx+\gamma\tilde{V}_0(s)+\beta\tilde{R}_0(s) \tag{2.19}$$

$$\tilde{P}_n(0,s)=(1-p)\int_0^{\infty}\tilde{P}_{n+1}(x,s)\mu(x)dx+\gamma\tilde{V}_{n+1}(s)+\beta\tilde{R}_{n+1}(s)+\lambda c_{n+1}\tilde{Q}(s), \qquad n \geq 0 \tag{2.20}$$

Then, we multiply equation (2.13) by z^n and take the summation over n from 1 to ∞ we get

$$\sum_{n=1}^{\infty} z^n \frac{\partial}{\partial x}\tilde{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\sum_{n=1}^{\infty} z^n \tilde{P}_n(x,s) = \lambda \sum_{n=1}^{\infty} z^n \sum_{l=1}^{n-1} c_l \tilde{P}_{n-l}(x,s)$$

Adding the result to equation (2.14) yields

$$\sum_{n=0}^{\infty} z^n \frac{\partial}{\partial x}\tilde{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\sum_{n=0}^{\infty} z^n \tilde{P}_n(x,s) = \lambda \sum_{n=1}^{\infty} z^n \sum_{l=1}^{n-1} c_l \tilde{P}_{n-l}(x,s)$$

Using the generating functions defined in (2.12) we get

$$\frac{\partial}{\partial x}\tilde{P}_q(x,z,s) + (s+\lambda-\lambda C(z)+\mu(x)+\alpha)\tilde{P}_q(x,z,s) = 0 \tag{2.21}$$

Similarly multiplying equation (2.15) by z^n, taking summation over n from 1 to ∞, adding the result to equation (2.16) and using the probability generating functions defined in (2.12) we get

$$(s+\lambda-\lambda C(z)+\gamma)\tilde{V}_q(z,s) = p\int_0^{\infty} \tilde{P}_q(x,z,s)\mu(x)dx \tag{2.22}$$

Again, multiplying equation (2.17) by z^n, take summation over n from 1 to ∞ as follows, adding the result to equation (2.18), and using the probability generating functions defined in (2.12) we get

$$(s+\lambda-\lambda C(z)+\beta)\tilde{R}_q(z,s) = \alpha z\int_0^{\infty} \tilde{P}_q(x,z,s)dx \tag{2.23}$$

For the boundary condition, we multiply equation (2.20) by z^{n+1}, take the summation over n from 0 to ∞ and use the probability generating functions defined in (2.12). Thus we obtain

$$\begin{aligned} z\tilde{P}_q(0,z,s) &= (1-p)\int_0^{\infty} \tilde{P}_q(x,z,s)\mu(x)dx + \gamma\tilde{V}_q(z,s) + \beta\tilde{R}_q(z,s) + \lambda C(z)\tilde{Q}(s) \\ &\quad - \left((1-p)\int_0^{\infty} \tilde{P}_0(x,s)\mu(x)dx + \gamma\tilde{V}_0(s) + \beta\tilde{R}_0(s)\right) \end{aligned} \tag{2.24}$$

Equation (2.19) can be rewritten in the form

$$1-(s+\lambda)\tilde{Q}(s) = -\left((1-p)\int_0^{\infty} \tilde{P}_0(x,s)\mu(x)dx + \gamma\tilde{V}_0(s) + \beta\tilde{R}_0(s)\right)$$

Utilizing the above equation in (2.24) we get

$$z\tilde{P}_q(0,z,s) = 1 + (1-p)\int_0^\infty \tilde{P}_q(x,z,s)\mu(x)dx + \gamma\tilde{V}_q(z,s) + \beta\tilde{R}_q(z,s)$$
$$+ [\lambda(C(z)-1)-s]\tilde{Q}(s) \tag{2.25}$$

Now we integrate equation (2.21) between 0 and x and use the boundary conditions we have

$$\tilde{P}_q(x,z,s) = \tilde{P}_q(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x - \int_0^x \mu(t)dt} \tag{2.26}$$

where $\tilde{P}_q(0,z,s)$ is given by (2.25). Again integrating (2.26) with respect to x by parts and using (2.2), we get

$$\tilde{P}_q(z,s) = \tilde{P}_q(0,z,s)\left[\frac{1-\bar{G}[s+\lambda-\lambda C(z)+\alpha]}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{2.27}$$

where $\bar{G}[s+\lambda-\lambda C(z)+\alpha] = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x} \cdot dG(x)$ is the Laplace-Stieltjes transform of the service time. Now, using the result obtained in (2.27), equation (2.23) becomes

$$(s+\lambda-\lambda C(z)+\beta)\tilde{R}_q(z,s) = \alpha z\tilde{P}_q(0,z,s)\left[\frac{1-\bar{G}[s+\lambda-\lambda C(z)+\alpha]}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{2.28}$$

Multiplying both sides of (2.26) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \tilde{P}_q(x,z,s)\mu(x)dx = \tilde{P}_q(0,z,s)\bar{G}[s+\lambda-\lambda C(z)+\alpha] \tag{2.29}$$

Then, using equation (2.29), equation (2.22) can be written as

$$(s+\lambda-\lambda C(z)+\gamma)\tilde{V}_q(z,s) = p\tilde{P}_q(0,z,s)\bar{G}[s+\lambda-\lambda C(z)+\alpha] \tag{2.30}$$

Next, using equation (2.29) in equation (2.25) and simplifying we get

$$[z-(1-p)\bar{G}[s+\lambda-\lambda C(z)+\alpha]]\tilde{P}_q(0,z,s) = 1+\gamma\tilde{V}_q(z,s)+\beta\tilde{R}(z,s)$$
$$+[\lambda(C(z)-1)-s]\tilde{Q}(s) \tag{2.31}$$

We now substitute the expressions for $\tilde{V}_q(z,s)$ and $\tilde{R}_q(z,s)$ from (2.28) and (2.30) in equation (2.31) and solve for $\tilde{P}_q(0,z,s)$, we get

$$\tilde{P}_q(0,z,s) = \frac{f_1(z,s)f_2(z,s)f_3(z,s)\{1+[\lambda(C(z)-1)-s]\tilde{Q}(s)\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s) - \alpha\beta z f_3(z,s) - \bar{G}[f_1(z,s)]\{\gamma p f_1(z,s)f_2(z,s) - \alpha\beta z f_3(z,s)\}} \tag{2.32}$$

where

$f_1(z,s) = s+\lambda-\lambda C(z)+\alpha$

$f_2(z,s) = s + \lambda - \lambda C(z) + \beta$

$f_3(z,s) = s + \lambda - \lambda C(z) + \gamma$

$f_4(z,s) = z - (1-p)\overline{G}[s + \lambda - \lambda C(z) + \alpha]$

Next, using (2.32) in equations (2.27), (2.28), and (2.30) we obtain

$$\widetilde{P}_q(z,s) = \frac{f_2(z,s) f_3(z,s)\left(1-\overline{G}[f_1(z,s)]\right)\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}} \tag{2.33}$$

$$\widetilde{V}_q(z,s) = \frac{p f_1(z,s) f_2(z,s)\overline{G}[f_1(z,s)]\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}} \tag{2.34}$$

$$\widetilde{R}_q(z,s) = \frac{\alpha z f_3(z,s)\left(1-\overline{G}[f_1(z,s)]\right)\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}} \tag{2.35}$$

Let $\widetilde{W}_q(z,s)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (2.33), (2.34) and (2.35) we obtain

$$\widetilde{W}_q(z,s) = \widetilde{P}_q(z,s) + \widetilde{V}_q(z,s) + \widetilde{R}_q(z,s) \tag{2.36}$$

$$\begin{aligned}\widetilde{W}_q(z,s) = {} & \frac{\left[f_2(z,s) f_3(z,s)\left(1-\overline{G}[f_1(z,s)]\right) + p f_1(z,s) f_2(z,s)\overline{G}[f_1(z,s)]\right]\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}} \\ & + \frac{\alpha z f_3(z,s)\left(1-\overline{G}[f_1(z,s)]\right)\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}}{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}}\end{aligned} \tag{2.37}$$

If we let $z = 1$ in equation (2.37), we can easily verify that $\widetilde{Q}(s) + \widetilde{W}_q(z,s) = 1/s$ as it should be. Further, it can be shown that the denominator of the right hand side of (2.37) has one zero inside the unit circle $|z|=1$ which is sufficient to determine the only unknown $\widetilde{Q}(s)$ appearing in the numerator. Therefore, $\widetilde{P}_q(z,s)$, $\widetilde{V}_q(z,s)$, $\widetilde{R}_q(z,s)$ and for that matter $\widetilde{W}_q(z,s)$ can be completely determined.

We note that many particular cases of interest such as $M/M/1$, $M/E_k/1$, $M/D/1$, etc., can be derived from the results given in (2.33), (3.34), (2.35), and (2.37), by using the appropriate values for $\overline{G}[s+\lambda-\lambda C(z)+\alpha]$.

2.5 The Steady State Results

To define the steady state probabilities and the corresponding probability generating functions we drop the argument t and for that matter, the argument s, wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by applying the well-known Tauberian property

$$\lim_{s\to0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

Thus, multiplying both sides of equation (2.33) by s, taking the limit as s approaches zero, applying the Tauberian property, and after some simplification we get

$$P_q(z) = \lim_{s\to0} s\widetilde{P}_q(z,s) =$$

$$\frac{\lim\limits_{s\to0}\{sf_2(z,s)f_3(z,s)(1-\overline{G}[f_1(z,s)])\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\}\}}{\lim\limits_{s\to0}\{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta zf_3(z,s)-\overline{G}[f_1(z,s)]\{\gamma pf_1(z,s)f_2(z,s)-\alpha\beta zf_3(z,s)\}\}}$$

$$=\frac{\lim\limits_{s\to0}\{f_2(z,s)f_3(z,s)(1-\overline{G}[f_1(z,s)])\{s+s[\lambda(C(z)-1)-s]\widetilde{Q}(s)\}\}}{\lim\limits_{s\to0}\{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta zf_3(z,s)-\overline{G}[f_1(z,s)]\{\gamma pf_1(z,s)f_2(z,s)-\alpha\beta zf_3(z,s)\}\}}$$

$$P_q(z)=\frac{f_2(z)f_3(z)(1-\overline{G}[f_1(z)])\lambda(C(z)-1)Q}{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta zf_3(z)-\overline{G}[f_1(z)]\{\gamma pf_1(z)f_2(z)-\alpha\beta zf_3(z)\}} \qquad (2.38)$$

where

$f_1(z)=\lambda-\lambda C(z)+\alpha$

$f_2(z)=\lambda-\lambda C(z)+\beta$

$f_3(z)=\lambda-\lambda C(z)+\gamma$

$f_4(z)=z-(1-p)\overline{G}[\lambda-\lambda C(z)+\alpha]$

Performing similar operations to equations (2.34) and (2.35) leads to

$$V_q(z) = \lim_{s\to0} s\widetilde{V}_q(z,s) =$$

$$\frac{\lim\limits_{s\to0}\{spf_1(z,s)f_2(z,s)\overline{G}[f_1(z,s)]\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\}\}}{\lim\limits_{s\to0}\{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta zf_3(z,s)-\overline{G}[f_1(z,s)]\{\gamma pf_1(z,s)f_2(z,s)-\alpha\beta zf_3(z,s)\}\}}$$

$$V_q(z)=\frac{pf_1(z)f_2(z)\overline{G}[f_1(z)]\lambda(C(z)-1)Q}{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta zf_3(z)-\overline{G}[f_1(z)]\{\gamma pf_1(z)f_2(z)-\alpha\beta zf_3(z)\}} \qquad (2.39)$$

$$R_q(z) = \lim_{s\to 0} s\widetilde{R}_q(z,s) =$$
$$\frac{\lim\limits_{s\to 0}\left\{s\alpha z f_3(z,s)\left(1-\overline{G}[f_1(z,s)]\right)\left\{1+[\lambda(C(z)-1)-s]\widetilde{Q}(s)\right\}\right\}}{\lim\limits_{s\to 0}\left\{f_1(z,s)f_2(z,s)f_3(z,s)f_4(z,s)-\alpha\beta z f_3(z,s)-\overline{G}[f_1(z,s)]\left\{\gamma p f_1(z,s)f_2(z,s)-\alpha\beta z f_3(z,s)\right\}\right\}}$$

$$R_q(z) = \frac{\alpha z f_3(z)\left(1-\overline{G}[f_1(z)]\right)\lambda(C(z)-1)Q}{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta\, z f_3(z)-\overline{G}[f_1(z)]\left\{\gamma p f_1(z)f_2(z)-\alpha\beta\, z f_3(z)\right\}} \tag{2.40}$$

Now, for the steady states, we add equations (2.38), (2.39), and (2.40) we get

$$W_q(z) = P_q(z)+V_q(z)+R_q(z) =$$
$$\frac{\left\{f_3(z)\left(f_2(z)+\alpha z\right)+\overline{G}[f_1(z)]\left(pf_1(z)f_2(z)-f_2(z)f_3(z)-\alpha z f_3(z)\right)\right\}[\lambda(C(z)-1)]\lambda Q}{f_1(z)f_2(z)f_3(z)f_4(z)-\alpha\beta\, z f_3(z)-\overline{G}[f_1(z)]\left\{\gamma p f_1(z)f_2(z)-\alpha\beta\, z f_3(z)\right\}} \tag{2.41}$$

According to the normalization condition we have

$$Q + W_q(1) = 1$$

We see that for $z = 1$, $W_q(z)$ is indeterminate of 0/0 form. Therefore, we apply L'Hopital's Rule on equation (2.41), where we differentiate both the numerator and denominator with respect to z. Accordingly, we get

$$W_q(1) = \frac{\lambda Q E(I)\left\{\alpha\gamma+\beta\gamma+\overline{G}[\alpha]\left(p\alpha\beta-\alpha\gamma-\beta\gamma\right)\right\}}{\overline{G}[\alpha]\left(\lambda\beta\gamma E(I)+\lambda\alpha\gamma E(I)-\lambda\alpha\beta p E(I)+\alpha\beta\gamma\right)-\lambda\gamma E(I)(\alpha+\beta)} \tag{2.42}$$

where $C(1)=1$, $C'(1)=E(I)$ is the mean batch size of the arriving customers. Adding Q to (2.42), equating to 1 and solving for Q we get

$$Q = \frac{\overline{G}[\alpha]\left(\alpha\beta\gamma+\lambda\beta\gamma E(I)+\lambda\alpha\gamma E(I)-\lambda\alpha\beta p E(I)\right)-\lambda\gamma E(I)(\alpha+\beta)}{\alpha\beta\gamma\overline{G}[\alpha]} \tag{2.43}$$

After simplification, equation (2.43) can be written as

$$Q = 1-\lambda E(I)\left(\frac{1}{\beta\overline{G}[\alpha]}+\frac{1}{\alpha\overline{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right) \tag{2.44}$$

where $\lambda E(I)\left(\frac{1}{\beta\overline{G}[\alpha]}+\frac{1}{\alpha\overline{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right)<1$ emerges out to be the stability condition under which the steady states exits. Hence

$$\rho = \lambda E(I)\left(\frac{1}{\beta\overline{G}[\alpha]}+\frac{1}{\alpha\overline{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right) \tag{2.45}$$

Equation (2.44) gives the probability that the server is idle. Substituting for Q from (2.44) in (2.41), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size at a random epoch.

2.6 The Mean Queue Size and the Mean Waiting Time

To find the mean number of customers in the queueing system considered in this chapter, we write $W_q(z)$ obtained in (2.41) as $W_q(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (2.41), respectively. Then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{2.46}$$

which was derived in equation (1.30) in chapter 1. Note that primes and double primes in (2.46) denote first and second derivatives at $z = 1$, respectively. Then carrying out the first and second derivatives for the expression for $W_q(z)$ obtained in (2.41) and letting $z = 1$ we get the following

$$N'(1) = \lambda Q E(I)\{\alpha\gamma + \beta\gamma + \overline{G}[\alpha](p\alpha\beta - \alpha\gamma - \beta\gamma)\} \tag{2.47}$$

$$\begin{aligned} N''(1) = 2Q(\lambda E(I))^2 \Bigg\{ & \left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right) \\ & + \overline{G}[\alpha]\left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)} - p\beta - p\alpha\right) \\ & + \overline{G}'[\alpha](\alpha\gamma + \beta\gamma - p\alpha\beta) \Bigg\} \\ & + \lambda Q E(I(I-1))\{\alpha\gamma + \beta\gamma + \overline{G}[\alpha](p\alpha\beta - \alpha\gamma - \beta\gamma)\} \end{aligned} \tag{2.48}$$

$$D'(1) = -\lambda\gamma E(I)(\alpha + \beta) + \overline{G}[\alpha](\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) - \lambda\alpha\beta p E(I) + \alpha\beta\gamma) \tag{2.49}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\gamma\beta}{\lambda E(I)}\right) \\ & + \overline{G}[\alpha]\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\lambda E(I)}\right) \\ & + \overline{G}'[\alpha]\left(\alpha\beta p - \alpha\gamma - \beta\gamma - \frac{\alpha\beta\gamma}{\lambda E(I)}\right) \Bigg\} \\ & - \lambda E(I(I-1))\{\alpha\gamma + \beta\gamma + \overline{G}[\alpha](p\alpha\beta - \alpha\gamma - \beta\gamma)\} \end{aligned} \tag{2.50}$$

where $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers, and Q has been found in (2.44). Then if we substitute for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from (2.47) – (2.50) in equation (2.46) we obtain L_q in a closed form. Further, the mean waiting time of a customer can be found using Little's laws discussed in chapter 1, specifically; equation (1.3) can be used to find the mean waiting time, knowing the mean queue size.

2.7 Particular Cases

2.7.1 Single Poisson Arrivals

When customers arrive at the system one by one, then $c_1 = 1$ and $c_i = 0$ for $i \neq 1$. Consequently, $C(z) = z$, $E(I) = 1$, $E(I(I\text{-}1)) = 0$. Using these substitutions in (2.41) we get,

$$W_q(z) = \frac{\{f_2(z)f_3(z)(1-\bar{G}[f_1(z)]) + pf_1(z)f_2(z)\bar{G}[f_1(z)] + \alpha z f_3(z)(1-\bar{G}[f_1(z)])\}[\lambda(z-1)]Q}{f_1(z)f_2(z)f_3(z)f_4(z) - p\gamma f_1(z)f_2(z)\bar{G}[f_1(z)] - \alpha\beta z f_3(z)(1-\bar{G}[f_1(z)])} \tag{2.51}$$

where

$f_1(z) = \lambda - \lambda z + \alpha$

$f_2(z) = \lambda - \lambda z + \beta$

$f_3(z) = \lambda - \lambda z + \gamma$

$f_4(z) = z - (1-p)\bar{G}[\lambda - \lambda z + \alpha]$

and Q is given by

$$Q = 1 - \lambda\left(\frac{1}{\beta\bar{G}[\alpha]} + \frac{1}{\alpha\bar{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{2.52}$$

and hence,

$$\rho = \lambda\left(\frac{1}{\beta\bar{G}[\alpha]} + \frac{1}{\alpha\bar{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{2.53}$$

Using the same substitutions in (2.47) – (2.50) yields

$$N'(1) = \lambda Q\{(\alpha\gamma + \beta\gamma) + (p\alpha\beta - \alpha\gamma - \beta\gamma)\bar{G}[\alpha]\} \tag{2.54}$$

$$\begin{aligned} N''(1) = 2\lambda Q\{&(-\lambda\gamma - \lambda\beta - \lambda\alpha + \alpha\gamma) \\ &+ \bar{G}[\alpha](\lambda\gamma + \lambda\beta + \lambda\alpha - \alpha\gamma - p\lambda\beta - p\lambda\alpha) \\ &+ \bar{G}'[\alpha](\lambda\alpha\gamma + \lambda\beta\gamma - \lambda p\alpha\beta)\} \end{aligned} \tag{2.55}$$

$$D'(1) = -\lambda\gamma(\alpha + \beta) + \bar{G}[\alpha](\alpha\beta\gamma + \lambda\beta\gamma + \lambda\alpha\gamma - \lambda\alpha\beta p) \tag{2.56}$$

$$\begin{aligned} D''(1) = 2\{&(\lambda^2\alpha + \lambda^2\beta + \lambda^2\gamma - \lambda\gamma\alpha - \lambda\gamma\beta) \\ &+ \bar{G}[\alpha](p\lambda^2\alpha + p\lambda^2\beta - \lambda^2\alpha - \lambda^2\beta - \lambda^2\gamma - \alpha\beta\lambda) \\ &+ \bar{G}'[\alpha](\lambda^2\alpha\beta p - \lambda^2\alpha\gamma - \lambda^2\beta\gamma - \alpha\beta\gamma\lambda)\} \end{aligned} \tag{2.57}$$

Then if we substitute for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ in equation (2.46) we obtain L_q in a closed form. Further, the mean waiting time of a customer can be found by using equation (1.3).

2.7.2 *k*-Erlang Service Time

In this case, the service time has a *k*-Erlang distribution, and hence

$$g(x)=\frac{\mu^k x^{k-1}e^{-\mu x}}{(k-1)!},\quad \mu>0,\ k=1,2,3\ldots$$

$$\overline{G}[\lambda-\lambda C(z)+\alpha]=\frac{\mu^k}{(\lambda-\lambda C(z)+\alpha+\mu)^k},\quad \overline{G}[\alpha]=\frac{\mu^k}{(\alpha+\mu)^k}$$

Substituting for $\overline{G}[\lambda-\lambda C(z)+\alpha]$ in equation (2.41) we have

$$W_q(z)=$$

$$\frac{(\lambda-\lambda C(z)+\alpha+\mu)^k f_3(z)(f_2(z)+\alpha z)\lambda(C(z)-1)Q}{(\lambda-\lambda C(z)+\alpha+\mu)^k f_3(z)\{f_1(z)f_2(z)f_4(z)-\alpha\beta z\}-\mu^k\{\gamma p f_1(z)f_2(z)-\alpha\beta z f_3(z)\}}$$
$$+\frac{\{\mu^k(pf_1(z)f_2(z)-f_2(z)f_3(z)-\alpha z f_3(z))\}\lambda(C(z)-1)Q}{(\lambda-\lambda C(z)+\alpha+\mu)^k f_3(z)\{f_1(z)f_2(z)f_4(z)-\alpha\beta z\}-\mu^k\{\gamma p f_1(z)f_2(z)-\alpha\beta z f_3(z)\}}\tag{2.58}$$

where

$f_1(z)=\lambda-\lambda C(z)+\alpha$

$f_2(z)=\lambda-\lambda C(z)+\beta$

$f_3(z)=\lambda-\lambda C(z)+\gamma$

$$f_4(z)=z-\frac{(1-p)\mu^k}{(\lambda-\lambda C(z)+\alpha+\mu)^k}$$

and Q and ρ are given by

$$Q=1-\lambda E(I)\left(\frac{(\alpha+\mu)^k}{\beta\mu^k}+\frac{(\alpha+\mu)^k}{\alpha\mu^k}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right)\tag{2.59}$$

$$\rho=\lambda E(I)\left(\frac{(\alpha+\mu)^k}{\beta\mu^k}+\frac{(\alpha+\mu)^k}{\alpha\mu^k}-\frac{1}{\alpha}-\frac{1}{\beta}+\frac{p}{\gamma}\right)\tag{2.60}$$

Further, L_q can be found using (2.46) and then W_q is obtained using (1.3). Thus, substituting for $\overline{G}[\alpha]=\frac{\mu^k}{(\alpha+\mu)^k}$ and $\overline{G}'[\alpha]=\frac{-k\,\mu^k}{(\alpha+\mu)^{k+1}}$ in equations (2.47) – (2.50) we get

$$N'(1)=\lambda Q E(I)\left\{\alpha\gamma+\beta\gamma+\frac{\mu^k}{(\alpha+\mu)^k}(p\alpha\beta-\alpha\gamma-\beta\gamma)\right\}\tag{2.61}$$

$$\begin{aligned}
N''(1) = 2Q(\lambda E(I))^2 \Bigg\{ & \left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right) \\
& + \frac{\mu^k}{(\alpha+\mu)^k}\left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)} - p\beta - p\alpha\right) \\
& - \frac{k\mu^k}{(\alpha+\mu)^{k+1}}(\alpha\gamma + \beta\gamma - p\alpha\beta) \Bigg\} \\
& + \lambda Q E(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{\mu^k}{(\alpha+\mu)^k}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\}
\end{aligned} \tag{2.62}$$

$$D'(1) = -\lambda\gamma E(I)(\alpha+\beta) + \frac{\mu^k}{(\alpha+\mu)^k}(\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) - \lambda\alpha\beta p E(I) + \alpha\beta\gamma) \tag{2.63}$$

$$\begin{aligned}
D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\gamma\beta}{\lambda E(I)}\right) \\
& + \frac{\mu^k}{(\alpha+\mu)^k}\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\lambda E(I)}\right) \\
& - \frac{k\mu^k}{(\alpha+\mu)^{k+1}}\left(\alpha\beta p - \alpha\gamma - \beta\gamma - \frac{\alpha\beta\gamma}{\lambda E(I)}\right) \Bigg\} \\
& - \lambda E(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{\mu^k}{(\alpha+\mu)^k}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\}
\end{aligned} \tag{2.64}$$

2.7.3 Exponential Service Time

The most common distribution function for the service time is the exponential distribution. For this distribution, the rate of service is $\mu > 0$. The equations for an exponential service time could be found by letting $k = 1$ in the k-Erlang service time results. Thus

$$W_q(z) = \frac{\{f_2(z)f_3(z) + \mu p f_2(z) + \alpha z f_3(z)\}\lambda(C(z)-1)Q}{(f_1(z)+\mu)f_2(z)f_3(z)f_4(z) - \mu\gamma p f_2(z) - \alpha\beta z f_3(z)} \tag{2.65}$$

where

$f_1(z) = \lambda - \lambda C(z) + \alpha$

$f_2(z) = \lambda - \lambda C(z) + \beta$

$f_3(z) = \lambda - \lambda C(z) + \gamma$

$$f_4(z) = z - \frac{(1-p)\mu}{(\lambda - \lambda C(z) + \alpha + \mu)}$$

and Q simplifies to

$$Q = 1 - \lambda E(I)\left(\frac{\alpha}{\beta\mu} + \frac{1}{\mu} + \frac{p}{\gamma}\right) \tag{2.66}$$

and hence

$$\rho = \lambda E(I)\left(\frac{\alpha}{\beta\mu} + \frac{1}{\mu} + \frac{p}{\gamma}\right) \tag{2.67}$$

Now we let $k = 1$ in equations (2.47) – (2.50), we obtain

$$N'(1) = \lambda QE(I)\left\{\alpha\gamma + \beta\gamma + \frac{\mu}{(\alpha+\mu)}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \tag{2.68}$$

$$\begin{aligned} N''(1) = 2Q(\lambda E(I))^2 \Bigg\{ & \left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right) \\ & + \frac{\mu}{(\alpha+\mu)}\left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)} - p\beta - p\alpha\right) - \frac{\mu}{(\alpha+\mu)^2}(\alpha\gamma + \beta\gamma - p\alpha\beta) \Bigg\} \\ & + \lambda QE(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{\mu}{(\alpha+\mu)}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \end{aligned} \tag{2.69}$$

$$D'(1) = \frac{\mu}{(\alpha+\mu)}(\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) - \lambda\alpha\beta p E(I) + \alpha\beta\gamma) - \lambda\gamma E(I)(\alpha + \beta) \tag{2.70}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\gamma\beta}{\lambda E(I)}\right) \\ & + \frac{\mu}{(\alpha+\mu)}\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\lambda E(I)}\right) - \frac{\mu}{(\alpha+\mu)^2}\left(\alpha\beta p - \alpha\gamma - \beta\gamma - \frac{\alpha\beta\gamma}{\lambda E(I)}\right) \Bigg\} \\ & - \lambda E(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{\mu}{(\alpha+\mu)}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \end{aligned} \tag{2.71}$$

Utilizing equations (2.68) – (2.71) in equation (2.46) we can find L_q, and hence W_q can be found using equation (1.3).

2.7.4 Deterministic Service Time

In this case we assume that the service time is constant of length $b > 0$. Then,

$$\overline{G}[\lambda - \lambda C(z) + \alpha] = \frac{b}{\lambda - \lambda C(z) + \alpha}, \quad \overline{G}[\alpha] = \frac{b}{\alpha}, \quad \overline{G}'[\alpha] = -\frac{b}{\alpha^2} \tag{2.72}$$

Using these substitutions, equation (2.41) can be written as

$$W_q(z) = \frac{\{(f_1(z) - b)(f_2(z) + \alpha z)f_3(z) + bpf_1(z)f_2(z)\}\lambda(C(z)-1)Q}{f_1^2(z)f_2(z)f_3(z)f_4(z) - \alpha\beta\, zf_1(z)f_3(z) - b\{\gamma pf_1(z)f_2(z) - \alpha\beta\, zf_3(z)\}} \tag{2.73}$$

where

$f_1(z) = \lambda - \lambda C(z) + \alpha$

$f_2(z) = \lambda - \lambda C(z) + \beta$

$f_3(z) = \lambda - \lambda C(z) + \gamma$

$$f_4(z) = z - \frac{(1-p)b}{\lambda - \lambda C(z) + \alpha}$$

and Q is expressed as

$$Q = 1 - \lambda E(I)\left(\frac{\alpha}{\beta b} + \frac{1}{b} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{2.74}$$

Thus, the utilization factor becomes

$$\rho = \lambda E(I)\left(\frac{\alpha}{\beta b} + \frac{1}{b} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{2.75}$$

For the mean number of customers in the queue and the mean waiting time, we need to use (2.72) in equations (2.47) – (2.50) which gives

$$N'(1) = \lambda Q E(I)\left\{\alpha\gamma + \beta\gamma + \frac{b}{\alpha}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \tag{2.76}$$

$$\begin{aligned} N''(1) = 2Q(\lambda E(I))^2 & \left\{ \left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right) + \frac{b}{\alpha}\left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)} - p\beta - p\alpha\right) \right. \\ & \left. - \frac{b}{\alpha^2}(\alpha\gamma + \beta\gamma - p\alpha\beta) \right\} + \lambda Q E(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{b}{\alpha}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \end{aligned} \tag{2.77}$$

$$D'(1) = \frac{b}{\alpha}(\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) - \lambda\alpha\beta p E(I) + \alpha\beta\gamma) - \lambda\gamma E(I)(\alpha + \beta) \tag{2.78}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I))^2 & \left\{ \left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\gamma\beta}{\lambda E(I)}\right) + \frac{b}{\alpha}\left(p\alpha + p\beta - \alpha - \beta - \gamma - \frac{\alpha\beta}{\lambda E(I)}\right) \right. \\ & \left. - \frac{b}{\alpha^2}\left(\alpha\beta p - \alpha\gamma - \beta\gamma - \frac{\alpha\beta\gamma}{\lambda E(I)}\right) \right\} - \lambda E(I(I-1))\left\{\alpha\gamma + \beta\gamma + \frac{b}{\alpha}(p\alpha\beta - \alpha\gamma - \beta\gamma)\right\} \end{aligned} \tag{2.79}$$

2.7.5 No Server Vacations

When the server has no option to take a vacation, we have $p = 0$. Using this in equation (2.41), $W_q(z)$ can be written in the form

$$W_q(z) = \frac{((\lambda - \lambda C(z) + \beta) + \alpha z)(1 - \bar{G}[\lambda - \lambda C(z) + \alpha])\lambda(C(z) - 1)Q}{(\lambda - \lambda C(z) + \alpha)(\lambda - \lambda C(z) + \beta)(z - \bar{G}[\lambda - \lambda C(z) + \alpha]) - \alpha\beta z(1 - \bar{G}[\lambda - \lambda C(z) + \alpha])} \tag{2.80}$$

where Q could be found by setting $p = 0$ in (2.44)

$$Q = 1 - \lambda E(I)\left(\frac{1}{\beta\bar{\bar{G}}[\alpha]} + \frac{1}{\alpha\bar{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta}\right) \tag{2.81}$$

Therefore,

$$\rho = \lambda E(I)\left(\frac{1}{\beta\bar{G}[\alpha]} + \frac{1}{\alpha\bar{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta}\right) \tag{2.82}$$

Further, to find the mean number in the queue and the mean waiting time in the queue we find $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$

$$N'(1) = \lambda QE(I)\{\alpha\gamma + \beta\gamma - \bar{G}[\alpha](\alpha\gamma + \beta\gamma)\} \tag{2.83}$$

$$\begin{aligned} N''(1) = 2Q(\lambda E(I))^2 \Bigg\{ & \left(-\gamma - \beta - \alpha + \frac{\alpha\gamma}{\lambda E(I)}\right) + \left(\gamma + \beta + \alpha - \frac{\alpha\gamma}{\lambda E(I)}\right)\bar{G}[\alpha] \\ & + (\alpha\gamma + \beta\gamma)\bar{G}'[\alpha] \Bigg\} + \lambda QE(I(I-1))\{\alpha\gamma + \beta\gamma - \bar{G}[\alpha](\alpha\gamma + \beta\gamma)\} \end{aligned} \tag{2.84}$$

$$D'(1) = \bar{G}[\alpha](\lambda\beta\gamma E(I) + \lambda\alpha\gamma E(I) + \alpha\beta\gamma) - \lambda\gamma E(I)(\alpha + \beta) \tag{2.85}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(\alpha + \beta + \gamma - \frac{\gamma\alpha}{\lambda E(I)} - \frac{\gamma\beta}{\lambda E(I)}\right) - \bar{G}[\alpha]\left(\alpha + \beta + \gamma + \frac{\alpha\beta}{\lambda E(I)}\right) \\ & - \bar{G}'[\alpha]\left(\alpha\gamma + \beta\gamma + \frac{\alpha\beta\gamma}{\lambda E(I)}\right) \Bigg\} - \lambda E(I(I-1))\{\alpha\gamma + \beta\gamma - \bar{G}[\alpha](\alpha\gamma + \beta\gamma)\} \end{aligned} \tag{2.86}$$

Using the results obtained in (2.83) – (2.86) in (2.46) we can find L_q, and hence W_q.

2.7.6 No System Breakdowns

In this case, the system will never fail and hence $\alpha = 0$ which gives $R_q(z) = 0$. Then equation (2.41) becomes

$$W_q(z) = \frac{\{(\lambda - \lambda C(z) + \gamma)(1 - \bar{G}[\lambda - \lambda C(z)]) + p(\lambda - \lambda C(z))\bar{G}[\lambda - \lambda C(z)]\}Q}{p\gamma\bar{G}[\lambda - \lambda C(z)] - (\lambda - \lambda C(z) + \gamma)(z - (1-p)\bar{G}[\lambda - \lambda C(z)])} \tag{2.87}$$

To find Q, we let $\alpha = 0$ in (2.43), get the 0/0 form, and then we use L'Hopitals rule. Differentiating both the numerator and denominator of the right hand side of (2.43) with respect to α, we get

$$Q = 1 - \lambda E(I)\left(\frac{p}{\gamma} + E(S)\right) \tag{2.88}$$

where $\bar{G}[0] = 1$ and $-\bar{G}'[0] = E(S)$ is the mean service time. Hence

$$\rho = \lambda E(I)\left(\frac{p}{\gamma} + E(S)\right) \tag{2.89}$$

Further, we compute $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ using the expression obtained for $W_q(z)$ in (2.87).

$$N'(1) = -\lambda Q E(I)\left(\gamma E(S) + p\right) \tag{2.90}$$

$$\begin{aligned} N''(1) = Q\big\{&2(1-p)E(S)\left(\lambda E(I)\right)^2 - \gamma\left(\lambda E(I)\right)^2 E(S^2) \\ &- \lambda E(I(I-1))\left[p + \gamma E(S)\right]\big\} \end{aligned} \tag{2.91}$$

$$D'(1) = \gamma\lambda E(I)E(S) + \lambda p E(I) - \gamma \tag{2.92}$$

$$\begin{aligned} D''(1) = {}&\gamma\left(\lambda E(I)\right)^2 E(S^2) - 2(1-p)\left(\lambda E(I)\right)^2 E(S) + 2\lambda E(I) \\ &+ \lambda E(I(I-1))\left[p + \gamma E(S)\right] \end{aligned} \tag{2.93}$$

where $\overline{G}''[0] = E(S^2)$ is the second moment of the service time. Utilizing equations (2.90) – (2.93) in equation (2.46) we can easily find L_q, and hence W_q.

If customers arrive one by one instead of batches, then $C(z) = z$, and hence $W_q(z)$ becomes

$$W_q(z) = \frac{\left\{\left(\lambda - \lambda z + \gamma\right)\left(1 - \overline{G}[\lambda - \lambda z]\right) + p\left(\lambda - \lambda z\right)\overline{G}[\lambda - \lambda z]\right\}Q}{p\gamma\overline{G}[\lambda - \lambda z] - \left(\lambda - \lambda z + \gamma\right)\left(z - (1-p)\overline{G}[\lambda - \lambda z]\right)} \tag{2.94}$$

For single arrivals we let $E(I) = 1$ in (2.88), hence Q becomes

$$Q = 1 - \lambda\left(\frac{p}{\gamma} + E(S)\right) \tag{2.95}$$

Thus, the stability condition is given by

$$\rho = \lambda\left(\frac{p}{\gamma} + E(S)\right) \tag{2.96}$$

The results obtained in (2.94) and (2.95) appeared in the literature of queueing theory (Madan, Abu-Dayyeh, & Saleh, 2002).

Letting $E(I) = 1$, and $E(I(I\text{-}1)) = 0$ in (2.90), (2.91), (2.92), and (2.93) we compute $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$

$$N'(1) = -\lambda Q\left(\gamma E(S) + p\right) \tag{2.97}$$

$$N''(1) = \lambda^2 Q\left\{2(1-p)E(S) - \gamma E(S^2)\right\} \tag{2.98}$$

$$D'(1) = \gamma\lambda E(S) - \gamma + \lambda p \tag{2.99}$$

$$D''(1) = \lambda^2\gamma E(S^2) - 2\lambda^2(1-p)E(S) + 2\lambda \tag{2.100}$$

Substituting for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from equations (2.97) – (2.100) in equation (2.46) and using (2.95) we get the following closed form of the mean number of customers in the queue

$$L_q = \frac{\lambda^2\left(\gamma^2 E(S^2) + \gamma p E(S) + p\right)}{2\left(\gamma\lambda E(S) - \gamma + \lambda p\right)^2}\left(1 - \lambda\left(\frac{p}{\gamma} + E(S)\right)\right) \qquad (2.101)$$

which gives the following expression for the mean waiting time in the queue

$$W_q = \frac{\lambda\left(\gamma^2 E(S^2) + \gamma p E(S) + p\right)}{2\left(\gamma\lambda E(S) - \gamma + \lambda p\right)^2}\left(1 - \lambda\left(\frac{p}{\gamma} + E(S)\right)\right) \qquad (2.102)$$

2.7.7 No Server Vacations, No System Breakdowns

In this case we let $p = 0 = \alpha$, hence $V_q(z) = 0 = R_q(z)$. Using these substitutions in (2.41) and after some simplifications we get

$$W_q(z) = \frac{\left(\overline{G}[\lambda - \lambda C(z)] - 1\right)Q}{z - \overline{G}[\lambda - \lambda C(z)]} \qquad (2.103)$$

where Q could be found by letting $p = 0$ in (2.95)

$$Q = 1 - \lambda E(I)E(S) \qquad (2.104)$$

which gives

$$\rho = \lambda E(I)E(S) \qquad (2.105)$$

Then, (2.103) can be written as

$$W_q(z) = \frac{\left(\overline{G}[\lambda - \lambda C(z)] - 1\right)\left(1 - \lambda E(I)E(S)\right)}{z - \overline{G}[\lambda - \lambda C(z)]} \qquad (2.106)$$

Further, we can find the mean number of customers in the queue by setting $p = 0$ in (2.97) – (2.100)

$$N'(1) = -\lambda\gamma Q E(I)E(S) \qquad (2.107)$$

$$N''(1) = Q\left\{2E(S)(\lambda E(I))^2 - \lambda\gamma E(S)E(I(I-1)) - \gamma(\lambda E(I))^2 E(S^2)\right\} \qquad (2.108)$$

$$D'(1) = \gamma\lambda E(I)E(S) - \gamma \qquad (2.109)$$

$$D''(1) = \gamma(\lambda E(I))^2 E(S^2) - 2(\lambda E(I))^2 E(S) + 2\lambda E(I) + \lambda\gamma E(S)E(I(I-1)) \qquad (2.110)$$

Utilizing equations (2.107) – (2.110) in equation (2.46), we get

$$L_q = \frac{(\lambda E(I))^2 E(S^2) + \lambda E(S)E(I(I-1))}{2\left(1 - \lambda E(I)E(S)\right)} \qquad (2.111)$$

and hence the mean waiting time in the queue is given by

$$W_q = \frac{\lambda(E(I))^2 E(S^2) + E(S)E(I(I-1))}{2\left(1 - \lambda E(I)E(S)\right)} \qquad (2.112)$$

For single Poisson arrivals, we let $C(z) = z$, $E(I) = 1$, and $E(I(I-1)) = 0$ in the main results, we get

$$W_q(z) = \frac{\left(\overline{G}[\lambda - \lambda z] - 1\right)\left(1 - \lambda E(S)\right)}{z - \overline{G}[\lambda - \lambda z]} \tag{2.113}$$

where

$$Q = 1 - \lambda E(S) \tag{2.114}$$

$$\rho = \lambda E(S) \tag{2.115}$$

Using same substitutions in (2.111) and (2.112) gives

$$L_q = \frac{\lambda^2 E(S^2)}{2\left(1 - \lambda E(S)\right)} \tag{2.116}$$

$$W_q = \frac{\lambda E(S^2)}{2\left(1 - \lambda E(S)\right)} \tag{2.117}$$

The result obtained in (2.116) is given by Bunday (1996).

2.8 A Numerical Illustration

For the purpose of a numerical illustration, we obtain some tables and graphs for the first three particular cases. We choose the following arbitrary values $\lambda = 2$, $\mu = 20$, $E(I) = 1$, and $E(I(I-1)) = 0$.

All the tables give the computed values of various states of the server, the proportion of idle time, the utilization factor, the mean queue size, the mean waiting time, the probability that the server is working irrespective of the number of customers in the queue, the probability that the server is on vacation irrespective of the number of customers in the queue, the probability that the system is in repair due to breakdown irrespective of the number of customers in the queue, and the probability that the server is not idle.

In Tables 2.1 – 2.6 and graphs 2.1 – 2.7, we consider the special case discussed in section 2.7.2 which is a queueing system with Bernoulli vacations and random breakdowns. Service time follows k-Erlang distribution (values of k vary from 4 to 2), while vacation times and repair times both have exponential distributions.

In Table 2.1 values of β and γ are fixed to be 10 and 7, respectively, while α varies from 1 to 4 and p takes the values 0.25, 0.5, and 0.75. All the values were chosen such that the steady state condition is satisfied.

Table 2.1 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time, k = 4, $\beta = 10$, $\gamma = 7$*

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.25	0.4544	0.5456	0.5211	0.2606	0.431	0.0714	0.0431	0.5455
1	0.5	0.383	0.617	0.752	0.376	0.431	0.1428	0.0431	0.6169
1	0.75	0.3116	0.6884	1.0887	0.5444	0.431	0.2143	0.0431	0.6884
2	0.25	0.3716	0.6284	0.9187	0.4594	0.4641	0.0714	0.0928	0.6283
2	0.5	0.3002	0.6998	1.3158	0.6579	0.4641	0.1429	0.0928	0.6998
2	0.75	0.2288	0.7712	1.9611	0.9806	0.4642	0.2143	0.0928	0.7713
3	0.25	0.2796	0.7204	1.6655	0.8328	0.4993	0.0714	0.1498	0.7205
3	0.5	0.2081	0.7919	2.5058	1.2529	0.4992	0.1428	0.1498	0.7918
3	0.75	0.1367	0.8633	4.2258	2.1129	0.4992	0.2143	0.1498	0.8633
4	0.25	0.1772	0.8228	3.4507	1.7254	0.5367	0.0714	0.2147	0.8228
4	0.5	0.1058	0.8942	6.3386	3.1693	0.5369	0.1429	0.2148	0.8946
4	0.75	0.0343	0.9657	21.2071	10.6036	0.5361	0.2141	0.2145	0.9647

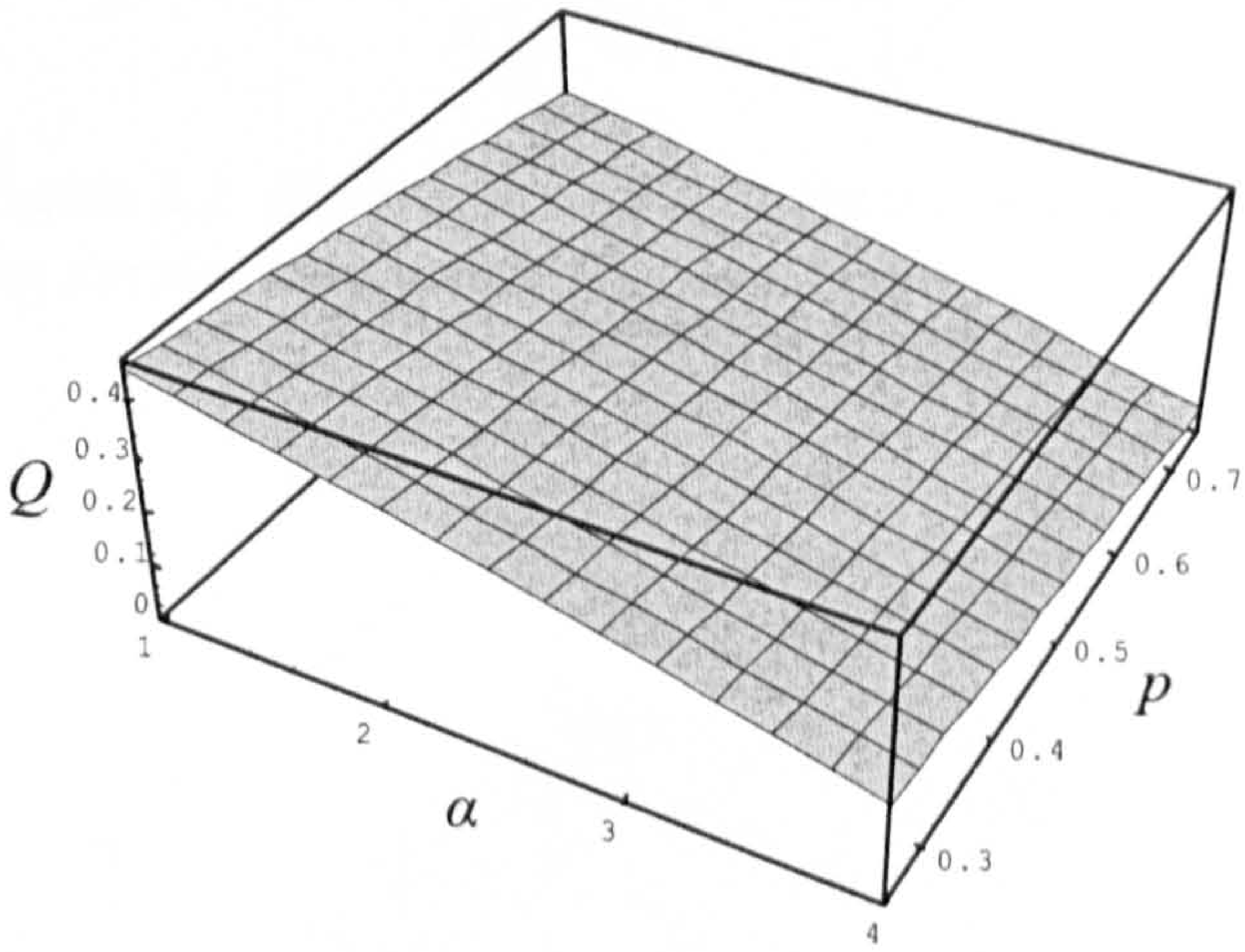

Figure 2.1 *Effect of α and p on the proportion of time that the server is idle Q (k-Erlang service time with k = 4, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)*

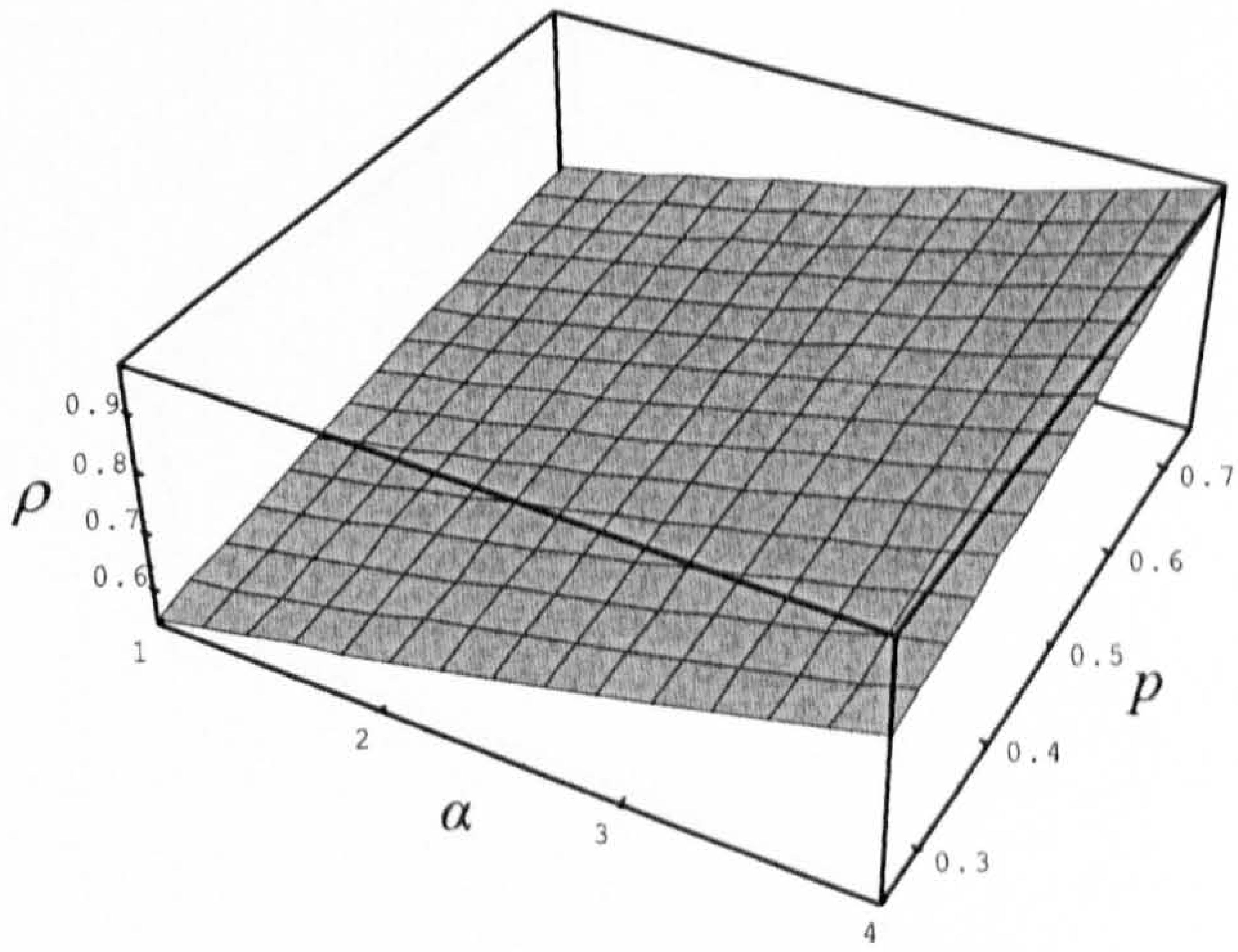

Figure 2.2 *Effect of α and p on the utilization factor ρ*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)

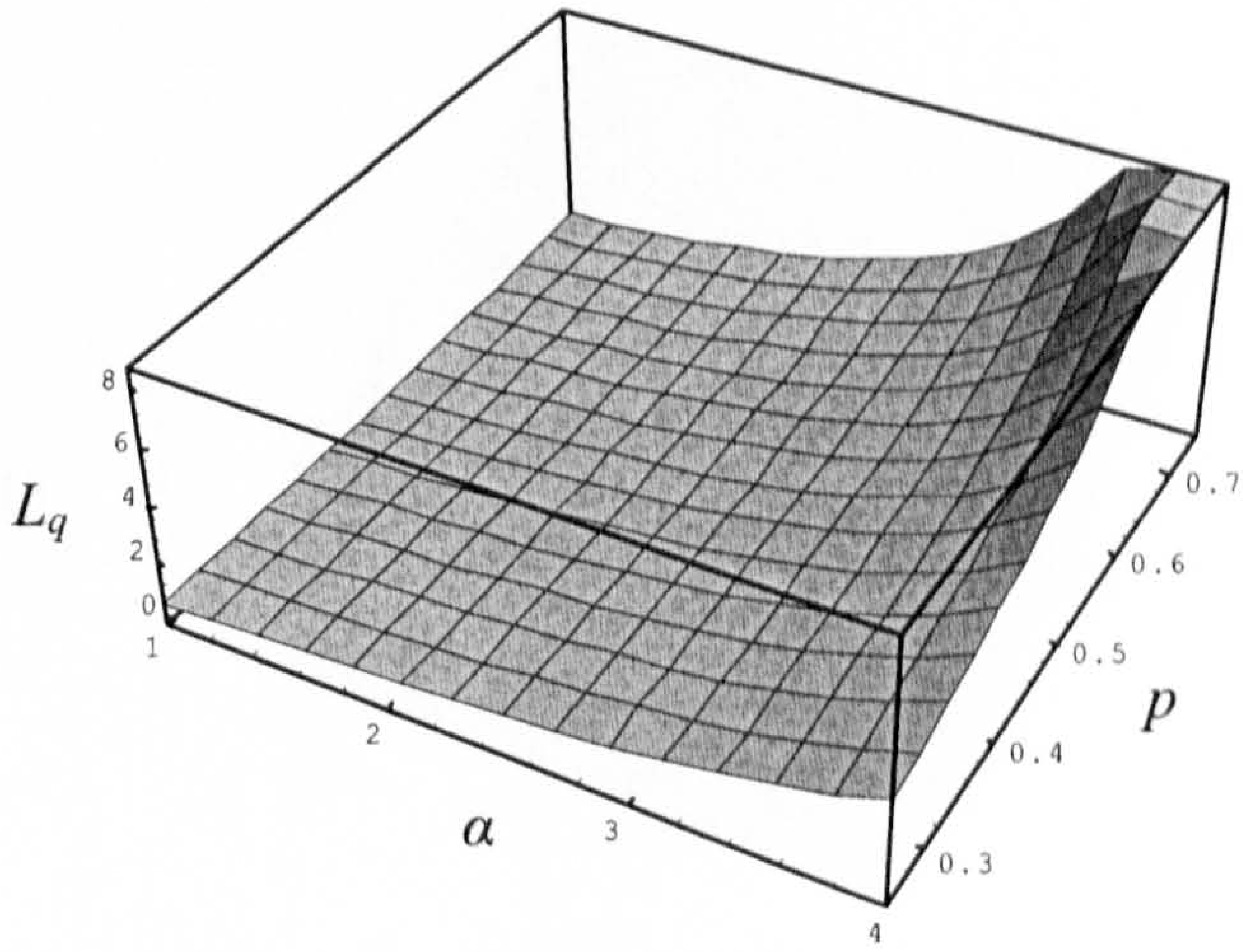

Figure 2.3 *Effect of α and p on the mean queue size L_q*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)

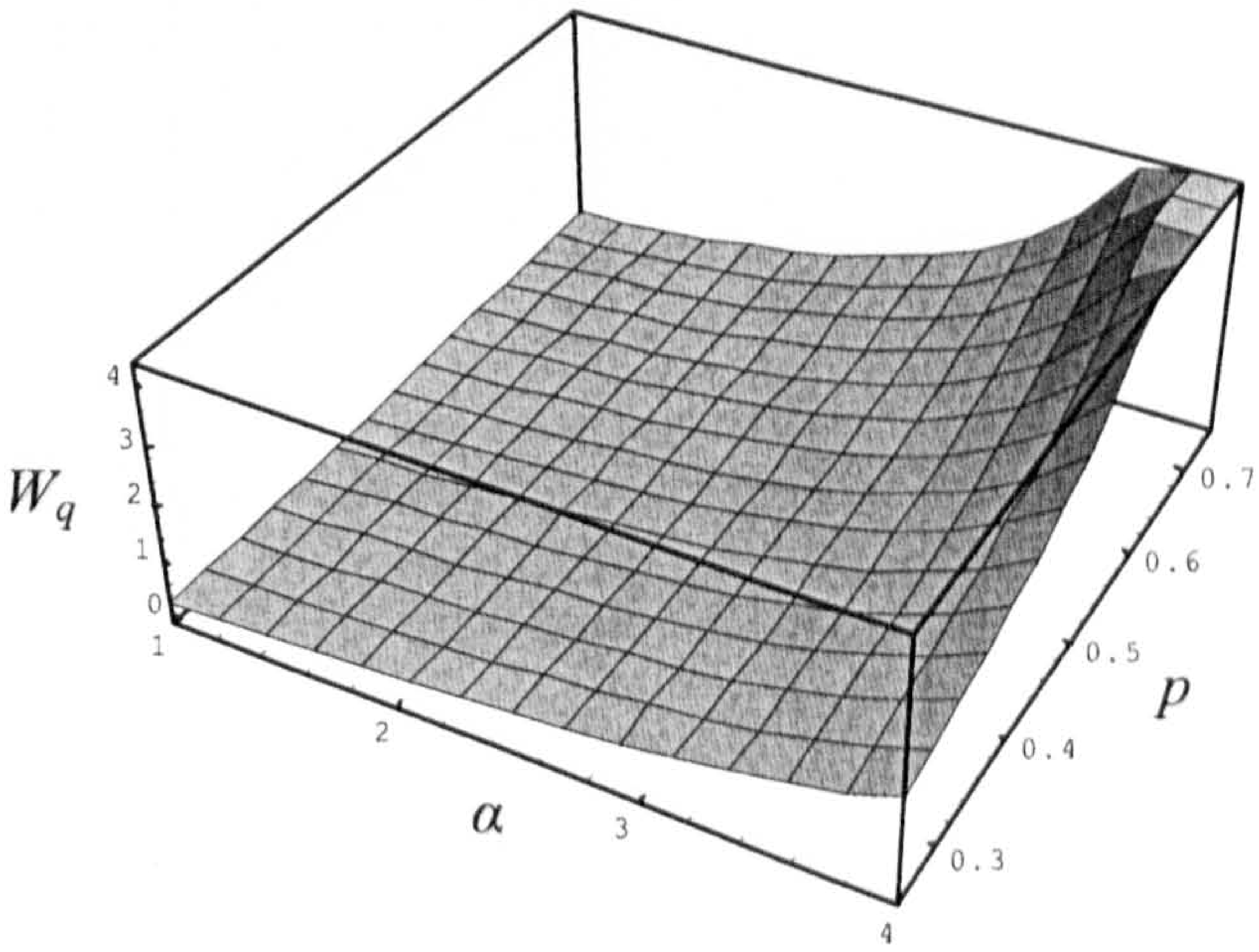

Figure 2.4 *Effect of α and p on the mean waiting time W_q*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)

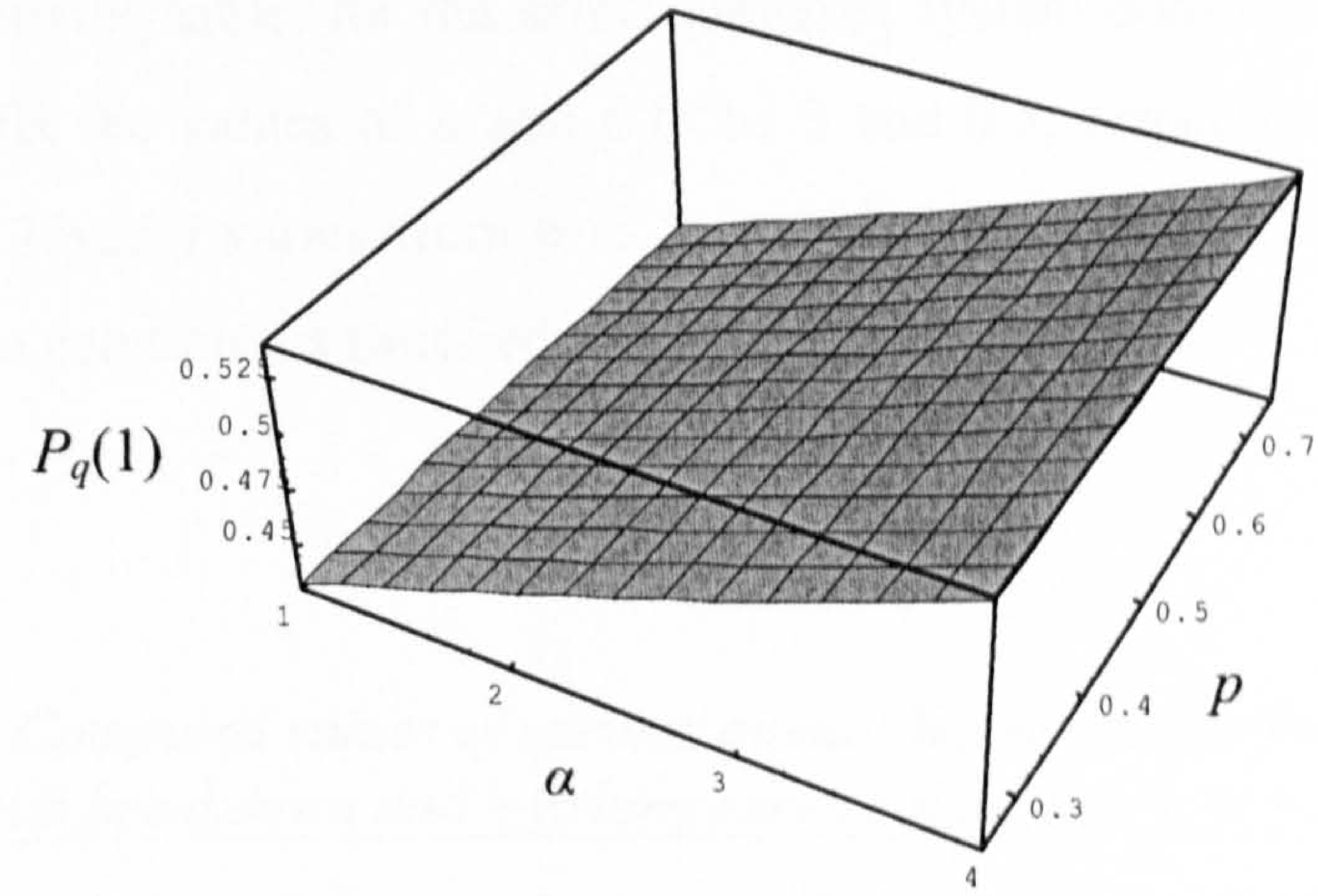

Figure 2.5 *Effect of α and p on the probability that the server is working (k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)*

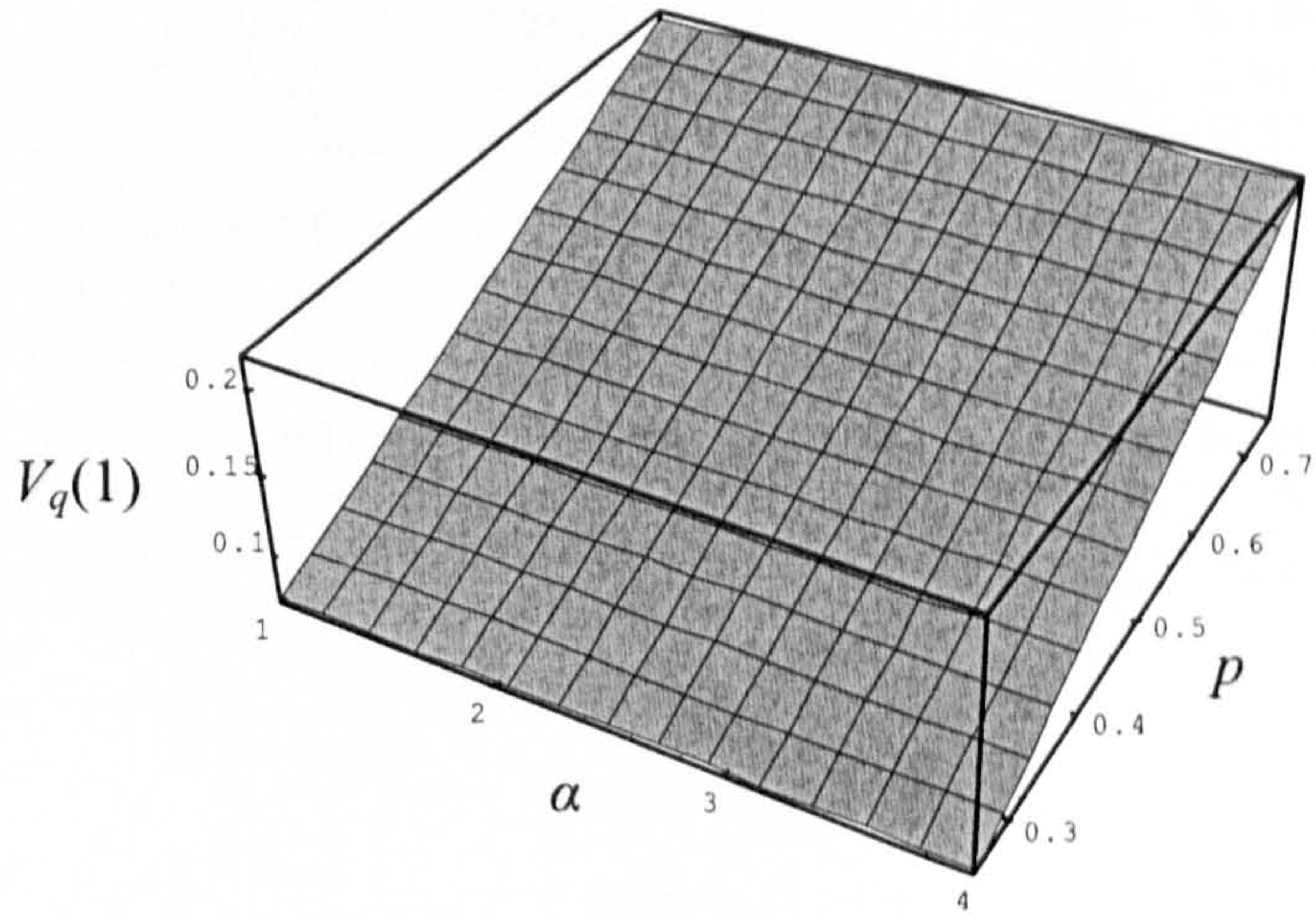

Figure 2.6 *Effect of α and p on the probability that the server is on vacation (k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)*

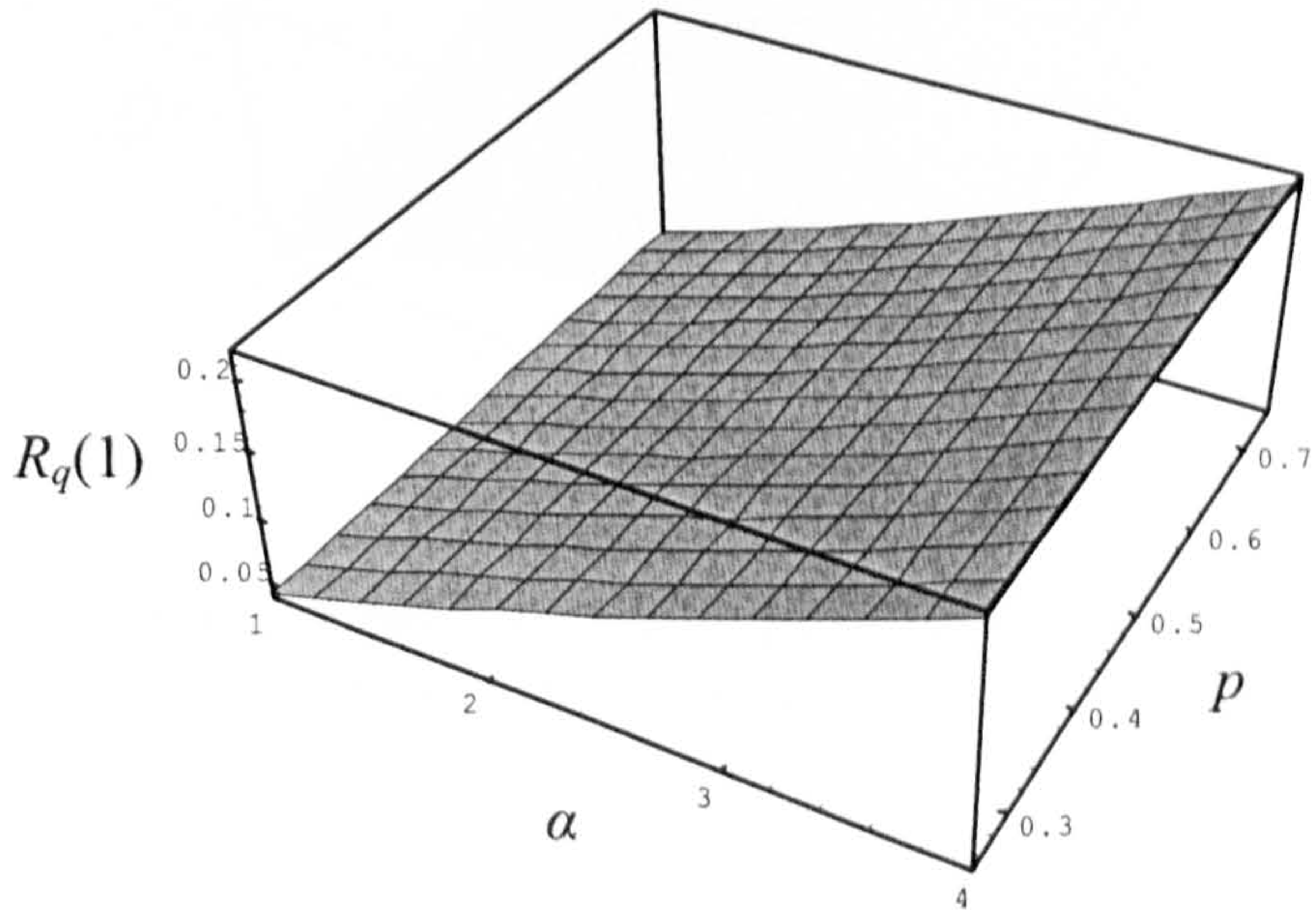

Figure 2.7 *Effect of α and p on the probability that the system is under repair (k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\beta = 10$, and $\gamma = 7$)*

In the following table, for the same queueing system considered in the previous table, we fix the values of α and p to be 3 and 0.5, respectively, while β varies from 6 to 9 and γ varies from 6 to 8. All the values were chosen such that the steady state condition is satisfied.

Table 2.2 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time, k = 4, $\alpha = 3$, p = 0.5*

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	6	0.0845	0.9155	8.7241	4.3621	0.4994	0.1667	0.2497	0.9158
6	7	0.1083	0.8917	6.5587	3.2794	0.4993	0.1429	0.2497	0.8919
6	8	0.1261	0.8739	5.481	2.7405	0.4991	0.125	0.2495	0.8736
7	6	0.1201	0.8799	5.5451	2.7726	0.4991	0.1666	0.2139	0.8796
7	7	0.1439	0.8561	4.4428	2.2214	0.4991	0.1428	0.2139	0.8558
7	8	0.1618	0.8382	3.8394	1.9197	0.4993	0.125	0.214	0.8383
8	6	0.1469	0.8531	4.1931	2.0966	0.4993	0.1667	0.1872	0.8532
8	7	0.1707	0.8293	3.4519	1.726	0.4993	0.1429	0.1872	0.8294
8	8	0.1885	0.8115	3.026	1.513	0.4991	0.125	0.1872	0.8113
9	6	0.1677	0.8323	3.4462	1.7231	0.4993	0.1667	0.1664	0.8324
9	7	0.1915	0.8085	2.8785	1.4393	0.4993	0.1429	0.1664	0.8086
9	8	0.2093	0.7907	2.5439	1.272	0.4991	0.125	0.1664	0.7905

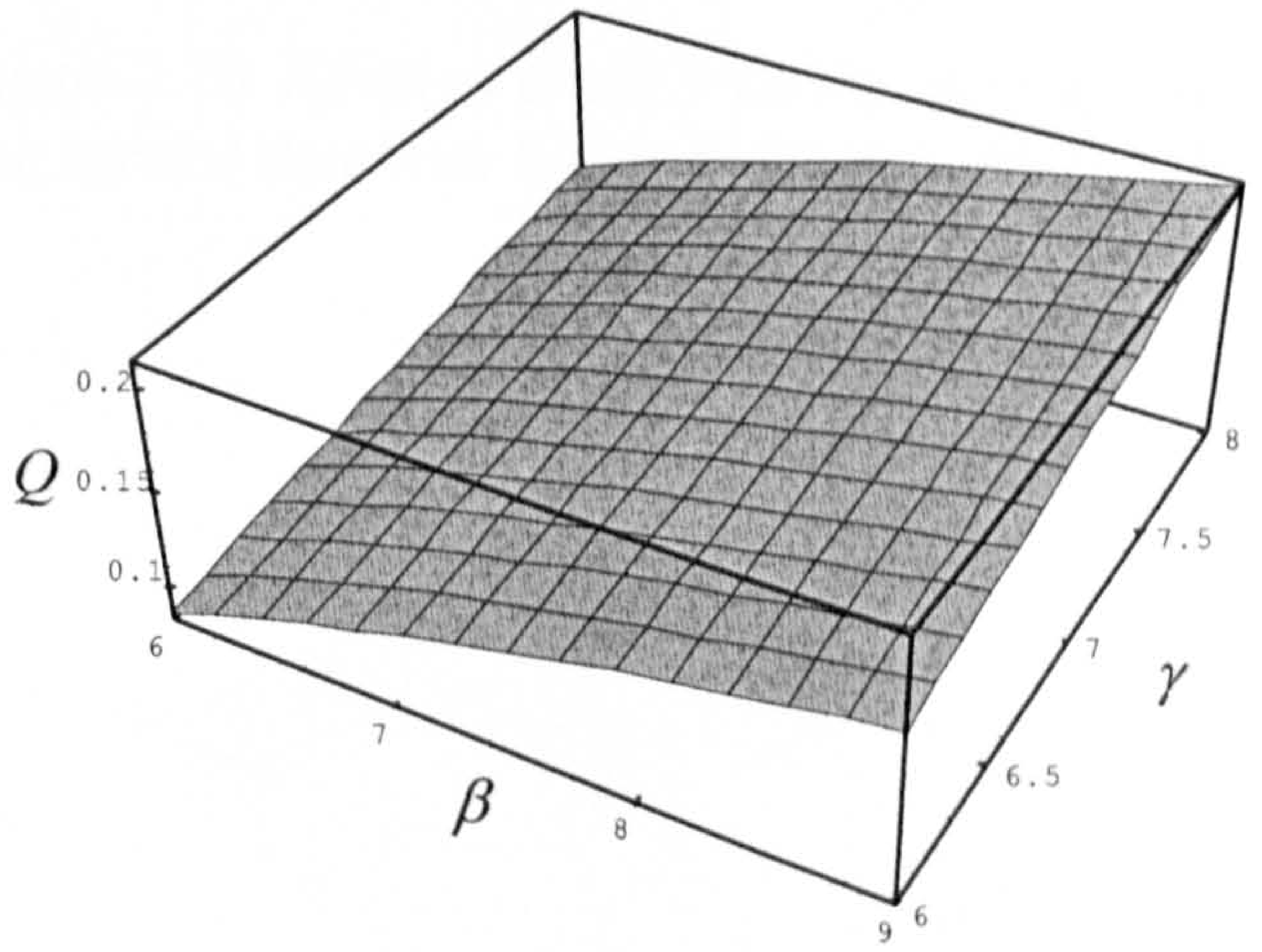

Figure 2.8 *Effect of β and γ on the proportion of time that the server is idle Q (k-Erlang service time with k = 4, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and p = 0.5)*

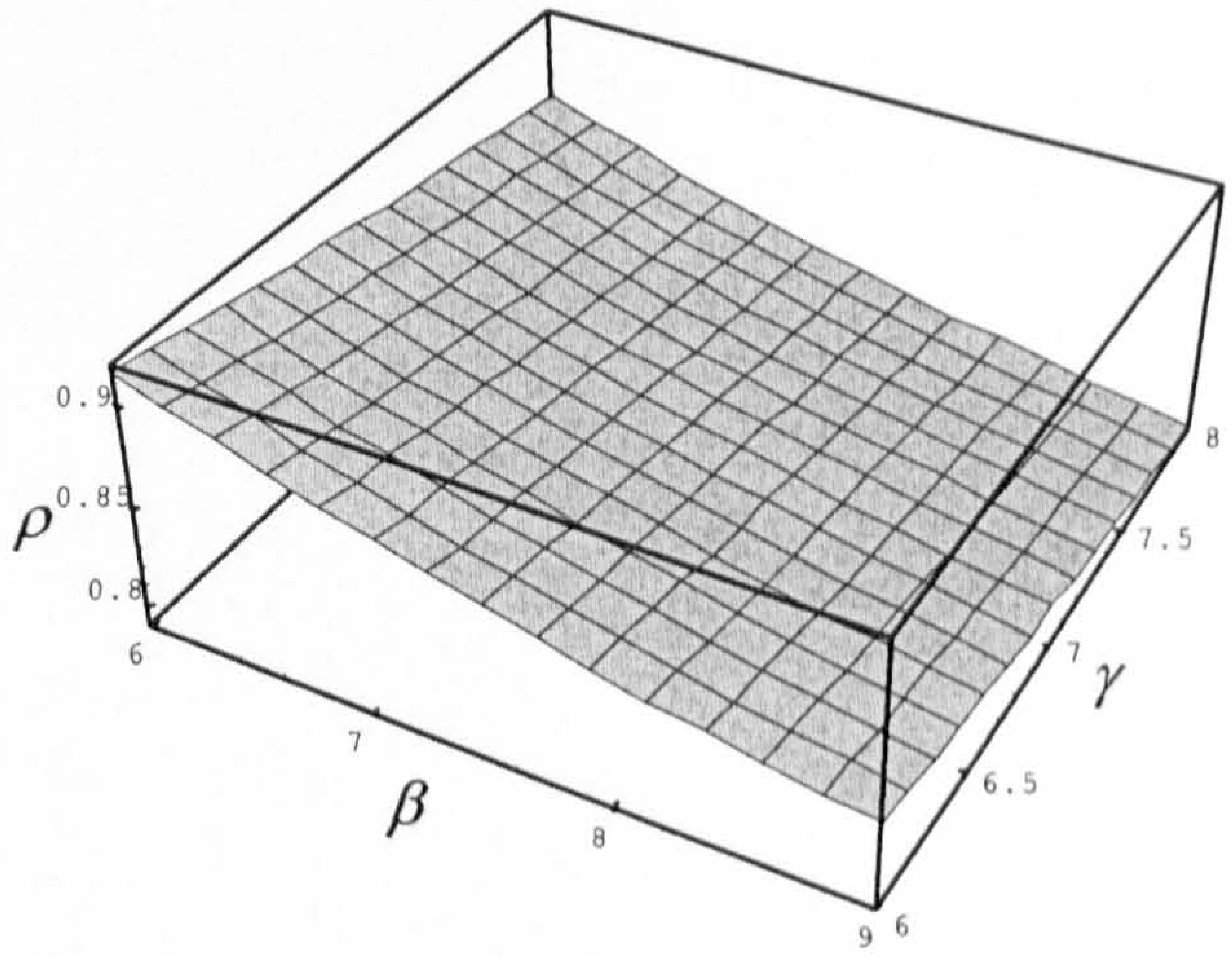

Figure 2.9 *Effect of β and γ on the utilization factor ρ*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and $p = 0.5$)

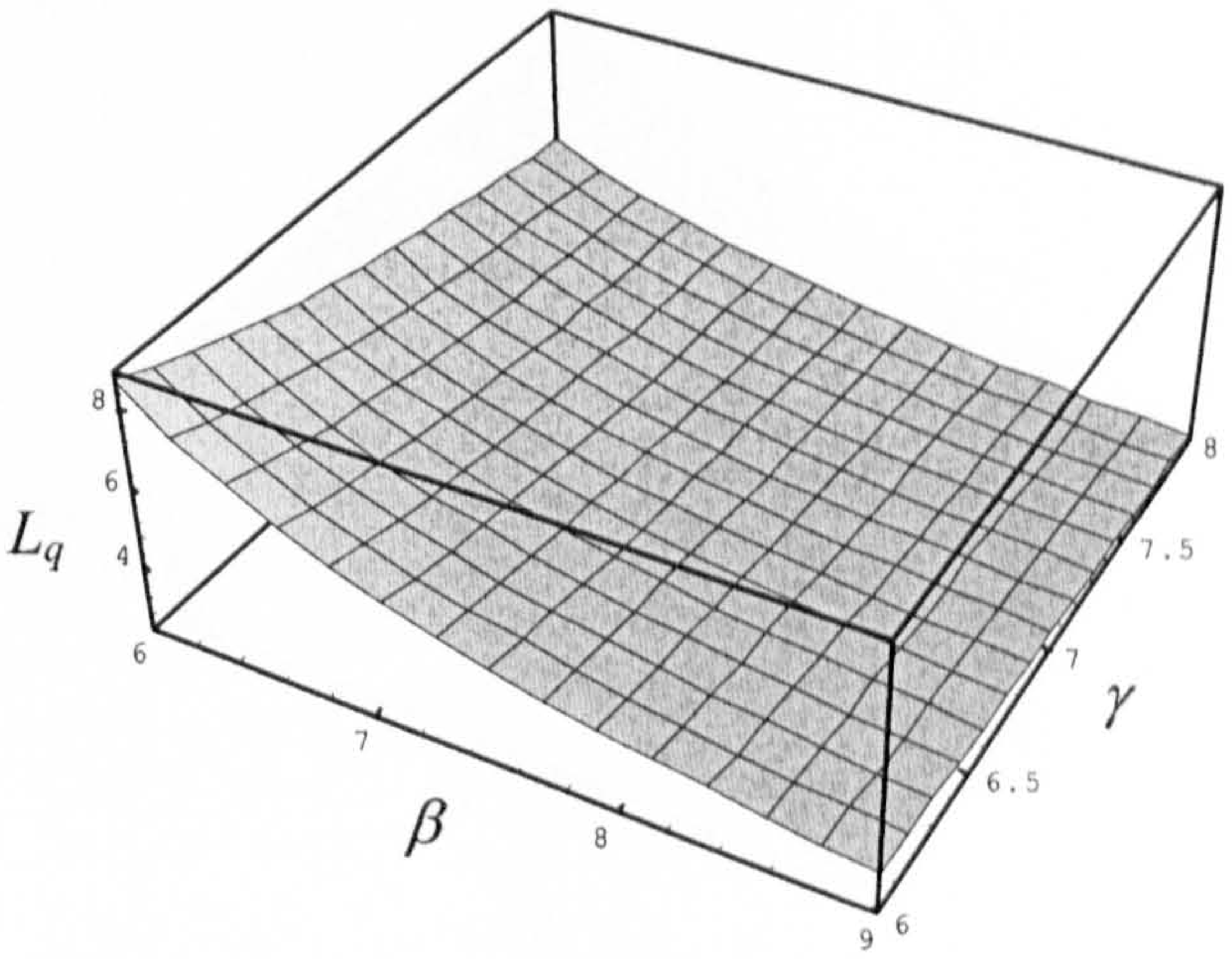

Figure 2.10 *Effect of β and γ on the mean queue size L_q*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and $p = 0.5$)

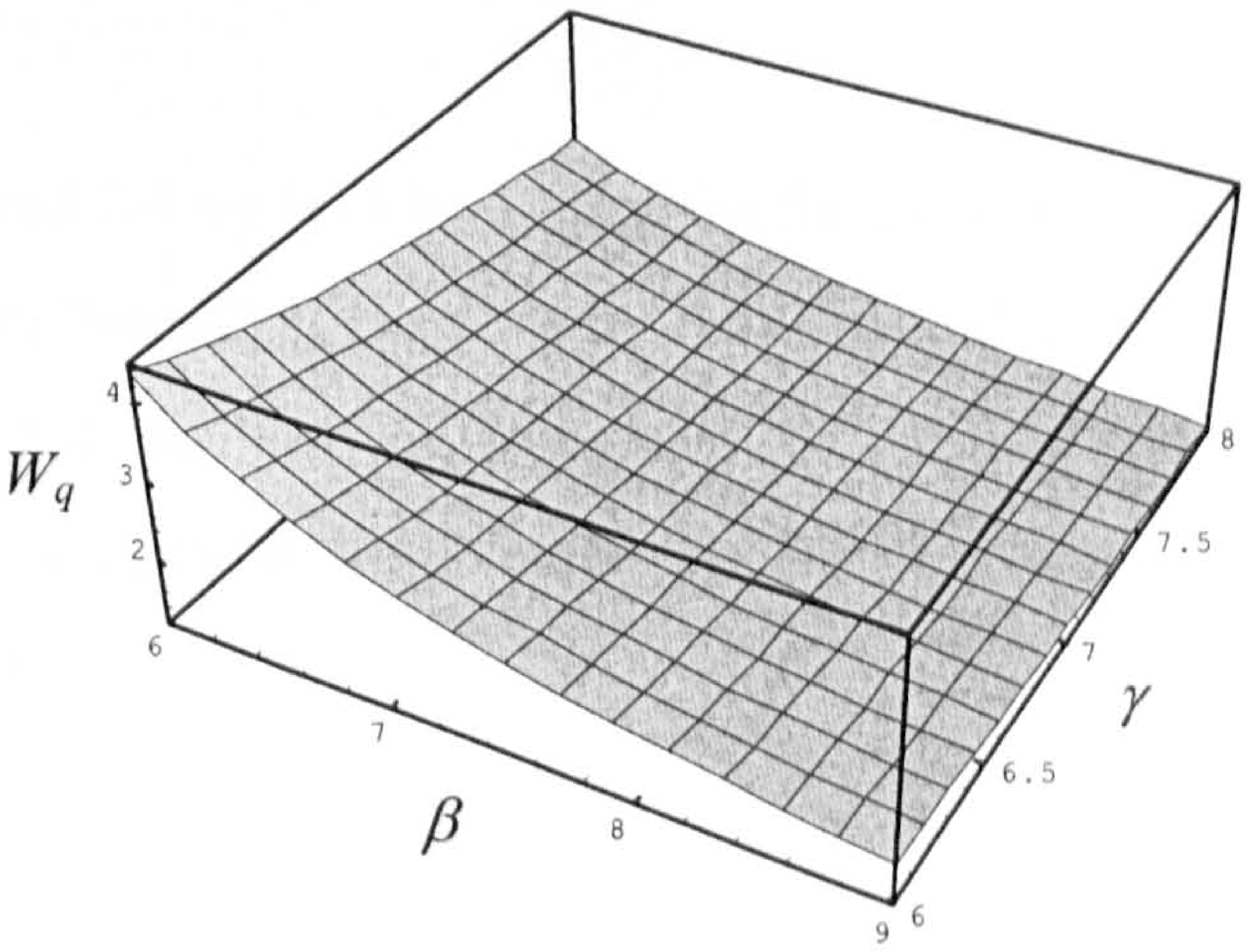

Figure 2.11 *Effect of β and γ on the mean waiting time W_q*
(k-Erlang service time with $k = 4$, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and $p = 0.5$)

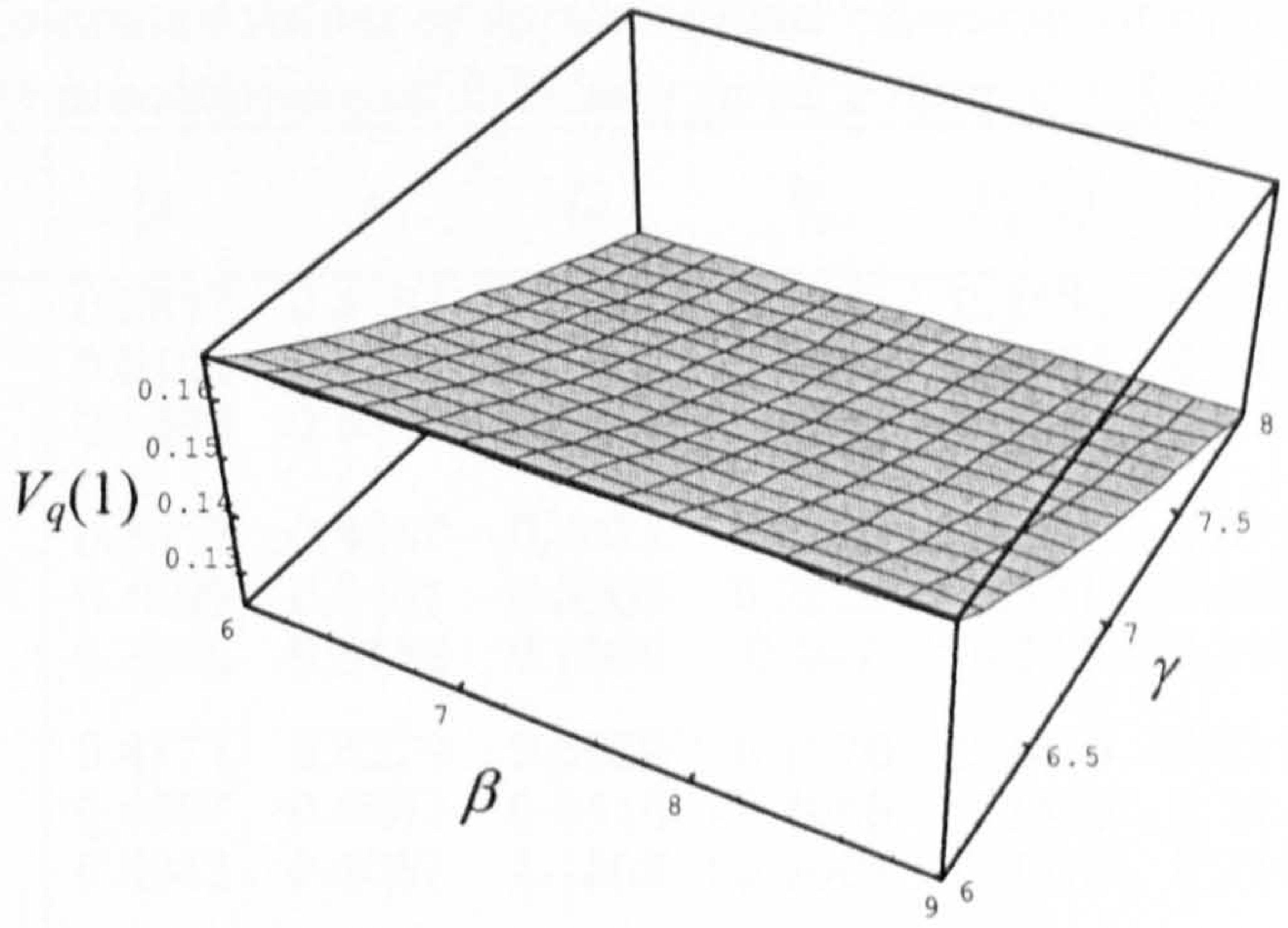

Figure 2.12 *Effect of β and γ on the probability that the server is on vacation (k-Erlang service time with k = 4, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and p = 0.5)*

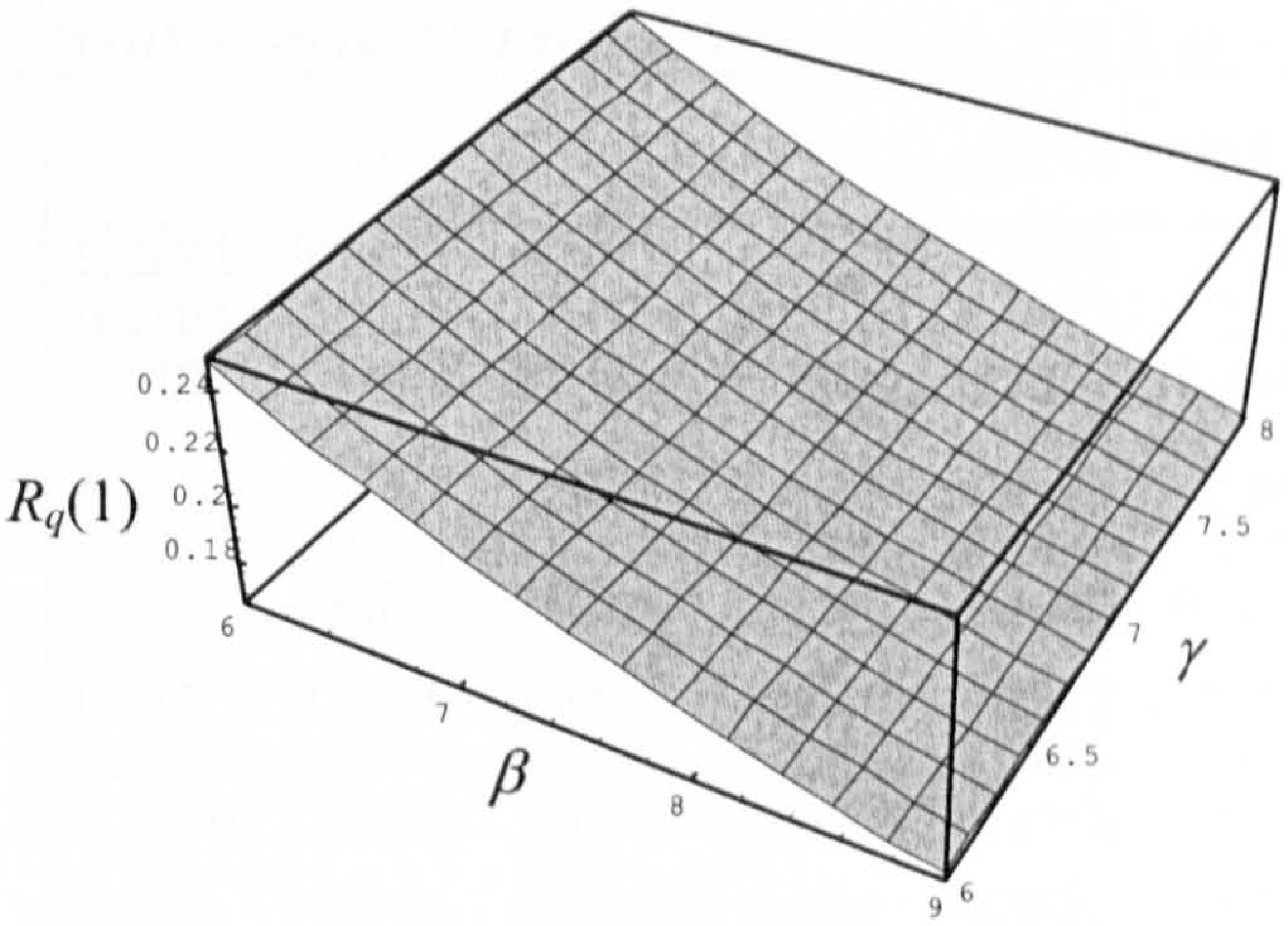

Figure 2.13 *Effect of β and γ on the probability that the system is under repair (k-Erlang service time with k = 4, $\lambda = 2$, $\mu = 20$, $\alpha = 3$, and p = 0.5)*

In Tables 2.3 and 2.4 we let k be 3. First we fix the values of β and γ while α and p are varying in Table 2.3. Next we fix the values of α and p and vary both of β and γ in Table 2.4. Similarly, in Tables 2.5 and 2.6 for k equal to 2, and in Tables 2.7 and 2.8 for k equal to 1 which corresponds to an exponential distribution for the service time.

Table 2.3 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time, $k = 3$, $\beta = 10$, $\gamma = 7$*

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.25	0.5817	0.4183	0.2686	0.1343	0.3154	0.0714	0.0315	0.4183
1	0.5	0.5103	0.4897	0.3904	0.1952	0.3154	0.1429	0.0315	0.4898
1	0.75	0.4388	0.5612	0.5517	0.2759	0.3153	0.2143	0.0315	0.5611
2	0.25	0.5313	0.4687	0.4072	0.2036	0.331	0.0714	0.0662	0.4686
2	0.5	0.4599	0.5401	0.5663	0.2832	0.331	0.1429	0.0662	0.5401
2	0.75	0.3885	0.6115	0.7839	0.392	0.331	0.2143	0.0662	0.6115
3	0.25	0.4771	0.5229	0.5952	0.2976	0.3473	0.0714	0.1042	0.5229
3	0.5	0.4057	0.5943	0.8115	0.4058	0.3473	0.1429	0.1042	0.5944
3	0.75	0.3343	0.6657	1.1202	0.5601	0.3473	0.2143	0.1042	0.6658
4	0.25	0.419	0.581	0.8574	0.4287	0.364	0.0714	0.1456	0.581
4	0.5	0.3475	0.6525	1.167	0.5835	0.364	0.1428	0.1456	0.6524
4	0.75	0.2761	0.7239	1.637	0.8185	0.364	0.2143	0.1456	0.7239

Table 2.4 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time, $k = 3$, $\alpha = 3$, $p = 0.5$*

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	4	0.2291	0.7709	2.5082	1.2541	0.3473	0.25	0.1736	0.7709
6	5	0.2791	0.7209	1.8354	0.9177	0.3473	0.2	0.1736	0.7209
6	6	0.3124	0.6876	1.5241	0.7621	0.3473	0.1667	0.1736	0.6876
7	4	0.2539	0.7461	2.0803	1.0402	0.3473	0.25	0.1488	0.7461
7	5	0.3039	0.6961	1.5328	0.7664	0.3473	0.2	0.1488	0.6961
7	6	0.3372	0.6628	1.2743	0.6372	0.3472	0.1667	0.1488	0.6627
8	4	0.2725	0.7275	1.8162	0.9081	0.3473	0.25	0.1302	0.7275
8	5	0.3225	0.6775	1.3412	0.6706	0.3473	0.2	0.1302	0.6775
8	6	0.3558	0.6442	1.1143	0.5572	0.3472	0.1667	0.1302	0.6441
9	4	0.287	0.713	1.6379	0.819	0.3473	0.25	0.1158	0.7131
9	5	0.337	0.663	1.2097	0.6049	0.3473	0.2	0.1158	0.6631
9	6	0.3703	0.6297	1.0034	0.5017	0.3473	0.1667	0.1158	0.6298

Table 2.5 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time, $k = 2$, $\beta = 10$, $\gamma = 7$*

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.25	0.703	0.297	0.1351	0.0676	0.2051	0.0714	0.0205	0.297
1	0.5	0.6316	0.3684	0.2059	0.103	0.2051	0.1429	0.0205	0.3685
1	0.75	0.5601	0.4399	0.2947	0.1474	0.2051	0.2143	0.0205	0.4399
2	0.25	0.6765	0.3235	0.1877	0.0939	0.2101	0.0714	0.042	0.3235
2	0.5	0.6051	0.3949	0.2684	0.1342	0.2101	0.1429	0.042	0.395
2	0.75	0.5336	0.4664	0.3706	0.1853	0.2101	0.2143	0.042	0.4664
3	0.25	0.649	0.351	0.2468	0.1234	0.215	0.0714	0.0645	0.3509
3	0.5	0.5776	0.4224	0.3393	0.1697	0.2151	0.1429	0.0645	0.4225
3	0.75	0.5061	0.4939	0.4578	0.2289	0.215	0.2143	0.0645	0.4938
4	0.25	0.6205	0.3795	0.315	0.1575	0.22	0.0714	0.088	0.3794
4	0.5	0.5491	0.4509	0.4218	0.2109	0.2201	0.1429	0.088	0.451
4	0.75	0.4776	0.5224	0.5604	0.2802	0.22	0.2143	0.088	0.5223

Table 2.6 *Computed values of various queue characteristics for a vacation queue with breakdown and k-Erlang service time,* $k = 2$, $\alpha = 3$, $p = 0.5$

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	4	0.4274	0.5726	0.8805	0.4403	0.215	0.25	0.1075	0.5725
6	5	0.4774	0.5226	0.6715	0.3358	0.215	0.2	0.1075	0.5225
6	6	0.5108	0.4892	0.5659	0.283	0.2151	0.1667	0.1075	0.4893
7	4	0.4428	0.5572	0.788	0.394	0.2151	0.25	0.0922	0.5573
7	5	0.4928	0.5072	0.5949	0.2975	0.2151	0.2	0.0922	0.5073
7	6	0.5261	0.4739	0.4971	0.2486	0.215	0.1667	0.0922	0.4739
8	4	0.4543	0.5457	0.7248	0.3624	0.215	0.25	0.0806	0.5456
8	5	0.5043	0.4957	0.5424	0.2712	0.215	0.2	0.0806	0.4956
8	6	0.5376	0.4624	0.4499	0.225	0.215	0.1667	0.0806	0.4623
9	4	0.4633	0.5367	0.6791	0.3396	0.2151	0.25	0.0717	0.5368
9	5	0.5133	0.4867	0.5043	0.2522	0.2151	0.2	0.0717	0.4868
9	6	0.5466	0.4534	0.4157	0.2079	0.2151	0.1667	0.0717	0.4535

Table 2.7 *Computed values of various queue characteristics for a vacation queue with breakdown and exponential service time,* $\beta = 10$, $\gamma = 7$

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.25	0.8186	0.1814	0.0613	0.0307	0.1	0.0714	0.01	0.1814
1	0.5	0.7472	0.2528	0.1041	0.0521	0.1	0.1429	0.01	0.2529
1	0.75	0.6758	0.3242	0.1558	0.0779	0.1	0.2143	0.01	0.3243
2	0.25	0.8086	0.1914	0.0787	0.0394	0.1	0.0714	0.02	0.1914
2	0.5	0.7372	0.2628	0.1237	0.0619	0.1	0.1429	0.02	0.2629
2	0.75	0.6657	0.3343	0.1783	0.0892	0.1	0.2143	0.02	0.3343
3	0.25	0.7986	0.2014	0.0958	0.0479	0.1	0.0714	0.03	0.2014
3	0.5	0.7272	0.2728	0.1431	0.0716	0.1	0.1429	0.03	0.2729
3	0.75	0.6558	0.3442	0.2007	0.1004	0.1	0.2143	0.03	0.3443
4	0.25	0.7885	0.2115	0.1137	0.0569	0.1	0.0714	0.04	0.2114
4	0.5	0.7171	0.2829	0.1634	0.0817	0.1	0.1429	0.04	0.2829
4	0.75	0.6457	0.3543	0.2242	0.1121	0.1	0.2143	0.04	0.3543

Table 2.8 *Computed values of various queue characteristics for a vacation queue with breakdown and exponential service time,* $\alpha = 3$, $p = 0.5$

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	4	0.6	0.4	0.386	0.193	0.1	0.25	0.05	0.4
6	5	0.65	0.35	0.2794	0.1397	0.1	0.2	0.05	0.35
6	6	0.6834	0.3166	0.2251	0.1126	0.1	0.1667	0.05	0.3167
7	4	0.6072	0.3928	0.3613	0.1807	0.1	0.25	0.0428	0.3928
7	5	0.6572	0.3428	0.2577	0.1289	0.1	0.2	0.0428	0.3428
7	6	0.6905	0.3095	0.205	0.1025	0.1	0.1667	0.0428	0.3095
8	4	0.6125	0.3875	0.3438	0.1719	0.1	0.25	0.0375	0.3875
8	5	0.6625	0.3375	0.2424	0.1212	0.1	0.2	0.0375	0.3375
8	6	0.6959	0.3041	0.1909	0.0955	0.1	0.1667	0.0375	0.3042
9	4	0.6167	0.3833	0.3309	0.1655	0.1	0.25	0.0333	0.3833
9	5	0.6667	0.3333	0.231	0.1155	0.1	0.2	0.0333	0.3333
9	6	0.7	0.3	0.1804	0.0902	0.1	0.1667	0.0333	0.3

The results obtained for the special case discussed in section 2.7.4 are used to calculate various queue characteristics in Tables 2.9 – 2.12. In this case we consider a queueing system with Bernoulli vacations and random breakdowns where the service times follow a deterministic distribution while both the vacation time and repair time are exponentially distributed. For Table 2.9 we select the arbitrary value b to be equal to 3, the values of β and γ are fixed to be 10 and 7, respectively, while α varies from 1 to 4 and p takes the values 0.25, 0.5, and 0.75. All the values of queue parameters were chosen such that the steady state condition is satisfied.

Table 2.9 *Computed values of various queue characteristics for a vacation queue with breakdown and deterministic service time, b = 3, β = 10, γ = 10*

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	0.25	0.4167	0.5833	0.6404	0.3202	0.3334	0.05	0.2	0.5834
6	0.5	0.3667	0.6333	0.8005	0.4003	0.3334	0.1	0.2	0.6334
6	0.75	0.3167	0.6833	1.0111	0.5056	0.3334	0.15	0.2	0.6834
7	0.25	0.3025	0.6975	1.5113	0.7557	0.381	0.05	0.2667	0.6977
7	0.5	0.2525	0.7475	1.9257	0.9629	0.381	0.1	0.2667	0.7477
7	0.75	0.2025	0.7975	2.5449	1.2725	0.381	0.15	0.2667	0.7977
8	0.25	0.2	0.8	3.1534	1.5767	0.4167	0.05	0.3333	0.8
8	0.5	0.15	0.85	4.41	2.205	0.4167	0.1	0.3333	0.85
8	0.75	0.1	0.9	6.9234	3.4617	0.4167	0.15	0.3333	0.9
9	0.25	0.1054	0.8946	7.5305	3.7653	0.4444	0.05	0.4	0.8944
9	0.5	0.0554	0.9446	14.9009	7.4505	0.4443	0.0999	0.3999	0.9441
9	0.75	0.0054	0.9946	157.3123	78.6562	0.4421	0.1492	0.3979	0.9892

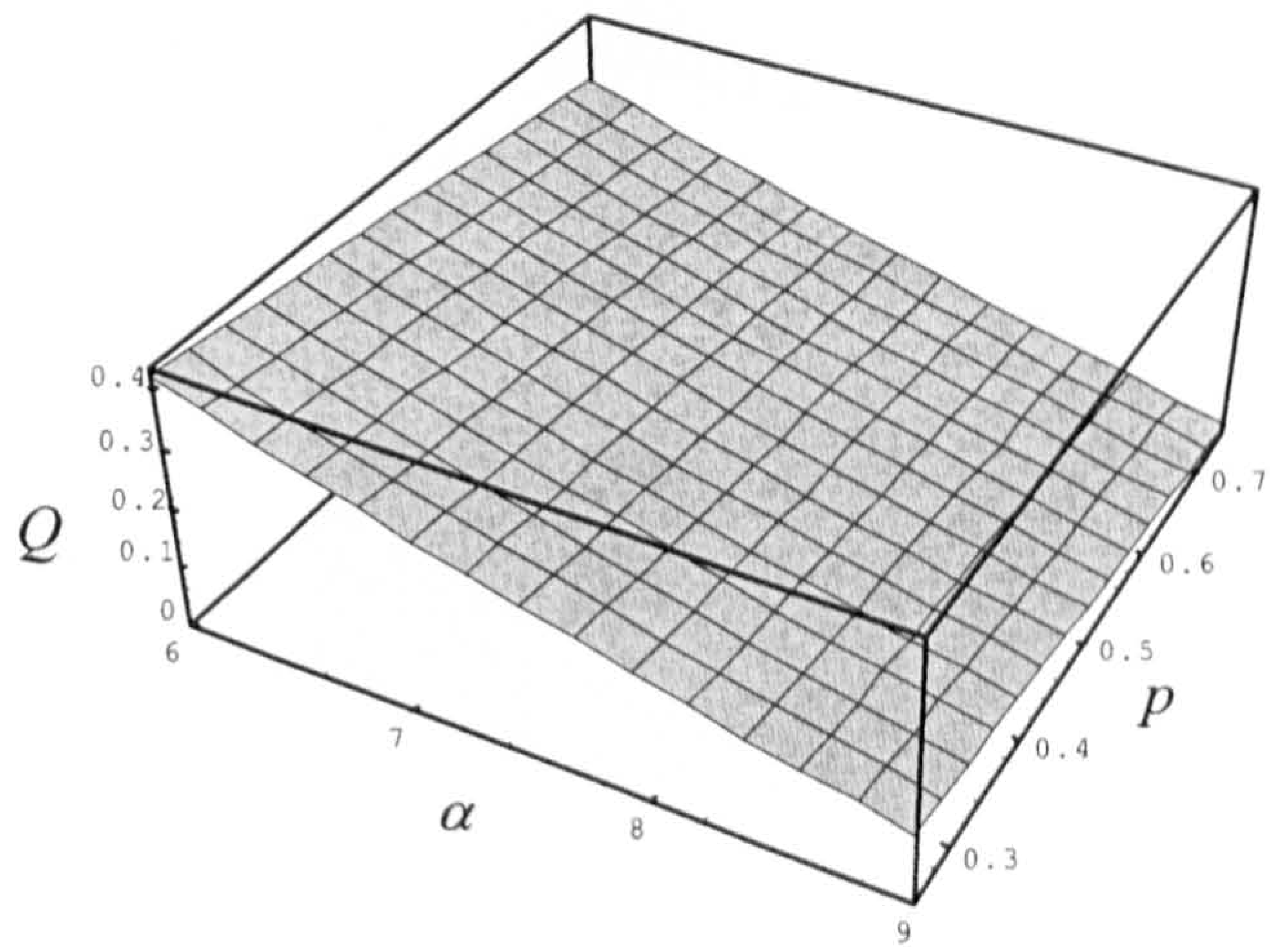

Figure 2.14 *Effect of α and p on the proportion of time that the server is idle Q (Deterministic service time with b = 3, λ = 2, β = 10, and γ = 10)*

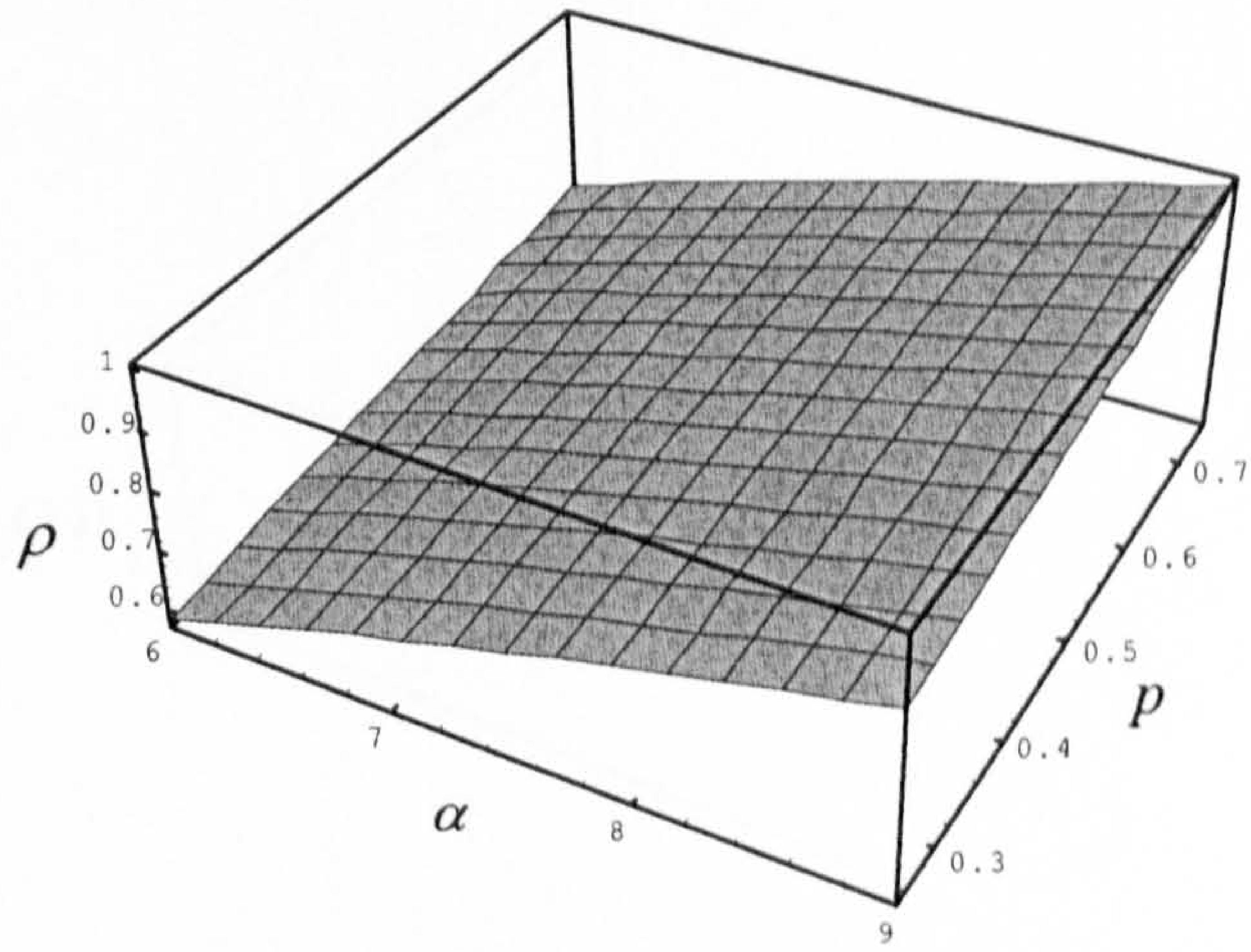

Figure 2.15 *Effect of α and p on the utilization factor ρ*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)

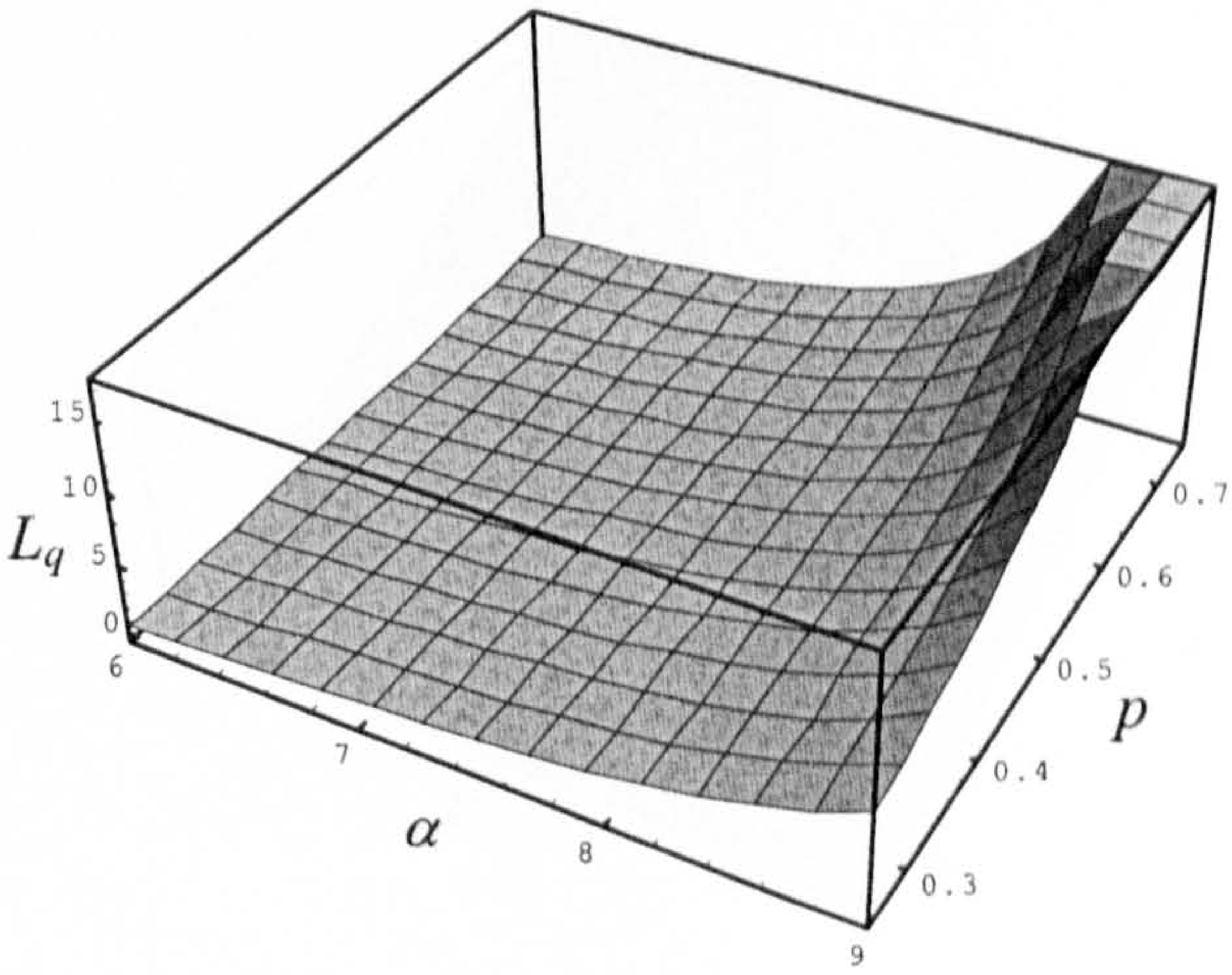

Figure 2.16 *Effect of α and p on the mean queue size L_q*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)

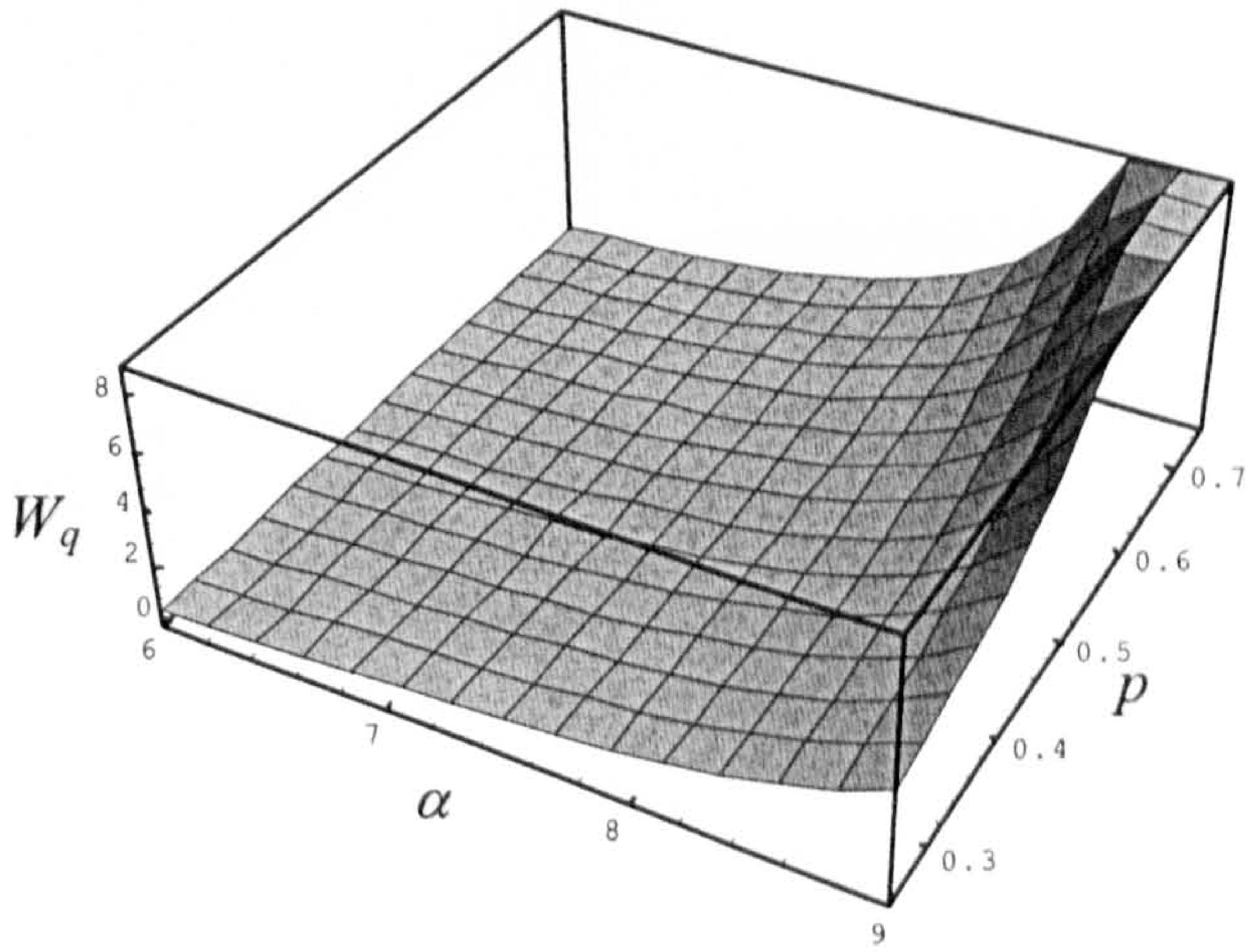

Figure 2.17 *Effect of α and p on the mean waiting time W_q*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)

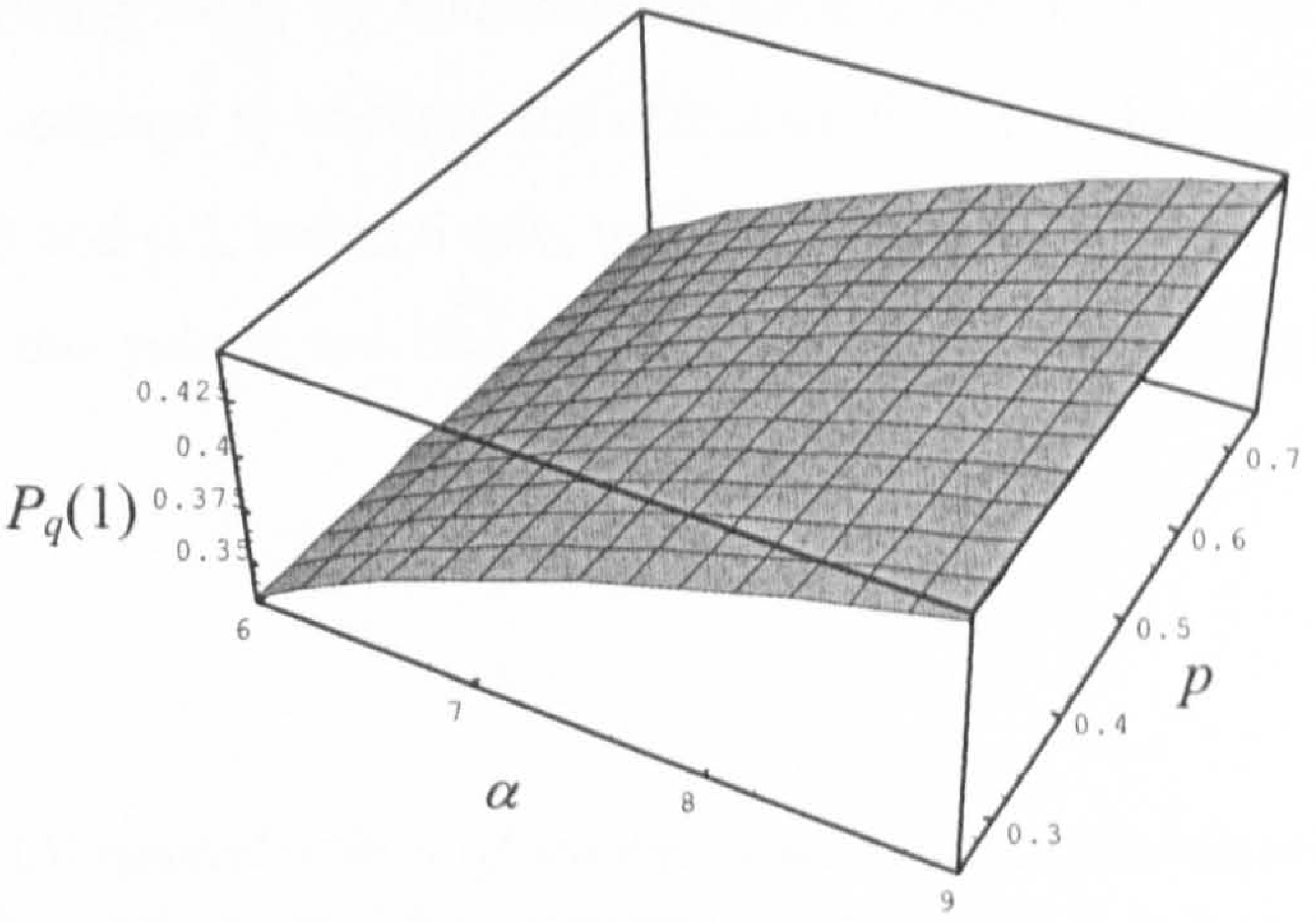

Figure 2.18 *Effect of α and p on the probability that the server is working (Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)*

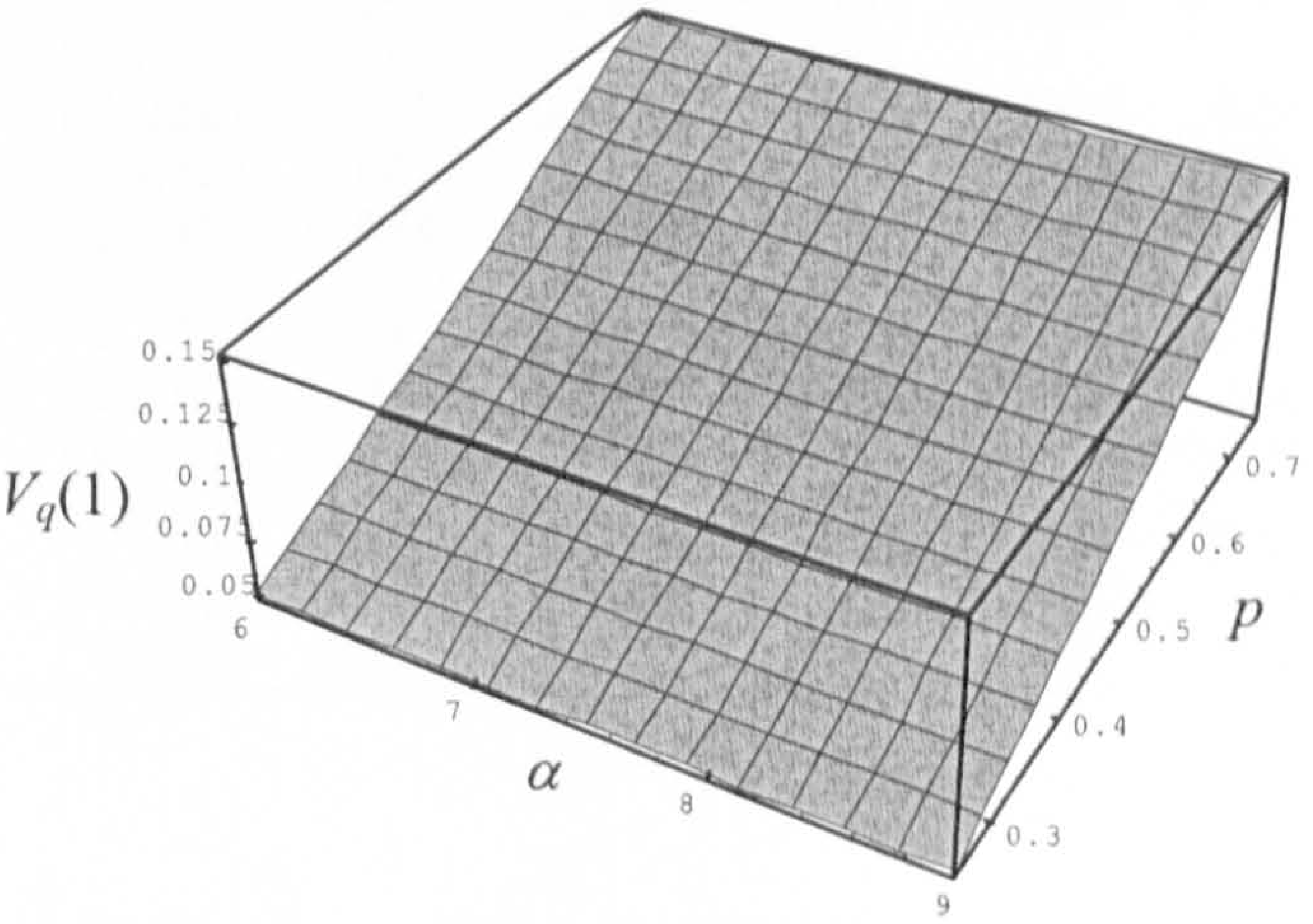

Figure 2.19 *Effect of α and p on the probability that the server is on vacation (Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)*

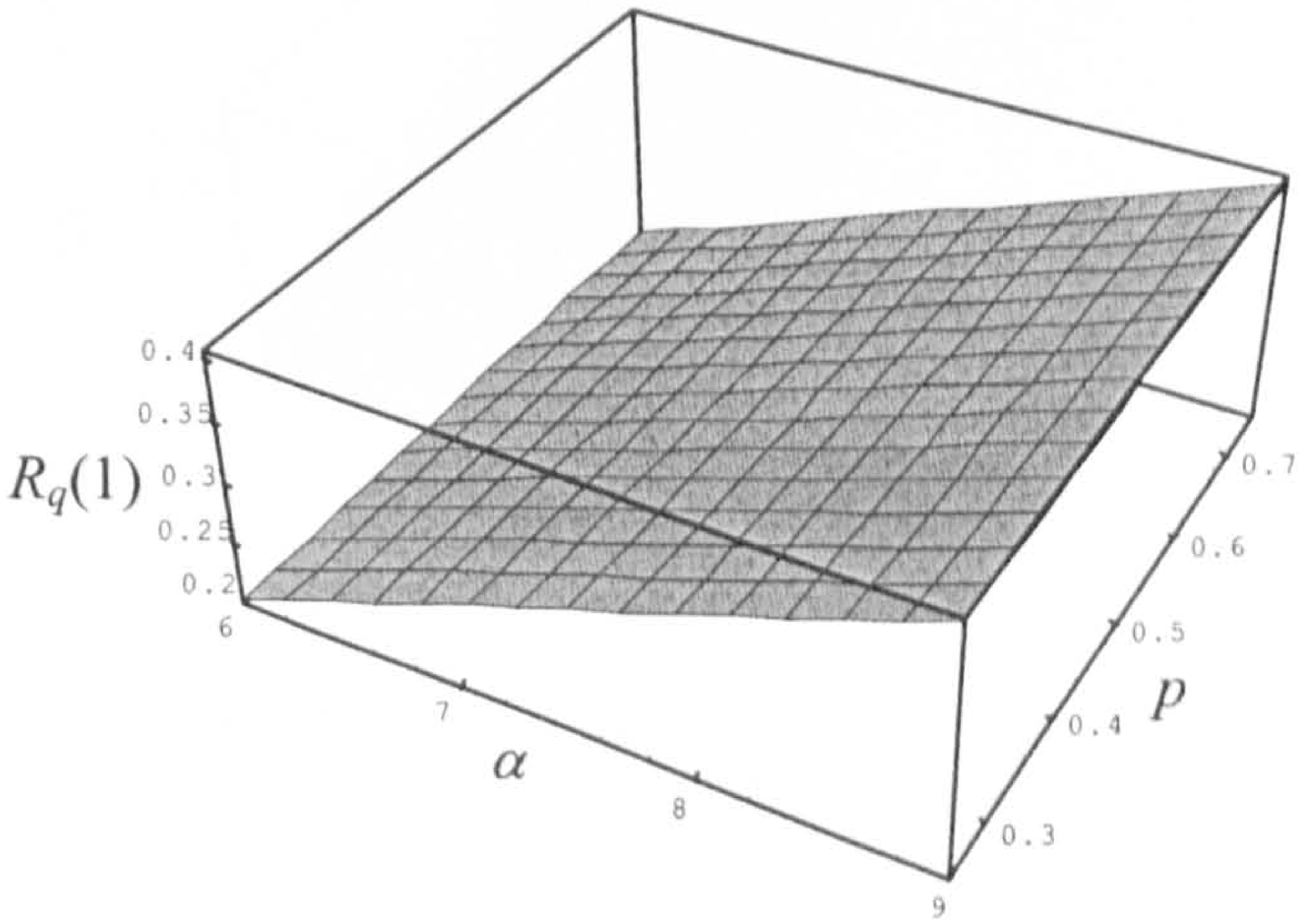

Figure 2.20 *Effect of α and p on the probability that the system is under repair (Deterministic service time with $b = 3$, $\lambda = 2$, $\beta = 10$, and $\gamma = 10$)*

For the following table, we consider the same queueing system used in Table 2.9, but here we attempt to observe the effect of β and γ by fixing the values of α and p to be 3 and 0.5, respectively, while β varies from 10 to 13 and γ varies from 4 to 6. All the values are chosen in such away that satisfies the steady state condition.

Table 2.10 *Computed values of various queue characteristics for vacation queue with breakdown and deterministic service time, $b = 3$, $\alpha = 5$, $p = 0.5$*

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
10	4	0.35	0.65	0.6238	0.3119	0.2667	0.25	0.1333	0.65
10	5	0.4	0.6	0.4	0.2	0.2667	0.2	0.1333	0.6
10	6	0.4333	0.5667	0.2923	0.1462	0.2666	0.1667	0.1333	0.5666
11	4	0.3621	0.6379	0.5678	0.2839	0.2667	0.25	0.1212	0.6379
11	5	0.4121	0.5879	0.3574	0.1787	0.2667	0.2	0.1212	0.5879
11	6	0.4455	0.5545	0.2558	0.1279	0.2667	0.1667	0.1212	0.5546
12	4	0.3722	0.6278	0.5248	0.2624	0.2667	0.25	0.1111	0.6278
12	5	0.4222	0.5778	0.3245	0.1623	0.2667	0.2	0.1111	0.5778
12	6	0.4556	0.5444	0.2277	0.1139	0.2667	0.1667	0.1111	0.5445
13	4	0.3808	0.6192	0.491	0.2455	0.2667	0.25	0.1026	0.6193
13	5	0.4308	0.5692	0.2986	0.1493	0.2667	0.2	0.1026	0.5693
13	6	0.4641	0.5359	0.2053	0.1027	0.2667	0.1667	0.1026	0.536

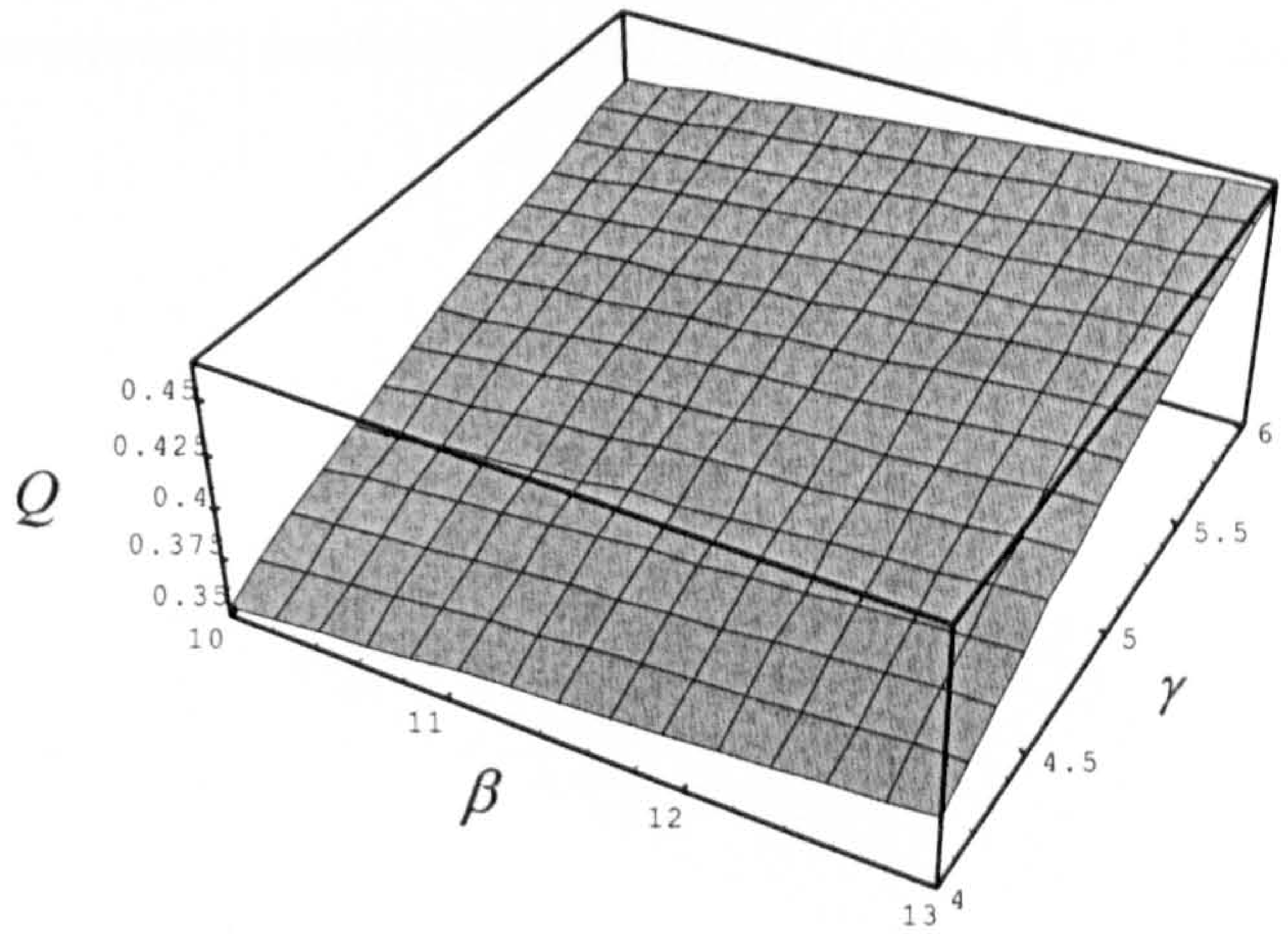

Figure 2.21 *Effect of β and γ on the proportion of time that the server is idle Q (Deterministic service time with $b = 3$, $\lambda = 2$, $\alpha = 5$, and $p = 0.5$)*

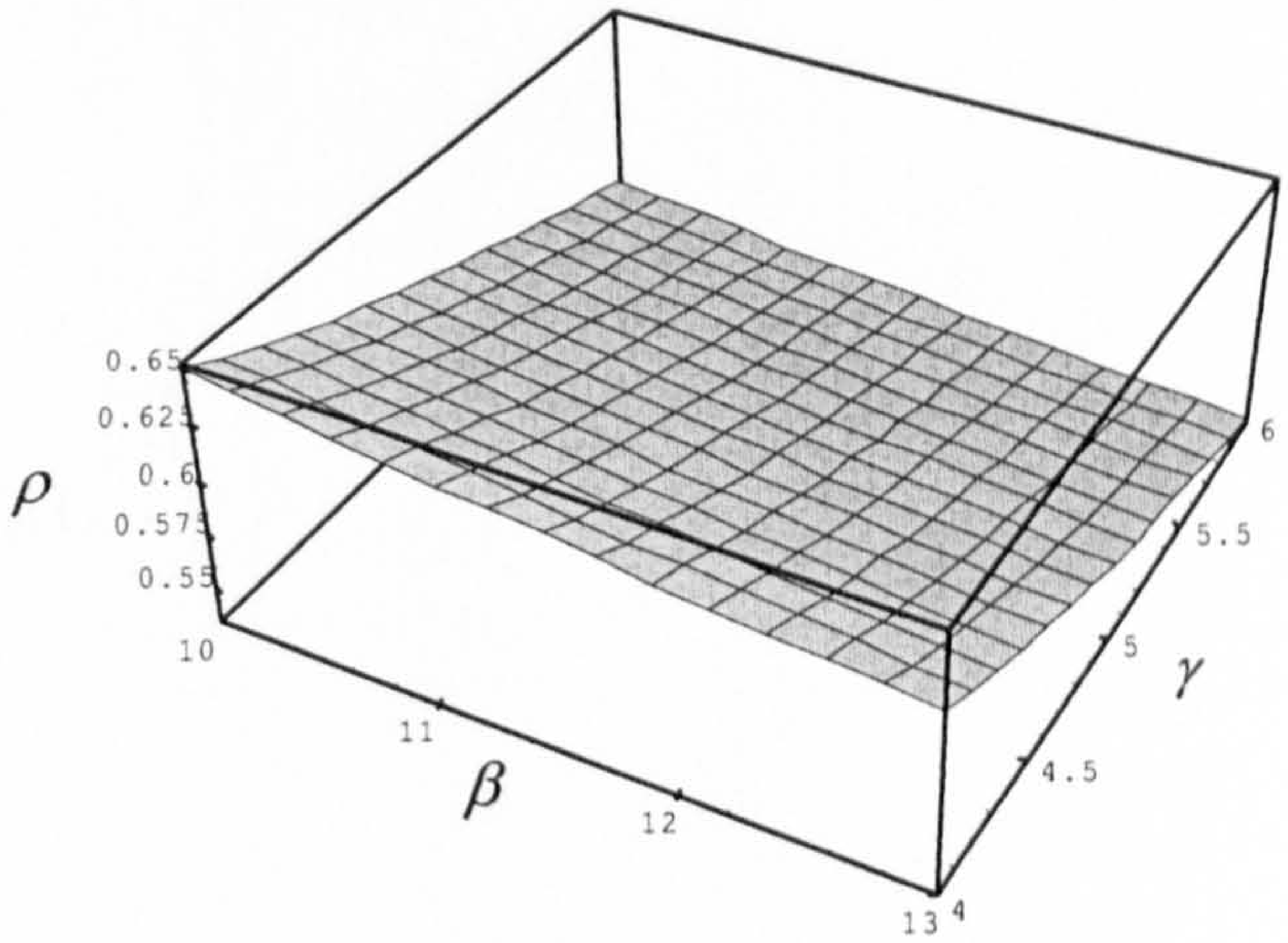

Figure 2.22 *Effect of β and γ on the utilization factor ρ*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\alpha = 5$, and $p = 0.5$)

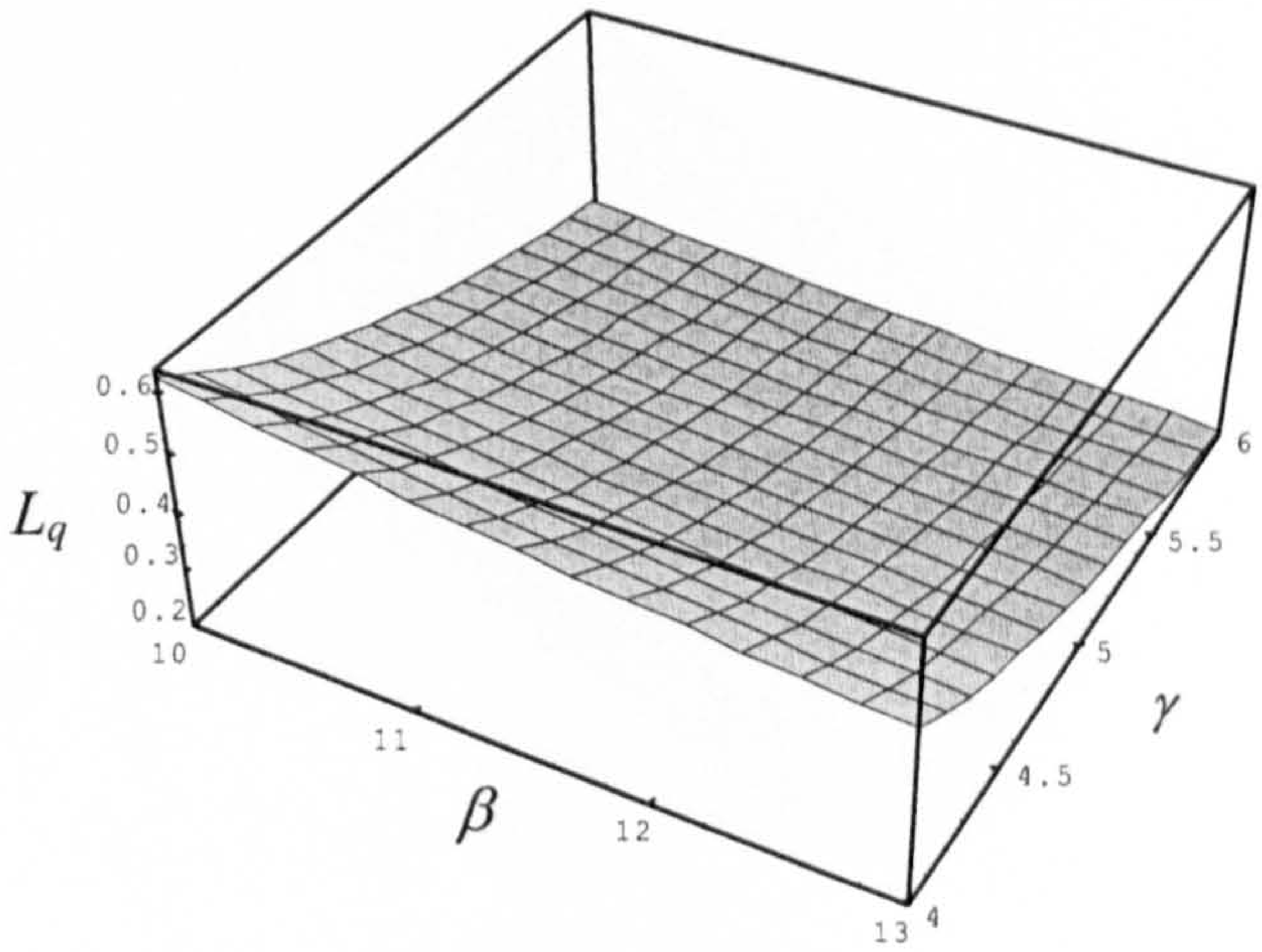

Figure 2.23 *Effect of β and γ on the mean queue size L_q*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\alpha = 5$, and $p = 0.5$)

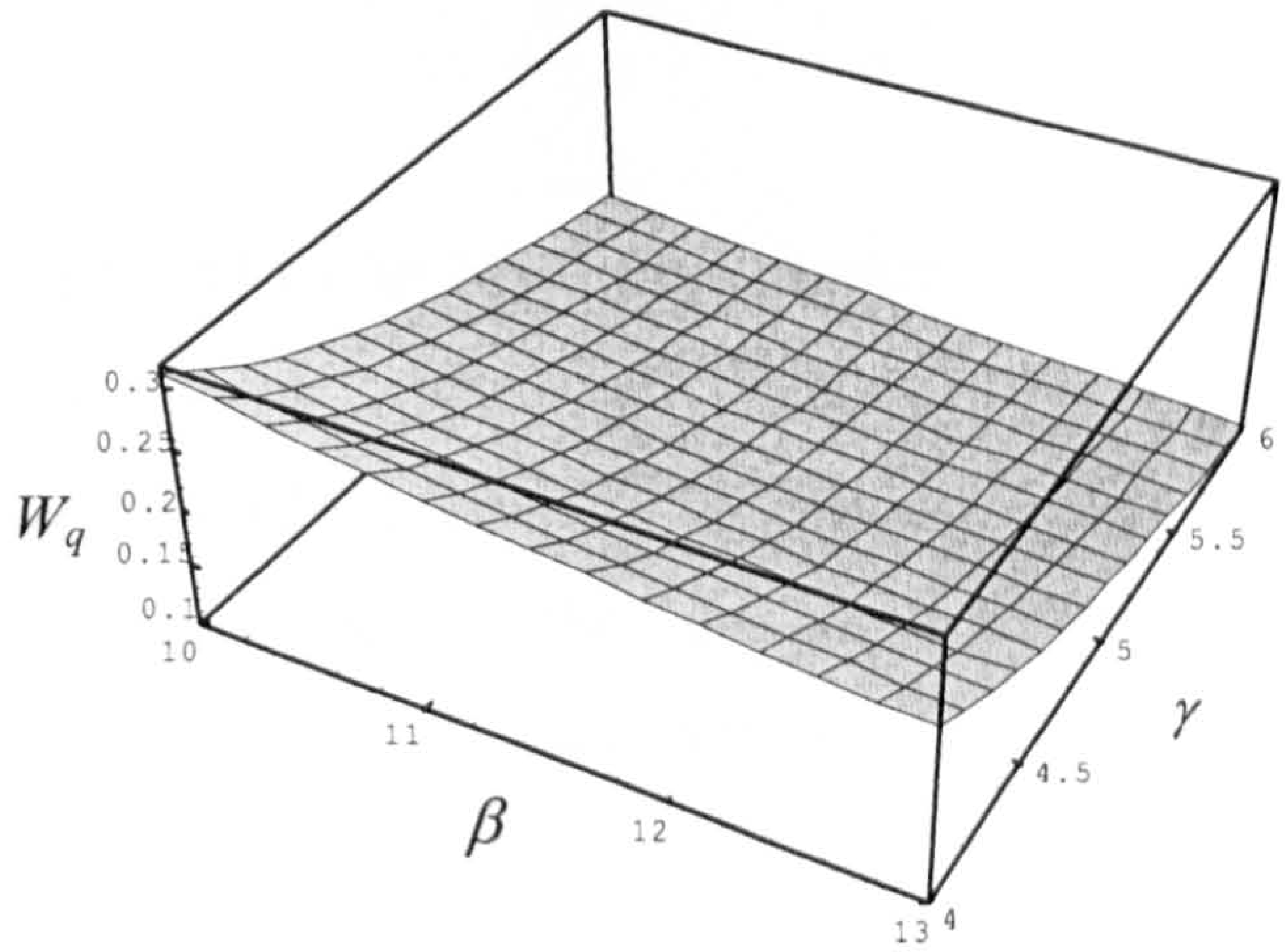

Figure 2.24 *Effect of β and γ on the mean waiting time W_q*
(Deterministic service time with $b = 3$, $\lambda = 2$, $\alpha = 5$, and $p = 0.5$)

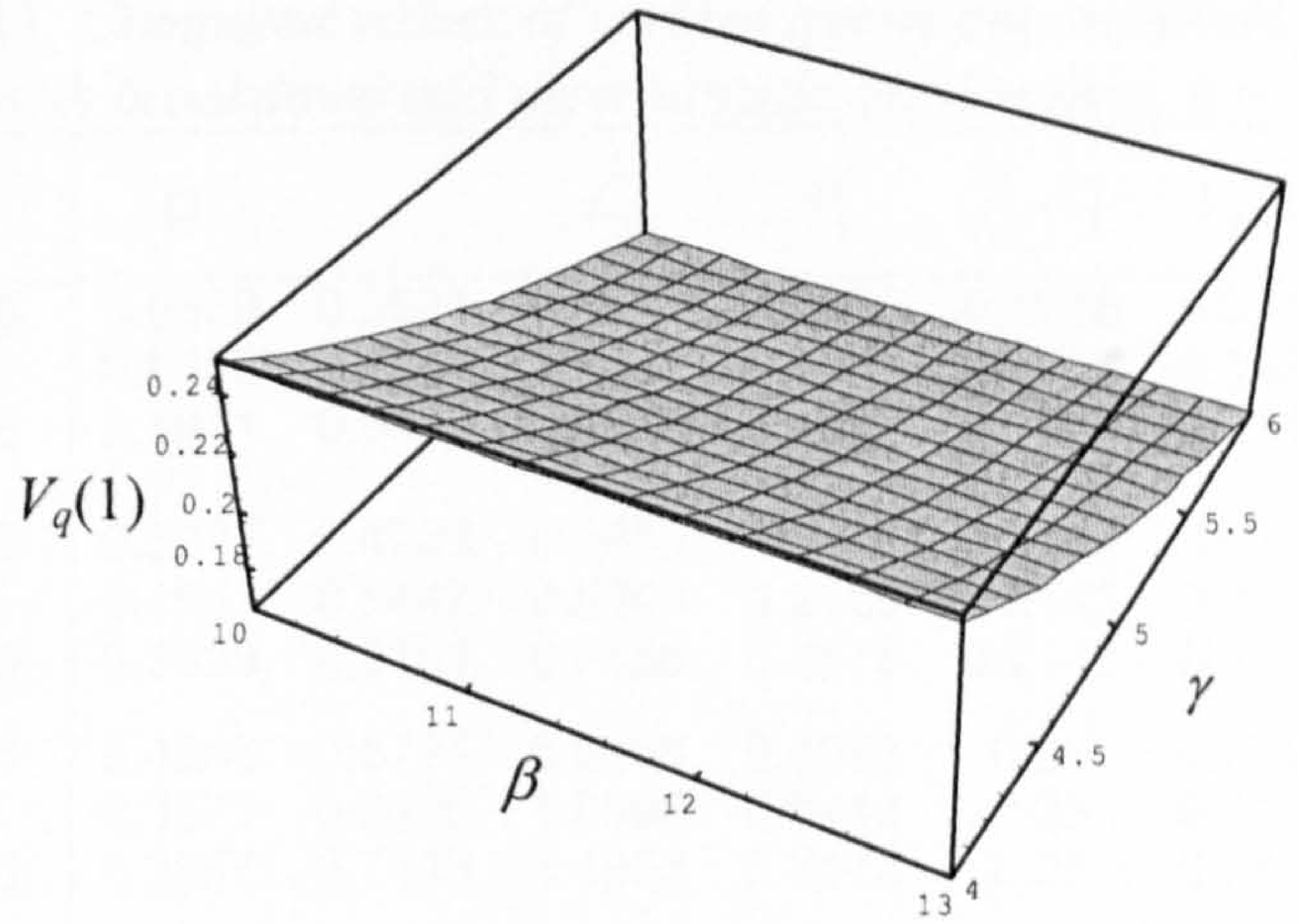

Figure 2.25 *Effect of* β *and* γ *on the probability that the server is on vacation (Deterministic service time with* $b = 3$, $\lambda = 2$, $\alpha = 5$, *and* $p = 0.5$*)*

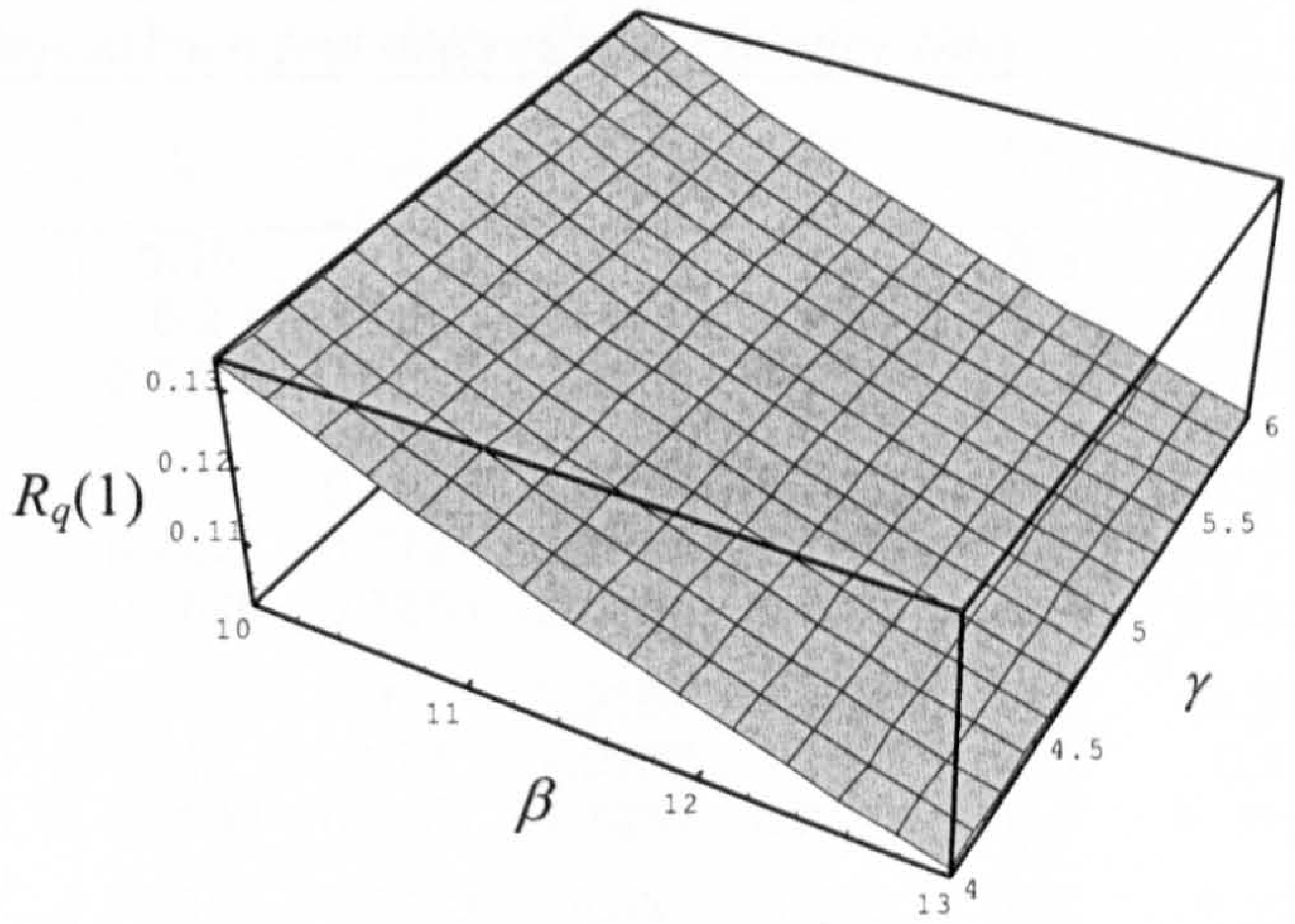

Figure 2.26 *Effect of* β *and* γ *on the probability that the system is under repair (Deterministic service time with* $b = 3$, $\lambda = 2$, $\alpha = 5$, *and* $p = 0.5$*)*

In Tables 2.11 and 2.12 we let b be 3 for the deterministic service time where first we fix the values of β and γ to be 8 and 7, respectively, while α varies from 6 to 9 and p takes the values 0.25, 0.5 and 0.75. Then, we fix the values of α and p to be 10 and 0.5, respectively, and vary the value of β from 10 to 13 and γ from 4 to 6. The selected values for different parameters of the queueing system satisfy the stability condition.

Table 2.11 *Computed values of various queue characteristics for a vacation queue with breakdown and deterministic service time, $b = 4$, $\beta = 8$, $\gamma = 7$*

α	p	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	0.25	0.6369	0.3631	0.0671	0.0336	0.1666	0.0714	0.125	0.363
6	0.5	0.5655	0.4345	0.1327	0.0664	0.1666	0.1429	0.125	0.4345
6	0.75	0.4941	0.5059	0.2173	0.1087	0.1666	0.2143	0.125	0.5059
7	0.25	0.5267	0.4733	0.3859	0.193	0.2143	0.0714	0.1875	0.4732
7	0.5	0.4553	0.5447	0.5249	0.2625	0.2143	0.1429	0.1875	0.5447
7	0.75	0.3839	0.6161	0.7156	0.3578	0.2143	0.2143	0.1875	0.6161
8	0.25	0.4286	0.5714	0.8185	0.4093	0.25	0.0714	0.25	0.5714
8	0.5	0.3571	0.6429	1.0892	0.5446	0.25	0.1428	0.25	0.6428
8	0.75	0.2857	0.7143	1.4955	0.7478	0.25	0.2143	0.25	0.7143
9	0.25	0.3382	0.6618	1.4487	0.7244	0.2778	0.0714	0.3126	0.6618
9	0.5	0.2668	0.7332	1.9878	0.9939	0.2779	0.1429	0.3126	0.7334
9	0.75	0.1953	0.8047	2.9199	1.46	0.2778	0.2143	0.3125	0.8046

Table 2.12 *Computed values of various queue characteristics for vacation queue with breakdown and deterministic service time, $b = 4$, $\alpha = 10$, $p = 0.5$*

β	γ	Q	ρ	L_q	W_q	$P_q(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
10	4	0.15	0.85	4.4	2.2	0.3	0.25	0.3	0.85
10	5	0.2	0.8	3	1.5	0.3	0.2	0.3	0.8
10	6	0.2333	0.7667	2.4235	1.2118	0.3	0.1666	0.3	0.7666
11	4	0.1773	0.8227	3.4854	1.7427	0.3	0.25	0.2728	0.8228
11	5	0.2273	0.7727	2.4545	1.2273	0.3	0.2	0.2728	0.7728
11	6	0.2606	0.7394	2.0081	1.0041	0.3	0.1667	0.2727	0.7394
12	4	0.2	0.8	2.9167	1.4584	0.3	0.25	0.25	0.8
12	5	0.25	0.75	2.0933	1.0467	0.3	0.2	0.25	0.75
12	6	0.2833	0.7167	1.7253	0.8627	0.3	0.1666	0.25	0.7166
13	4	0.2192	0.7808	2.53	1.265	0.3	0.25	0.2307	0.7807
13	5	0.2692	0.7308	1.8374	0.9187	0.3	0.2	0.2307	0.7307
13	6	0.3026	0.6974	1.5215	0.7608	0.3	0.1667	0.2308	0.6975

The above tables 2.1 – 2.12 and graphs 2.1 – 2.26 clearly show if we increase either the value of p or α, the server idle time decreases, while the utilization factor, the mean queue size and the mean waiting time of customers, all increase. On the other hand, increasing β or γ increases the server idle time and decreases the utilization factor, the mean number of customers in the queue and the mean waiting time. Also the tables and graphs show that either increasing p or decreasing γ increases the probability that the server is on vacation. Similarly, increasing α or decreasing β increases the probability that the system is under repair. The trends shown by the tables and graphs are as expected.

* * *

Chapter 3

An $M^{[X]}/G/1$ Queue with Bernoulli Schedule General Vacations, Random Breakdowns and General Repair Times

3.1 Introduction

In this chapter we generalise the results obtained in chapter 2 by assuming a general (arbitrary) distribution for vacation times and a general (arbitrary) distribution for repair times instead of the exponential special case assumed in chapter 2. These general assumptions make the results applicable to a wider range of queueing systems in which the service times, vacation times and repair times can be exponential, hyperexponential, deterministic, k-Erlang, etc.

Numerous research results have been published regarding queueing systems with generalised vacations. Sharda and Indra (1996) assumed general distributions for the service time and vacation time of a two-state queueing model. Takagi (1990) obtained time dependent results of $M/G/1$ with general vacation time based on single and multiple vacation policies, while Hur and Ahn (2005) obtained steady state results for batch arrival queues with general vacations and server setup. Madan (1991) assumed general vacations for a queueing system with bulk input and bulk output. In studies of a finite capacity queue with exhaustive vacation, close-down, setup times and Markovian arrival process, general distribution for vacation times were assumed by the authors (Niu & Takahashi, 1999; Niu, Shu & Takahashi, 2003). Chae, Lee and Ahn (2001) proposed the arrival time approach of finding the queue length distributions for $M/G/1$ type queues with generalised server vacations. Madan and Saleh (2001) studied the $M/D/1$ queue with general server vacations. An $M^{[X]}/G/1$ queue with feedback and optional server vacations based on general vacation times and a single vacation policy was investigated by Madan and Al-Rawwash (2005). The optional vacation assumed by Choudhury (2006) for an $M/G/1$ queue was generally distributed as well.

In the current research, first we assumed a special case of exponential distribution for vacation time and repair time, as appears in chapter 2, then we attempt to generalise vacation time and repair time to general (arbitrary) distributions in this chapter. The same sequence was followed by Altman and Yechiali (2006) who first assumed exponential distribution for the multiple vacations considered for an $M/M/1$ queue with impatient times, and then they generalised the vacation time and service time to an arbitrary distribution.

In other studies, general distributions were assumed for repair times when the queueing model is subject to breakdown and repair process. Cao (1994) considered the $M/G/1$ queueing system where the service station consists of r units and it operates if and only if all of the r units operate, assuming that each unit has a constant failure rate and arbitrary repair time distribution. Wang, Cao and Li (2001) studied the reliability analysis of the retrial queue with server breakdowns and repairs where a general distribution was assumed for repair times. Similarly, when the system breaks down it enters a repair process of random length and the repair periods are generally distributed for the $M/G/1$ retrial queue considered by Atencia *et al.* (2006).

From the above discussion on the literature on queues, we see that most of the papers are concerned with either vacations or breakdowns with different assumptions underlying the queue model. One of the few authors who investigated both vacations and breakdowns in the queueing model is Ke (2007a). He studied the operating characteristics of an $M^{[X]}/G/1$ queueing system under vacation policies with startup/closedown times and server breakdowns where the vacation time, the startup time, closedown time and repair time are generally distributed. In this model, it was assumed that the server is not allowed to take a vacation unless the system becomes empty, that is, the server may take a vacation only when all the customers are served in the system exhaustively. In this chapter, we assume a batch arrival queueing system with server vacations and random breakdowns where the service times, vacation times, and repair times are all assumed to follow general distributions, but contrary to Ke's work, the server has the option to takea vacation after any service completion. We obtain steady state solutions for our model.

Maraghi, Madan and Darby-Dowman (2009a) analysed the steady state behavior of a queueing system with Bernoulli vacations and random breakdowns where vacation times are generally distributed and repair times are exponentially distributed. In another study, they generalised the repair time, while the vacation times were assumed to be exponential (Maraghi, Madan and Darby-Dowman, 2009b). The results obtained in these studies may be considered as special cases of the results of this chapter, since in this chapter we consider the most general case where the service times, vacation times and repair times are assumed to have arbitrary distributions.

The remaining part of this chapter is organised as follows: The assumptions underlying the mathematical model are detailed in section 3.2. In section 3.3 we formulate the steady state equations governing the system in which the solutions to these equations are obtained in section 3.4. Some performance measures are obtained in section 3.5. In section 3.6 we present some important particular cases.

3.2 The Mathematical Model

The mathematical model of this chapter is described by the following assumptions:

a) Batches of customers of variable size arrive to the system in a compound Poisson process. We let $\lambda c_i dt$ $(i = 1, 2, 3, \ldots)$ to be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

b) The service is provided to customers one by one on a "first come, first served" basis by a single server. The service time follows a general (arbitrary) distribution with distribution function $G(s)$ and density function $g(s)$. Let $\mu(x)\mathrm{d}x$ be the conditional probability density of service completion during the interval $(x, x + \mathrm{d}x]$, given that the elapsed time is x, so that

$$\mu(x) = \frac{g(x)}{1 - G(x)} \tag{3.1}$$

which gives

$$g(s) = \mu(s)e^{-\int_0^s \mu(x)dx} \tag{3.2}$$

c) When a service completes, then with probability p the server may take a vacation of random length, or with probability 1-p he may stay in the system providing service, where $0 \le p \le 1$.

d) Vacation time follows a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. We let $\gamma(x)\mathrm{d}x$ be the conditional probability of a completion of a vacation during the interval $(x, x + \mathrm{d}x]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{b(x)}{1 - B(x)} \tag{3.3}$$

Accordingly,

$$b(v) = \gamma(v)e^{-\int_0^v \gamma(x)dx} \tag{3.4}$$

e) The queueing system is subject to breakdowns which are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. We also assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue waiting for the service to resume.

f) Once the system breaks down, it enters a repair process immediately. The repair time follows a general (arbitrary) distribution with distribution function $\Phi(r)$ and density function $\varphi(r)$. Let $\beta(x)\mathrm{d}x$ be the conditional probability of a repair completion during the interval $(x, x + \mathrm{d}x]$ given that the elapsed repair time is x, so that

$$\beta(x) = \frac{\varphi(x)}{1 - \Phi(x)} \tag{3.5}$$

and, hence

$$\varphi(r)=\beta(r)e^{-\int_0^r \beta(x)dx} \tag{3.6}$$

g) Different stochastic processes involved in the system are independent of each other.

3.3 Steady State Equations Governing the System

We assume that the steady state solutions exist. Hence, we define the following limits as the corresponding steady state probabilities.

$$\lim_{t\to\infty} P_n(x,t)=P_n(x), \qquad \lim_{t\to\infty} P_n(t)=\lim_{t\to\infty}\int_0^\infty P_n(x,t)dx=P_n$$

$$\lim_{t\to\infty} V_n(x,t)=V_n(x), \qquad \lim_{t\to\infty} V_n(t)=\lim_{t\to\infty}\int_0^\infty V_n(x,t)dx=V_n$$

$$\lim_{t\to\infty} R_n(x,t)=R_n(x), \qquad \lim_{t\to\infty} R_n(t)=\lim_{t\to\infty}\int_0^\infty R_n(x,t)dx=R_n$$

$$\lim_{t\to\infty} Q(t)=Q$$

Then, connecting states of the system at time t + dt with those at time t and taking the limit as $t \to \infty$, we obtain the following set of steady state equations governing the system

$$\frac{d}{dx}P_n(x)+(\lambda+\mu(x)+\alpha)P_n(x)=\lambda\sum_{i=1}^{n-1}c_iP_{n-i}(x), \qquad n\geq 1 \tag{3.7}$$

$$\frac{d}{dx}P_0(x)+(\lambda+\mu(x)+\alpha)P_0(x)=0 \tag{3.8}$$

$$\frac{d}{dx}V_n(x)+(\lambda+\gamma(x))V_n(x)=\lambda\sum_{i=1}^{n-1}c_iV_{n-i}(x), \qquad n\geq 1 \tag{3.9}$$

$$\frac{d}{dx}V_0(x)+(\lambda+\gamma(x))V_0(x)=0 \tag{3.10}$$

$$\frac{d}{dx}R_n(x)+(\lambda+\beta(x))R_n(x)=\lambda\sum_{i=1}^{n-1}c_iR_{n-i}, \qquad n\geq 1 \tag{3.11}$$

$$\frac{d}{dx}R_0(x)+(\lambda+\beta(x))R_0(x)=0 \tag{3.12}$$

$$\lambda Q=(1-p)\int_0^\infty P_0(x)\mu(x)dx+\int_0^\infty V_0(x)\gamma(x)dx+\int_0^\infty R_0(x)\beta(x)dx \tag{3.13}$$

The differential equations (3.7) – (3.13) will be solved with the following boundary conditions

$$P_n(0)=(1-p)\int_0^\infty P_{n+1}(x)\mu(x)dx+\int_0^\infty V_{n+1}(x)\gamma(x)dx+\int_0^\infty R_{n+1}(x)\beta(x)dx+\lambda c_{n+1}Q,\quad n\geq 0 \tag{3.14}$$

$$V_n(0)=p\int_0^\infty P_n(x)\mu(x)dx,\qquad n\geq 0 \tag{3.15}$$

$$R_n(0)=\alpha\int_0^\infty P_{n-1}(x)dx,\qquad n\geq 1 \tag{3.16}$$

$$R_0(0)=0 \tag{3.17}$$

3.4 Queue Size Distribution at a Random Epoch

To solve the differential equations obtained in the previous section, we need to define the following probability generating functions for different states of the system

$$P_q(x,z)=\sum_{n=0}^\infty z^nP_n(x),\quad P_q(z)=\sum_{n=0}^\infty z^nP_n,$$

$$V_q(x,z)=\sum_{n=0}^\infty z^nV_n(x),\quad V_q(z)=\sum_{n=0}^\infty z^nV_n,$$

$$R_q(x,z)=\sum_{n=0}^\infty z^nR_n(x),\quad R_q(z)=\sum_{n=0}^\infty z^nR_n,$$

$$C(z)=\sum_{i=1}^\infty z^ic_i \tag{3.18}$$

Now, multiplying equation (3.7) by z^n, summing over n from 1 to ∞, adding to (3.8) and using the generating functions defined in (3.18), we get

$$\frac{d}{dx}P_q(x,z)+\left(\lambda-\lambda C(z)+\mu(x)+\alpha\right)P_q(x,z)=0 \tag{3.19}$$

Performing similar operations to equations (3.9) and (3.10), (3.11) and (3.12), (3.14), (3.15), (3.16), and (3.17) we get

$$\frac{d}{dx}V_q(x,z)+\left(\lambda-\lambda C(z)+\gamma(x)\right)V_q(x,z)=0 \tag{3.20}$$

$$\frac{d}{dx}R_q(x,z)+\left(\lambda-\lambda C(z)+\beta(x)\right)R_q(x,z)=0 \tag{3.21}$$

$$zP_q(0,z) = (1-p)\int_0^\infty P_q(x,z)\mu(x)dx + \int_0^\infty V_q(x,z)\gamma(x)dx + \int_0^\infty R_q(x,z)\beta(x)dx + \lambda(C(z)-1)Q \qquad (3.22)$$

$$V_q(0,z) = p\int_0^\infty P_q(x,z)\mu(x)dx \qquad (3.23)$$

$$R_q(0,z) = \alpha z\int_0^\infty P_q(x,z)dx \qquad (3.24)$$

Now, we solve the differential equations (3.19) – (3.21) we obtain

$$P_q(x,z) = P_q(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt} \qquad (3.25)$$

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \qquad (3.26)$$

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \qquad (3.27)$$

where $P_q(0,z)$, $V_q(0,z)$, and $R_q(0,z)$ are given by (3.22), (3.23) and (3.24), respectively. Integrating equations (3.25) – (3.27) by parts with respect to x we obtain

$$P_q(z) = P_q(0,z)\left[\frac{1-\bar{G}[\lambda-\lambda C(z)+\alpha]}{\lambda-\lambda C(z)+\alpha}\right] \qquad (3.28)$$

$$V_q(z) = V_q(0,z)\left[\frac{1-\bar{B}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \qquad (3.29)$$

$$R_q(z) = R_q(0,z)\left[\frac{1-\bar{\Phi}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \qquad (3.30)$$

where $\bar{G}[\lambda-\lambda C(z)+\alpha]=\int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}\cdot dG(x)$, $\bar{B}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot dB(x)$, and $\bar{\Phi}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot d\Phi(x)$ are the Laplace-Stieltjes transform of the service time, vacation time and repair time, respectively.

Now, multiplying both sides of equation (3.25) by $\mu(x)$, equation (3.26) by $\gamma(x)$ and equation (3.27) by $\beta(x)$ and integrating the resulting equations over x we obtain

$$\int_0^\infty P_q(x,z)\mu(x)dx = P_q(0,z)\bar{G}[\lambda-\lambda C(z)+\alpha] \qquad (3.31)$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx = V_q(0,z)\bar{B}[\lambda-\lambda C(z)] \qquad (3.32)$$

$$\int_0^\infty R_q(x,z)\beta(x)dx = R_q(0,z)\overline{\Phi}[\lambda-\lambda C(z)] \tag{3.33}$$

Using equation (3.31), equation (3.23) can be written in the form

$$V_q(0,z) = pP_q(0,z)\overline{G}[\lambda-\lambda C(z)+\alpha] \tag{3.34}$$

And using equation (3.28), equation (3.24) will take the form

$$R_q(0,z) = \alpha z P_q(0,z)\left[\frac{1-\overline{G}[\lambda-\lambda C(z)+\alpha]}{\lambda-\lambda C(z)+\alpha}\right] \tag{3.35}$$

Substituting for $V_q(0,z)$ from (3.34) in equations (3.29) and (3.32), and for $R_q(0,z)$ from (3.35) in equations (3.30) and (3.33), we obtain

$$V_q(z) = pP_q(0,z)\left[\frac{\overline{G}[\lambda-\lambda C(z)+\alpha]\left(1-\overline{B}[\lambda-\lambda C(z)]\right)}{\lambda-\lambda C(z)}\right] \tag{3.36}$$

$$R_q(z) = \alpha z P_q(0,z)\left[\frac{\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)\left(1-\overline{\Phi}[\lambda-\lambda C(z)]\right)}{\left(\lambda-\lambda C(z)+\alpha\right)\left(\lambda-\lambda C(z)\right)}\right] \tag{3.37}$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx = pP_q(0,z)\overline{G}[\lambda-\lambda C(z)+\alpha]\overline{B}[\lambda-\lambda C(z)] \tag{3.38}$$

$$\int_0^\infty R_q(x,z)\beta(x)dx = \alpha z P_q(0,z)\left[\frac{\overline{\Phi}[\lambda-\lambda C(z)]\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)}{\lambda-\lambda C(z)+\alpha}\right] \tag{3.39}$$

Now, using equations (3.31), (3.38) and (3.39), equation (3.22) becomes

$$\begin{aligned} zP_q(0,z) &= (1-p)P_q(0,z)\overline{G}[\lambda-\lambda C(z)+\alpha] + pP_q(0,z)\overline{G}[\lambda-\lambda C(z)+\alpha]\overline{B}[\lambda-\lambda C(z)] \\ &\quad + \alpha z P_q(0,z)\left[\frac{\overline{\Phi}[\lambda-\lambda C(z)]\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)}{\lambda-\lambda C(z)+\alpha}\right] + \lambda\left(C(z)-1\right)Q \end{aligned}$$

Solving this equation for $P_q(0,z)$ gives

$$P_q(0,z) = \frac{(\lambda-\lambda C(z)+\alpha)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{3.40}$$

where

$$\begin{aligned} D(z) &= \left(\lambda-\lambda C(z)+\alpha\right)\left\{z-(1-p)\overline{G}[\lambda-\lambda C(z)+\alpha]-p\overline{G}[\lambda-\lambda C(z)+\alpha]\overline{B}[\lambda-\lambda C(z)]\right\} \\ &\quad -\alpha z\overline{\Phi}[\lambda-\lambda C(z)]\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right) \end{aligned}$$

Substituting for $P_q(0,z)$ in (3.28), (3.36), and (3.37) we have

$$P_q(z) = \frac{\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{3.41}$$

$$V_q(z) = \frac{p(\lambda-\lambda C(z)+\alpha)\overline{G}[\lambda-\lambda C(z)+\alpha]\left(\overline{B}[\lambda-\lambda C(z)]-1\right)Q}{D(z)} \tag{3.42}$$

$$R_q(z)=\frac{\alpha z\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)\left(\overline{\Phi}[\lambda-\lambda C(z)]-1\right)Q}{D(z)} \tag{3.43}$$

To find the probability generating function of the queue size irrespective of the state of the system, we let $W_q(z)=P_q(z)+V_q(z)+R_q(z)$. Then adding equations (3.41), (3.42), and (3.43) we obtain

$$\begin{aligned}W_q(z)=&\frac{\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)}\\&+\frac{p\left(\lambda-\lambda C(z)+\alpha\right)\overline{G}[\lambda-\lambda C(z)+\alpha]\left(\overline{B}[\lambda-\lambda C(z)]-1\right)Q}{D(z)}\\&+\frac{\alpha z\left(1-\overline{G}[\lambda-\lambda C(z)+\alpha]\right)\left(\overline{\Phi}[\lambda-\lambda C(z)]-1\right)Q}{D(z)}\end{aligned} \tag{3.44}$$

We need to determine Q which appeared in the expression of $W_q(z)$ given by equation (3.44). Using the normalization condition $W_q(1)+Q=1$. We see that for z = 1, $W_q(z)$ is indeterminate of 0/0 form. Applying L'Hopitals Rule on equation (3.44), we obtain

$$W_q(1)=\frac{\lambda E(I)Q\left\{\left(1+\alpha E(R)\right)+\overline{G}[\alpha]\left(p\alpha E(V)-1-\alpha E(R)\right)\right\}}{-\lambda E(I)\left(1+\alpha E(R)\right)+\overline{G}[\alpha]\left\{\alpha+\lambda E(I)\left(1+\alpha E(R)-p\alpha E(V)\right)\right\}} \tag{3.45}$$

where $C(1)=1$, $C'(1)=E(I)$ is the mean batch size of the arriving customers, $\overline{B}[0]=1$, $-\overline{B}'[0]=E(V)$ is the mean vacation time, $\overline{\Phi}[0]=1$, and $-\overline{\Phi}'[0]=E(R)$ is the mean repair time. Therefore, adding Q to equation (3.45), equating to 1 and simplifying, we get

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\overline{G}[\alpha]}-E(R)+pE(V)\right) \tag{3.46}$$

and hence, the utilization factor, ρ of the system is given by

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\overline{G}[\alpha]}-E(R)+pE(V)\right) \tag{3.47}$$

where $\rho<1$ is the stability condition under which the steady states exits.

We derived the probability that the server is idle, Q as expressed by equation (3.46). Substituting for Q from (3.46) in (3.44), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size at a random epoch.

3.5 The Mean Queue Size and the Mean Waiting Time

To find L_q, the mean number of customers in the queue under the steady state we write $W_q(z) = N(z)/D(z)$ and then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{3.48}$$

Carrying out the derivatives at $z = 1$ we get

$$N'(1) = \lambda E(I)Q\{(1+\alpha E(R)) + \overline{G}[\alpha](p\alpha E(V) - 1 - \alpha E(R))\} \tag{3.49}$$

$$\begin{aligned} N''(1) = 2Q(\lambda E(I))^2 \Bigg\{ & \left(\frac{\alpha E(R)}{\lambda E(I)} + \frac{1}{2}\alpha E(R^2)\right) \\ & + \overline{G}[\alpha]\left(-pE(V) - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2) + \frac{1}{2}p\alpha E(V^2)\right) \\ & + \overline{G}'[\alpha](1 - p\alpha E(V) + \alpha E(R)) \Bigg\} \\ & + Q\lambda E(I(I-1))\{(1+\alpha E(R)) + \overline{G}[\alpha](p\alpha E(V) - 1 - \alpha E(R))\} \end{aligned} \tag{3.50}$$

$$D'(1) = -\lambda E(I)(1+\alpha E(R)) + \overline{G}[\alpha]\{\alpha + \lambda E(I)(1 + \alpha E(R) - p\alpha E(V))\} \tag{3.51}$$

$$\begin{aligned} D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(-\frac{1}{\lambda E(I)} - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2)\right) \\ & + \overline{G}[\alpha]\left(pE(V) + \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}p\alpha E(V^2) + \frac{1}{2}\alpha E(R^2)\right) \\ & + \overline{G}'[\alpha]\left(p\alpha E(V) - 1 - \frac{\alpha}{\lambda E(I)} - \alpha E(R)\right)\Bigg\} \\ & + \lambda E(I(I-1))\{(-1 - \alpha E(R)) + \overline{G}[\alpha](1 + \alpha E(R) - p\alpha E(V))\} \end{aligned} \tag{3.52}$$

where $\overline{B}''[0] = E(V^2)$ and $\overline{\Phi}''[0] = E(R^2)$ are the second moment of the vacation time and repair time, respectively, $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers, and Q has been found in (3.46). Then if we substitute for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from (3.49) – (3.52) in (3.48) we obtain L_q in a closed form. Dividing L_q by λ we can find W_q, the mean waiting time in the queue.

3.6 Particular Cases

3.6.1 Exponential Vacation Time

In this case we assume that the vacation times are exponentially distributed with rate $\gamma > 0$ and hence mean vacation time $1/\gamma$, so we have

$$\bar{B}[\lambda-\lambda C(z)]=\frac{\gamma}{\lambda-\lambda C(z)+\gamma},\quad E(V)=\frac{1}{\gamma},\quad E(V^2)=\frac{2}{\gamma^2} \tag{3.53}$$

Using (3.53) in the main results obtained in this chapter we get

$$W_q(z)=$$

$$\frac{f_3(z)\left(1-\bar{G}[f_1(z)]\right)\lambda(C(z)-1)Q+pf_1(z)\bar{G}[f_1(z)]\lambda(C(z)-1)Q}{f_1(z)\left\{f_3(z)f_4(z)-\gamma p\bar{G}[f_1(z)]\right\}-\alpha zf_3(z)\Phi[\lambda-\lambda C(z)]\left(1-\bar{G}[f_1(z)]\right)}+$$

$$\frac{\alpha zf_3(z)\left(1-\bar{G}[f_1(z)]\right)\left(\Phi[\lambda-\lambda C(z)]-1\right)Q}{f_1(z)\left\{f_3(z)f_4(z)-\gamma p\bar{G}[f_1(z)]\right\}-\alpha zf_3(z)\Phi[\lambda-\lambda C(z)]\left(1-\bar{G}[f_1(z)]\right)} \tag{3.54}$$

where

$f_1(z)=\lambda-\lambda C(z)+\alpha$

$f_3(z)=\lambda-\lambda C(z)+\gamma$

$f_4(z)=z-(1-p)\bar{G}[\lambda-\lambda C(z)+\alpha]$

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\bar{G}[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\bar{G}[\alpha]}-E(R)+\frac{p}{\gamma}\right) \tag{3.55}$$

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\bar{G}[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\bar{G}[\alpha]}-E(R)+\frac{p}{\gamma}\right) \tag{3.56}$$

$$N'(1)=\lambda E(I)Q\left\{\left(1+\alpha E(R)\right)+\bar{G}[\alpha]\left(\frac{p\alpha}{\gamma}-1-\alpha E(R)\right)\right\} \tag{3.57}$$

$$\begin{aligned}N''(1)=2Q\left(\lambda E(I)\right)^2&\left\{\left(\frac{\alpha E(R)}{\lambda E(I)}+\frac{1}{2}\alpha E(R^2)\right)\right.\\&\left.+\bar{G}[\alpha]\left(-\frac{p}{\gamma}-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)+\frac{p\alpha}{\gamma^2}\right)+\bar{G}'[\alpha]\left(1-\frac{p\alpha}{\gamma}+\alpha E(R)\right)\right\}\\&+Q\lambda E(I(I-1))\left\{\left(1+\alpha E(R)\right)+\bar{G}[\alpha]\left(\frac{p\alpha}{\gamma}-1-\alpha E(R)\right)\right\}\end{aligned} \tag{3.58}$$

$$D'(1)=-\lambda E(I)\left(1+\alpha E(R)\right)+\bar{G}[\alpha]\left\{\alpha+\lambda E(I)\left(1+\alpha E(R)-\frac{p\alpha}{\gamma}\right)\right\} \tag{3.59}$$

$$\begin{aligned}D''(1)=2\left(\lambda E(I)\right)^2&\left\{\left(-\frac{1}{\lambda E(I)}-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)\right)\right.\\&+\bar{G}[\alpha]\left(\frac{p}{\gamma}-\frac{p\alpha}{\gamma^2}+\frac{\alpha E(R)}{\lambda E(I)}+\frac{1}{2}\alpha E(R^2)\right)\\&\left.+\bar{G}'[\alpha]\left(\frac{p\alpha}{\gamma}-1-\frac{\alpha}{\lambda E(I)}-\alpha E(R)\right)\right\}\\&+\lambda E(I(I-1))\left\{\left(-1-\alpha E(R)\right)+\bar{G}[\alpha]\left(1+\alpha E(R)-\frac{p\alpha}{\gamma}\right)\right\}\end{aligned} \tag{3.60}$$

The results obtained in this special case agree with those obtained by Maraghi, Madan and Darby-Dowman (2009a).

3.6.2 Exponential Repair Time

If the repair time has an exponential distribution with parameter β, then

$$\overline{\Phi}[\lambda-\lambda C(z)]=\frac{\beta}{\lambda-\lambda C(z)+\beta},\quad E(R)=\frac{1}{\beta},\quad E(R^2)=\frac{2}{\beta^2} \tag{3.61}$$

Using (3.61) in the main results obtained in this chapter we get

$$\begin{aligned}&W_q(z)=\\&\frac{f_2(z)\left\{\left(1-\overline{G}[f_1(z)]\right)\lambda\left(C(z)-1\right)+pf_1(z)\overline{G}[f_1(z)]\left(\overline{B}[\lambda-\lambda C(z)]-1\right)\right\}Q}{f_1(z)f_2(z)\left\{f_4(z)-p\overline{G}[f_1(z)]\overline{B}[\lambda-\lambda C(z)]\right\}-\alpha\beta z\left(1-\overline{G}[f_1(z)]\right)}\\&+\frac{\alpha z\left(1-\overline{G}[f_1(z)]\right)\lambda(C(z)-1)Q}{f_1(z)f_2(z)\left\{f_4(z)-p\overline{G}[f_1(z)]\overline{B}[\lambda-\lambda C(z)]\right\}-\alpha\beta z\left(1-\overline{G}[f_1(z)]\right)}\end{aligned} \tag{3.62}$$

where

$f_1(z)=\lambda-\lambda C(z)+\alpha$

$f_2(z)=\lambda-\lambda C(z)+\beta$

$f_4(z)=z-(1-p)\overline{G}[\lambda-\lambda C(z)+\alpha]$

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]}+\frac{1}{\beta\overline{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+pE(V)\right) \tag{3.63}$$

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]}+\frac{1}{\beta\overline{G}[\alpha]}-\frac{1}{\alpha}-\frac{1}{\beta}+pE(V)\right) \tag{3.64}$$

$$N'(1)=\lambda E(I)Q\left\{\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left(p\alpha E(V)-1-\frac{\alpha}{\beta}\right)\right\} \tag{3.65}$$

$$\begin{aligned}N''(1)=2Q\left(\lambda E(I)\right)^2&\left\{\left(\frac{\alpha}{\beta\lambda E(I)}+\frac{\alpha}{\beta^2}\right)\right.\\&+\overline{G}[\alpha]\left(-pE(V)-\frac{\alpha}{\beta\lambda E(I)}-\frac{\alpha}{\beta^2}+\frac{1}{2}p\alpha E(V^2)\right)\\&\left.+\overline{G}'[\alpha]\left(1-p\alpha E(V)+\frac{\alpha}{\beta}\right)\right\}\\&+Q\lambda E(I(I-1))\left\{\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left(p\alpha E(V)-1-\frac{\alpha}{\beta}\right)\right\}\end{aligned} \tag{3.66}$$

$$D'(1)=-\lambda E(I)\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left\{\alpha+\lambda E(I)\left(1+\frac{\alpha}{\beta}-p\alpha E(V)\right)\right\} \tag{3.67}$$

$$\begin{aligned}D''(1) = 2(\lambda E(I))^2 \Bigg\{ & \left(-\frac{1}{\lambda E(I)} - \frac{\alpha}{\beta\lambda E(I)} - \frac{\alpha}{\beta^2}\right) \\ & + \overline{G}[\alpha]\left(pE(V) + \frac{\alpha}{\beta\lambda E(I)} - \frac{1}{2}p\alpha E(V^2) + \frac{\alpha}{\beta^2}\right) \\ & + \overline{G}'[\alpha]\left(p\alpha E(V) - 1 - \frac{\alpha}{\lambda E(I)} - \frac{\alpha}{\beta}\right)\Bigg\} \\ & + \lambda E(I(I-1))\left\{\left(-1-\frac{\alpha}{\beta}\right) + \overline{G}[\alpha]\left(1+\frac{\alpha}{\beta} - p\alpha E(V)\right)\right\} \end{aligned} \tag{3.68}$$

Equations (3.62) – (3.68) agree with the results obtained by Maraghi, Madan and Darby-Dowman (2009b).

3.6.3 Exponential Vacation Time and Repair Time

In this case we assume that the vacation time and the repair time both have exponential distributions with parameters γ and β, respectively, which gives

$$\left.\begin{aligned} \overline{B}[\lambda-\lambda C(z)] &= \frac{\gamma}{\lambda-\lambda C(z)+\gamma}, \quad E(V)=\frac{1}{\gamma}, \quad E(V^2)=\frac{2}{\gamma^2} \\ \overline{\Phi}[\lambda-\lambda C(z)] &= \frac{\beta}{\lambda-\lambda C(z)+\beta}, \quad E(R)=\frac{1}{\beta}, \quad E(R^2)=\frac{2}{\beta^2} \end{aligned}\right\} \tag{3.69}$$

Using the substitutions defined in (3.69) in the main results obtained in this chapter we get

$$\begin{aligned} & W_q(z) = \\ & \frac{\{f_2(z)f_3(z)(1-\bar{G}[f_1(z)]) + pf_1(z)f_2(z)\bar{G}[f_1(z)]\}\lambda(C(z)-1)Q}{f_1(z)f_2(z)f_3(z)f_4(z) - p\gamma f_1(z)f_2(z)\bar{G}[f_1(z)] - \alpha\beta z f_3(z)(1-\bar{G}[f_1(z)])} + \\ & \frac{\alpha z f_3(z)(1-\bar{G}[f_1(z)])\lambda(C(z)-1)Q}{f_1(z)f_2(z)f_3(z)f_4(z) - p\gamma f_1(z)f_2(z)\bar{G}[f_1(z)] - \alpha\beta z f_3(z)(1-\bar{G}[f_1(z)])} \end{aligned} \tag{3.70}$$

where

$f_1(z) = \lambda - \lambda C(z) + \alpha$

$f_2(z) = \lambda - \lambda C(z) + \beta$

$f_3(z) = \lambda - \lambda C(z) + \gamma$

$f_4(z) = z - (1-p)\overline{G}[\lambda - \lambda C(z) + \alpha]$

$$Q = 1 - \lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]} + \frac{1}{\beta\overline{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{3.71}$$

$$\rho = \lambda E(I)\left(\frac{1}{\alpha\overline{G}[\alpha]} + \frac{1}{\beta\overline{G}[\alpha]} - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{p}{\gamma}\right) \tag{3.72}$$

$$N'(1)=\lambda E(I)Q\left\{\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left(\frac{p\alpha}{\gamma}-1-\frac{\alpha}{\beta}\right)\right\} \tag{3.73}$$

$$\begin{aligned}N''(1)=2Q(\lambda E(I))^2&\left\{\left(\frac{\alpha}{\beta\lambda E(I)}+\frac{\alpha}{\beta^2}\right)\right.\\&+\overline{G}[\alpha]\left(-\frac{p}{\gamma}-\frac{\alpha}{\beta\lambda E(I)}-\frac{\alpha}{\beta^2}+\frac{p\alpha}{\gamma^2}\right)\\&\left.+\overline{G}'[\alpha]\left(1-\frac{p\alpha}{\gamma}+\frac{\alpha}{\beta}\right)\right\}\\&+Q\lambda E(I(I-1))\left\{\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left(\frac{p\alpha}{\gamma}-1-\frac{\alpha}{\beta}\right)\right\}\end{aligned} \tag{3.74}$$

$$D'(1)=-\lambda E(I)\left(1+\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left\{\alpha+\lambda E(I)\left(1+\frac{\alpha}{\beta}-\frac{p\alpha}{\gamma}\right)\right\} \tag{3.75}$$

$$\begin{aligned}D''(1)=2(\lambda E(I))^2&\left\{\left(-\frac{1}{\lambda E(I)}-\frac{\alpha}{\beta\lambda E(I)}-\frac{\alpha}{\beta^2}\right)\right.\\&+\overline{G}[\alpha]\left(\frac{p}{\gamma}+\frac{\alpha}{\beta\lambda E(I)}-\frac{p\alpha}{\gamma^2}+\frac{\alpha}{\beta^2}\right)\\&\left.+\overline{G}'[\alpha]\left(\frac{p\alpha}{\gamma}-1-\frac{\alpha}{\lambda E(I)}-\frac{\alpha}{\beta}\right)\right\}\\&+\lambda E(I(I-1))\left\{\left(-1-\frac{\alpha}{\beta}\right)+\overline{G}[\alpha]\left(1+\frac{\alpha}{\beta}-\frac{p\alpha}{\gamma}\right)\right\}\end{aligned} \tag{3.76}$$

The results obtained in (3.70) through (3.72) agree with the results obtained in chapter 2. Furthermore, using equations (3.73) – (3.76) in (3.48) gives the mean number of customers in the queue L_q which agrees with the expressions obtained in chapter 2.

* * *

Chapter 4

Batch Arrival Queue with Two-Stage Heterogeneous Service, Bernoulli Schedule General Vacations, Random Breakdowns and General Repair Times

4.1 Introduction

A number of papers have appeared recently in the queueing literature in which the server provides to each customer two stages (phases) of heterogeneous service in succession. Doshi (1991) studied a queueing system in which customers receive a batch service in the first stage and individual services in the second stage. A similar model was considered by Kim and Park (2003) who further assumed that when the system becomes empty at the moment of the completion of the second stage service, it is turned off and after an idle period, the server is turned on when the queue length reaches N (threshold). Bocharov, Manzo and Pechinkin (2005) analysed a two-stage queue with Markov arrival process and losses in which there are buffers of finite capacity in each phases. Selvam and Sivasankaran (1994) extended the model studied by Doshi (1991) by introducing vacations. They assumed that the server takes vacations as soon as the system becomes empty.

Two-stage queueing systems with server vacations have been studied extensively. Madan (2000b) obtained steady state solutions for a single server queue with a two-stage heterogeneous service and Binomial schedule server vacations having exponential vacation times. He assumed that in both stages of service, customers are served individually. In another paper, Madan (2001) assumed deterministic server vacations. Later, Madan and Choudhury (2005) generalised these results by deriving the steady state queue size distributions at a random epoch as well as at a departure epoch for generalised vacation time. They have also studied a two-stage batch arrival queueing system with a modified Bernoulli schedule vacation under *N*-Policy (Choudhury & Madan, 2005), and with restricted admissibility and random setup time (Madan & Choudhury, 2006). More recently, Choudhury (2008) investigated this model in depth by considering different vacation models.

Choi and Kim (2003) introduced Bernoulli feedback in a two-phase queueing system with vacations. Katayama and Kobayashi (2006) studied the sojourn time analysis of a queueing system with two-phase service and server vacations. Choudhury (2007) and Kumar and Arumuganathan (2008) investigated batch arrival retrial queues with two stages of service and Bernoulli schedule server vacations.

The motivation of two-stage queues with vacations comes from wide applications of these models in real situations. An example may well be found in some transportation system in which a ferry driver or a locomotive driver may like to go on a vacation after every round trip which essentially involves two phases of service that is a trip to a particular destination and back to a starting point. Another example is a production system where the machine producing certain items may require two phases of service in succession such as periodic checking followed by a usual process to complete the processing of raw materials. The machines need to be stopped once in a while for overhauling after these two phases of service. This overhauling may be utilized as a vacation time. Also, in some computer networks and telecommunication systems messages are processed in two stages by a single server (Madan & Choudhury, 2005).

Although some aspects of a two-stage heterogeneous service systems with server vacations have been discussed in the literature of queueing theory as shown in the papers discussed earlier in this chapter, some questions still need to be addressed. What if the bus, locomotive, machines, or telecommunication systems in the examples discussed above have suddenly broken down? The breakdowns will occur randomly and hence a repair process must immediately start in order to continue functioning. This will affect the queue size and the customers waiting time. All the above papers ignored the fact that in real situations a system might breakdown. Thus the purpose of this chapter is to consider a more general problem by deriving the steady state queue size distribution at a random epoch of queueing systems with two-stage heterogeneous service, Bernoulli schedule vacations and random breakdowns with rate $\alpha > 0$, in which the system enters a repair process once it breaks down and the customer whose service is interrupted goes back to the head of the queue waiting for the repair process to complete. We assume that the system might break down at either stages of service. Vacation times, repair times and both service times are all assumed to be generally distributed. The rest of this chapter is organised in the same manner as in chapter 3.

4.2 The Mathematical Model

The following assumptions describe the mathematical model underlying the queueing system of this chapter.

a) Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$ $(i = 1, 2, 3, \ldots)$ to be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

b) A single server provides service to one customer at a time. Each customer undergoes a two-stage service on a first come, first served basis. The service time of the two stages follow different general (arbitrary) distributions with distribution functions $G_j(s)$ and density function $g_j(s)$, $j = 1, 2$. Let $\mu_j(x)\mathrm{d}x$ be the conditional probability density of the completion of the j^{th} stage of service during the time interval $(x, x + \mathrm{d}x]$, given that the elapsed time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1 - G_j(x)}, \qquad j = 1,2 \tag{4.1}$$

thus

$$g_j(s) = \mu_j(s) e^{-\int_0^s \mu_j(x)dx}, \qquad j = 1,2 \tag{4.2}$$

c) As soon as the second stage service of a customer is complete, then with probability p the server may decide to go on a vacation of random length, or with probability $1-p$ he may continue to be available for the next service, where $0 \leq p \leq 1$.

d) The server's vacation time follows a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{b(x)}{1 - B(x)} \tag{4.3}$$

thus

$$b(v) = \gamma(v) e^{-\int_0^v \gamma(x)dx} \tag{4.4}$$

e) The system may break down while service of either stage is going on, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further, we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue whether the breakdown occurred in the first or second stage. In either case, the customer restarts with the first service after the repairs are complete.

f) When the system breaks down, it enters a repair process immediately. The repair time follows a general (arbitrary) distribution with distribution function $\Phi(r)$ and density function $\varphi(r)$. Let $\beta(x)dx$ be the conditional probability of a repair completion during the interval $(x, x + dx]$ given that the elapsed repair time is x, so that

$$\beta(x) = \frac{\varphi(x)}{1 - \Phi(x)} \tag{4.5}$$

and, hence

$$\varphi(r) = \beta(r) e^{-\int_0^r \beta(x)dx} \tag{4.6}$$

g) The stochastic processes involved in the system are assumed to be independent of each other.

4.3 Steady State Equations Governing the System

Assuming that the steady state solutions exist, we define the following limits as the corresponding steady state probabilities.

$$\lim_{t\to\infty} P_n^{(j)}(x,t) = P_n^{(j)}(x), \quad \lim_{t\to\infty} P_n^{(j)}(t) = \lim_{t\to\infty}\int_0^{\infty} P_n^{(j)}(x,t)dx = P_n^{(j)}, \quad j = 1,2$$

$$\lim_{t\to\infty} V_n(x,t) = V_n(x), \quad \lim_{t\to\infty} V_n(t) = \lim_{t\to\infty}\int_0^{\infty} V_n(x,t)dx = V_n$$

$$\lim_{t\to\infty} R_n(x,t) = R_n(x), \quad \lim_{t\to\infty} R_n(t) = \lim_{t\to\infty}\int_0^{\infty} R_n(x,t)dx = R_n$$

$$\lim_{t\to\infty} Q(t) = Q$$

To find the steady state equations governing the system, we connect states of the system at time $t + dt$ with those at time t and take the limit as $t \to \infty$, we obtain the following set of differential equations

$$\frac{d}{dx}P_n^{(1)}(x) + (\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x) = \lambda\sum_{i=1}^{n-1} c_i P_{n-i}^{(1)}(x), \qquad n \geq 1 \tag{4.7}$$

$$\frac{d}{dx}P_0^{(1)}(x) + (\lambda + \mu_1(x) + \alpha)P_0^{(1)}(x) = 0 \tag{4.8}$$

$$\frac{d}{dx}P_n^{(2)}(x) + (\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x) = \lambda\sum_{i=1}^{n-1} c_i P_{n-i}^{(2)}(x), \qquad n \geq 1 \tag{4.9}$$

$$\frac{d}{dx}P_0^{(2)}(x) + (\lambda + \mu_2(x) + \alpha)P_0^{(2)}(x) = 0 \tag{4.10}$$

$$\frac{d}{dx}V_n(x) + (\lambda + \gamma(x))V_n(x) = \lambda\sum_{i=1}^{n-1} c_i V_{n-i}(x), \qquad n \geq 1 \tag{4.11}$$

$$\frac{d}{dx}V_0(x) + (\lambda + \gamma(x))V_0(x) = 0 \tag{4.12}$$

$$\frac{d}{dx}R_n(x) + (\lambda + \beta(x))R_n(x) = \lambda\sum_{i=1}^{n-1} c_i R_{n-i}, \qquad n \geq 1 \tag{4.13}$$

$$\frac{d}{dx}R_0(x) + (\lambda + \beta(x))R_0(x) = 0 \tag{4.14}$$

$$\lambda Q = (1-p)\int_0^{\infty} P_0^{(2)}(x)\mu_2(x)dx + \int_0^{\infty} V_0(x)\gamma(x)dx + \int_0^{\infty} R_0(x)\beta(x)dx \tag{4.15}$$

These equations are to be solved according to the following boundary conditions

$$P_n^{(1)}(0) = (1-p)\int_0^{\infty} P_{n+1}^{(2)}(x)\mu_2(x)dx + \int_0^{\infty} R_{n+1}(x)\beta(x)dx + \int_0^{\infty} V_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q, \quad n \geq 0 \tag{4.16}$$

$$P_n^{(2)}(0)=\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx, \quad n\ge 0 \tag{4.17}$$

$$V_n(0)=p\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx, \quad n\ge 0 \tag{4.18}$$

$$R_n(0)=\alpha\int_0^\infty P_{n-1}^{(1)}(x)dx+\alpha\int_0^\infty P_{n-1}^{(2)}(x)dx, \quad n\ge 1 \tag{4.19}$$

$$R_0(0)=0 \tag{4.20}$$

4.4 Queue Size Distribution at a Random Epoch

First we need to define the probability generating functions for different states of the system

$$P_q^{(j)}(x,z)=\sum_{n=0}^\infty z^n P_n^{(j)}(x), \quad P_q^{(j)}(z)=\sum_{n=0}^\infty z^n P_n^{(j)}, \quad j=1,2,$$

$$V_q(x,z)=\sum_{n=0}^\infty z^n V_n(x), \quad V_q(z)=\sum_{n=0}^\infty z^n V_n,$$

$$R_q(x,z)=\sum_{n=0}^\infty z^n R_n(x), \quad R_q(z)=\sum_{n=0}^\infty z^n R_n,$$

$$C(z)=\sum_{i=1}^\infty z^i c_i \tag{4.21}$$

Now, multiplying equation (4.7) by z^n, summing over n from 1 to ∞, adding to (4.8), and using the generating functions defined in (4.21), we get

$$\frac{d}{dx}P_q^{(1)}(x,z)+\left(\lambda-\lambda C(z)+\mu_1(x)+\alpha\right)P_q^{(1)}(x,z)=0 \tag{4.22}$$

Proceeding in the same manner, equations (4.9) and (4.10), (4.11) and (4.12), and (4.13) and (4.14) yield

$$\frac{d}{dx}P_q^{(2)}(x,z)+\left(\lambda-\lambda C(z)+\mu_2(x)+\alpha\right)P_q^{(2)}(x,z)=0 \tag{4.23}$$

$$\frac{d}{dx}V_q(x,z)+\left(\lambda-\lambda C(z)+\gamma(x)\right)V_q(x,z)=0 \tag{4.24}$$

$$\frac{d}{dx}R_q(x,z)+\left(\lambda-\lambda C(z)+\beta(x)\right)R_q(x,z)=0 \tag{4.25}$$

For the boundary conditions, we multiply equation (4.16) by z^{n+1}, sum over n from 0 to ∞, use the generating functions defined in (4.21) and use equation (4.15) we get

$$zP_q^{(1)}(0,z) = (1-p)\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx + \int_0^\infty R_q(x,z)\beta(x)dx$$
$$+\int_0^\infty V_q(x,z)\gamma(x)dx + \lambda\big(C(z)-1\big)Q \tag{4.26}$$

Performing similar operations to equation (4.17), (4.18), (4.19) and (4.20) we get

$$P_q^{(2)}(0,z) = \int_0^\infty P_q^{(1)}(x,z)\mu_1(x)dx \tag{4.27}$$

$$V_q(0,z) = p\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx \tag{4.28}$$

$$R_q(0,z) = \alpha z\int_0^\infty P_q^{(1)}(x,z)dx + \alpha z\int_0^\infty P_q^{(2)}(x,z)dx \tag{4.29}$$

We integrate equations (4.22) – (4.25) between the limits 0 and x and obtain the following

$$P_q^{(1)}(x,z) = P_q^{(1)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_1(t)dt} \tag{4.30}$$

$$P_q^{(2)}(x,z) = P_q^{(2)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_2(t)dt} \tag{4.31}$$

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{4.32}$$

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{4.33}$$

where $P_q^{(1)}(0,z)$, $P_q^{(2)}(0,z)$, $V_q(0,z)$ and $R_q(0,z)$ are defined in equations (4.26), (4.27), (4.28) and (4.29), respectively. Again integrating equations (4.30) – (4.33) with respect to x gives

$$P_q^{(1)}(z) = P_q^{(1)}(0,z)\left[\frac{1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{4.34}$$

$$P_q^{(2)}(z) = P_q^{(2)}(0,z)\left[\frac{1-\overline{G}_2[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{4.35}$$

$$V_q(z) = V_q(0,z)\left[\frac{1-\overline{B}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{4.36}$$

$$R_q(z) = R_q(0,z)\left[\frac{1-\overline{\Phi}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{4.37}$$

where $\bar{G}_1[\lambda-\lambda C(z)+\alpha]=\int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}\cdot dG_1(x)$, $\bar{G}_2[\lambda-\lambda C(z)+\alpha]=\int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}\cdot dG_2(x)$, $\bar{B}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot dB(x)$ and $\Phi[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot d\Phi(x)$ are the Laplace-Stieltjes transform of the first stage service time, second stage service time, vacation time and repair time, respectively.

Now, we need to determine the integrals $\int_0^\infty P_q^{(1)}(x,z)\mu_1(x)dx$, $\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx$, $\int_0^\infty V_q(x,z)\gamma(x)dx$ and $\int_0^\infty R_q(x,z)\beta(x)dx$ which appeared in the right hand sides of equations (4.26) – (4.28). To do so, we multiply equation (4.30) by $\mu_1(x)$, equation (4.31) by $\mu_2(x)$, equation (4.32) by $\gamma(x)$ and equation (4.33) by $\beta(x)$, integrate with respect to x and use equations (4.2), (4.4) and (4.6), we get

$$\int_0^\infty P_q^{(1)}(x,z)\mu_1(x)dx=P_q^{(1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{4.38}$$

$$\int_0^\infty P_q^{(2)}(x,z)\mu_2(x)dx=P_q^{(2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha] \tag{4.39}$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx=V_q(0,z)\bar{B}[\lambda-\lambda C(z)] \tag{4.40}$$

$$\int_0^\infty R_q(x,z)\beta(x)dx=R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)] \tag{4.41}$$

Using (4.34), (4.35) and (4.38) – (4.41), equations (4.26) – (4.29) become

$$\begin{aligned} zP_q^{(1)}(0,z)=(1-p)P_q^{(2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha]+R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)] \\ +V_q(0,z)\bar{B}[\lambda-\lambda C(z)]+\lambda(C(z)-1)Q \end{aligned} \tag{4.42}$$

$$P_q^{(2)}(0,z)=P_q^{(1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{4.43}$$

$$V_q(0,z)=pP_q^{(2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha] \tag{4.44}$$

$$R_q(0,z)=\alpha zP_q^{(1)}(0,z)\left[\frac{1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right]+\alpha zP_q^{(2)}(0,z)\left[\frac{1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{4.45}$$

Using equation (4.43), equations (4.44) and (4.45) become

$$V_q(0,z)=pP_q^{(1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha] \tag{4.46}$$

$$R_q(0,z)=\frac{\alpha zP_q^{(1)}(0,z)}{(\lambda-\lambda C(z)+\alpha)}\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right) \tag{4.47}$$

Using (4.43) in (4.35), (4.46) in (4.36) and (4.47) in (4.37) we get

$$P_q^{(2)}(z)=\frac{P_q^{(1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)}{\lambda-\lambda C(z)+\alpha} \tag{4.48}$$

$$V_q(z)=\frac{pP_q^{(1)}(0,z)\left(1-\bar{B}[\lambda-\lambda C(z)]\right)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]}{\lambda-\lambda C(z)} \tag{4.49}$$

$$R_q(z)=\frac{\alpha zP_q^{(1)}(0,z)\left(1-\bar{\Phi}[\lambda-\lambda C(z)]\right)\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)}{\left(\lambda-\lambda C(z)+\alpha\right)\left(\lambda-\lambda C(z)\right)} \tag{4.50}$$

Substituting for $P_q^{(2)}(0,z)$, $V_q(0,z)$ and $R_q(0,z)$ from equations (4.43), (4.46) and (4.47) in equation (4.42) and solving the resulting equation for $P_q^{(1)}(0,z)$ we get

$$P_q^{(1)}(0,z)=\frac{\left(\lambda-\lambda C(z)+\alpha\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.51}$$

where

$$\begin{aligned}D(z)=&\left(\lambda-\lambda C(z)+\alpha\right)\left\{z-(1-p)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right\}\\&-p\left(\lambda-\lambda C(z)+\alpha\right)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\bar{B}[\lambda-\lambda C(z)]\\&-\alpha z\bar{\Phi}[\lambda-\lambda C(z)]\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\end{aligned}$$

Substituting for $P_q^{(1)}(0,z)$ from (4.51) in (4.34), (4.48), (4.49) and (4.50), we get the following probability generating functions for the number of customers in the queue at a random epoch

$$P_q^{(1)}(z)=\frac{\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.52}$$

$$P_q^{(2)}(z)=\frac{\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.53}$$

$$V_q(z)=\frac{p\left(\lambda-\lambda C(z)+\alpha\right)\left(\bar{B}[\lambda-\lambda C(z)]-1\right)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]Q}{D(z)} \tag{4.54}$$

$$R_q(z)=\frac{\alpha z\left(\bar{\Phi}[\lambda-\lambda C(z)]-1\right)\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)Q}{D(z)} \tag{4.55}$$

Accordingly, we can find the probability generating function of the queue size irrespective of the state of the system, $W_q(z)$, by adding equations (4.52) – (4.55) as the following equation describes

$$W_q(z)=P_q^{(1)}(z)+P_q^{(2)}(z)+V_q(z)+R_q(z) \tag{4.56}$$

To find Q, we use the normalization condition

$$W_q(1)+Q=1$$

However, as in the previous chapters, for $z = 1$, $W_q(z)$ is indeterminate and hence L'Hopital's Rule will be used. Applying this rule on equation (4.56), we obtain

$$W_q(1)=\frac{\lambda E(I)Q\left\{(1+\alpha E(R))+\overline{G}_1[\alpha]\overline{G}_2[\alpha](p\alpha E(V)-1-\alpha E(R))\right\}}{-\lambda E(I)(1+\alpha E(R))+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\{\alpha+\lambda E(I)(1+\alpha E(R)-p\alpha E(V))\}} \tag{4.57}$$

where $C(1)=1$, $C'(1)=E(I)$ is the mean batch size of the arriving customers, $\overline{B}[0]=1$, $-\overline{B}'[0]=E(V)$ is the mean vacation time, $\overline{\Phi}[0]=1$, and $-\overline{\Phi}'[0]=E(R)$ is the mean repair time. Therefore, it follows from the normalization condition that

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-E(R)+pE(V)\right) \tag{4.58}$$

which gives

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-E(R)+pE(V)\right) \tag{4.59}$$

where $\rho<1$ is the stability condition for the steady states to exist.

Substituting for Q from (4.58) in (4.56), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size at a random epoch.

4.5 The Mean Queue Size and the Mean Waiting Time

We find the mean queue size using the probability generating function of the number of customers in the queue obtained in equation (4.56) and the following formula

$$L_q=\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2} \tag{4.60}$$

where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (4.56), respectively, and primes and double primes in (4.60) denote first and second derivatives at $z = 1$, respectively. Then finding the derivatives at $z = 1$ we have

$$N'(1)=\lambda E(I)Q\left\{(1+\alpha E(R))+\overline{G}_1[\alpha]\overline{G}_2[\alpha](p\alpha E(V)-1-\alpha E(R))\right\} \tag{4.61}$$

$$
\begin{aligned}
N''(1) = 2Q\left(\lambda E(I)\right)^2 \Bigg\{ & \left(\frac{\alpha E(R)}{\lambda E(I)} + \frac{1}{2}\alpha E(R^2)\right) \\
& + \overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-pE(V) - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2) + \frac{1}{2}p\alpha E(V^2)\right) \\
& + \left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha] + \overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(1 - p\alpha E(V) + \alpha E(R)\right) \Bigg\} \\
& + Q\lambda E(I(I-1))\left\{\left(1 + \alpha E(R)\right) + \overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(p\alpha E(V) - 1 - \alpha E(R)\right)\right\}
\end{aligned}
\tag{4.62}
$$

$$
D'(1) = -\lambda E(I)\left(1 + \alpha E(R)\right) + \overline{G}_1[\alpha]\overline{G}_2[\alpha]\left\{\alpha + \lambda E(I)\left(1 + \alpha E(R) - p\alpha E(V)\right)\right\}
\tag{4.63}
$$

$$
\begin{aligned}
D''(1) = 2\left(\lambda E(I)\right)^2 \Bigg\{ & \left(-\frac{1}{\lambda E(I)} - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2)\right) \\
& + \overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(pE(V) + \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}p\alpha E(V^2) + \frac{1}{2}\alpha E(R^2)\right) \\
& + \left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha] + \overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(p\alpha E(V) - 1 - \frac{\alpha}{\lambda E(I)} - \alpha E(R)\right)\Bigg\} \\
& + \lambda E(I(I-1))\left\{\left(-1 - \alpha E(R)\right) + \overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(1 + \alpha E(R) - p\alpha E(V)\right)\right\}
\end{aligned}
\tag{4.64}
$$

where Q is given by (4.58), $\overline{B}''[0] = E(V^2)$ and $\overline{\Phi}''[0] = E(R^2)$ are the second moment of the vacation time and repair time, respectively, and $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then, utilizing (4.61) – (4.64) in (4.60) we obtain L_q, and hence the mean queueing time, W_q can be found using equation (1.3).

4.6 Particular cases

4.6.1 Exponential Vacation Time and Repair Time

In this case we assume that the vacation times are exponentially distributed with rate $\gamma > 0$ and the repair time are also exponentially distributed but with rate $\beta > 0$, so we have

$$
\left.
\begin{aligned}
&\overline{B}[\lambda - \lambda C(z)] = \frac{\gamma}{\lambda - \lambda C(z) + \gamma}, \quad E(V) = \frac{1}{\gamma}, \quad E(V^2) = \frac{2}{\gamma^2} \\
&\overline{\Phi}[\lambda - \lambda C(z)] = \frac{\beta}{\lambda - \lambda C(z) + \beta}, \quad E(R) = \frac{1}{\beta}, \quad E(R^2) = \frac{2}{\beta^2}
\end{aligned}
\right\}
\tag{4.65}
$$

Using the substitutions defined in (4.65) in the main results obtained in this chapter we get

$$W_q(z)=P_q^{(1)}(z)+P_q^{(2)}(z)+V_q(z)+R_q(z)$$

where

$$P_q^{(1)}(z)=\frac{\left(\lambda-\lambda C(z)+\gamma\right)\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.66}$$

$$P_q^{(2)}(z)=\frac{\left(\lambda-\lambda C(z)+\gamma\right)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\left(1-\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.67}$$

$$V_q(z)=\frac{p\left(\lambda-\lambda C(z)+\alpha\right)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.68}$$

$$R_q(z)=\frac{\alpha z\left(\dfrac{\lambda-\lambda C(z)+\gamma}{\lambda-\lambda C(z)+\beta}\right)\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{4.69}$$

$$\begin{aligned}D(z)={}&\left(\lambda-\lambda C(z)+\gamma\right)\left(\lambda-\lambda C(z)+\alpha\right)\left\{z-(1-p)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right\}\\&-\frac{\alpha\beta z}{\left(\lambda-\lambda C(z)+\beta\right)}\left(\lambda-\lambda C(z)+\gamma\right)\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\\&-p\gamma\left(\lambda-\lambda C(z)+\alpha\right)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\end{aligned} \tag{4.70}$$

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\alpha}+\frac{1}{\beta\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\beta}+\frac{p}{\gamma}\right) \tag{4.71}$$

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\alpha}+\frac{1}{\beta\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\beta}+\frac{p}{\gamma}\right) \tag{4.72}$$

Similarly, to find the mean number of customers in the queue, we find $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ using equations (4.61) – (4.64) and the substitutions defined in (4.65)

$$N'(1)=\lambda E(I)Q\left\{\gamma\left(1+\frac{\alpha}{\beta}\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(p\alpha-\gamma-\frac{\alpha\gamma}{\beta}\right)\right\} \tag{4.73}$$

$$\begin{aligned}N''(1)={}&2Q\left(\lambda E(I)\right)^2\Bigg[-1+\frac{\alpha\gamma}{\lambda\beta E(I)}-\frac{\alpha}{\beta}+\frac{\alpha\gamma}{\beta^2}\\&+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(1-p-\frac{\alpha\gamma}{\lambda\beta E(I)}+\frac{\alpha}{\beta}-\frac{\alpha\gamma}{\beta^2}\right)\\&+\left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha]+\overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(\gamma+\frac{\alpha\gamma}{\beta}-p\alpha\right)\Bigg]\\&+\lambda QE(I(I-1))\left[\gamma+\frac{\gamma\alpha}{\beta}+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-\gamma-\frac{\alpha\gamma}{\beta}+p\alpha\right)\right]\end{aligned} \tag{4.74}$$

$$D'(1)=\lambda E(I)\left\{-\gamma\left(1+\frac{\alpha}{\beta}\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-p\alpha+\gamma+\frac{\alpha\gamma}{\beta}+\frac{\alpha\gamma}{\lambda E(I)}\right)\right\} \tag{4.75}$$

$$\begin{aligned}D''(1)=2(\lambda E(I))^2\Bigg[&\left(1-\frac{\gamma}{\lambda E(I)}-\frac{\alpha\gamma}{\lambda\beta E(I)}+\frac{\alpha}{\beta}-\frac{\alpha\gamma}{\beta^2}\right)\\&+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-1+p-\frac{\alpha}{\lambda E(I)}+\frac{\alpha\gamma}{\lambda\beta E(I)}-\frac{\alpha}{\beta}+\frac{\alpha\gamma}{\beta^2}\right)\\&+\left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha]+\overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(-\gamma-\frac{\alpha\gamma}{\lambda E(I)}-\frac{\alpha\gamma}{\beta}+p\alpha\right)\Bigg]\\&+\lambda E(I(I-1))\left[\left(-\gamma-\frac{\gamma\alpha}{\beta}\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(\gamma+\frac{\alpha\gamma}{\beta}-p\alpha\right)\right]\end{aligned}\tag{4.76}$$

4.6.2 No Server Vacations

If the server can not take vacations, then we drop the assumption of server vacations by letting $p = 0$ in the equations of this chapter. This gives $V_q(z) = 0$, and we have

$$P_q^{(1)}(z)=\frac{\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)}\tag{4.77}$$

$$P_q^{(2)}(z)=\frac{\overline{G}_1[\lambda-\lambda C(z)+\alpha]\left(1-\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)}\tag{4.78}$$

$$R_q(z)=\frac{\alpha z\left(\overline{\Phi}[\lambda-\lambda C(z)]-1\right)\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)Q}{D(z)}\tag{4.79}$$

$$\begin{gathered}W_q(z)=P_q^{(1)}(z)+P_q^{(2)}(z)+R_q(z)=\\\frac{\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\left\{\alpha z\left(\overline{\Phi}[\lambda-\lambda C(z)]-1\right)+\lambda\left(C(z)-1\right)\right\}Q}{D(z)}\end{gathered}\tag{4.80}$$

$$\begin{aligned}D(z)=&\left(\lambda-\lambda C(z)+\alpha\right)\left\{z-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right\}\\&-\alpha z\overline{\Phi}[\lambda-\lambda C(z)]\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)+\alpha]\right)\end{aligned}\tag{4.81}$$

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-\frac{1}{\alpha}+\frac{E(R)}{\overline{G}_1[\alpha]\overline{G}_2[\alpha]}-E(R)\right)\tag{4.82}$$

$$N'(1)=\lambda E(I)Q\left\{\left(1+\alpha E(R)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-1-\alpha E(R)\right)\right\}\tag{4.83}$$

$$\begin{aligned}N''(1)=2Q(\lambda E(I))^2\Bigg[&\left(\frac{\alpha E(R)}{\lambda E(I)}+\frac{1}{2}\alpha E(R^2)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)\right)\\&+\left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha]+\overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(1+\alpha E(R)\right)\Bigg]\\&+\lambda QE(I(I-1))\left[\left(1+\alpha E(R)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(-1-\alpha E(R)\right)\right]\end{aligned}\tag{4.84}$$

$$D'(1)=\lambda E(I)\left\{\left(-1-\alpha E(R)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(1+\alpha E(R)+\frac{\alpha}{\lambda E(I)}\right)\right\}\tag{4.85}$$

$$D''(1)=2(\lambda E(I))^2\Bigg[\left(-\frac{1}{\lambda E(I)}-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(\frac{\alpha E(R)}{\lambda E(I)}+\frac{1}{2}\alpha E(R^2)\right)$$
$$+\left(\overline{G}_1[\alpha]\overline{G}_2'[\alpha]+\overline{G}_1'[\alpha]\overline{G}_2[\alpha]\right)\left(-1-\frac{\alpha}{\lambda E(I)}-\alpha E(R)\right)\Bigg]$$
$$+\lambda E(I(I-1))\left[\left(-1-\alpha E(R)\right)+\overline{G}_1[\alpha]\overline{G}_2[\alpha]\left(1+\alpha E(R)\right)\right] \qquad (4.86)$$

4.6.3 No System Breakdowns and Exponential Vacation Time

In this case we assume that the system is not subject to breakdowns, i.e. $\alpha = 0$ and hence $R_q(z) = 0$. Also we assume an exponential distribution for vacation times. Using these assumptions in the main results of the chapter we get

$$P_q^{(1)}(z)=$$
$$\frac{(\lambda-\lambda C(z)+\gamma)\left(1-\overline{G}_1[\lambda-\lambda C(z)]\right)Q}{\left[(\lambda-\lambda C(z)+\gamma)-p(\lambda-\lambda C(z))\right]\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z(\lambda-\lambda C(z)+\gamma)} \qquad (4.87)$$

$$P_q^{(2)}(z)=$$
$$\frac{(\lambda-\lambda C(z)+\gamma)\overline{G}_1[\lambda-\lambda C(z)]\left(1-\overline{G}_2[\lambda-\lambda C(z)]\right)Q}{\left[(\lambda-\lambda C(z)+\gamma)-p(\lambda-\lambda C(z))\right]\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z(\lambda-\lambda C(z)+\gamma)} \qquad (4.88)$$

$$V_q(z)=$$
$$\frac{p(\lambda-\lambda C(z))\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]Q}{\left[(\lambda-\lambda C(z)+\gamma)-p(\lambda-\lambda C(z))\right]\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z(\lambda-\lambda C(z)+\gamma)} \qquad (4.89)$$

$$W_q(z)=P_q^{(1)}(z)+P_q^{(2)}(z)+V_q(z)=$$
$$\frac{\left\{\left[p(\lambda-\lambda C(z))-(\lambda-\lambda C(z)+\gamma)\right]\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]+(\lambda-\lambda C(z)+\gamma)\right\}Q}{\left[(\lambda-\lambda C(z)+\gamma)-p(\lambda-\lambda C(z))\right]\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z(\lambda-\lambda C(z)+\gamma)} \qquad (4.90)$$

To find Q, if we let $\alpha = 0$ in (4.58), we get 0/0 form, hence we apply L'Hopital's rule on (4.58), and obtain

$$Q=1-\lambda E(I)\left(\frac{p}{\gamma}+E(S_1)+E(S_2)\right) \qquad (4.91)$$

where $\overline{G}_1[0]=1$, $\overline{G}_2[0]=1$, $-\overline{G_1}'[0]=E(S_1)$ is the mean for the first service time, and $-\overline{G_2}'[0]=E(S_2)$ is the mean for the second service time.

Further, we compute $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ using the expression obtained for $W_q(z)$ in (4.90)

$$N'(1)=-\lambda E(I)Q\left[\gamma\left(E(S_1)+E(S_2)\right)+p\right] \qquad (4.92)$$

$$\begin{aligned} N''(1) &= 2Q(\lambda E(I))^2(1-p)(E(S_1)+E(S_2)) \\ &\quad -\gamma Q(\lambda E(I))^2\left(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\right) \\ &\quad -\lambda QE(I(I-1))[\gamma(E(S_1)+E(S_2))+p] \end{aligned} \tag{4.93}$$

$$D'(1) = \gamma\lambda E(I)(E(S_1)+E(S_2))+\lambda pE(I)-\gamma \tag{4.94}$$

$$\begin{aligned} D''(1) &= 2\lambda E(I)-2(1-p)(\lambda E(I))^2(E(S_1)+E(S_2)) \\ &\quad +\gamma(\lambda E(I))^2\left(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\right) \\ &\quad +\lambda E(I(I-1))[\gamma(E(S_1)+E(S_2))+p] \end{aligned} \tag{4.95}$$

where $\overline{G}_1''[0]=E(S_1^2)$ and $\overline{G}_2''[0]=E(S_2^2)$ are the second moment of the first service time and the second service time, respectively. Using equations (4.92) – (4.95) in equation (4.60) we can easily find L_q, and hence W_q.

The results obtained in this special case agree with the results obtained by Madan (2000b).

4.6.4 No Server Vacations, No System Breakdowns

In this case the server has no option to take a vacation and the system will never break down, hence $V_q(z)=0=R_q(z)$. Letting $p=0$ in the results obtained in (4.87) – (4.91) we get the following results for the simple batch arrival queueing system with two stages of service

$$P_q^{(1)}(z)=\frac{\left(1-\overline{G}_1[\lambda-\lambda C(z)]\right)Q}{\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z} \tag{4.96}$$

$$P_q^{(2)}(z)=\frac{\overline{G}_1[\lambda-\lambda C(z)]\left(1-\overline{G}_2[\lambda-\lambda C(z)]\right)Q}{\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z} \tag{4.97}$$

$$W_q(z)=P_q^{(1)}(z)+P_q^{(2)}(z) \tag{4.98}$$

$$Q=1-\lambda E(I)(E(S_1)+E(S_2)) \tag{4.99}$$

Equations (4.96) and (4.97) agree with equations (56) and (57) derived by Madan (2000b). Now, using equations (4.96) – (4.97), the probability generating function for the queue size takes the form

$$W_q(z)=\frac{\left(1-\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]\right)\left[1-\lambda E(I)(E(S_1)+E(S_2))\right]}{\overline{G}_1[\lambda-\lambda C(z)]\overline{G}_2[\lambda-\lambda C(z)]-z} \tag{4.100}$$

Further, we let $p=0$ in equations (4.92) – (4.95) to find $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$, and obtain

$$N'(1) = -\lambda E(I) Q\gamma\big(E(S_1)+E(S_2)\big) \tag{4.101}$$

$$\begin{aligned} N''(1) = {} & 2Q(\lambda E(I))^2\big(E(S_1)+E(S_2)\big) \\ & - \gamma Q(\lambda E(I))^2\big(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\big) \\ & - \lambda Q E(I(I-1))\gamma\big(E(S_1)+E(S_2)\big) \end{aligned} \tag{4.102}$$

$$D'(1) = \gamma\lambda E(I)\big(E(S_1)+E(S_2)\big) - \gamma \tag{4.103}$$

$$\begin{aligned} D''(1) = {} & 2\lambda E(I) - 2(\lambda E(I))^2\big(E(S_1)+E(S_2)\big) \\ & + \gamma(\lambda E(I))^2\big(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\big) \\ & + \lambda E(I(I-1))\gamma\big(E(S_1)+E(S_2)\big) \end{aligned} \tag{4.104}$$

Substituting the above derivatives in (4.60) and simplifying, we get the mean queue size in the following closed form

$$L_q = \frac{(\lambda E(I))^2\big(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\big)+\lambda E(I(I-1))\big(E(S_1)+E(S_2)\big)}{2\big[1-\lambda E(I)\big(E(S_1)+E(S_2)\big)\big]} \tag{4.105}$$

and hence, using Little's formula discussed in chapter 1, the average waiting time in the queue is given by

$$W_q = \frac{\lambda(E(I))^2\big(E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)\big)+E(I(I-1))\big(E(S_1)+E(S_2)\big)}{2\big[1-\lambda E(I)\big(E(S_1)+E(S_2)\big)\big]} \tag{4.106}$$

4.7 A Numerical Illustration

In order to see the effect of different parameters of the system, especially the vacation and breakdown parameter, on various states of the server, the proportion of idle time, the utilization factor, the mean queue size and the mean waiting time, we compute some numerical results along with some graphs. For the sake of convenience, we use the first special case in this chapter where both the vacations and repair times have exponential distributions. Further, we assume that arrivals come to the system one by one, i.e., $E(I)=1$ and $E(I(I-1))=0$, with arrival rate $\lambda = 2$, and that both service times have exponential distributions with rates $\mu_1 = 8$ and $\mu_2 = 16$.

To monitor how the breakdown rate α and the probability that the server takes a vacation p affect the behavior of the queueing model, we fix the values of β and γ to be 10 and 7, respectively, while α varies from 1 to 4 and p takes the values

0.25, 0.5, and 0.75. These numerical illustrations are given in Table 4.1. All parameters were selected such that the steady state condition is satisfied.

Table 4.1 *Computed values of various queue characteristics for vacation queue with breakdown and two-stage service,* $\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$

α	p	Q	ρ	L_q	W_q	$P_q^{(1)}(1)$	$P_q^{(2)}(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
1	0.25	0.499	0.501	0.4569	0.2285	0.2656	0.1249	0.0714	0.0391	0.501
1	0.5	0.4276	0.5724	0.6464	0.3232	0.2656	0.125	0.1429	0.0391	0.5726
1	0.75	0.3561	0.6439	0.9115	0.4558	0.2656	0.1249	0.2143	0.0391	0.6439
2	0.25	0.4411	0.5589	0.6968	0.3484	0.2812	0.125	0.0714	0.0812	0.5588
2	0.5	0.3697	0.6303	0.9653	0.4827	0.2813	0.125	0.1429	0.0813	0.6305
2	0.75	0.2982	0.7018	1.362	0.681	0.2812	0.125	0.2143	0.0812	0.7017
3	0.25	0.3802	0.6198	1.0344	0.5172	0.2969	0.125	0.0714	0.1266	0.6199
3	0.5	0.3087	0.6913	1.4371	0.7186	0.2968	0.125	0.1428	0.1265	0.6911
3	0.75	0.2373	0.7627	2.0828	1.0414	0.2968	0.125	0.2143	0.1265	0.7626
4	0.25	0.3161	0.6839	1.5359	0.768	0.3124	0.125	0.0714	0.175	0.6838
4	0.5	0.2447	0.7553	2.1955	1.0978	0.3124	0.125	0.1429	0.175	0.7553
4	0.75	0.1733	0.8267	3.399	1.6995	0.3125	0.125	0.2143	0.175	0.8268

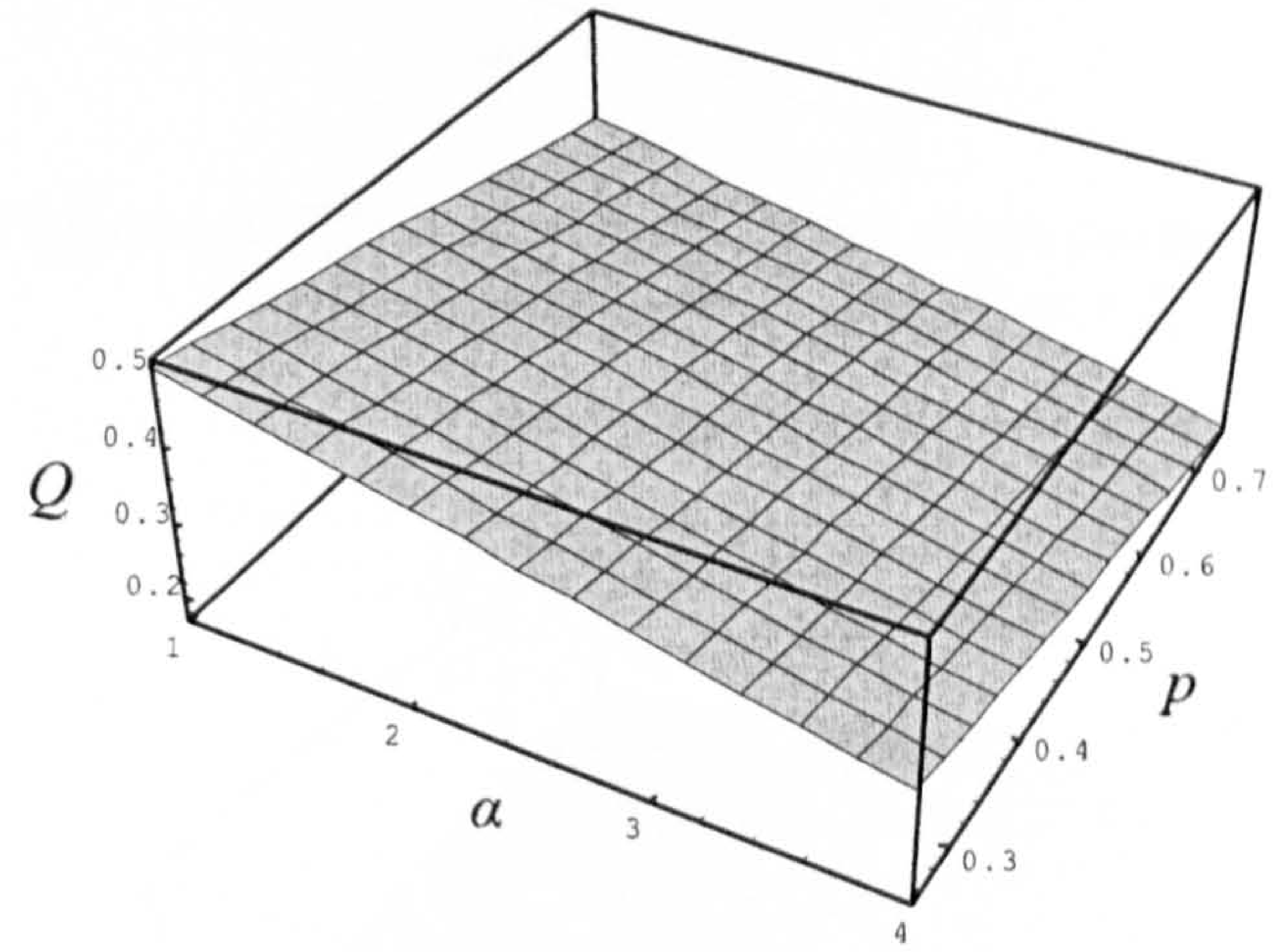

Figure 4.1 *Effect of* α *and* p *on the proportion of time that the server is idle* Q ($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

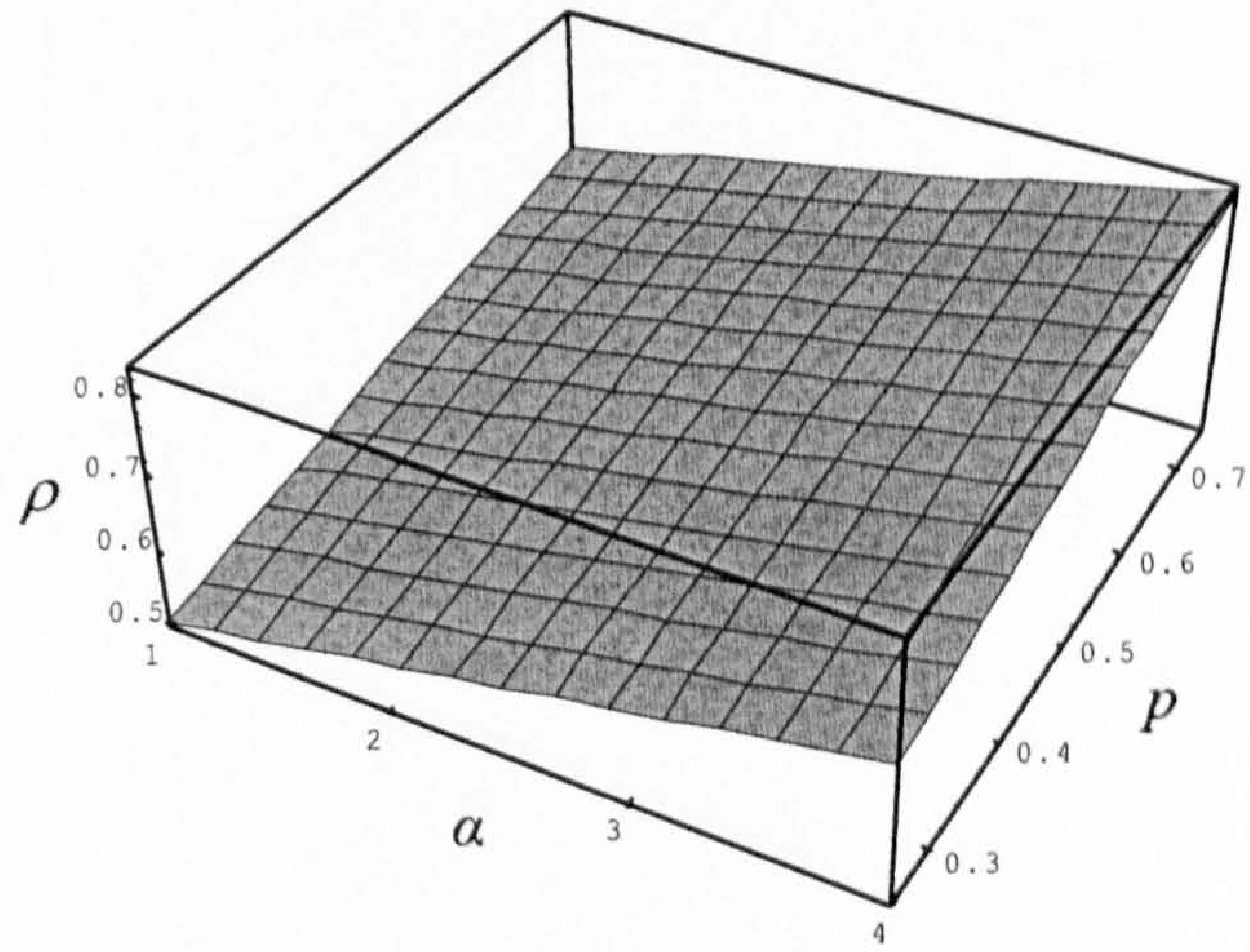

Figure 4.2 *Effect of α and p on the utilization factor ρ*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

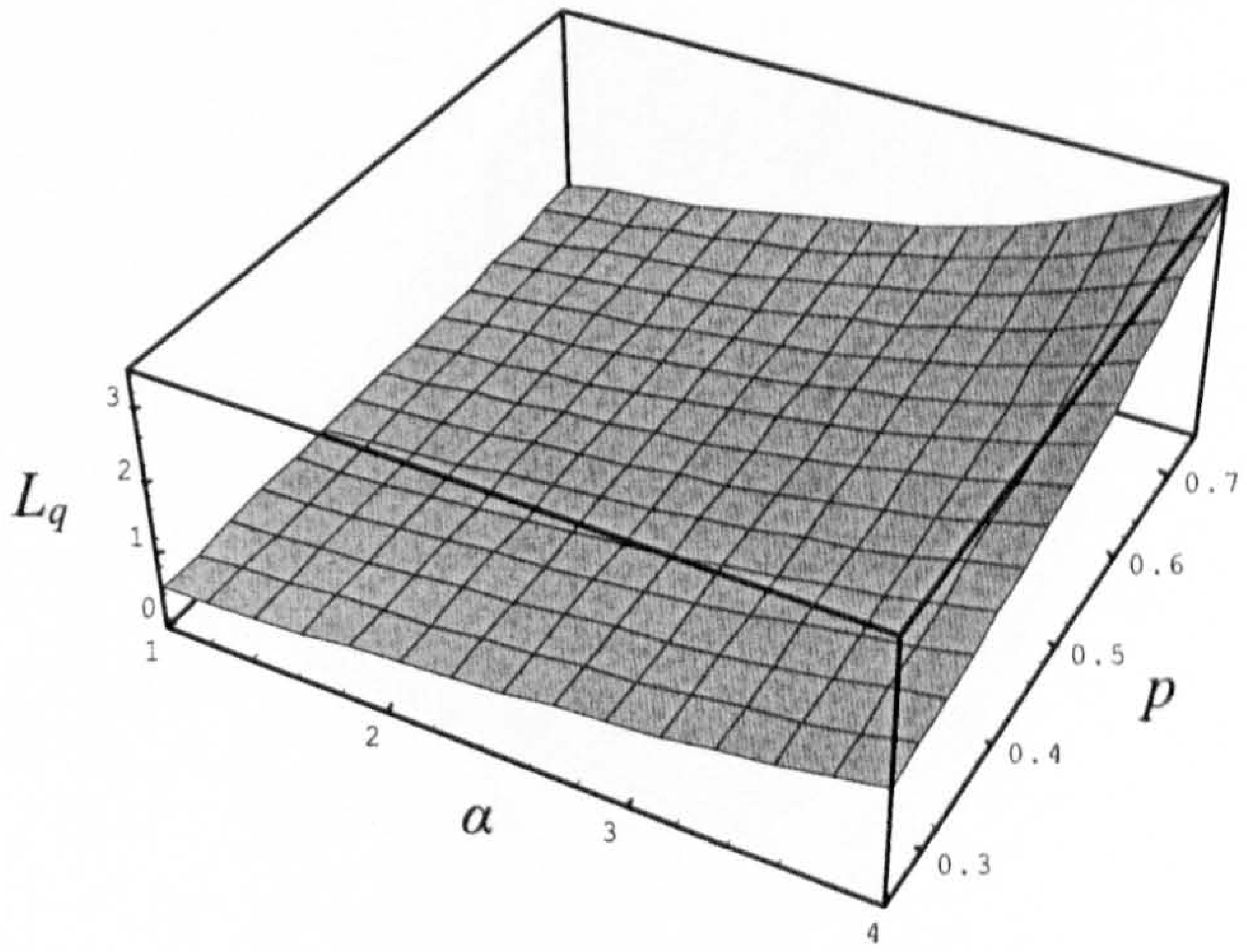

Figure 4.3 *Effect of α and p on the mean queue size L_q*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

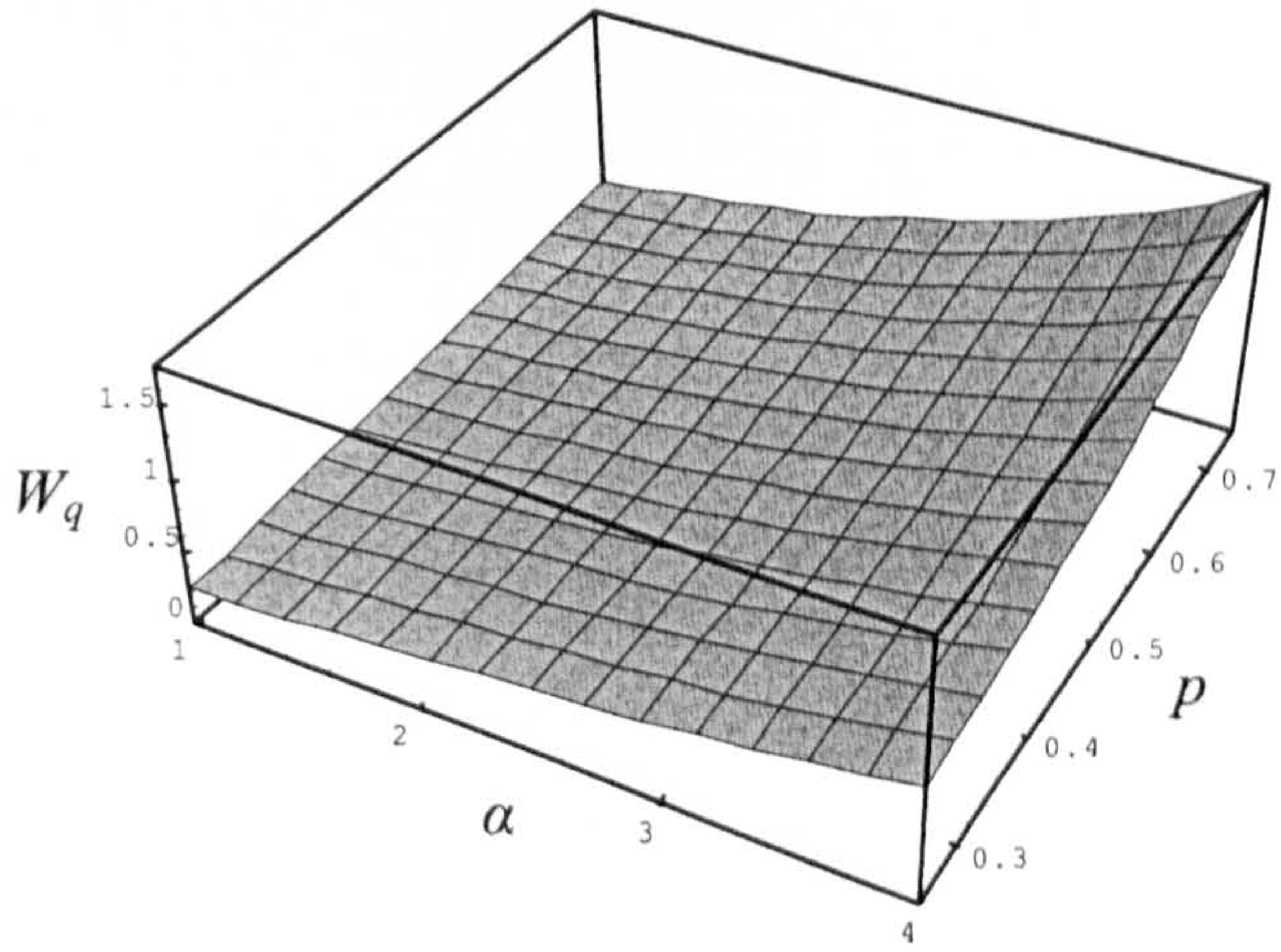

Figure 4.4 *Effect of α and p on the mean waiting time W_q*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

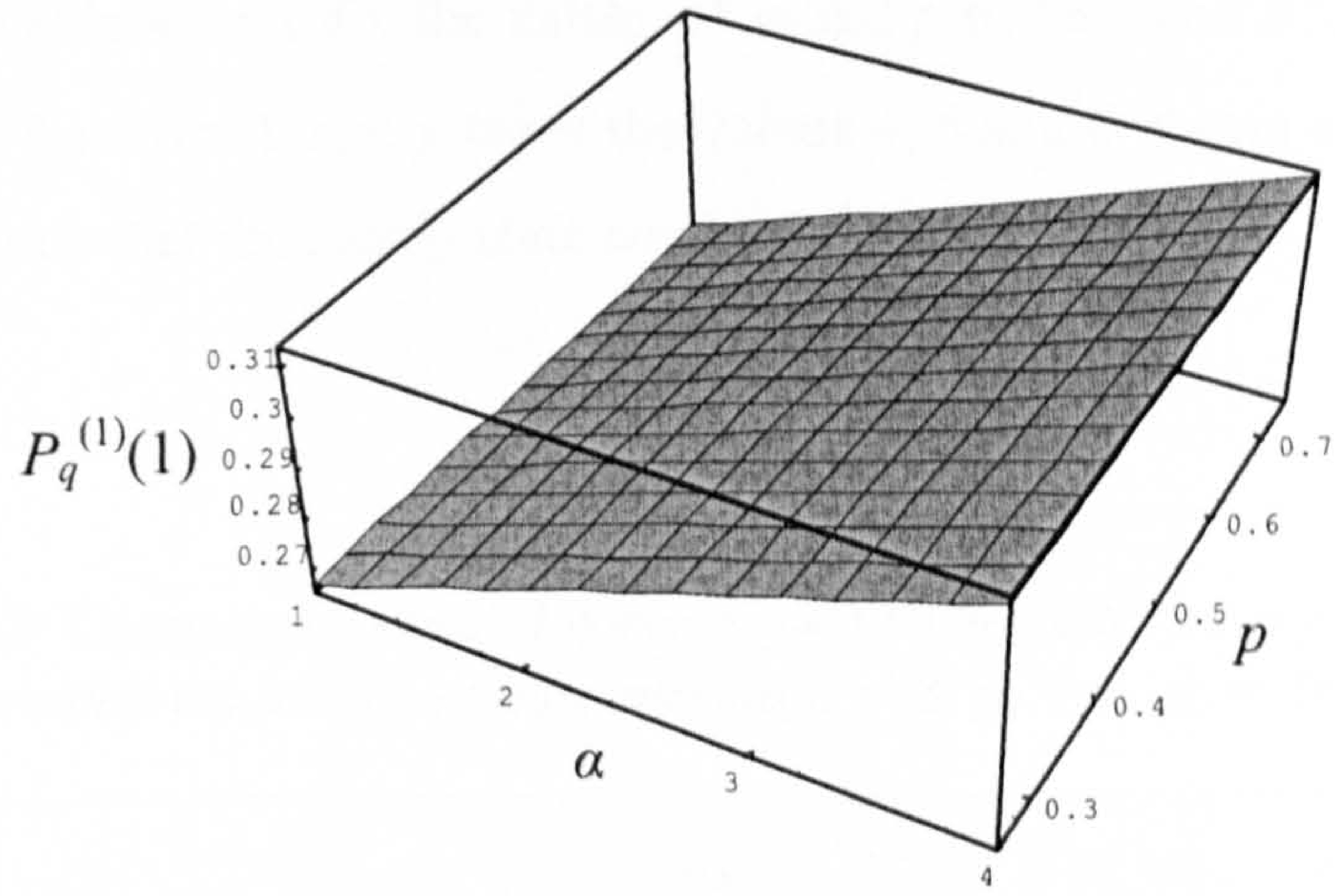

Figure 4.5 *Effect of α & p on the probability that the server is providing the first stage of service* ($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

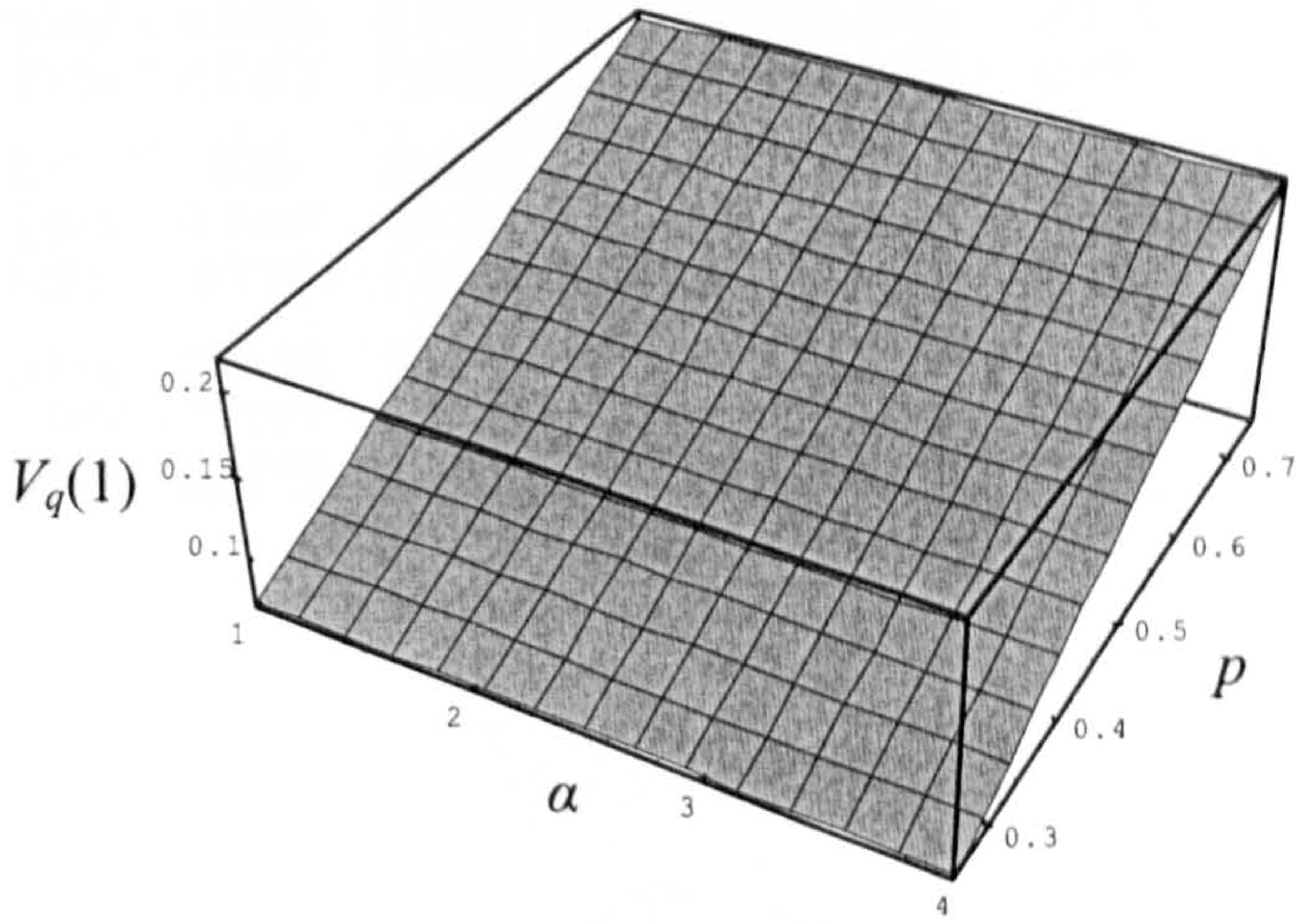

Figure 4.6 *Effect of α and p on the probability that the server is on vacation* ($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

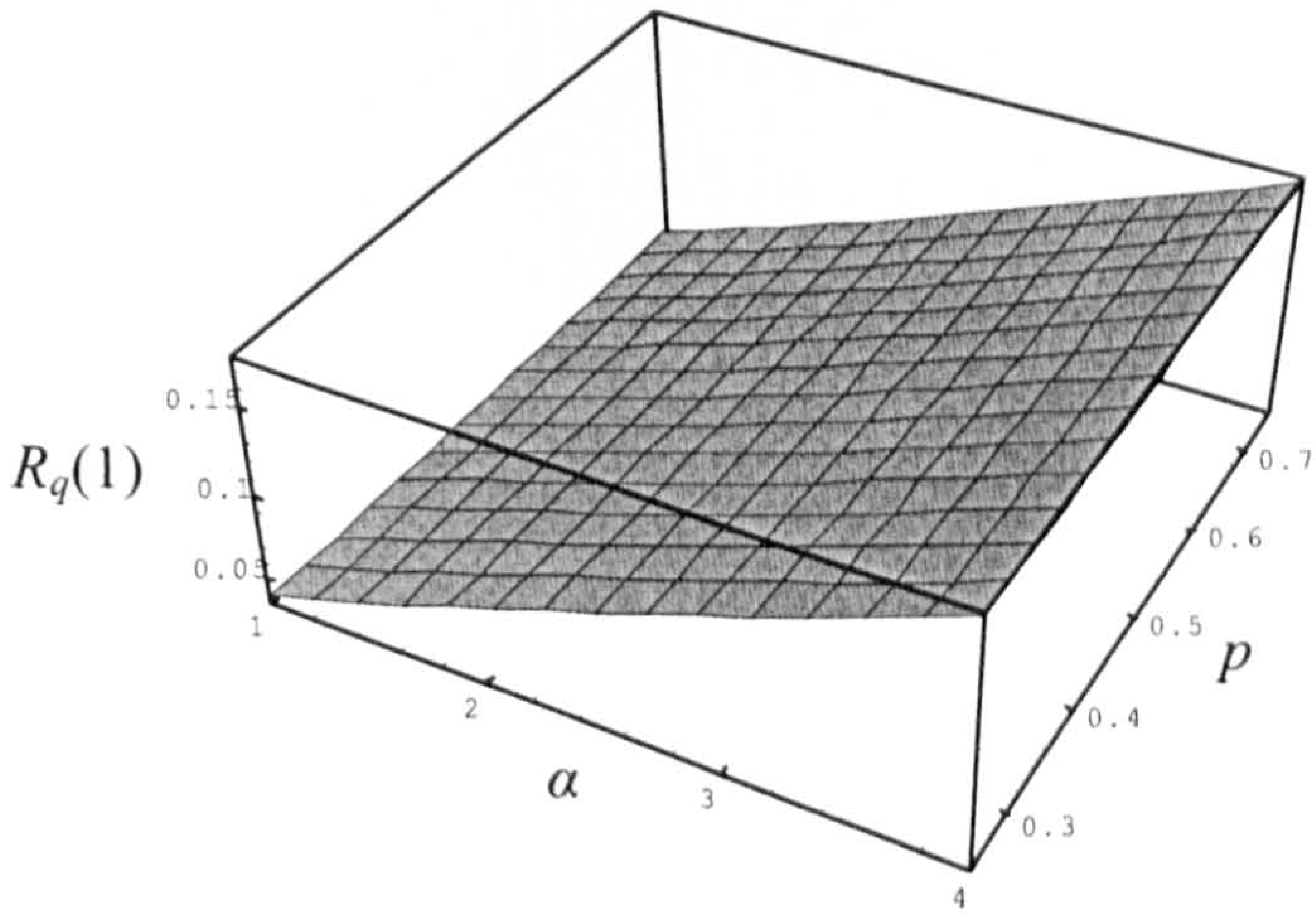

Figure 4.7 *The effect of α and p on the probability that the system is under repair* ($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\beta = 10$, $\gamma = 7$)

Now, in Table 4.2 we fix the values of α and p to be 3 and 0.5, respectively, and let β vary from 6 to 9 and γ takes the values 4, 6 and 8. Again all parameters were selected such that the steady state condition is satisfied.

Table 4.2 *Computed values of various queue characteristics for vacation queue with breakdown and two-stage service,* $\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$

β	γ	Q	ρ	L_q	W_q	$P_q^{(1)}(1)$	$P_q^{(2)}(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
6	4	0.1172	0.8828	6.2413	3.1207	0.2968	0.125	0.2499	0.2109	0.8826
6	6	0.2006	0.7994	3.1278	1.5639	0.2969	0.125	0.1667	0.2109	0.7995
6	8	0.2422	0.7578	2.4162	1.2081	0.2968	0.125	0.125	0.2109	0.7577
7	4	0.1474	0.8526	4.565	2.2825	0.2969	0.125	0.2501	0.1808	0.8528
7	6	0.2307	0.7693	2.4621	1.2311	0.2968	0.125	0.1667	0.1808	0.7693
7	8	0.2724	0.7276	1.9319	0.966	0.2969	0.125	0.125	0.1808	0.7277
8	4	0.17	0.83	3.7074	1.8537	0.2969	0.125	0.25	0.1582	0.8301
8	6	0.2533	0.7467	2.0743	1.0372	0.2968	0.125	0.1667	0.1582	0.7467
8	8	0.295	0.705	1.6395	0.8198	0.2969	0.125	0.125	0.1582	0.7051
9	4	0.1875	0.8125	3.1879	1.594	0.2968	0.125	0.2499	0.1406	0.8123
9	6	0.2709	0.7291	1.8219	0.911	0.2969	0.125	0.1667	0.1406	0.7292
9	8	0.3125	0.6875	1.4446	0.7223	0.2968	0.125	0.125	0.1406	0.6874

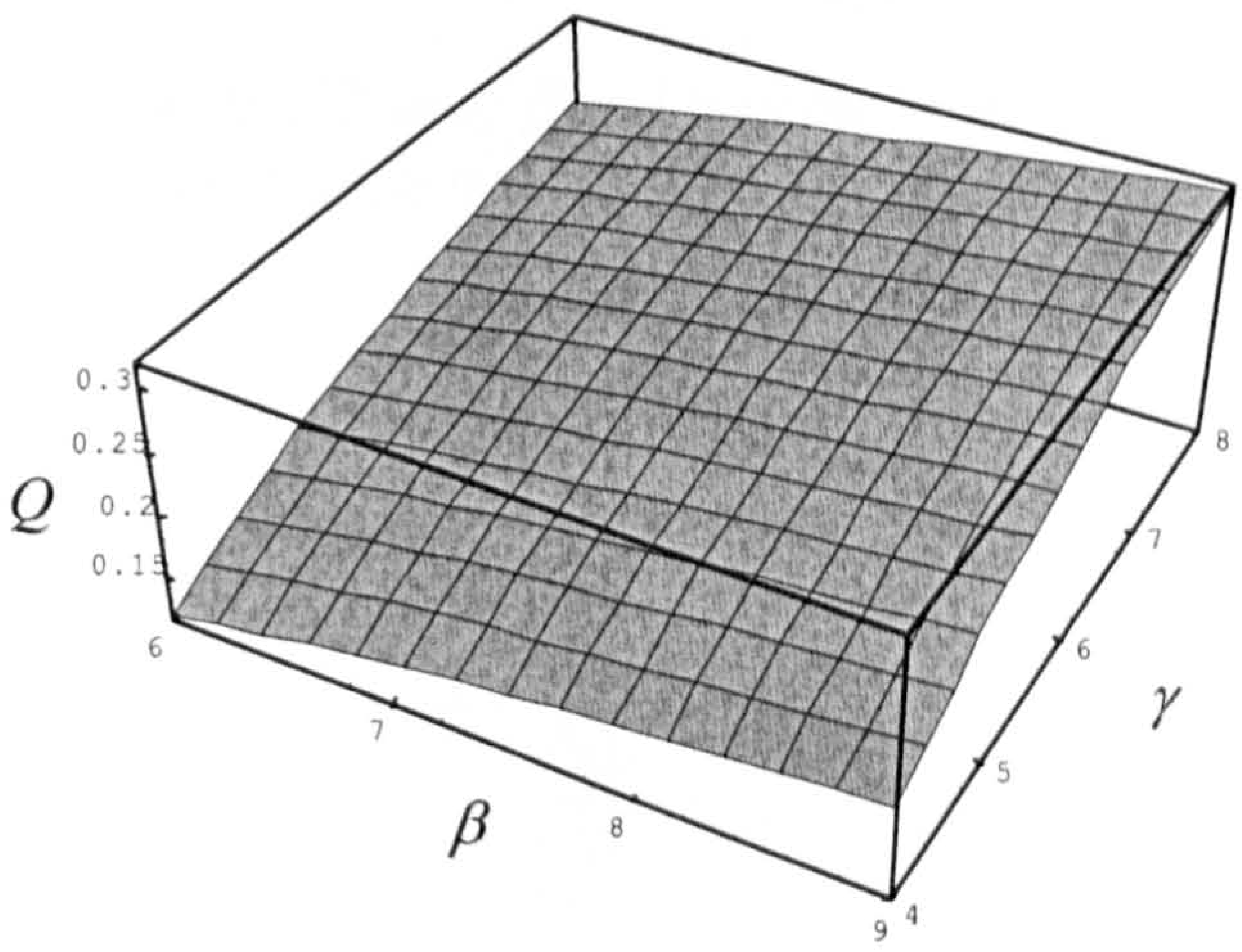

Figure 4.8 *Effect of* β *and* γ *on the proportion of time that the server is idle* Q ($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)

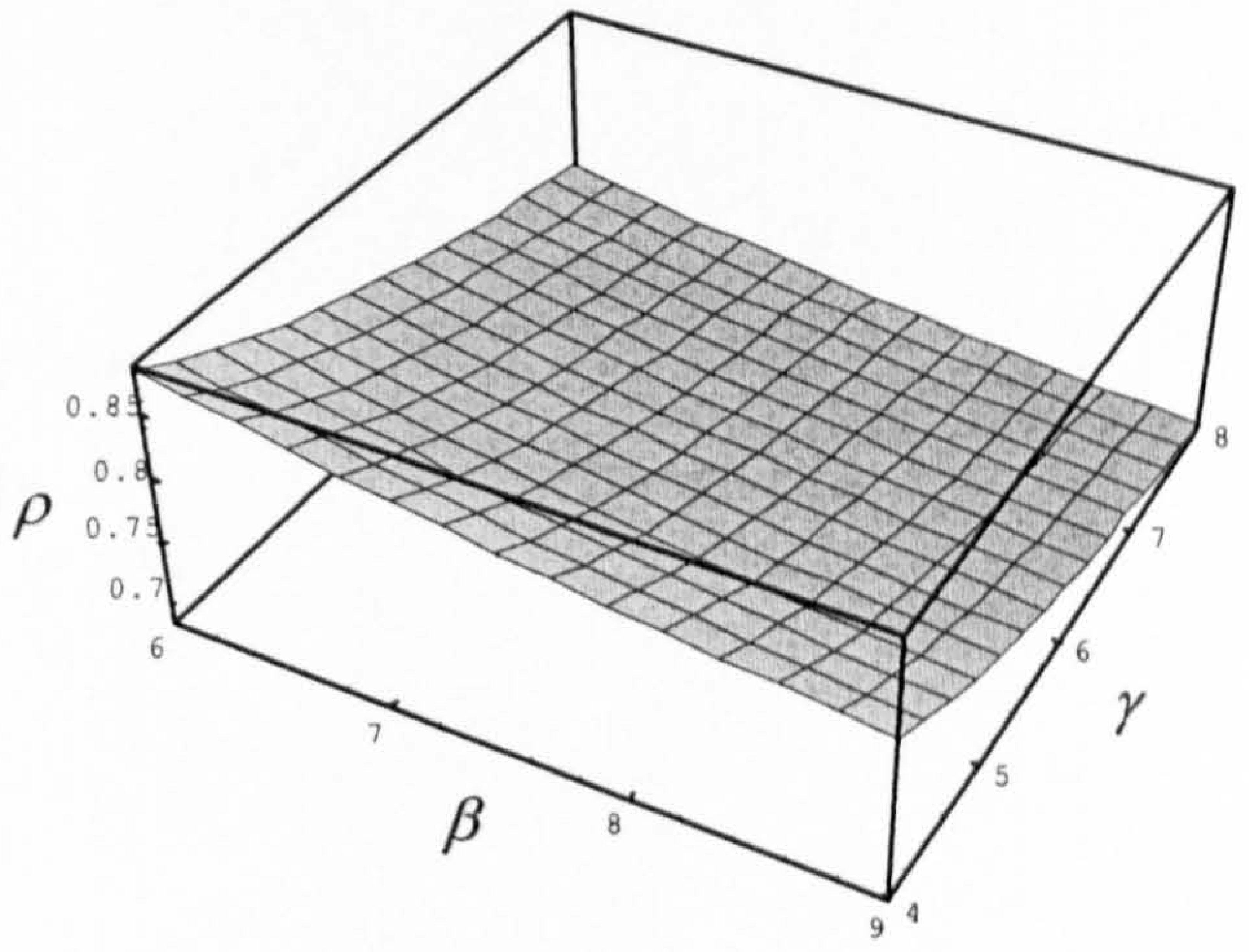

Figure 4.9 *Effect of β and γ on the utilization factor ρ*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)

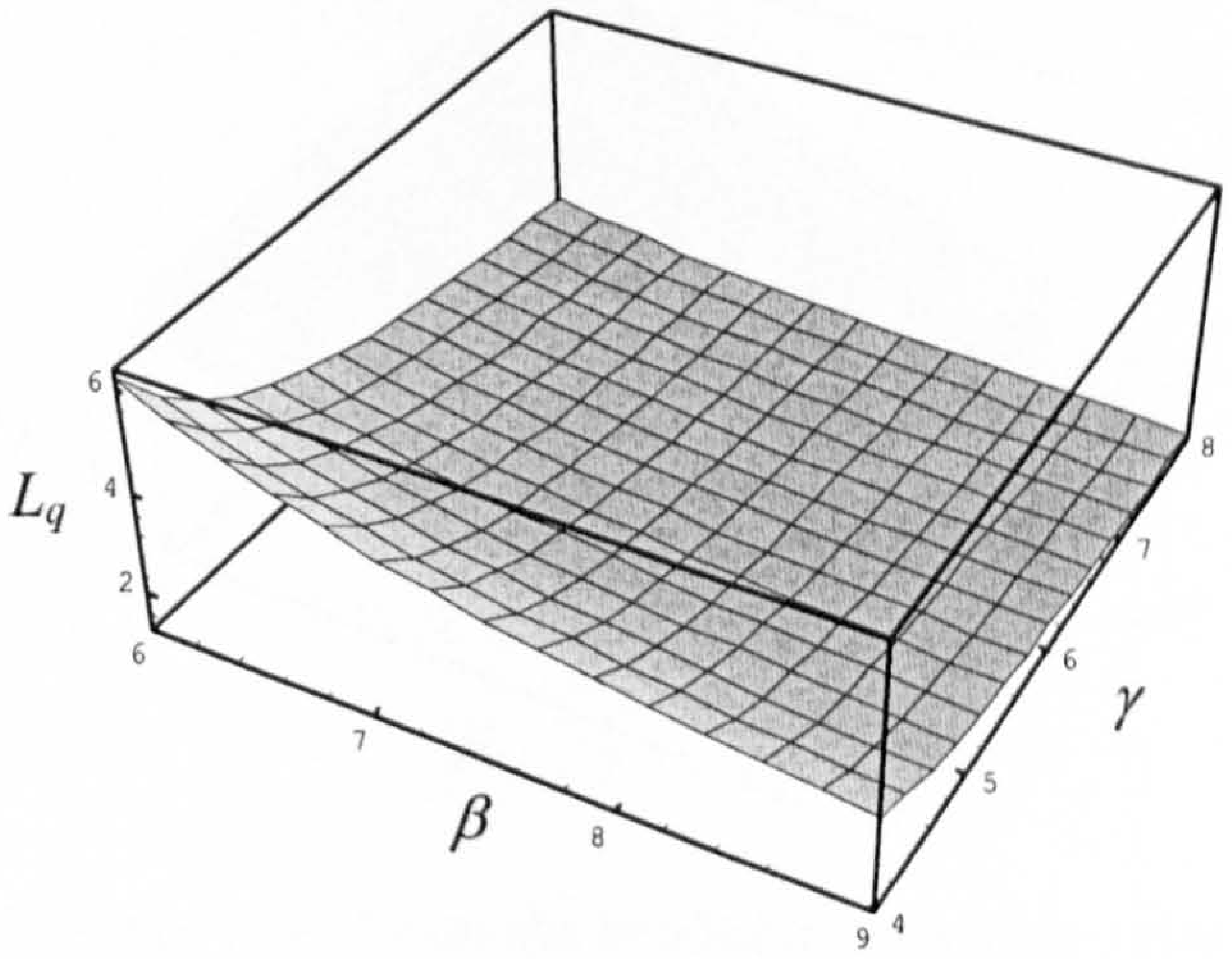

Figure 4.10 *Effect of β and γ on the mean queue size L_q*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)

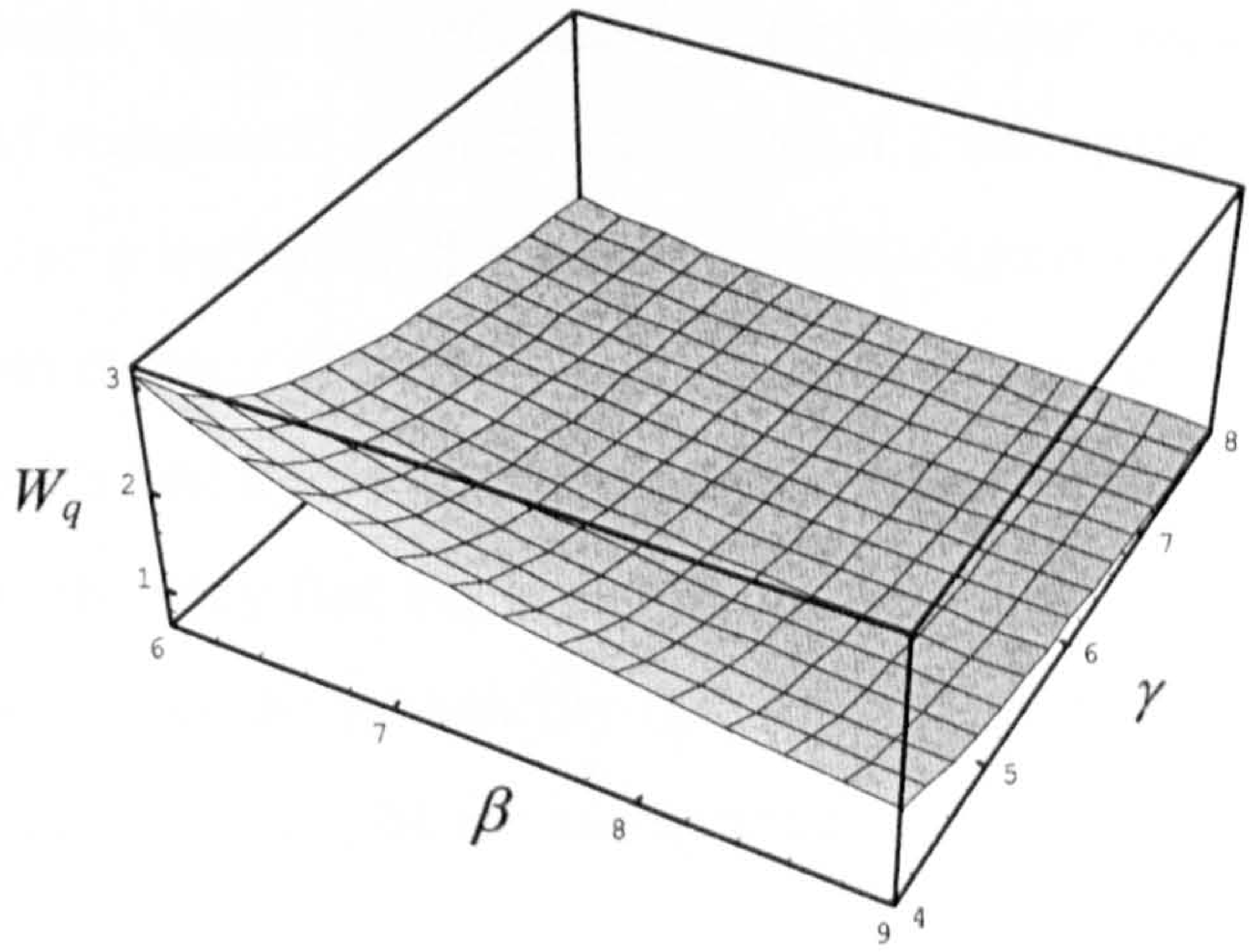

Figure 4.11 *Effect of β and γ on the mean waiting time W_q*
($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)

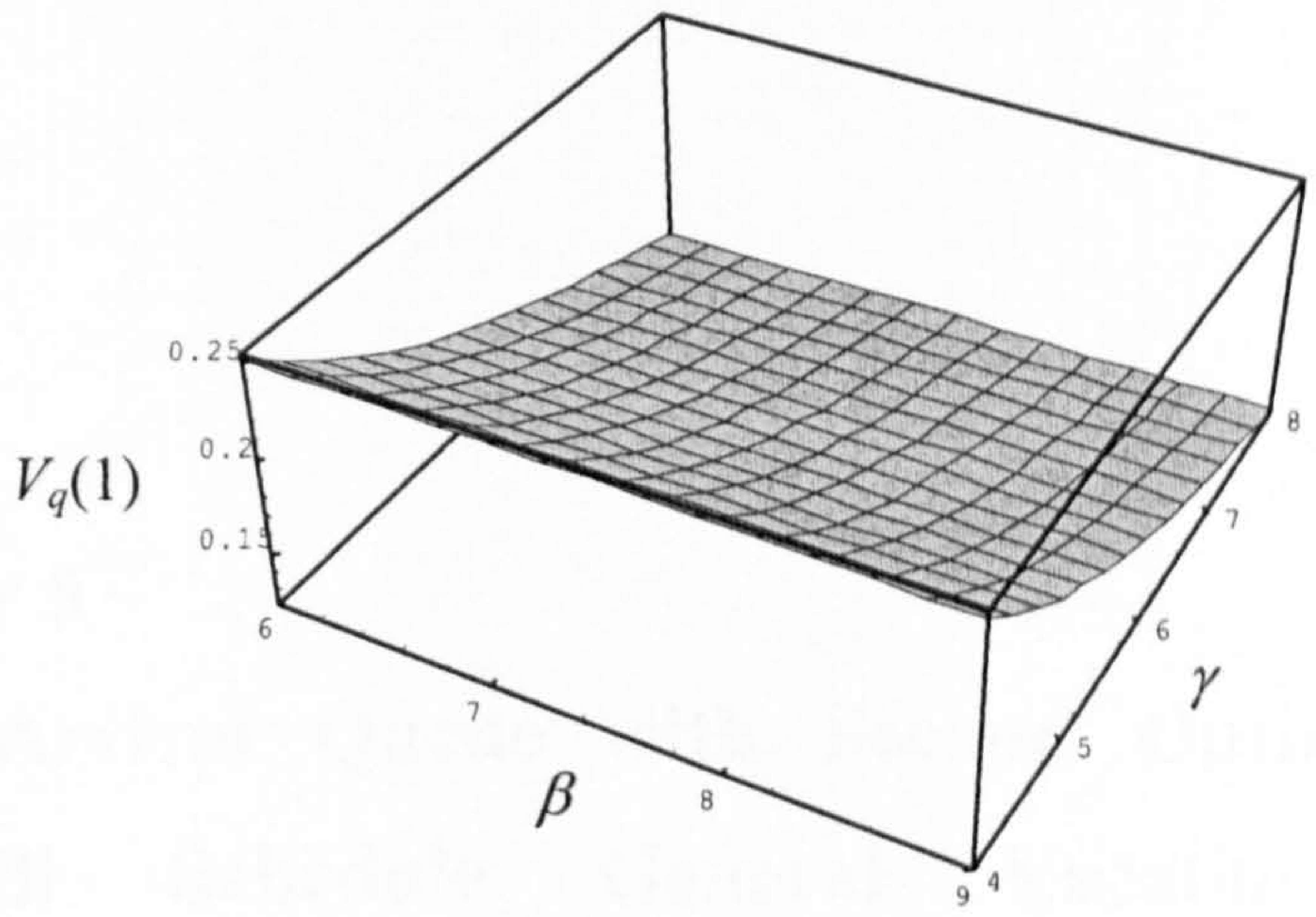

Figure 4.12 *Effect of β and γ on the probability that the server is on vacation* *($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)*

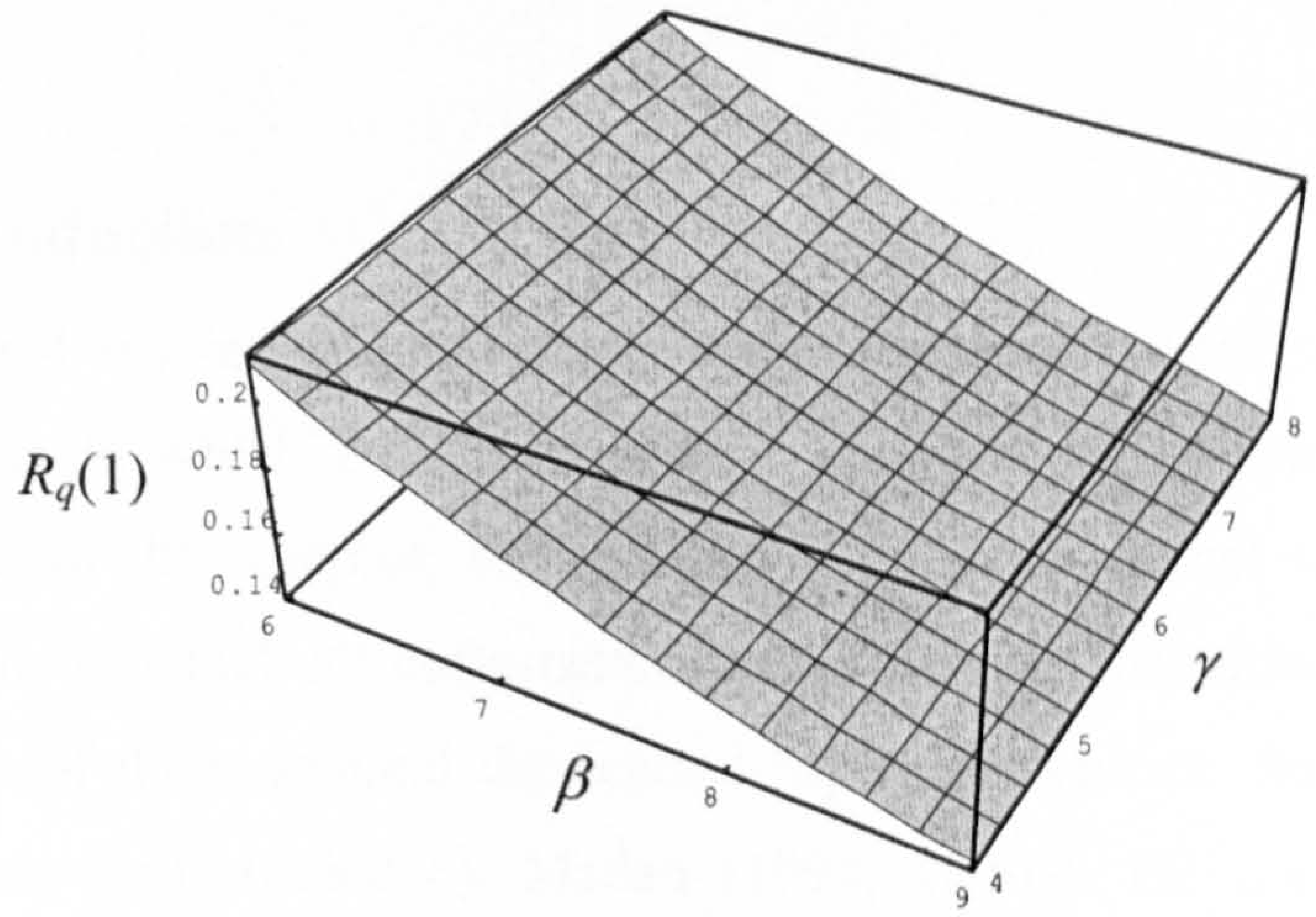

Figure 4.13 *Effect of β and γ on the probability that the system is under repair* *($\lambda = 2$, $\mu_1 = 8$, $\mu_2 = 16$, $\alpha = 3$, $p = 0.5$)*

Table 4.1 and graphs 4.1 to 4.4 clearly show that as α or p increases, the server idle time decreases, while the utilization factor, the mean queue size and the mean waiting time of customers, all increase. Table 4.2 and graphs 4.8 to 4.11 clearly show that as β or γ increases, the server idle time increases, while the utilization factor, the mean number of customers in the queue and the mean waiting time all decrease. Also the tables and graphs show that either increasing p or decreasing γ increases the probability that the server is on vacation. Similarly, increasing α or decreasing β increases the probability that the system is under repair. The trends shown by the tables and graphs are as expected.

* * *

Chapter 5

Batch Arrival Queue with Second Optional Service, Bernoulli Schedule General Vacations, Random Breakdowns and General Repair Times

5.1 Introduction

In chapter 4 we investigated a two-stage queueing system with vacations and breakdowns in which both service were essential and provided to all arriving customers. In this chapter, we consider a queueing model with vacations and breakdowns in which all customers demand the first "essential" service, whereas only some of them demand the second "optional" service. Some aspects of this model were first studied by Madan (1994; 2000a). He cited some important applications of this model in many real life situations. Some of these applications are mentioned as follows:

- At a gifts shop, every customer may need to buy a gift, but only some of these customers may ask for wrapping.
- In a small town one finds many shops, which sell coffee beans and grains of various kinds. All such shop-keepers normally have a grinding machine. All customers coming to such a shop buy grains or coffee beans, but only some of these customers want to utilize the grinding facility.
- All passengers wish to travel to a big town or a metropolitan city on a particular airline, but only some of these customers take the airline's further flight to an interior destination of tourist's interest.
- All students joining a particular department of a university want to complete their undergraduate program of study, but only some of them may join the

postgraduate programme soon after completing the undergraduate programme.

- All ships arriving at a port may need an unloading service on arrival, but only some of them may require a re-loading service soon after the unloading.

Choudhury (2003b) derived the queue size distribution at a stationary point of time and waiting time distribution for an *M*/*G*/1 queueing system with optional second service where both services are provided by the same server. In another study, he assumed that the optional service is provided in an additional service channel in which case another customer at the head of queue is taken up for the first essential service (Choudhury, 2003a). Later, this work was generalised to a batch arrival queueing system with the second optional service channel under *D*-policy where the server is turned off after the service facility becomes free, and it is turned on when the cumulative service times of the primary customer at the service facility exceed the level *D* (Choudhury, 2005), and under *N*-policy where the server is turned on when the queue size reaches the number *N* (Choudhury & Paul, 2006).

The queueing system with a second optional service was further investigated. Madan and Baklizi (2002) and Choudhury and Paul (2005) studied this model with the assumption that after receiving the first phase or second phase of unsuccessful service, a customer may immediately join the end of the original queue as feed back customer to have another regular service. Artalejo and Choudhury (2004) and Atencia and Moreno (2006) studied retrial queues with an additional second phase of service.

Although queues with a second optional service have been studied in various forms by numerous authors, one important fact has been overlooked is that perfectly reliable servers are virtually nonexistent. There might be times of unavailability of service due to scheduled vacations or unexpected breakdowns.

To the best of our knowledge, very few authors considered server vacations or breakdowns in queues with a second optional service. Madan, Abu-Dayyeh and Saleh (2002) studied *M*/*G*/1 queue with second a optional service where after completion of a service, the server takes Bernoulli schedule vacations. They assumed a general distribution for the first essential service times and exponential distributions for both the optional service times and vacation times. They obtained time dependent probability generating functions and their corresponding steady state results.

On the other hand, Kumar, Arivudainambi & Vijayakumar (2002) investigated an $M/G/1$ queue with second optional service and server breakdowns, where breakdowns may occur with a fixed rate while servicing customers in either phase. They assumed that the customer leaves the system upon breakdown. All time periods were generally distributed in this study.

Wang (2004) obtained transient and steady state solutions for both the queueing and reliability measures of a queueing model similar to the one considered by Kumar, Arivudainambi & Vijayakumar (2002) with the following differences: the breakdown rates are different for the two phases of service, exponential distribution for the second optional service times, and the customer just being served before server breakdown waits for the server to complete its remaining service.

In this chapter we consider a batch arrival queueing system with a single server, a second optional service, Bernoulli schedule server vacation and random breakdowns. We assume general (arbitrary) distributions for the first essential service time, second optional service time, vacation time, and repair time. This extends the work done by Madan, Abu-Dayyeh and Saleh (2002) by adding the assumption of breakdowns to their queueing model, generalising the second optional service time and the vacation time and considering batch arrival. The current chapter also extends the work done by Kumar, Arivudainambi & Vijayakumar (2002) and Wang (2004) by adding the assumption of server vacation to the queueing model. The model of this chapter generalises the classical $M^{[X]}/G/1$ queue with Bernoulli vacations, the $M^{[X]}/G/1$ queue with random breakdowns and the $M^{[X]}/G/1$ queue with a second optional service. The detailed assumptions underlying the queueing model considered in this chapter are presented in the following section.

5.2 The Mathematical Model

In this chapter, the assumptions of batch arrivals, vacations and breakdowns are similar to those considered in the previous chapter, but the introduction of an optional service makes the model differ from those considered earlier. The detailed assumptions of this chapter are as follows:

a) Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$ (i = 1, 2, 3, ...) to be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

b) There is a single server who provides both the first essential service and the second optional service. The first essential service is provided to all arriving customers. As soon as the essential service of a customer is completed, then with probability k, he may opt for the second service, in which case his second service will immediately commence or else with probability 1- k, he may opt to leave the system, in which case another customer at the head of the queue (if any) is taken up for his first essential service.

c) The service time of the two services (essential and optional) follow different general (arbitrary) distributions with distribution functions $G_j(s)$ and density function $g_j(s)$, j = 1, 2. Let $\mu_j(x)\mathrm{d}x$ be the conditional probability density of the completion of j^{th} service during the time interval $(x, x + \mathrm{d}x]$, given that the elapsed time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1 - G_j(x)}, \qquad j = 1,2 \tag{5.1}$$

and, therefore

$$g_j(s) = \mu_j(s) e^{-\int_0^s \mu_j(x)dx}, \qquad j = 1,2 \tag{5.2}$$

d) As soon as a service is completed, no matter whether it is the first essential service or the second optional service, the server takes a vacation with probability p, and decides to stay in the system with probability 1-p, where $0 \le p \le 1$. We further assume that whenever a customer requires a second optional service, the server is not allowed to take a vacation unless the optional service of the customer is completed.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{b(x)}{1 - B(x)} \tag{5.3}$$

and, therefore

$$b(v) = \gamma(v) e^{-\int_0^v \gamma(x)dx} \tag{5.4}$$

f) The system may break down at random while the first essential service is going on. It is assumed that the optional service never breaks down (e.g. a manual service). Breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$.

g) Once the system breaks down, the system enters a repair process immediately, and the customer whose service is interrupted comes back to the head of the queue waiting for the service to resume. The repair time follows a general (arbitrary) distribution with distribution function $\Phi(r)$ and density function $\varphi(r)$. Let $\beta(x)dx$ be the conditional probability of a repair completion during the interval $(x, x + dx]$ given that the elapsed repair time is x, so that

$$\beta(x) = \frac{\varphi(x)}{1 - \Phi(x)} \tag{5.5}$$

and, therefore

$$\varphi(r) = \beta(r) e^{-\int_0^r \beta(x)dx} \tag{5.6}$$

h) The stochastic processes involved in the system are assumed to be independent of each other.

5.3 Steady State Equations Governing the System

We assume that the steady state solutions exist. Accordingly, we define the following limits as the corresponding steady state probabilities.

$$\lim_{t\to\infty} P_n^{(e)}(x,t) = P_n^{(e)}(x), \quad \lim_{t\to\infty} P_n^{(e)}(t) = \lim_{t\to\infty} \int_0^\infty P_n^{(e)}(x,t)dx = P_n^{(e)}$$

$$\lim_{t\to\infty} P_n^{(o)}(x,t) = P_n^{(o)}(x), \quad \lim_{t\to\infty} P_n^{(o)}(t) = \lim_{t\to\infty} \int_0^\infty P_n^{(o)}(x,t)dx = P_n^{(o)}$$

$$\lim_{t\to\infty} V_n(x,t) = V_n(x), \quad \lim_{t\to\infty} V_n(t) = \lim_{t\to\infty} \int_0^\infty V_n(x,t)dx = V_n$$

$$\lim_{t\to\infty} R_n(x,t) = R_n(x), \quad \lim_{t\to\infty} R_n(t) = \lim_{t\to\infty} \int_0^\infty R_n(x,t)dx = R_n$$

$$\lim_{t\to\infty} Q(t) = Q$$

Then, we formulate the equations governing the system in the same manner we did in previous chapters, and get the following set of steady state differential equations

$$\frac{d}{dx}P_n^{(e)}(x) + (\lambda + \mu_1(x) + \alpha)P_n^{(e)}(x) = \lambda \sum_{l=1}^{n-1} c_l P_{n-l}^{(e)}(x), \qquad n \geq 1 \tag{5.7}$$

$$\frac{d}{dx}P_0^{(e)}(x) + (\lambda + \mu_1(x) + \alpha)P_0^{(e)}(x) = 0 \tag{5.8}$$

$$\frac{d}{dx}P_n^{(o)}(x) + (\lambda + \mu_2(x))P_n^{(o)}(x) = \lambda \sum_{l=1}^{n-1} c_l P_{n-l}^{(o)}(x), \qquad n \geq 1 \tag{5.9}$$

$$\frac{d}{dx}P_0^{(o)}(x) + (\lambda + \mu_2(x))P_0^{(o)}(x) = 0 \tag{5.10}$$

$$\frac{d}{dx}V_n(x) + (\lambda + \gamma(x))V_n(x) = \lambda \sum_{l=1}^{n-1} c_l V_{n-l}(x), \qquad n \geq 1 \tag{5.11}$$

$$\frac{d}{dx}V_0(x) + (\lambda + \gamma(x))V_0(x) = 0 \tag{5.12}$$

$$\frac{d}{dx}R_n(x) + (\lambda + \beta(x))R_n(x) = \lambda \sum_{l=1}^{n-1} c_l R_{n-l}, \qquad n \geq 1 \tag{5.13}$$

$$\frac{d}{dx}R_0(x) + (\lambda + \beta(x))R_0(x) = 0 \tag{5.14}$$

$$\lambda Q = (1-p)\int_0^\infty P_0^{(o)}(x)\mu_2(x)dx + (1-k)(1-p)\int_0^\infty P_0^{(e)}(x)\mu_1(x)dx$$
$$+ \int_0^\infty V_0(x)\gamma(x)dx + \int_0^\infty R_0(x)\beta(x)dx \tag{5.15}$$

The above equations are to be solved subject to the boundary conditions

$$P_n^{(e)}(0) = (1-p)(1-k)\int_0^\infty P_{n+1}^{(e)}(x)\mu_1(x)dx + (1-p)\int_0^\infty P_{n+1}^{(o)}(x)\mu_2(x)dx$$

$$+\int_0^\infty R_{n+1}(x)\beta(x)dx + \int_0^\infty V_{n+1}(x)\gamma(x)dx + \lambda c_{n+1}Q, \quad n \ge 0 \tag{5.16}$$

$$P_n^{(o)}(0) = k\int_0^\infty P_n^{(e)}(x)\mu_1(x)dx, \quad n \ge 0 \tag{5.17}$$

$$V_n(0) = p(1-k)\int_0^\infty P_n^{(e)}(x)\mu_1(x)dx + p\int_0^\infty P_n^{(o)}(x)\mu_2(x)dx, \quad n \ge 0 \tag{5.18}$$

$$R_n(0) = \alpha\int_0^\infty P_{n-1}^{(e)}(x)dx, \quad n \ge 1 \tag{5.19}$$

$$R_0(0) = 0 \tag{5.20}$$

5.4 Queue Size Distribution at a Random Epoch

We define the following probability generating functions

$$P_q^{(e)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(e)}(x), \quad P_q^{(e)}(z) = \sum_{n=0}^{\infty} z^n P_n^{(e)},$$

$$P_q^{(o)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(o)}(x), \quad P_q^{(o)}(z) = \sum_{n=0}^{\infty} z^n P_n^{(o)},$$

$$V_q(x,z) = \sum_{n=0}^{\infty} z^n V_n(x), \quad V_q(z) = \sum_{n=0}^{\infty} z^n V_n,$$

$$R_q(x,z) = \sum_{n=0}^{\infty} z^n R_n(x), \quad R_q(z) = \sum_{n=0}^{\infty} z^n R_n,$$

$$C(z) = \sum_{i=1}^{\infty} z^i c_i \tag{5.21}$$

Now, we multiply the differential equations in (5.7) – (5.20) by appropriate powers of z, sum over all possible values of n and use the generating functions defined in (5.21), we obtain

$$\frac{d}{dx}P_q^{(e)}(x,z) + \left(\lambda - \lambda C(z) + \mu_1(x) + \alpha\right)P_q^{(e)}(x,z) = 0 \tag{5.22}$$

$$\frac{d}{dx}P_q^{(o)}(x,z) + \left(\lambda - \lambda C(z) + \mu_2(x)\right)P_q^{(o)}(x,z) = 0 \tag{5.23}$$

$$\frac{d}{dx}V_q(x,z) + \left(\lambda - \lambda C(z) + \gamma(x)\right)V_q(x,z) = 0 \tag{5.24}$$

$$\frac{d}{dx}R_q(x,z)+\left(\lambda-\lambda C(z)+\beta(x)\right)R_q(x,z)=0 \tag{5.25}$$

$$zP_q^{(e)}(0,z)=(1-p)\int_0^\infty P_q^{(o)}(x,z)\mu_2(x)dx+(1-p)(1-k)\int_0^\infty P_q^{(e)}(x,z)\mu_1(x)dx$$
$$+\int_0^\infty R_q(x,z)\beta(x)dx+\int_0^\infty V_q(x,z)\gamma(x)dx+\lambda(C(z)-1)Q \tag{5.26}$$

$$P_q^{(o)}(0,z)=k\int_0^\infty P_q^{(e)}(x,z)\mu_1(x)dx \tag{5.27}$$

$$V_q(0,z)=p\int_0^\infty P_q^{(o)}(x,z)\mu_2(x)dx+p(1-k)\int_0^\infty P_q^{(e)}(x,z)\mu_1(x)dx \tag{5.28}$$

$$R_q(0,z)=\alpha z\int_0^\infty P_q^{(e)}(x,z)dx \tag{5.29}$$

We solve the differential equations (5.22) – (5.25) by integrating between the limits 0 and x and obtain

$$P_q^{(e)}(x,z)=P_q^{(e)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_1(t)dt} \tag{5.30}$$

$$P_q^{(o)}(x,z)=P_q^{(o)}(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x\mu_2(t)dt} \tag{5.31}$$

$$V_q(x,z)=V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x\gamma(t)dt} \tag{5.32}$$

$$R_q(x,z)=R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x\beta(t)dt} \tag{5.33}$$

where $P_q^{(e)}(0,z)$, $P_q^{(o)}(0,z)$, $V_q(0,z)$ and $R_q(0,z)$ are given by (5.26), (5.27), (5.28) and (5.29), respectively. Again integrating equations (5.30) – (5.33) by parts with respect to x yields

$$P_q^{(e)}(z)=P_q^{(e)}(0,z)\left[\frac{1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{5.34}$$

$$P_q^{(o)}(z)=P_q^{(o)}(0,z)\left[\frac{1-\bar{G}_2[\lambda-\lambda C(z)]}{(\lambda-\lambda C(z))}\right] \tag{5.35}$$

$$V_q(z)=V_q(0,z)\left[\frac{1-\bar{B}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{5.36}$$

$$R_q(z)=R_q(0,z)\left[\frac{1-\bar{\Phi}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{5.37}$$

where $\bar{G}_1[\lambda-\lambda C(z)+\alpha]=\int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}\cdot dG_1(x)$, $\bar{G}_2[\lambda-\lambda C(z)+\alpha]=\int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x}\cdot dG_2(x)$, $\bar{B}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot dB(x)$ and $\bar{\Phi}[\lambda-\lambda C(z)]=\int_0^\infty e^{-(\lambda-\lambda C(z))x}\cdot d\Phi(x)$ are the Laplace-Stieltjes transform of the first essential service time, second optional service time, vacation time and repair time, respectively.

Now, we determine the integrals appearing in the right hand sides of equations (5.26) – (5.28). We multiply equations (5.30), (5.31), (5.32) and (5.33) by $\mu_1(x)$, $\mu_2(x)$, $\gamma(x)$ and $\beta(x)$, respectively, integrate with respect to x and use equations (5.2), (5.4) and (5.6), we obtain

$$\int_0^\infty P_q^{(e)}(x,z)\mu_1(x)dx = P_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{5.38}$$

$$\int_0^\infty P_q^{(o)}(x,z)\mu_2(x)dx = P_q^{(o)}(0,z)\bar{G}_2[\lambda-\lambda C(z)] \tag{5.39}$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx = V_q(0,z)\bar{B}[\lambda-\lambda C(z)] \tag{5.40}$$

$$\int_0^\infty R_q(x,z)\beta(x)dx = R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)] \tag{5.41}$$

Using equations (5.34) and (5.38) – (5.41), we can write equations (5.26) – (5.29) in the following forms

$$\begin{aligned} zP_q^{(e)}(0,z) &= (1-p)P_q^{(o)}(0,z)\bar{G}_2[\lambda-\lambda C(z)]+(1-p)(1-k)P_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \\ &\quad +R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)]+V_q(0,z)\bar{B}[\lambda-\lambda C(z)]+\lambda(C(z)-1)Q \end{aligned} \tag{5.42}$$

$$P_q^{(o)}(0,z) = kP_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{5.43}$$

$$V_q(0,z) = pP_q^{(o)}(0,z)\bar{G}_2[\lambda-\lambda C(z)]+p(1-k)P_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{5.44}$$

$$R_q(0,z) = \alpha z P_q^{(e)}(0,z)\left[\frac{1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{5.45}$$

Using equation (5.43) in (5.44) and simplifying, we get

$$V_q(0,z) = pP_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(k\bar{G}_2[\lambda-\lambda C(z)]+(1-k)\right) \tag{5.46}$$

Using (5.43) in (5.35), (5.46) in (5.36) and (5.45) in (5.37) yields

$$P_q^{(o)}(z) = \frac{kP_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(1-\bar{G}_2[\lambda-\lambda C(z)]\right)}{(\lambda-\lambda C(z))} \tag{5.47}$$

$$V_q(z)=$$
$$\frac{pP_q^{(e)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(k\bar{G}_2[\lambda-\lambda C(z)+\alpha]+(1-k)\right)\left(1-\bar{B}[\lambda-\lambda C(z)]\right)}{(\lambda-\lambda C(z))} \tag{5.48}$$

$$R_q(z)=\frac{\alpha zP_q^{(e)}(0,z)\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)\left(1-\bar{\Phi}[\lambda-\lambda C(z)]\right)}{(\lambda-\lambda C(z)+\alpha)(\lambda-\lambda C(z))} \tag{5.49}$$

We now substitute the expressions for $P_q^{(o)}(0,z)$, $V_q(0,z)$ and $R_q(0,z)$ from equations (5.43), (5.46) and (5.45) in (5.42) and after some algebraic manipulations, we obtain the expression for $P_q^{(e)}(0,z)$ as follows

$$P_q^{(e)}(0,z)=\frac{(\lambda-\lambda C(z)+\alpha)\lambda(C(z)-1)Q}{D(z)} \tag{5.50}$$

where

$$\begin{aligned}D(z)=&(\lambda-\lambda C(z)+\alpha)\{z-k(1-p)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\overline{G}_2[\lambda-\lambda C(z)]\}\\&-(1-p)(1-k)(\lambda-\lambda C(z)+\alpha)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\\&-p(\lambda-\lambda C(z)+\alpha)\overline{B}[\lambda-\lambda C(z)]\overline{G}_1[\lambda-\lambda C(z)+\alpha]\left(k\overline{G}_2[\lambda-\lambda C(z)]+(1-k)\right)\\&-\alpha z\overline{\Phi}[\lambda-\lambda C(z)]\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\right)\end{aligned}$$

Substituting for $P_q^{(e)}(0,z)$ from (5.50) in (5.34), (5.47), (5.48), and (5.49), we get the probability generating functions of the queue size for the different states of the system as follows

$$P_q^{(e)}(z)=\frac{\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda(C(z)-1)Q}{D(z)} \tag{5.51}$$

$$P_q^{(o)}(z)=\frac{k\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(\bar{G}_2[\lambda-\lambda C(z)]-1\right)(\lambda-\lambda C(z)+\alpha)Q}{D(z)} \tag{5.52}$$

$$V_q(z)=\frac{p(\lambda-\lambda C(z)+\alpha)\left(\overline{B}[\lambda-\lambda C(z)]-1\right)\overline{G}_1[\lambda-\lambda C(z)+\alpha]\left(k\overline{G}_2[\lambda-\lambda C(z)]+(1-k)\right)Q}{D(z)} \tag{5.53}$$

$$R_q(z)=\frac{\alpha z\left(\overline{\Phi}[\lambda-\lambda C(z)]-1\right)\left(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]\right)Q}{D(z)} \tag{5.54}$$

Adding equations (5.51) – (5.54) we obtain the probability generating function of the queue size at a random epoch irrespective of the state of the system

$$W_q(z)=P_q^{(e)}(z)+P_q^{(o)}(z)+V_q(z)+R_q(z) \tag{5.55}$$

It remains to derive the probability that the server is idle, Q. Using the normalization condition

$$W_q(1)+Q=1$$

we get

$$Q = 1 - \lambda E(I)\left(kE(S_2) + pE(V) + \frac{1}{\alpha\overline{G}_1[\alpha]} - \frac{1}{\alpha} + \frac{E(R)}{\overline{G}_1[\alpha]} - E(R)\right) \tag{5.56}$$

where $C(1) = 1$, $C'(1) = E(I)$ is the mean batch size of the arriving customers, $\overline{G}_2[0] = 1$, $-\overline{G}_2'[0] = E(S_2)$ is the mean service time for the second optional service, $\overline{B}[0] = 1$, $-\overline{B}'[0] = E(V)$ is the mean vacation time, $\overline{\Phi}[0] = 1$, and $-\overline{\Phi}'[0] = E(R)$ is the mean repair time. Using the expression for Q obtained in (5.56), the utilization factor, ρ of the system is given by

$$\rho = \lambda E(I)\left(kE(S_2) + pE(V) + \frac{1}{\alpha\overline{G}_1[\alpha]} - \frac{1}{\alpha} + \frac{E(R)}{\overline{G}_1[\alpha]} - E(R)\right) \tag{5.57}$$

where $\rho < 1$ is the stability condition under which the steady state exits.

To this point, we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size at a random epoch, and this will be used in the next section to derive some performance measures of the system.

5.5 The Mean Queue Size and the Mean Waiting Time

Using the expression for $W_q(z)$ obtained in (5.55) and the formula

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{5.58}$$

which was proved in chapter 1, we derive the mean queue size for the system considered in this chapter. Carrying out the derivatives required in (5.58) at $z = 1$ we get

$$N'(1) = \lambda E(I)Q\left\{1 + \alpha E(R) + \overline{G}_1[\alpha]\left(k\alpha E(S_2) + \alpha pE(V) - \alpha E(R) - 1\right)\right\} \tag{5.59}$$

$$\begin{aligned} N''(1) = {} & \left(\lambda E(I)\right)^2 Q\left\{\left(\frac{2\alpha E(R)}{\lambda E(I)} + \alpha E(R^2)\right)\right. \\ & + \overline{G}_1[\alpha]\left(p\alpha E(V^2) - 2pE(V) - \frac{2\alpha E(R)}{\lambda E(I)} - \alpha E(R^2) + k\alpha E(S_2^2) + 2kE(S_2)\left(p\alpha E(V) - 1\right)\right) \\ & \left. + 2\overline{G}_1[\alpha]\left(1 + \alpha E(R) - p\alpha E(V) - k\alpha E(S_2)\right)\right\} \\ & + \lambda E(I(I-1))Q\left\{1 + \alpha E(R) + \overline{G}_1[\alpha]\left(p\alpha E(V) - 1 - \alpha E(R) + \alpha kE(S_2)\right)\right\} \end{aligned} \tag{5.60}$$

$$D'(1) = \lambda E(I)\left\{-1-\alpha E(R)+\overline{G}_1[\alpha]\left(\frac{\alpha}{\lambda E(I)}-k\alpha E(S_2)-\alpha pE(V)+\alpha E(R)+1\right)\right\} \tag{5.61}$$

$$\begin{aligned} D''(1) = (\lambda E(I))^2 &\left\{\left(\frac{-2}{\lambda E(I)}-\frac{2\alpha E(R)}{\lambda E(I)}-\alpha E(R^2)\right)\right. \\ &+\overline{G}_1[\alpha]\left(2pE(V)-\alpha pE(V^2)+\frac{2\alpha E(R)}{\lambda E(I)}+\alpha E(R^2)+k\alpha E(S_2^2)-2kE(S_2)(p\alpha E(V)-1)\right) \\ &\left.+2\overline{G}_1[\alpha]\left(\alpha pE(V)-1-\frac{\alpha}{\lambda E(I)}-\alpha E(R)-k\alpha E(S_2)\right)\right\} \\ &+\lambda E(I(I-1))\{-1-\alpha E(R)+\overline{G}_1[\alpha](1-p\alpha E(V)+\alpha E(R)+\alpha kE(S_2))\} \end{aligned} \tag{5.62}$$

where Q has been found in (5.56), $\overline{G_2}''[0]=E(S_2^2), \overline{B}''[0]=E(V^2)$ and $\overline{\Phi}''[0]=E(R^2)$ are the second moment of the optional service time, vacation time, and repair time, respectively, and $C''(1)=E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Using the results obtained in (5.59) – (5.62) and equation (5.58), we can find L_q, the mean number of customers in the queue, and hence W_q, the mean waiting time in the queue.

5.6 Particular Cases

5.6.1 No Customer Requires the Second Optional Service

In this case, we assume that all customers require only the first essential service and that they have no option to opt for the second service. Accordingly, we set $k=0$ in the main results of this chapter, and obtain

$$P_q^{(e)}(z)=\frac{(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha])\lambda(C(z)-1)Q}{D(z)} \tag{5.63}$$

$$P_q^{(o)}(z)=0 \tag{5.64}$$

$$V_q(z)=\frac{p(\lambda-\lambda C(z)+\alpha)(\overline{B}[\lambda-\lambda C(z)]-1)\overline{G}_1[\lambda-\lambda C(z)+\alpha]Q}{D(z)} \tag{5.65}$$

$$R_q(z)=\frac{\alpha z(\overline{\Phi}[\lambda-\lambda C(z)]-1)(1-\overline{G}_1[\lambda-\lambda C(z)+\alpha])Q}{D(z)} \tag{5.66}$$

$$\begin{aligned} D(z)=&(\lambda-\lambda C(z)+\alpha)\{z-(1-p)\overline{G}_1[\lambda-\lambda C(z)+\alpha]-p\overline{B}[\lambda-\lambda C(z)]\overline{G}_1[\lambda-\lambda C(z)+\alpha]\} \\ &-\alpha z\overline{\Phi}[\lambda-\lambda C(z)](1-\overline{G}_1[\lambda-\lambda C(z)+\alpha]) \end{aligned} \tag{5.67}$$

$$W_q(z)=P_q^{(e)}(z)+V_q(z)+R_q(z) \tag{5.68}$$

$$Q=1-\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]}-\frac{1}{\alpha}+pE(V)+\frac{E(R)}{\overline{G}_1[\alpha]}-E(R)\right) \tag{5.69}$$

$$\rho=\lambda E(I)\left(\frac{1}{\alpha\overline{G}_1[\alpha]}-\frac{1}{\alpha}+pE(V)+\frac{E(R)}{\overline{G}_1[\alpha]}-E(R)\right) \tag{5.70}$$

$$N'(1)=\lambda E(I)Q\left\{1+\alpha E(R)+\overline{G}_1[\alpha]\left(\alpha pE(V)-\alpha E(R)-1\right)\right\} \tag{5.71}$$

$$\begin{aligned} N''(1)=\left(\lambda E(I)\right)^2 Q\Bigg\{&\left(\frac{2\alpha E(R)}{\lambda E(I)}+\alpha E(R^2)\right)\\ &+\overline{G}_1[\alpha]\left(p\alpha E(V^2)-2pE(V)-\frac{2\alpha E(R)}{\lambda E(I)}-\alpha E(R^2)\right)\\ &+2\overline{G}_1[\alpha]\left(1+\alpha E(R)-p\alpha E(V)\right)\Bigg\}\\ &+\lambda E(I(I-1))Q\left\{1+\alpha E(R)+\overline{G}_1[\alpha]\left(p\alpha E(V)-1-\alpha E(R)\right)\right\} \end{aligned} \tag{5.72}$$

$$D'(1)=\lambda E(I)\left\{-1-\alpha E(R)+\overline{G}_1[\alpha]\left(\frac{\alpha}{\lambda E(I)}-\alpha pE(V)+\alpha E(R)+1\right)\right\} \tag{5.73}$$

$$\begin{aligned} D''(1)=\left(\lambda E(I)\right)^2\Bigg\{&\left(\frac{-2}{\lambda E(I)}-\frac{2\alpha E(R)}{\lambda E(I)}-\alpha E(R^2)\right)\\ &+\overline{G}_1[\alpha]\left(2pE(V)-\alpha pE(V^2)+\frac{2\alpha E(R)}{\lambda E(I)}+\alpha E(R^2)\right)\\ &+2\overline{G}_1[\alpha]\left(\alpha pE(V)-1-\frac{\alpha}{\lambda E(I)}-\alpha E(R)\right)\Bigg\}\\ &+\lambda E(I(I-1))\left\{-1-\alpha E(R)+\overline{G}_1[\alpha]\left(1-p\alpha E(V)+\alpha E(R)\right)\right\} \end{aligned} \tag{5.74}$$

The results obtained in (5.63) – (5.74) agree with the results obtained in chapter 3 in which we studied a queueing system providing single service.

5.6.2 No System Breakdowns

In this case, we assume that the system does not break down, and hence $\alpha = 0$ and $R_q(z) = 0$. Using this assumption in the main results of the chapter we get

$$P_q^{(e)}(z)=\frac{\left(1-\bar{G}_1[\lambda-\lambda C(z)]\right)Q}{-z-\left[p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)-1\right]\left[(1-k)\bar{G}_1[\lambda-\lambda C(z)]+k\bar{G}_1[\lambda-\lambda C(z)]\bar{G}_2[\lambda-\lambda C(z)]\right]} \tag{5.75}$$

$$P_q^{(o)}(z)=\frac{k\bar{G}_1[\lambda-\lambda C(z)]\left(1-\bar{G}_2[\lambda-\lambda C(z)]\right)Q}{-z-\left[p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)-1\right]\left[(1-k)\bar{G}_1[\lambda-\lambda C(z)]+k\bar{G}_1[\lambda-\lambda C(z)]\bar{G}_2[\lambda-\lambda C(z)]\right]} \tag{5.76}$$

$$V_q(z)=\frac{p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)\bar{G}_1[\lambda-\lambda C(z)]\left(k\bar{G}_2[\lambda-\lambda C(z)]+(1-k)\right)Q}{-z-\left[p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)-1\right]\left[(1-k)\bar{G}_1[\lambda-\lambda C(z)]+k\bar{G}_1[\lambda-\lambda C(z)]\bar{G}_2[\lambda-\lambda C(z)]\right]} \tag{5.77}$$

$$W_q(z)=P_q^{(e)}(z)+P_q^{(o)}(z)+V_q(z)=$$

$$\frac{\left\{1+\left[p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)-1\right]\left[(1-k)\bar{G}_1[\lambda-\lambda C(z)]+k\bar{G}_1[\lambda-\lambda C(z)]\bar{G}_2[\lambda-\lambda C(z)]\right]\right\}Q}{-z-\left[p\left(1-\bar{B}[\lambda-\lambda C(z)]\right)-1\right]\left[(1-k)\bar{G}_1[\lambda-\lambda C(z)]+k\bar{G}_1[\lambda-\lambda C(z)]\bar{G}_2[\lambda-\lambda C(z)]\right]} \tag{5.78}$$

To find Q, we use $W_q(1)+Q=1$ and the expression for $W_q(z)$ obtained in (5.78). We see that for z = 1, $W_q(z)$ is indeterminate of 0/0 form. Therefore, we apply L'Hopital's Rule on equation (5.78). Consequently, we get

$$Q=1-\lambda E(I)[E(S_1)+kE(S_2)+pE(V)] \tag{5.79}$$

where $\overline{G}_1[0]=1$ and $-\overline{G}_1'[0]=E(S_1)$ is the mean for the first essential service time. Further, we compute $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ using the expression obtained for $W_q(z)$ in (5.78)

$$N'(1)=-\lambda E(I)Q(E(S_1)+kE(S_2)+pE(V)) \tag{5.80}$$

$$\begin{aligned} N''(1)=&Q(\lambda E(I))^2[-2pE(V)(E(S_1)+kE(S_2))-E(S_1^2)-kE(S_2^2)-2kE(S_1)E(S_2)-pE(V^2)] \\ &+Q\lambda E(I(I-1)(-E(S_1)-kE(S_2)+pE(V)) \end{aligned} \tag{5.81}$$

$$D'(1)=-1+\lambda E(I)(E(S_1)+kE(S_2)+pE(V)) \tag{5.82}$$

$$\begin{aligned} D''(1)=&-(\lambda E(I))^2[-2pE(V)(E(S_1)+kE(S_2))-E(S_1^2)-kE(S_2^2)-2kE(S_1)E(S_2)-pE(V^2)] \\ &-\lambda E(I(I-1)(-E(S_1)-kE(S_2)+pE(V)) \end{aligned} \tag{5.83}$$

where $\overline{G}_1''[0]=E(S_1^2)$ is the second moment of the first essential service time. Using equations (5.80) – (5.83) in (5.58) we can easily find L_q, and hence W_q.

Madan, Abu-Dayyeh and Saleh (2002) studied a special case of this model; vacation queue with a second optional service where the optional service times and the vacation times were assumed to be exponential and the arrivals were single.

If the server has no option to take vacations; that is p = 0, and the arrivals were single, then the model will be reduced to the one considered by Al-Jaraha and Madan (2003) and we get similar results.

5.7 A Numerical Illustration

For the numerical illustration purpose, we use the general results obtained in equations (5.51) through (5.62). We assume that the essential service time, optional service time, vacation time and repair time are all exponential with rates μ_1 , μ_2, γ and β, respectively. Accordingly, we use the following substitutions in equations (5.51) through (5.62)

$$\overline{G}_1[\alpha] = \frac{\mu_1}{\alpha + \mu_1}, \quad \overline{G}_1'[\alpha] = \frac{-\mu_1}{(\alpha + \mu_1)^2}$$

$$\overline{G}_2[\lambda - \lambda C(z)] = \frac{\mu_2}{\lambda - \lambda C(z) + \mu_2}, \quad E(S_2) = \frac{1}{\mu_2}, \quad E(S_2^2) = \frac{2}{\mu_2^2}$$

$$\overline{B}[\lambda - \lambda C(z)] = \frac{\gamma}{\lambda - \lambda C(z) + \gamma}, \quad E(V) = \frac{1}{\gamma}, \quad E(V^2) = \frac{2}{\gamma^2}$$

$$\overline{\Phi}[\lambda - \lambda C(z)] = \frac{\beta}{\lambda - \lambda C(z) + \beta}, \quad E(R) = \frac{1}{\beta}, \quad E(R^2) = \frac{2}{\beta^2}$$

Further, we assume $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$ and $\gamma = 7$, while p and k both taking the values 0.25, 0.5 and 0.75. Also we assume single arrivals with rate $\lambda = 2$, hence

$$C(z) = z, \quad E(I) = 1, \quad E(I(I-1)) = 0$$

All the values of system parameters were chosen such that the steady state condition is satisfied.

In chapters 2 and 4, we presented numerical tables and graphs showing how vacation and breakdown parameters affect the queue characteristics for different queue models, whereas in this chapter we fix most of those parameters and seek the effect of the two parameters p and k; the probability of taking a vacation and the probability that a customer requires the optional service. Table 5.1 gives the numerical values of some queue characteristics for the chosen values of parameter.

Table 5.1 *Computed values of various queue characteristics for vacation queue with breakdown and optional service,* $\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$

p	k	Q	ρ	L_q	W_q	$P_q^{(e)}(1)$	$P_q^{(o)}(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
0.25	0.25	0.2786	0.7214	2.0443	1.0222	0.3332	0.05	0.0714	0.2666	0.7212
0.25	0.5	0.2286	0.7714	2.5001	1.2501	0.3332	0.1	0.0714	0.2666	0.7712
0.25	0.75	0.1786	0.8214	3.1927	1.5964	0.3332	0.15	0.0714	0.2666	0.8212
0.5	0.25	0.2072	0.7928	2.9679	1.484	0.3333	0.05	0.1428	0.2666	0.7927
0.5	0.5	0.1572	0.8428	3.9315	1.9658	0.3332	0.1	0.1428	0.2666	0.8426
0.5	0.75	0.1072	0.8928	5.7641	2.8821	0.3333	0.15	0.1428	0.2666	0.8927
0.75	0.25	0.1358	0.8642	4.8635	2.4318	0.3333	0.05	0.2143	0.2667	0.8643
0.75	0.5	0.0858	0.9142	7.752	3.876	0.3334	0.1	0.2144	0.2667	0.9145
0.75	0.75	0.0358	0.9642	18.6077	9.3039	0.3334	0.15	0.2144	0.2667	0.9646

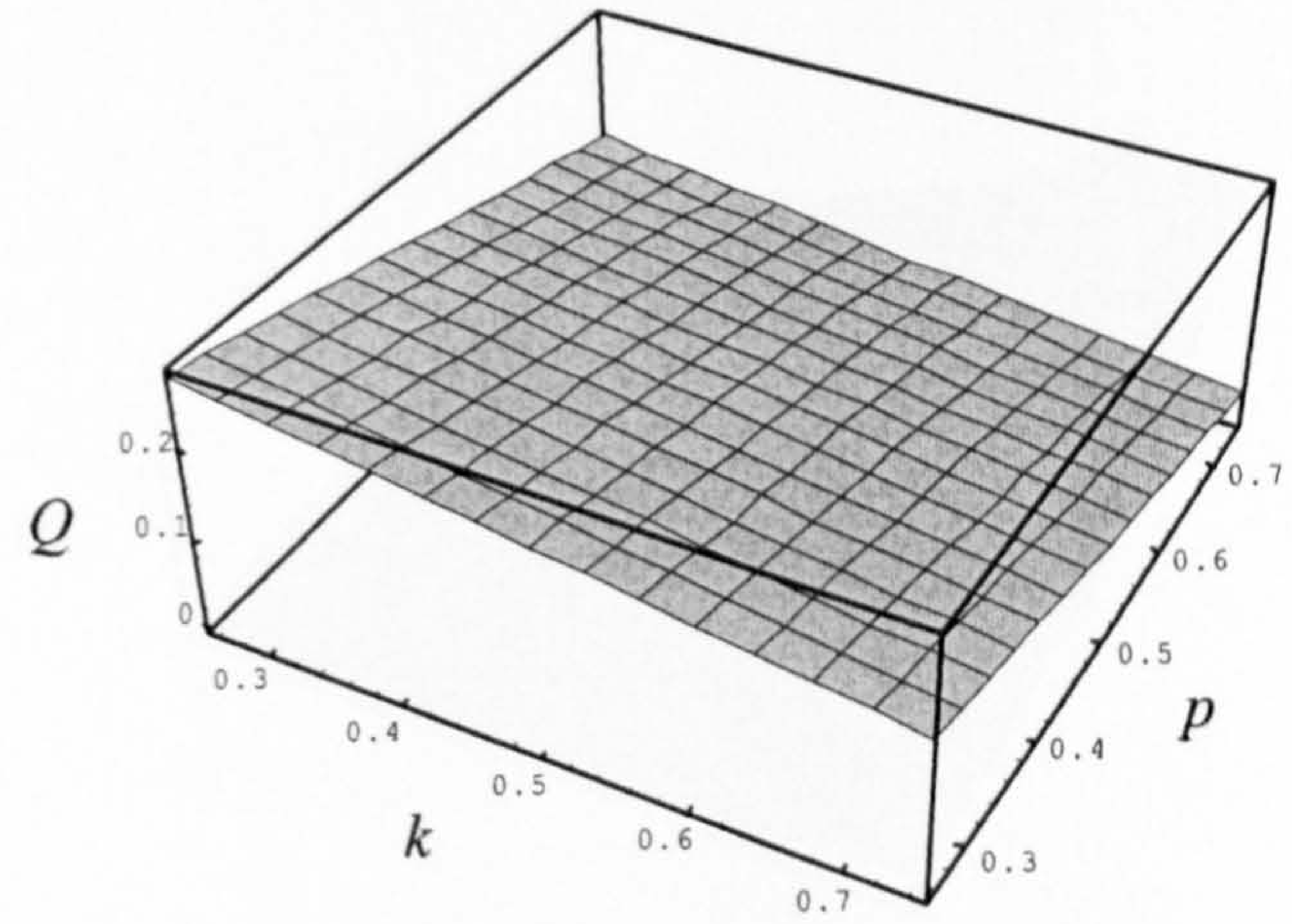

Figure 5.1 *Effect of p and k on the proportion of time that the server is idle Q*
($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)

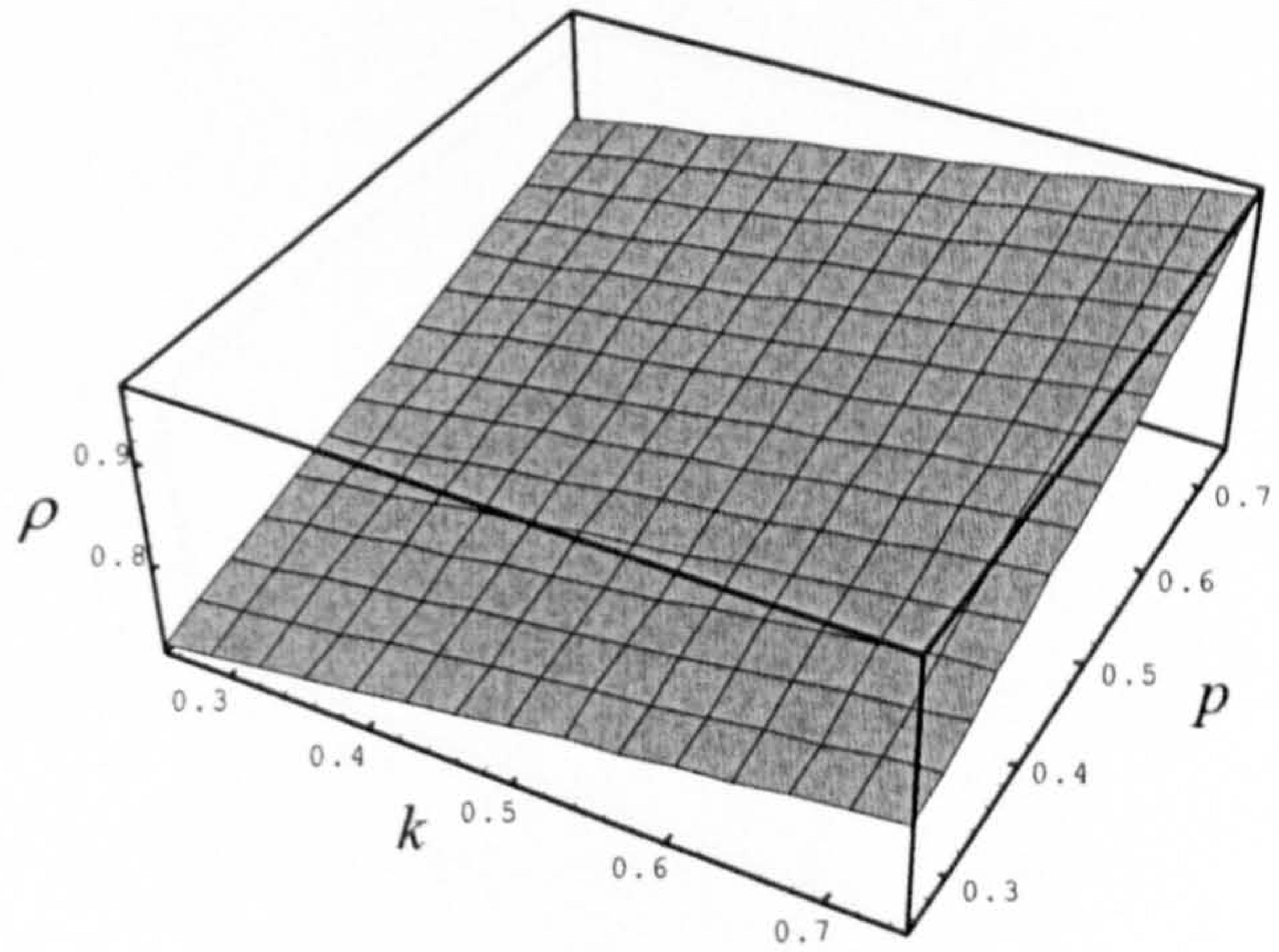

Figure 5.2 *Effect of p and k on the utilization factor ρ*
($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)

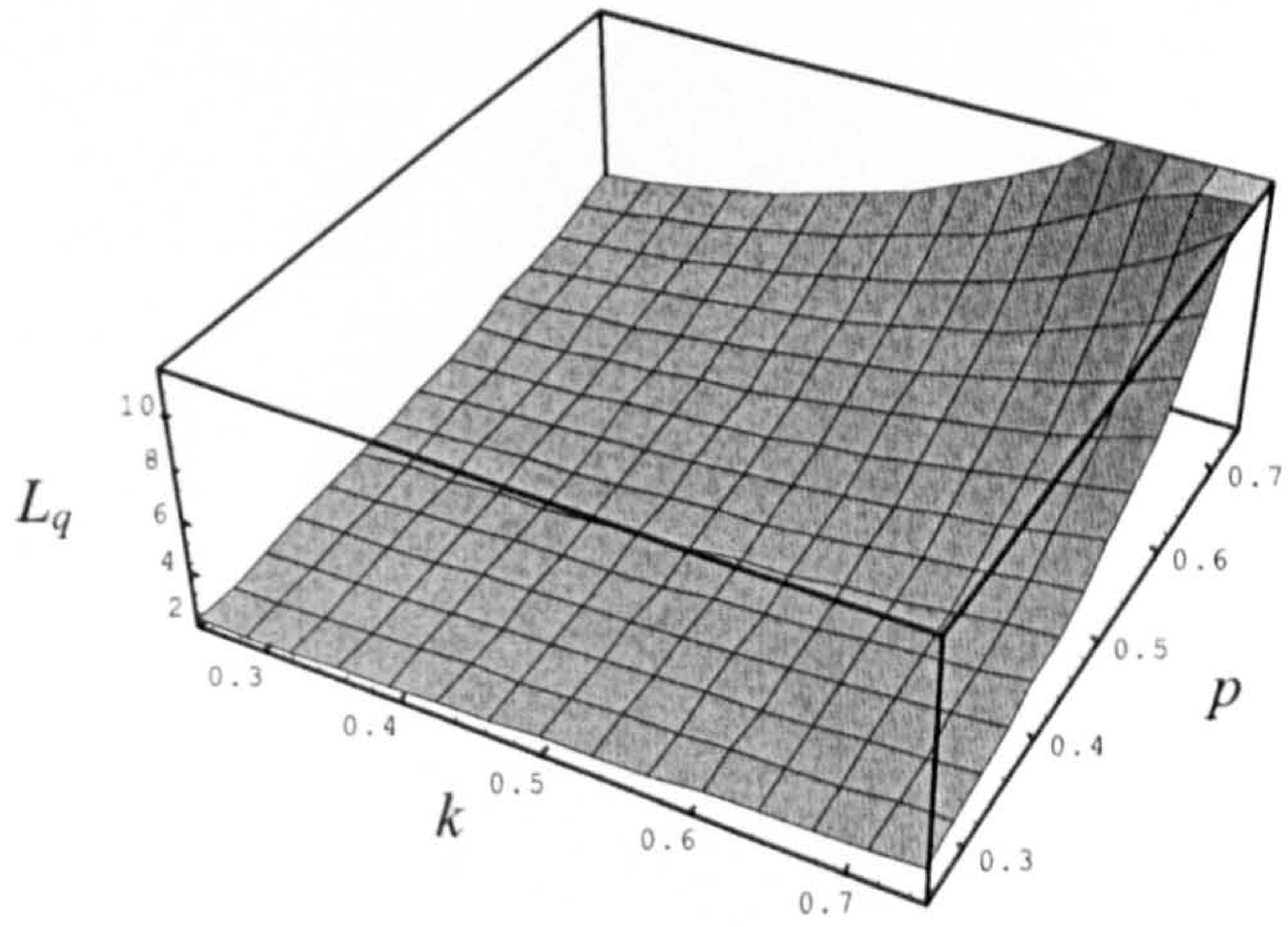

Figure 5.3 *Effect of p and k on the mean queue size L_q*
($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)

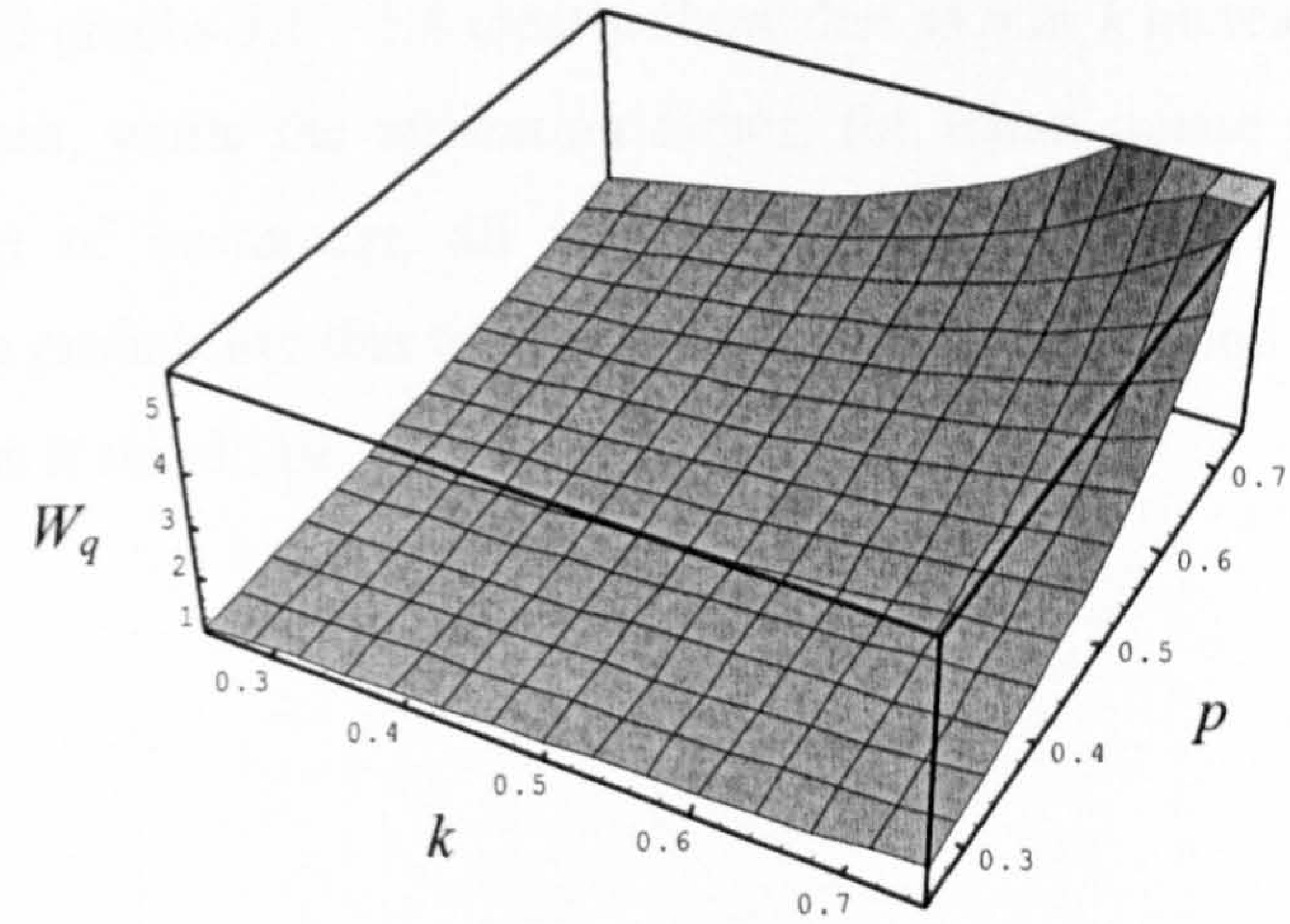

Figure 5.4 *Effect of p and k on the mean waiting time* W_q *($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

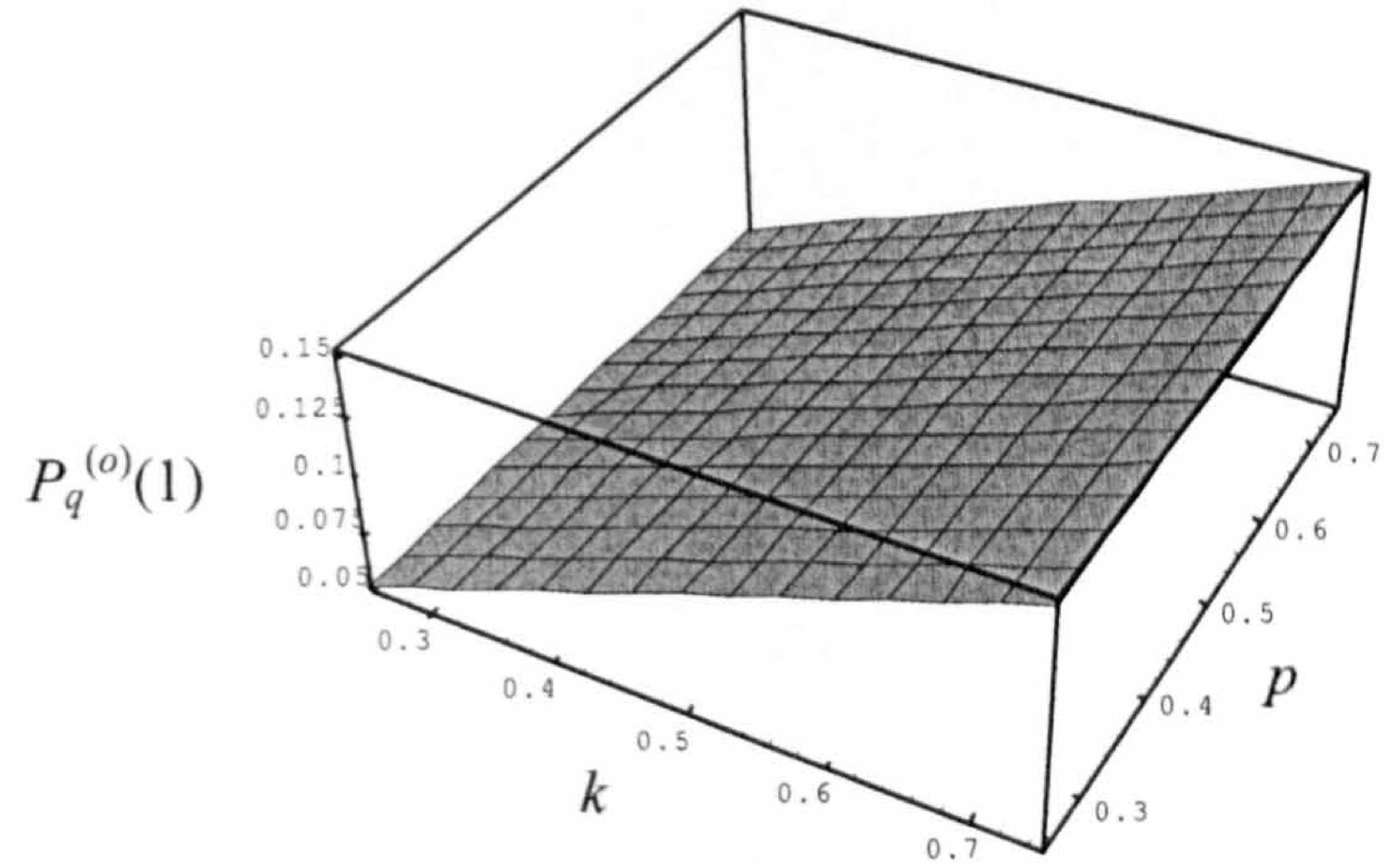

Figure 5.5 *Effect of p and k on the probability that the server is providing the second optional service ($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

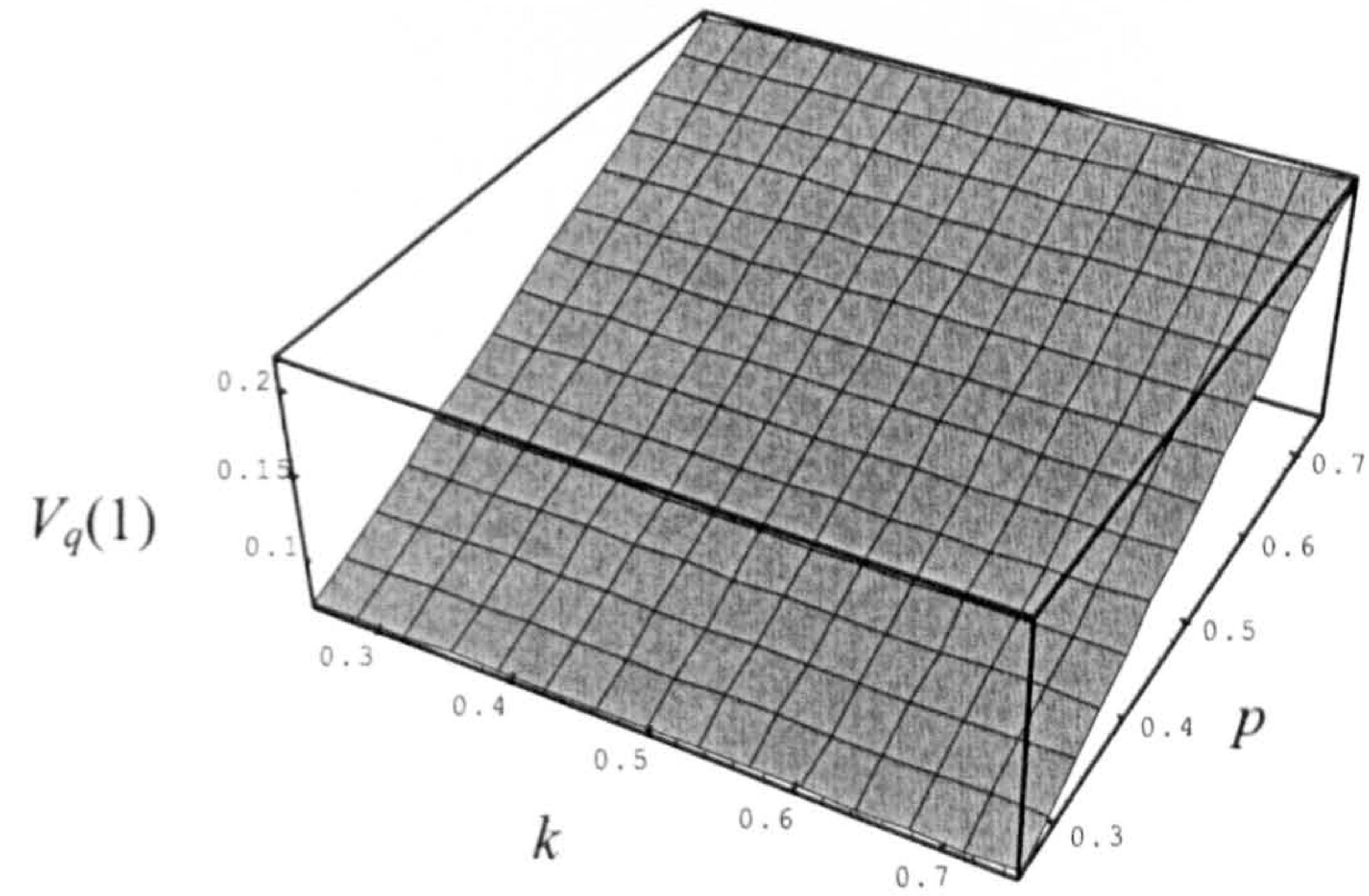

Figure 5.6 *Effect of p & k on the probability that the server is on vacation ($\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 10$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

Table 5.1 and graphs 5.1 – 5.4 clearly show that as p or k increases, the server idle time decreases, while the utilization factor, the mean queue size and the mean waiting time of customers, all increase. Figure 5.5 shows that increasing k increases the probability that the server is providing the second optional service to customers, as it should be.

* * *

Chapter 6

Batch Arrival Queue with Two Kinds of General Heterogeneous Service, Bernoulli Schedule General Vacations, Random Breakdowns and General Repair Times

6.1 Introduction

Some service stations provide more than one kind of service. Just before a service starts, a customer has the option to choose one of the different kinds of service provided. Such a model may find applications in many day-to-day life queueing situations encountered at post offices, banks, automobile stations, computer centres, beauty centres, and so forth. Such queueing systems were investigated by Beja and Teller (1975). They studied a single station service system with bounded queue capacity operates in discrete time and many types of service. They assumed K possible types of service in which an arriving customer has the option to choose one. The expected cost per served customer in each kind of service was compared to the expense of losing him.

Anabosi and Madan (2003) studied a single server queue with two types of service, optional server vacations based on Bernoulli schedule and a single vacation policy. They assumed that the server provides two types of heterogeneous exponential service and a customer may choose either type of service. Under the single vacation policy, it is assumed that whenever the server takes a vacation, it is always a single vacation with exponentially distributed vacation period. Explicit steady state results for the probability generating functions of the queue size and the system size, the expected number of customers and the expected waiting time in the queue and in the system have been derived. This work was generalised by Madan, Al-Rawi and Al-Nasser (2005) in which they assumed general arbitrary distributions for both types of service times and for vacation times. In this chapter, we further generalise the work done by Madan, Al-Rawi and Al-Nasser (2005).

In chapter 4 we analysed a single server batch arrival queueing system with two-stage heterogeneous service, server vacations and random breakdowns where each arriving customer undergoes both essential services. In chapter 5, the arriving customers are provided an essential service, and once this service completes, she/he has the option to take the second additional service. It was assumed that the server takes Bernoulli vacations and the system may break down while providing the first essential service to a customer. In the current chapter, we consider a single server batch arrival queue in which the server provides two types of service to arriving customers. A customer may choose the first kind of service with probability θ, or the second kind of service with probability $1-\theta$. The times of both kinds of service are generally distributed. It is also assumed that after the service completion of any kind of service, the server may leave for a vacation of random length. Vacations are assumed to follow Bernoulli schedule under a single vacation policy and the vacation times are generally distributed. We further assume that the system may break down at random during any kind of service. This is followed by a repair process of random length and general repair time distribution.

The model of this chapter extends the work done by Madan, Al-Rawi and Al-Nasser (2005) in which a more realistic queueing system is considered by adding the assumption of server breakdowns.

6.2 The Mathematical Model

The mathematical model of this chapter is described by the following assumptions:

a) Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt$ (i = 1, 2, 3, ...) to be the first order probability that a batch of i customers arrives at the system during a short interval of time (t, t + dt], where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

b) There is a single server who provides two kinds of general heterogeneous (one by one) service to customers on a first come, first served basis. Before his service starts, each customer has the option to choose the first service with probability θ or the second service with probability $1-\theta$.

c) The service time of the two kinds of services follow different general (arbitrary) distributions with distribution functions $G_j(s)$ and density function $g_j(s)$, $j = 1, 2$. Let $\mu_j(x)$dx be the conditional probability density of the completion of j^{th} kind of service during the time interval (x, x + dx], given that the elapsed time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}, \qquad j=1,2 \tag{6.1}$$

And, therefore

$$g_j(s) = \mu_j(s) e^{-\int_0^s \mu_j(x)dx}, \qquad j=1,2 \tag{6.2}$$

d) As soon as the service is completed, the server takes a vacation with probability p, or decides to stay in the system with probability $1-p$, where $0 \le p \le 1$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $B(v)$ and density function $b(v)$. Let $\gamma(x)$dx be the conditional probability of a completion of a vacation during the interval (x, x + dx] given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{b(x)}{1-B(x)} \tag{6.3}$$

And, therefore

$$b(v) = \gamma(v) e^{-\int_0^v \gamma(x)dx} \tag{6.4}$$

f) The system may break down at random while the service of either kind is going on. Breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further, we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue.

g) Once the system breaks down, it enters a repair process immediately. The repair time follows a general (arbitrary) distribution with distribution function $\Phi(r)$ and density function $\varphi(r)$. Let $\beta(x)$dx be the conditional probability of a repair completion during the interval $(x, x + \mathrm{d}x]$ given that the elapsed repair time is x, so that

$$\beta(x) = \frac{\varphi(x)}{1-\Phi(x)} \tag{6.5}$$

And, therefore

$$\varphi(r) = \beta(r)e^{-\int_0^r \beta(x)dx} \tag{6.6}$$

h) The stochastic processes involved in the system are assumed to be independent of each other.

6.3 Steady State Equations Governing the System

We shall find the steady state equations governing the system, but first we define the following limits which correspond to steady state probabilities.

$$\lim_{t\to\infty} P_n^{(\kappa j)}(x,t) = P_n^{(\kappa j)}(x), \quad \lim_{t\to\infty} P_n^{(\kappa j)}(t) = \lim_{t\to\infty} \int_0^\infty P_n^{(\kappa j)}(x,t)dx = P_n^{(\kappa j)}, \quad j=1,2$$

$$\lim_{t\to\infty} V_n(x,t) = V_n(x), \quad \lim_{t\to\infty} V_n(t) = \lim_{t\to\infty} \int_0^\infty V_n(x,t)dx = V_n$$

$$\lim_{t\to\infty} R_n(x,t) = R_n(x), \quad \lim_{t\to\infty} R_n(t) = \lim_{t\to\infty} \int_0^\infty R_n(x,t)dx = R_n$$

$$\lim_{t\to\infty} Q(t) = Q$$

Connecting states of the system at time t + dt with those at time t we obtain the following set of steady state equations

$$\frac{d}{dx}P_n^{(\kappa 1)}(x) + (\lambda + \mu_1(x) + \alpha)P_n^{(\kappa 1)}(x) = \lambda\sum_{i=1}^{n-1} c_i P_{n-i}^{(\kappa 1)}(x), \qquad n \ge 1 \tag{6.7}$$

$$\frac{d}{dx}P_0^{(\kappa 1)}(x) + (\lambda + \mu_1(x) + \alpha)P_0^{(\kappa 1)}(x) = 0 \tag{6.8}$$

$$\frac{d}{dx}P_n^{(\kappa 2)}(x)+\left(\lambda+\mu_2(x)+\alpha\right)P_n^{(\kappa 2)}(x)=\lambda\sum_{i=1}^{n-1}c_i P_{n-i}^{(\kappa 2)}(x), \qquad n\geq 1 \tag{6.9}$$

$$\frac{d}{dx}P_0^{(\kappa 2)}(x)+\left(\lambda+\mu_2(x)+\alpha\right)P_0^{(\kappa 2)}(x)=0 \tag{6.10}$$

$$\frac{d}{dx}V_n(x)+\left(\lambda+\gamma(x)\right)V_n(x)=\lambda\sum_{i=1}^{n-1}c_i V_{n-i}(x), \qquad n\geq 1 \tag{6.11}$$

$$\frac{d}{dx}V_0(x)+\left(\lambda+\gamma(x)\right)V_0(x)=0 \tag{6.12}$$

$$\frac{d}{dx}R_n(x)+\left(\lambda+\beta(x)\right)R_n(x)=\lambda\sum_{i=1}^{n-1}c_i R_{n-i}, \qquad n\geq 1 \tag{6.13}$$

$$\frac{d}{dx}R_0(x)+\left(\lambda+\beta(x)\right)R_0(x)=0 \tag{6.14}$$

$$\lambda Q=(1-p)\left(\int_0^\infty P_0^{(\kappa 1)}(x)\mu_1(x)dx+\int_0^\infty P_0^{(\kappa 2)}(x)\mu_2(x)dx\right)+\int_0^\infty V_0(x)\gamma(x)dx+\int_0^\infty R_0(x)\beta(x)dx \tag{6.15}$$

Equations (6.7) – (6.15) should be solved subject to the following boundary conditions

$$\begin{aligned}P_n^{(\kappa 1)}(0)=&(1-p)\theta\left(\int_0^\infty P_{n+1}^{(\kappa 1)}(x)\mu_1(x)dx+\int_0^\infty P_{n+1}^{(\kappa 2)}(x)\mu_2(x)dx\right)\\&+\theta\int_0^\infty V_{n+1}(x)\gamma(x)dx+\theta\int_0^\infty R_{n+1}(x)\beta(x)dx+\lambda\theta c_{n+1}Q, \qquad n\geq 0\end{aligned} \tag{6.16}$$

$$\begin{aligned}P_n^{(\kappa 2)}(0)=&(1-p)(1-\theta)\left(\int_0^\infty P_{n+1}^{(\kappa 1)}(x)\mu_1(x)dx+\int_0^\infty P_{n+1}^{(\kappa 2)}(x)\mu_2(x)dx\right)\\&+(1-\theta)\int_0^\infty V_{n+1}(x)\gamma(x)dx+(1-\theta)\int_0^\infty R_{n+1}(x)\beta(x)dx+\lambda(1-\theta)c_{n+1}Q, \quad n\geq 0\end{aligned} \tag{6.17}$$

$$V_n(0)=p\left(\int_0^\infty P_n^{(\kappa 1)}(x)\mu_1(x)dx+\int_0^\infty P_n^{(\kappa 2)}(x)\mu_2(x)dx\right), \qquad n\geq 0 \tag{6.18}$$

$$R_n(0)=\alpha\left(\int_0^\infty P_{n-1}^{(\kappa 1)}(x)dx+\int_0^\infty P_{n-1}^{(\kappa 2)}(x)dx\right), \qquad n\geq 1 \tag{6.19}$$

$$R_0(0)=0 \tag{6.20}$$

6.4 Queue Size Distribution at a Random Epoch

Defining the probability generating functions of different states of the system, we have

$$P_q^{(\kappa j)}(x,z)=\sum_{n=0}^{\infty} z^n P_n^{(\kappa j)}(x), \quad P_q^{(\kappa j)}(z)=\sum_{n=0}^{\infty} z^n P_n^{(\kappa j)}, \quad j=1,2,$$

$$V_q(x,z)=\sum_{n=0}^{\infty} z^n V_n(x), \quad V_q(z)=\sum_{n=0}^{\infty} z^n V_n,$$

$$R_q(x,z)=\sum_{n=0}^{\infty} z^n R_n(x), \quad R_q(z)=\sum_{n=0}^{\infty} z^n R_n,$$

$$C(z)=\sum_{i=1}^{\infty} z^i c_i \tag{6.21}$$

Proceeding in the same manner as in the previous chapters we get the following differential equations involving the probability generating functions

$$\frac{d}{dx}P_q^{(\kappa 1)}(x,z)+\left(\lambda-\lambda C(z)+\mu_1(x)+\alpha\right)P_q^{(\kappa 1)}(x,z)=0 \tag{6.22}$$

$$\frac{d}{dx}P_q^{(\kappa 2)}(x,z)+\left(\lambda-\lambda C(z)+\mu_2(x)+\alpha\right)P_q^{(\kappa 2)}(x,z)=0 \tag{6.23}$$

$$\frac{d}{dx}V_q(x,z)+\left(\lambda-\lambda C(z)+\gamma(x)\right)V_q(x,z)=0 \tag{6.24}$$

$$\frac{d}{dx}R_q(x,z)+\left(\lambda-\lambda C(z)+\beta(x)\right)R_q(x,z)=0 \tag{6.25}$$

$$\begin{aligned} zP_q^{(\kappa 1)}(0,z)=&(1-p)\theta\left(\int_0^\infty P_q^{(\kappa 1)}(x,z)\mu_1(x)dx+\int_0^\infty P_q^{(\kappa 2)}(x,z)\mu_2(x)dx\right)\\ &+\theta\int_0^\infty V_q(x,z)\gamma(x)dx+\theta\int_0^\infty R_q(x,z)\beta(x)dx+\lambda\theta\left(C(z)-1\right)Q \end{aligned} \tag{6.26}$$

$$\begin{aligned} zP_q^{(\kappa 2)}(0,z)=&(1-p)(1-\theta)\left(\int_0^\infty P_q^{(\kappa 1)}(x,z)\mu_1(x)dx+\int_0^\infty P_q^{(\kappa 2)}(x,z)\mu_2(x)dx\right)\\ &+(1-\theta)\int_0^\infty V_q(x,z)\gamma(x)dx+(1-\theta)\int_0^\infty R_q(x,z)\beta(x)dx+\lambda(1-\theta)\left(C(z)-1\right)Q \end{aligned} \tag{6.27}$$

$$V_q(0,z)=p\left(\int_0^\infty P_q^{(\kappa 1)}(x,z)\mu_2(x)dx+\int_0^\infty P_q^{(\kappa 2)}(x,z)\mu_1(x)dx\right) \tag{6.28}$$

$$R_q(0,z)=\alpha z\left(\int_0^\infty P_q^{(\kappa 1)}(x,z)dx+\int_0^\infty P_q^{(\kappa 2)}(x,z)dx\right) \tag{6.29}$$

We integrate equations (6.22) – (6.25) from 0 to x and obtain

$$P_q^{(\kappa 1)}(x,z)=P_q^{(\kappa 1)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_1(t)dt} \tag{6.30}$$

$$P_q^{(\kappa 2)}(x,z)=P_q^{(\kappa 2)}(0,z)e^{-(\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_2(t)dt} \tag{6.31}$$

$$V_q(x,z) = V_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{6.32}$$

$$R_q(x,z) = R_q(0,z)e^{-(\lambda-\lambda C(z))x-\int_0^x \beta(t)dt} \tag{6.33}$$

where $P_q^{(\kappa 1)}(0,z)$, $P_q^{(\kappa 2)}(0,z)$, $V_q(0,z)$ and $R_q(0,z)$ are given by (6.26), (6.27), (6.28) and (6.29), respectively. Again integrating equations (6.30) – (6.33) by parts with respect to x gives

$$P_q^{(\kappa 1)}(z) = P_q^{(\kappa 1)}(0,z)\left[\frac{1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{6.34}$$

$$P_q^{(\kappa 2)}(z) = P_q^{(\kappa 2)}(0,z)\left[\frac{1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right] \tag{6.35}$$

$$V_q(z) = V_q(0,z)\left[\frac{1-\bar{B}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{6.36}$$

$$R_q(z) = R_q(0,z)\left[\frac{1-\bar{\Phi}[\lambda-\lambda C(z)]}{\lambda-\lambda C(z)}\right] \tag{6.37}$$

where $\bar{G}_1[\lambda-\lambda C(z)+\alpha] = \int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x} \cdot dG_1(x)$, $\bar{G}_2[\lambda-\lambda C(z)+\alpha] = \int_0^\infty e^{-(\lambda-\lambda C(z)+\alpha)x} \cdot dG_2(x)$, $\bar{B}[\lambda-\lambda C(z)] = \int_0^\infty e^{-(\lambda-\lambda C(z))x} \cdot dB(x)$ and $\bar{\Phi}[\lambda-\lambda C(z)] = \int_0^\infty e^{-(\lambda-\lambda C(z))x} \cdot d\Phi(x)$ are the Laplace-Stieltjes transform of kind 1 service time, kind 2 service time, vacation time and repair time, respectively.

Now, we multiply equations (6.30), (6.31), (6.32) and (6.33) by $\mu_1(x)$, $\mu_2(x)$, $\gamma(x)$ and $\beta(x)$, respectively, integrate with respect to x and use equations (6.2), (6.4) and (6.6), we obtain

$$\int_0^\infty P_q^{(\kappa 1)}(x,z)\mu_1(x)dx = P_q^{(\kappa 1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha] \tag{6.38}$$

$$\int_0^\infty P_q^{(\kappa 2)}(x,z)\mu_2(x)dx = P_q^{(\kappa 2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha] \tag{6.39}$$

$$\int_0^\infty V_q(x,z)\gamma(x)dx = V_q(0,z)\bar{B}[\lambda-\lambda C(z)] \tag{6.40}$$

$$\int_0^\infty R_q(x,z)\beta(x)dx = R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)] \tag{6.41}$$

Using equations (6.34), (6.35) and (6.38) – (6.41) in equations (6.26) through

(6.29) we get

$$zP_q^{(\kappa 1)}(0,z)=(1-p)\theta\left(P_q^{(\kappa 1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]+P_q^{(\kappa 2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)$$
$$+\theta V_q(0,z)\bar{B}[\lambda-\lambda C(z)]+\theta R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)]+\lambda\theta\left(C(z)-1\right)Q \tag{6.42}$$

$$zP_q^{(\kappa 2)}(0,z)=(1-p)(1-\theta)\left(P_q^{(\kappa 1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]+P_q^{(\kappa 2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)$$
$$+(1-\theta)V_q(0,z)\bar{B}[\lambda-\lambda C(z)]+(1-\theta)R_q(0,z)\bar{\Phi}[\lambda-\lambda C(z)]+\lambda(1-\theta)\left(C(z)-1\right)Q \tag{6.43}$$

$$V_q(0,z)=pP_q^{(\kappa 1)}(0,z)\bar{G}_1[\lambda-\lambda C(z)+\alpha]+pP_q^{(\kappa 2)}(0,z)\bar{G}_2[\lambda-\lambda C(z)+\alpha] \tag{6.44}$$

$$R_q(0,z)=\alpha z\left(P_q^{(\kappa 1)}(0,z)\left[\frac{1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right]+P_q^{(\kappa 2)}(0,z)\left[\frac{1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]}{(\lambda-\lambda C(z)+\alpha)}\right]\right) \tag{6.45}$$

Equations (6.42) – (6.45) need to be solved simultaneously. We use Cramer's Rule to solve these equations. First we rewrite the four equations as follows

$$\left(z-(1-p)\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)P_q^{(\kappa 1)}(0,z)-(1-p)\theta\bar{G}_2[\lambda-\lambda C(z)+\alpha]P_q^{(\kappa 2)}(0,z)$$
$$-\theta\bar{B}[\lambda-\lambda C(z)]V_q(0,z)-\theta\bar{\Phi}[\lambda-\lambda C(z)]R_q(0,z)=\lambda\theta\left(C(z)-1\right)Q$$

$$-(1-p)(1-\theta)\bar{G}_1[\lambda-\lambda C(z)+\alpha]P_q^{(\kappa 1)}(0,z)+\left(z-(1-p)(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)P_q^{(\kappa 2)}(0,z)$$
$$-(1-\theta)\bar{B}[\lambda-\lambda C(z)]V_q(0,z)-(1-\theta)\bar{\Phi}[\lambda-\lambda C(z)]R_q(0,z)=\lambda(1-\theta)\left(C(z)-1\right)Q$$

$$p\bar{G}_1[\lambda-\lambda C(z)+\alpha]P_q^{(\kappa 1)}(0,z)+p\bar{G}_2[\lambda-\lambda C(z)+\alpha]P_q^{(\kappa 2)}(0,z)-V_q(0,z)=0$$

$$\alpha z\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)P_q^{(\kappa 1)}(0,z)+\alpha z\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)P_q^{(\kappa 2)}(0,z)$$
$$-\left(\lambda-\lambda C(z)+\alpha\right)R_q(0,z)=0$$

Using the coefficients appearing in the previous equations, we form the following matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\begin{pmatrix} P_q^{(\kappa 1)}(0,z) \\ P_q^{(\kappa 2)}(0,z) \\ V_q(0,z) \\ R_q(0,z) \end{pmatrix}=\begin{pmatrix} \lambda\theta\left(C(z)-1\right)Q \\ \lambda(1-\theta)\left(C(z)-1\right)Q \\ 0 \\ 0 \end{pmatrix}$$

where

$$a_{11}=z-(1-p)\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]$$

$$a_{21}=-(1-p)(1-\theta)\bar{G}_1[\lambda-\lambda C(z)+\alpha]$$

$$a_{31}=p\bar{G}_1[\lambda-\lambda C(z)+\alpha]$$

$$a_{41}=\alpha z\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)$$

$$a_{12}=-(1-p)\theta\bar{G}_2[\lambda-\lambda C(z)+\alpha]$$

$$a_{22}=z-(1-p)(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]$$

$$a_{32}=p\bar{G}_2[\lambda-\lambda C(z)+\alpha]$$

$$a_{42}=\alpha z\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)$$

$$a_{13}=-\theta\bar{B}[\lambda-\lambda C(z)]$$

$$a_{23}=-(1-\theta)\bar{B}[\lambda-\lambda C(z)]$$

$$a_{33}=-1$$

$$a_{43}=0$$

$$a_{14}=-\theta\bar{\Phi}[\lambda-\lambda C(z)]$$

$$a_{24}=-(1-\theta)\bar{\Phi}[\lambda-\lambda C(z)]$$

$$a_{34}=0$$

$$a_{44}=-\left(\lambda-\lambda C(z)+\alpha\right)$$

Solving the system, we get

$$P_q^{(\kappa 1)}(0,z)=\frac{\lambda\theta z\left(\lambda-\lambda C(z)+\alpha\right)\left(C(z)-1\right)Q}{D(z)} \tag{6.46}$$

$$P_q^{(\kappa 2)}(0,z)=\frac{\lambda(1-\theta)z\left(\lambda-\lambda C(z)+\alpha\right)\left(C(z)-1\right)Q}{D(z)} \tag{6.47}$$

$$V_q(0,z)=\frac{\lambda pz\left(\lambda-\lambda C(z)+\alpha\right)\left\{\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]+(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right\}\left(C(z)-1\right)Q}{D(z)} \tag{6.48}$$

$$R_q(0,z)=\frac{\lambda\alpha z^2\left\{\theta\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)+(1-\theta)\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\right\}\left(C(z)-1\right)Q}{D(z)} \tag{6.49}$$

where

$$\begin{aligned}D(z)={}&z\left(\lambda-\lambda C(z)+\alpha\right)\left\{z-\left(\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]+(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\left(p\bar{B}[\lambda-\lambda C(z)]+(1-p)\right)\right\}\\&-\alpha z^2\bar{\Phi}[\lambda-\lambda C(z)]\left\{\theta\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)+(1-\theta)\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\right\}\end{aligned}$$

Substituting for $P_q^{(\kappa 1)}(0,z)$, $P_q^{(\kappa 2)}(0,z)$, $V_q(0,z)$, and $R_q(0,z)$ from equations (6.46) – (6.49) in equations (6.34) – (6.37) we get the probability generating functions of queue size for different states of the system

$$P_q^{(\kappa 1)}(z) = \frac{\theta\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{6.50}$$

$$P_q^{(\kappa 2)}(z) = \frac{(1-\theta)\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{6.51}$$

$$V_q(z) = \frac{p\left(\lambda-\lambda C(z)+\alpha\right)\left(\bar{B}[\lambda-\lambda C(z)]-1\right)\left\{\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]+(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right\}Q}{D(z)} \tag{6.52}$$

$$R_q(z) = \frac{\alpha z\left(\bar{\Phi}[\lambda-\lambda C(z)]-1\right)\left\{\theta\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)+(1-\theta)\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\right\}Q}{D(z)} \tag{6.53}$$

where

$$\begin{aligned} D(z) = {} & \left(\lambda-\lambda C(z)+\alpha\right)\left\{z-\left(\theta\bar{G}_1[\lambda-\lambda C(z)+\alpha]+(1-\theta)\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\left(p\bar{B}[\lambda-\lambda C(z)]+(1-p)\right)\right\} \\ & -\alpha z\bar{\Phi}[\lambda-\lambda C(z)]\left\{\theta\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)+(1-\theta)\left(1-\bar{G}_2[\lambda-\lambda C(z)+\alpha]\right)\right\} \end{aligned}$$

Irrespective of the state of the system, the probability generating function of queue size is given by

$$W_q(z) = P_q^{(\kappa 1)}(z) + P_q^{(\kappa 2)}(z) + V_q(z) + R_q(z) \tag{6.54}$$

Using the normalization condition $W_q(1)+Q=1$ to find the proportion of server's idle time, Q, we get

$$Q = 1-\lambda E(I)\left(\frac{1+\alpha E(R)}{\alpha\left(\theta\bar{G}_1[\alpha]+(1-\theta)\bar{G}_2[\alpha]\right)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{6.55}$$

where $C(1)=1$, $C'(1)=E(I)$ is the mean batch size of the arriving customers, $\bar{B}[0]=1$, $-\bar{B}'[0]=E(V)$ is the mean vacation time, $\bar{\Phi}[0]=1$, and $-\bar{\Phi}'[0]=E(R)$ is the mean repair time. Hence, the utilization factor, ρ of the system is given by

$$\rho = \lambda E(I)\left(\frac{1+\alpha E(R)}{\alpha\left(\theta\bar{G}_1[\alpha]+(1-\theta)\bar{G}_2[\alpha]\right)}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{6.56}$$

where $\rho < 1$ is the stability condition under which the steady states exits.

Equation (6.55) gives the probability that the server is idle. Substituting for Q from (6.55) in (6.54), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

6.5 The Mean Queue Size and the Mean Waiting Time

To find L_q, the mean number of customers in the queue under the steady state we write $W_q(z)$ obtained in (6.54) in the form $W_q(z) = N(z)/D(z)$ and then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{6.57}$$

Carrying out the derivatives at $z = 1$ we get

$$N'(1) = \lambda E(I)Q\left\{(1+\alpha E(R)) + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left(p\alpha E(V) - 1 - \alpha E(R)\right)\right\} \tag{6.58}$$

$$\begin{aligned}
N''(1) = 2Q(\lambda E(I))^2 & \left\{\left(\frac{\alpha E(R)}{\lambda E(I)} + \frac{1}{2}\alpha E(R^2)\right)\right. \\
& + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left(-pE(V) - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2) + \frac{1}{2}p\alpha E(V^2)\right) \\
& \left. + \left(\theta\bar{G}_1'[\alpha] + (1-\theta)\bar{G}_2'[\alpha]\right)\left(1 - p\alpha E(V) + \alpha E(R)\right) \right\} \\
& + Q\lambda E(I(I-1))\left\{(1+\alpha E(R)) + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left(p\alpha E(V) - 1 - \alpha E(R)\right)\right\}
\end{aligned} \tag{6.59}$$

$$D'(1) = -\lambda E(I)\left(1+\alpha E(R)\right) + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left\{\alpha + \lambda E(I)\left(1 + \alpha E(R) - p\alpha E(V)\right)\right\} \tag{6.60}$$

$$\begin{aligned}
D''(1) = 2(\lambda E(I))^2 & \left\{\left(-\frac{1}{\lambda E(I)} - \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}\alpha E(R^2)\right)\right. \\
& + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left(pE(V) + \frac{\alpha E(R)}{\lambda E(I)} - \frac{1}{2}p\alpha E(V^2) + \frac{1}{2}\alpha E(R^2)\right) \\
& \left. + \left(\theta\bar{G}_1'[\alpha] + (1-\theta)\bar{G}_2'[\alpha]\right)\left(p\alpha E(V) - 1 - \frac{\alpha}{\lambda E(I)} - \alpha E(R)\right)\right\} \\
& + \lambda E(I(I-1))\left\{(-1-\alpha E(R)) + \left(\theta\bar{G}_1[\alpha] + (1-\theta)\bar{G}_2[\alpha]\right)\left(1 + \alpha E(R) - p\alpha E(V)\right)\right\}
\end{aligned} \tag{6.61}$$

where $E(R^2)$ is the second moment of the repair time, $E(V^2)$ is the second moment of the vacation time, $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers, and Q has been found in (6.55). Then, substituting for $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ from (6.58) – (6.61) in (6.57) we obtain L_q in a closed form. Dividing L_q by λ we can find the mean waiting time in the queue.

6.6 Particular cases

6.6.1 No Customer Chooses the Second kind of Service

If all customers choose the first kind of service and no one chooses the second, then $\theta = 1$ and $1 - \theta = 0$. Using this in the main results of the chapter, we get

$$P_q^{(\kappa 1)}(z) = \frac{\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)\lambda\left(C(z)-1\right)Q}{D(z)} \tag{6.62}$$

$$P_q^{(\kappa 2)}(z) = 0 \tag{6.63}$$

$$V_q(z) = \frac{p\left(\lambda-\lambda C(z)+\alpha\right)\left(\bar{B}[\lambda-\lambda C(z)]-1\right)\bar{G}_1[\lambda-\lambda C(z)+\alpha]Q}{D(z)} \tag{6.64}$$

$$R_q(z) = \frac{\alpha z\left(\bar{\Phi}[\lambda-\lambda C(z)]-1\right)\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right)Q}{D(z)} \tag{6.65}$$

$$\begin{aligned} D(z) = {} & \left(\lambda-\lambda C(z)+\alpha\right)\left\{z-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\left(p\bar{B}[\lambda-\lambda C(z)]+(1-p)\right)\right\} \\ & -\alpha z\,\bar{\Phi}[\lambda-\lambda C(z)]\left(1-\bar{G}_1[\lambda-\lambda C(z)+\alpha]\right) \end{aligned} \tag{6.66}$$

$$W_q(z) = P_q^{(\kappa 1)}(z) + V_q(z) + R_q(z) \tag{6.67}$$

$$Q = 1-\lambda E(I)\left(\frac{1+\alpha E(R)}{\alpha\bar{G}_1[\alpha]}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{6.68}$$

$$\rho = \lambda E(I)\left(\frac{1+\alpha E(R)}{\alpha\bar{G}_1[\alpha]}-\frac{1}{\alpha}-E(R)+pE(V)\right) \tag{6.69}$$

$$N'(1) = \lambda E(I)Q\left\{\left(1+\alpha E(R)\right)+\bar{G}_1[\alpha]\left(p\alpha E(V)-1-\alpha E(R)\right)\right\} \tag{6.70}$$

$$\begin{aligned} N''(1) = {} & 2Q\left(\lambda E(I)\right)^2\left\{\left(\frac{\alpha E(R)}{\lambda E(I)}+\frac{1}{2}\alpha E(R^2)\right)\right. \\ & +\bar{G}_1[\alpha]\left(-pE(V)-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)+\frac{1}{2}p\alpha E(V^2)\right) \\ & \left.+\bar{G}_1'[\alpha]\left(1-p\alpha E(V)+\alpha E(R)\right)\right\} \\ & +Q\lambda E(I(I-1))\left\{\left(1+\alpha E(R)\right)+\bar{G}_1[\alpha]\left(p\alpha E(V)-1-\alpha E(R)\right)\right\} \end{aligned} \tag{6.71}$$

$$D'(1) = -\lambda E(I)\left(1+\alpha E(R)\right)+\bar{G}_1[\alpha]\left\{\alpha+\lambda E(I)\left(1+\alpha E(R)-p\alpha E(V)\right)\right\} \tag{6.72}$$

$$\begin{aligned} D''(1) = {} & 2\left(\lambda E(I)\right)^2\left\{\left(-\frac{1}{\lambda E(I)}-\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}\alpha E(R^2)\right)\right. \\ & +\bar{G}_1[\alpha]\left(pE(V)+\frac{\alpha E(R)}{\lambda E(I)}-\frac{1}{2}p\alpha E(V^2)+\frac{1}{2}\alpha E(R^2)\right) \\ & \left.+\bar{G}_1'[\alpha]\left(p\alpha E(V)-1-\frac{\alpha}{\lambda E(I)}-\alpha E(R)\right)\right\} \\ & +\lambda E(I(I-1))\left\{\left(-1-\alpha E(R)\right)+\bar{G}_1[\alpha]\left(1+\alpha E(R)-p\alpha E(V)\right)\right\} \end{aligned} \tag{6.73}$$

The results obtained in (6.62) through (6.73) agree with the results obtained in chapter 3 in which we studied a queueing system providing single service.

6.6.2 No System Breakdowns

Assuming that the system never fails implies $\alpha = 0$ and hence $R_q(z) = 0$. Using this assumption in the main results of the chapter we get

$$P_q^{(\kappa 1)}(z) = \frac{\theta\left(\bar{G}_1[\lambda - \lambda C(z)] - 1\right)Q}{z - \left(\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right)\left(p\bar{B}[\lambda - \lambda C(z)] + (1-p)\right)} \tag{6.74}$$

$$P_q^{(\kappa 2)}(z) = \frac{(1-\theta)\left(\bar{G}_2[\lambda - \lambda C(z)] - 1\right)Q}{z - \left(\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right)\left(p\bar{B}[\lambda - \lambda C(z)] + (1-p)\right)} \tag{6.75}$$

$$V_q(z) = \frac{p\left(\bar{B}[\lambda - \lambda C(z)] - 1\right)\left\{\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right\}Q}{z - \left(\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right)\left(p\bar{B}[\lambda - \lambda C(z)] + (1-p)\right)} \tag{6.76}$$

$$W_q(z) = P_q^{(\kappa 1)}(z) + P_q^{(\kappa 2)}(z) + V_q(z) =$$
$$\frac{\left\{-1 + \left(\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right)\left(p\bar{B}[\lambda - \lambda C(z)] + (1-p)\right)\right\}Q}{z - \left(\theta\bar{G}_1[\lambda - \lambda C(z)] + (1-\theta)\bar{G}_2[\lambda - \lambda C(z)]\right)\left(p\bar{B}[\lambda - \lambda C(z)] + (1-p)\right)} \tag{6.77}$$

We find Q using the expression for $W_q(z)$ obtained in (6.77) and the normalization condition. Hence we get

$$Q = 1 - \lambda E(I)\left[\theta E(S_1) + (1-\theta)E(S_2) + pE(V)\right] \tag{6.78}$$

where $\bar{G}_1[0] = 1$, $-\bar{G}_1'[0] = E(S_1)$ is the mean time for the first kind of service and $-\bar{G}_2'[0] = E(S_2)$ is the mean time for the second kind of service. Further, we compute $N'(1)$, $N''(1)$, $D'(1)$, and $D''(1)$ using $W_q(z)$ as appeared in (6.77)

$$N'(1) = \lambda E(I)Q\left(\theta E(S_1) + (1-\theta)E(S_2) + pE(V)\right) \tag{6.79}$$

$$\begin{aligned} N''(1) = {} & Q\left(\lambda E(I)\right)^2\left[2pE(V)\left(\theta E(S_1) + (1-\theta)E(S_2)\right) + pE(V^2) + \theta E(S_1^2) + (1-\theta)E(S_2^2)\right] \\ & + Q\lambda E(I(I-1)\left(pE(V) + \theta E(S_1) + (1-\theta)E(S_2)\right) \end{aligned} \tag{6.80}$$

$$D'(1) = 1 - \lambda E(I)\left(\theta E(S_1) + (1-\theta)E(S_2) + pE(V)\right) \tag{6.81}$$

$$\begin{aligned} D''(1) = {} & -\left(\lambda E(I)\right)^2\left[2pE(V)\left(\theta E(S_1) + (1-\theta)E(S_2)\right) + pE(V^2) + \theta E(S_1^2) + (1-\theta)E(S_2^2)\right] \\ & - \lambda E(I(I-1)\left(pE(V) + \theta E(S_1) + (1-\theta)E(S_2)\right) \end{aligned} \tag{6.82}$$

where $\bar{G}_1''[0] = E(S_1^2)$ and $\bar{G}_2''[0] = E(S_2^2)$ is the second moment of kind 1 and kind 2 service times, respectively. Using equations (6.79) – (6.82) in equation (6.57) we can obtain L_q, and therefore W_q.

This special case, where no breakdowns occur, was the model studied by Madan, Al-Rawi and Al-Nasser (2004).

6.7 A Numerical Illustration

For the numerical illustration purpose, we use the general results obtained in equations (6.50) through (6.61). We assume that the service times, vacation time and repair time are all exponential with rates μ_1 , μ_2, γ and β, respectively. Accordingly, we use the following substitutions in equations (6.50) through (6.61)

$$\bar{G}_1[\alpha]=\frac{\mu_1}{\alpha+\mu_1},\quad \bar{G}_1'[\alpha]=\frac{-\mu_1}{(\alpha+\mu_1)^2},\quad \bar{G}_2[\alpha]=\frac{\mu_2}{\alpha+\mu_2},\quad \bar{G}_2'[\alpha]=\frac{-\mu_2}{(\alpha+\mu_2)^2}$$

$$E(V)=\frac{1}{\gamma},\quad E(V^2)=\frac{2}{\gamma^2},\quad E(R)=\frac{1}{\beta},\quad E(R^2)=\frac{2}{\beta^2}$$

Further, we assume $\mu_1=5$, $\mu_2=6$, $\alpha=8$, $\beta=10$ and $\gamma=7$, while p and θ both taking the values 0.25, 0.5 and 0.75. Also we assume single arrivals with rate $\lambda=2$, which requires the following substitutions

$$C(z)=z,E(I)=1 \text{ and } E(I(I-1))=0.$$

We select values of system parameters such that the steady state condition holds.

Table 6.1 *Computed values of various queue characteristics for vacation queue with breakdown & two kinds of service,* $\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$

p	θ	Q	ρ	L_q	W_q	$P_q^{(\kappa 1)}(1)$	$P_q^{(\kappa 2)}(1)$	$V_q(1)$	$R_q(1)$	$W_q(1)$
0.25	0.25	0.301	0.699	1.9907	0.9954	0.0921	0.2566	0.0714	0.2789	0.699
0.25	0.5	0.2718	0.7282	2.3419	1.171	0.1892	0.1756	0.0714	0.2919	0.7281
0.25	0.75	0.2411	0.7589	2.8097	1.4049	0.2917	0.0903	0.0714	0.3056	0.759
0.5	0.25	0.2296	0.7704	2.8078	1.4039	0.0921	0.2566	0.1429	0.279	0.7706
0.5	0.5	0.2004	0.7996	3.4084	1.7042	0.1892	0.1757	0.1428	0.2919	0.7996
0.5	0.75	0.1696	0.8304	4.2732	2.1366	0.2917	0.0903	0.1428	0.3055	0.8303
0.75	0.25	0.1581	0.8419	4.3606	2.1803	0.0921	0.2565	0.2142	0.2789	0.8417
0.75	0.5	0.129	0.871	5.6589	2.8295	0.1892	0.1757	0.2143	0.292	0.8712
0.75	0.75	0.0982	0.9018	7.8663	3.9332	0.2917	0.0903	0.2143	0.3055	0.9018

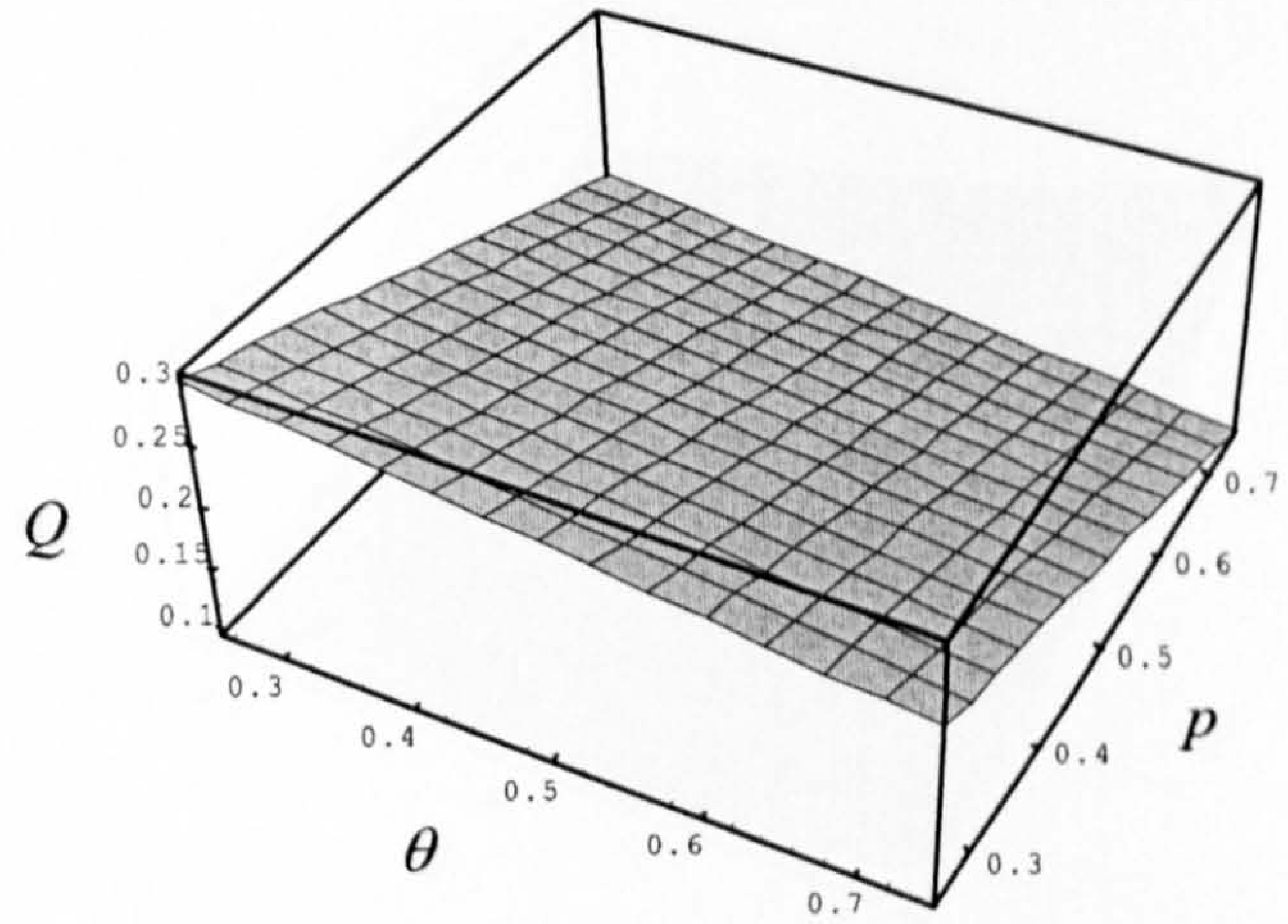

Figure 6.1 *Effect of p and θ on the proportion of time that the server is idle Q ($\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

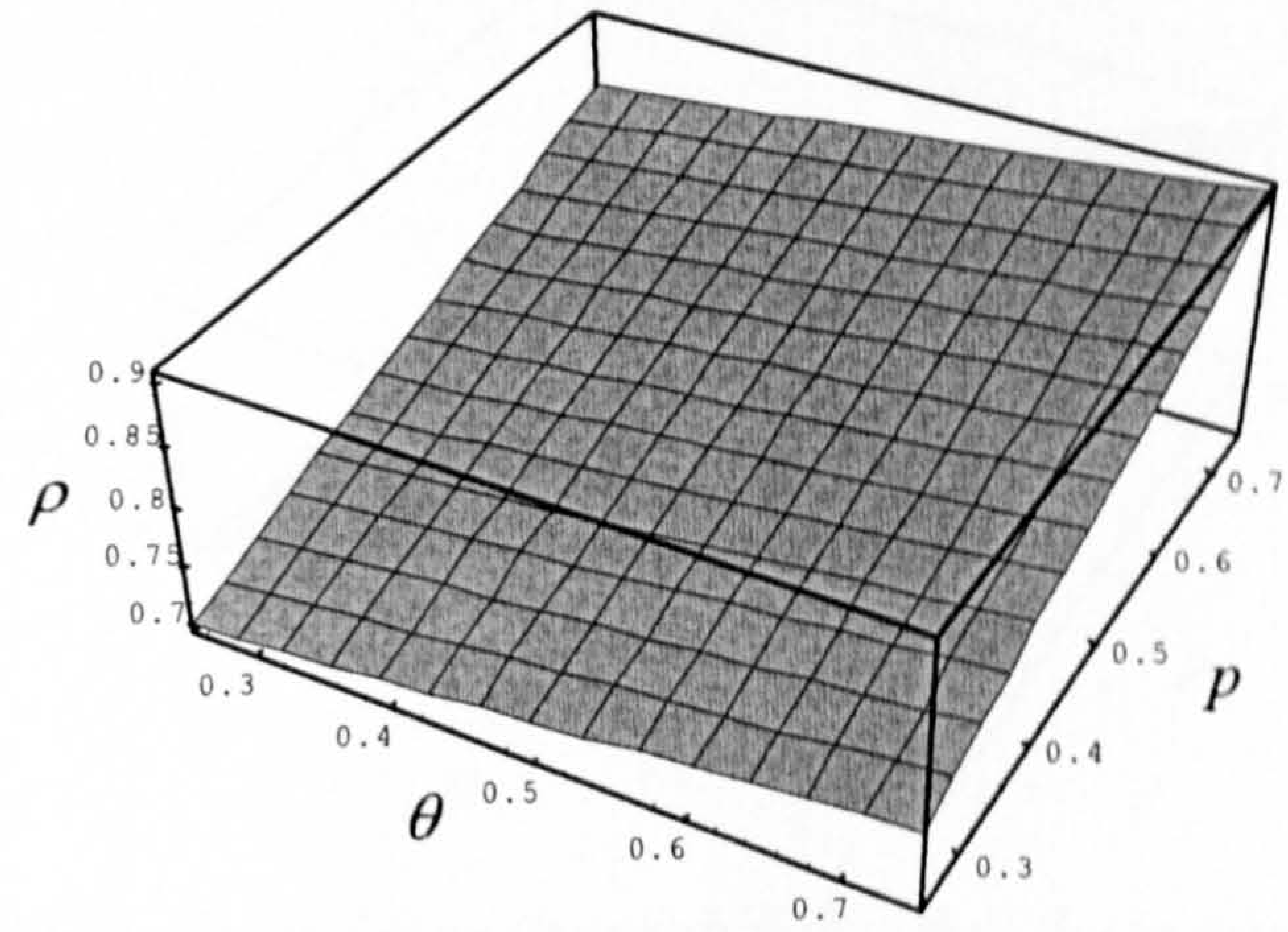

Figure 6.2 *Effect of p and θ on the utilization factor ρ ($\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

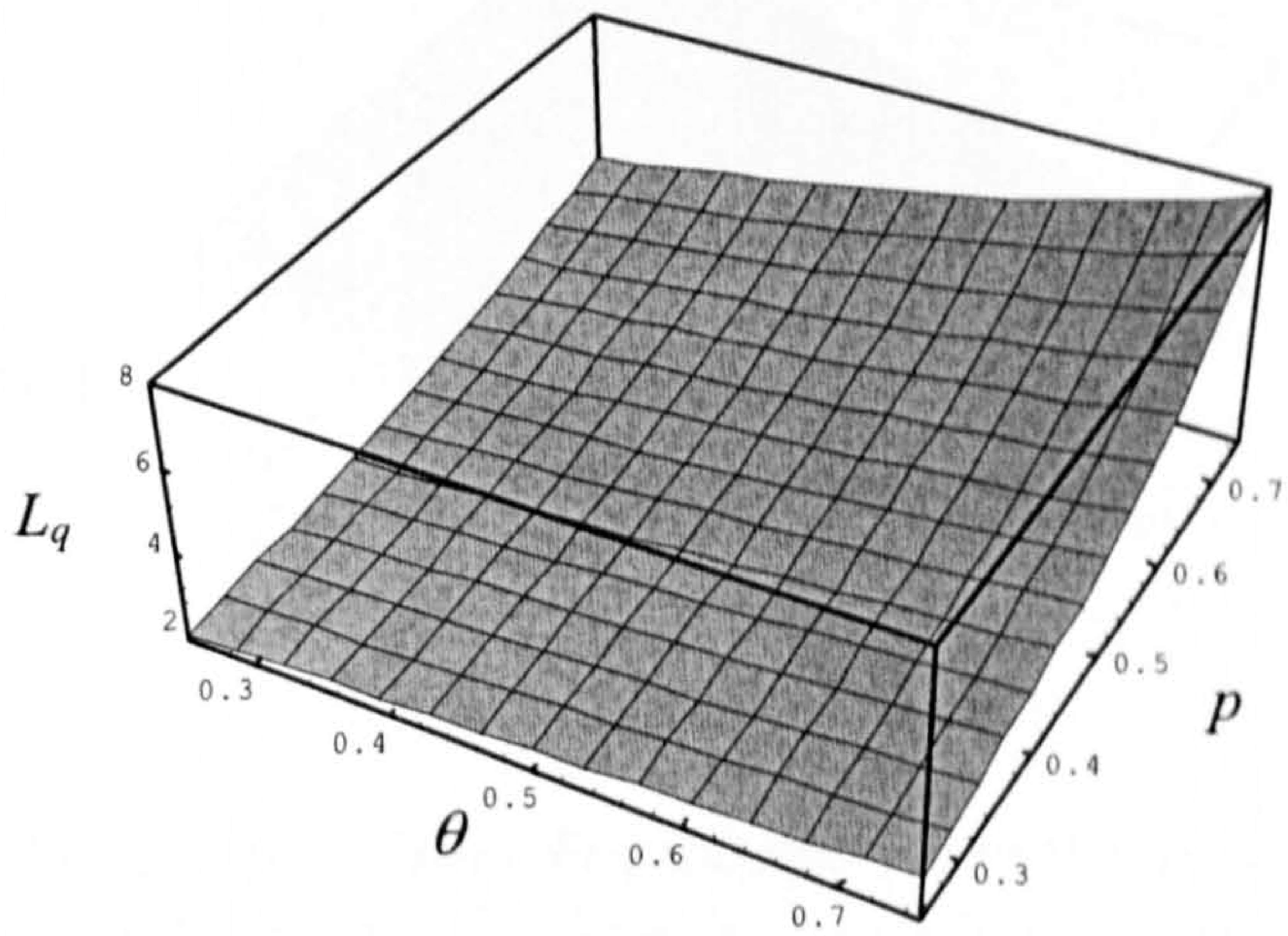

Figure 6.3 *Effect of p and θ on the mean queue size L_q ($\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

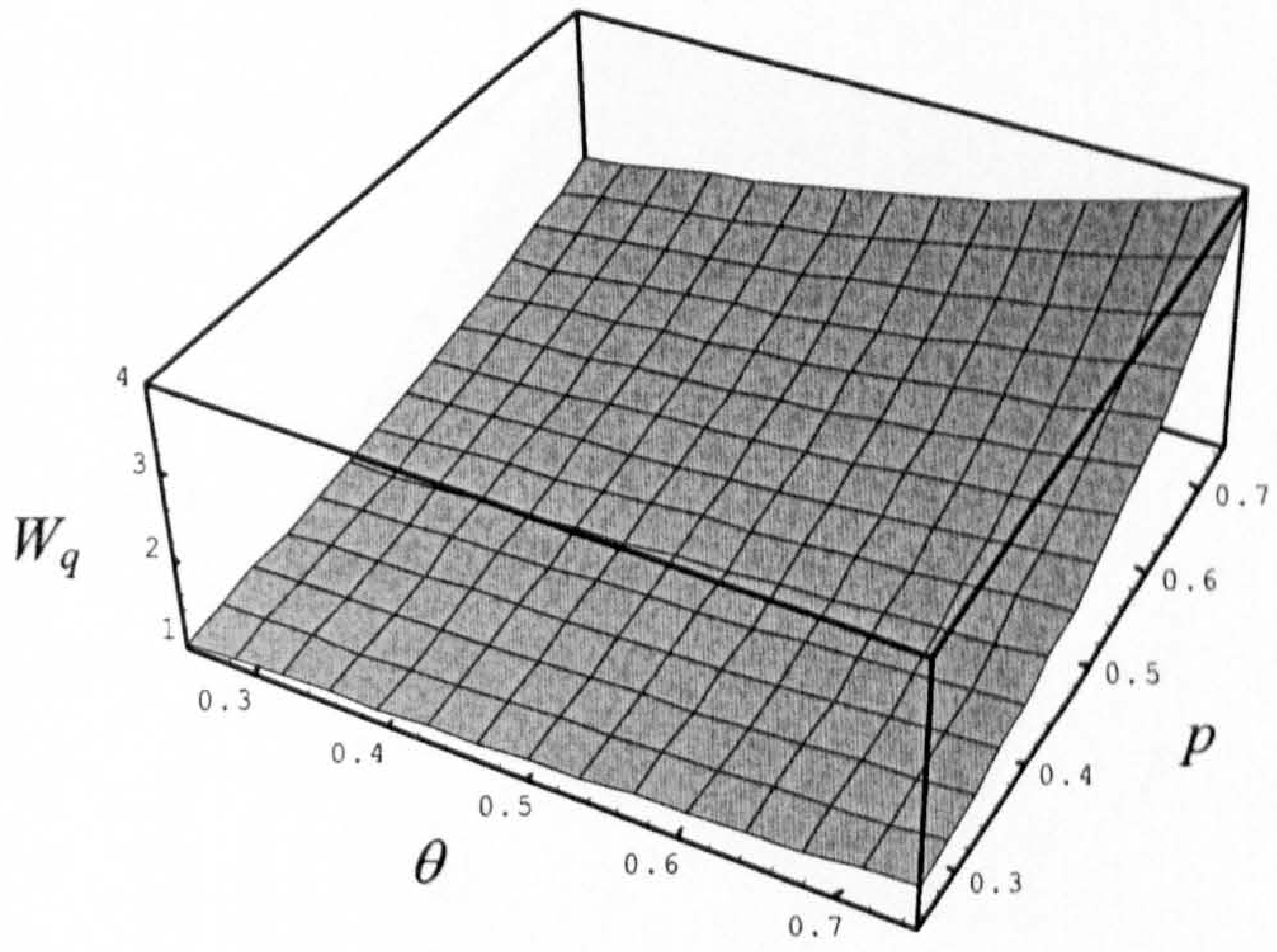

Figure 6.4 *Effect of p and θ on the mean waiting time* W_q
($\lambda = 2, \mu_1 = 5, \mu_2 = 6, \alpha = 8, \beta = 10, \gamma = 7$)

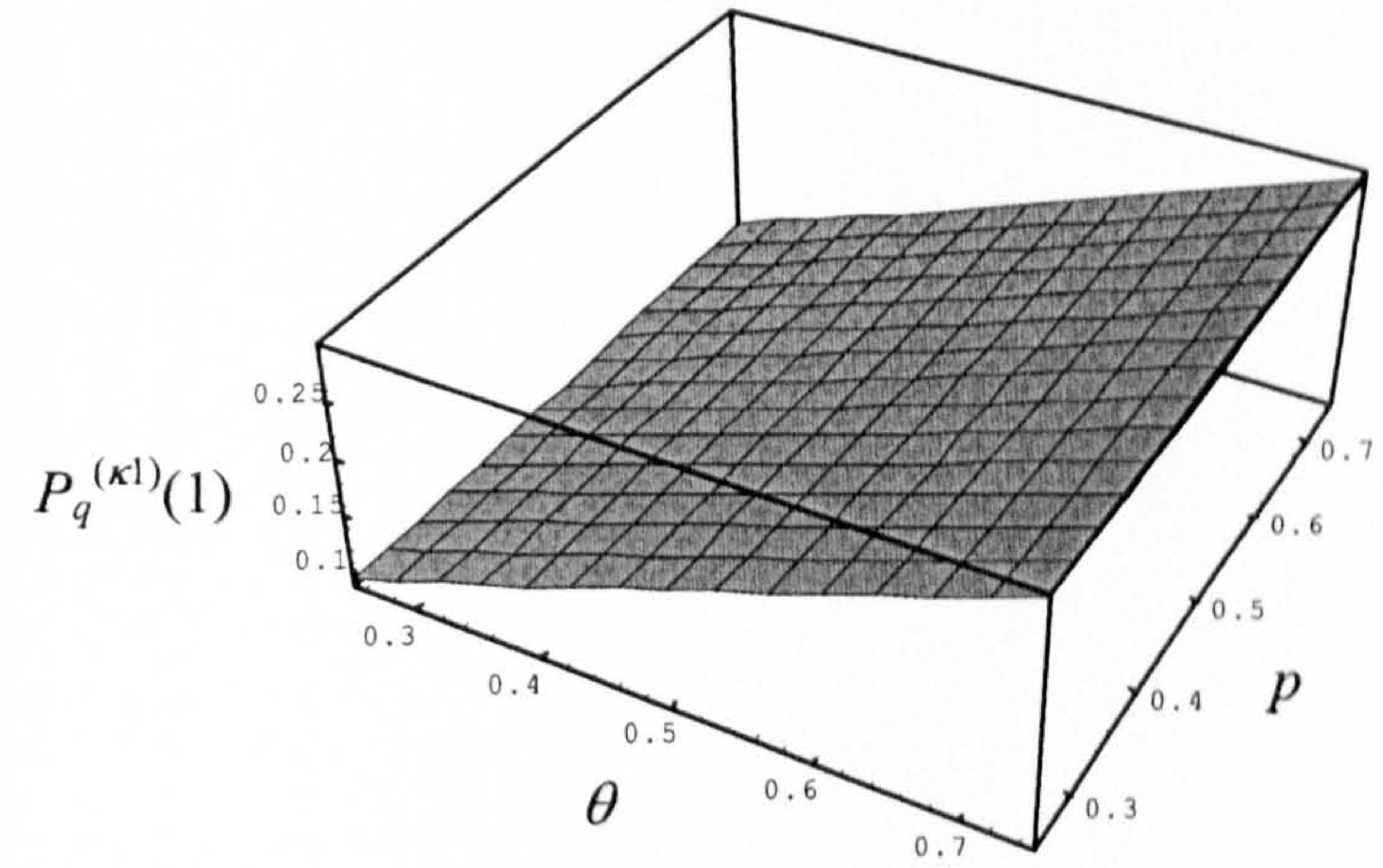

Figure 6.5 *Effect of p and θ on the probability that the server is providing the first kind of service* ($\lambda = 2, \mu_1 = 5, \mu_2 = 6, \alpha = 8, \beta = 10, \gamma = 7$)

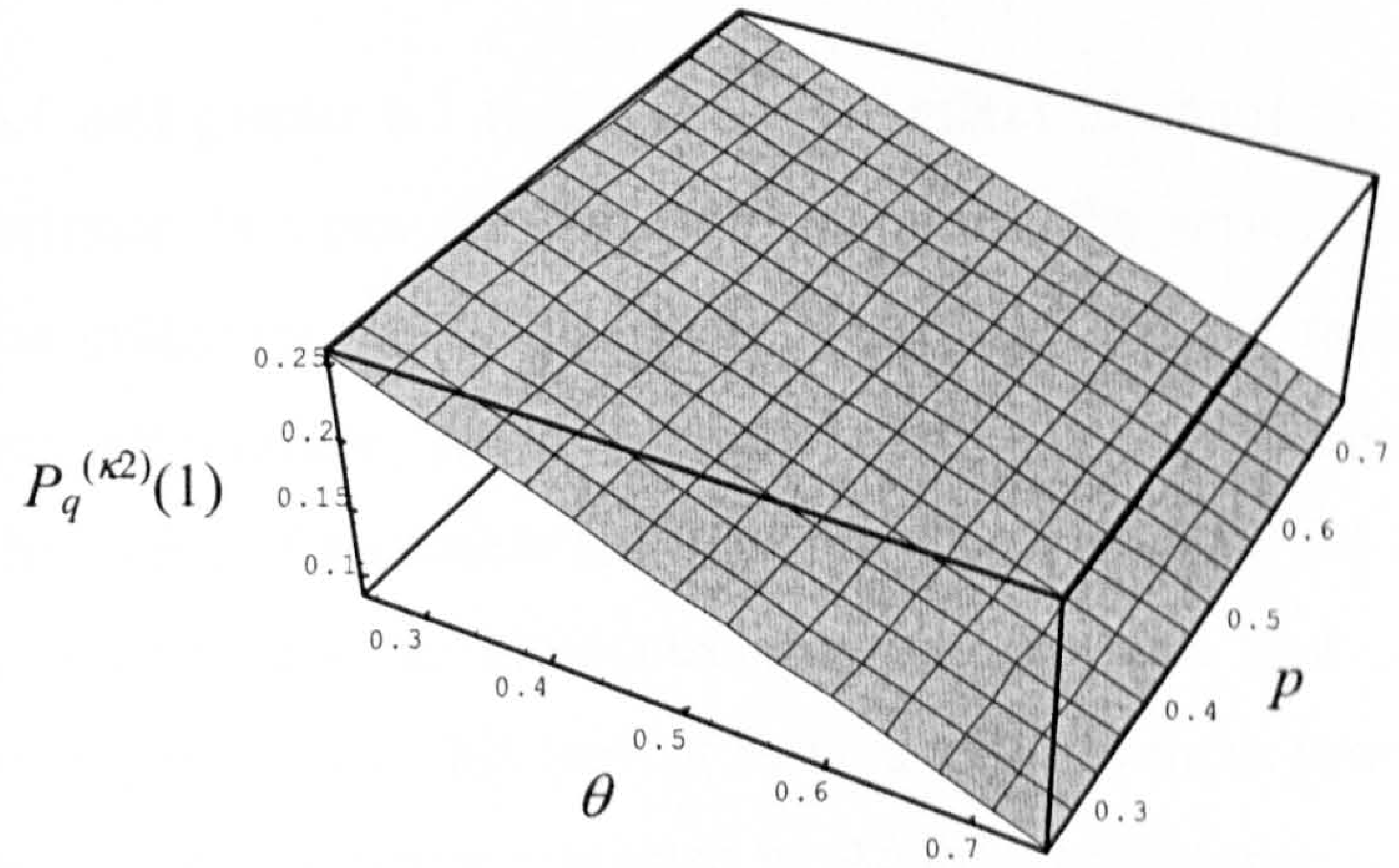

Figure 6.6 *Effect of p and θ on the probability that the server is providing the second kind of service*($\lambda = 2, \mu_1 = 5, \mu_2 = 6, \alpha = 8, \beta = 10, \gamma = 7$)

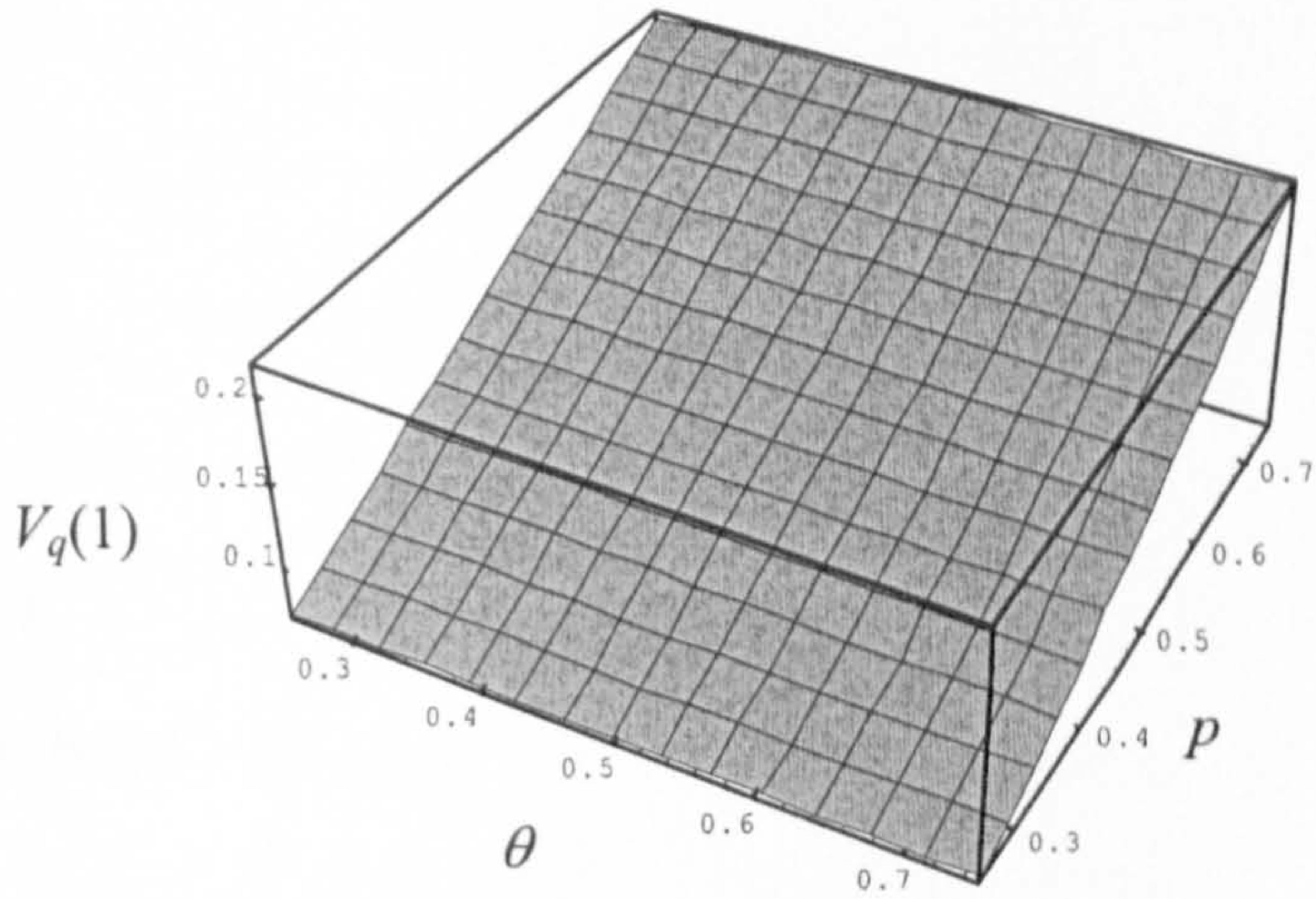

Figure 6.7 *Effect of p & θ on the probability that the server is on vacation ($\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

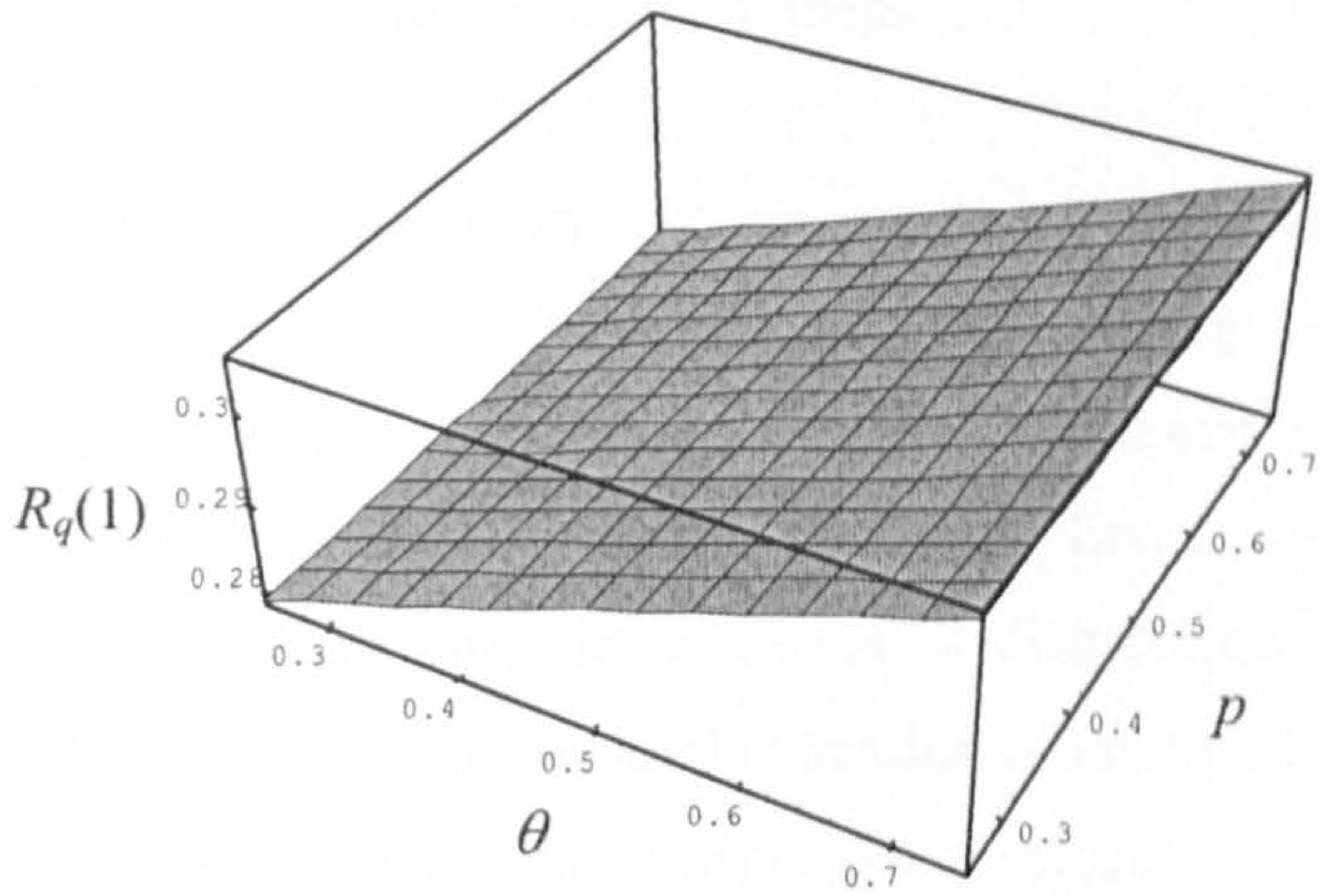

Figure 6.8 *Effect of p and θ on the probability that the system is under repair ($\lambda = 2$, $\mu_1 = 5$, $\mu_2 = 6$, $\alpha = 8$, $\beta = 10$, $\gamma = 7$)*

Table 6.1 and graphs 6.1 to 6.8 show the effect of changing p and θ on queue characteristics. We can see that as θ increases, the server idle time decreases, while the utilization factor, the mean queue size and the mean waiting time of customers, all increase. This is acceptable knowing that the assumed service rate for the first kind of service is smaller than the service rate for the second kind of service. Accordingly, as more customers require the first service, the queue becomes larger because this type of service requires more time. These numerical trends agree with the tables presented by Anabosi and Madan (2003).

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Chapter 7

Conclusions

In this research, the classical $M^{[X]}/G/1$ vacation queue and the classical $M^{[X]}/G/1$ with breakdowns have been extended by considering an $M^{[X]}/G/1$ queueing system which takes into consideration both phenomena.

A batch arrival queueing system with Bernoulli schedule server vacations and random breakdowns has been analysed. In this queueing system, it has been assumed that the customers arrive to the system in batches of variable size, but are served individually by a single server in a first come, first served basis. It has been assumed that the service times have a general distribution. After any service completion, the server may take a single vacation of random length. On the other hand, it has also been assumed that the system is subject to random breakdowns. Whenever the system breaks down, the customer whose service is interrupted comes back to the head of the queue and the system undergoes a repair process of variable length.

Introducing the elapsed service time as a supplementary variable enabled us to obtain a set of time dependent differential equations. It has been shown how to solve these equations to obtain the queue length at an arbitrary point of time. The solutions to these equations were derived in equations (2.33) – (2.37). These results have been used to obtain steady state solutions given in equations (2.38) – (2.41).

This queueing model has been generalised by assuming arbitrary distributions for vacation times and repair times in which extra supplementary variables are needed to represent the elapsed vacation time and the elapsed repair time. Steady state probability generating functions for queue length have been obtained as expressed by equations (3.41) – (3.44).

Assuming the service consists of two heterogeneous stages extends the above mentioned problem to a wider range of queues. In this case, it has been assumed that both stages of service are subject to breakdowns, and the server is allowed to take a vacation only after serving a customer, that is after the second stage of service. Steady state results of such a queueing system have been derived in equations (4.52) – (4.56).

A batch arrival queueing system with a second optional service, Bernoulli vacations and random breakdowns have been studied as well. In this model, the customers are provided with an essential service followed by an optional service in which only a portion of arriving customers require the second optional service. Equations (5.51) – (5.56) have been derived to provide the steady state queue size distribution at a random epoch for this queueing model.

In a different queueing system, but under similar assumptions of vacations and breakdowns, it has been assumed that the server provides two types of heterogeneous service in which both services are subject to random breakdowns. For this model, equations (6.50) – (6.55) have been derived for the steady state queue size distribution.

For each of the above mentioned queueing models, the necessary and sufficient condition for the system to be stable and some useful performance measures such as the mean queue size and the mean waiting time in the queue have been derived. Also some important particular cases have been discussed especially when dropping the assumption of vacations or breakdowns which reduces the problem to some models investigated earlier by queueing theorists, in an attempt to connect the current research with the literature on queueing theory. Numerical results have been calculated and some three-dimensional graphs have been presented for each of the queueing models discussed. These illustrations show the effect of queue parameters, especially vacation and breakdown parameters, on different queue measures such as the mean queue size, the mean waiting time, the utilization factor, the proportion of time that the server is idle, etc. Overall these results illustrate the general point that introducing vacations and breakdowns in the queueing model affects the queue behavior. When the probability that the server takes a vacation or the rate of breakdowns increase, the utilization factor, the mean queue size and the mean waiting time all increase while the proportion of time that the server is idle decreases. On the other hand, it has been shown that when the rate of vacation completion or repair completion increases, the proportion of server's idle time increases, while the utilization factor, the mean queue size and the mean waiting time all decrease. The trends that have been discovered were not unexpected.

The queueing systems have been investigated in this research and the results obtained, can model many practical problems where the server are not continuously available for providing service for arriving customers. These results provide vital information useful for management, manufacturing industry, communication networks and other fields in which the need to take decisions on systems having queues is essential.

Based on the findings of this research, the researcher suggests further studies to be conducted on the following queueing systems:

- Batch arrival queueing systems with random breakdowns and server vacations based on multiple vacation policy or *N*-policy. These vacation models are more complicated than the single vacation policy.
- Single server vacation queue with batch arrival and breakdowns in which the system does not enter a repair process once it fails, but it needs to wait for some time of random length till it starts being repaired. This problem requires introducing a random variable for the waiting time of a system to be repaired.
- Batch arrival queueing systems with random breakdowns and Bernoulli server vacations in which the server is not allowed to take vacations until the system becomes empty. In this problem, the server provides an exhaustive service.
- Batch arrival queueing systems with random breakdowns, Bernoulli schedule vacations and two-stage heterogeneous service in which the

breakdown rates are different for the two stages of service. Having different breakdown rates makes the problem a more complicated one than the one discussed in chapter 4. However, this is a more generic model.

- Batch arrival queueing systems with random breakdowns, Bernoulli schedule vacation based on a single vacation policy and a second optional service in which both services are subjected to breakdowns. This will be a generalization of the results obtained in chapter 5.
- Batch arrival queueing systems with random breakdowns, Bernoulli schedule vacation based on a single vacation policy and a second optional service provided in an additional service channel. This is not a single server queueing model anymore.
- Single server finite capacity queue with vacation time and repair time. Having finite system capacity makes it a more complicated model because we have to keep tracking of when the service facility becomes full.
- Single server queueing system with server vacations and random breakdowns in which customers arrive to the system in batches of variable size and the service is provided to the customer in batches of variable size. In this case, a new random variable should be introduced for the size of batches of customers in the service.
- Queueing systems with server vacations, random breakdowns and impatient customers. It could be assumed that a customer leaves the service station upon system breakdown or when vacation time or repair time exceeds a given limit.
- A batch arrival queueing system with server vacations and random breakdowns in which the stochastic processes involved in the system are not independent of each other.

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