HYPERFINITE CONSTRUCTION OF G-EXPECTATION

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ABSTRACT. The *hyperfinite G-expectation* is a nonstandard discrete analogue of G-expectation (in the sense of Robinsonian nonstandard analysis). A *lifting* of a continuous-time G-expectation operator is defined as a hyperfinite G-expectation which is infinitely close, in the sense of nonstandard topology, to the continuous-time G-expectation. We develop the basic theory for hyperfinite G-expectations and prove an existence theorem for liftings of (continuous-time) G-expectation. For the proof of the lifting theorem, we use a new discretization theorem for the G-expectation (also established in this paper, based on the work of Dolinsky, Nutz and Soner [Stoch. Proc. Appl. 122, (2012), 664–675]).
Keywords: G-expectation; Volatility uncertainty; Weak limit theorem; Lifting theorem; Nonstandard analysis; Hyperfinite discretization.

1. INTRODUCTION

Dolinsky et al. [8] showed a Donsker-type result for G-Brownian motion by introducing a notion of volatility uncertainty in discrete time and defined a discrete version of *Peng's* G-expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to G-expectation. In their discretization, Dolinsky et al. [8] allow for martingale laws whose support is the whole set of reals in a d-dimensional setting. In other words, they only discretize the time line, but not the state space of the canonical process. Now for certain applications, for example, a hyperfinite construction of G-expectation in the sense of Robinsonian nonstandard analysis, a discretization of the state space would be necessary. Thus, we develop a modification of the construction by Dolinsky et al. [8] which even ensures that the sublinear expectation operator for the discrete-time canonical process corresponding to this discretization of the state space (whence the martingale laws are supported by a finite lattice only) converges to the G-expectation. Further, we prove a lifting theorem, in the sense of Robinsonian nonstandard analysis, for the G-expectation. Herein, we use the discretization result for the G-expectation.

Nonstandard analysis makes consistent use of infinitesimals in mathematical analysis based on techniques from mathematical logic. This approach is very promising because it also allows, for instance, to study continuous-time stochastic processes as formally finite objects. Many authors have applied nonstandard analysis to problems in measure theory, probability theory and mathematical economics (see for example, Anderson and Raimondo [3] and the references therein or the contribution in Berg [4]), especially after Loeb [20] converted nonstandard

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measures (i.e. the images of standard measures under the nonstandard embedding *) into real-valued, countably additive measures, by means of the standard part operator and *Caratheodory*'s extension theorem. One of the main ideas behind these applications is the extension of the notion of a finite set known as *hyperfinite set* or more causally, a formally finite set. Very roughly speaking, hyperfinite sets are sets that can be formally enumerated with both standard and nonstandard natural numbers up to a (standard or nonstandard, i.e. unlimited) natural number.

Anderson [2], Keisler [16], Lindstrøm [19], Hoover and Perkins [14], a few to mention, used Loeb's [20] approach to develop basic nonstandard stochastic analysis and in particular, the nonstandard Itô calculus. Loeb [20] also presents the construction of a Poisson processes using nonstandard analysis. Anderson [2] showed that Brownian motion can be constructed from a hyperfinite number of coin tosses, and provides a detailed proof using a special case of Donsker's theorem. Anderson [2] also gave a nonstandard construction of stochastic integration with respect to his construction of Brownian motion. Keisler [16] uses Anderson's [2] result to obtain some results on stochastic differential equations. Lindstrøm [19] gave the hyperfinite construction (*lifting*) of L^2 standard martingales. Using nonstandard stochastic analysis, Perkins [24] proved a global characterization of (standard) Brownian local time. In this paper, we do not work on the Loeb space because the G-expectation and its corresponding G-Brownian motion are not based on a classical probability measure, but on a set of martingale laws.

The aim of this paper is to give two approximation results on G-expectation. First, to refine the discretization of G-expectation by Dolinsky et al. [8], in order to obtain a discretization of the sublinear expectation where the martingale laws are defined on a finite lattice rather than the whole set of reals. Second, to give an alternative, combinatorially inspired construction of the G-expectation based on the discretization result. We hope that this result may eventually become useful for applications in financial economics (especially existence of equilibrium on continuous-time financial markets with volatility uncertainty) and provides additional intuition for *Peng*'s G-stochastic calculus. We begin the nonstandard treatment of the G-expectation by defining a notion of S-continuity, a standard part operator, and proving a corresponding lifting (and pushing down) theorem. Thereby, we show that our hyperfinite construction is the appropriate nonstandard analogue of the G-expectation.

The rest of this paper is divided into two parts: in the first part, Section 2, we define Peng's G-expectation and introduce a discrete-time analogue of a G-expectation in the spirit of Dolinsky et al. [8]. Unlike in Dolinsky et al. [8], we require the discretization of the martingale laws to be defined on a finite lattice rather than the whole set of reals. In the continuous-time limit, the resulting sublinear expectation converges weakly to the continuous-time G-expectation. In the second part, Section 3, we develop the basic theory for hyperfinite G-expectations and prove an existence theorem for liftings of (continuous-time) G-expectation. We extend the discrete time analogue of the G-expectation in Section 2 to a hyperfinite time analogue. Then, we use the characterization of convergence in nonstandard analysis to prove that the hyperfinite discrete-time analogue of the G-expectation is infinitely close in the sense of nonstandard topology to the continuous-time G-expectation.

2. Weak approximation of G-expectation with discrete state space

Peng [23] introduced a sublinear expectation on a well-defined space $\mathbb{L}^1_G$, the completion of $\mathrm{Lip}_{b.cyl}(\Omega)$ (bounded and Lipschitz cylinder function) under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, under which the increments of the canonical process $(B_t)_{t>0}$ are zero-mean, independent and stationary and can be proved to be (G)-normally distributed. This type of process is called *G-Brownian motion* and the corresponding sublinear expectation is called *G-expectation*.

The G-expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear operator defined on a class of random variables on Ω. The symbol G refers to a given function

$$G(\gamma) := \frac{1}{2}\sup_{c\in\mathbf{D}} c\gamma : \mathbb{R} \to \mathbb{R} \tag{1}$$

where $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$ is a nonempty, compact and convex set, and $0 \le r_{\mathbf{D}} \le R_{\mathbf{D}} < \infty$ are fixed numbers. The construction of the G-expectation is as follows. Let $\xi = f(B_T)$, where B_T is the G-Brownian motion and f a sufficiently regular function. Then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0,0)$ of the solution of the nonlinear backward heat equation,

$$-\partial_t u - G(\partial^2_{xx} u) = 0,$$

with terminal condition $u(\cdot, T) = f$, Pardoux and Peng [22]. The mapping $\mathcal{E}^G$ can be extended to random variables of the form $\xi = f(B_{t_1}, \cdots, B_{t_n})$ by a stepwise evaluation of the PDE and then to the completion $\mathbb{L}^1_G$ of the space of all such random variables (cf. Dolinsky et al. [8]). Denis et al. [7] showed that $\mathbb{L}^1_G$ is the completion of $\mathcal{C}_b(\Omega)$ and $\mathrm{Lip}_{b.cyl}(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, and that $\mathbb{L}^1_G$ is the space of the so-called quasi-continuous function and contains all bounded continuous functions on the canonical space Ω, but not all bounded measurable functions are included. Ruan [27] introduced the invariance principle of G-Brownian motion using the theory of sublinear expectation. There also exists an equivalent alternative representation of the G-expectation known as the *dual view on G-expectation via volatility uncertainty*, see Denis et al. [7]:

$$\mathcal{E}^G(\xi) = \sup_{P\in\mathcal{P}^G} \mathbb{E}^P[\xi], \quad \xi = f(B_T), \tag{2}$$

where $\mathcal{P}^G$ is defined as the set of probability measures on Ω such that, for any $P \in \mathcal{P}^G$, B is a martingale with the volatility $d\langle B\rangle_t/dt \in \mathbf{D}$ $P\otimes dt$ a.e.

2.1. Continuous-time construction of sublinear expectation.

Let $\Omega = \{\omega \in \mathcal{C}([0,T];\mathbb{R}) : \omega_0 = 0\}$ be the canonical space endowed with the uniform norm $\|\omega\|_\infty = \sup_{0\le t\le T}|\omega_t|$, where $|\cdot|$ denotes the absolute value on $\mathbb{R}$. Let B be the canonical process $B_t(\omega) = \omega_t$, and $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ the filtration generated by B. A probability measure P on Ω is a martingale law provided B is a P-martingale and $B_0 = 0$ P a.s. Then, $\mathcal{P}_{\mathbf{D}}$ is the set of martingale laws on Ω and the volatility takes values in $\mathbf{D}$, $P\otimes dt$ a.e;

$$\mathcal{P}_{\mathbf{D}} = \{P \text{ martingale law on } \Omega\text{: } d\langle B\rangle_t/dt \in \mathbf{D},\ P\otimes dt \text{ a.e.}\}.$$

2.2. Discrete-time construction of sublinear expectation.

We denote

$$\mathcal{L}_n = \left\{\frac{j}{n\sqrt{n}}, \quad -n^2\sqrt{R_{\mathbf{D}}} \le j \le n^2\sqrt{R_{\mathbf{D}}}, \quad \text{for } j \in \mathbb{Z}\right\},$$

and $\mathcal{L}_n^{n+1} = \mathcal{L}_n \times \cdots \times \mathcal{L}_n (n+1 \text{ times})$, for $n \in \mathbb{N}$. Let $X^n = (X_k^n)_{k=0}^n$ be the canonical process $X_k^n(x) = x_k$ defined on $\mathcal{L}_n^{n+1}$ and $(\mathcal{F}_k^n)_{k=0}^n = \sigma(X_l^n, l = 0, \dots, k)$ be the filtration generated by X^n. We note that $R_{\mathbf{D}} = \sup_{\alpha \in \mathbf{D}} |\alpha|$.

$$\mathbf{D}'_n = \mathbf{D} \cap \left(\frac{1}{n}\mathbb{N}\right)^2$$

is a nonempty bounded set of volatilities. A probability measure P on $\mathcal{L}_n^{n+1}$ is a martingale law provided X^n is a P-martingale and $X_0^n = 0$ P a.s. The increment $\Delta X_k^n = X_k^n - X_{k-1}^n$. Let $\mathcal{P}_{\mathbf{D}}^n$ be the set of martingale laws of X^n on $\mathbb{R}^{n+1}$, i.e.,

$$\mathcal{P}_{\mathbf{D}}^n = \left\{ P \text{ martingale law on } \mathbb{R}^{n+1} \colon\, r_{\mathbf{D}} \leq |\Delta X_k^n|^2 \leq R_{\mathbf{D}},\, P \text{ a.s.} \right\},$$

such that for all n, $\mathcal{L}_n^{n+1} \subseteq \mathbb{R}^{n+1}$.
. In order to establish a relation between the continuous-time and discrete-time settings, we obtained a continuous-time process $\widehat{x}_t \in \Omega$ from any discrete path $x \in \mathcal{L}_n^{n+1}$ by linear interpolation. i.e.,

$$\widehat{x}_t := (\lfloor nt/T \rfloor + 1 - nt/T) x_{\lfloor nt/T \rfloor} + (nt/T - \lfloor nt/T \rfloor) x_{\lfloor nt/T \rfloor + 1}$$

where $\widehat{\ }\colon \mathcal{L}_n^{n+1} \to \Omega$ is the linear interpolation operator, $x = (x_0, \dots, x_n) \mapsto \widehat{x} = \{(\widehat{x})_{0 \leq t \leq T}\}$, and $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. If X^n is the canonical process on $\mathcal{L}_n^{n+1}$ and ξ is a random variable on Ω, then $\xi(\widehat{X}^n)$ defines a random variable on $\mathcal{L}_n^{n+1}$.

2.3. **Strong formulation of volatility uncertainty.** We consider martingale laws generated by stochastic integrals with respect to a fixed Brownian motion as in Dolinsky et al. [8], Nutz [21] and a fixed random walk as in Dolinsky et al. [8]. Continuous-time construction; let $\mathcal{Q}_{\mathbf{D}}$ be the set of martingale laws:

$$\mathcal{Q}_{\mathbf{D}} = \left\{ P_0 \circ (M)^{-1};\ M = \int f(t, B) dB_t,\ \text{and } f \in \mathcal{C}([0,T] \times \Omega; \sqrt{\mathbf{D}}) \text{ is adapted} \right\}.$$

B is the canonical process under the Wiener measure P_0.
Discrete-time construction; we fix $n \in \mathbb{N}$, $\Omega_n = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \{\pm 1\}, \quad i = 1, \dots, n\}$ equipped with the power set and let

$$P_n = \underbrace{\frac{\delta_{-1} + \delta_{+1}}{2} \otimes \cdots \otimes \frac{\delta_{-1} + \delta_{+1}}{2}}_{\text{n times}}$$

be the product probability associated with the uniform distribution where $\delta_x(A)$ is a Dirac measure for any $A \subseteq \mathbb{R}$ and a given $x \in A$. Let $\xi_1, \dots, \xi_n$ be an i.i.d sequence of $\{\pm 1\}$-valued random variables. The components of ξ_k are orthonormal in $L^2(P_n)$ and the associated scaled random walk is

$$\mathbb{X} = \frac{1}{\sqrt{n}} \sum_{l=1}^{k} \xi_l.$$

We denote by $\mathcal{Q}_{\mathbf{D}'_n}^n$ the set of martingale laws of the form:

$$\mathcal{Q}_{\mathbf{D}'_n}^n = \left\{ P_n \circ (M^{f,\mathbb{X}})^{-1};\ f : \{0, \dots, n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n} \text{ is } \mathcal{F}^n\text{-adapted.} \right\} \tag{3}$$

where $M^{f,\mathbb{X}} = \left(\sum_{l=1}^{k} f(l-1, \mathbb{X}) \Delta \mathbb{X}_l \right)_{k=0}^{n}$.

2.4. **Results and proofs.** Theorem 1 states that a sublinear expectation with discrete-time volatility uncertainty on our finite lattice converges to the G-expectation.

Lemma 2.1. $\mathcal{Q}^n_{\mathbf{D}} = \left\{ P_n \circ \left(M^{f,\mathbb{X}}\right)^{-1};\ f : \{0,\dots,n\} \times \mathbb{R}^{n+1} \to \sqrt{\mathbf{D}} \text{ is adapted} \right\}$. *Then* $\mathcal{Q}^n_{\mathbf{D}} \subseteq \mathcal{P}^n_{\mathbf{D}}$.

Proposition 2.2. *Let* $\xi : \Omega \to \mathbb{R}$ *be a continuous function satisfying* $|\xi(\omega)| \leq a(1+ \parallel \omega \parallel_\infty)^b$ *for some constants* $a, b > 0$. *Then,*

(i)

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{4}$$

(ii)

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \max_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{5}$$

To prove (4), we prove two separate inequalities together with a density argument. The left-hand side of (5) can be written as

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \sup_{f\in\mathcal{A}} \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)],$$

where $\mathcal{A} = \left\{ f : \{0,\dots,n\} \times \mathcal{L}^{n+1}_n \to \sqrt{\mathbf{D}'_n/n} \text{ is } \mathcal{F}^n\text{-adapted.} \right\}$. We prove that $\mathcal{A}$ is a compact subset of a finite-dimensional vector space, and that $f \mapsto \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ is continuous. Before then, we introduce a smaller space $\mathbb{L}^1_*$ that is defined as the completion of $\mathcal{C}_b(\Omega;\mathbb{R})$ under the norm (cf. Dolinsky et al. [8])

$$\parallel \xi \parallel_* := \sup_{Q\in\mathcal{Q}} \mathbb{E}^Q|\xi|, \quad \mathcal{Q} := \mathcal{P}_{\mathbf{D}} \cup \{P \circ (\widehat{X}^n)^{-1}; P \in \mathcal{P}^n_{\mathbf{D}/n}, n \in \mathbb{N}.\}.$$

This is because Proposition 2.2 will not hold if ξ just belong to $\mathbb{L}^1_G$, which is the completion of $\mathcal{C}_b(\Omega;\mathbb{R})$ under the norm

$$\parallel \xi \parallel_{\mathbb{L}^1_G} := \sup_{P\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[|\xi|]. \tag{6}$$

Proof of Proposition 2.2. First inequality (for $\leq$ in (4))*:*

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{7}$$

For all n, $\sqrt{\mathbf{D}'_n/n} \subseteq \sqrt{\mathbf{D}/n}$ and $\mathcal{Q}^n_{\mathbf{D}'_n} \subseteq \mathcal{Q}^n_{\mathbf{D}}$. It is shown in Dolinsky et al. [8] that

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{P}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

Since $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{P}_{\mathbf{D}}$ (see Dolinsky et al. [8, Remark 3.6]) and $\mathcal{Q}^n_{\mathbf{D}} \subseteq \mathcal{P}^n_{\mathbf{D}}$ (see Lemma 2.1), (7) follows.

Second inequality (for $\geq$ in (4))*:* It remains to show that

$$\liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

For arbitrary $P \in \mathcal{Q}_{\mathbf{D}}$, we construct a sequence $(P^n)_n$ such that for all n,

$$P^n \in \mathcal{Q}^n_{\mathbf{D}'_n/n}, \tag{8}$$

and

$$\mathbb{E}^{P}[\xi] \leq \liminf_{n\to\infty} \mathbb{E}^{P^n}[\xi(\widehat{X}^n)]. \tag{9}$$

For fixed n, we want to construct martingales M^n whose laws are in $\mathcal{Q}^n_{\mathbf{D}'_n/n}$ and the laws of their interpolations tend to P. Thus, we introduce a scaled random walk with the piecewise constant càdlàg property,

$$W^n_t := \frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt/T \rfloor} \xi_l = \frac{1}{\sqrt{n}} Z^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T, \tag{10}$$

and we denote the continuous version of (10) obtained by linear interpolation by

$$\widehat{W}^n_t := \frac{1}{\sqrt{n}} \widehat{Z}^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T. \tag{11}$$

By the central limit theorem; $(W^n, \widehat{W}^n) \Rightarrow (W, W)$ as $n \to \infty$ on $D([0,T];\mathbb{R}^2)$ ($\Rightarrow$ implies convergence in distribution). i.e., the law (P_n) converges to the law P_0 on the Skorohod space $D([0,T];\mathbb{R}^2)$ Billingsley [5, Theorem 27.1]. Let $g \in \mathcal{C}([0,T] \times \Omega, \sqrt{\mathbf{D}})$ such that

$$P = P_0 \circ \Bigg(\underbrace{\int g(t,W) dW_t}_{M} \Bigg)^{-1}.$$

Since g is continuous and $\widehat{W}^n_t$ is the interpolated version of (10),

$$\left(W^n, \left(g\left(\lfloor nt/T \rfloor T/n, \widehat{W}^n_t \right) \right)_{t\in[0,T]} \right) \Rightarrow \left(W, (g(t,W_t))_{t\in[0,T]} \right) \text{ as } n \to \infty \text{ on } D([0,T];\mathbb{R}^2).$$

We introduce martingales with discrete-time integrals,

$$M^n_k := \sum_{l=1}^{k} g\left((l-1)T/n, \widehat{W}^n \right) \widehat{W}^n_{lT/n} - \widehat{W}^n_{(l-1)T/n}. \tag{12}$$

In order to construct M^n which is "close" to M and also is such that $P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}$. We choose $\widetilde{h}_n : \{0, \cdots, n\} \times \Omega \to \sqrt{\mathbf{D}'_n/n}$ such that

$$d_{J_1}\left(\left(\widetilde{h}_n(\lfloor nt/T \rfloor T/n, \widehat{W}^n_t) \right)_{t\in[0,T]}, \left(g(\lfloor nt/T \rfloor T/n, \widehat{W}^n_t) \right)_{t\in[0,T]} \right)$$

is minimal (this is possible because there are only finitely many choices for $\left(\widetilde{h}_n(\lfloor nt/T \rfloor T/n, \widehat{W}^n_t) \right)_{t\in[0,T]}$) and d_{J_1} is the Kolmogorov metric for the Skorohod J_1 topology. From Billingsley [6, Theorem 4.3 and Definition 4.1], it follows that

$$\left(W^n, \left(\widetilde{h}_n\left(\lfloor nt/T \rfloor T/n, \widehat{W}^n_t \right) \right)_{t\in[0,T]} \right) \Rightarrow \left(W, g(t,W_t)_{t\in[0,T]} \right) \text{ on } D([0,T];\mathbb{R}^2).$$

We then define $g_n : \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n/n}$ by $g_n : (\ell, \vec{\mathbb{X}}) \mapsto \widetilde{h}_n(\ell, \widehat{\vec{X}})$. Let M^n be defined by

$$M^n_k = \sum_{l=1}^{k} g_n\left(l-1, \frac{1}{\sqrt{n}} Z^n \right) \frac{1}{\sqrt{n}} \Delta Z^n_l, \quad \forall k \in \{0, \cdots, n\}.$$

By stability of stochastic integral (see Duffie and Protter [9, Theorem 4.3 and Definition 4.1]),

$$\left(M^n_{\lfloor nt/T\rfloor}\right)_{t\in[0,T]} \Rightarrow M \quad \text{as } n\to\infty \text{ on } D([0,T];\mathbb{R})$$

because

$$M^n_{\lfloor nt/T\rfloor} = \sum_{l=1}^{\lfloor nt/T\rfloor} \widetilde{h}_n\left((l-1)T/n, \left(\widehat{W}_{kT/n}\right)_{k=0}^{n}\right)\Delta\widehat{W}_{lT/n}.$$

In addition, as n goes to ∞, the increments of M^n uniformly tend to 0. Thus, $\widehat{M}^n \Rightarrow M$ on Ω. Since ξ is bounded and continuous,

$$\lim_{n\to\infty}\mathbb{E}^{P_n\circ(M^n)^{-1}}[\xi(\widehat{X}^n)] = \mathbb{E}^{P_0\circ M^{-1}}[\xi]. \tag{13}$$

Therefore, (8) is satisfied for $P^n = P_n\circ(M^n)^{-1}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}$. Taking the lim inf as n tends to ∞ and the supremum over $P\in\mathcal{Q}_{\mathbf{D}}$, (13) becomes

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}}\mathbb{E}^P[\xi] \le \liminf_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}}\mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{14}$$

Combining (7) and (14),

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}}\mathbb{E}^P[\xi] \ge \limsup_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}}\mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \ge \liminf_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}}\mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \ge \sup_{P\in\mathcal{Q}_{\mathbf{D}}}\mathbb{E}^P[\xi].$$

Therefore,

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}}\mathbb{E}^P[\xi] = \lim_{n\to\infty}\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}}\mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{15}$$

Density argument: (4) is established for all $\xi\in\mathcal{C}_b(\Omega,\mathbb{R})$. Since $\mathcal{Q}_{\mathbf{D}}\subseteq\mathcal{P}_{\mathbf{D}}$ (see Dolinsky et al. [8, Remark 3.6]) and $\mathcal{Q}^n_{\mathbf{D}}\subseteq\mathcal{P}^n_{\mathbf{D}}$ (see Lemma 2.1), $\mathcal{Q}^n_{\mathbf{D}'_n}\subseteq\mathcal{Q}$ and $\mathcal{Q}_{\mathbf{D}}\subseteq\mathcal{Q}$. Thus, (4) holds for all $\xi\in\mathbb{L}^1_*$, and hence, holds for all ξ that satisfy condition of Proposition 2.2.

First part of 5: $\mathcal{A}$ is closed and obviously bounded with respect to the norm $\|\cdot\|_\infty$ as $\mathbf{D}'_n$ is bounded. By Heine-Borel theorem, $\mathcal{A}$ is a compact subset of a $N(n,n)$-dimensional vector space[1] equipped with the norm $\|\cdot\|_\infty$.

Second part of 5: Here, we show that $F: f\mapsto\mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ is continuous. From Proposition 2.2 we know that ξ is continuous, $\widehat{X}^n$ is the interpolated canonical process, i.e., $\widehat{X}:\mathcal{L}^{n+1}_n\to\Omega$, thus $\widehat{X}^n$ is continuous and P_n takes it values from the set of real numbers. For $F: f\mapsto\mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ to be continuous, $\psi: f\mapsto M^{f,\mathbb{X}}$ has to be continuous. Since $\mathcal{A}$ is a compact subset of a $N(n,n)$-dimensional vector space for fixed $n\in\mathbb{N}$ and $M^{f,\mathbb{X}}:\Omega_n\to\mathcal{L}^{n+1}_n$, for all $f,g\in\mathcal{A}$,

$$|M^{f,\mathbb{X}} - M^{g,\mathbb{X}}| = |\|f\|_\infty - \|g\|_\infty| \le \|f-g\|_\infty.$$

Thus, ψ is continuous with respect to the norm $\|\cdot\|_\infty$. Hence F is continuous with respect to any norm on $\mathbb{R}^{N(n,n)}$.

□

[1] The cardinality of $\mathcal{L}_n$, $\#\mathcal{L}_n = 2n+1$, $\#\mathcal{L}^{n+1}_n = (2n+1)^{n+1}$, and $\#(\{0,\dots,n\}\times\mathcal{L}^{n+1}_n) = (n+1)(2n+1)^{n+1} = N(n,n)$.

Theorem 1. Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$. Then,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] = \lim_{n\to\infty} \max_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{16}$$

Proof. The proof follows directly from Proposition 2.2. □

3. Nonstandard construction of G-expectation

3.1. **Hyperfinite-time setting.** Here we present the nonstandard version of the discrete-time setting of the sublinear expectation and the strong formulation of volatility uncertainty on the hyperfinite timeline.

Definition 3.1. ${}^*\Omega$ is the *-image of Ω endowed with the *-extension of the maximum norm ${}^*\|\cdot\|_\infty$.

${}^*\mathbf{D} = {}^*[r_{\mathbf{D}}, R_{\mathbf{D}}]$ is the *-image of $\mathbf{D}$, and as such it is *internal.*
It is important to note that $st : {}^*\Omega \to \Omega$ is the standard part map, and $st(\omega)$ will be referred to as the *standard part* of ω, for every $\omega \in {}^*\Omega$. ${}^\circ z$ denotes the standard part of a hyperreal z.

Definition 3.2. For every $\omega \in \Omega$, if there exists $\widetilde{\omega} \in {}^*\Omega$ such that $\|\widetilde{\omega} - {}^*\omega\|_\infty \simeq 0$, then $\widetilde{\omega}$ is a *nearstandard point* in ${}^*\Omega$. This will be denoted as $ns(\widetilde{\omega}) \in {}^*\Omega$.

For all hypernatural N, let

$$\mathcal{L}_N = \left\{ \frac{K}{N\sqrt{N}}, \quad -N^2\sqrt{R_{\mathbf{D}}} \leq K \leq N^2\sqrt{R_{\mathbf{D}}}, \quad K \in {}^*\mathbb{Z} \right\}, \tag{17}$$

and the hyperfinite timelime

$$\mathbb{T} = \left\{ 0, \frac{T}{N}, \cdots, -\frac{T}{N} + T, T \right\}. \tag{18}$$

We consider $\mathcal{L}_N^{\mathbb{T}}$ as the canonical space of paths on the hyperfinite timeline, and $X^N = (X^N_k)_{k=0}^N$ as the canonical process denoted by $X^N_k(\bar{\omega}) = \bar{\omega}_k$ for $\bar{\omega} \in \mathcal{L}_N^{\mathbb{T}}$. $\mathcal{F}^N$ is the internal filtration generated by X^N. The linear interpolation operator can be written as

$$\widetilde{\ } : \widehat{\ } \circ \iota^{-1} \to {}^*\Omega, \quad \text{for } \widetilde{\mathcal{L}_N^{\mathbb{T}}} \subseteq {}^*\Omega,$$

where

$$\widehat{\omega}(t) := (\lfloor Nt/T \rfloor + 1 - Nt/T)\omega_{\lfloor Nt/T \rfloor} + (Nt/T - \lfloor Nt/T \rfloor)\omega_{\lfloor Nt/T \rfloor + 1},$$

for $\omega \in \mathcal{L}_N^{N+1}$ and for all $t \in {}^*[0, T]$. $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y and $\iota : \mathbb{T} \to \{0, \cdots, N\}$ for $\iota : t \mapsto Nt/T$.

For the hyperfinite strong formulation of the volatility uncertainty, fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Consider $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$, and let P_N be the uniform counting measure on $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$. P_N can also be seen as a measure on $\mathcal{L}_N^{\mathbb{T}}$, concentrated on $\left\{\pm\frac{1}{\sqrt{N}}\right\}^{\mathbb{T}}$. Let $\Omega_N = \{\underline{\omega} = (\underline{\omega}_1, \cdots, \underline{\omega}_N); \underline{\omega}_i = \{\pm 1\}, i = 1, \cdots, N\}$, and let $\Xi_1, \cdots, \Xi_N$ be a *-independent sequence of $\{\pm 1\}$-valued random variables on Ω_N and the components of Ξ_k are orthonormal in $L^2(P_N)$. We denote the hyperfinite random walk by

$$\mathbb{X}_t = \frac{1}{\sqrt{N}} \sum_{l=1}^{Nt/T} \Xi_l \quad \text{for all } t \in \mathbb{T}.$$

The hyperfinite-time stochastic integral of some $F : \mathbb{T} \times \mathcal{L}_N^{\mathbb{T}} \to {}^*\mathbb{R}$ with respect to the hyperfinite random walk is given by

$$\sum_{s=0}^{t} F(s, \mathbb{X})\Delta\mathbb{X}_s : \Omega_N \to {}^*\mathbb{R}, \quad \underline{\omega} \in \Omega_N \mapsto \sum_{s=0}^{t} F(s, \mathbb{X}(\underline{\omega}))\Delta\mathbb{X}_s(\underline{\omega}).$$

Thus, the hyperfinite set of martingale laws can be defined by

$$\bar{\mathcal{Q}}^N_{\mathbf{D}'_N} = \left\{ \; P_N \circ (M^{F,\mathbb{X}})^{-1}; \; F : \mathbb{T} \times \mathcal{L}_N^{\mathbb{T}} \to \sqrt{\mathbf{D}'_N} \; \right\}$$

where

$$\mathbf{D}'_N = {}^*\mathbf{D} \cap \left(\frac{1}{N}{}^*\mathbb{N}\right)^2$$

and

$$M^{F,\mathbb{X}} = \left(\sum_{s=0}^{t} F(s, \mathbb{X})\Delta\mathbb{X}_s\right)_{t\in\mathbb{T}}.$$

Remark 3.1. *Up to scaling,* $\bar{\mathcal{Q}}^N_{\mathbf{D}'_N} = \mathcal{Q}^n_{\mathbf{D}'_n}$.

3.2. **Results and proofs.**

Definition 3.3 ((Uniform lifting of ξ))**.** Let $\Xi : \mathcal{L}_N^{\mathbb{T}} \to {}^*\mathbb{R}$ be an internal function, and let $\xi : \Omega \to \mathbb{R}$ be a continuous function. Ξ is said to be a *uniform lifting* of ξ if and only if

$$\forall \bar{\omega} \in \mathcal{L}_N^{\mathbb{T}} \left(\tilde{\bar{\omega}} \in ns({}^*\Omega) \Rightarrow {}^\circ\Xi(\bar{\omega}) = \xi(st(\tilde{\bar{\omega}})) \right),$$

where $st(\tilde{\bar{\omega}})$ is defined with respect to the topology of uniform convergence on Ω.

In order to construct the hyperfinite version of the G-expectation, we need to show that the *-image of ξ, ${}^*\xi$, with respect to $\tilde{\bar{\omega}} \in ns({}^*\Omega)$, is the canonical lifting of ξ with respect to $st(\tilde{\bar{\omega}}) \in \Omega$. i.e., for every $\tilde{\bar{\omega}} \in ns({}^*\Omega)$, ${}^\circ\left({}^*\xi(\tilde{\bar{\omega}})\right) = \xi(st(\tilde{\bar{\omega}}))$. To do this, we need to show that ${}^*\xi$ is S-continuous in every nearstandard point $\tilde{\bar{\omega}}$.

It is easy to prove that there are two equivalent characteristics of S-continuity on ${}^*\Omega$.

Remark 3.2. *The following are equivalent for an internal function* $\Phi : {}^*\Omega \to {}^*\mathbb{R}$*;*

(1) $\forall \omega' \in {}^*\Omega \left({}^*\|\omega - \omega'\|_\infty \simeq 0 \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega')| \simeq 0 \right)$.

(2) $\forall \varepsilon \gg 0, \exists \delta \gg 0 : \forall \omega' \in {}^*\Omega \left({}^*\|\omega - \omega'\|_\infty < \delta \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega')| < \varepsilon \right)$.

(The case of Remark 3.2 where $\Omega = \mathbb{R}$ is well known and proved in Stroyan and Luxemburg [28, Theorem 5.1.1])

Definition 3.4. Let $\Phi : {}^*\Omega \to {}^*\mathbb{R}$ be an internal function. We say Φ is *S-continuous* in $\omega \in {}^*\Omega$, if and only if it satisfies one of the two equivalent conditions of Remark 3.2.

Proposition 3.3. *If* $\xi : \Omega \to \mathbb{R}$ *is a continuous function satisfying* $|\xi(\omega)| \le a(1 + \|\omega\|_\infty)^b$*, for* $a, b > 0$*, then,* $\Xi = {}^*\xi \circ \tilde{\cdot}$ *is a uniform lifting of* ξ.

Proof. Fix $\omega \in \Omega$. By definition, ξ is continuous on Ω. i.e., for all $\omega \in \Omega$, and for every $\varepsilon \gg 0$, there is a $\delta \gg 0$, such that for every $\omega^{'} \in \Omega$, if

$$\|\omega - \omega^{'}\|_{\infty} < \delta, \text{ then } |\xi(\omega) - \xi(\omega^{'})| < \varepsilon. \tag{19}$$

By the Transfer Principle: For all $\omega \in \Omega$, and for every $\varepsilon \gg 0$, there is a $\delta \gg 0$, such that for every $\omega^{'} \in {}^*\Omega$, (19) becomes,

$$^*\|{}^*\omega - \omega^{'}\|_{\infty} < \delta, \text{ and } {}^*|{}^*\xi({}^*\omega) - {}^*\xi(\omega^{'})| < \varepsilon. \tag{20}$$

So, ${}^*\xi$ is S-continuous in ${}^*\omega$ for all $\omega \in \Omega$. Applying the equivalent characterization of S-continuity, Remark 3.2, (20) can be written as

$$^*\|{}^*\omega - \omega^{'}\|_{\infty} \simeq 0, \text{ and } {}^*|{}^*\xi({}^*\omega) - {}^*\xi(\omega^{'})| \simeq 0.$$

We assume $\tilde{\tilde{\omega}}$ to be a nearstandard point. By Definition 3.2, this simply implies,

$$\forall \tilde{\tilde{\omega}} \in ns({}^*\Omega), \ \exists \omega \in \Omega : {}^*\|\tilde{\tilde{\omega}} - {}^*\omega\|_{\infty} \simeq 0. \tag{21}$$

Thus, by S-continuity of ${}^*\xi$ in ${}^*\omega$,

$$^*|{}^*\xi(\tilde{\tilde{\omega}}) - {}^*\xi({}^*\omega)| \simeq 0.$$

Using the triangle inequality, if $\omega^{'} \in {}^*\Omega$ with ${}^*\|\tilde{\tilde{\omega}} - \omega^{'}\|_{\infty} \simeq 0$,

$$^*\|{}^*\omega - \omega^{'}\|_{\infty} \leq {}^*\|{}^*\omega - \tilde{\tilde{\omega}}\|_{\infty} + {}^*\|\tilde{\tilde{\omega}} - \omega^{'}\|_{\infty} \simeq 0$$

and therefore again by the S-continuity of ${}^*\xi$ in ${}^*\omega$,

$$^*|{}^*\xi({}^*\omega) - {}^*\xi(\omega^{'})| \simeq 0.$$

And so,

$$^*|{}^*\xi(\tilde{\tilde{\omega}}) - {}^*\xi(\omega^{'})| \leq {}^*|{}^*\xi(\tilde{\tilde{\omega}}) - {}^*\xi({}^*\omega)| + {}^*|{}^*\xi({}^*\omega) - {}^*\xi(\omega^{'})| \simeq 0.$$

Thus, for all $\tilde{\tilde{\omega}} \in ns({}^*\Omega)$ and $\omega^{'} \in {}^*\Omega$, if ${}^*\|\tilde{\tilde{\omega}} - \omega^{'}\|_{\infty} \simeq 0$, then,

$$^*|{}^*\xi(\tilde{\tilde{\omega}}) - {}^*\xi(\omega^{'})| \simeq 0.$$

Hence, ${}^*\xi$ is S-continuous in $\tilde{\tilde{\omega}}$. Equation (21) also implies

$$\tilde{\tilde{\omega}} \in m(\omega) \left(m(\omega) = \bigcap \{{}^*\mathcal{O}; \mathcal{O} \text{ is an open neighbourhood of } \omega\} \right)$$

such that ω is unique, and in this case $st(\tilde{\tilde{\omega}}) = \omega$.
Therefore,

$$^\circ\left({}^*\xi(\tilde{\tilde{\omega}})\right) = \xi(st(\tilde{\tilde{\omega}})).$$

□

Definition 3.5. Let $\bar{\mathcal{E}} : {}^*\mathbb{R}^{\mathcal{L}_N^{\mathbb{T}}} \to {}^*\mathbb{R}$. We say that $\bar{\mathcal{E}}$ lifts $\mathcal{E}^G$ if and only if for every $\xi : \Omega \to \mathbb{R}$ that satisfies $|\xi(\omega)| \leq a(1 + \|\omega\|_{\infty})^b$ for some $a, b > 0$,

$$\bar{\mathcal{E}}({}^*\xi \circ \tilde{\cdot}) \simeq \mathcal{E}^G(\xi).$$

Theorem 2.

$$\max_{\bar{Q} \in \bar{\mathcal{Q}}^N_{\mathbf{D}'_N}} \mathbb{E}^{\bar{Q}}[\cdot] \text{ lifts } \mathcal{E}^G(\xi). \tag{22}$$

Proof. From Theorem 1,

$$\max_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \to \mathcal{E}^G(\xi), \quad \text{as } n\to\infty. \tag{23}$$

For all $N \in {}^*\mathbb{N}\setminus\mathbb{N}$, we know that (23) holds if and only if

$$\max_{Q\in{}^*\mathcal{Q}^N_{\mathbf{D}'_N}} \mathbb{E}^{Q}[{}^*\xi(\widehat{X}^N)] \simeq \mathcal{E}^G(\xi), \tag{24}$$

(see Albeverio et al. [1], Proposition 1.3.1). Now, we want to express (24) in term of $\bar{\mathcal{Q}}^N_{\mathbf{D}'_\mathbf{N}}$. i.e., to show that

$$\max_{\bar{Q}\in\bar{\mathcal{Q}}^N_{\mathbf{D}'_N}} \mathbb{E}^{\bar{Q}}[{}^*\xi\circ\tilde{\cdot}] \simeq \mathcal{E}^G(\xi).$$

To do this, use

$$\mathbb{E}^{Q}[{}^*\xi\circ\hat{\cdot}] = \mathbb{E}^{Q}[{}^*\xi\circ\hat{\cdot}\circ\iota^{-1}\circ\iota]$$

and

$$\begin{aligned}
\mathbb{E}^{Q}[{}^*\xi\circ\hat{\cdot}\circ\iota^{-1}\circ\iota] &= \mathbb{E}^{Q}[{}^*\xi\circ\tilde{\cdot}\circ\iota] \\
&= \int_{{}^*\mathbb{R}^{N+1}} {}^*\xi\circ\tilde{\cdot}\circ\iota\, dQ, \quad \text{(transforming measure)} \\
&= \int_{{}^*\mathbb{R}^{\mathbb{T}}} {}^*\xi\circ\tilde{\cdot}\, d(Q\circ j), \\
&= \mathbb{E}^{Q\circ j}[{}^*\xi\circ\tilde{\cdot}]
\end{aligned}$$

for $j : {}^*\mathbb{R}^{\mathbb{T}} \to {}^*\mathbb{R}^{N+1}$, $(xt)_{t\in\mathbb{T}} \mapsto \left(\frac{xNt}{T}\right)_{t\in\mathbb{R}^{N+1}}$.
Thus,

$$\bar{\mathcal{Q}}^N_{\mathbf{D}'_N} = \{Q\circ j : Q\in{}^*\mathcal{Q}^N_{\mathbf{D}'_N}\}.$$

This implies,

$$\max_{\bar{Q}\in\bar{\mathcal{Q}}^N_{\mathbf{D}'_N}} \mathbb{E}^{\bar{Q}}[{}^*\xi\circ\tilde{\cdot}] = \max_{Q\in{}^*\mathcal{Q}^N_{\mathbf{D}'_N}} \mathbb{E}^{Q}[{}^*\xi\circ\hat{\cdot}].$$

□

Appendix

Proof of Lemma 2.1. From the above equation, we can say that $\Delta M^f_k = f(k,\mathbb{X})\xi_k$. And by the orthonormality property of ξ_k, we have

$$\mathbb{E}^{P_n}[f(k,\mathbb{X})^2\xi_k^2|\mathcal{F}^n_k] = \mathbb{E}^{P_n}[f(k,\mathbb{X})^2|\mathcal{F}^n_k] \le \mathbb{E}^{P_n}[(\sqrt{R_\mathbf{D}})^2|\mathcal{F}^n_k] = R_\mathbf{D} \quad P_n \text{ a.s.},$$

as $|\xi_k| = 1$, $f(\cdots)^2 \in \mathbf{D}$ implies

$$|(\Delta M^f_k)^2| = |f(k,\mathbb{X})|^2 \in [r_\mathbf{D}, R_\mathbf{D}] \quad P_n \text{ a.s.}$$

□

Density argument verification. Let

$$f : \xi \mapsto \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]$$

and

$$g : \xi \mapsto \lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)].$$

From (15), we know that for all $\xi \in \mathcal{C}_b(\Omega, \mathbb{R}),\ f(\xi) = g(\xi)$. Since $\mathbb{L}^1_*$ is the completion of $\mathcal{C}_b(\Omega, \mathbb{R})$ under the norm $\|\cdot\|_*$, $\mathcal{C}_b(\Omega, \mathbb{R})$ is dense in $\mathbb{L}^1_*$; and we want to prove for all $\xi \in \mathbb{L}^1_*$, $f(\xi) = g(\xi)$. To prove this, it is sufficient to show that f and g are continuous with respect to the norm $\|\cdot\|_*$.

For continuity of f: For all $P \in \mathcal{Q}_{\mathbf{D}}$ and $\xi, \xi^{'} \in \mathbb{L}^1_*$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi^{'}] \leq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[|\xi - \xi^{'}|].$$

Since, $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi^{'}] \leq \|\xi - \xi^{'}\|_*. \tag{25}$$

Interchanging ξ and $\xi^{'}$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi^{'}] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] \leq \|\xi^{'} - \xi\|_*. \tag{26}$$

Adding (25) and (26), we have $|f(\xi) - f(\xi^{'})| \leq \|\xi - \xi^{'}\|_*$.

For continuity of g: We follow the same argument as above.

Proof of Remark 3.2. Let Φ be an internal function such that condition (1) holds. To show that (1) $\Rightarrow$ (2), fix $\varepsilon \gg 0$. We shall show there exists a δ for this ε as in condition (2). Since Φ is internal, the set

$$I = \left\{ \delta \in {}^*\mathbb{R}_{>0} :\ \forall \omega^{'} \in {}^*\Omega\ ({}^*\|\omega - \omega^{'}\|_\infty < \delta \Rightarrow {}^*|\Phi(\omega) - \Phi(\omega^{'})| < \varepsilon) \right\},$$

is internal by the Internal Definition Principle and also contains every positive infinitesimal. By Overspill (cf. Albeverio et al. [1, Proposition 1.27]) I must then contain some positive $\delta \in \mathbb{R}$.
Conversely, suppose condition (1) does not hold, that is, there exists some $\omega^{'} \in {}^*\Omega$ such that

$${}^*\|\omega - \omega^{'}\|_\infty \simeq 0 \text{ and } {}^*|\Phi(\omega) - \Phi(\omega^{'})| \text{ is not infinitesimal.}$$

If $\varepsilon = \min(1, {}^*|\Phi(\omega) - \Phi(\omega^{'})|/2)$, we know that for each standard $\delta > 0$, there is a point $\omega^{'}$ within δ of ω at which $\Phi(\omega^{'})$ is farther than ε from $\Phi(\omega)$. This shows that condition (2) cannot hold either. □

References

[1] Albeverio, S., R. Høegh-Krohn, J. Fenstad, and T. Lindstrøm (1986). *Nonstandard methods in stochastic analysis and mathematical physics.*, Volume 122 of *Pure and Applied Mathematics.* Orlando: Academic Press.

[2] Anderson, R. (1976). A nonstandard representation for Brownian motion and Itô integration. *Bulletin of the American Mathematical Society 82*, 99–101.

[3] Anderson, R. and R. Raimondo (2008). Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets. *Econometrica 76*(4), 841–907.

[4] Berg, I. v. d. (2007). *The strength of nonstandard analysis.* Vienna: Springer-Wien.

[5] Billingsley, P. (1995). *Probability and measure* (Third ed.). Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons.

[6] Billingsley, P. (1999). *Convergence of probability measures* (Second ed.). Wiley Series in Probability and Statistics. New York: John Wiley & Sons.

[7] Denis, L., M. Hu, and S. Peng (2011). Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Analysis 34*(2), 139–161.

[8] Dolinsky, Y., M. Nutz, and M. Soner (2012). Weak approximation of G-expectations. *Stochastic Processes and their Applications 122*(2), 664–675.

[9] Duffie, D. and P. Protter (1992). From discrete to continuous-time finance: Weak convergence of the financial gain process. *Mathematical Finance 2*(1), 1–15.

[10] Duffie, D. and W. Shafer (1986a). Equilibrium in incomplete markets. I: A basic model of generic existence. *Journal of Mathematical Economics 14*, 285–300.

[11] Duffie, D. and W. Shafer (1986b). Equilibrium in incomplete markets. II: Generic existence in stochastic economies. *Journal of Mathematical Economics 15*, 199–216.

[12] Epstein, L. and S. Ji (2013). Ambiguous volatility and asset pricing in continuous time. *The Review of Financial Studies 26*(7), 1740–1786.

[13] Herzberg, F. S. (2013, Aug). First steps towards an equilibrium theory for lévy financial markets. *Annals of Finance 9*(3), 543–572.

[14] Hoover, D. and E. Perkins (1983). Nonstandard construction of the stochastic integral and applications to stochastic differential equations. I., II. *Transactions of the American Mathematical Society 275*, 1–58.

[15] Hugonnier, J., S. Malamud, and E. Trubowitz (2012). Endogenous completeness of diffusion driven equilibrium markets. *Econometrica 80*(3), 1249–1270.

[16] Keisler, H. (1977). Hyperfinite model theory. *Logic Colloqium. 76*, 5–110.

[17] Kramkov, D. (2015). Existence of an endogenously complete equilibrium driven by a diffusion. *Finance and Stochastics 19*(1), 1–22.

[18] Lindstrøm, T. (1980). Hyperfinite stochastic integration. I, II, III: The nonstandard theory. *Mathematica Scandinavica 46*, 265–333.

[19] Lindstrøm, T. (1988). An invitation to nonstandard analysis. In N. Cutland (Ed.), *Nonstandard analysis and its applications*, pp. 1–99. Cambridge: Cambridge University Press.

[20] Loeb, P. (1975). Conversion from nonstandard to standard measure spaces and applications in probability theory. *Bulletin of the American Mathematical*

Society 211, 113–122.

[21] Nutz, M. (2013). Random G-expectations. *The Annals of Applied Probability 23*(5), 1755–1777.

[22] Pardoux, É. and S. Peng (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Letters 14*(1), 55–61.

[23] Peng, S. (2010). Nonlinear expectations and stochastic calculus under uncertainty. arXiv:1002.4546.

[24] Perkins, E. (1981). A global intrinsic characterization of Brownian local time. *The Annals of Probability 9*, 800–817.

[25] Radner, R. (1972). Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. *Econometrica 40*, 289–303.

[26] Riedel, F. and F. Herzberg (2013). Existence of financial equilibria in continuous time with potentially complete markets. *Journal of Mathematical Economics 49*(5), 398 – 404.

[27] Ruan, C. (2011). *The construction of G-Brownian motion and relative financial application*. Master's dissertation. Jinan, China: School of Mathematics, Shandong University.

[28] Stroyan, K. and W. Luxemburg (1976). *Introduction to the theory of infinitesimals*, Volume 72 of *Pure and Applied Mathematics*. New York: Academic Press.

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