Regularity and Approximation of a Hyperbolic-Elliptic Coupled Problem

A Thesis submitted for the degree of Doctor of Philosophy

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December 2010

Abstract

In this thesis, we investigate the regularity and approximation of a hyperbolic-elliptic coupled problem. In particular, we consider the Poisson and the transport equation where both are assigned nonhomogeneous Dirichlet boundary conditions. The coupling of the two problems is executed as follows. The right hand side function of the Poisson equation is the solution ρ of the transport equation whereas the gradient field $E = -\nabla u$, with u being solution of the Poisson problem, is the convective field for the transport equation. The analysis is done throughout on a nonconvex, not simply connected domain Ω that is supposed to be homeomorph to an annular domain.

In the first part of this thesis, we will focus on the existence and uniqueness of a classical solution to this highly nonlinear problem using the framework of Hölder continuous functions. Herein, we distinguish between a time dependent and time independent formulation. In both cases, we investigate the streamline functions defined by the convective field E. These are used in the time dependent case to derive an operator equation whose fixed point is the streamline function to the gradient of the classical solution u. In the time independent setting, we formulate explicitly the solution operators L for the Poisson and T for the transport equation and show with a fixed point argument the existence and uniqueness of a classical solution (u, ρ).

The second part of this thesis deals with the approximation of the coupled problem in Sobolev spaces. First, we show that the nonlinear transport equation can be formulated equivalently as variational inequality and analyse its Galerkin finite element discretization. Due to the nonlinearity of the coupled problem, it is necessary to use iterative solvers. We will introduce the *staggered algorithm* which is an iterative method solving alternating the Poisson and transport equation until convergence is obtained. Assuming that $L \circ T$ is a contraction in the Sobolev space $H^1(\Omega)$, we will investigate the convergence of the discrete staggered algorithm and obtain an error estimate. Subsequently, we present numerical results in two and three dimensions. Beside the staggered algorithm, we will introduce other iterative solvers that are based on linearizing the coupled problem by Newton's method. We illustrate that all iterative solvers converge satisfactorily to the solution (u, ρ).

Acknowledgements

First at foremost, I wish to thank my supervisor Dr. Matthias Maischak for all the time he devoted to give advice on numerous mathematical problems. Without him, this thesis would have never been possible. Furthermore, I am grateful for the suggestion of my topic which turned out to be very interesting and challenging.
Second, I wish to thank the Department of Mathematical Sciences of Brunel University for the scholarship over the three years. Without this financial background, I would have not been able to undertake the studies.
I would like to thank all my friends here in London and back home and for the enjoyable three years. In particular I wish to mention my colleagues Dominik Hollmann and Matthias Gläfke with whom I spent several nice lunch breaks. Furthermore, I want to thank Matthias and my aunt Marlies for patiently proofreading my thesis and Till Truschinski for never leaving me without material to read.
I would like to thank Antonio for never stop believing in me and his invaluable support over all the three years.
Last but not least, I want to thank my brother for giving me so many nice reasons for coming home and my parents for the steady moral support and, especially, for being available at night and day time if needed.

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List of Symbols

$C^m(\Omega)$	Space of m-times continuous differentiable functions on Ω. 17
$C^{m,\alpha}(I)$	Space of Hölder continuous functions on interval I. 18
$C^{m,\alpha}(\Omega)$	Space of Hölder continuous functions on Ω. 17
$C^{m,\alpha;k,\gamma}(\Omega)$	Space of Hölder continuous functions in space and time. 18
$C^{m,\alpha}(Q_t)$	Classical function space on Q_t. 48
c_{mv}	Constant for mean-value-like estimate. 24
$c_S(\Omega)$	Constant in Schauder a priori estimate. 52
Γ_t^{IF}	Interface of Φ_1 and Φ_2 at time t. 56
$H_{convex}(\Omega)$	Convex hull of Ω. 46
I_{Γ_-}	Interval for parametrization of Γ_-. 16
Σ_t	Initial set. 46
$\|f\|_{\alpha,\Omega}$	Hölder coefficient of f. 17
$\\|f\\|_{m,\Omega}$	Classical norm of f. 17
$\|f\|_\infty$	Maximum norm of f. 16
$\\|f\\|_{0,\Omega}$	Sup-norm for f. 16
$\\|f\\|_{m,\alpha;Q_t}$	Classical norm on Q_t. 47
Q	Parameter set of streamline function Φ. 136
Q_t	Inflow set. 46
Γ	Boundary of Ω. 15
Γ_-	Inflow Boundary. 15
Γ_+	Outlow Boundary. 15
Ω	Nonconvex open domain Ω. 15
Ω_-	Convex domain enclosed by Γ_-. 16
φ	Paramtrization of Γ_-. 16

List of Problem Definitions

Chapter 1

Introduction

In this thesis we investigate a mathematical model of corona discharge. Corona discharge occurs in the electrostatic spray painting process. To paint a metallic work piece, the spray gun releasing the colour particles contains a negative electrode. At the same time, the work piece is grounded and thus represents the positive electrode. In practice, multiple mechanisms exist for the guns to atomize the paint particles, see for example [47] for one particular realization. In all of them, a high voltage is applied to the electrode to maintain the potential difference and, in particular, to produce ions by corona discharge at the negative electrode. The atomized paint particles are then attached to the ions. Consequently, the now charged paint particles are accelerated by the electrical field towards the work piece.

The modeling of the corona discharge has been considered by several authors in recent years [42, 1, 7]. The governing equations are first the Poisson equation that models the electrostatic potential u between the electrode and a plate

$$-\epsilon_0 \Delta u = \rho$$

where ϵ_0 is the permittivity constant of the gas present in the gap space and ρ is the space charge density. Due to the high applied voltage, ions are emitted at the electrode by corona discharge. To model the movement of the ions towards the plate of lower electric potential, the space charge density ρ should satisfy the transport equation which in the steady state case is given by

$$\operatorname{div}(E\rho) = 0$$

with $E = -\nabla u$. The boundary conditions for the Poisson equation are constant, in other words the boundary forms equipotential curves [7]. Usually, they are chosen as $u = u_- > 0$ at the emitting electrode and $u = 0$ at the collector plate. For hyperbolic partial differential equations such as the transport equation, a boundary condition is only required at the emitting electrode (also called inflow boundary). Although both problems are linear, the coupled system is a strongly nonlinear problem. The coupling is obtained as follows: The solution u of the Poisson equation is due to the electrical field E the coefficient function of the transport equation. On the other hand, the solution ρ of the transport equation is the right-hand side function of the Poisson equation.

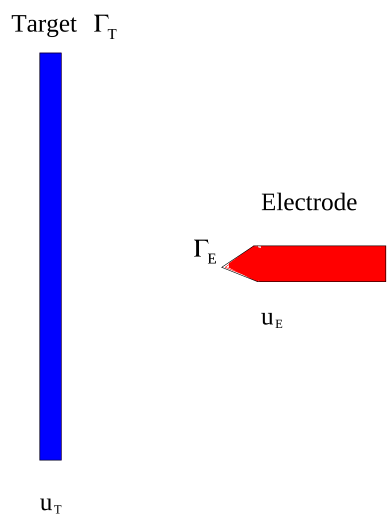


Figure 1.1: Possible problem setting with $u(x) = u_T$ for $x \in \Gamma_T$ and $u(x) = u_E$ for $x \in \Gamma_E$

Several works have been published on the numerical modeling of the ion current and the distribution of the charge density at the target. We will now introduce some of them and explain the various discretization methods for the Poisson as well as transport equation. Adamiak et al. [1, 7] investigated the electrical corona discharge in a point-plane configuration in two dimensions. In this work, an equivalent representation of the continuous coupled problem is used by replacing the linear transport equation by a nonlinear one. Substituting $\frac{1}{\epsilon_0} \operatorname{div} E = -\frac{1}{\epsilon_0}\Delta u = \frac{1}{\epsilon_0}\rho$ into the linear transport equation, we get

$$\frac{1}{\epsilon_0}\rho^2 + E \cdot \nabla\rho = 0. \tag{1.1}$$

The electrode and the target are modeled by wires of distinct radii of curvature, wherein the tiny electrode is assigned to a large radius of curvature. In the previous works and [8], the authors use a hybrid Finite Element - Boundary Element - Method of Characteristics technique to simulate the current density on the ground plate and ion current for different radii of the emitting electrode and gap space between the electrode and the target. Their idea is to use the linearity of the Dirichlet problem of the Poisson equation: It can be decomposed into a Laplace problem with non-homogeneous boundary conditions plus a Poisson problem with homogeneous boundary conditions. The Laplace equation thus describes the strong electrical field around the electrode and is solved by the Boundary Element Method (BEM). Using the BEM solution, the mesh for the Finite Element Method (FEM) is created by choosing the intersection of the solution with the equipotential lines as nodes. The Poisson equation is then solved by FEM. The Method Of Characteristics (MOC) was implemented for the transport equation solving the problem on the trajectories of the ions. Since the Laplace equation depends only on the given non-homogeneous boundary data, it is solved only once at the beginning of the algorithm in which the Poisson and transport equations are solved alternating until convergence is obtained. Another modified model is to introduce a stabilization term to the transport equation. Feng [29] replaces the nonlinear transport equation (1.1) by

$$-\frac{1}{Pe_E}\Delta\rho + E\cdot\nabla\rho + \rho^2 = 0$$

with $Pe_E > 0$ being the electric Peclet number. He then examines the corona discharge numerically in a configuration where a wire is enclosed by different collector geometries, that are a cylinder, square and rectangular shield. The discretization method chosen is the Galerkin Finite Element Method for the Poisson as well as transport equation. The obtained nonlinear variational system is solved by Newton's method with an accuracy of 10^{-6}.
In the project [59], Maischak et al. worked on numerical methods to solve the electrostatic problem in the spray painting process in three dimensions. The aim was to measure the ion current in the domain and the charge density on the target. The particularity in comparison to the previous works is that a more sophisticated implementation of the electrode was used. Further, the domain of computation is enclosed by a frame on which the Poisson equation was assigned additional boundary conditions. They distinguish between two kinds of boundary conditions. In the first case, homogeneous boundary conditions are used (this is a Faraday cage) and in the second case, the problem is formulated as a transmission problem. To solve the coupled problem, the authors used a *staggered algorithm*. Therein, the Poisson and transport equations are solved alternating until convergence is obtained. To solve each of them, several discretization methods are applied. The Poisson equation is discretized in case of the Faraday cage using the Least Squares and Galerkin methods while for the transmission problem, a symmetrical FEM-BEM coupling is applied. In case of the linear transport equation, the authors use the Streamline Upwind Petrov Galerkin (SUPG) and Least Squares methods (for information see [12, 15, 16]) whereas in the nonlinear case, the Least Squares and Newton Methods are applied.

Deliége [21] uses a different approach to model the coupled problem. Considering the time dimension in the painting process, he replaces the transport equation by the time dependent transport equation

$$\partial_t\rho + \mathrm{div}(\mu\rho E) = 0$$

with $\mu = 1.8\cdot 10^{-4}V/s$, along with an initial condition for ρ at $t = 0$. The Poisson solution is thus time dependent through the variation of ρ in time. Also in this paper, the results are only of numerical nature: the authors compare different time stepping schemes and their accuracy.

The time dependent coupled problem is closely related to another research field that is the study of mean field models for superconducting vortices. The equations in the mean field model are obtained by setting the permittivity constant ϵ_0 and μ to 1. Moreover, the Poisson equation is replaced by

$$-\Delta u + u = \rho$$

with boundary condition for u at infinity.
This topic has received much attention in recent years. Several results have been proved for

the existence and uniqueness of a solution on a time interval $[0, T]$ as well as for the discretization. The literature is divided into two approaches. The first one focuses on the existence of a classical solution [44, 43, 55]. Here, the linear transport equation is used with an initial charge distribution $\rho_0 \in C^\alpha(\Omega_0)$ with Ω_0 being defined as the support of the charge distribution at time $t = 0$. The proof is based on the streamline function Φ indicating the particle trajectory with respect to time. The domain Ω_t is moving, that is it changes for every time $t \leq T$. The technique is to reformulate the coupled problem as an integro-differential operator A by combining the Poisson, transport and streamline equations. The operator A is applied to functions of a set $W(M,T) \subset C^{1,\alpha}(\Omega_t)$. All functions in $W(M,T)$ are invertible with respect to the space variable at every fixed instant t and are bounded with M being the boundedness constant. By a compactness argument, it is shown that a unique fixed point $\Phi \in W(M,T)$ of the operator A exists. Conclusively, the existence of a classical solution (u, ρ) follows. In a slightly different setting of equations but with the same techniques, the method is also applied in [33, 32]. After proving short time existence, all previous works aim to show global existence in time [11]. Using $\Phi(x, T)$ as new initial distribution, the authors prove with the same method that Φ also exists in a subsequent small time interval $[T, T_2]$. In fact, by proving that the streamline function is a priori bounded in the $C^{1,\alpha}(\Omega_t)$ norm, global existence to the mean field problem is obtained.
The apparent similarities in the equations of the mean field problem and the time dependent formulation of the electrostatic spray painting process suggest an application of the method. Nevertheless, the ideas are not immediately transferable. The absence of inflow boundary conditions for ρ simplifies the setting markedly.

Beside the approach of [44, 43, 55] in which an integro-differential operator is used, a second approach exists and is presented in several works. In [56], Schätzle and Styles analyze the time dependent coupled problem on a fixed domain Ω for a given time interval $[0, T]$, denoted by Ω_T. Here, the Poisson equation is assigned constant boundary conditions on $\partial\Omega_T$ while the inflow condition is assumed to be homogeneous. The idea is to use the regularized time dependent transport equation

$$\partial_t \rho^\epsilon = -\operatorname{div}(\rho^\epsilon E^\epsilon) + \epsilon \Delta \rho^\epsilon$$

and to prove first the existence of a solution $(u^\epsilon, \rho^\epsilon) \in H^{2+\alpha, 1+\alpha/2}(\Omega_T)$. With various estimates, it is shown in a second step that in the limit for $\epsilon \to 0$ a weak solution exists to the problem satisfying the prescribed boundary conditions. The work closest to the electrostatic spray painting process is [5]. Therein, the authors consider non homogeneous inflow boundary conditions for the transport equation and Neumann boundary conditions for the Poisson equation. Their method is based on [56] and it is proved that a weak solution exists to the coupled problem. The authors mention that the method is transferable to Dirichlet and Robin boundary conditions in case of the potential u.
Beside the works on the existence of a continuous solution using asymptotic analysis, discretization results have been also obtained in an analogous manner. In [27], the authors study the approximation of a steady state coupled problem by an upwind finite volume method for the transport equation and the finite element method for the elliptic equation. They prove the existence of a sequence $(u_h, \rho_h) \in H^1(\Omega) \times L^\infty(\Omega)$ that fulfils the variational discrete formulation. In [18, 22] the convergence of the discrete solution of the time dependent coupled problem arising

in the mean field model to a weak limit is shown by a compactness argument.

1.1 Outline of Thesis

We will now give an outline of the thesis. In the first part, we will focus on the existence and uniqueness of a classical solution to the continuous coupled problem using the nonlinear transport equation. Therein, we will distinguish between the time dependent and steady state formulation. The analysis is done (apart from Chapter 3) on a domain homeomorph to an annular domain for which the inner and outer boundaries are convex curves. This domain is nonconvex and additionally not simply connected. The inner boundary simulates the electrode and the outer boundary the collector plate. We use nonhomogeneous Dirichlet boundary conditions for the Poisson equation and nonhomogeneous inflow boundary conditions for the transport equation. We thus generalize the above discussed results not only in terms of boundary conditions but also by the choice of a non convex domain. In the second part of this thesis, we will focus on approximation methods for the coupled problem in a theoretical and numerical manner.

In Chapter 3, we begin our analysis with the one-dimensional time dependent coupled problem on an interval $I = [0, 1]$. We will present an approach of modeling the inflow of charge by introducing a so-called inflow set Q_t for $t \in [0, T]$ on which we define a streamline function Φ. Following [44], we derive an integro-differential operator A that is then applied to a set W of all bounded and invertible streamline functions. By the Banach fixed point theorem, we prove that a unique fixed point Φ exists to the operator A for a short time interval $[0, T]$ and consequently also a classical solution (u, ρ). This chapter shall explain the modeling of the charge inflow and motivate the methods used to prove the existence of a solution of the general two-dimensional setting in Chapter 4.

In Chapter 4, we consider the two-dimensional time dependent coupled problem. Having an initial distribution ρ_0 with compact support in Ω, we prove the existence of a classical solution (u, ρ) for a short time T. The challenge and novelty in comparison to [44, 55] is to model the inflow of charge in combination with the transport of the initial charge distribution. We therefore introduce two distinct streamline functions Φ_1 and Φ_2 where Φ_1 corresponds to the transport of the initial distribution and Φ_2 to the transport of the inflow set. We thus generalize the work [44] in which only the streamline function Φ_1 is considered. Next, we define two operators A_1 and A_2 that are applied to the product of streamline functions $\Phi = (\Phi_1, \Phi_2) \in W \subset W_1 \times W_2$ where W_1 and W_2 are the sets of all those streamline functions Φ_1 and Φ_2 that are invertible and feasible for the problem. We are able to prove the existence of a unique fixed point $A(\Phi) = \Phi$ and conclusively the existence of a classical solution (u, ρ) for small times T. Further, we show that the solution can be continued in time until the support of ρ reaches the outflow boundary.

Chapters 5 and 6 deal with the two-dimensional steady state coupled problem. In Chapter 5, we consider a radially symmetric problem on an annular domain for which the Poisson and transport equation reduce to one-dimensional boundary value problems with variable coefficients. The approach is to define the solution operators L for the Poisson problem, i.e. $L\rho = u'$ and the solution operator T for the transport problem, i.e. $Tu' = \rho$. By the Banach fixed point theorem we prove that a unique fixed point $\rho = T \circ L\rho$ exists on a set R of bounded functions ρ provided the inflow boundary data ρ_A is small enough. Since $T \circ L$ is the solution operator for the transport problem, it follows immediately that also a unique classical solution exists to the radially symmetric problem with $\rho \in R$.

In Chapter 6, we examine the general steady state two-dimensional coupled problem. The approach is different to Chapter 4 and motivated by the technique of Chapter 5. Again, we define the solution operators L for the Poisson problem and T for the transport problem and reformulate the coupled problem as fixed point problem. As the nonlinear transport equation is solvable on a streamline, we obtain an explicit representation of the operator T. To prove existence and uniqueness of a fixed point to $L \circ T$, it is sufficient to use only existence results and a priori estimates for the Poisson problem in Hölder spaces (see e.g. [34]). The main emphasis in this Chapter will be given on analyzing the transport solution operator. We obtain restrictions for the electrical field E that have to be fulfilled to obtain existing streamline functions. Eventually, we prove by the Banach Fixed Point Theorem that a unique fixed point to the composite operator $L \circ T$ exists on a set $W \subset C^{1,\alpha}(\bar{\Omega})$ and conclusively also a unique classical solution (u, ρ) with $-\nabla u \in W$. Beside the existence result, we obtain that in the continuous case the Banach fixed point iterations are the staggered algorithm.

The remaining part of the thesis deals with approximation results for the coupled problem. The advantage of the presented method in Chapter 6 in comparison to the technique given by [5] is illustrated in Chapter 7. Since $L \circ T$ is a contraction on $C^{1,\alpha}(\Omega)$ and due to numerical evidence in Chapter 8, we assume that the continuous composite operator $L \circ T$ is also a contraction on $H^1(\Omega)$. We will show that an error estimate for the approximate solution to the coupled problem is an immediate consequence.

In Chapter 8, we present numerical results for the approximation of the steady state coupled problem. While it is equivalent to use the linear or nonlinear transport equation in the continuous formulation, we need to distinguish the two approaches in the discretization. Next to the staggered algorithm, we introduce three more algorithms based on Newton's method to deal with the nonlinearity of the problem setting. We will use the radially symmetric model problem of Chapter 5 and investigate whether the algorithms converge. As outlook for further research, we also present a three-dimensional radially symmetric coupled problem on a hollow sphere. As last example, we go back to the spray painting process and investigate the convergence behavior of the coupled problem on a non smooth domain.

Chapter 2

Foundations

Let us begin with describing the domain we will consider in the analysis. Throughout this thesis, let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. We assume that Ω is homeomorphic to an annular domain. We thus deal with a domain that is neither convex nor simply connected. The boundary Γ is decomposed into a convex inner closed curve Γ_- and a convex outer closed curve Γ_+ . In the context of the transport equation, the boundary parts Γ_- and Γ_+ are referred to as inflow and outflow boundary. For a convective field E in $\mathbb{R}^2$ and the outward normal vector $\vec{n}$, the *inflow boundary* is defined by

$$\Gamma_- = \{x \in \Gamma : \vec{n} \cdot E < 0\}$$

and the *outflow boundary* is defined by

$$\Gamma_+ = \{x \in \Gamma : \vec{n} \cdot E \geq 0\}.$$

We will only consider those electrical fields E such that the previous definitions coincide, i.e. the inner boundary Γ_- is the inflow and the outer boundary Γ_+ is the outflow boundary.

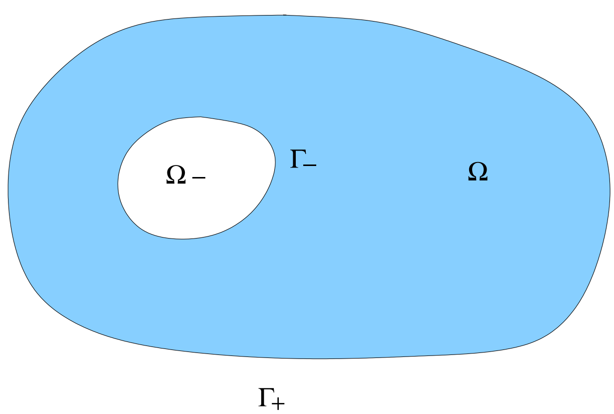


Figure 2.1: One Possible Configuration for Ω

We will also refer to Γ_- by its parametrization φ with respect to the arc length L_{Γ_-}, i.e.

$$\Gamma_- = \left\{ x \,:\, x = \varphi(t), t \in [0, L_{\Gamma_-}] \right\}. \tag{2.1}$$

In the following we will abbreviate $I_{\Gamma_-} := [0, L_{\Gamma_-}]$. φ is assumed to be positively oriented. The outward normal vector $\vec{n}(x)$ to $x \in \Gamma_-$ is thus given for every point $\varphi(t) = x \in \Gamma_-$ by

$$\vec{n}(\varphi(t)) = \begin{pmatrix} -\varphi_2'(t) \\ \varphi_1'(t) \end{pmatrix}.$$

In certain cases, it will become necessary to refer to the domain that is enclosed by Γ_-. Let us denote this domain by Ω_-.

2.1 Classical Norms and Function Spaces

We will carry out the analysis in the Chapters 3 to 6 in the classical function spaces. We begin by defining the maximums norm for a vector valued function and the corresponding induced matrix norm.

Definition 2.1. *For a vector valued function $f = (f_1, f_2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, we define the* maximum norm *for all $x \in \Omega$ by*

$$|f(x)|_\infty := \max \left\{ |f_1(x)|, |f_2(x)| \right\}.$$

where $|\cdot|$ denotes the absolute value.
Let A be a 2×2 matrix. Then the induced matrix norm *is given by the maximum absolute row sum*

$$|A(x)|_\infty := \left| \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \right|_\infty = \max \left\{ |a_{11}(x)| + |a_{12}(x)|, |a_{21}(x)| + |a_{22}(x)| \right\}.$$

Next, we define the usual sup-norm for a function f.

Definition 2.2. *We define the* sup-norm *for a scalar or vector valued function f defined on Ω by*

$$\|f\|_{0,\Omega} := \sup_{x \in \Omega} |f(x)|_\infty.$$

We can now define the space of continuously differentiable functions.

Definition 2.3 (Continuous differentiable functions)**.** *Let $\Omega \in \mathbb{R}^2$ be a bounded open domain with closure $\bar{\Omega}$. Let $\beta = (\beta_1, \beta_2)$ be a multi-index with $m = |\beta|$. Then we define*

$$C^m(\Omega) := \left\{ f : \Omega \to \mathbb{R}^2 : \partial^m f \textit{is continuous and } \|\partial^\beta f\|_{0,\Omega} < \infty, \forall \beta \leq m \right\}$$

as the space of continuously differentiable functions up to order m. *We equip the space with the norm*

$$\|f\|_{m,\Omega} := \sum_{\beta \le |m|} \|\partial^\beta f\|_{0,\Omega}.$$

For the closed domain $\bar{\Omega}$, we define

$$C^m(\bar{\Omega}) := \left\{ f \in C^m(\Omega) : \partial^\beta f \text{ is continuously extendable up to the boundary } \partial\Omega, \forall \beta \le m \right\}.$$

We equip the space with the norm

$$\|f\|_{m,\bar{\Omega}} = \sum_{\beta \le |m|} \max_{x \in \bar{\Omega}} |\partial^\beta f(x)|_\infty.$$

The following corollary shows that for a function in $C^m(\bar{\Omega})$, it is sufficient to work with the sup-norm on the open domain Ω.

Corollary 2.4. *[6, p.30] It holds for the norms on $C^m(\Omega)$ and $C^m(\bar{\Omega})$*

$$\|f\|_{m,\bar{\Omega}} = \|f\|_{m,\Omega}. \tag{2.2}$$

The classical theory of partial differential equations uses the space of Hölder continuous functions. The functions in this space are uniformly continuous with exponent $0 < \alpha < 1$.

Definition 2.5 (Hölder continuous functions)**.** *Let Ω be a bounded open domain with boundary $\partial\Omega$. For any multi-index β and $0 < \alpha < 1$, define the* Hölder coefficient *by*

$$|f|_{m,\alpha;\bar{\Omega}} = \sup_{\substack{x \ne y \in \bar{\Omega}, \\ |\beta| = m}} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|_\infty}{|x - y|_\infty^\alpha}$$

We define the space of Hölder continuous functions *by*

$$C^{m,\alpha}(\bar{\Omega}) := \left\{ f \in C^m(\bar{\Omega}) : |f|_{m,\alpha,\bar{\Omega}} < \infty \right\}.$$

We equip the spaces with the norm (see [58])

$$\|f\|_{m,\alpha;\bar{\Omega}} = \sum_{\beta \le |m|} \|\partial^\beta f\|_{0,\bar{\Omega}} + |f|_{m,\alpha;\bar{\Omega}}. \tag{2.3}$$

The normed space $C^{m,\alpha}(\Omega)$ is complete. We obtain the well-known result

Lemma 2.6. *[3, p. 44] The space $C^{m,\alpha}(\Omega)$ is a Banach space.*

In Chapters 3 and 4, we investigate the time dependent coupled problem. We follow the definition of [43, p.514] to introduce the time dependent classical function space.

Definition 2.7. *Let Ω be an open bounded domain and $f : \Omega \times [0,T] \to \mathbb{R}^2$. Let $\beta = (\beta_1, \beta_2)$ be a multi-index with $m = |\beta|$. Let further $l \leq k$ be integers and $0 < \alpha \leq 1$, $0 < \gamma \leq 1$. Then we denote*

$$\|f\|_{m,\alpha,\Omega;k,\gamma,[0,T]} := \sup_{t\in T} \sum_{l\leq k} \|\partial_t^l f(t)\|_{m,\alpha;\Omega} + \sup_{x\in\Omega} \sum_{|\beta|\leq m} \|\partial_x^\beta f(x)\|_{k,\gamma;[0,T]}. \tag{2.4}$$

We define the function space of (m, α)-Hölder functions in space and (k, γ)-Hölder functions in time *by*

$$C^{m,\alpha;k,\gamma}(\bar{\Omega}, [0,T]) := \left\{ f : \bar{\Omega} \times [0,T] \to \mathbb{R}^2 : \|f\|_{m,\alpha,\Omega;k,\gamma,[0,T]} < \infty \right\}. \tag{2.5}$$

With this definition, it follows that $\partial_t^l f(\cdot, t) \in C^{k,\alpha}(\bar{\Omega})$ for all $l \leq k$ and $\partial_x^\beta f(x, \cdot) \in C^{m,\gamma}([0,T])$ for all $|\beta| \leq m$. Throughout this thesis, we use the following convention. If the Hölder norm is only applied in space for a time dependent function $f(x,t)$, then we will write for every $t \in [0,T]$

$$\|f(t)\|_{m,\alpha,\Omega} := \sum_{|\beta|\leq m} \sup_{x\in\Omega} |\partial^\beta f(x,t)|_\infty + |f(\cdot,t)|_{\alpha,\Omega}.$$

We use the argument t to emphasize that $\|f(t)\|_{m,\alpha}$ is a function in t.
Analogously, if the Hölder norm is only applied in time, then we write for $x \in \Omega$

$$\|f(x)\|_{k,\alpha,[0,T]} := \sum_{l\leq k} \sup_{t\in[0,T]} |\partial_t^l f(x,t)|_\infty + |f(x,\cdot)|_{\alpha,[0,T]}.$$

Lemma 2.8. *[3, p. 44] The space $C^{m,\alpha;k,\gamma}(\Omega, [0,T])$ is a Banach space.*

In the Chapters 3 and 5, we will investigate the one-dimensional coupled problem. The previous definitions are adapted analogously.

Definition 2.9. *Let $I = [a,b]$ an interval. Then we define the space of* continuously differentiable functions on I *by*

$$C^k(I) := \left\{ f : \partial^l f \text{ is continuous } \forall x \in I, \forall l = 0,..,k \right\}. \tag{2.6}$$

We equip the space with the norm

$$\|u\|_{k,I} := \sum_{l=0}^{k} \|\partial^l u\|_{0,I}.$$

Let now $f : I \times [0,T] \to \mathbb{R}$ be a time dependent function. Then we denote

$$\|u\|_{C^{m;k}(I,[0,T])} := \sum_{l=0}^{k} \sup_{0\leq t\leq T} \|\partial_t^l u(t)\|_{0,I} + \sum_{s=0}^{m} \sup_{x\in I} \|\partial_x^m u(x)\|_{0,[0,T]}.$$

We define the space of l-times continuously differentiable functions in space and m times continuously differentiable functions in space *by*

$$C^{m;k}(I, [0,T]) := \{ f : \|f\|_{m;k} < \infty \}. \tag{2.7}$$

So far, we have not yet defined the regularity of the domain Ω. In later Chapters, we assume that Ω is a domain of $C^{k,\alpha}$ regularity.

Definition 2.10. *[34, p. 94] A bounded domain $\Omega \subset \mathbb{R}^2$ and its boundary are of class $C^{k,\alpha}$, $0 \le \alpha \le 1$, if at each point $x_0 \in \partial\Omega$ there is an $\epsilon > 0$, a ball $B := B_\epsilon(x_0)$ around x_0 and a one-to-one mapping Ψ of $B_\epsilon(x_0)$ onto $D \subset \mathbb{R}^2$ such that*

- $\Psi(B \cap \Omega) \subset \mathbb{R} \times \mathbb{R}_+$
- $\Psi(B \cap \partial\Omega) \subset \partial(\mathbb{R} \times \mathbb{R}_+)$
- $\Psi \in C^{k,\alpha}(B)$, $\Psi^{-1} \in C^{k,\alpha}(D)$.

We can conclude that whenever the domain Ω is of class $C^{k,\alpha}$ also holds $\varphi \in C^{k,\alpha}([0, L_{\Gamma_-}])$.

2.2 Properties of Hölder Continuous Functions

We introduce some basic results for Hölder continuous functions that are used frequently in the following analysis. We begin with an analogue of the product and chain rule.

Lemma 2.11. *Let U and V be bounded open domains in $\mathbb{R}^2$ and V convex. Let $W \subset \mathbb{R}$ be an open interval. Let $f, p : V \to W$ with $f, p \in C^\alpha(V)$ and $h : U \to V$ with $h \in C^1(U)$ and $0 < \alpha \le 1$. Then we have*

$$|fp|_{\alpha,V} \le \|f\|_{0,V}|p|_{\alpha,V} + |f|_{\alpha,V}\|p\|_{0,V}, \tag{2.8}$$

$$|f(h)|_{\alpha,U} \le |f|_{\alpha,V}\|\nabla h\|_{0,U}^{\alpha}. \tag{2.9}$$

Proof. Using the definition of the Hölder norm gives

$$\begin{aligned}
|f\,p|_{\alpha,V} &= \sup_{x_1,x_2\in V} \frac{|f(x_1)p(x_1) - f(x_2)p(x_2)|}{|x_1 - x_2|^\alpha} \\
&= \sup_{x_1,x_2\in V} \frac{|f(x_1)p(x_1) - f(x_1)p(x_2) + f(x_1)p(x_2) - f(x_2)p(x_2)|}{|x_1 - x_2|^\alpha} \\
&\le \sup_{x_1,x_2\in V} \frac{|f(x_1)p(x_1) - f(x_1)p(x_2)|}{|x_1 - x_2|^\alpha} + \sup_{x_1,x_2\in V} \frac{|f(x_1)p(x_2) - f(x_2)p(x_2)|}{|x_1 - x_2|^\alpha} \\
&\le \|f\|_{0,V}|p|_{\alpha,V} + |f|_{\alpha,V}\|p\|_{0,V}.
\end{aligned}$$

Next, we prove (2.9). Since V is a convex domain, we may apply the mean value theorem. It holds

$$
\begin{aligned}
|f(h)|_{\alpha,U} &= \sup_{x_1,x_2\in U} \frac{|f(h(x_1)) - f(h(x_2))|}{|x_1 - x_2|^\alpha} \\
&= \sup_{x_1,x_2\in U} \left[\frac{|f(h(x_1)) - f(h(x_2))|}{|h(x_1) - h(x_2)|_\infty^\alpha} \frac{|h(x_1) - h(x_2)|_\infty^\alpha}{|x_1 - x_2|^\alpha} \right] \\
&\leq \sup_{y_1,y_2\in V} \frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|^\alpha} \sup_{x_1,x_2\in U} \frac{|h(x_1) - h(x_2)|_\infty^\alpha}{|x_1 - x_2|^\alpha} \\
&\leq |f|_{\alpha,V} \|\nabla h\|_{0,U}^\alpha.
\end{aligned}
$$

□

Let h be a vector valued function. The next Lemma yields a bound for the Jacobian matrix of h.

Lemma 2.12. *Let $V \subset \mathbb{R}^2$ be a bounded open domain. For a vector valued function $h : V \to \mathbb{R}^2$ holds*

$$
\begin{aligned}
\|\nabla h\|_{0,V} &\leq \|\partial_{x_1} h\|_{0,V} + \|\partial_{x_2} h\|_{0,V} \\
|\nabla h|_{\alpha,V} &\leq |\partial_{x_1} h|_{\alpha,V} + |\partial_{x_2} h|_{\alpha,V}.
\end{aligned}
$$

Proof. $\nabla h(x)$ is the 2×2 Jacobian matrix of h for every $x \in V$. It holds for the sup-norm

$$
\begin{aligned}
|\nabla h|_{0,V} &= \sup_{x\in\Omega} \left| \begin{pmatrix} \partial_{x_1} h_1 & \partial_{x_2} h_1 \\ \partial_{x_1} h_2 & \partial_{x_2} h_2 \end{pmatrix} \right|_\infty \\
&= \sup_{x\in V} \max \left\{ |\partial_{x_1} h_1(x)| + |\partial_{x_2} h_1(x)|\, , \, |\partial_{x_1} h_2(x)| + |\partial_{x_2} h_2(x)| \right\} \\
&\leq \sup_{x\in V} \max \left\{ |\partial_{x_1} h(x)|_\infty + |\partial_{x_2} h(x)|_\infty\, , \, |\partial_{x_1} h(x)|_\infty + |\partial_{x_2} h(x)|_\infty \right\} \\
&= \sup_{x\in V} \left(|\partial_{x_1} h(x)|_\infty + |\partial_{x_2} h(x)|_\infty \right) \\
&= \|\partial_{x_1} h\|_{0,V} + \|\partial_{x_2} h\|_{0,V}.
\end{aligned}
$$

For the α-Hölder norm holds

$$
\begin{aligned}
|\nabla h|_{\alpha,V} &\leq \sup_{x,y\in V} \frac{1}{|x-y|_\infty^\alpha} \left| \begin{pmatrix} \partial_{x_1} h_1(x) - \partial_{y_1} h_1(y) & \partial_{x_2} h_1(x) - \partial_{y_2} h_1(y) \\ \partial_{x_1} h_2(x) - \partial_{y_1} h_2(y) & \partial_{x_2} h_2(x) - \partial_{y_2} h_2(y) \end{pmatrix} \right|_\infty \\
&= \sup_{x,y\in V} \max \left\{ \frac{|\partial_{x_1} h_1(x) - \partial_{y_1} h_1(y)| + |\partial_{x_2} h_1(x) - \partial_{y_2} h_1(y)|}{|x-y|_\infty^\alpha}, \right. \\
&\qquad \left. \frac{|\partial_{x_1} h_2(x) - \partial_{y_1} h_2(y)| + |\partial_{x_2} h_2(x) - \partial_{y_2} h_2(y)|}{|x-y|_\infty^\alpha} \right\} \\
&\leq \sup_{x,y\in V} \left(\frac{|\partial_{x_1} h(x) - \partial_{y_1} h(y)|_\infty + |\partial_{x_2} h(x) - \partial_{y_2} h(y)|_\infty}{|x-y|_\infty^\alpha} \right) \\
&\leq |\partial_{x_1} h|_{\alpha,V} + |\partial_{x_2} h|_{\alpha,V}.
\end{aligned}
$$

□

Many results connected to differentiability, like for example the implicit function theorem, are only defined on an open domain. However, the Hölder continuity allows us to extend these functions uniformly up to the boundary.

Lemma 2.13. *[60, p.35] Let V, W be open bounded domains in $\mathbb{R}^2$.*
a) *Let $f : V \to W$ be uniformly continuous. Then f can be continuously extended up to the boundary, i.e. there exists a continuous extension $\hat{f} : \bar{V} \to W$.*

b) *Let $f \in C^\alpha(V)$ with Hölder constant M. Then the extension is also Hölder continuous, i.e. $\hat{f} \in C^\alpha(\bar{V})$. $\hat{f}$ has the same Hölder constant M.*

We apply this result to a function $f \in C^{1,\alpha}(V)$.

Corollary 2.14. *Let V be an open bounded domain in $\mathbb{R}^2$. If $f \in C^{1,\alpha}(V)$, then f and ∇f are continuously extendable up to the boundary ∂V with*

$$\nabla \hat{f} \in C^\alpha(\bar{V}).$$

∇f and $\nabla \hat{f}$ have the same Hölder constant M.

Proof. As f is continuously differentiable, it is uniformly continuous on V. By Lemma 2.13, there exists an extension $\hat{f}$ up to the boundary ∂V. For ∇f, the assertion follows immediately by Lemma 2.13b). □

2.3 Geometry and Mean Value Theorem

In the following analysis, we will deal with vector valued functions in $\mathbb{R}^2$. For vector valued function, the usual mean value theorem is not applicable. We will need the following modified version.

Theorem 2.15. *(Mean Value Theorem) [30] Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ be a continuously differentiable function. Let $x \in U$ and $\xi \in \mathbb{R}^2$, such that $x + t\xi \in U$ for all $0 \le t \le 1$. Then we have*

$$f(x+\xi) - f(x) = \left(\int_0^1 \nabla f(x + t\xi)\, dt \right) \cdot \xi. \tag{2.10}$$

It follows pointwise for $x \in U$

$$|f(x+\xi) - f(x)|_\infty \le \|\nabla f\|_{0,U} |\xi|_\infty. \tag{2.11}$$

The representation (2.10) is simply an integration over the line segment $x + t\xi$, $t \in [0,1]$. For a vector valued function, the integral is taken over the Jacobian matrix of f. In the following Lemma, we show that if $f \in C^{1,\alpha}(V)$ then the matrix of the right hand side of (2.10) is Hölder continuous with exponent α.

Lemma 2.16. *Let $V, W \in \mathbb{R}^2$ be a bounded convex domain. Let $f \in C^{1,\alpha}(V)$ and $g_1, g_2 : W \to V$ with $g_1 \in C^1(W)$ and $g_2 \in C^1(W)$ be vector valued functions in $\mathbb{R}^2$. For the matrix*

$$A(x) = \int_0^1 \nabla f(g_1(x) + \tau(g_2(x) - g_1(x)))\, d\tau$$

holds

$$|A(x) - A(y)|_\infty \le |\nabla f|_{\alpha,V}(\|\nabla g_1\|_{0,W} + \|\nabla g_2\|_{0,W})^\alpha |x - y|_\infty^\alpha.$$

Proof. Set $h(x,\tau) := g_1(x) + \tau(g_2(x) - g_1(x))$. We get

$$\begin{aligned} |A(x) - A(y)|_\infty &= \left| \int_0^1 \nabla f(h(x,\tau)) - \nabla f(h(y,\tau))\, d\tau \right|_\infty \\ &\le \int_0^1 |\nabla f(h(x,\tau)) - \nabla f(h(y,\tau))|_\infty \, d\tau. \end{aligned}$$

As $f \in C^{1,\alpha}(V)$ holds $|\nabla f(x) - \nabla f(y)|_\infty \le |\nabla f|_{\alpha,V}|x - y|_\infty^\alpha$. We get

$$\begin{aligned} \int_0^1 |\nabla f(h(x,\tau)) - \nabla f(h(y,\tau))|_\infty \, dt &\le \int_0^1 |\nabla f|_{\alpha,V} \, |h(x,\tau) - h(y,\tau)|_\infty^\alpha \, d\tau \\ &\le |\nabla f|_{\alpha,V} \sup_{0\le\tau\le 1} |h(x,\tau) - h(y,\tau)|_\infty^\alpha . \end{aligned} \tag{2.12}$$

With $g_1 \in C^1(W)$, $g_2 \in C^1(W)$ and $\tau \in [0,1]$, we get

$$\begin{aligned} |h(x,\tau) - h(y,\tau)|_\infty &= |g_1(x) + \tau(g_2(x) - g_1(x)) - g_1(y) - \tau(g_2(y) - g_1(y))|_\infty \\ &= |(1-\tau)(g_1(x) - g_1(y)) + \tau(g_2(x) - g_2(y))|_\infty \\ &\le (1-\tau)\, |g_1(x) - g_1(y)|_\infty + \tau\, |g_2(x) - g_2(y)|_\infty \\ &\le (1-\tau)\|\nabla g_1\|_{0,W}|x-y|_\infty + \tau\|\nabla g_2\|_{0,W}|x-y|_\infty \\ &\le ((1-\tau)\|\nabla g_1\|_{0,W} + \tau\|\nabla g_2\|_{0,W})|x-y|_\infty. \end{aligned}$$

Substituting the latter equation into (2.12), we have

$$\begin{aligned} |A(x) - A(y)|_\infty &\le |\nabla f|_{\alpha,V} \sup_{0\le\tau\le 1} |h(x,\tau) - h(y,\tau)|_\infty^\alpha \\ &\le |\nabla f|_{\alpha,V} \sup_{0\le\tau\le 1} ((1-\tau)\|\nabla g_1\|_{0,W} + \tau\|\nabla g_2\|_{0,W})^\alpha |x-y|_\infty^\alpha \\ &\le |\nabla f|_{\alpha,V}(\|\nabla g_1\|_{0,W} + \|\nabla g_2\|_{0,W})^\alpha |x-y|_\infty^\alpha. \end{aligned}$$

□

It is generally impossible to apply the mean value theorem to vector valued functions $f \in C^1(\Omega)$ as the line segment connecting two arbitrary points in Ω might intersect the convex domain Ω_-. However, it is possible to prove a mean value like estimate for functions defined on the nonconvex domain Ω. In [62], Whitney introduced the domains of P-property in which the length of a curve connecting any two points is bounded by a finite constant $c \ge 1$ and their euclidean distance. Referring to this work, Wienholtz uses the concept of *domains of finite length* in [63]. Yet, the

actual size of the constant c remains unclear. We will show that Ω fulfils a similar condition and prove that the constant c only depends on the size and shape of Ω_-.
To bound the distance of a function evaluated at any two arbitrary points , we use the so-called *geometric dilation* which is a concept introduced in [23, 24, 25]. For two points a and b on a closed convex curve γ, it gives the ratio of the shortest detour on γ in relation to the actual euclidean distance of a and b. Let us denote for every $a, b \in \gamma$ the minimum length of one of the

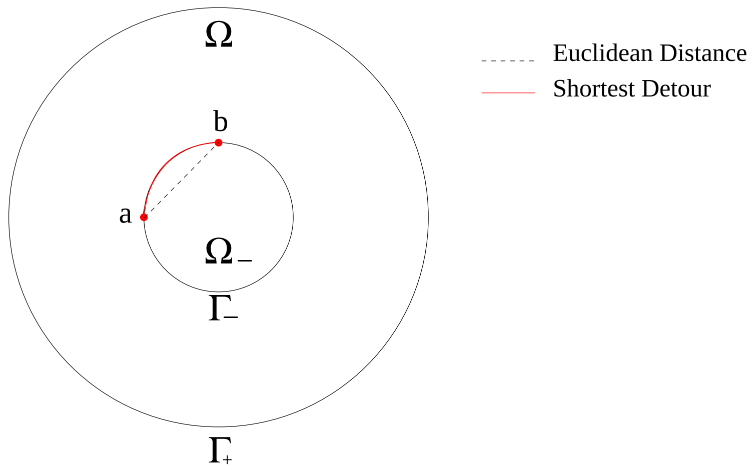


Figure 2.2: The shortest detour for two points a and b on Γ_-

two curve segments by $d_\gamma(a, b)$. The euclidean distance of a and b is denoted by $\|a - b\|_2$.

Definition 2.17. *[23, 25] The* geometric dilation *of the closed convex curve γ is defined by*

$$\delta(\gamma) := \sup_{\substack{a,b\in\gamma,\\ a\neq b}} \frac{d_\gamma(a, b)}{\|a - b\|_2}. \tag{2.13}$$

The geometric dilation of a convex closed curve is bounded in terms of its geometry.

Theorem 2.18. *[23, Theorem 2] Let γ be a convex closed curve, D the diameter of the enclosed domain V and w the minimum distance of two parallel lines enclosing V. Then holds*

$$\delta(\gamma) \leq 2\left(\frac{D}{w}\arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}\right). \tag{2.14}$$

Applying the previous theorem to our setting results in the following Corollary.

Corollary 2.19. *Let $D_{\Omega_-} := \operatorname{diam}\Omega_-$ be the diameter of Ω_- and w_{Ω_-} the minimum distance of two parallel lines enclosing Ω_-. Then the geometric dilation for the parametrization φ of Γ_- is bounded by*

$$\delta(\varphi) \leq 3\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi \tag{2.15}$$

Proof. Ω_- is convex and so is the boundary curve φ. The assertion thus follows immediately by Theorem 2.18. $\square$

The previous result opens the door for a mean-value-like estimate for a function $f \in C^1(\Omega)$.

Lemma 2.20. *Let Ω be a bounded domain. Let $f : \Omega \to \mathbb{R}^2$ be a vector valued differentiable function. Then holds the mean-value-like estimate*

$$|f(x) - f(y)|_\infty \leq c_{mv} \|\nabla f\|_{0,\Omega} |x - y|_\infty \tag{2.16}$$

with $c_{mv} := 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi$.

Proof. We have to distinguish two cases due to the nonconvexity of Ω.
Case 1:
Let $x, y \in \Omega$, such that the line segment $\overrightarrow{xy}$ is contained in Ω. Then holds by Theorem 2.15

$$|f(x) - f(y)|_\infty \leq \|\nabla f\|_{0,\Omega} |x - y|_\infty.$$

Case 2:

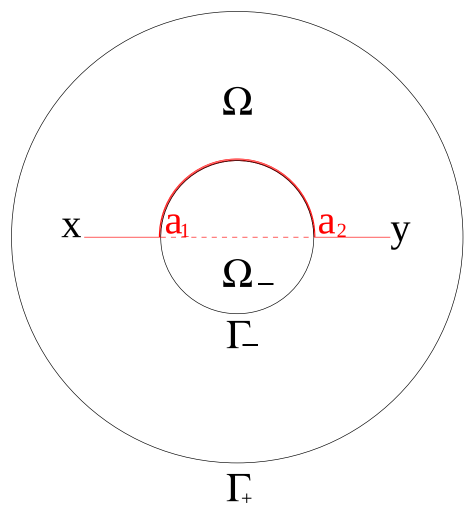


Figure 2.3: Choice of points a_1 and a_2

Let $x, y \in \Omega$, such that the line segment $\overrightarrow{xy}$ intersects Ω_-. The application of Theorem 2.15 is impossible as the line $z = x + \tau(y - x)$, $\tau \in [0, 1]$, is not contained in Ω. Denote the points of intersection of z and Γ_- by a_1 and a_2 such that $\|a_1 - x\|_2 \leq \|x - y\|_2$. The line segment $\overrightarrow{xy}$ is divided into three disjoint segments $\overrightarrow{xa_1}$, $\overrightarrow{a_1a_2}$ and $\overrightarrow{a_2y}$. By the triangle inequality holds

$$\begin{aligned} |f(x) - f(y)|_\infty &= |f(x) - f(a_1) + f(a_1) - f(a_2) + f(a_2) - f(y)|_\infty \\ &\leq |f(x) - f(a_1)|_\infty + |f(a_1) - f(a_2)|_\infty + |f(a_2) - f(y)|_\infty . \end{aligned} \tag{2.17}$$

The line segments $\overrightarrow{a_1x}$ and $\overrightarrow{a_2y}$ are contained in Ω which allows the application of case 1. For $|f(a_1) - f(a_2)|$, we have again to distinguish two cases. Since a_1 and a_2 are both elements of the boundary Γ_-, there exist $s_1, s_2 \in [0, L_{\Gamma_-}]$, such that $\varphi(s_1) = a_1$ and $\varphi(s_2) = a_2$. Γ_- is a closed curve, the two points a_1 and a_2 thus define two curve segments on Γ_-.

a) Let the curve segment between $\varphi(s_1)$ and $\varphi(s_2)$ in line with the orientation of the parametrization be the shorter curve segment. Then follows

$$\begin{aligned} |f(a_1) - f(a_2)|_\infty &= |f(\varphi(s_1)) - f(\varphi(s_2))|_\infty \\ &= \left| \int_{s_1}^{s_2} \frac{d}{ds} f(\varphi(s))\, ds \right|_\infty \\ &\leq \left| \int_{s_1}^{s_2} |\nabla f(\varphi(s))|_\infty \left| \frac{d}{ds}\varphi(s) \right|_\infty ds \right| \\ &\leq \|\nabla f\|_{0,\Omega} \left| \int_{s_1}^{s_2} \left| \frac{d}{ds}\varphi(s) \right|_\infty ds \right| \end{aligned}$$

By the equivalence of the maximum and euclidean norm, we bound $\left|\frac{d}{ds}\varphi(s)\right|_\infty \leq \left\|\frac{d}{ds}\varphi(s)\right\|_2 \leq \sqrt{2}\left|\frac{d}{ds}\varphi(s)\right|_\infty$ and obtain due to the parametrization w.r.t. the arc length of φ

$$\left| \int_{s_1}^{s_2} \left| \frac{d}{ds}\varphi(s) \right|_\infty ds \right| \leq \int_{s_1}^{s_2} \left\| \frac{d}{ds}\varphi(s) \right\|_2 ds = |s_2 - s_1| = d_{\Gamma_-}(\varphi(s_1), \varphi(s_2)).$$

By Corollary 2.19, we obtain

$$d_{\Gamma_-}(\varphi(s_1), \varphi(s_2)) \leq 3\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi\|\varphi(s_1) - \varphi(s_2)\|_2 \leq 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi|\varphi(s_1) - \varphi(s_2)|_\infty = 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi|a_1 - a_2|_\infty.$$

b) Let the curve segment between $\varphi(s_1)$ and $\varphi(s_2)$ against the orientation of the parametrization be the shorter curve segment. Let us assume that $s_1 < s_2$. Then holds as φ is a closed curve

$$\begin{aligned} |f(a_1) - f(a_2)|_\infty &= \left|f(\varphi(s_1)) - f(\varphi(0)) + f(\varphi(L_{\Gamma_-}) - f(\varphi(s_2))\right|_\infty \\ &\leq |f(\varphi(s_1)) - f(\varphi(0))|_\infty + \left|f(\varphi(L_{\Gamma_-}) - f(\varphi(s_2))\right|_\infty . \end{aligned} \tag{2.18}$$

We do the same calculations as in case a) and obtain

$$\begin{aligned} |f(a_1) - f(a_2)|_\infty &\leq 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi(|s_1 - 0| + |L_{\Gamma_-} - s_2|) \\ &= 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi\, d_{\Gamma_-}(a_1, a_2). \end{aligned}$$

The last step is based on the parametrization of φ with respect to the arc length. $s_1 + |L_{\Gamma_-} - s_2|$ is nothing else than the length of the sought curve segment between $\varphi(s_1)$ and $\varphi(s_2)$. In case of $s_2 < s_1$, the calculations are done analogously with the difference that in (2.18), $f(\varphi(s_1))$ has to be compared to $f(\varphi(L_{\Gamma_-})$ and $f(\varphi(s_2))$ to $f(\varphi(0))$.

Now we reassemble case 1 and case 2. Since $\overrightarrow{xa_1}$, $\overrightarrow{a_1a_2}$ and $\overrightarrow{a_2y}$ is a disjoint decomposition of the interval $\overrightarrow{xy}$, it holds

$|x - a_1|_\infty + |a_1 - a_2|_\infty + |a_2 - y|_\infty = |x - y|_\infty$. We obtain for (2.17) for every $x,\, y \in \Omega$ with $\overrightarrow{xy} \cap \Omega_- \neq \emptyset$ and since $3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi > 1$

$$\begin{aligned} |f(x) - f(y)|_\infty &\leq \|\nabla f\|_{0,\Omega}|x - a_1|_\infty + 3\sqrt{2}\frac{D_{\Omega_-}}{w_{\Omega_-}}\pi\|\nabla f\|_{0,\Omega}|a_1 - a_2|_\infty + \|\nabla f\|_{0,\Omega}|a_2 - y|_\infty \\ &\leq c_{mv}\|\nabla f\|_{0,\Omega}|x - y|_\infty. \end{aligned}$$

The assertion follows by combining case 1 and 2. $\square$

Chapter 3

Time Dependent Coupled Problem in 1d

We begin the analysis of the time dependent coupled problem with the one-dimensional case. The problem is given by

Problem (CP 3.1). *Let $I = [0, 1]$ and $[0, T]$ a time interval. Find (u, ρ) with $u \in C^{2;0}(I, [0, T])$ and $\rho(\cdot, t) \in C^1(I)$, $\rho(x, \cdot) \in C^1([0, T])$ such that*

$$-\partial_x^2 u(x,t) = \rho(x,t) \qquad (x,t) \in I \times [0,T] \tag{3.1a}$$

$$u(0,t) = u_{A_1} \qquad t \in [0,T] \tag{3.1b}$$

$$u(1,t) = u_{A_2} \qquad t \in [0,T] \tag{3.1c}$$

$$\frac{d}{dt}\rho(x,t) - \partial_x u(x,t)\partial_x \rho(x,t) + \rho^2(x,t) = 0 \qquad (x,t) \in I \times [0,T] \tag{3.1d}$$

$$\rho(x,0) = 0 \qquad x \in I \tag{3.1e}$$

$$\rho(0,t) = \rho_A(t) \qquad t \in [0,T]. \tag{3.1f}$$

where $u_{A_1} > u_{A_2}$ constant and $\rho_A \in C^1([-\infty, T])$ with $\rho_A(t) = 0$ for $t \in [-\infty, 0]$.

Several authors [32, 44, 43, 55] used a particle trajectory approach to model the dynamics of vortex patches in two or three dimensions. The set of equations considered in their works differ from (3.1a)-(3.1f) by using a different elliptic equation. In addition, none of the publications [32, 44, 43, 55] deal either with non-zero boundary conditions for the potential u or a non-zero inflow condition ρ_A at the inflow boundary, such as it is necessary for the electrostatic spray painting process. This chapter is based on and generalizes the previous publications in a one-dimensional setting. As novelty, the model problem (CP 3.1) focuses only on modeling the charge inflow into the domain of which we assume that it is initially free of charge. Moreover, we introduce non-homogeneous boundary conditions for the potential u representing an applied voltage difference of $u_{A_1} - u_{A_2}$. This section is also understood as a preparatory result for the two-dimensional case in Chapter 4 wherein we follow the same ideas.

The main task and difficulty is to model the inflow of charge given by the boundary condition ρ_A. The present variables x and t are not sufficient to express the emission of charge at $x = 0$. We introduce the following set of points.

Definition 3.1. *For a given time $t \in [0, T]$, the set*

$$Q_t = \{t_x : t_x \in [0, t]\} \tag{3.2}$$

is called the inflow set*. t_x is understood as the time a charge particle is emitted at the inflow $x = 0$ into the interval $I = [0, 1]$.*

For every t, the elements of the inflow set are the times when a charge particle flew into the domain since $t = 0$. Due to the continuity in time, the inflow set is therefore the time interval $[0, t]$. The usefulness of the set Q_t will become clear immediately when formulating the streamline function Φ.

As in [44], we choose the particle trajectory approach to show existence and uniqueness of a classical solution (u, ρ) to (CP 3.1). As hyperbolic differential equation, the transport equation (3.1d) gives rise to a streamline function $\Phi(t_x, t)$, that is drawing the path of a charge particle through the space time domain. Although streamlines are defined on the interval I, as the potential u is given therein, we restrict our view to the streamlines starting from the inflow boundary $x = 0$. Hence, we focus only on the transport of present charges, in other words on the expansion of the support of ρ.

Definition 3.2. *[45, Section 9] The* streamline function Φ *for the transport equation (3.1d) is given as solution of*

$$\frac{d}{dt}\Phi(t_x, t) = -\partial_x u(\Phi(t_x, t), t) \qquad t_x \in [0, t], t \in [0, T] \tag{3.3}$$

$$\Phi(t_x, t_x) = 0. \qquad t_x \in [0, T] \tag{3.4}$$

$\Phi(t_x, t)$ describes the position of a particle emitted at t_x in $x = 0$ at time t.

With the streamline function, we are able to determine the position of a charge particle at time t that was emitted at $t_x < t$. Φ thus maps Q_t into the subset of I called I_t. As Q_t is the set of all inflowing charge, I_t is thus the support of ρ at time t. We therefore define

$$I_t := \operatorname*{supp}_{x \in I} \{\rho(x, t)\}.$$

For small t, it yields that $I_t \subset I$. The range of Φ is thus given by

$$\Phi(\cdot, t) : Q_t \to I_t \subset I. \tag{3.5}$$

We give a short overview about this Chapter. In section 3.2, we derive an integro-differential operator A by means of (3.3), an explicit representation of the solutions u in terms of Green's functions and the solution ρ evaluated on the streamlines by

$$\begin{aligned} A(\Phi) = &- \int_{t_x}^{t} \int_{t_x}^{\mu} \Phi(t_y, \mu)\partial_{t_y}\Phi(t_y, \mu) \frac{\rho_A(t_y)}{1 + (\mu - t_y)\rho_A(t_y)}\, dt_y\, d\mu \\ &+ \int_{t_x}^{t} \left[\int_{0}^{t_x} (1 - \Phi(t_y, \mu))\partial_{t_y}\Phi(t_y, \mu) \frac{\rho_A(t_y)}{1 + (\mu - t_y)\rho_A(t_y)}\, dt_y + u_{A_1} - u_{A_2} \right] d\mu \end{aligned} \tag{3.6}$$

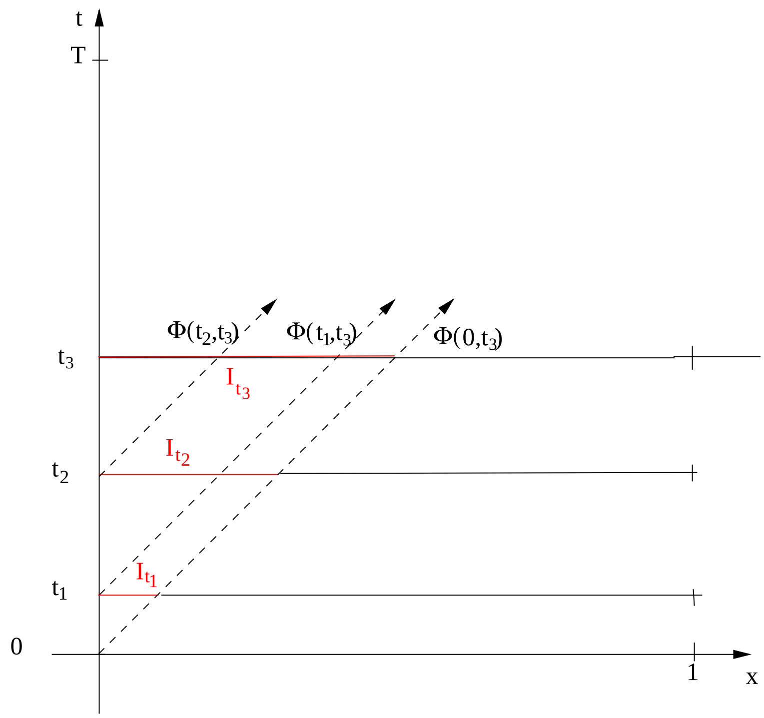


Figure 3.1: Streamline Function $\Phi(t_x, t)$

In section 3.3, we define a set $W(M, T, \delta) \subset C^{1;0}(I, [0, T])$ of all such streamline functions that are feasible for the model problem, i.e. they fulfil the requirements of the model. The operator A is applied to a streamline function $\Phi \in W(M, T, \delta)$. The question we need to answer is whether there exists a unique fixed point $\Phi \in W(M, T, \delta)$ to (3.6), i.e.

$$A\Phi = \Phi. \tag{3.7}$$

A fundamental convergence theorem is the fixed point theorem of Banach.

Theorem 3.3 (Banach Fixed Point Theorem)**.** *([65, p.19]) Let W be a closed nonempty subspace of a Banach space X and assume that the operator $A : W \to W$ is contractive, i.e.*

$$\|Au - Aw\| \leq k\|u - w\|, \quad \forall u, w \in W,\ 0 < k < 1.$$

Then there exists a unique fixed point for A in the set W, i.e. there exists a unique $v \in W$, such that $Av = v$.

In section 3.3, we prove that A is a selfmap and a contraction on the set $W(M, T, \delta)$. The main result of this Chapter is presented in Theorem 3.15 where we obtain the existence of a unique fixed point Φ in $W(M, T, \delta)$ to (3.6) with the Banach fixed point theorem. Our method is different to [44, 43] where a compactness argument is used to show existence and uniqueness. The found fixed point is then the streamline function corresponding to the solution of the coupled problem (CP 3.1). With the dependence of u and ρ on the streamline function derived in section 3.1, we will eventually prove the existence of a unique classical solution (u, ρ) to (CP 3.1) in

Theorem 3.16.

Next to the classical one-dimensional function space $C^{1;0}(I,[0,T])$ defined in Chapter 2, we define $C^{1;0}$ functions for $t_x \in Q_t$ and $t \in [0,T]$.

$$C^{1;0}(Q_t,[0,T]) := \{\Phi : \Phi \text{ and } \partial_{t_x}\Phi \text{ continuous } \forall (t_x,t) \text{ with } t_x \in Q_t, t \in [0,T]\}\,. \tag{3.8}$$

We equip $C^{1;0}(Q_t,[0,T])$ with the norm

$$\begin{aligned}\|\Phi\|_{1,Q_t;0,[0,T]} = \sup_{0\le t\le T}\left(\sup_{0\le t_x\le t}|\Phi(t_x,t)| + \sup_{0\le t_x\le t}|\partial_{t_x}\Phi(t_x,t)|\right)\\ + \sup_{0\le t_x\le T}\left(\sup_{t_x\le t\le T}|\partial_{t_x}\Phi(t_x,t)| + \sup_{t_x\le t\le T}|\Phi(t_x,t)|\right).\end{aligned}$$

To simplify the notation in the estimates in section 3.3, we will denote the sup-norm over all $0 \le t_x \le t$ for every $t \in [0,T]$ as

$$\|\Phi(t)\|_{0,Q_t} := \sup_{0\le t_x\le t}|\Phi(t_x,t)|\,.$$

and the sup-norm for every $t_x \in Q_T$ by

$$\|\Phi(t_x)\|_{0,[t_x,T]} := \sup_{t_x\le t\le T}|\Phi(t_x,t)|\,.$$

Note that the previous expressions are still variable in t and t_x respectively. We abbreviate the sup-norm in space and time by

$$\|\Phi\|_{0,Q_t;0,[0,T]} := \sup_{0\le t\le T}\|\Phi(t)\|_{0,Q_t} + \sup_{0\le t_x\le T}\|\Phi(t_x)\|_{0,[t_x,T]}\,. \tag{3.9}$$

Theorem 3.4. *[3, p. 42]*
$C^{1;0}(Q_t,[0,T])$ *equipped with the norm (3.9) is a Banach space.*

3.1 Explicit Solutions for the Partial Problems

In the next two sections, we will derive explicit representations of the solutions of the Poisson equation (3.1a)-(3.1c) and the transport equation (3.1d)-(3.1f). In case of the Poisson equation, we will use Green's function to obtain an integral representation of u. The transport equation is solved on a streamline and depends thereon only on the inflow function ρ_A. These representations are used for the derivation of the integro-differential operator A in section 3.2.

3.1.1 Poisson Equation

One method to transform the solution of the Poisson equation into an integral equation is the usage of Green's function. For every fixed time $t \in [0,T]$, the one-dimensional Dirichlet problem for the Poisson equation is given by

Problem (Po 3.2). *For a continuous right hand side function $\rho(\cdot,t) \in C^0(I)$ and a fixed time $t \in [0,T]$, find $u(\cdot,t) \in C^2([0,1])$, such that*

$$-\partial_x^2 u(x,t) = \rho(x,t) \qquad (x,t) \in I \times [0,T] \tag{3.10a}$$
$$u(0,t) = u_{A_1} \qquad t \in [0,T] \tag{3.10b}$$
$$u(1,t) = u_{A_2} \qquad t \in [0,T]. \tag{3.10c}$$

with $u_{A_1} > u_{A_2}$ constant.

For this standard one-dimensional problem, the solution is given in terms of Green's function.

Lemma 3.5. *[35, p.13] Let $\rho \in C^{0,0}([0,1],[0,T])$, then the Green's function for (3.10a)-(3.10c) is given by*

$$G(x,y) = \begin{cases} x(1-y) & 0 \leq x \leq y \leq 1 \\ y(1-x) & 0 \leq y \leq x \leq 1 \end{cases}.$$

For the solution $u(x,t)$ yields

$$u(x,t) = \int_0^x y(1-x)\rho(y,t)\,dy + \int_x^1 x(1-y)\rho(y,t)\,dy + u_{A_1}(1-x) + u_{A_2}x. \tag{3.11}$$

It holds $u \in C^{2;0}(I,[0,T])$.

Proof. We easily verify that (3.11) fulfils the boundary value problem (Po 3.2), as $\partial_x^2 u(x,t) = \rho(x,t)$, $u(0,t) = u_{A_1}$ and $u(1,t) = u_{A_2}$. u is thus twice continuously differentiable w.r.t x. Also, u is continuous in time, as $\rho \in C^{0,0}(I,[0,T])$. □

The right hand side function of the differential equation (3.3) is $\partial_x u$. As a consequence, the integro-differential operator A is based on the integral representation of $\partial_x u$. The next Corollary is a simple implication of Lemma 3.5 but it is worth noticing as $\partial_x u$ is used frequently.

Corollary 3.6. *The derivative w.r.t. x of (3.11) is given by*

$$\partial_x u(x,t) = -\int_0^x y\rho(y,t)\,dy + \int_x^1 (1-y)\rho(y,t)\,dy + u_{A_2} - u_{A_1}. \tag{3.12}$$

Proof. By differentiation of (3.11), we get

$$\begin{aligned}\partial_x u(x,t) &= x(1-x)\rho(x,t) - \int_0^x y\rho(y,t)\,dy - x(1-x)\rho(x,t) + \int_x^1 (1-y)\rho(y,t)\,dy + u_{A_2} - u_{A_1} \\ &= -\int_0^x y\rho(y,t)\,dy + \int_x^1 (1-y)\rho(y,t)\,dy + u_{A_2} - u_{A_1}.\end{aligned}$$

□

3.1.2 Transport Equation

The second component of the coupled problem is the transport problem.

Problem (Tr 3.3). *For a given function $\partial_x u \in C^{1;0}(I,[0,T])$, find ρ with $\rho(\cdot,t) \in C^1([0,1])$ and $\rho(x,\cdot) \in C^1([0,T])$ such that*

$$\partial_t \rho(x,t) - \partial_x u(x,t)\partial_x \rho(x,t) + \rho^2(x,t) = 0 \qquad (x,t) \in [0,1]\times[0,T] \tag{3.13a}$$
$$\rho(x,0) = 0 \qquad x \in [0,1] \tag{3.13b}$$
$$\rho(0,t) = \rho_A(t) \qquad t \in [0,T] \tag{3.13c}$$

with $\rho_A \in C^1([-\infty,T])$ and $\rho_A(t) = 0$ for $t \in [-\infty,0]$.

The transport equation is reduced to an ordinary differential equation on a streamline. For completeness, we will demonstrate this well known result for hyperbolic partial differential equations, see e.g. [45, p.169]. In fact, the ordinary differential equation is easily solved and we obtain an explicit representation of the solution ρ on a streamline.

Lemma 3.7. *The boundary value problem (Tr 3.3) reduces on a streamline Φ for all (t_x,t) with $t_x \in [0,t], t \in [0,T]$ to*

$$\frac{d}{dt}\rho(\Phi(t_x,t),t) = -\rho^2(\Phi(t_x,t),t) \tag{3.14}$$

with initial condition

$$\rho(\Phi(t_x,t_x),t_x) = \rho_A(t_x). \tag{3.15}$$

Proof. With the chain rule and (3.3), we obtain

$$\begin{aligned}\frac{d}{dt}\rho(\Phi(t_x,t),t) &= \partial_x\rho(\Phi(t_x,t),t)\frac{d}{dt}\Phi(t_x,t) + \partial_t\rho(\Phi(t_x,t),t)\\ &= -\partial_x\rho(\Phi(t_x,t),t)\partial_x u(\Phi(t_x,t),t) + \partial_t\rho(\Phi(t_x,t),t).\end{aligned} \tag{3.16}$$

Using (3.13a) with $x = \Phi(t_x,t)$ gives

$$\partial_t\rho(\Phi(t_x,t),t) - \partial_x u(\Phi(t_x,t),t)\partial_x\rho(\Phi(t_x,t),t) + \rho^2(\Phi(t_x,t),t) = 0. \tag{3.17}$$

Substituting (3.16) into (3.17) gives

$$\frac{d}{dt}\rho(\Phi(t_x,t),t) = -\rho^2(\Phi(t_x,t),t).$$

□

The next result is the solution of the nonlinear boundary value problem (3.14)-(3.15) and we obtain the solution $\rho(\Phi(t_x,t),t)$ of (Tr 3.3). The solution is given on a streamline and depends thereon only on the inflow boundary data.

Lemma 3.8. *The solution ρ of (Tr 3.3) is given on the streamlines by*

$$\rho(\Phi(t_x,t),t) = \frac{\rho_A(t_x)}{1+(t-t_x)\rho_A(t_x)} \tag{3.18}$$

for every (t_x,t) with $t_x \in [0,t], t \in [0,T]$.

Proof. Set

$$p(t) = \rho^{-1}(\Phi(t_x,t),t) \tag{3.19}$$

Then holds by the chain rule and (3.14)

$$\begin{aligned}\frac{d}{dt}p(t) &= -\rho^{-2}(\Phi(t_x,t),t)\frac{d}{dt}\rho(\Phi(t_x,t),t)\\ &= \frac{\rho^2(\Phi(t_x,t),t)}{\rho^2(\Phi(t_x,t),t)}\\ &= 1.\end{aligned}$$

Integration with respect to t gives

$$p(t) = t + c(t_x).$$

It follows by (3.19)

$$\rho(\Phi(t_x,t),t) = \frac{1}{t+c(t_x)}.$$

We determine $c(t_x)$ by using the initial condition (3.15)

$$\rho(\Phi(t_x,t_x),t_x) = \frac{1}{t_x+c(t_x)} = \rho_A(t_x).$$

Eventually,

$$\rho(\Phi(t_x,t),t) = \frac{\rho_A(t_x)}{1+(t-t_x)\rho_A(t_x)}.$$

□

The result of the previous Lemma defines the value of ρ on I_t for every t. Moreover, it leads to a global solution $\rho(x,t)$ on the interval I. The decisive component is the inflow boundary function ρ_A which is chosen to be 0 for $t \leq 0$ but differentiable on the interval $[-\infty, T]$. This choice ensures a smooth transition of ρ into the interval $I \backslash I_t$.

Lemma 3.9. *Let $\Phi(\cdot,t) \in C^1(Q_t)$ for every $t \in [0,T]$ and $\Phi(t_x,\cdot) \in C^1([0,T])$. Let $|\partial_{t_x}\Phi(t_x,t)| \geq \delta$ for some $\delta > 0$. Then the solution of (Tr 3.3) is given by*

$$\rho(x,t) = \begin{cases} \frac{\rho_A(\Phi^{-1}(x,t))}{1+(t-\Phi^{-1}(x,t))\rho_A(\Phi^{-1}(x,t))} & x \in I_t,\ t \in [0,T] \\ 0 & x \in I \backslash I_t,\ t \in [0,T] \end{cases}. \tag{3.20}$$

It holds $\rho(\cdot,t) \in C^1(I)$ and $\rho(x,\cdot) \in C^1([0,T])$.

Proof. By the implicit function theorem and $|\partial_{t_x}\Phi(t_x,t)| \geq \delta$, the inverse function $\Phi^{-1}(y,t)$ exists and is differentiable. $\rho(x,t)$ is thus given by (3.18) on I_t. We extend ρ continuously into $I \backslash I_t$ by 0 due to the assumptions on $\rho_A(t)$ in (Tr 3.3). Since $\rho_A \in C^1(-\infty,T)$, we also obtain the differentiability of ρ in x. $\Phi^{-1}(y,t)$ is the inverse function for a fixed $t \in [0,T]$ with respect to the space variable. Since Φ is differentiable in space and time and due to

$$\partial_t\Phi^{-1}(\Phi(t_x,t),t) = -\nabla\Phi^{-1}(\Phi(t_x,t),t)\frac{d}{dt}\Phi(t_x,t)$$

we also obtain that Φ^{-1} is differentiable with respect to t. Since ρ_A is differentiable holds $\rho(x,\cdot) \in C^1([0,T])$. □

3.2 Derivation of the Operator A

In this section, we present the derivation of the intego-differential operator $A(\Phi)$ by combining the definition of Φ in Definition 3.2, the Green's function representation of $\partial_x u$ in Corollary 3.6 and the explicit solution ρ in Lemma 3.9. The derivation is based on [44, 55].

Integrating the streamline differential equation (3.3) over $[t_x,t]$ gives

$$\Phi(t_x,t) = \Phi(t_x,t_x) - \int_{t_x}^{t} \partial_x u(\Phi(t_x,\mu),\mu)\,d\mu.$$

With (3.4), i.e. $\Phi(t_x,t_x) = 0$ and (3.12), we obtain

$$\begin{aligned}\Phi(t_x,t) &= -\int_{t_x}^{t} \partial_x u(\Phi(t_x,\mu),\mu)\,d\mu \\ &= \int_{t_x}^{t}\left[\int_0^{\Phi(t_x,\mu)} y\rho(y,\mu)\,dy - \int_{\Phi(t_x,\mu)}^{1}(1-y)\rho(y,\mu)\,dy + u_{A_1} - u_{A_2}\right]d\mu. \qquad (3.21)\end{aligned}$$

To substitute $\rho(x,t)$ by (3.20), we need additional information about the interval I_t. According to the definition, $\Phi(\cdot,t)$ maps the set Q_t onto the interval I_t for every $t \in [0,T]$, i.e.

$$\Phi(\cdot,t) : [0,t] \to I_t.$$

For the model to be meaningful, we have to exclude that any two streamline functions intersect in the space time domain. Using this assumption, it is possible to determine the point that is mapped with the greatest distance to the origin, i.e. $\sup_{0\leq t_x\leq t}|\Phi(t_x,t)|$. The charge particle that flew in first still needs to be in front, since otherwise a later particle would have 'overtaken', in other words the streamlines had crossed. We thus determine

$$\sup_{0\leq t_x\leq t}|\Phi(t_x,t)| = \Phi(0,t) \in I.$$

We adapt the integral limits to the support of ρ and substitute ρ by (3.20). We obtain for (3.21)

$$\begin{aligned}
\Phi(t_x,t) &= \int_{t_x}^{t}\int_0^{\Phi(t_x,\mu)} y\rho(y,\mu)\,dy - \int_{\Phi(t_x,\mu)}^{\Phi(0,\mu)} (1-y)\rho(y,\mu)\,dy + u_{A_1} - u_{A_2}\,d\mu \\
&= \int_{t_x}^{t}\int_0^{\Phi(\Phi^{-1}(y,\mu),\mu)} y\frac{\rho_A(\Phi^{-1}(y,\mu))}{1+(\mu-\Phi^{-1}(y,\mu))\rho_A(\Phi^{-1}(y,\mu))}\,dy\,d\mu \\
&\quad - \int_{t_x}^{t}\left[\int_{\Phi(t_x,\mu)}^{\Phi(0,\mu)} (1-y)\frac{\rho_A(\Phi^{-1}(y,\mu))}{1+(\mu-\Phi^{-1}(y,\mu))\rho_A(\Phi^{-1}(y,\mu))}\,dy + u_{A_2} - u_{A_1}\right]d\mu. \quad (3.22)
\end{aligned}$$

In (3.22), we obtained an integral equation for Φ. One way of solving the integral equation is to reformulate it as integro-differential operator A and seek for its fixed point. In the next section we will therefore introduce a set $W(M,T,\delta) \subset C^{1;0}(Q_t,[0,T])$ in which we will apply a fixed point argument.

Define the operator A applied to all invertible $\Phi \in C^{1;0}(Q_t,[0,T])$ by

$$\begin{aligned}
A(\Phi)(t_x,t) &= \int_{t_x}^{t}\int_0^{\Phi(t_x,\mu)} y\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\bigg|_{t_y=\Phi^{-1}(y,\mu)}\,dy\,d\mu \\
&\quad - \int_{t_x}^{t}\int_{\Phi(t_x,\mu)}^{\Phi(0,\mu)} (1-y)\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\bigg|_{t_y=\Phi^{-1}(y,\mu)}\,dy + u_{A_2} - u_{A_1}\,d\mu. \quad (3.23)
\end{aligned}$$

The integral limits in (3.23) are variable in Φ. With the substitution $y = \Phi(t_y,\mu)$, we allow to conduct the calculations on a set independent of the choice of Φ. This is advantageous as soon as we compare the distance of the operator applied to two distinct streamline functions.

With the substitution $y = \Phi(t_y,\mu)$, (3.23) reduces to

$$\begin{aligned}
A(\Phi)(t_x,t) &= \int_{t_x}^{t}\int_{\mu}^{t_x} \Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu)\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\,dt_y\,d\mu \\
&\quad - \int_{t_x}^{t}\int_{t_x}^{0} (1-\Phi(t_y,\mu))\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\partial_{t_y}\Phi(t_y,\mu)\,dt_y + u_{A_2} - u_{A_1}\,d\mu \\
&= -\int_{t_x}^{t}\int_{t_x}^{\mu} \Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu)\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\,dt_y\,d\mu \\
&\quad + \int_{t_x}^{t}\int_0^{t_x} (1-\Phi(t_y,\mu))\partial_{t_y}\Phi(t_y,\mu)\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\,dt_y + u_{A_1} - u_{A_2}\,d\mu. \quad (3.24)
\end{aligned}$$

The fixed point $\Phi(t_x,t)$ is sought for in a subset of $C^{1;0}(Q_t,[0,T])$. To show that the operator A is a selfmap, we thus need the derivative of $A(\Phi)$ w.r.t. t_x.

Lemma 3.10. *Let $\Phi(\cdot,t) \in C^1(Q_t)$ and $\inf_{0\le t_x\le t} |\partial_{t_x}\Phi(t_x,t)| \ge \delta$ for some $\delta > 0$ and $t \in [0,T]$.*

Then the derivative of $A(\Phi)(t_x,t)$ w.r.t. t_x is given by

$$\begin{aligned}\partial_{t_x} A(\Phi)(t_x,t) = &\int_0^{t_x} (\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y \\ &+\int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu + u_{A_2}-u_{A_1}. \end{aligned} \tag{3.25}$$

Proof. By differentiation of (3.24), we get

$$\begin{aligned}\partial_{t_x} A(\Phi)(t_x,t) = &-\int_0^{t_x} (1-\Phi(t_y,t_x))\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y + u_{A_2}-u_{A_1} \\ &+\int_{t_x}^{t} \Phi(t_x,\mu)\partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu \\ &+\int_{t_x}^{t} (1-\Phi(t_x,\mu))\partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu \\ = &-\int_0^{t_x} (1-\Phi(t_y,t_x))\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y \\ &+\int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu + u_{A_2}-u_{A_1}. \end{aligned}$$

□

3.3 Existence of a Fixed Point for A

In this section, we apply the Banach fixed point theorem to the operator A on the set $W(M,T,\delta) \subset C^{1;0}(Q_t,[0,T])$, where

$$\begin{aligned}W(M,T,\delta) := \Big\{&\Phi \in C^{1;0}(Q_t,[0,T]) : \Phi(\cdot,t) : Q_t \to I,\ \Phi(t_x,t_x)=0,\ \|\Phi\|_{0,Q_t;0,[0,T]} \le 1, \\ &\|\partial_{t_x}\Phi(t)\|_{0,Q_t;0,[0,T]} \le M,\ \inf_{0\le t_x\le T} |\partial_{t_x}\Phi(t_x,t)| \ge \delta > 0\,\forall t\in[0,T]\Big\}.\end{aligned}$$

There are five conditions on the functions Φ in the set $W(M,T,\delta)$. The first two conditions ensure that every $\Phi \in W(M,T,\delta)$ maps into the interval I and fulfils the initial condition. The second one is the boundedness in the sup-norm $C^{0;0}(Q_t,[0,T])$ where the bound 1 expresses that $\Phi(t_x,t) \in W(M,T,\delta)$ maps into I. The derivative is bounded by a constant M which will be determined in Lemma 3.13. The last condition provides the invertibility for the streamline function $\Phi(t_x,t)$ which is important in multiple ways. First, the lower bound δ permits a bound for the inverse function

$$\left|\partial_y \Phi^{-1}(y,t)\right| = \left|\left(\partial_{t_x}\Phi(\Phi^{-1}(y,t),t)\right)^{-1}\right| \le \left\|\left(\partial_{t_x}\Phi(t)\right)^{-1}\right\|_{0,Q_t} \le \frac{1}{\delta}.$$

Second, the invertibility condition on $\partial_{t_x}\Phi(t_x,t)$ also ensures that the streamlines exist for every time t. If the invertibility of Φ was violated, then Φ could not be a bijective map such that

intersections of streamlines could exist. Last, it guarantees the expansion of I_t in time, i.e. a particle increases the distance to $x = 0$ for increasing t.
Excluding intersections of the streamline functions is important as otherwise, it would lead to shock fronts as described in [52]. We will obtain conditions on the choice of the constant δ while examining whether A is a selfmap.

We first prove that W is a non-empty set and is convex.

Lemma 3.11. *Let $M \geq 2(u_{A_1} - u_{A_2})$, $\delta \leq u_{A_1} - u_{A_2}$ and $T \leq \frac{1}{2(u_{A_1}-u_{A_2})}$. Then $W(M,T,\delta)$ is a non-empty set.*

Proof. We show that the streamline function corresponding to the gradient of the solution of the Laplace equation

$$\begin{aligned} -\Delta u_0 &= 0 \\ u_0(0) &= u_{A_1} \\ u_0(1) &= u_{A_2} \end{aligned}$$

is contained in $W(M,T,\delta)$.

The solution of the Laplace equation is given by

$$u_0(x,t) = u_{A_1}(1-x) + u_{A_2}x.$$

We thus define the streamline function Φ as solution of the equation

$$\begin{aligned} \frac{d}{dt}\Phi(t_x,t) &= -\partial_x u_0(\Phi(t_x,t),t) = u_{A_1} - u_{A_2} \\ \Phi(t_x,t_x) &= 0. \end{aligned}$$

We obtain by integration over $[t_x,t]$

$$\begin{aligned} \Phi(t_x,t) &= \Phi(t_x,t_x) + \int_{t_x}^{t} u_{A_1} - u_{A_2}\, d\tau \\ &= (u_{A_1} - u_{A_2})(t-t_x). \end{aligned}$$

We will now show that all the restrictions in the space $W(M,T,\delta)$ are fulfilled for M,T and δ as chosen in the assertion. First, we see $\Phi(t_x,t_x) = 0$ and since $u_{A_1} > u_{A_2}$ also $\Phi(t_x,t) \to \mathbb{R}_+$. Second, we have

$$\begin{aligned} \|\Phi\|_{0,Q_t;0,[0,T]} &= \sup_{0\leq t_x\leq T}\|\Phi(t_x)\|_{0,[t_x,T]} + \sup_{0\leq t\leq T}\|\Phi(t)\|_{0,Q_t} \leq 2T(u_{A_1}-u_{A_2}), \\ \|\partial_{t_x}\Phi\|_{0,Q_t;0,[0,T]} &= \sup_{0\leq t_x\leq T}\|\partial_{t_x}\Phi(t_x)\|_{0,[t_x,T]} + \sup_{0\leq t\leq T}\|\partial_{t_x}\Phi(t)\|_{0,Q_t} = 2(u_{A_1}-u_{A_2}), \end{aligned}$$

and for all $t \in [0,T]$

$$\inf_{0\leq t_x\leq t}|\partial_{t_x}\Phi(t_x,t)| = u_{A_1} - u_{A_2}.$$

With $M \geq 2(u_{A_1} - u_{A_2})$, $\delta \leq u_{A_1} - u_{A_2}$ and $T \leq \frac{1}{2(u_{A_1}-u_{A_2})}$, all restrictions on Φ in the set $W(M,T,\delta)$ are fulfilled. □

Lemma 3.12. *The set $W(M, T, \delta)$ is closed in the $C^{1;0}(Q_t, [0, T])$-norm.*

Proof. We need to show that every convergent sequence $\{\Phi_n\}_{n\in\mathbb{N}} \in W(M, T, \delta)$ converges to a $\Phi \in W(M, T, \delta)$. Let Φ_n be such a sequence with

$$\|\Phi_n - \Phi\|_{1,0} \to 0 \text{ for } n \to \infty.$$

The uniform convergence in $C^{1;0}(I, [0, T])$ implies pointwise convergence. Then the initial condition is fulfilled by Φ, as

$$|\Phi(t_x, t_x)| = \lim_{n\to\infty} |\Phi_n(t_x, t_x)| = 0.$$

It follows that $\Phi(t_x, t_x) = 0$.

As Φ_n converges uniformly in the $C^{1;0}(I, [0, T])$-norm, it converges even pointwise to Φ. We obtain for all (s_x, s) and (t_x, t)

$$\begin{aligned}
|\Phi(s_x, s)| + |\Phi(t_x, t)| &= \lim_{n\to\infty} (|\Phi_n(s_x, s)| + |\Phi_n(t_x, t)|) \\
&\leq \lim_{n\to\infty} (\sup_{0\leq s\leq T} \|\Phi_n(s)\|_{0,Q_s} + \sup_{0\leq t_x\leq T} \|\Phi_n(t_x)\|_{0,[t_x,T]}) \\
&= \lim_{n\to\infty} \|\Phi_n\|_{0,Q_t;0,[0,T]} \leq 1.
\end{aligned}$$

Hence $\|\Phi\|_{0,Q_t;0,[0,T]} \leq 1$. The remaining restrictions are shown likewise and are therefore omitted. □

Since $W(M, T, \delta)$ is a closed subset of the Banach space $C^{1;0}(Q_t, [0, T])$, it is complete and thus complies with the requirement of Banach's fixed point theorem. The remaining part of this section is concerned with checking whether A is a selfmap and contraction. First, it is shown that, with a sensible choice of M, T and δ, A maps $W(M, T, \delta)$ into itself. Second, we prove that A is a contraction for small T.

Lemma 3.13. *Let $\Phi \in W(M, T, \delta)$ and choose $(u_{A_1} - u_{A_2}) > \delta + \|\rho_A\|_{0,[0,T]}$, $M = 2 + 2(u_{A_1} - u_{A_2})$ and $T = \inf\left\{\frac{1}{\sqrt{8M\|\rho_A\|_{0,[0,T]}}}, \frac{1}{2M\|\rho_A\|_{0,[0,T]}}, \frac{1}{2M}, \frac{1}{4(u_{A_1}-u_{A_2})}\right\}$. Then holds*

$$A : W(M, T, \delta) \to W(M, T, \delta). \tag{3.26}$$

Proof. $A(\Phi)$ fulfils the inflow condition

$$A(\Phi)(t_x, t_x) = 0.$$

First we show that $A(\Phi)$ maps into the positive real numbers. Recall that $A(\Phi)$ is given by

$$\begin{aligned}
A(\Phi)(t_x, t) = &- \int_{t_x}^{t} \int_{t_x}^{\mu} \Phi(t_y, \mu)\partial_{t_y}\Phi(t_y, \mu)\frac{\rho_A(t_y)}{1 + (\mu - t_y)\rho_A(t_y)}\, dt_y\, d\mu \\
&+ \int_{t_x}^{t} \int_{0}^{t_x} (1 - \Phi(t_y, \mu))\partial_{t_y}\Phi(t_y, \mu)\frac{\rho_A(t_y)}{1 + (\mu - t_y)\rho_A(t_y)}\, dt_y + u_{A_1} - u_{A_2}\, d\mu
\end{aligned}$$

We will show that the two integrals are less than $u_{A_1} - u_{A_2} > 0$ for small T which proves $A(\Phi) \geq 0$. We obtain for $\mu \in [0,T]$ and with the choice of T and $u_{A_1} - u_{A_2}$ as in the assertion

$$
\begin{aligned}
u_{A_1} - u_{A_2} - &\left| \int_{t_x}^{\mu} \Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y \right| \\
&- \left| \int_0^{t_x} (1-\Phi(t_y,\mu))\partial_{t_y}\Phi(t_y,\mu) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y \right| \\
\geq\ & u_{A_1} - u_{A_2} - T\left\|\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\partial_{t_y}\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\rho_A\right\|_{0,[0,T]} \\
&- T\left\|1-\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\partial_{t_y}\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\rho_A\right\|_{0,[0,T]} \\
\geq\ & u_{A_1} - u_{A_2} - 2TM\left\|\rho_A\right\|_{0,[0,T]} \\
\geq\ & \delta + \left\|\rho_A\right\|_{0,[0,T]} - \left\|\rho_A\right\|_{0,[0,T]} = \delta > 0.
\end{aligned}
$$

In the next step, we investigate the remaining restrictions on Φ in the set $W(M,T,\delta)$. By showing that $\|A(\Phi)\|_{0,Q_t;0,[0,T]} \leq 1$, we have proved together with the previous result that $A(\Phi)(\cdot,t) : Q_t \to I$.

We show the boundedness of $A(\Phi)$. We obtain pointwise by the triangle inequality and the definition of $W(M,T,\delta)$

$$
\begin{aligned}
|A(\Phi)(t_x,t)| \leq\ & \left| \int_{t_x}^{t} \int_{t_x}^{\mu} \Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y\, d\mu \right| \\
&+ \left| \int_{t_x}^{t} \int_0^{t_x} (1-\Phi(t_y,\mu))\partial_{t_y}\Phi(t_y,\mu) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y\, d\mu \right| + \left| \int_{t_x}^{t} u_{A_2} - u_{A_1}\, d\mu \right| \\
\leq\ & \int_0^t \int_0^{\mu} |\Phi(t_y,\mu)| \left|\partial_{t_y}\Phi(t_y,\mu)\right| \left| \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)} \right| dt_y\, d\mu \\
&+ \int_0^t \int_0^t |(1-\Phi(t_y,\mu))| \left|\partial_{t_y}\Phi(t_y,\mu)\right| \left| \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)} \right| dt_y\, d\mu + t\left|u_{A_2} - u_{A_1}\right| \\
\leq\ & t^2\left\|\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\partial_{t_y}\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\rho_A\right\|_{0,[0,T]} \\
&+ t^2\left\|1-\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\partial_{t_y}\Phi\right\|_{0,Q_t;0,[0,T]} \left\|\rho_A\right\|_{0,[0,T]} + t\left|u_{A_2} - u_{A_1}\right| \\
\leq\ & 2t^2M\left\|\rho_A\right\|_{0,[0,T]} + t\left|u_{A_2} - u_{A_1}\right| \\
\leq\ & 2T^2M\left\|\rho_A\right\|_{0,[0,T]} + T\left|u_{A_2} - u_{A_1}\right|.
\end{aligned}
$$

It holds pointwise for the derivative (3.25) with the definition of $W(M,T,\delta)$

$$\begin{aligned}
|\partial_{t_x} A(\Phi)(t_x,t)| &\leq \left|\int_0^{t_x} (\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y\right| \\
&\quad + \left|\int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu\right| + |u_{A_2}-u_{A_1}| \\
&\leq \int_0^{t} |\Phi(t_y,t_x)-1|\,|\partial_{t_y}\Phi(t_y,t_x)|\left|\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\right|\,dt_y \\
&\quad + \int_{t_x}^{t} |\partial_{t_x}\Phi(t_x,t)|\left|\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\right|\,d\mu + |u_{A_2}-u_{A_1}| \\
&\leq 2tM\,\|\rho_A\|_{0,[0,t]} + |u_{A_1}-u_{A_2}| \\
&\leq 2TM\,\|\rho_A\|_{0,[0,T]} + |u_{A_1}-u_{A_2}|.
\end{aligned}$$

Finally, we need to show that $A(\Phi)$ is invertible. Using again (3.25)

$$\begin{aligned}
&|\partial_{t_x} A(\Phi)(t_x,t)| \\
&= \left|\int_0^{t_x} (\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y + \int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu + u_{A_2}-u_{A_1}\right|
\end{aligned}$$

The inverse triangle inequality gives

$$\begin{aligned}
|\partial_{t_x} A(\Phi)(t_x,t)| &\geq |u_{A_1}-u_{A_2}| - \left|\int_0^{t_x} (\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x)\frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\,dt_y\right| \\
&\quad - \left|\int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\,d\mu\right| \\
&\geq |u_{A_1}-u_{A_2}| - TM\|\rho_A\|_{0,[0,T]} - TM\,\|\rho_A\|_{0,[0,T]} \\
&\geq |u_{A_1}-u_{A_2}| - 2TM\|\rho_A\|_{0,[0,T]}.
\end{aligned}$$

With the choice of T, M and $u_{A_1}-u_{A_2}$ as given in the assumptions of this Lemma, we obtain

$$\begin{aligned}
\|A(\Phi)\|_{0,Q_t;0,[0,T]} &= \sup_{0\leq t\leq T}\|A(\Phi)(t)\|_{0,Q_t} + \sup_{0\leq t_x\leq T}\|A(\Phi)(t_x)\|_{0,[t_x,T]} \leq \frac{1}{2}+\frac{1}{2} = 1, \\
\|\partial_{t_x}A(\Phi)(t_x,t)\|_{0,Q_t;0,[0,T]} &= \sup_{0\leq t\leq T}\|\partial_{t_x}A(\Phi)(t)\|_{0,Q_t} + \sup_{0\leq t_x\leq T}\|\partial_{t_x}A(\Phi)(t_x)\|_{0,[t_x,T]} \\
&\leq 2 + 2|u_{A_2}-u_{A_1}| =: M
\end{aligned}$$

and for $t\in[0,T]$

$$\inf_{0\leq t_x\leq t}|\partial_{t_x}\Phi(t_x,t)| \geq \delta + \|\rho_A\|_{0,[0,T]} - \|\rho_A\|_{0,[0,T]} = \delta.$$

$A(\Phi)$ is thus bounded in $W(M,T,\delta)$ and the last estimate guarantees the invertibility of Φ which is equivalent to the existence of the streamlines for every $0\leq t\leq T$. □

The second requirement on A in Banach's fixed point Theorem is the contraction property. The next theorem shows that A is Lipschitz continuous in $W(M,T,\delta)$ with a Lipschitz constant depending on T, $\|\rho_A\|_{0,[0,T]}$ and a constant $c(M)$ depending only on M. With T sufficiently small, A is even a contraction.

Theorem 3.14. *$A : W(M,T,\delta) \to W(M,T,\delta)$ is continuous in $C^{1,0}(Q_t,[0,T])$, i.e.*

$$\left\|A(\Phi) - A(\tilde{\Phi})\right\|_{1,Q_t;0,[0,T]} \leq (5M+4)T\|\rho_A\|_{1,Q_t;0,[0,T]} \left\|\Phi - \tilde{\Phi}\right\|_{1,Q_t;0,[0,T]}$$

for $\Phi, \tilde{\Phi} \in W(M,T,\delta)$.

Proof. It holds

$$\begin{aligned}\left\|A(\Phi) - A(\tilde{\Phi})\right\|_{1,Q_t,0,[0,T]} &= \sup_{0\leq t\leq T} \left\|A(\Phi)(t) - A(\tilde{\Phi})(t)\right\|_{1,Q_t} \\ + \sup_{0\leq t_x\leq T} \left\|\partial_{t_x} A(\Phi)(t_x) - \partial_{t_x} A(\tilde{\Phi})(t_x)\right\|_{0,[t_x,T]} &+ \sup_{0\leq t_x\leq T} \left\|A(\Phi)(t_x) - A(\tilde{\Phi})(t_x)\right\|_{0,[t_x,T]}. \end{aligned} \tag{3.27}$$

We use the representation (3.24) of the operator A, as the integral limits are the same for Φ and $\tilde{\Phi}$. We begin with a pointwise estimate for $|A(\Phi) - A(\tilde{\Phi})|$. For every (t_x, t) with $t_x \in [0,t], t \in [0,T]$ holds

$$\begin{aligned}
&\left|A(\Phi)(t_x,t) - A(\tilde{\Phi})(t_x,t)\right| \\
&\leq \Bigg| \int_{t_x}^{t} \int_{t_x}^{\mu} \frac{\Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu)\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y - \int_0^{t_x} \frac{(1-\Phi(t_y,\mu))\partial_{t_y}\Phi(t_y,\mu)\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y d\mu \\
&\quad - \int_{t_x}^{t} \int_{t_x}^{\mu} \frac{\tilde{\Phi}(t_y,\mu)\partial_{t_y}\tilde{\Phi}(t_y,\mu)\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y - \int_0^{t_x} \frac{(1-\tilde{\Phi}(t_y,\mu))\partial_{t_y}\tilde{\Phi}(t_y,\mu)\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y\, d\mu \Bigg| \\
&\leq \left| \int_{t_x}^{t} \int_{t_x}^{\mu} \left(\Phi(t_y,\mu)\partial_{t_y}\Phi(t_y,\mu) - \tilde{\Phi}(t_y,\mu)\partial_{t_y}\tilde{\Phi}(t_y,\mu)\right) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y\, d\mu \right| \\
&\quad + \left| \int_{t_x}^{t} \int_0^{t_x} \left((1-\Phi(t_y,\mu))\partial_{t_y}\Phi(t_y,\mu) - (1-\tilde{\Phi}(t_y,\mu))\partial_{t_y}\tilde{\Phi}(t_y,\mu)\right) \frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\, dt_y d\mu \right| \\
&\leq \int_0^t \int_0^{\mu} \left|\Phi(t_y,\mu) - \tilde{\Phi}(t_y,\mu)\right| |\partial_{t_y}\Phi(t_y,\mu)| \left|\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\right|\, dt_y\, d\mu \\
&\quad + \int_0^t \int_0^{\mu} \left|\partial_{t_y}\Phi(t_y,\mu) - \partial_{t_y}\tilde{\Phi}(t_y,\mu)\right| |\Phi(t_y,\mu)| \left|\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\right|\, dt_y\, d\mu \\
&\quad + \int_0^t \int_0^{t} \left|\Phi(t_y,\mu) - \tilde{\Phi}(t_y,\mu)\right| |\partial_{t_y}\Phi(t_y,\mu)| \left|\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\right|\, dt_y\, d\mu \\
&\quad + \int_0^t \int_0^{t} \left|\partial_{t_y}\Phi(t_y,\mu) - \partial_{t_y}\tilde{\Phi}(t_y,\mu)\right| \left(|1-\tilde{\Phi}(t_y,\mu)|\right) \left|\frac{\rho_A(t_y)}{1+(\mu-t_y)\rho_A(t_y)}\right|\, dt_y d\mu \\
&\leq t^2 2(M+1)\|\rho_A\|_{0,[0,T]} \left(\left\|\Phi - \tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} + \left\|\partial_{t_y}\Phi - \partial_{t_y}\tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} \right) \\
&\leq T^2 2(M+1)\|\rho_A\|_{0,[0,T]} \left(\left\|\Phi - \tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} + \left\|\partial_{t_y}\Phi - \partial_{t_y}\tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} \right)
\end{aligned}$$

where we took the supremum over $0 \leq t \leq T$ in the last step. Conclusively, we have for the first and third term of (3.27)

$$\begin{aligned}
&\sup_{0\leq t\leq T} \left\|A(\Phi)(t) - A(\tilde{\Phi})(t)\right\|_{0,Q_t} \\
&\quad \leq T^2 2(M+1)\|\rho_A\|_{0,[0,T]} \left(\left\|\Phi - \tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} + \left\|\partial_{t_y}\Phi - \partial_{t_y}\tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} \right)
\end{aligned}$$

and

$$\sup_{0\le t_x\le T} \left\| A(\Phi)(t_x) - A(\tilde{\Phi})(t_x) \right\|_{0,[t_x,T]}$$
$$\le T^2 2(M+1)\|\rho_A\|_{0,[0,T]} \left(\left\| \Phi - \tilde{\Phi} \right\|_{0,Q_t;0,[0,T]} + \left\| \partial_{t_y}\Phi - \partial_{t_y}\tilde{\Phi} \right\|_{0,Q_t;0,[0,T]} \right).$$

We bound the first and second term $|\partial_{t_x} A(\Phi) - \partial_{t_x} A(\tilde{\Phi})|$ of (3.27) pointwise by

$$\begin{aligned}
&\left|\partial_{t_x} A(\Phi)(t_x,t) - \partial_{t_x} A(\tilde{\Phi})(t_x,t)\right| \\
&= \left| \int_0^{t_x} \frac{(\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x)\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\, dt_y - \int_0^{t_x} \frac{(\tilde{\Phi}(t_y,t_x)-1)\partial_{t_y}\tilde{\Phi}(t_y,t_x)\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\, dt_y \right| \\
&\quad + \left| \int_{t_x}^{t} \partial_{t_x}\Phi(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\, d\mu - \int_{t_x}^{t} \partial_{t_x}\tilde{\Phi}(t_x,\mu)\frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\, d\mu \right| \\
&\le \left| \int_0^{t_x} \left((\Phi(t_y,t_x)-1)\partial_{t_y}\Phi(t_y,t_x) - (\tilde{\Phi}(t_y,t_x)-1)\partial_{t_y}\tilde{\Phi}(t_y,t_x) \right) \frac{\rho_A(t_y)}{1+(t_x-t_y)\rho_A(t_y)}\, dt_y \right| \\
&\quad + \left| \int_{t_x}^{t} \left(\partial_{t_x}\Phi(t_x,\mu) - \partial_{t_x}\tilde{\Phi}(t_x,\mu) \right) \frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)}\, d\mu \right| \\
&\le \int_0^{t_x} \left|\Phi(t_y,t_x) - \tilde{\Phi}(t_y,t_x)\right| \left|\partial_{t_y}\Phi(t_y,t_x)\right| \left| \frac{\rho_A(t_x)}{1+(t_x-t_y)\rho_A(t_x)} \right| dt_y \\
&\quad + \int_{t_x}^{t} \left|\partial_{t_x}\Phi(t_x,\mu) - \partial_{t_x}\tilde{\Phi}(t_x,\mu)\right| \left|1-\tilde{\Phi}(t_x,\mu)\right| \left| \frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)} \right| d\mu \\
&\quad + \int_{t_x}^{t} \left|\partial_{t_x}\Phi(t_x,\mu) - \partial_{t_x}\tilde{\Phi}(t_x,\mu)\right| \left| \frac{\rho_A(t_x)}{1+(\mu-t_x)\rho_A(t_x)} \right| d\mu \\
&\le (M+2)T\left\|\rho_A\right\|_{0,[0,T]} \left(\left\|\Phi - \tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} + \left\|\partial_{t_x}\Phi - \partial_{t_x}\tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} \right).
\end{aligned}$$

We have for (3.27)

$$\|A(\Phi) - A(\tilde{\Phi})\|_{1,Q_t;0,[0,T]}$$
$$\le (4M+8)T\left\|\rho_A\right\|_{0,[0,T]} \left(\left\|\Phi - \tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} + \left\|\partial_{t_x}\Phi - \partial_{t_x}\tilde{\Phi}\right\|_{0,Q_t;0,[0,T]} \right).$$

□

The continuity constant of A depends on the maximum time T. For small T and in combination with Lemma 3.13, we show that A is a contraction.

Theorem 3.15. *With the assumptions and notations of Lemma 3.13 and $T \le \frac{1}{2(6M+8)\|\rho_A\|_{0,[0,T]}}$, the operator $A : W(M,T,\delta) \to W(M,T,\delta)$ is a contraction, i.e.*

$$\left\|A(\Phi) - A(\tilde{\Phi})\right\|_{1,Q_t;0,[0,T]} \le \frac{1}{2}\left\|\Phi - \tilde{\Phi}\right\|_{1,Q_t;0,[0,T]}.$$

Proof. The continuity constant of A is given in Theorem 3.14 as

$$c_L = (6M+8)T\left\|\rho_A\right\|_{0,[0,T]}.$$

for $T \leq \frac{1}{2(5M+4)\|\rho_A\|_{[0,T]}}$, we thus obtain

$$c_L \leq \frac{1}{2}.$$

$\square$

We arrive at the main result of this Chapter. By Lemma 3.13 and Theorem 3.15, $A : W(M,T,\delta) \to W(M,T,\delta)$ is a selfmap as well as a contraction. Hence, it fulfils both requirements in Banach's fixed point theorem. Using the definition of the operator A, we define a sequence $\{\Phi_n\}_n$ in the space $W(M,T,\delta)$

$$\Phi_{n+1} = A(\Phi_n) \in W(M,T,\delta), \qquad \Phi_0 \in W(M,T,\delta) \tag{3.28}$$

with Φ_0 being an arbitrary element of $W(M,T,\delta)$.

Theorem 3.16 (Existence of a Unique Fixed Point)**.** *Let*

$$T := \inf\left\{\frac{1}{\sqrt{8M\|\rho_A\|_{0,[0,T]}}}, \frac{1}{2M\|\rho_A\|_{0,[0,T]}}, \frac{1}{2M}, \frac{1}{2(6M+8)\|\rho_A\|_{0,[0,T]}}, \frac{1}{4|u_{A_2} - u_{A_1}|}\right\},$$

$M := 2 + 2(u_{A_1} - u_{A_2})$ *and* $|u_{A_1} - u_{A_2}| > \delta + \|\rho_A\|_0$*. Then the sequence (3.28) has a unique fixed point in* $W(M,T,\delta)$*.*

Proof. We use the Banach fixed point theorem to prove the unique existence of a fixed point to the sequence (3.28). Lemma 3.13 shows that A is a selfmap and thus $\Phi_n \in W(M,T,\delta)$ for all $n \in \mathbb{N}$. Theorem 3.15 proves that for T small enough, the operator A is a contraction. It follows immediately with the Banach fixed point theorem that there exists a unique fixed point to the sequence (3.28) in the set $W(M,T,\delta)$. $\square$

The next theorem now gives the existence of a classical solution to (CP 3.1) for small time T.

Theorem 3.17. *Let T, M and δ be as chosen in Theorem 3.16. Then a classical solution (u,ρ) exists to (CP 3.1).*

Proof. Theorem 3.16 proves the existence of a unique fixed point of $\Phi = A(\Phi) \in W(M,T,\delta)$. With the restrictions in $W(M,T,\delta)$, we conclude that there exists $\Phi^{-1}(y,t) \in C^{1,0}(I,[0,T])$. Moreover, since Φ is the fixed point to A, we know by the fundamental theorem of calculus that Φ is differentiable with respect to t. It thus holds for the fixed point $\Phi(t_x,\cdot) \in C^1([t_x,T])$. Hence, $\rho(\cdot,t) \in C^1(I)$ and $\rho(x,\cdot) \in C^1([0,T])$ by Lemma 3.9. By Lemma 3.5, we obtain $u \in C^{2;0}(I,[0,T])$. We thus found the classical solution (u,ρ) to (CP 3.1). $\square$

3.4 Remarks about the Chapter

In this chapter, we have shown that for the model problem (CP 3.1) exists a classical solution (u, ρ) with $u \in C^{2;0}(I, [0,T])$ and $\rho(\cdot, t) \in C^1(I)$ and $\rho(x, \cdot) \in C^1([0,T])$ for small times T, whenever the Poisson boundary data u_{A_1}, u_{A_2} and inflow function ρ_A fulfil certain requirements. The potential difference $|u_{A_1} - u_{A_2}|$ and ρ_A are dependent, since $|u_{A_1} - u_{A_2}| \geq \delta + \|\rho_A\|_{0,[0,T]}$ for some $\delta > 0$. There are two strategies to follow:

1. For a given boundary function ρ_A and $\delta > 0$, the potential difference must be chosen big enough.

2. For $\delta > 0$ and a given potential difference $|u_{A_1} - u_{A_2}| \geq \delta$, choose ρ_A such that $\|\rho_A\|_{0,[0,T]}$ small enough.

The time interval of existence depends on the applied potential difference $|u_{A_2} - u_{A_1}|$ and the supremum of the inflow charge $\|\rho_A\|_{0,[0,T]}$. The greater the potential difference and $\|\rho_A\|_{0,[0,T]}$, the smaller T becomes.

This chapter focused only on the modeling of the inflow of charge into the domain. We did not address the question of long time existence. A simple continuation argument is not immediately applicable as it would require the initial distribution $\rho_0 \neq 0$. However, in the following chapter, we will examine the general two-dimensional setting on a domain that is not convex and not simply connected. Therein, we use the ideas presented in the previous analysis and apply them in two dimensions.

Chapter 4

Time dependent 2d

4.1 The Time Dependent Case

In this chapter, we analyze the existence and uniqueness of a solution to the two-dimensional time dependent coupled problem. With the notations of Chapter 2, the model problem is given by

Problem (CP 4.1). *Let Ω be a $C^{2,\alpha}$ domain and $[0,T]$ a time interval. Find $(u,\rho) \in C^{2,\alpha;0}(\Omega,[0,T]) \times C^{1;1}(\Omega,[0,T])$ such that*

$$
\begin{aligned}
-\Delta u(x,t) &= \rho(x,t) & (x,t) &\in \Omega \times [0,T] & \text{(4.1a)}\\
u(x,t) &= u_A(x) & x &\in \Gamma \times [0,T] & \text{(4.1b)}\\
\frac{d}{dt}\rho(x,t) + E(x,t)\cdot\nabla\rho(x,t) + \rho^2(x,t) &= 0 & (x,t) &\in \Omega \times [0,T] & \text{(4.1c)}\\
\rho(x,0) &= \rho_0(x) & x &\in \Omega & \text{(4.1d)}\\
\rho(x,t) &= \rho_A(x,t) & (x,t) &\in \Gamma_- \times [0,T] & \text{(4.1e)}\\
-\nabla u(x,t) &= E(x,t) & (x,t) &\in \Omega \times [0,T]. & \text{(4.1f)}
\end{aligned}
$$

with $\rho_A \in C^{1;1}(\Gamma_-,[0,T])$, $\rho_0 \in C^1(\Omega)$, $u_A \in C^{2,\alpha}(\Gamma)$ with $u_A|_{\Gamma_-}$ and $u_A|_{\Gamma_+}$ constant with $u_A|_{\Gamma_-} > u_A|_{\Gamma_+}$.

Many authors have been studying similar problems in connection with vortex patches. Huang and Svobodny proved in [44] the existence of a unique classical solution to a problem using an integrated transport equation. This work and the three dimensional model of [55] provide the basis for our analysis. Yet, our problem differs from the previous ones in several aspects. First of all, we use the Poisson equation (4.1a) while in the study of vortex patches an additional mass term is required in the elliptic equation, i.e. $-\Delta u + u = \rho$. Furthermore, to pay tribute to the model of corona discharge, we have to model the applied potential and the inflowing charge. In this Chapter, we will present a generalization of [44, 55] by using nonhomogeneous inflow boundary conditions (4.1e) and Dirichlet boundary conditions (4.1b) for the Poisson equation on the bounded non-convex, not simply connected domain Ω. However, as in [44], we do not find a classical solution to (CP 4.1). Instead, we first have to reformulate the coupled problem and substitute the transport equation by an integrated formulation. Let us start with an overview about this chapter and define the set of equations for which we prove the existence of a classical solution.

In section 4.3, we introduce the streamline functions of the transport equation that play a crucial role in the analysis. The main idea, presented in section 4.4, is to reformulate the coupled problem as system of integro-differential operators A_1 and A_2 that use the streamline function as their argument. We strictly divide the analysis of the streamline functions into the study of the transport of the inflowing charge and the transport of the initial distribution ρ_0 in time. Let us define an initial domain Ω_0 by $\Omega_0 := (H_{convex}(\operatorname{supp}\rho_0))\backslash\Omega_-$ where $H_{convex}(\operatorname{supp}\rho_0)$ is the convex hull of ρ_0. Then the transport of the initial distribution is described by the streamline function $\Phi_1 : \Omega_0 \times [0,T] \to \Omega$ given as solution of

$$\frac{d\Phi_1}{dt}(x,t) = E(\Phi_1(x,t),t) \qquad (x,t) \in \Omega_0 \times [0,T] \tag{4.2a}$$

$$\Phi_1(x,0) = x \qquad x \in \Omega_0. \tag{4.2b}$$

The charge distribution ρ at a time t is decomposed into two sets. On one hand, it is the image set $\Omega_t^1 \subset \Omega$ of Ω_0 under $\Phi_1(\cdot,t)$ and on the other hand it is the set Ω_t^2 of charge that flew in the domain up to the time t. We also want to define the transport of the inflowing charge and need therefore a second streamline function Φ_2. As in Chapter 3, we first define an inflow set.

Definition 4.1. *Let L_{Γ_-} be the arc length of the closed boundary curve φ as defined in (2.1) and $I_{\Gamma_-} := [0, L_{\Gamma_-}]$. For a $t \in [0,T]$, we denote the* inflow set *by*

$$Q_t := [0, L_{\Gamma_-}] \times [0,t]. \tag{4.3}$$

An element $(s,t_x) \in Q_t$ defines the starting point $\varphi(s) \in \Gamma_-$ and the time of emission t_x for a particle. The set

$$\Sigma_t = \left\{Q_t, \bar{\Omega}_0\right\}$$

is called the initial set *for time $t \in [0,T]$. Its elements are given by*

$$\tau \in \Sigma_t.$$

The elements of Σ_t represent the initial positions and emission times for all charge present at a time $t < T$. Σ_t is a set of two disjoint sets and is disconnected.

With this definition, we obtain analogously to Chapter 3 the streamline function Φ_2. It describes the particle trajectories starting from the inflow boundary and is given as solution of

$$\frac{d\Phi_2}{dt}(s,t_x,t) = E(\Phi_2(s,t_x,t),t) \qquad (s,t_x,t) \text{ with } (s,t_x) \in Q_t, t \in [0,T] \tag{4.4a}$$

$$\Phi_2(s,t_x,t_x) = \varphi(s) \qquad (s,t_x) \in Q_T. \tag{4.4b}$$

With the streamline functions Φ_1 and Φ_2, we find the position of a charge particle at a time t that was initially present or flew into the domain. Hyperbolic partial differential equations such as the transport equation reduce to ordinary differential equations on the streamlines. We use this observation and obtain in section 4.3 the solution ρ to the coupled problem by

$$\rho(y,t) = \begin{cases} 0, & y \in \Omega \backslash \Omega_t \\ \frac{\rho_0(\Phi_1^{-1}(y,t))}{1+t\rho_0(\Phi_1^{-1}(y,t))} & y \in \Omega_t^1 \\ \frac{\rho_A(\Phi_2^{-1}(y,t))}{1+(t-[\Phi_2^{-1}(y,t))]_2)\rho_A(\Phi_2^{-1}(y,t))} & y \in \Omega_t^2 \end{cases} \tag{4.5}$$

where Ω_t^1 is the range of Φ_1, Ω_t^2 the range of Φ_2 and $\Omega_t = \Omega_t^1 \cup \Omega_t^2$ and $\Phi_1^{-1}(y,t)$ and $\Phi_2^{-1}(y,t)$ are understood as inverse functions with respect to the space variable for a fixed time t. In this Chapter, we will seek for a classical solution in the following sense.

Problem (CP 4.2). *Let Ω be a $C^{2,\alpha}$ domain and $u_A \in C^{1,\alpha}(\Gamma)$. Let further $\rho_A \in C^{0,1}(Q_T)$, $\rho_0 \in C^{0,1}(\Omega_0)$ with $\Omega_0 := (H_{convex}(\operatorname{supp}\rho_0))\backslash\Omega_- \subset\subset \Omega$ and $u_A \in C^{2,\alpha}(\Gamma)$ with $u_A|_{\Gamma_-} = u_{A_1}$ and $u_A|_{\Gamma_+} = u_{A_2}$ constant with $u_{A_1} > u_{A_2}$. Moreover let the transition between ρ_0 and ρ_A be continuous, i.e. $\rho_0(x) = \rho_A(x,0)$ for $x \in \Gamma_-$.*

Then find $(u,\rho) \in C^{2,\alpha;0}(\Omega,[0,T]) \times C^{0,1;\alpha}(\Omega,[0,T])$, such that (4.1a), (4.1b), (4.5) and (4.1f) hold pointwise.

We now proceed as follows. In section 4.2, we find a representation of the solution u of the Poisson equation in terms of Green's function. In section 4.3, we derive the solution (4.5) in more detail and discuss requirements for the streamline functions Φ_1 and Φ_2 such that ρ is well defined. Combining the Green's function representation of u with (4.2a)-(4.2b) and (4.4a)-(4.4b), we obtain a system of integro-differential operators $A = (A_1, A_2)$. In section 4.4, we will define function sets W_1 and W_2 of all streamline functions that lead to a possible solution of (CP 4.2) and in particular whose elements are invertible. The remaining part of this Chapter then deals with proving the existence of a unique fixed point to A in a set $W \subset W_1 \times W_2$. Unfortunately, the existence of a fixed point $A(\Phi_1,\Phi_2) = (\Phi_1,\Phi_2)$ cannot be proved with the Banach fixed point theorem as A lacks the contraction property. We will instead use a compactness argument to show that a classical solution to (CP 4.2) indeed exists for a small time interval T and small data ρ_A and ρ_0. Eventually, we show that the solution can be extended in time until the support of ρ reaches the outflow boundary Γ_+.

Beside the classical norms defined in Chapter 2, we need to define the $C^{1,\alpha}(Q_t)$ norm on the inflow set.

Definition 4.2. *Let Q_t be defined as in (4.3) with $t \in [0,T]$. For any multi-index $\beta = (\beta_1,\beta_2)$, $m = |\beta|$, $0 < \alpha \leq 1$, and any function $f : Q_t \to \mathbb{R}^2$, denote by*

$$|f|_{m,\alpha;Q_t} = \sup_{\substack{(s_1,t_1)\neq(s_2,t_2)\in Q_t, \\ |\beta|=m}} \frac{|\partial^\beta f(s_1,t_1) - \partial^\beta f(s_2,t_2)|_\infty}{|(s_1,t_2)-(s_2,t_2)|_\infty^\alpha}$$

the Hölder seminorm *and by*

$$\|f\|_{m,\alpha;Q_t} = \sup_{\substack{(s,t_x)\in Q_t,\\ |\beta|\le m}} |\partial^\beta f(s,t_x)|_\infty + |f|_{m,\alpha;Q_t}$$

the Hölder norm.
We define the corresponding function spaces

$$C^{m,\alpha}(Q_t) = \{f \in C^m(Q_t) \,:\, \|f\|_{m,\alpha;Q_t} < \infty\}\,. \tag{4.6}$$

The normed space $C^{m,\alpha}(Q_t)$ is complete. We obtain the well-known result

Lemma 4.3. *[3, p. 44] The space $C^{m,\alpha}(Q_t)$ is a Banach spaces.*

Analogously, we define space-time Hölder functions on Q_t and $t \in [0,T]$. Note that here we need to carefully work with the dependence of Q_t on $t \in [0,T]$.

Definition 4.4. *Let Q_t be as defined in (4.3) and $f : Q_t \times [0,T] \to \mathbb{R}^2$. For any multi-index $\beta = (\beta_1, \beta_2)$, $m = |\beta|$, integer $l \le k$ and $0 < \alpha \le 1$, $0 < \gamma \le 1$, we denote*

$$\|f\|_{m,\alpha,Q_t;k,\gamma,[0,T]} = \sup_{0\le t\le T} \sum_{l\le k} \|\partial_t^l f(t)\|_{m,\alpha;Q_t} + \sup_{(s,t_x)\in Q_T} \sum_{|\beta|\le m} \|\partial_x^\beta f(s,t_x)\|_{k,\gamma;[t_x,T]}. \tag{4.7}$$

We define the function space of (m,α)-Hölder functions in space and (k,γ)-Hölder functions in time by

$$C^{m,\alpha;k,\gamma}(\Omega,[0,T]) = \left\{f : \Omega\times[0,T] \to \mathbb{R}^2 \,:\, \|f\|_{m,\alpha,\Omega;k,\gamma,[0,T]} < \infty\right\}. \tag{4.8}$$

If the norm is taken only w.r.t. to Q_t, then we write for every $0 \le t \le T$

$$\|f(t)\|_{m,\alpha,Q_t} := \sup_{(s,t_x)\in Q_t} |f(s,t_x,t)|_\infty$$

to emphasize the variability in t. Analogously, we write for every $(s,t_x) \in Q_T$

$$\|f(s,t_x)\|_{k,\gamma,[t_x,T]} := \sup_{t_x\le t\le T} |f(s,t_x,t)|_\infty$$

to emphasize the variability in (s,t_x).

Lemma 4.5. *[3, p. 44] The space $C^{m,\alpha;k,\gamma}(Q_t,[0,T])$ is a Banach space.*

4.2 Poisson Equation

In this section, we will investigate the existence and uniqueness of a classical solution to the Poisson equation. In contrast to the one-dimensional problem of Chapter 3, a continuous right-hand side function ρ is not sufficient to prove existence and uniqueness of a classical solution.

The classical theory for elliptic partial differential equations is done in the space of Hölder continuous functions. In this section, we will first use the linearity of the Laplacian to decompose the Dirichlet problem for the Poisson equation into two subproblems. We then introduce the Green's function representation of u and examine some of its properties.

Problem (Po 4.3). *Let Ω be a $C^{2,\alpha}$ domain. Let $\rho(\cdot,t) \in C^{\alpha}(\bar{\Omega})$, $u_A \in C^{2,\alpha}(\Gamma)$ and $t \in [0,T]$. Then find $u(\cdot,t) \in C^{2,\alpha}(\bar{\Omega})$ such that*

$$-\Delta u(x,t) = \rho(x,t) \qquad x \in \Omega \tag{4.9a}$$

$$u(x,t) = u_A(x) \qquad x \in \Gamma. \tag{4.9b}$$

The existence and uniqueness of a solution to (Po 4.3) is given by the following Theorem.

Theorem 4.6. *[34, Theorem 6.14] Let Ω be a bounded $C^{2,\alpha}$ domain, $\rho(\cdot,t) \in C^{\alpha}(\bar{\Omega})$ and $u_A \in C^{2,\alpha}(\Gamma)$. Then the Dirichlet problem*

$$\begin{aligned} -\Delta u &= \rho \qquad && x \in \Omega \\ u &= u_A && x \in \partial\Omega \end{aligned}$$

has a (unique) solution u lying in $C^{2,\alpha}(\bar{\Omega})$.

We will now investigate some properties of the Poisson equation. The linearity of the Laplacian allows us to decompose (Po 4.3) into an equivalent formulation of two subproblems. With the decomposition $u = u_0 + u_1$, we obtain for $t \in [0,T]$

$$-\Delta u_1(x,t) = \rho(x,t) \qquad x \in \Omega \tag{4.10a}$$

$$u_1(x,t) = 0 \qquad x \in \Gamma \tag{4.10b}$$

and

$$-\Delta u_0(x) = 0 \qquad x \in \Omega \tag{4.11a}$$

$$u_0(x) = u_A(x) \qquad x \in \Gamma. \tag{4.11b}$$

By Theorem 4.6, we immediately conclude that unique solutions u_1 and u_0 to (4.10a)-(4.10b) and (4.11a)-(4.11b) exist.

Further, we observe that the Dirichlet problem for the Laplace equation (4.11a)-(4.11b) is independent of time, as the solution only depends on the domain Ω and the time independent boundary function u_A. The time dependence of u is thus introduced by u_1 through the time dependent right-hand side function ρ. We will now introduce the Green's function representation of u_1.

Definition 4.7. *[9, p. 117 ff.] Consider the Poisson equation $-\Delta u = \rho$ in Ω with Dirichlet boundary data $u(x) = 0$ for $x \in \partial\Omega$. Let δ be the Dirac delta function. A function $G(x,y)$ with*

$$\begin{aligned} -\Delta G(x,y) &= \delta(x-y) \qquad && x \in \Omega,\ y \in \Omega \\ G(x,y) &= 0 && x \in \partial\Omega,\ y \in \Omega \end{aligned}$$

is called Green's function.

The definition shows that Green's function $G(x, y)$ depends on the shape of the domain. The next theorem confirms that Green's function always exists for regular domains and is thus applicable to the problem (Po 4.3).

Theorem 4.8. *[63, p. 119] Let Ω be a $C^{0,1}$ domain. Then the Green's function exists for Ω.*

We can now rewrite u_1 as a singular integral equation.

Lemma 4.9. *The solution u_1 of the Poisson equation $-\Delta u_1 = \rho$ in Ω with Dirichlet boundary data $u_1(x, t) = 0$ for $x \in \partial\Omega$, $t \in [0, T]$ is given by*

$$u_1(x,t) = \int_\Omega G(x,y)\rho(y,t)\,dy \qquad\qquad x \in \bar{\Omega},\ t \in [0,T]. \tag{4.12}$$

Proof. To verify the assumption, we show that u_1 given in (4.12) is a solution of the Poisson equation. Applying the Laplacian gives

$$-\Delta u_1(x,t) = \int_\Omega -\Delta G(x,y)\rho(y,t)\,dy = \int_\Omega \delta(x-y)\rho(y,t)\,dy = \rho(x,t).$$

The boundary conditions are fulfilled due to the boundary conditions on $G(x, y)$. By Definition 4.7 holds for $x_0 \in \Gamma$

$$u_1(x_0,t) = \int_\Omega G(x_0,y)\rho(y)\,dy = 0.$$

□

With the previous result, we now define the singular integral operator G_1 for the Poisson problem (4.10a)-(4.10b), i.e. $G_1\rho = u_1$ for a given right-hand side function $\rho(\cdot, t) \in C^\alpha(\bar{\Omega})$.

Theorem 4.10. *Let Ω be a bounded $C^{2,\alpha}$ domain and $\rho(\cdot, t) \in C^\alpha(\bar{\Omega})$. Then for every $t \in [0, T]$ the linear operator $G_1 : C^\alpha(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ defined by*

$$G_1\rho(x,t) = \int_\Omega G(x,y)\rho(y,t)\,dy \tag{4.13}$$

is the solution operator to the Dirichlet problem for the Poisson equation (4.10a)-(4.10b), that is

$$u_1 = G_1\rho. \tag{4.14}$$

Proof. By Lemma 4.9, we obtain the integral operator G_1. It is left to prove that $G_1\rho(\cdot, t) \in C^{2,\alpha}(\bar{\Omega})$. As $u_1 \in C^{2,\alpha}(\bar{\Omega})$, we obtain by Theorem 4.6

$$G_1\rho = u_1(\cdot,t) \in C^{2,\alpha}(\bar{\Omega}).$$

□

We now obtain a representation for the solution u of (Po 4.3) that will be the basis for the following analysis. We therefore sum up the previous results in the following Lemma.

Lemma 4.11. *Let Ω be a $C^{2,\alpha}$ domain, $\rho \in C^{\alpha;0}(\bar{\Omega}, [0,T])$ and $u_0 \in C^{2,\alpha}(\Gamma)$. Then the solution $u \in C^{2,\alpha}(\bar{\Omega})$ of (Po 4.3) is given for every $t \in [0,T]$ by*

$$u(x,t) = G_1\rho(x,t) + u_0(x) \tag{4.15}$$

with $G_1\rho \in C^{2,\alpha;0}(\bar{\Omega}, [0,T])$ and $u_0 \in C^{2,\alpha}(\bar{\Omega})$. Moreover holds $u \in C^{2,\alpha;0}(\bar{\Omega}, [0,T])$.

Proof. As the Laplacian is linear, we split (Po 4.3) into $u(x,t) = u_1(x,t) + u_0(x)$. By Theorem 4.6 and Theorem 4.10, we obtain $G_1\rho(\cdot,t) \in C^{2,\alpha}(\bar{\Omega})$ and $u_0 \in C^{2,\alpha}(\bar{\Omega})$. As $\rho \in C^{\alpha;0}(\bar{\Omega}, [0,T])$, we can conclude that also $G_1\rho \in C^{2,\alpha;0}(\bar{\Omega}, [0,T])$, as the time dependence of $G_1\rho$ is only introduced by ρ. Since u_0 is constant in time, it is also continuous in time. We thus obtain $u \in C^{2,\alpha;0}(\bar{\Omega}, [0,T])$. $\square$

A helpful tool in the upcoming analysis is the continuous dependence of u on the right hand side and boundary data. In case of the sup-norm, a first a priori bound is given by the maximum principle.

Theorem 4.12. *(Maximum Principle)[34, Theorem 3.1]*
Let Ω be a bounded domain and $-\Delta u = 0$ in Ω with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then the maximum of u is achieved on $\partial\Omega$, that is

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x).$$

A consequence is the following a priori bound for the Poisson equation.

Theorem 4.13. *[34, Theorem 3.7] Let Ω be a bounded domain, and $-\Delta u = \rho$ with $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Then*

$$\sup_{x\in\Omega} |u(x)| \leq \sup_{x\in\partial\Omega} |u(x)| + c(\Omega) \sup_{x\in\Omega} |\rho(x)|$$

where $c(\Omega)$ is a constant depending only on the diameter of Ω.

Schauder developed a priori estimates for classical solutions of elliptic partial differential equations in the $C^{2,\alpha}(\Omega)$-norm in [57, 58]. It shows that the solution of the Poisson equation is bounded by its right-hand side and boundary data, if all quantities are sufficiently regular. The standard references for this topic are [34, 51].

Theorem 4.14. *(Schauder's A priori Estimate)[34, Theorem 6.6]*
Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^2$. Let $u \in C^{2,\alpha}(\bar{\Omega})$ be the solution of $-\Delta u = \rho, u|_\Gamma = u_A$ with $\rho \in C^\alpha(\bar{\Omega})$ and $u_A \in C^{2,\alpha}(\bar{\Omega})$. Then holds

$$\|u\|_{2,\alpha;\Omega} \leq c_0(\Omega, \alpha) \left(\|u\|_{0;\Omega} + \|\rho\|_{\alpha;\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right)$$

where $c_0(\Omega, \alpha)$ is a constant depending on α and the domain Ω.

With the previous results, we are able to obtain a priori bounds for u_0 and u_1 and consequently also for the solution u of (Po 4.3). The a priori bound for u_1 implies the continuity of the operator G_1.

Lemma 4.15. *Let Ω be a $C^{2,\alpha}$ domain, $\rho(\cdot,t) \in C^\alpha(\bar{\Omega})$ and $u_A \in C^{2,\alpha}(\Gamma)$. The mappings*

$$G_1 : C^0(\bar{\Omega}) \to C^0(\bar{\Omega}) \tag{4.16}$$

$$G_1 : C^\alpha(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega}) \tag{4.17}$$

are continuous for every $t \in [0,T]$ with continuity constant $c_S(\alpha,\Omega)$. Further holds for u_0

$$\|u_0\|_{2,\alpha;\Omega} \leq c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}.$$

Proof. Since $u_1(x,t) = G_1\rho(x,t)$ and $u_1(x,t) = 0$ for $x \in \Gamma$, $t \in [0,T]$, application of Theorem 4.13 gives

$$\sup_{x\in\Omega} |G_1\rho(x,t)| \leq c(\Omega) \sup_{x\in\Omega} |\rho(x,t)|.$$

By Theorem 4.14 and 4.13, we obtain

$$\begin{aligned}\|G_1\rho(t)\|_{2,\alpha;\Omega} &\leq c_0(\alpha,\Omega)\left(\|u_1(t)\|_{0,\Omega} + \|\rho(t)\|_{\alpha,\Omega}\right)\\ &\leq c_0(\alpha,\Omega)(c(\Omega)+1)\|\rho(t)\|_{\alpha,\Omega}.\end{aligned}$$

By Theorem 4.13 and 4.12, we have

$$\begin{aligned}\|u_0\|_{2,\alpha;\Omega} &\leq c_0(\Omega,\alpha)\left(\|u_0\|_{0,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right)\\ &\leq 2c_0(\alpha,\Omega)\|u_A\|_{2,\alpha;\Gamma}.\end{aligned}$$

With $c_S(\Omega,\alpha) = \max\{2c_0(\Omega,\alpha), c_0(\Omega,\alpha)(c(\Omega)+1), c(\Omega)\}$, the proof is complete. □

The constant $c_S(\Omega,\alpha)$ that was defined in the previous Lemma depends only on α and Ω and is thus independent of time. The exact size of the constant is unknown. It is only important for applications in section 4.10 that $c_S(\Omega,\alpha)$ is the same for every $t \in [0,T]$ and depends on quantities that are a priori known.

With Theorem 4.14, we can bound $\|\nabla G_1\rho(t)\|_{0,\Omega}$. However, it is bounded by $\|\rho(t)\|_{\alpha,\Omega}$ which is not always convenient in the following analysis. Let us therefore investigate ∇G_1 further and find a bound in terms of the sup-norm of ρ.

Lemma 4.16. *Let Ω be a $C^{2,\alpha}$ domain and $\rho(\cdot,t) \in C^0(\Omega)$. Then holds for every $t \in [0,T]$*

$$\sup_{x\in\Omega}\left|\int_\Omega \nabla G(x,y)\rho(y,t)\,dy\right|_\infty \leq c(\mathrm{diam}(\Omega))\,\|\rho(t)\|_{0,\Omega}\,. \tag{4.18}$$

Proof. It holds pointwise

$$\left|\int_\Omega \nabla G(x,y)\rho(y,t)\,dy\right|_\infty \le \int_\Omega |\nabla G(x,y)|_\infty\,|\rho(y,t)|_\infty\,dy$$
$$\le \|\rho(t)\|_{0,\Omega}\int_\Omega |\nabla G(x,y)|_\infty\,dy$$
$$\le \|\rho(t)\|_{0,\Omega}\int_\Omega \|\nabla G(x,y)\|_2\,dy, \tag{4.19}$$

with $\|\cdot\|_2$ being the Euclidean norm. Since Ω is a $C^{2,\alpha}$ domain, we may apply [63, p.128] to bound $\|\nabla G(x,y)\|_2$. It holds

$$\|\nabla G(x,y)\|_2 \le c(\mathrm{diam}(\Omega))\log\left(\frac{\|x-y\|_2}{6\,\mathrm{diam}(\Omega)}\right)\|x-y\|_2^{-1}.$$

Hence, we bound the integral (4.19) by using polar coordinates

$$\int_\Omega \|\nabla G(x,y)\|_2\,dy \le c(\mathrm{diam}(\Omega))\int_\Omega \log\left(\frac{\|x-y\|_2}{6\,\mathrm{diam}(\Omega)}\right)\|x-y\|_2^{-1}\,dy$$
$$\le c(\mathrm{diam}(\Omega))\int_0^{\mathrm{diam}(\Omega)}\log\left(\frac{r}{6\,\mathrm{diam}(\Omega)}\right)\frac{1}{r}r\,dr$$
$$\le c(\mathrm{diam}(\Omega))\left|\left[r\log\left(\frac{r}{6\,\mathrm{diam}(\Omega)}\right)-r\right]_0^{\mathrm{diam}(\Omega)}\right|$$
$$\le c(\mathrm{diam}(\Omega))\left|\mathrm{diam}(\Omega)\log\left(\frac{1}{6}\right)-\mathrm{diam}(\Omega)\right|.$$
$$\le c(\mathrm{diam}(\Omega)).$$

The assertion follows with (4.19). □

4.3 Transport Equation

We will now explain in detail how the integrated transport equation (4.5) is obtained and therefore start off with the second subproblem of (CP 4.1).

Problem (Tr 4.4). *Let Ω be a $C^{2,\alpha}$ domain. For a given vector field $E \in C^{1,\alpha;\alpha}(\bar{\Omega},[0,T])$, find $\rho \in C^{0,1;\alpha}(\bar{\Omega},[0,T])$, such that*

$$\partial_t\rho(x,t) + E(x,t)\cdot\nabla\rho(x,t) + \rho^2(x,t) = 0 \qquad (x,t)\in\Omega\times[0,T] \tag{4.20a}$$
$$\rho(x,0) = \rho_0(x) \qquad x\in\Omega_0 \tag{4.20b}$$
$$\rho(x,t) = \rho_A(x,t) \qquad (x,t)\in\Gamma_-\times[0,T] \tag{4.20c}$$

with $\rho_0 \in C^{0,1}(\bar{\Omega})$ and $\rho_0 \equiv 0$ for $x \in \Omega\backslash\Omega_0$, $\rho_A \in C^{0,1}(Q_t)$ and $\rho_0(x_0) = \rho_A(s_0,0)$ for all $x_0 = \varphi(s_0)$.

One method to solve partial differential equations of hyperbolic type such as (4.20a) are the *streamline functions* or *characteristic curves*. In our model case, we consider the transport of two disconnected sets, that is on one hand the initial distribution Ω_0 and on the other hand the inflow set Q_t. It is therefore necessary to obtain a piecewise definition of the streamline function and to distinguish those that are defined on Ω_0 and Q_t.

For the initial domain Ω_0, the streamlines shall start from every point $x \in \bar{\Omega}_0$ as every point is transported in time. We obtain the streamline function $\Phi_1 : \Omega_0 \times [0, T] \to \Omega_t^1 \subset \Omega$ as solution of the initial value problem

$$\frac{d}{dt}\Phi_1(x,t) = E(\Phi_1(x,t),t) \qquad (x,t) \in \bar{\Omega}_0 \times [0,T] \tag{4.21a}$$

$$\Phi_1(x,0) = x \qquad x \in \bar{\Omega}_0. \tag{4.21b}$$

In the above definition, the time interval $[0,T]$ must be chosen small enough such that the range of Φ_1, i.e. Ω_t^1, is subset of Ω for every $t \in T$. Ω_t^1 shall be understood as the image set of Ω_0 under Φ_1 at a time t. Φ_1 is a tool to determine the position of a point $x \in \bar{\Omega}_0$ at a time $t \leq T$. We are thus able to follow the path of a particle in time.

As Q_t describes all points flowing into the domain through the inflow boundary Γ_-, the corresponding streamlines need to start on Γ_-. We obtain the streamline function $\Phi_2 : Q_t \times [0,T] \to \Omega_t^2$ as solution of the initial value problem

$$\frac{d}{dt}\Phi_2(s,t_x,t) = E(\Phi_2(s,t_x,t),t) \qquad (s,t_x) \in Q_t,\ t \in [0,T] \tag{4.22a}$$

$$\Phi_2(s,t_x,t_x) = \varphi(s) \qquad (s,t_x) \in Q_T \tag{4.22b}$$

where φ is the parametrization of the inflow boundary Γ_-. Again, Ω_t^2 denotes the image of Q_t under Φ_2 for a time $t \leq T$. $\Phi_2(s,t_x,t)$ gives the position of a charge particle at time t that flew into the domain at $x = \varphi(s)$ at a time $t_x \leq t$. The corresponding one-dimensional case has been discussed in Chapter 3.

To simplify the notation in some cases, we will refer to the piecewise defined streamline function $\Phi : \Sigma_t \to \Omega_t = \Omega_t^1 \cup \Omega_t^2 \subset \Omega$ if the computations are the same for $\Phi_1(x,t)$ and $\Phi_2(s,t_x,t)$. We define

$$\Phi(\tau,t) := \begin{cases} \Phi_1(x,t) & \tau = x \in \Omega_0 \\ \Phi_2(s,t_x,t) & \tau = (s,t_x) \in Q_t. \end{cases} \tag{4.23}$$

Later on, we will also use the 2-tupel notation $\Phi = (\Phi_1, \Phi_2)$ which shall be understood as in (4.23). As $\Phi(\tau,t)$ describes the particle trajectories, we conclude for the charge density ρ that

$$\operatorname*{supp}_x \{\rho(x,t)\} \subset \Omega_t.$$

The streamline functions $\Phi_1(x,t)$ and $\Phi_2(s,t_x)$ play a crucial role in the procedure to prove existence and uniqueness of a solution to (CP 4.2). Following [44], they are the key to formulate the integro-differential operator A in section 4.4. Further, the streamline function $\Phi(\tau)$ enables us to find the solution ρ to (Tr 4.4). Given a velocity field E, the hyperbolic partial differential equation (4.20a) reduces to an ordinary differential equation on the streamlines.

Lemma 4.17. *Let $\Phi(\tau, t)$ be defined as in (4.23). Then the solution ρ of (Tr 4.4) solves*

$$\frac{d\rho(\Phi(\tau,t),t)}{dt} = -\rho^2(\Phi(\tau,t),t). \tag{4.24}$$

Proof. The total derivative of $\rho(\Phi(\tau,t),t)$ with respect to t gives

$$\begin{aligned}\frac{d}{dt}\rho(\Phi(\tau,t),t) &= \frac{\partial\rho(\Phi(\tau,t),t)}{\partial[\Phi(\tau,t)]_1}\frac{d[\Phi(\tau,t)]_1}{dt} + \frac{\partial\rho(\Phi(\tau,t),t)}{\partial[\Phi(\tau,t)]_2}\frac{d[\Phi(\tau,t)]_2}{dt} + \frac{\partial\rho(\Phi(\tau,t),t)}{\partial t}\\ &= \nabla_x\rho(\Phi(\tau,t),t)\cdot\frac{d}{dt}\vec{\Phi}(\tau,t) + \partial_t\rho(\Phi(\tau,t),t).\end{aligned} \tag{4.25}$$

(4.20a) evaluated at $x = \Phi(\tau, t)$ together with (4.21a) and (4.22a) gives

$$\begin{aligned}\partial_t\rho(\Phi(\tau,t),t) + \rho^2(\Phi(\tau,t),t) &= -E(\Phi)\cdot\nabla\rho(\Phi(\tau,t),t)\\ &= -\frac{d}{dt}\Phi(\tau,t)\cdot\nabla\rho(\Phi(\tau,t),t).\end{aligned} \tag{4.26}$$

Substituting (4.26) into (4.25), we obtain

$$\begin{aligned}\frac{d}{dt}\rho(\Phi(\tau,t),t) &= -\partial_t\rho(\Phi(\tau,t),t) - \rho^2(\Phi(\tau,t),t) + \partial_t\rho(\Phi(\tau,t),t)\\ &= -\rho^2(\Phi(\tau,t),t).\end{aligned}$$

□

The nonlinear differential equation for $\rho(\Phi)$ obtained in Lemma 4.17 is explicitly solvable subject to the initial conditions ρ_0 and ρ_A. The piecewise definition of $\Phi(\tau)$ is reflected in the solution ρ on a streamline.

Lemma 4.18. *Let Ω be a $C^{2,\alpha}$ domain. Then the solution ρ for the initial value problem (Tr 4.4) is given on the streamlines by*

$$\rho(\Phi(\tau,t),t) = \begin{cases}\frac{\rho_0(x)}{1+t\rho_0(x)} & \tau = x \in \Omega_0\\ \frac{\rho_A(s,t_x)}{1+(t-t_x)\rho_A(s,t_x)} & \tau = (s,t_x) \in Q_t.\end{cases} \tag{4.27}$$

Proof. Set

$$p(t) = \frac{1}{\rho(\Phi(\tau,t),t)}. \tag{4.28}$$

Then holds with the chain rule and (4.24)

$$\begin{aligned}\frac{d}{dt}p(t) &= -\frac{\frac{d}{dt}\rho(\Phi(\tau,t),t)}{\rho^2(\Phi(\tau,t),t)}\\ &= \frac{\rho^2(\Phi(\tau,t),t)}{\rho^2(\Phi(\tau,t),t)}\\ &= 1.\end{aligned}$$

Integration with respect to t gives

$$p(t) = t + c(\tau).$$

It follows with (4.28)

$$\rho(\Phi(\tau, t), t) = \frac{1}{t + c(\tau)}.$$

We now have to distinguish the cases $\tau \in \Omega_0$ and $\tau \in Q_t$. We use the initial conditions (4.21b) and (4.22b) for $\Phi_1(x,t)$ and $\Phi_2(s, t_x, t)$. With $\Phi_1(x, 0) = x$, we get

$$\rho(\Phi_1(x, 0), 0) = \frac{1}{c(x)} = \rho_0(x)$$

and thus

$$\rho(\Phi_1(x, t), t) = \frac{\rho_0(x)}{1 + t\rho_0(x)}.$$

Analogously, we obtain for Φ_2

$$\rho(\Phi_2(s, t_x, t_x), t_x) = \frac{1}{t_x + c(s, t_x)} = \rho_A(x, t_x)$$

resulting in

$$\rho(\Phi_2(x, t_x, t), t) = \frac{\rho_A(x, t_x)}{1 + (t - t_x)\rho_A(x)}.$$

□

The piecewise representation of ρ in (4.27) must not be understood as a global solution in the domain Ω. As (4.27) is defined on the streamlines, it only gives the solution in the time dependent domain Ω_t. However, due to the choice of ρ_0, extending (4.27) by 0 into $\Omega \backslash \Omega_t$ results in a continuous solution ρ for $x \in \Omega$. To do so, it is necessary to use the inverse streamline functions Φ_1^{-1} and Φ_2^{-1} where the inversion is understood with respect to the space variable, i.e. $x \in \Omega$ and $(s, t_x) \in Q_t$, for every fixed time t. We will need to define an interface between the images of Φ_1 and Φ_2 and exclude any overlapping of the open domains Ω_t^1 and Ω_t^2 in order to obtain well defined inverse functions.

Definition 4.19. *We say that Φ_1 and Φ_2 fulfil an* interface condition *if the image domains intersect only in boundary points, i.e. the open domains Ω_t^1 and Ω_t^2 are disjoint and*

$$\Phi_1(x_0, t) = \Phi_2(s_0, 0, t), \text{ for all } x_0 = \varphi(s_0). \tag{4.29}$$

The closed curve given by

$$\Gamma_t^{IF} := \{y : \Phi_1(x_0, t) = y = \Phi_2(s_0, 0, t), \text{ for all } x_0 = \varphi(s_0) \in \Gamma_-\} \tag{4.30}$$

is called interface *of $\Phi_1(x,t)$ and $\Phi_2(s, t_x, t)$.*

Theorem 4.20. *Let Ω be a $C^{2,\alpha}$ domain, $\bar{\Omega}_t^1 \subset \Omega$, $\bar{\Omega}_t^2 \subset \Omega$ with Ω_t^1 and Ω_t^2 disjoint. Further assume that the interface condition*

$$\Phi_1(x_0, t) = \Phi_2(s_0, 0, t), \text{ for all } x_0 = \varphi(s_0),$$

holds and let $\Phi_1(x,t)$ and $\Phi_2(s, t_x, t)$ be invertible for every fixed $t \in [0,T]$ with respect to the space variable. Then holds
a) *The solution ρ of (Tr 4.4) is given by*

$$\rho(y,t) = \begin{cases} 0, & y \in \Omega \backslash \Omega_t \\ \frac{\rho_0(\Phi_1^{-1}(y,t))}{1+t\rho_0(\Phi_1^{-1}(y,t))} & y \in \Omega_t^1 \\ \frac{\rho_A(\Phi_2^{-1}(y,t))}{1+(t-[\Phi_2^{-1}(y,t)]_2)\rho_A(\Phi_2^{-1}(y,t))} & y \in \Omega_t^2 \end{cases}. \tag{4.31}$$

b) *ρ is continuous for $x \in \Omega$.*

Proof. **a)** By Lemma 4.18, we know the solution ρ in Ω_t. Due to the definition of Ω_t as $\Omega_t \supseteq \text{supp}_{y\in\Omega}\{\rho(y,t)\}$, it holds further that ρ is continuously extendable by 0 into $\Omega\backslash\Omega_t$.

b) By a), it is immediately true that $\rho|_{\Omega_t^1}$ is continuously extendable by 0 into $\Omega\backslash\Omega_t$. It is left to show that we have a continuous interface between $\rho|_{\Omega_t^1}$ and $\rho|_{\Omega_t^2}$. For every $y \in \Gamma_t^{IF}$, there exist s_0 and x_0 due to the definition of the interface, such that
$\Phi_1(x_0,t) = y = \Phi_2(s_0, 0, t)$ with $\varphi(s_0) = x_0$. We now use the invertibility of the streamline functions and the continuity condition for ρ_0 and ρ_A claimed in (Tr 4.4). Hence we obtain by (4.27) for $y \in \Gamma_t^{IF}$

$$\begin{aligned} \frac{\rho_0(\Phi_1^{-1}(y,t))}{1+t\rho_0(\Phi_1^{-1}(y,t))} &= \frac{\rho_0(\Phi_1^{-1}(\Phi_1(x_0),t))}{1+t\rho_0(\Phi_1^{-1}(\Phi_1(x_0,t),t))} \\ &= \frac{\rho_0(x_0)}{1+t\rho_0(x_0)} \\ &= \frac{\rho_A(s_0,0)}{1+t\rho_A(s_0,0)} \\ &= \frac{\rho_A(\Phi_2^{-1}(y,t))}{1+t\rho_A(\Phi_2^{-1}(y,t))} \end{aligned}$$

and thus the continuous transition of ρ at Γ_t^{IF}. □

4.4 Derivation of the Integro-Differential Operator A

We are now going to derive the integro-differential operators A_1 and A_2 following the method of [44, 43]. We begin with manipulating Φ_1 and Φ_2 and to formulate these as integral equations. With the differential equation (4.21a), (4.1f) and the representation (4.15) of u, we obtain

$$\frac{d\Phi_1(x,t)}{dt} = -\nabla u(\Phi_1(x,t),t) = -\nabla_{\Phi_1}\int_\Omega G(\Phi_1(x,t),y)\rho(y,t)\,dy + E_0(\Phi_1(x,t)) \tag{4.32}$$

where $E_0 = -\nabla u_0$ and ∇_{Φ_1} shall be understood as gradient with respect to the argument Φ_1. Integration with respect to t over $[0,T]$ and using the initial condition $\Phi_1(x,0) = x$ in (4.21b), we obtain

$$\begin{aligned}\Phi_1(x,t) &= x - \int_0^t \nabla_{\Phi_1} \int_\Omega G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_1(x,\mu))\,d\mu \\ &= x - \int_0^t \int_\Omega \nabla_{\Phi_1} G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_1(x,\mu))\,d\mu. \end{aligned}\tag{4.33}$$

Analogously, we derive with (4.22a), (4.1f) and (4.15) the following integral equation for $\Phi_2(s,t_x,t)$

$$\begin{aligned}\frac{d\Phi_2(s,t_x,t)}{dt} &= -\nabla_{\Phi_2} u(\Phi_2(s,t_x,t),t) \\ &= -\nabla_{\Phi_2} \int_\Omega G(\Phi_2(s,t_x,t),y)\rho(y,t)\,dy + E_0(\Phi_2(s,t_x,t))\end{aligned}\tag{4.34}$$

with again $E_0 = -\nabla u_0$. By integration over $[t_x,t]$ and the initial condition $\Phi(s,t_x,t_x) = \varphi(s)$ in (4.22b), we get

$$\begin{aligned}\Phi_2(s,t_x,t) &= \varphi(s) - \int_{t_x}^t \nabla_{\Phi_2} \int_\Omega G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_2(s,t_x,\mu))\,d\mu \\ &= \varphi(s) - \int_{t_x}^t \int_\Omega \nabla_{\Phi_2} G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_2(s,t_x,\mu))\,d\mu.\end{aligned}\tag{4.35}$$

At the first glance, it might appear that the right hand side of (4.33) only depends on the streamline function Φ_1 and that the analysis for Φ_1 could be done without the knowledge of Φ_2. In fact, this is not the case as ρ defined in (4.27) depends on both streamline functions Φ_1 and Φ_2. We now reformulate the presented integral equations for Φ_1 and Φ_2 as system of integro-differential operators that are applied to 2-tuple $\Phi = (\Phi_1, \Phi_2)$ by

$$A_1(\Phi)(x,t) = x - \int_0^t \int_\Omega \nabla_{\Phi_1} G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_1(x,\mu))\,d\mu \tag{4.36}$$

and

$$A_2(\Phi)(s,t_x,t) = \varphi(s) - \int_{t_x}^t \int_\Omega \nabla_{\Phi_2} G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_2(s,t_x,\mu))\,d\mu \tag{4.37}$$

where ρ is defined in (4.31).

Lemma 4.21. *Let Ω be a $C^{2,\alpha}$ domain. If solutions Φ_1 and Φ_2 of (4.33) and (4.35) exist, then they are given as fixed point $(\Phi_1, \Phi_2) = (A_1(\Phi_1, \Phi_2), A_2(\Phi_1, \Phi_2))$.*

Proof. The right hand side of the integral equations (4.33) and (4.35) depend on Φ_1 and Φ_2 due to the function ρ defined in (4.31). To find the solutions Φ_1 and Φ_2, we therefore need to find the fixed point of the integral operators (4.36)-(4.37). □

To obtain a classical solution (u,ρ), the fixed point Φ of $A := (A_1, A_2)$ has to be a $C^{1,\alpha}$-function in space on the respective domain of definition. The next Lemma states the gradients of both operators as they will be used frequently in the following.

Lemma 4.22. *It holds*

$$\begin{aligned}\nabla A_1(\Phi)(x,t) = I - \int_0^t \nabla_{\Phi_1} \int_\Omega \nabla_{\Phi_1} G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_1(x,\mu)\,d\mu \\ + \int_0^t \nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}\nabla A_2(\Phi)(s,t_x,t) = D_A(s,t_x) - \int_{t_x}^t \nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\,\nabla\Phi_2(s,t_x,\mu)\,d\mu \\ + \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\,\nabla\Phi_2(s,t_x,\mu)\,d\mu\end{aligned} \tag{4.39}$$

with I being the identity matrix and

$$D_A(s,t_x) = \begin{pmatrix} [\varphi'(s)]_1 & \left[\int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\,dy - E_0(\varphi(s))\right]_1 \\ [\varphi'(s)]_2 & \left[\int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\,dy - E_0(\varphi(s))\right]_2 \end{pmatrix} \tag{4.40}$$

where $[\cdot]_1, [\cdot]_2$ denote the first and second components of the vector.

Proof. In case of $\nabla A_1(\Phi)$, the Lemma is a simple application of the chain rule. For $\nabla A_2(\Phi)$, we first find the derivative with respect to s.

$$\begin{aligned}\partial_s A_2(\Phi)(s,t_x,t) = \varphi'(s) - \int_{t_x}^t \nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\ \partial_s\Phi_2(s,t_x,\mu)\,d\mu \\ + \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\ \partial_s\Phi_2(s,t_x,\mu)\,d\mu.\end{aligned}$$

For the partial derivative with respect to t_x, we also have to differentiate the integral limits.

$$\begin{aligned}\partial_{t_x} A_2(\Phi)(s,t_x,t) = \int_\Omega \nabla G(\Phi_2(s,t_x,t_x),y)\rho(y,t_x)\,dy - E_0(\Phi_2(s,t_x,t_x)) \\ - \int_{t_x}^t \nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\,\partial_{t_x}\Phi_2(s,t_x,\mu)\,d\mu \\ + \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\partial_{t_x}\Phi_2(s,t_x,\mu)\,d\mu.\end{aligned}$$

By (4.22b), i.e. $\Phi_2(s,t_x,t_x) = \varphi(s)$, we obtain the assertion. □

4.4.1 Definition of the Set $W(M,T,K,\delta)$

We will now define subspaces W_1 and W_2 of $C^{1,\alpha;\alpha}(\bar\Omega,[0,T])$ and $W \subset W_1 \times W_2$ in which we will seek for the fixed point $\Phi = (\Phi_1,\Phi_2)$ of (4.36) and (4.37). The elements Φ_1 and Φ_2 of W_1 and W_2 obey certain restrictions in order to be possible solutions to (CP 4.2). While W_1 is chosen as in [44], we have to do some more considerations for W_2.

Definition 4.23. *Define the sets*

$$W_1(M,T) := \Big\{\Phi_1 \in C^{1,\alpha;\alpha}(\bar{\Omega}_0,[0,T]) : \Phi_1(\cdot,t) : \Omega_0 \to \Omega,\ \Phi_1(x,0) = x,$$
$$\|\Phi_1\|_{1,\alpha,\Omega_0;\alpha,[0,T]} \leq M,\ \sup_{0\leq t\leq T} \|I - \nabla\Phi_1(t)\|_{0,\Omega_0} \leq 0.5\Big\} \tag{4.41}$$

and

$$W_2(M,T,K,\delta) := \Big\{\Phi_2 \in C^{1,\alpha;\alpha}(Q_t,[0,T]) : \Phi_2(\cdot,t) : Q_t \to \Omega \text{ for } t \in [0,T],$$
$$\Phi_2(s,t_x,t_x) = \varphi(s),\ \|\Phi_2\|_{1,Q_t;\alpha,[0,T]} \leq M,$$
$$\sup_{0\leq t\leq T} \|D - \nabla\Phi_2(t)\|_{0,Q_t} \leq \frac{\delta}{4M},\ \sup_{0\leq t\leq T} |\nabla\Phi_2(t)|_{\alpha,Q_t} \leq K$$
$$\inf_{(s,t_x)\in Q_t} \left|\partial_{t_x}\Phi_2(s,t_x,t=t_x)\cdot(-\varphi_2'(s),\varphi_1'(s))\right| \geq \delta > 0\Big\} \tag{4.42}$$

with

$$D(s,t_x) = \begin{pmatrix} [\varphi'(s)]_1 & [\partial_{t_x}\Phi_2(s,t_x,t)|_{t=t_x}]_1 \\ [\varphi'(s)]_2 & [\partial_{t_x}\Phi_2(s,t_x,t)|_{t=t_x}]_2 \end{pmatrix}. \tag{4.43}$$

Choose the constant δ in the set $W_2(M,T,K,\delta)$ such that

$$|E_0(\varphi(s))\cdot\varphi(s)^{\perp}| \geq 2\delta,$$
$$\inf_{x\in\Omega} |E_0(x)|_\infty \geq 2\delta.$$

with $E_0 = -\nabla u_0$ being the solution of (4.11a)-(4.11b).

Remark 4.24. *The field E_0 as solution of the Laplace equation (4.11a)-(4.11b) is constant in time. By defining the boundary condition u_A and the geometry of Ω, we thus define the constant δ. δ depends on the difference $u_A|_{\Gamma_-} - u_A|_{\Gamma_+}$. The bigger the difference $u_A|_{\Gamma_-} - u_A|_{\Gamma_+}$, the larger δ is.*

The initial condition on $\Phi_1 \in W_1(M,T)$ ensures that the streamlines start from every point $x \in \Omega_0$. In case of W_2, the initial condition states that every streamline starts from the inflow boundary. While Φ_1 is bounded by a common constant M in the $C^{1,\alpha,\alpha}(\Omega_0\times[0,T])$ norm, we need to distinguish two cases for Φ_2. For technical reasons, the $C^\alpha(Q_t)$-semi norm of $\nabla\Phi_2$ is bounded by a second constant K. The remaining restrictions on Φ_1 and Φ_2 ensure the invertibility of the streamline functions. We therefore introduce the following result based on Neumann series and show that any $\Phi_1 \in W_1$ and $\Phi_2 \in W_2$ are invertible with respect to the space variables.

Lemma 4.25. *Let K be a continuous endomorphism on a Banach space V. Let $I : V \to V$ be the identity operator. If*

$$|K|_V \leq q < 1,$$

then $I - K$ is invertible. $(I-K)^{-1}$ is bounded by

$$\|(I-K)^{-1}\| \leq \frac{1}{1-q}. \tag{4.44}$$

Proof. The proof uses the Neumann series, compare [61, p.56]. $\square$

With the previous Lemma, we conclude that to any $\Phi_1 \in W_1(M,T)$ exists a bounded inverse function Φ_1^{-1}.

Lemma 4.26. *Let $\Phi_1 \in W_1(M,T)$ with $\Phi_1(\cdot,t) : \Omega_0 \to \Omega_t^1$. Then for every fixed $t \in [0,T]$, there exists the inverse function $\Phi_1^{-1} \in C^1(\Omega_t^1)$ with*

$$\left\|\Phi_1^{-1}(t)\right\|_{0,\Omega_1^t} \leq \|x\|_{0,\Omega_0} \tag{4.45}$$

$$\left\|\nabla\Phi_1^{-1}(t)\right\|_{0,\Omega_1^t} \leq 2. \tag{4.46}$$

Proof. $\nabla\Phi_1(x,t)$ is the Jacobian matrix of Φ_1 and as such, an endomorphism on $\mathbb{R}^2$. Since $\|I - \nabla\Phi_1(t)\|_{0,\Omega_0} < \frac{1}{2}$ there exists the inverse matrix $(\nabla\Phi_1(x,t))^{-1}$ for every $x \in \Omega$ by Lemma 4.25. By the implicit function theorem, there exists conclusively

$$\Phi_1^{-1} : \Omega_t^1 \to \Omega_0.$$

As Φ_1^{-1} maps Ω_t^1 into Ω_0, (4.45) is immediate. Since $|I - \nabla\Phi_1(t)|_{0,\Omega_0} < \frac{1}{2} =: q$ it holds by (4.44)

$$\|(I - I - \nabla\Phi_1(t))^{-1}\|_{0,\Omega_0} = \sup_{x\in\Omega_0} |(\nabla\Phi_1(x,t))^{-1}|_\infty \leq \frac{1}{1-\frac{1}{2}} = 2. \tag{4.47}$$

By the implicit function theorem and the previous equation holds

$$\sup_{y\in\Omega_t^1} |\nabla\Phi_1^{-1}(y,t)|_\infty = \sup_{y\in\Omega_t^1} |(\nabla\Phi_1(\Phi_1^{-1}(y,t),t))^{-1}|_\infty \leq \sup_{x\in\Omega_0} |\nabla\Phi_1(y,t)^{-1}|_\infty \leq 2.$$

$\square$

Lemma 4.25 does not immediately apply to $\nabla\Phi_2$, as we compare $\nabla\Phi_2$ to a variable matrix $D(s,t_x)$ defined in the space $W_2(M,T,K,\delta)$. However, with the following generalization of Lemma 4.25 and provided $D(s,t_x)$ is invertible, we can also prove the invertibility of $\nabla\Phi_2(s,t_x,t)$.

Lemma 4.27. *compare [3, p. 147, 3.8]. Let V and V' be Banach spaces and $S : V \to V'$ be an invertible operator. If for an operator $T : V \to V'$ holds*

$$\|S - T\| \leq q\|S^{-1}\|^{-1}, \qquad 0 < q < 1, \tag{4.48}$$

then T has a continuous inverse. The inverse operator is bounded by

$$\|T^{-1}\| \leq \frac{1}{1-q}\|S^{-1}\|. \tag{4.49}$$

Proof. As S is invertible, it holds

$$T = S(I - (I - S^{-1}T)). \tag{4.50}$$

The product on the right hand side is invertible iff both factors are invertible. Applying Lemma 4.25 to the second factor and using the assumption (4.48), we obtain

$$\|I - (I - (I - S^{-1}T))\| = \|I - S^{-1}T\| = \|S^{-1}S - S^{-1}T\| \leq \|S^{-1}\|\|S - T\| \leq q < 1.$$

It follows that $I - (I - S^{-1}T)$ is invertible and thus is T.
To obtain (4.49), we first use (4.50). It holds

$$\|T^{-1}\| \leq \|(I - (I - S^{-1}T))^{-1}\|\|S^{-1}\|.$$

As $(I - I - (I - S^{-1}T)) = (I - S^{-1}T) = S^{-1}(S - T)$ and due the assumption holds $\|S^{-1}\|\|S - T\| \leq q$. We apply Lemma 4.25 and obtain the bound for the inverse

$$\|(I - (I - S^{-1}T))^{-1}\| \leq \frac{1}{1-q}.$$

The bound for T^{-1} is thus given by

$$\|T^{-1}\| \leq \frac{1}{1-q}\|S^{-1}\|.$$

□

Before applying the previous Lemma to $\nabla\Phi_2$, we have to ensure that the matrix $D(s,t_x)$ is invertible for every $(s,t_x) \in Q_t$. Whenever a matrix norm is used in the following, it is the $|\cdot|_\infty$-norm as defined in Definition 2.1.

Lemma 4.28. *Let $\Phi_2 \in W_2(M,T,K,\delta)$. Then the matrix $D(s,t_x)$ defined in (4.43) is invertible for every $(s,t_x) \in Q_t$. Further holds for every $t \in [0,T]$*

$$\|D^{-1}\|_{0,Q_t} \leq \frac{2M}{\delta}. \tag{4.51}$$

Proof. Recall that $D(s,t_x)$ is a 2×2 matrix. Since $\Phi_2 \in W_2(M,T,K,\delta)$, it holds for the determinant of $D(s,t_x)$

$$\inf_{(s,t_x)\in Q_t} |\det D(s,t_x)| = \left|\varphi_1'(s)[\partial_{t_x}\Phi_2(s,t_x,t=t_x)]_2 - [\partial_{t_x}\Phi_2(s,t_x,t=t_x)]_1\varphi_2'(s)\right| \geq \delta > 0.$$

It follows that $D(s_1,t_x)$ is invertible for every $(s,t_x) \in Q_t$. The inverse matrix is given for every $(s,t_x) \in Q_t$ by

$$D(s,t_x)^{-1} = \frac{1}{\det D}\begin{pmatrix} [\partial_{t_x}\Phi_2(s,t_x,t=t_x)]_2 & -[\partial_{t_x}\Phi_2(s,t_x,t=t_x)]_1 \\ -\varphi_2(s) & \varphi_1(s) \end{pmatrix}.$$

As $\Phi_2 \in W_2(M,T,K,\delta)$ and $\|\varphi\|_{0,Q_t} \leq M$ and $\|\partial_{t_x}\Phi_2(s,t_x,t=t_x)\|_{0,Q_t} \leq M$, follows

$$\|D^{-1}\|_{0,Q_t} \leq \frac{2M}{\delta}.$$

□

We can now prove that every $\Phi_2 \in W_2(M,T,K,\delta)$ is invertible.

Lemma 4.29. *Let $\Phi_2 \in W_2(M,T,K,\delta)$ with $\Phi_2(\cdot,t) : Q_t \to \Omega_t^2$. Then for every $t \in [0,T]$ exists the inverse function $\Phi_2^{-1} \in C^1(\Omega_t^2)$ with*

$$\left\|\Phi_2^{-1}\right\|_{0,\Omega_t^2} \leq \max\left\{\|\varphi\|_{0,[0,L_{\Gamma\_}]}, T\right\} \tag{4.52}$$

$$\left\|\nabla\Phi_2^{-1}\right\|_{0,\Omega_t^2} \leq \frac{4M}{\delta}. \tag{4.53}$$

Proof. Since $\Phi_2 \in W_2(M,T,K,\delta)$ and by (4.51), we obtain

$$\|D - \nabla\Phi_2\|_{0,Q_t}\|D^{-1}\|_{0,Q_t} \leq \frac{\delta}{4M}\frac{2M}{\delta} = \frac{1}{2} =: q. \tag{4.54}$$

Thus the inverse matrix $(\nabla\Phi_2(y,t))^{-1}$ on Ω_t^2 exists by Lemma 4.27. Due to the implicit function theorem, there exists for every $t \in [0,T]$

$$\Phi_2^{-1} : \Omega_t^2 \to Q_t.$$

(4.52) is an immediate consequence. By the implicit function theorem, Lemma 4.27 with $q = \frac{1}{2}$ and (4.51), we obtain

$$\begin{aligned}\sup_{y\in\Omega_t^2}\left|\nabla\Phi_2^{-1}(y,t)\right|_\infty &= \sup_{y\in\Omega_t^2}\left|(\nabla\Phi_2(\Phi^{-1}(y,t),t))^{-1}\right|_\infty \leq \sup_{(s,t_x)\in Q_t}\left|(\nabla\Phi_2(s,t_x,t))^{-1}\right|_\infty \\ &\leq 2\left\|D^{-1}\right\|_{0,Q_t} \leq \frac{4M}{\delta}.\end{aligned}$$

□

Remark 4.30. *We have seen in Lemma 4.20 that the invertibility of $\Phi = (\Phi_1, \Phi_2)$ is necessary to obtain the solution ρ to (Tr 4.4). If the invertibility was violated, then two streamline functions would cross over and the solution would blow up, in other words it would cease to exist. We thus ensure the existence of a solution to (CP 4.2) by demanding that Φ must be invertible.*

So far, we have defined the function sets $W_1(M,T)$ and $W_2(M,T,K,\delta)$. We can now specify the set in which we seek for the fixed point $\Phi = (\Phi_1,\Phi_2)$ to (4.38)-(4.39) as subset of the product space $W_1 \times W_2$. Not every combination of any two functions Φ_1 and Φ_2 leads to a classical solution of (CP 4.2). Recall that to obtain ρ in (4.31) we require the existence of the global inverse function $\Phi^{-1} = (\Phi_1^{-1}, \Phi_2^{-1})$. Additionally, to obtain a continuous ρ, holes between Ω_t^1 and Ω_t^2 must be excluded. We thus conclude, that the range of Φ_1 and Φ_2 must not intersect (in more than boundary points) and the interface between Ω_t^1 and Ω_t^2 has to be continuous. We will formulate an interface condition ensuring that Ω_t^1 and Ω_t^2 only intersect at the interface Γ_t^{IF} and obtain the following set of functions

$$\begin{aligned}W(M,T,K,\delta) = \{&\Phi = (\Phi_1,\Phi_2) : \Phi_1 \in W_1(M,T), \Phi_2 \in W_2(M,T,K,\delta), \\ &\Phi_1(x_0,t) = \Phi_2(s,0,t) \text{ for all } x_0 = \varphi(s)\}.\end{aligned} \tag{4.55}$$

Clearly, $W(M,T,K,\delta) \subset W_1(M,T) \times W_2(M,T,K,\delta)$. The solution Φ is considered as a 2-tuple consisting of Φ_1 and Φ_2 that fulfil the interface conditions. The norm on $W(M,T,K,\delta)$ is defined due to the natural convention for product spaces

$$\|\Phi(t)\|_{1,\alpha,\Sigma_t} = \|\Phi_1(t)\|_{1,\alpha,\Omega_0} + \|\Phi_2(t)\|_{1,\alpha,Q_t}.$$

Lemma 4.31. *The set $W(M,T,K,\delta)$ is non-empty a small time T_{E_0}, $M \geq 2\|x\|_{0,\Omega_0} + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma} + 3$ and $K \geq c_{mv}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} L_{\Gamma_-}^{1-\alpha} + 1$.*

Proof. We show that the streamline functions Φ_1 and Φ_2 generated by E_0 are in $W(M, T_{E_0}, K, \delta)$. The proof is given in the appendix A.1. □

We proceed to find the following result on the image domains Ω_t^1 and Ω_t^2.

Lemma 4.32. *Let Ω be a $C^{2,\alpha}$ domain, $\Phi \in W(M,T,K,\delta)$. Then $\bar{\Omega}_t^1$ and $\bar{\Omega}_t^2$ intersect only in the interface Γ_t^{IF}.*

Proof. By the interface condition

$$\Phi_1(x_0,t) = \Phi_2(s,0,t) \text{ for all } x_0 = \varphi(s),$$

we know that Φ_1 and Φ_2 intersect at boundary points. We have to show that these are the only points of intersection.

$\Phi_1 \in W(M,T,K,\delta)$ is invertible with respect to the space variable for every fixed $t \in [0,T]$ due to Lemma 4.26. Further, the initial condition is the identity map $\Phi(x,0) = x$ for all $x \in \Omega_0$ and $t = 0$. With these two statements, we can conclude that the boundary $\partial\Omega_0$ is mapped onto the boundary $\partial\Omega_t^1$ for every $t \in [0,T]$. For $t \in [0,T]$, no two streamlines can intersect because otherwise the invertibility of Φ_1 was violated. It follows that no inner point of Ω_0 can ever be a boundary point of Ω_t^1 since holes in Ω_t^1 are excluded due to the continuity of Φ_1.

$\Phi_2(s,t_x,t) \in W(M,T,K,\delta)$ is invertible with respect to the variables (s,t_x) for every fixed $t \in [0,T]$ due to Lemma 4.29. The initial condition is given by $\Phi_2(s,t_x,t_x) = \varphi(s)$ stating that every inflowing point starts from the inflow boundary. The expansion of $\Phi_2(s,t_x,t)$ is thus described by the interface condition in $W(M,T,K,\delta)$. Since at $t=0$ holds $\Gamma_0^{IF} = \Gamma_-$, all inflowing points for $t \in (0,T)$ must be mapped in between Γ_t^{IF} and Γ_-. It is not possible that any points inflowing at $t \in (0,T)$ are outside these two boundaries as then the invertibility condition must have been violated.
We conclude that the open domains Ω_t^1 and Ω_t^2 are disjoint. □

The remaining part of this chapter is concerned with the question whether there exists a unique fixed point $\Phi = (\Phi_1, \Phi_2)$ such that

$$\Phi(\tau,t) := (A_1(\Phi)(x,t), A_2(\Phi)(s,t_x,t)). \tag{4.56}$$

We proceed as follows. In section 4.6, we first show that A is a selfmap on the set $W(M,T,K,\delta)$. We start by considering the regularity of ρ in the next section and then in section 4.7, we show that A is a contraction in the sup-norm. In contrast to the one-dimensional case in Chapter 3, we are not able to prove the contraction property for A in the $C^{1,\alpha,\alpha}(\Sigma_t,[0,T])$ norm and thus we can not apply the Banach fixed point theorem. Instead, we will use a compactness argument to show the existence of a unique fixed point in section 4.8.

4.5 Regularity of ρ

To show that A is a selfmap, we have to investigate whether the functions $A_1(\Phi)$ and $A_2(\Phi)$ fulfil the restrictions in $W_1(M,T)$ and $W_2(M,T,K,\delta)$. The main work in bounding $A_1(\Phi)$ and $A_2(\Phi)$ is caused by ρ. At this point, the non convexity of the domain Ω hits in and makes the proofs lengthy and technical. In this section, we examine ρ with respect to its regularity in space and time. The first Lemmas deal with the Hölder continuity of ρ in space. We need to distinguish several cases due to the piecewise definition of Φ and the nonconvexity of the domain Ω.

Lemma 4.33. *Let Ω be a $C^{2,\alpha}$ domain and let L_{ρ_0}, L_{ρ_A} be the Lipschitz constants for ρ_0 and ρ_A. Let ρ be defined in (4.31) and $\Phi \in W(M,T,K,\delta)$.*
***a)** Let $y_1, y_2 \in \Omega_t^1$ such that the line segment $\overrightarrow{y_1 y_2} \subset \Omega_t^1$. Then we have for $t \in [0,T]$*

$$|\rho(y_1,t) - \rho(y_2,t)| \le L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1} |y_1 - y_2|_\infty. \tag{4.57}$$

***b)** Let $y_1, y_2 \in \Omega_t^2$, such that the line segment $\overrightarrow{y_1 y_2} \subset \Omega_t^2$. Then we have for $t \in [0,T]$*

$$|\rho(y_1,t) - \rho(y_2,t)| \le \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} |y_1 - y_2|_\infty. \tag{4.58}$$

Proof. **a)** As the line segment $\overrightarrow{y_1 y_2}$ is fully contained in Ω_t^1, we apply the mean value theorem (Theorem 2.15) to the streamline function Φ_1^{-1}. It follows with the Lipschitz continuity of ρ_0 and by (4.31)

$$\begin{aligned}
|\rho(y_1,t) - \rho(y_2,t)| &= \left| \frac{\rho_0(\Phi_1^{-1}(y_1,t))}{1 + t\rho_0(\Phi_1^{-1}(y_1,t))} - \frac{\rho_0(\Phi_1^{-1}(y_2,t))}{1 + t\rho_0(\Phi_1^{-1}(y_2,t))} \right| \\
&= \left| \frac{\rho_0(\Phi_1^{-1}(y_1,t)) - \rho_0(\Phi_1^{-1}(y_2,t))}{(1 + t\rho_0(\Phi_1^{-1}(y_2,t)))(1 + t\rho_0(\Phi_1^{-1}(y_2,t)))} \right| \\
&\le \left| \rho_0(\Phi_1^{-1}(y_1,t)) - \rho_0(\Phi_1^{-1}(y_2,t)) \right| \\
&\le L_{\rho_0} \left| \Phi_1^{-1}(y_1,t) - \Phi_1^{-1}(y_2,t) \right|_\infty \\
&\le L_{\rho_0} \left\| \nabla \Phi_1^{-1}(t) \right\|_{0,\Omega_t^1} |y_1 - y_2|_\infty.
\end{aligned}$$

b) We proceed the same way as in a). Since the line segment $\overrightarrow{y_1 y_2}$ is fully contained in Ω_t^2, we apply the mean value theorem (Theorem 2.15) to the streamline function Φ_2^{-1}. It follows with the Lipschitz continuity of ρ_A and by (4.31)

$$\begin{aligned}
&|\rho(y_1,t) - \rho(y_2,t)| \\
&= \left| \frac{\rho_A(\Phi_2^{-1}(y_1,t))}{1 + (t - [\Phi_2^{-1}(y_1,t)]_2)\rho_A(\Phi_2^{-1}(y_1,t))} - \frac{\rho_A(\Phi_2^{-1}(y_2,t))}{1 + (t - [\Phi_2^{-1}(y_2,t)]_2)\rho_A(\Phi_2^{-1}(y_2,t))} \right| \\
&= \left| \frac{\rho_A(\Phi_2^{-1}(y_1,t)) - \rho_A(\Phi_2^{-1}(y_2,t)) + ([\Phi_2^{-1}(y_2,t)]_2 - [\Phi_2^{-1}(y_1,t)]_2)\rho_A(\Phi_2^{-1}(y_1,t))\rho_A(\Phi_2^{-1}(y_2,t))}{(1 + (t - [\Phi_2^{-1}(y_1,t)]_2)\rho_A(\Phi_2^{-1}(y_1,t)))(1 + (t - [\Phi_2^{-1}(y_2,t)]_2)\rho_A(\Phi_2^{-1}(y_2,t)))} \right| \\
&\le \left| \rho_A(\Phi_2^{-1}(y_1,t)) - \rho_A(\Phi_2^{-1}(y_2,t)) + ([\Phi_2^{-1}(y_2,t)]_2 - [\Phi_2^{-1}(y_1,t)]_2)\rho_A(\Phi_2^{-1}(y_1,t))\rho_A(\Phi_2^{-1}(y_2,t)) \right| \\
&\le \left| \rho_A(\Phi_2^{-1}(y_1,t)) - \rho_A(\Phi_2^{-1}(y_2,t)) \right| + \left| [\Phi_2^{-1}(y_2,t)]_2 - [\Phi_2^{-1}(y_1,t)]_2 \right| \left| \rho_A(\Phi_2^{-1}(y_1,t)) \right| \left| \rho_A(\Phi_2^{-1}(y_2,t)) \right| \\
&\le L_{\rho_A} \left| \Phi_2^{-1}(y_1,t) - \Phi_2^{-1}(y_2,t) \right|_\infty + \left| \Phi_2^{-1}(y_2,t) - \Phi_2^{-1}(y_1,t) \right| \left| \rho_A(\Phi_2^{-1}(y_1,t)) \right|_\infty \left| \rho_A(\Phi_2^{-1}(y_2,t)) \right| \\
&\le L_{\rho_A} \left| \nabla \Phi_2^{-1}(y_1,t) \right|_\infty |y_1 - y_2| + \left| \nabla \Phi_2^{-1}(y_2,t) \right|_\infty |y_1 - y_2|_\infty \left| \rho_A(\Phi_2^{-1}(y_1,t)) \right| \left| \rho_A(\Phi_2^{-1}(y_2,t)) \right| \\
&\le \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} |y_1 - y_2|_\infty.
\end{aligned}$$

□

We next show that ρ is Hölder continuous for every $0 < \beta \leq 1$. In particular, we obtain the Lipschitz continuity of ρ in Ω.

Lemma 4.34. *Let Ω be a $C^{2,\alpha}$ domain, ρ be defined in (4.31) and $\Phi \in W(M,T,K,\delta)$. Then for $t \in [0,T]$ and $x, y \in \Omega$ follows*

$$|\rho(x,t) - \rho(y,t)| \leq c_\rho |x-y|_\infty.$$

with

$$c_\rho = \max\left\{4c_{mv}(L_{\rho_A} + \|\rho_A\|^2_{0,Q_T})\frac{M}{\delta},\ 2L_{\rho_0}\right\}. \tag{4.59}$$

We have further for $t \in [0,T]$

$$\|\rho(t)\|_{\beta,\Omega} \leq \kappa_\rho(\beta)$$

with

$$\kappa_\rho(\beta) := \max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\} + c_\rho \operatorname{diam}(\Omega)^{1-\beta} \tag{4.60}$$

and L_{ρ_0}, L_{ρ_A} Lipschitz constants for ρ_0 and ρ_A and c_{mv} defined in Theorem 2.20.

It follows that $\rho(t) \in C^\beta(\bar{\Omega})$ for $0 \leq \beta \leq 1$ and $t \in [0,T]$.

Proof. The β-norm is given by

$$\|\rho(t)\|_{\beta,\Omega} = \|\rho(t)\|_{0,\Omega} + |\rho(t)|_{\beta,\Omega}. \tag{4.61}$$

By (4.31), we know that $\rho(y,t) = 0$ for $y \in \Omega \backslash \Omega_t^1 \cup \Omega_t^2$. Consequently, we obtain for the sup-norm of (4.61)

$$\sup_{y\in\Omega} |\rho(y,t)| = \max\left\{\sup_{y\in\Omega_t^1} |\rho(y,t)|,\ \sup_{y\in\Omega_t^2} |\rho(y,t)|\right\}. \tag{4.62}$$

For the first term of (4.62) follows

$$\sup_{y\in\Omega_t^1} |\rho(y,t)| = \sup_{y\in\Omega_t^1} \left|\frac{\rho_0(\Phi_1^{-1}(y,t))}{1 + t\rho_0(\Phi_1^{-1}(y,t))}\right| \leq \|\rho_0\|_{0,\Omega_0}$$

and for the second term of (4.62) follows

$$\sup_{y\in\Omega_t^2} |\rho(y,t)| = \sup_{y\in\Omega_t^2} \left|\frac{\rho_A(\Phi_2^{-1}(y,t))}{1 + (t - [\Phi_2^{-1}(y,t)]_2)\rho_A(\Phi_2^{-1}(y,t))}\right| \leq \|\rho_A\|_{0,Q_t}.$$

To bound the β semi norm, we first bound the difference $|\rho(x,t) - \rho(y,t)|$. We have to distinguish eight cases that are caused on one hand by the non convexity of the domain and on the other hand by the piecewise definition of Φ. Recall that the domain Ω is composed of the three

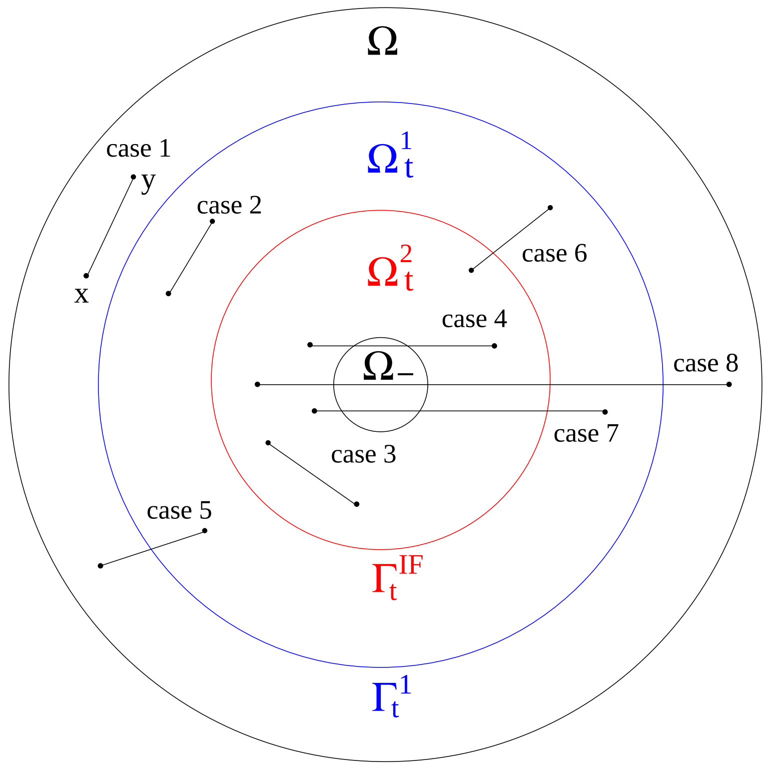


Figure 4.1: Possible configurations for Lemma 4.34

disjoint sets $\Omega = \Omega_t^1 \cup \Omega_t^2 \cup (\Omega \backslash \Omega_t)$. Moreover, it will become important, if a line segment $\overrightarrow{xy}$ of two points $x, y \in \Omega$ intersects the domain Ω_- which is disjoint to Ω. In the following, we will denote the outer boundary of Ω_t^1 by Γ_t^1.

Case 1: $x, y \in \Omega \backslash \Omega_t$ This case is trivial, as $\rho(y,t) = 0$ for $y \in \Omega \backslash \Omega_t$. This is the only case for which it is negligable whether the line segment $\overrightarrow{xy}$ intersects any other subdomains of Ω. As long as the starting and endpoint are element of $\Omega \backslash \Omega_t$, this case is applicable.

Case 2: $\overrightarrow{xy} \subset \Omega_t^1$
By Lemma 4.33a) follows

$$|\rho(x,t) - \rho(y,t)| \leq L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1} |x - y|_\infty .$$

Case 3: $\overrightarrow{xy} \subset \Omega_t^2$
By Lemma 4.33b) follows

$$|\rho(x,t) - \rho(y,t)| \leq (L_{\rho_A} + \|\rho_A\|_{0,Q_t}) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} |x - y|_\infty .$$

Case 4: $\overrightarrow{xy} \subset \Omega_t^2 \cup \Omega_-$
As Ω_- is convex, the line segment $\overrightarrow{xy}$ intersects Γ_- exactly twice. Lemma 2.20 is not immediately applicable, as $\rho(y,t)$ is not differentiable. With the Lipschitz continuity of ρ_A, we get analogously

to the proof of 4.33b)

$$\begin{aligned}|\rho(x,t)-\rho(y,t)| &= \left|\frac{\rho_A(\Phi_2^{-1}(x,t))}{1+(t-[\Phi_2^{-1}(x,t)]_2)\rho_A(\Phi_2^{-1}(x,t))} - \frac{\rho_A(\Phi_2^{-1}(y,t))}{1+(t-[\Phi_2^{-1}(y,t)]_2)\rho_A(\Phi_2^{-1}(y,t))}\right| \\ &\leq \left(L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}\right) \left|\Phi_2^{-1}(x,t) - \Phi_2^{-1}(y,t)\right|_\infty .\end{aligned}$$

Applying Lemma 2.20 to Φ_2^{-1}, we get

$$|\rho(x,t)-\rho(y,t)| \leq c_{mv} \left(L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}\right) \|\nabla\Phi_2^{-1}(t)\|_{0,\Omega_t^2} |x-y|_\infty.$$

Case 5: $\overrightarrow{xy} \subset \Omega_t^1 \cup (\Omega \backslash \Omega_t)$

Let $x \in \Omega_1^t$ and $y \in (\Omega\backslash\Omega_t)$. Since the boundary Γ_t^1 is not necessarily convex, the line segment $\overrightarrow{xy}$ might intersect Γ_t^1 more than once. Choose the intersection point a_1, such that the line segment $\overrightarrow{a_1 x} \subset \Omega_1^t$. Then we have by the triangle inequality

$$|\rho(x,t)-\rho(y,t)| \leq |\rho(x,t)-\rho(a_1,t)| + |\rho(a_1,t)-\rho(y,t)|. \tag{4.63}$$

Since $a_1 \in \Gamma_t^1$ follows due to the assumptions on ρ_0 in (CP 4.2) that $\rho(a_1,t) = 0$, and thus the second term vanishes. It is left to bound the first term of (4.63). a_1 was chosen, such that $\overrightarrow{a_1 x} \subset \Omega_1^t$. We apply case 2 and get

$$\begin{aligned}|\rho(x,t)-\rho(y,t)| &\leq |\rho(x,t)-\rho(a_1,t)| \\ &\leq L_{\rho_0} |\nabla\Phi_1^{-1}(t)|_{0,\Omega_t^1} \left|x-a_1\right|_\infty \\ &\leq L_{\rho_0} |\nabla\Phi_1^{-1}(t)|_{0,\Omega_t^1} \left|x-y\right|_\infty .\end{aligned}$$

Case 6: $\overrightarrow{xy} \subset \Omega_t^1 \cup \Omega_t^2$

Let $x \in \Omega_t^1$ and $y \in \Omega_t^2$. Since the interface Γ_t^{IF} is not necessarily convex, the line segment $\overrightarrow{xy}$ might intersect Γ_t^{IF} more than once. Denote the n points of intersection by a_i for $i = 1,..,n$, with n odd. The points are assumed to be ordered while following the line segment from x to y, i.e. $|x-a_1| \leq |x-a_2| \leq ... \leq |x-a_n|$. By adding 0 and the triangle inequality, we obtain

$$|\rho(x,t)-\rho(y,t)| \leq |\rho(x,t)-\rho(a_1,t)| + \sum_{i=1}^{n-1} |\rho(a_i,t)-\rho(a_{i+1},t)| + |\rho(a_n,t)-\rho(y,t)|$$

Due to the continuity of ρ and Φ at the interface, we obtain on one hand line segments in Ω_t^1 and on the other hand line segments in Ω_t^2. We then obtain for all indices $j \in J_1$, such that $\overrightarrow{a_j a_{j+1}} \subset \Omega_t^1$ by case 2

$$|\rho(x,t)-\rho(a_1,t)| + \sum_{j\in J_1} |\rho(a_j,t)-\rho(a_{j-1},t)| \leq L_{\rho_0} \|\nabla\Phi_1^{-1}(t)\|_{0,\Omega_t^1} \left(|x-a_1|_\infty + \sum_{j\in J_1} |a_j - a_{j+1}|_\infty \right).$$

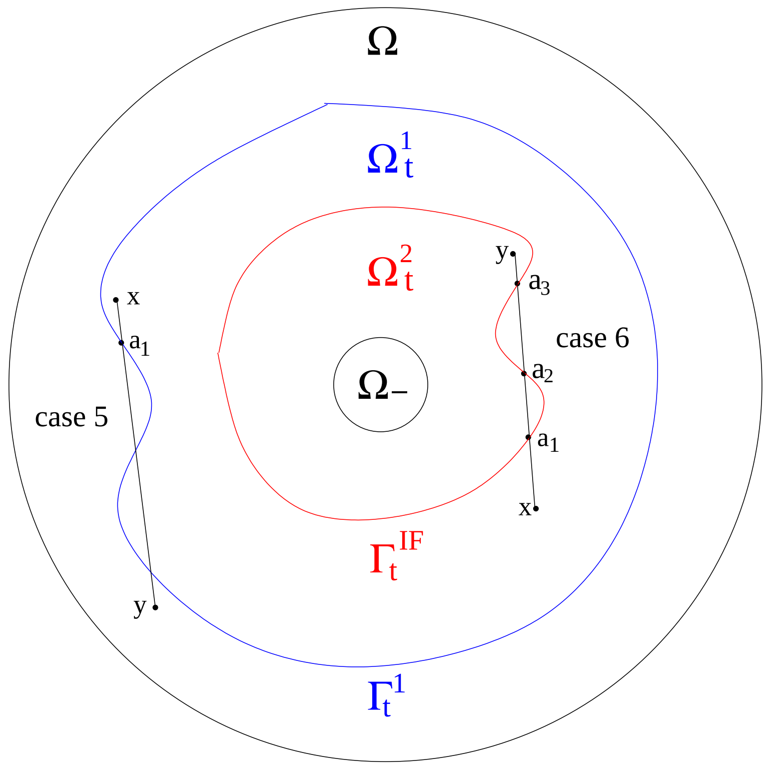


Figure 4.2: Choice of a_i for cases 5 and 6

For all those indices $j \in J_2$ with $\overrightarrow{a_j a_{j+1}} \subset \Omega_t^2$, we obtain by case 3

$$\sum_{j \in J_1} |\rho(a_j,t) - \rho(a_{j-1},t)| + |\rho(a_n,t) - \rho(y,t)|$$
$$\leq (L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} \left(\sum_{j \in J_2} |a_j - a_{j+1}|_\infty + |y - a_n|_\infty \right).$$

Since all the line segments are disjoint due to their construction, the sum of the lengths of all segments is the length of the interval (x,y). We thus get

$$|\rho(x,t) - \rho(y,t)| \leq |\rho(x,t) - \rho(a_1,t)| + \sum_{i=1}^{n-1} |\rho(a_i,t) - \rho(a_{i+1},t)| + |\rho(a_n,t) - \rho(y,t)|$$
$$\leq \max \left\{ L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}, (L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} \right\}$$
$$\cdot \left(|x - a_1|_\infty + \sum_{i=1}^{n-1} |a_i - a_{i+1}|_\infty + |a_n - y|_\infty \right)$$
$$\leq \max \left\{ L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}, (L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2} \right\} |x - y|_\infty.$$

Case 7: $\overrightarrow{xy} \subset \Omega_t^1 \cup \Omega_t^2 \cup \Omega_-$
This case is a combination of cases 4 and 6. Let us assume that $x \in \Omega_t^1$ and $y \in \Omega_t^2$. Further,

assume that there is one intersection points of $\overrightarrow{xy}$ with Ω_t^1 and call it a_1. We obtain with case 2 since $\overrightarrow{xa_1} \subset \Omega_t^1$

$$|\rho(x,t) - \rho(a_1,t)| \leq L_{\rho 0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1} |x - a_1|_\infty.$$

The line segment $\overrightarrow{a_1 y}$ is contained in $\Omega_t^2 \cup \Omega_-$ due to the assumption. By case 4 follows

$$|\rho(a_1,t) - \rho(y,t)| \leq c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega} |a_1 - y|_\infty.$$

Thus, we obtain

$$|\rho(x,t) - \rho(y,t)| \leq \max\left\{c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2}, L_{\rho 0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} |x - y|_\infty.$$

This case includes the combination of $x \in \Omega_t^1$ and $y \in \Omega_t^1$ with $\overrightarrow{xy}$ intersecting $\Omega_t^2 \cup \Omega_-$. Further, the case of multiple intersections of the interface Γ_t^{IF} is treated as in case 4 and leads to the same bound.

Case 8: $\overrightarrow{xy} \subset \Omega \cup \Omega_-$
In this last case, we consider a line segment intersecting all the inner boundaries. Let $x \in \Omega \backslash \Omega_t$, $y \in \Omega_t^2$. We assume that Γ_t^1 is intersected once by $\overrightarrow{xy}$ in a_1. Then the line segment $\overrightarrow{xa_1}$ is contained in $\Omega \backslash \Omega_t$. It follows due to case 1

$$|\rho(x,t) - \rho(a_1,t)| = 0.$$

The line segment $\overrightarrow{a_1 y}$ is subset of $\Omega_t \cup \Omega_-$. It is thus bounded with case 7 by

$$\begin{aligned}|\rho(a_1,t) - \rho(y,t)| &\leq \max\left\{c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2}, L_{\rho 0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} |a_1 - y|_\infty \\ &\leq \max\left\{c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega_t^2}, L_{\rho 0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} |x - y|_\infty.\end{aligned}$$

Case 8 includes those combinations of line segments with either start or endpoint in $\Omega \backslash \Omega_t$. Further, the bound is still valid for multiple intersections of $\overrightarrow{xy}$ with Γ_t^1 and Γ_t^{IF}.

Combining all the 8 cases, we obtain for any $x, y \in \Omega$ and $\Phi \in W(M,T,K,\delta)$

$$\begin{aligned}|\rho(x,t) - \rho(y,t)| &\leq \max\left\{c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \|\nabla \Phi_2^{-1}(t)\|_{0,\Omega}, L_{\rho 0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} |x - y|_\infty \\ &\leq \max\left\{4c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \frac{M}{\delta}, 2L_{\rho_0}\right\} |x - y|_\infty.\end{aligned}$$

Hence, we have for the β-norm (4.61) for all $t \in [0,T]$

$$\begin{aligned}\|\rho(t)\|_{\beta,\Omega} &= \sup_{x,y\in\Omega} \frac{|\rho(x,t) - \rho(y,t)|}{|x - y|_\infty^\beta} \\ &\leq \max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\right\} + \max\left\{4c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \frac{M}{\delta}, 2L_{\rho_0}\right\} \sup_{x,y\in\Omega} |x - y|_\infty^{1-\beta} \\ &\leq \max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\right\} + \max\left\{4c_{mv} \left(L_{\rho_A} + \|\rho_A\|_{0,Q_t}^2\right) \frac{M}{\delta}, 2L_{\rho_0}\right\} \operatorname{diam}(\Omega)^{1-\beta}.\end{aligned}$$

□

Next, we demonstrate that $\rho(x,\cdot) \in C^\alpha([0,T])$ for a fixed $x \in \Omega$. We will encounter the same technical difficulties in the proof as in Lemma 4.34. We will use the same idea and find intermediate points on the transition boundary whenever two points x, y are not in the same connected set. First we still need a result on the α-Hölder continuity of Φ_1^{-1} and Φ_2^{-1} in time.

Lemma 4.35. *Let Ω be a $C^{2,\alpha}$ domain and $\Phi \in W(M,T,K,\delta)$ and $t_1, t_2 \in [0,T]$.*
a) *Let $y \in \Omega^1_{t_1}, z \in \Omega^1_{t_2}$ and $z_1 = \Phi_1(\Phi_1^{-1}(y,t_1),t_2) \in \Omega^1_{t_2}$. If the line segment $\overrightarrow{z_1 z} \subset \Omega^1_{t_2}$, then follows*

$$\left|\Phi_1^{-1}(y,t_1) - \Phi_1^{-1}(z,t_2)\right|_\infty \le 2|z_1 - z|_\infty. \tag{4.64}$$

b) *Let $t_1 \le t_2$, $y \in \Omega^2_{t_1}, z \in \Omega^2_{t_2}$ and let $z_2 = \Phi_2(\Phi_2^{-1}(y,t_1),t_2) \in \Omega^2_{t_2}$. If the line segment $\overrightarrow{z_2 z} \subset \Omega^2_{t_2}$, then follows*

$$\left|\Phi_2^{-1}(y,t_1) - \Phi_1^{-1}(z,t_2)\right|_\infty \le \frac{4M}{\delta}|z_2 - z|_\infty. \tag{4.65}$$

Proof. **a)** Let $y \in \Omega^1_{t_1}, z \in \Omega^1_{t_2}$. As $\Phi_1^{-1}(y,t_1) \in \Omega_0$, the mapping $\Phi_1(\Phi_1^{-1}(y,t_1),t_2) =: z_1 \in \Omega^1_{t_2}$ is well-defined. We get by the identity $\Phi_1^{-1}(\Phi_1(x,t_2),t_2) = x$

$$\begin{aligned}\left|\Phi_1^{-1}(y,t_1) - \Phi_1^{-1}(z,t_2)\right|_\infty &= \left|\Phi_1^{-1}(\Phi_1(\Phi_1^{-1}(y,t_1),t_2),t_2) - \Phi_1^{-1}(z,t_2)\right|_\infty \\ &= \left|\Phi_1^{-1}(z_1,t_2) - \Phi_1^{-1}(z,t_2)\right|_\infty\end{aligned}$$

Since $\overrightarrow{z_1 z} \subset \Omega^1_{t_2}$, we may apply the mean value theorem (Theorem 2.15). With the bound $\|\nabla\Phi_1^{-1}(t)\|_{0,\Omega} \le 2$ for $\Phi_1 \in W_1(M,T)$, we obtain

$$\begin{aligned}\left|\Phi_1^{-1}(z_1,t_2) - \Phi_1^{-1}(z,t_2)\right|_\infty &\le \left\|\nabla\Phi_1^{-1}(t)\right\|_{0,\Omega_{t_2}} |z_1 - z|_\infty \\ &\le 2|z_1 - z|_\infty.\end{aligned}$$

The assertion follows.

b) For $y \in \Omega^2_{t_1} \cap \Omega^2_{t_2}$, we proceed analogously. The mapping $\Phi_2(\Phi_2^{-1}(y,t_1),t_2) =: z_2 \in \Omega^2_{t_2}$ is well-defined since $[0,t_1] \subset [0,t_2]$ and we get with the identity $\Phi_2^{-1}(\Phi_2(s,t_x,t_2),t_2) = (s,t_x)$

$$\begin{aligned}\left|\Phi_2^{-1}(y,t_1) - \Phi_2^{-1}(z,t_2)\right|_\infty &= \left|\Phi_2^{-1}(\Phi_2(\Phi_2^{-1}(y,t_1),t_2),t_2) - \Phi_2^{-1}(z,t_2)\right|_\infty \\ &= \left|\Phi_2^{-1}(z_2,t_2) - \Phi_2^{-1}(z,t_2)\right|_\infty.\end{aligned}$$

As the line segment $\overrightarrow{z z_2} \subset \Omega^2_{t_2}$, we may apply the mean value theorem (Theorem 2.15). With the bound for $\left\|\nabla\Phi_2^{-1}(t)\right\|_{0,\Omega^2_{t_2}} \le \frac{4M}{\delta}$ given in the space $W_2(M,T,K,\delta)$, we obtain

$$\begin{aligned}|\Phi_2^{-1}(z_2,t_2) - \Phi_2^{-1}(z,t_2)|_\infty &\le \left\|\nabla\Phi_2^{-1}(t)\right\|_{0,\Omega^2_{t_2}} |z_2 - z|_\infty \\ &\le \frac{4M}{\delta}|z_2 - z|_\infty.\end{aligned}$$

The assertion follows. □

We can now prove that $\rho(x,\cdot) \in C^\alpha([0,T])$.

Lemma 4.36. *Let Ω be a $C^{2,\alpha}$ domain, ρ be defined in (4.31) and $\Phi \in W(M, T, K, \delta)$. Then follows for $t_1, t_2 \in [0, T]$ with $T < 1$ and $y \in \Omega$*

$$|\rho(y, t_2) - \rho(y, t_1)| \leq 2 \max \left\{ c_\rho M, \|\rho_0\|^2_{0,\Omega_0}, \|\rho_A\|^2_{0,Q_T} \right\} |t_1 - t_2|^\alpha.$$

with c_ρ defined in Lemma 4.34.

Moreover follows $\rho \in C^\alpha([0, T])$.

Proof. We conduct a case analysis on y. Without loss of generality, let $t_1 < t_2$.

Case 1: $y \in \Omega^1_{t_1} \cap \Omega^1_{t_2}$
Denote $\Phi_1(\Phi_1^{-1}(y, t_1), t_2) =: z \in \Omega^1_{t_2}$. If the line segment $\overrightarrow{zy}$ intersects any boundary, then find the point of intersection a_1, such that the line segment $\overrightarrow{za_1}$ is fully contained in $\Omega^1_{t_2}$. We get by the triangle inequality

$$\begin{aligned} |\rho(y, t_1) - \rho(y, t_2)| &= |\rho(y, t_1) - \rho(a_1, t_2) + \rho(a_1, t_2) - \rho(y, t_2)| \\ &\leq |\rho(y, t_1) - \rho(a_1, t_2)| + |\rho(a_1, t_2) - \rho(y, t_2)|. \end{aligned} \tag{4.66}$$

In the second term of (4.66), ρ is evaluated at two points $a_1, y \in \Omega_{t_2}$ for a fixed time $t_2 \in [0, T]$. We can thus apply Lemma 4.34 and get

$$|\rho(a_1, t_2) - \rho(y, t_2)| \leq c_\rho |a_1 - y|_\infty.$$

For the first term of (4.66), we obtain with (4.31) and the Lipschitz continuity of ρ_0

$$\begin{aligned} |\rho(y, t_1) - \rho(a_1, t_2)| &= \left| \frac{\rho_0(\Phi_1^{-1}(y, t_1))}{1 + t_1 \rho_0(\Phi^{-1}(y, t_1))} - \frac{\rho_0(\Phi_1^{-1}(a_1, t_2))}{1 + t_2 \rho_0(\Phi^{-1}(a_1, t_2))} \right| \\ &= \left| \frac{\rho_0(\Phi_1^{-1}(y, t_1)) - \rho_0(\Phi_1^{-1}(a_1, t_2)) + (t_1 - t_2)\rho_0(\Phi_1^{-1}(y, t_1))\rho_0(\Phi_1^{-1}(a_1, t_2))}{(1 + t_1 \rho_0(\Phi^{-1}(y, t_1)))(1 + t_2 \rho_0(\Phi^{-1}(a_1, t_2)))} \right| \\ &\leq \left| \rho_0(\Phi_1^{-1}(y, t_1)) - \rho_0(\Phi_1^{-1}(a_1, t_2)) \right| + |t_1 - t_2| \left| \rho_0(\Phi^{-1}(y, t_1)) \right| \left| \rho_0(\Phi^{-1}(a_1, t_2) \right| \\ &\leq L_{\rho_0} \left| \Phi_1^{-1}(y, t_1) - \Phi_1^{-1}(a_1, t_2) \right|_\infty + |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0}. \end{aligned} \tag{4.67}$$

As $a_1 \in \Omega^1_{t_2}$, $y \in \Omega^1_{t_1}$, $z \in \Omega^1_{t_2}$ and $\overrightarrow{za_1} \subset \Omega^1_{t_2}$, we obtain with Lemma 4.35a)

$$\left| \Phi_1^{-1}(a_1, t_2) - \Phi_1^{-1}(y, t_1) \right|_\infty \leq 2 |a_1 - z|_\infty.$$

Hence, we get for (4.66)

$$\begin{aligned} |\rho(y, t_1) - \rho(y, t_2)| &\leq c_\rho |a_1 - y|_\infty + 2 L_{\rho_0} |a_1 - z|_\infty + |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} \\ &\leq c_\rho |z - y|_\infty + |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} \end{aligned}$$

By definition of z and the identity $y = \Phi_1(\Phi_1^{-1}(y, t_1), t_1)$, we obtain with the α-Hölder continuity of Φ_1 in time

$$|z - y|_\infty = |\Phi_1(\Phi_1^{-1}(y, t_1), t_2) - \Phi_1(\Phi_1^{-1}(y, t_1), t_1)|_\infty \leq M |t_1 - t_2|^\alpha.$$

Eventually, we have for (4.66) as $t_1, t_2 < 1$

$$\begin{aligned}|\rho(y,t_1)-\rho(y,t_2)| &\le c_\rho M|t_1-t_2|^\alpha + |t_1-t_2|\,\|\rho_0\|^2_{0,\Omega_0}\\ &\le \max\left\{c_\rho M, \|\rho_0\|^2_{0,\Omega_0}\right\}|t_1-t_2|^\alpha.\end{aligned}$$

Case 2: $y \in \Omega^2_{t_1} \cap \Omega^2_{t_2}$
Since $[0,t_1] \subset [0,t_2]$, $z := \Phi_2(\Phi_2^{-1}(y,t_1),t_2)$ is well defined. If the line segment $\overrightarrow{zy}$ intersects any boundary, then find the intersection point a_1, such that $\overrightarrow{a_1 z} \subset \Omega^2_{t_2}$. By the triangle inequality, we get

$$|\rho(y,t_1)-\rho(y,t_2)| \le |\rho(y,t_1)-\rho(a_1,t_2)| + |\rho(a_1,t_2)-\rho(y,t_2)|\,. \tag{4.68}$$

We apply Lemma 4.34 to the second term of (4.68) and get

$$|\rho(a_1,t_2)-\rho(y,t_2)| \le c_\rho|a_1-y|_\infty. \tag{4.69}$$

For the first term of (4.68), we obtain by (4.31), the Lipschitz continuity of ρ_A and Lemma 4.35b)

$$\begin{aligned}&|\rho(y,t_1)-\rho(a_1,t_2)|\\ &\quad= \left|\frac{\rho_A(\Phi_2^{-1}(y,t_1))}{1+(t_1-[\Phi_2^{-1}(y,t_1)]_2)\rho_A(\Phi_2^{-1}(y,t_1))} - \frac{\rho_A(\Phi_2^{-1}(a_1,t_2))}{1+(t_2-[\Phi_2^{-1}(a_1,t_2)]_2)\rho_A(\Phi_2^{-1}(a_1,t_2))}\right|\\ &\quad\le \left|\frac{\rho_A(\Phi_2^{-1}(y,t_1))-\rho_A(\Phi_2^{-1}(a_1,t_2))+(t_1-t_2)\rho_A(\Phi_2^{-1}(y,t_1)\rho_A(\Phi_2^{-1}(a_1,t_2)}{(1+t_1\rho_A(\Phi^{-1}(y,t_1)))(1+t_2\rho_A(\Phi^{-1}(a_1,t_2)))}\right|\\ &\qquad+\left|\frac{\left([\Phi_2^{-1}(a_1,t_2)]_2-[\Phi_2^{-1}(y,t_1)]_2\right)\rho_A(\Phi_2^{-1}(y,t_1)\rho_A(\Phi_2^{-1}(a_1,t_2)}{(1+t_1\rho_A(\Phi_2^{-1}(y,t_1)))(1+t_2\rho_A(\Phi_2^{-1}(a_1,t_2)))}\right|\\ &\quad\le \left|\rho_A(\Phi_2^{-1}(y,t_1))-\rho_A(\Phi_2^{-1}(a_1,t_2))\right| + |t_1-t_2|\left|\rho_A(\Phi_2^{-1}(y,t_1))\right|\left|\rho_A(\Phi_2^{-1}(a_1,t_2))\right|\\ &\qquad+\left|[\Phi_2^{-1}(a_1,t_2)]_2-[\Phi_2^{-1}(y,t_1)]_2\right|\left|\rho_A(\Phi_2^{-1}(y,t_1))\right|\left|\rho_A(\Phi_2^{-1}(a_1,t_2))\right|\\ &\quad\le (L_{\rho_A}+\|\rho_A\|^2_{0,Q_T})\left|\Phi_2^{-1}(y,t_1)-\Phi_2^{-1}(a_1,t_2)\right|_\infty + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\\ &\quad\le \frac{4M}{\delta}(L_{\rho_A}+\|\rho_A\|^2_{0,Q_T})\,|z-a_1|_\infty + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\,.\end{aligned}$$

We obtain for (4.68) by the definition of z, the identity $y = \Phi_2(\Phi_2^{-1}(y,t_1),t_1)$ and the α-Hölder continuity of Φ_2 in time

$$\begin{aligned}|\rho(y,t_1)-\rho(y,t_2)| &\le c_\rho|a_1-y|_\infty + \frac{4M}{\delta}(L_{\rho_A}+\|\rho_A\|^2_{0,Q_T})\,|z-a_1| + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\\ &\le c_\rho|z-y|_\infty + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\\ &= c_\rho|\Phi_2(\Phi_2^{-1}(y,t_1),t_2)-\Phi_2(\Phi_2^{-1}(y,t_1),t_1)|_\infty + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\\ &\le c_\rho M|t_1-t_2|^\alpha + |t_1-t_2|\,\|\rho_A\|^2_{0,Q_T}\\ &\le \max\left\{c_\rho M, \|\rho_A\|^2_{0,Q_T}\right\}|t_1-t_2|^\alpha.\end{aligned}$$

The last step follows as $t_1, t_2 < 1$.

Case 3: $y \in \Omega^1_{t_1} \cap \Omega^2_{t_2}$
The technique is the same as in the previous cases. The mapping $z := \Phi_1(\Phi_1^{-1}(y,t_1),t_2) \in \Omega^1_{t_2}$ is well defined. The line segment $\overrightarrow{yz}$ intersects the boundary $\Gamma^{IF}_{t_2}$ at least once. Choose the intersection point a_1 such that $\overrightarrow{a_1 z} \subset \Omega^1_{t_2}$. Hence,

$$|\rho(y,t_1) - \rho(y,t_2)| \leq |\rho(y,t_1) - \rho(a_1,t_2)| + |\rho(a_1,t_2) - \rho(y,t_2)| . \tag{4.70}$$

The second term of (4.70) is bounded with Lemma 4.34a). For the first term, since both points are element of $\Omega^1_{t_2}$, we apply case 1 starting from (4.67) and get

$$|\rho(y,t_1) - \rho(a_1,t_2)| \leq 2L_{\rho_0} |a_1 - z|_\infty + |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} .$$

Since a_1 was chosen as intermediate point on the line segment $\overrightarrow{xy}$, the sum of the length of the subintervals are the length of the interval, i.e. $|y - a_1|_\infty + |a_1 - z|_\infty = |y - z|_\infty$. Summing up, we obtain for (4.70) by the identity $\Phi_1(\Phi^{-1}(y,t_1),t_1) = y$ and the Hölder continuity in time of Φ_1

$$\begin{aligned}
|\rho(y,t_1) - \rho(y,t_2)| &\leq 2L_\rho |a_1 - z|_\infty + |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} + c_\rho |a_1 - y|_\infty \\
&\leq |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} + c_\rho |z - y|_\infty \\
&= |t_1 - t_2| \, \|\rho_0\|^2_{0,\Omega_0} + c_\rho |\Phi_1(\Phi_1^{-1}(y,t_1),t_2) - \Phi_1(\Phi^{-1}(y,t_1),t_1)|_\infty \\
&\leq \max \left\{ c_\rho M, \|\rho_0\|^2_{0,\Omega_0} \right\} |t_1 - t_2|^\alpha_\infty.
\end{aligned}$$

Case 4: $y \in \Omega^1_{t_2} \cap \Omega^2_{t_1}$
We set $z = \Phi_1(\Phi_1^{-1}(y,t_2),t_1)$ which is defined as the domain of definition of Φ_1 is Ω_0. We now conduct the computations analogously to case 3 leading to the result

$$|\rho(y,t_1) - \rho(y,t_2)| \leq \max \left\{ c_\rho, \|\rho_0\|^2_{0,\Omega_0} \right\} |t_1 - t_2|^\alpha_\infty.$$

Case 5: $y \in \Omega^1_{t_2} \cap (\Omega \backslash \Omega_{t_1})$
The mapping $\Phi_1(\Phi_1^{-1}(y,t_2),t_1) =: z \in \Omega^1_{t_1}$ is well defined. The line segment $\overrightarrow{zy}$ intersects the boundary $\Gamma^1_{t_1}$ at least once. Choose the intersection point a_1, such that $\overrightarrow{za_1} \subset \Omega^1_{t_1}$. Due to the compact support of ρ follows $\rho(a_1,t_1) = \rho(y,t_1) = 0$. We get

$$|\rho(y,t_2) - \rho(y,t_1)| = |\rho(y,t_2) - 0| = |\rho(y,t_2) - \rho(a_1,t_1)| .$$

Starting from (4.67) in case 1, we get

$$\begin{aligned}
|\rho(y,t_2) - \rho(y,t_1)| &= |\rho(y,t_2) - \rho(a_1,t_1)| \\
&\leq 2L_{\rho_0} M \, |t_1 - t_2|^\alpha + \|\rho_0\|^2_{0,\Omega_0} |t_1 - t_2| \\
&\leq \max \left\{ 2L_{\rho_0} M, \|\rho_0\|^2_{0,\Omega_0} \right\} |t_1 - t_2|^\alpha .
\end{aligned}$$

Case 6: $y \in \Omega^2_{t_2} \cap \Omega \backslash \Omega_{t_1}$
This case is more complicated than the previous ones. Since $t_1 \leq t_2$, the point $\Phi_2^{-1}(y,t_2)$ is not necessarily in the domain of definition of $\Phi_2(x,t_1)$. We therefore cannot apply the same

technique as before. However, due to the continuity of Φ_2 in time, there must exist a time $\tilde{t}$ with $t_1 \leq \tilde{t} \leq t_2$ such that y lays on the interface between $\Omega_1^{\tilde{t}}$ and $\Omega_2^{\tilde{t}}$, i.e.

$$y \in \Gamma_{\tilde{t}}^{IF}.$$

We now use the time $\tilde{t}$ and obtain

$$|\rho(y,t_1) - \rho(y,t_2)| \leq |\rho(y,t_1) - \rho(y,\tilde{t})| + |\rho(y,\tilde{t}) - \rho(y,t_2)|. \tag{4.71}$$

For the first term of (4.71) follows $y \in \Omega_{\tilde{t}}^1 \cap \Omega\backslash\Omega_{t_1}$. This is analogous to case 5 and we and obtain

$$\begin{aligned}|\rho(y,t_1) - \rho(y,\tilde{t})| &\leq \max\left\{2L_{\rho_0}M, \|\rho_0\|_{0,\Omega_0}^2\right\} |\tilde{t} - t_1|^\alpha \\ &\leq \max\left\{2L_{\rho_0}M, \|\rho_0\|_{0,\Omega_0}^2\right\} |t_2 - t_1|^\alpha.\end{aligned}$$

For the second term of (4.71) follows $y \in \Omega_{t_2}^2 \cap \Omega_{\tilde{t}}^2$. Since $\tilde{t} \leq t_2$, we may apply case 2 and obtain

$$\begin{aligned}|\rho(y,\tilde{t}) - \rho(y,t_2)| &\leq \max\left\{c_\rho M, \|\rho_A\|_{0,Q_T}^2\right\} |t_2 - \tilde{t}|^\alpha \\ &\leq \max\left\{c_\rho M, \|\rho_A\|_{0,Q_T}^2\right\} |t_2 - t_1|^\alpha.\end{aligned}$$

We thus obtain for (4.71)

$$|\rho(y,t_1) - \rho(y,t_2)| \leq 2\max\left\{c_\rho M, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_T}^2\right\} |t_1 - t_2|^\alpha. \tag{4.72}$$

Case 7: $y \in \Omega_{t_1}^1 \cap \Omega\backslash\Omega_{t_2}$
This case is dealt with analogously to case 5. The mapping $\Phi_1(\Phi_1^{-1}(y,t_1),t_2) =: z$ is well-defined. The line segment $\overrightarrow{zy}$ intersects the boundary $\partial\Omega_{t_2}^1$ at least once. Choose one intersection point $a_1 \in \Gamma_{t_1}^1$. We have

$$|\rho(y,t_2) - \rho(y,t_1)| = |0 - \rho(y,t_1)| = |\rho(a_1,t_2) - \rho(y,t_1)|\,.$$

Starting from (4.67) in case 1, we get

$$\begin{aligned}|\rho(y,t_2) - \rho(y,t_1)| &= |\rho(a_1,t_2) - \rho(y,t_1)| \\ &\leq 2L_{\rho_0}M\,|t_1 - t_2|^\alpha + \|\rho_0\|_{0,\Omega_0}^2 |t_1 - t_2| \\ &\leq \max\left\{2L_{\rho_0}M, \|\rho_0\|_{0,\Omega_0}^2\right\} |t_1 - t_2|^\alpha\,.\end{aligned}$$

Case 8: $y \in \Omega_{t_1}^2 \cap \Omega\backslash\Omega_{t_2}$
Since $t_1 \leq t_2$ and thus $[0,t_1] \subset [0,t_2]$, we may set $\Phi_2(\Phi_2^{-1}(y,t_1),t_2) =: z \in \Omega_{t_2}^2$. The line segment $\overrightarrow{yz}$ intersects the boundary $\Gamma_{t_2}^1$ at least once. Chose a point a_1 such that $\overrightarrow{a_1y} \subset \Omega_{t_1}^2$. We have

$$|\rho(y,t_2) - \rho(y,t_1)| = |0 - \rho(y,t_1)| = |\rho(a_1,t_2) - \rho(y,t_1)|.$$

We now have to do a second step. The line segment $\overrightarrow{a_1y}$ intersects the interface $\Gamma_{t_2}^{IF}$ at least once. Chose $a_2 \in \Gamma_{t_2}^{IF}$ such that $za_2 \subset \Omega_{t_2}^2$. Hence,

$$|\rho(a_1,t_2) - \rho(y,t_1)| \leq |\rho(a_1,t_2) - \rho(a_2,t_2)| + |\rho(a_2,t_2) - \rho(y,t_1)| \tag{4.73}$$

The first term of (4.73) is bounded with Lemma 4.34

$$|\rho(a_1, t_2) - \rho(a_2, t_2)| \leq c_\rho |a_1 - a_2|_\infty.$$

We use the computations of case 2 to bound the second term of (4.73) and obtain

$$|\rho(a_2, t_2) - \rho(y, t_1)| \leq \frac{4M}{\delta}(L_{\rho_A} + \|\rho_A\|^2_{0,Q_T})|a_2 - z|_\infty.$$

Summing up and using the identity, we obtain

$$\begin{aligned}
|\rho(y, t_2) - \rho(y, t_1)| &\leq c_\rho |a_2 - a_1|_\infty + \frac{4M}{\delta}(L_{\rho_A} + \|\rho_A\|^2_{0,Q_T})|a_2 - z|_\infty \\
&\leq c_\rho |z - a_1|_\infty \\
&\leq c_\rho |z - y|_\infty \\
&= c_\rho |\Phi_2(\Phi_2^{-1}(y, t_1), t_2) - \Phi_2(\Phi_2^{-1}(y, t_1), t_1)|_\infty \\
&\leq c_\rho M |t_1 - t_2|^\alpha.
\end{aligned}$$

Let us now summarize all eight cases. We obtain for all t_1, t_2 and $y \in \Omega$

$$|\rho(y, t_2) - \rho(y, t_1)| \leq 2 \max\left\{c_\rho M, \|\rho_0\|^2_{0,\Omega_0}, \|\rho_A\|^2_{0,Q_T}\right\} |t_1 - t_2|^\alpha.$$

□

4.6 A is a Selfmap

With the previous estimates for ρ, we may now proceed to prove that the operator

$$A = (A_1,\ A_2)$$

is a selfmap on the set $W(M, T, K, \delta)$ defined in (4.55). The following collection of Lemmas checks whether $A_1(\Phi)$ and $A_2(\Phi)$ fulfil the restrictions in $W_1(M, T)$ and $W_2(M, T, K, \delta)$ whenever the argument Φ is taken in $W(M, T, K, \delta)$. For a specific choice of M, T and K, we prove in Theorem 4.43 that the operator A maps the set $W(M, T, K, \delta)$ into itself. In comparison to Huang and Svobodny [44] who use potential theoretic estimates to bound $A_1(\Phi)$, we make use of the mapping properties obtained for the operator G_1 in Lemma 4.15 to bound $A_1(\Phi)$ and $A_2(\Phi)$.

Lemma 4.37. *Let Ω be a $C^{2,\alpha}$ domain with $0 < \alpha < 1$ and let $\Phi \in W(M, T, K, \delta)$. Let $u_A \in C^{2,\alpha}(\Gamma)$ and ρ be defined in (4.31). Then we have*
a) *for the operator A_1*

$$\sup_{0 \leq t \leq T} \|A_1(\Phi)(t)\|_{0,\Omega_0} \leq \|x\|_{0,\Omega_0} + T c_S(\Omega, \alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right). \tag{4.74}$$

b) *for the operator A_2*

$$\sup_{0 \leq t \leq T} \|A_2(\Phi)(t)\|_{0,Q_t} \leq \|\varphi\|_{0,I_{\Gamma_-}} + T c_S(\Omega, \alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right). \tag{4.75}$$

with $\kappa_\rho(\alpha)$ defined in (4.60) and $c_S(\Omega, \alpha)$ in Lemma 4.15.

Proof. First, we note that by Lemma 4.34 follows $\rho(\cdot,t) \in C^{\alpha}(\bar{\Omega})$.

a) We prove the assertion by applying Schauder's estimates in Theorem 4.14. By Lemma 4.15 and Lemma 4.34, we obtain

$$
\begin{aligned}
\|A_1(\Phi)(t)\|_{0,\Omega_0} &= \sup_{x\in\Omega_0} \left| x - \int_0^t \int_\Omega \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_1(x,\mu))\,d\mu \right|_\infty \\
&\le \|x\|_{0,\Omega_0} + \int_0^t \sup_{x\in\Omega_0} \left| \int_\Omega \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy \right|_\infty + \sup_{x\in\Omega_0} |E_0(\Phi_1(x,\mu))|_\infty \,d\mu \\
&\le \|x\|_{0,\Omega_0} + \int_0^t \sup_{x\in\Omega} \left| \int_\Omega \nabla G(x,y)\rho(y,\mu)\,dy \right|_\infty + \|E_0\|_{0,\Omega}\,d\mu \\
&\le \|x\|_{0,\Omega_0} + \int_0^t c_S(\Omega,\alpha)\,\|\rho(\mu)\|_{\alpha,\Omega} + c_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha;\Gamma}\,d\mu \\
&\le \|x\|_{0,\Omega_0} + Tc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}
$$

b) For (4.75), we obtain in a similar manner by Lemma 4.15 and Lemma 4.34

$$
\begin{aligned}
\|A_2(\Phi)(t)\|_{0,Q_t} &= \sup_{(s,t_x)\in Q_t} \left| \varphi(s) - \int_{t_x}^t \int_\Omega \nabla G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_2(s,t_x,\mu))\,d\mu \right|_\infty \\
&\le \|\varphi\|_{0,I_{\Gamma_-}} + \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t \left| \int_\Omega \nabla G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy \right|_\infty + |E_0(\Phi_2(s,t_x,\mu))|_\infty\,d\mu \right|_\infty \\
&\le \|\varphi\|_{0,I_{\Gamma_-}} + \int_0^t \sup_{x\in\Omega} \left| \int_\Omega \nabla G(x,y)\rho(y,\mu)\,dy \right|_\infty + |E_0|_\infty\,d\mu \\
&\le \|\varphi\|_{0,I_{\Gamma_-}} + \int_0^t c_S(\Omega,\alpha)\,\|\rho(t)\|_{\alpha,\Omega} + c_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha;\Gamma}\,d\mu \\
&\le \|\varphi\|_{0,I_{\Gamma_-}} + Tc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}
$$

□

In the next Lemma, we bound $\nabla A(\Phi)$ in the sup-norm.

Lemma 4.38. *Let Ω be a $C^{2,\alpha}$ domain, ρ be defined in (4.31), $u_A \in C^{2,\alpha}(\Gamma)$ and let $\Phi \in W(M,T,K,\delta)$. Then follows*
a) *for the operator A_1*

$$\sup_{0\le t\le T} \|\nabla A_1(\Phi)(t)\|_{0,\Omega_0} \le 1 + TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).$$

b) *for the operator A_2*

$$
\begin{aligned}
\sup_{0\le t\le T} \|\nabla A_2(\Phi)(t)\|_{0,Q_t} \le{}& 1 + c(\operatorname{diam}(\Omega)) \max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\
&+ TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right)
\end{aligned}
$$

with $\kappa_\rho(\alpha)$ defined in (4.60) and $c_S(\Omega,\alpha)$ in Lemma 4.15.

Proof. First, we note that Lemma 4.34 yields $\rho(\cdot,t) \in C^\alpha(\bar\Omega)$.

a) The gradient of $A(\Phi)$ is given in Lemma 4.22a). We obtain an upper bound by Lemma 4.15 and use that $|\nabla\Phi_1|_{0,\Omega_0} \le M$ since $\Phi_1 \in W_1(M,T)$,

$$\begin{aligned}
\|\nabla A_1(\Phi)(t)\|_{0,\Omega_0} &\le \|I\|_{0,\Omega} + \sup_{x\in\Omega_0}\left|\int_0^t \nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy\,\nabla\Phi_1(x,\mu)\,d\mu\right|_\infty \\
&\quad + \sup_{x\in\Omega_0}\left|\int_0^t \nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu.\right|_\infty \\
&\le 1 + \int_0^t \sup_{x\in\Omega_0}\left|\nabla_\Phi\int_\Omega \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy\right|_\infty \|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\,d\mu \\
&\quad + \int_0^t \|\nabla E_0(\Phi_1(\mu))\|_{0,\Omega_0}\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\,d\mu. \\
&\le 1 + \int_0^t c_S(\Omega,\alpha)\|\rho(t)\|_{\alpha,\Omega}\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\,d\mu + \int_0^t c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\,d\mu \\
&\le 1 + TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

b) With Lemma 4.22b), we have the gradient

$$\begin{aligned}
\nabla A_2(\Phi)(s,t_x,t) = D_A(s,t_x) &- \int_{t_x}^t \nabla_\Phi\int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\nabla\Phi_2(s,t_x,\mu) \\
&+ \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu
\end{aligned} \tag{4.76}$$

where

$$D_A(s,t_x) = \begin{pmatrix} [\varphi'(s)]_1 & \left[\int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\,dy - E_0(\varphi(s))\right]_1 \\ [\varphi'(s)]_2 & \left[\int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\,dy - E_0(\varphi(s))\right]_2 \end{pmatrix}$$

Let us first bound every component of the matrix D_A. We have pointwise for $i=1,2$ due to the arc length parametrization and the equivalence of the supremum and maximums norm

$$|[\varphi'(s)]_i| \le |\varphi'(s)|_\infty \le |\varphi'(s)|_2 = 1.$$

As $\varphi(s)$ is the parametrization of the inflow boundary, the remaining components are evaluated on Γ_-. This is allowed due to Lemma 4.15, as u_1 and u_0 and the derivatives are continuously extendable up to the boundary. By [6, p. 30] follows

$$\max_{x\in\bar\Omega_0}|u(x,t)|_\infty = \sup_{x\in\Omega_0}|u(x,t)|_\infty.$$

Hence,

$$\sup_{(s,t_x)\in Q_t}\left|\int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\,dy + E_0(\varphi(s))\right|_\infty \le \sup_{0\le t\le T}\sup_{x\in\Omega}\left|\int_\Omega \nabla G(x,y)\rho(y,t)\,dy + E_0(x)\right|_\infty.$$

By Lemma 4.15, Lemma 4.16 and the sup-norm estimate of Lemma 4.34, we get pointwise for every $x \in \Omega$

$$\begin{aligned}\left|\int_\Omega \nabla G(x,y)\rho(y,t)\,dy + E_0(x)\right|_\infty &\le \int_\Omega |\nabla G(x,y)|_\infty\,|\rho(y,t)|\;dy + |E_0(x)|_\infty \\ &\le c(\mathrm{diam}(\Omega))\,\|\rho(t)\|_{0,\Omega} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\ &\le c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}.\end{aligned}$$

Eventually, we obtain for (4.76) with Lemma 4.15 and Lemma 4.34

$$\begin{aligned}&\|A_2(\Phi)(t)\|_{0,Q_t} \le \|D_A\|_{0,Q_t} + \sup_{(s,t_x)\in Q_t}\left|\int_{t_x}^t \nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\;\nabla\Phi_2(s,t_x,\mu)\right|_\infty \\ &\quad + \sup_{(s,t_x)\in Q_t}\left|\int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu\right|_\infty \\ &\le 1 + c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\ &\quad + \int_0^t \sup_{(s,t_x)\in Q_t}\left|\nabla_\Phi \int_\Omega \nabla_\Phi G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\right|_\infty \|\nabla\Phi_2(\mu)\|_{0,Q_t}\;d\mu \\ &\quad + \int_0^t \|\nabla E_0(\Phi_2(\mu))\|_{0,Q_t}\,\|\nabla\Phi_2(\mu)\|_{0,Q_t}\;d\mu \\ &\le 1 + c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\ &\quad + M\int_0^t \sup_{x\in\Omega}\left|\nabla\int_\Omega \nabla G(x,y)\rho(y,\mu)\,dy\right|_\infty d\mu + M\int_0^t \|\nabla E_0\|_{0,\Omega}\;d\mu \\ &\le 1 + c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\ &\quad + TMc_S(\Omega,\alpha)\left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right) \\ &\le 1 + c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} + TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).\end{aligned}$$

□

Lemma 4.39. *Let Ω be a bounded $C^{2,\alpha}$ domain, ρ be defined in (4.31) and $u_A \in C^{2,\alpha}(\Gamma)$. Let $\Phi \in W(M,T,K,\delta)$ and $T < 1$. Then we have*
a) *for the operator A_1*

$$|\nabla A_1(\Phi)(t)|_{\alpha,\Omega_0} \le 2Tc_{mv}^\alpha c_S(\Omega,\alpha)M^{1+\alpha}\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).$$

b) *for the operator A_2*

$$\begin{aligned}&\sup_{0\le t\le T}|\nabla A_2(\Phi)(t)|_{\alpha,Q_t} \\ &\quad\le |\varphi'|_{\alpha,Q_t} + c_{mv}c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right)L_{\Gamma_-}^{1-\alpha} + 2\max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\} \\ &\qquad + T^{1-\alpha}c_S(\Omega,\alpha)(K + M + M^{1+\alpha})(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})\end{aligned}$$

with $\kappa_\rho(\alpha)$ defined in (4.60), c_ρ defined in (4.59) and $c_S(\Omega,\alpha)$ defined in Lemma 4.15.

Proof. First, we note that by Lemma 4.34 follows $\rho(\cdot,t) \in C^{\alpha}(\bar{\Omega})$ and by Lemma 4.36 follows $\rho(x,\cdot) \in C^{\alpha}([0,T])$.

a) The gradient of $A_1(\Phi)$ is given in Lemma 4.22a). We obtain pointwise

$$
\begin{aligned}
&|\nabla A_1(\Phi)(x,t) - \nabla A_1(\Phi)(z,t)|_{\infty} \\
&\le \left| \int_0^t \nabla \int_{\Omega} \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy \nabla\Phi_1(x,\mu) - \nabla \int_{\Omega} \nabla G(\Phi_1(z,\mu),y)\rho(y,\mu)\,dy \nabla\Phi_1(z,\mu)\,d\mu \right|_{\infty} \\
&\quad + \left| \int_0^t \nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu - \int_0^t \nabla E_0(\Phi_1(z,\mu))\nabla\Phi_1(z,\mu)\,d\mu \right|_{\infty} \\
&\le \int_0^t \left| \nabla \int_{\Omega} \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy \nabla\Phi_1(x,\mu) - \nabla \int_{\Omega} \nabla G(\Phi_1(z,\mu),y)\rho(y,\mu)\,dy \nabla\Phi_1(z,\mu) \right|_{\infty} d\mu \\
&\quad + \int_0^t |\nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu) - \nabla E_0(\Phi_1(z,\mu))\nabla\Phi_1(z,\mu)|_{\infty}\,d\mu \\
&\le \int_0^t \left| \nabla \int_{\Omega} \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy - \nabla \int_{\Omega} \nabla G(\Phi_1(z,\mu),y)\rho(y,\mu)\,dy \right|_{\infty} |\nabla\Phi_1(z,\mu)|_{\infty}\,d\mu \\
&\quad + \int_0^t \left| \nabla \int_{\Omega} \nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy \right|_{\infty} |\nabla\Phi_1(x,t) - \nabla\Phi_1(z,t)|_{\infty}\,d\mu \\
&\quad + \int_0^t |\nabla E_0(\Phi_1(x,\mu)) - \nabla E_0(\Phi_1(z,\mu))|_{\infty} |\nabla\Phi_1(z,\mu)|_{\infty}\,d\mu \\
&\quad + \int_0^t |\nabla E_0(\Phi_1(x,\mu))|_{\infty} |\nabla\Phi_1(x,\mu) - \nabla\Phi_1(z,\mu)|_{\infty}\,d\mu.
\end{aligned}
\tag{4.77}
$$

As $\Phi_1 \in W_1(M,T)$, $\nabla\Phi_1$ is α-Hölder continuous with constant M, i.e. $|\nabla\Phi_1(x,t) - \nabla\Phi_1(z,t)|_{\infty} \le M|x-y|_{\infty}^{\alpha}$. We bound (4.77) by Lemma 4.15.

$$
\begin{aligned}
&|\nabla A_1(\Phi)(x,t) - \nabla A_1(\Phi)(z,t)|_{\infty} \\
&\le \int_0^t Mc_S(\Omega,\alpha)\,|\Phi_1(x,\mu) - \Phi_1(z,\mu)|_{\infty}^{\alpha}\,\|\rho(\mu)\|_{\alpha,\Omega}\,d\mu + \int_0^t Mc_S(\Omega,\alpha)\|\rho(\mu)\|_{\alpha,\Omega}|x-z|_{\infty}^{\alpha}\,d\mu \\
&\quad + \int_0^t Mc_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}|\Phi_1(x,\mu) - \Phi_1(z,\mu)|_{\infty}^{\alpha}\,d\mu + \int_0^t Mc_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}|x-z|_{\infty}^{\alpha}\,d\mu.
\end{aligned}
\tag{4.78}
$$

Φ_1 is defined on the domain Ω_0 which is non convex. To bound $|\Phi_1(x,\mu) - \Phi_1(y,\mu)|_{\infty}^{\alpha}$, we apply Lemma 2.20

$$
\begin{aligned}
|\Phi_1(x,\mu) - \Phi_1(z,\mu)|_{\infty}^{\alpha} &\le c_{mv}^{\alpha}\|\nabla\Phi_1(t)\|_{0,\Omega_t^1}^{\alpha}|x-z|_{\infty}^{\alpha} \\
&\le c_{mv}^{\alpha}M^{\alpha}|x-z|_{\infty}^{\alpha}.
\end{aligned}
$$

It follows for (4.78) by Lemma 4.34

$$
|\nabla A_1(\Phi)(x,t) - \nabla A_1(\Phi)(z,t)|_{\infty} \le 2Tc_{mv}^{\alpha}c_S(\Omega,\alpha)M^{1+\alpha}\left(\kappa_{\rho}(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right)|x-z|_{\infty}^{\alpha}.
$$

Eventually, we have for the α-seminorm of $\nabla_x A_1(\Phi)$

$$\begin{aligned}
|\nabla A_1(\Phi)(x,t)|_{\alpha,\Omega_0} &\leq \sup_{x,y\in\Omega_0} \frac{|\nabla A_1(\Phi)(x,t) - \nabla A_1(\Phi)(z,t)|_\infty}{|x-z|_\infty^\alpha} \\
&\leq \sup_{x,y\in\Omega_0} \frac{2Tc_{mv}^\alpha c_S(\Omega,\alpha) M^{1+\alpha} \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right) |x-z|_\infty^\alpha}{|x-z|_\infty^\alpha} \\
&= T2c_{mv}^\alpha c_S(\Omega,\alpha) M^{1+\alpha} \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

b) It follows by Lemma 4.22b)

$$\begin{aligned}
&|\nabla A_2(\Phi)(s_1,t_1,t) - \nabla A_2(\Phi)(s_2,t_2,t)|_\infty \\
&\leq |D_A(s_1,t_1) - D_A(s_2,t_2)|_\infty + \left| \int_{t_1}^t \nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_2(s_1,t_1,\mu)\,d\mu \right. \\
&\quad \left. - \int_{t_2}^t \nabla \int_\Omega \nabla G(\Phi_2(s_2,t_2,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_2(s_2,t_2,\mu)\,d\mu \right|_\infty \\
&\quad + \left| \int_{t_1}^t \nabla E_0(\Phi_2(s_1,t_1,\mu))\nabla\Phi_2(s_1,t_1)\,d\mu - \int_{t_2}^t \nabla E_0(\Phi_2(s_2,t_2,\mu))\nabla\Phi_2(s_2,t_2)\,d\mu \right|_\infty.
\end{aligned} \tag{4.79}$$

We begin with bounding $|D_A(s_1,t_1) - D_A(s_2,t_2)|_\infty$. Due to the $C^{2,\alpha}$-regularity of Γ_-, we obtain

$$|\varphi'(s_1) - \varphi'(s_2)|_\infty \leq |\varphi'|_{\alpha,I_{\Gamma\_}Q_t} |s_1 - s_2|^\alpha.$$

Next we find a pointwise bound for the second component of D_A

$$\begin{aligned}
&\left| \int_\Omega \nabla G(\varphi(s_1),y)\rho(y,t_1)\,dy + E_0(\varphi(s_1)) - \int_\Omega \nabla G(\varphi(s_2),y)\rho(y,t_2)\,dy + E_0(\varphi(s_2)) \right|_\infty \\
&\quad \leq \left| \int_\Omega \left[\nabla G(\varphi(s_1),y) - \nabla G(\varphi(s_2),y)\right]\rho(y,t_1)\,dy \right|_\infty + |E_0(\varphi(s_1)) - E_0(\varphi(s_2))|_\infty \\
&\qquad + \left| \int_\Omega \nabla G(\varphi(s_1),y)\left[\rho(y,t_1) - \rho(y,t_2)\right]\,dy \right|_\infty.
\end{aligned} \tag{4.80}$$

We use the mean-value like estimate of Lemma 2.20 for the first and second term of (4.80). Then we get with Lemma 4.15 and Lemma 4.34

$$\begin{aligned}
&\left| \int_\Omega \left[\nabla G(\varphi(s_1),y) - \nabla G(\varphi(s_2),y)\right]\rho(y,t_1)\,dy \right|_\infty + |E_0(\varphi(s_1)) - E_0(\varphi(s_2))|_\infty \\
&\leq c_{mv} \left\| \nabla \int_\Omega \nabla G(x,y)\rho(y,t_1)\,dy \right\|_{0,\Omega} |\varphi(s_1) - \varphi(s_2)|_\infty + c_{mv} \|\nabla E_0\|_{0,\Omega} |\varphi(s_1) - \varphi(s_2)|_\infty \\
&\leq c_{mv} c_S(\Omega,\alpha) \left(\|\rho(t)\|_\alpha + \|u_A\|_{2,\alpha;\Gamma}\right) \|\varphi'\|_{0,I_{\Gamma_}} |s_1 - s_2| \\
&\leq c_{mv} c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right) |s_1 - s_2|.
\end{aligned}$$

For the third term of (4.80) follows with Lemma 4.16 and Lemma 4.36

$$\begin{aligned}
\left| \int_\Omega \nabla G(\varphi(s_1),y)\left[\rho(y,t_1) - \rho(y,t_2)\right]\,dy \right|_\infty &\leq \int_\Omega |\nabla G(\varphi(s_1),y)|_\infty |\rho(y,t_1) - \rho(y,t_2)|_\infty\,dy \\
&\leq \int_\Omega |\nabla G(\varphi(s_1),y)|_\infty\,dy \sup_{y\in\Omega} |\rho(y,t_1) - \rho(y,t_2)|_\infty \\
&\leq 2\max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\} |t_1 - t_2|_\infty^\alpha.
\end{aligned}$$

To sum up, we obtain

$$|D_A(s_1,t_1) - D_A(s_2,t_2)|_\infty \le |\varphi'|_{\alpha,I_{\Gamma_-}} |s_1 - s_2|^\alpha + c_{mv} c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right) |s_1 - s_2| \\ + 2\max\left\{Mc_\rho, \|\rho_0\|^2_{0,\Omega_0}, \|\rho_A\|^2_{0,Q_t}\right\} |t_1 - t_2|^\alpha_\infty . \tag{4.81}$$

To bound the second term of (4.79), we use Lemma 4.15 and Lemma 4.34. Moreover, we use $|\nabla\Phi_2|_{0,Q_t} \le M$ and $|\nabla\Phi_2|_{\alpha,Q_t} \le K$ since $\Phi_2 \in W_2(M,T,K,\delta)$ and $\int_{t_1}^t = \int_{t_1}^{t_2} + \int_{t_2}^t$

$$\begin{aligned}
&\left|\int_{t_1}^t \nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_2(s_1,t_1,\mu)\,d\mu \right.\\
&\quad \left. - \int_{t_2}^t \nabla \int_\Omega \nabla G(\Phi_2(s_2,t_2,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_2(s_2,t_2,\mu)\,d\mu\right|_\infty \\
&\le \left|\int_{t_1}^{t_2} \nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy\ \nabla\Phi_2(s_1,t_1,\mu)\,d\mu\right|_\infty \\
&\quad + \left|\int_{t_1}^t \left[\nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy - \nabla \int_\Omega \nabla G(\Phi_2(s_2,t_2,\mu),y)\rho(y,\mu)\,dy\right] \nabla\Phi_2\,d\mu\right|_\infty \\
&\quad + \left|\int_{t_1}^t \nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy \left[\nabla\Phi_2(s_1,t_1,\mu) - \nabla\Phi_2(s_2,t_2,\mu)\right]\,d\mu\right|_\infty \\
&\le |t_1 - t_2| \sup_{0\le t\le T} \left|\nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,t),y)\rho(y,t)\,dy\right|_\infty \sup_{0\le t\le T} |\nabla\Phi_2(s_1,t_1,t)|_\infty \\
&\quad + \int_0^t \left|\nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy - \nabla \int_\Omega \nabla G(\Phi_2(s_2,t_2,\mu),y)\rho(y,\mu)\,dy\right|_\infty |\nabla\Phi_2\,|_\infty\,d\mu \\
&\quad + \int_0^t \left|\nabla \int_\Omega \nabla G(\Phi_2(s_1,t_1,\mu),y)\rho(y,\mu)\,dy\right|_\infty |\nabla\Phi_2(s_1,t_1,\mu) - \nabla\Phi_2(s_2,t_2,\mu)|_\infty\,d\mu \\
&\le |t_1 - t_2| M c_S(\Omega,\alpha)\|\rho(t)\|_{\alpha,\Omega} + TMc_S(\Omega,\alpha)\|\rho(t)\|_{\alpha,\Omega}\,|\Phi_2(s_1,t_1,\mu) - \Phi_2(s_2,t_2,\mu)|^\alpha_\infty \\
&\quad + Tc_S(\Omega,\alpha)\|\rho(t)\|_{\alpha,\Omega}\,|\nabla\Phi_2(s_1,t_1,\mu) - \nabla\Phi_2(s_2,t_2,\mu)|_\infty \\
&\le |t_1 - t_2| M c_S(\Omega,\alpha)\kappa_\rho(\alpha) + TM^{1+\alpha} c_S(\Omega,\alpha)\kappa_\rho(\alpha)|(s_1,t_1) - (s_2,t_2)|^\alpha_\infty \\
&\quad + Tc_S(\Omega,\alpha)\kappa_\rho(\alpha)K|(s_1,t_1) - (s_2,t_2)|^\alpha_\infty
\end{aligned} \tag{4.82}$$

In the last step we used Theorem 2.15 to bound
$|\Phi_2(s_1,t_1,t) - \Phi_2(s_2,t_2,t)|_\infty \le M|(s_1,t_1) - (s_2,t_2)|^\alpha_\infty$. It is allowed since Q_t is a convex set.

We bound the third term of (4.79) analogously by Lemma 4.15 and Lemma 4.34

$$\begin{aligned}
&\left|\int_{t_1}^{t} \nabla E_0(\Phi_2(s_1,t_1,\mu))\nabla\Phi_2(s_1,t_1,\mu)\,d\mu - \int_{t_2}^{t} \nabla E_0(\Phi_2(s_2,t_2,\mu))\nabla\Phi_2(s_2,t_2,\mu)\,d\mu\right|_\infty \\
&\le \left|\int_{t_1}^{t_2} \nabla E_0(\Phi_2(s_1,t_1,\mu))\nabla\Phi_2(s_1,t_1,\mu)\,d\mu\right|_\infty \\
&\quad + \left|\int_{t_1}^{t} \left[\nabla E_0(\Phi_2(s_1,t_1,\mu)) - \nabla E_0(\Phi_2(s_2,t_2,\mu))\right]\nabla\Phi_2(s_1,t_1,\mu)\,d\mu\right|_\infty \\
&\quad + \left|\int_{t_1}^{t} \nabla E_0(\Phi_2(s_1,t_1,\mu))\left[\nabla\Phi_2(s_1,t_1,\mu) - \nabla\Phi_2(s_2,t_2,\mu)\right]d\mu\right|_\infty \\
&\le |t_1-t_2| c_S(\Omega,\alpha) M\|u_A\|_{2,\alpha;\Gamma} + c_S(\Omega,\alpha)Mt\|u_A\|_{2,\alpha;\Gamma}|\Phi_2(s_1,t_1,\mu)-\Phi_2(s_2,t_2,\mu)|_\infty^\alpha \\
&\quad + tc_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}K|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha \\
&\le |t_1-t_2| c_S(\Omega,\alpha) M\|u_A\|_{2,\alpha;\Gamma} + Tc_S(\Omega,\alpha)(M^{1+\alpha}+K)\|u_A\|_{2,\alpha;\Gamma}|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha. \quad (4.83)
\end{aligned}$$

Eventually, we get for the α-semi norm by (4.81), (4.82) and since $T<1$ (4.83)

$$\begin{aligned}
&|\nabla\Phi_2(t)|_{\alpha,Q_t} = \sup_{(s_1,t_1)\neq(s_2,t_2)\in Q_t} \frac{|\nabla\Phi_2(s_1,t_1,t)-\nabla\Phi_2(s_2,t_2,t)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\
&\le |\varphi'|_{\alpha,I_{\Gamma_-}} + c_{mv}c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)|s_1-s_2|^{1-\alpha} + 2\max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\} \\
&\quad + \sup_{0\le t_1,t_2\le t} |t_1-t_2|^{1-\alpha}Mc_S(\Omega,\alpha)\kappa_\rho(\alpha) + TM^{1+\alpha}c_S(\Omega,\alpha)\kappa_\rho(\alpha) + Tc_S(\Omega,\alpha)\kappa_\rho(\alpha)K \\
&\quad + \sup_{0\le t_1,t_2\le t} |t_1-t_2|^{1-\alpha}c_S(\Omega,\alpha)M\|u_A\|_{2,\alpha;\Gamma} + Tc_S(\Omega,\alpha)(M^{1+\alpha}+K)\|u_A\|_{2,\alpha;\Gamma} \\
&\le |\varphi'|_{\alpha,I_{\Gamma_-}} + c_{mv}c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)L_{\Gamma_-}^{1-\alpha} + 2\max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\} \\
&\quad + T^{1-\alpha}c_S(\Omega,\alpha)(K+M+M^{1+\alpha})(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}).
\end{aligned}$$

□

Next we bound the α-norm of $A(\Phi)$ with respect to time.

Lemma 4.40. *Let Ω be a bounded $C^{2,\alpha}$ domain, ρ be as defined in (4.31) and $u_A \in C^{2,\alpha}(\Gamma)$. Let $\Phi \in W(M,T,K,\delta)$. Then we have for $T<1$*

a) *for the operator A_1*

$$\begin{aligned}
&\sup_{x\in\Omega_0}\|A_1(\Phi)(x)\|_{\alpha,[0,T]} + \sup_{x\in\Omega_0}\|\nabla_x A_1(\Phi)(x)\|_{\alpha,[0,T]} \\
&\qquad \le 1 + \|x\|_{0,\Omega_0} + 2T^{1-\alpha}Mc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

b) *for the operator A_2*

$$\begin{aligned}
&\sup_{(s,t_x)\in Q_T}\|A_2(\Phi)(s,t_x)\|_{\alpha,[t_x,T]} + \sup_{(s,t_x)\in Q_T}\|\nabla A_2(\Phi)(s,t_x)\|_{\alpha,[t_x,T]} \\
&\quad \le 1 + \|\varphi\|_{0,I_{\Gamma_-}} + c(\mathrm{diam}(\Omega))\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\
&\qquad + 2T^{1-\alpha}(M+1)c_S(\alpha,\Omega)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)
\end{aligned}$$

with $\kappa_\rho(\alpha)$ defined in (4.60) and $c_S(\Omega,\alpha)$ defined in Lemma 4.15.

Proof. First, we note that by Lemma 4.34 follows $\rho(\cdot,t)\in C^\alpha(\bar\Omega)$.

a) Recall that the $C^{1,\alpha;\alpha}(\Omega,[0,T])$ norm is given by

$$\begin{aligned}&\|A_1(\Phi)\|_{1,\alpha,\Omega;\alpha,[0,T]}\\ &\quad=\sup_{0\le t\le T}\|A_1(\Phi)(x)\|_{1,\alpha,\Omega_0}+\sup_{x\in\Omega_0}\|A_1(\Phi)(x)\|_{\alpha,[0,T]}+\sup_{x\in\Omega_0}\|\nabla_x A_1(\Phi)(x)\|_{\alpha,[0,T]}.\end{aligned}\tag{4.84}$$

While the first term of (4.84) has been bounded in the previous Lemmas, we now have to deal with the second and third term. As $A(\Phi)$ is continuous in space and time, the sup-norm is interchangeable, i.e.

$$\sup_{x\in\Omega_0}\sup_{0\le t\le T}|A(\Phi)(x,t)|_\infty=\sup_{0\le t\le T}\sup_{x\in\Omega_0}|A(\Phi)(x,t)|_\infty.$$

Lemmas 4.37a) and 4.38a) thus give

$$\begin{aligned}&\sup_{x\in\Omega_0}\sup_{0\le t\le T}|A_1(\Phi)(x,t)|_\infty+\sup_{x\in\Omega_0}\sup_{0\le t\le T}|\nabla A_1(\Phi)(x,t)|_\infty\\ &\qquad\le 1+\|x\|_{0,\Omega_0}+T(1+M)c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right).\end{aligned}\tag{4.85}$$

In the following, we will bound the α- semi norm in time. By Lemma 4.15, we get the pointwise estimate for all $x\in\Omega_0$ and $t_1,t_2\in[0,T]$

$$\begin{aligned}&|A_1(\Phi)(x,t_1)-A_1(\Phi)(x,t_2)|_\infty=\left|\int_{t_1}^{t_2}\int_\Omega\nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy+E_0(\Phi_1(x,\mu))\,d\mu\right|_\infty\\ &\le|t_1-t_2|\sup_{0\le t\le T}\left|\int_\Omega\nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy+E_0(\Phi_1(x,\mu))\right|_\infty\\ &\le|t_1-t_2|c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right).\end{aligned}$$

It follows for the α- semi norm by Lemma 4.34

$$\begin{aligned}|A_1(\Phi)(x)|_{\alpha,[0,T]}&=\frac{|A_1(\Phi)(x,t_1)-A_1(\Phi)(x,t_2)|_\infty}{|t_1-t_2|^\alpha}\\ &\le|t_1-t_2|^{1-\alpha}c_S(\Omega,\alpha)\sup_{0\le t\le T}\left(\|\rho(t)\|_{\alpha,\Omega}+\|u_A\|_{2,\alpha;\Gamma}\right)\\ &\le T^{1-\alpha}c_S(\Omega,\alpha)\sup_{0\le t\le T}\left(\|\rho(t)\|_{\alpha,\Omega}+\|u_A\|_{2,\alpha;\Gamma}\right).\end{aligned}\tag{4.86}$$

Analogously, we obtain for $\nabla A_1(\Phi)$ by Lemma 4.15

$$\begin{aligned}&|\nabla A_1(\Phi)(x,t_1)-\nabla A_1(\Phi)(x,t_2)|_\infty\\ &\quad=\left|\int_{t_1}^{t_2}\nabla\int_\Omega\nabla G(\Phi_1(x,\mu),y)\rho(y,\mu)\,dy\nabla\Phi_1(x,\mu)+\nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu\right|_\infty\\ &\quad\le|t_1-t_2|\sup_{0\le t\le T}\left|\nabla\int_\Omega\nabla G(\Phi_1(x,t),y)\rho(y,\mu)\,dy\nabla\Phi_1(x,t)+\nabla E_0(\Phi_1(x,t))\nabla\Phi_1(x,t)\right|_\infty\\ &\quad\le|t_1-t_2|c_S(\Omega,\alpha)M\sup_{0\le t\le T}\left(\|\rho(t)\|_{\alpha,\Omega}+\|u_A\|_{2,\alpha;\Gamma}\right)\end{aligned}$$

and thus by Lemma 4.34

$$\begin{aligned}
|\nabla A_1(\Phi)(x)|_{\alpha,[0,T]} &= \sup_{t_1,t_2\in[0,T]} \frac{|\nabla A_1(\Phi)(x,t_1) - \nabla A_1(\Phi)(x,t_2)|_\infty}{|t_1-t_2|^\alpha} \\
&\leq \sup_{t_1,t_2\in[0,T]} |t_1-t_2|^{1-\alpha} c_S(\Omega,\alpha) M \sup_{0\leq t\leq T} \left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right) \\
&\leq T^{1-\alpha} M c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned} \tag{4.87}$$

Hence, we have with (4.85), (4.86), (4.87) and since $T < 1$

$$\begin{aligned}
&\sup_{x\in\Omega_0} \|A_1(\Phi)(x)\|_{\alpha,[0,T]} + \sup_{x\in\Omega_0} \|\nabla_x A_1(\Phi)(x)\|_{\alpha,[0,T]} \\
&\quad \leq 1 + \|x\|_{0,\Omega_0} + 2T^{1-\alpha} M c_S(\Omega,\alpha) \left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

b) Since $A_2(\Phi)$ and $\nabla A_2(\Phi)$ are continuous with respect to space and time follows

$$\sup_{(s,t_x)\in Q_T} \sup_{t_x\leq t\leq T} |A_2(\Phi)(s,t_x,t)| = \sup_{0\leq t\leq T} \sup_{(s,t_x)\in Q_t} |A_2(\Phi)(s,t_x,t)|.$$

Due to Lemmas 4.37b) and 4.38b), we get

$$\begin{aligned}
&\sup_{(s,t_x)\in Q_T} \sup_{t_x\leq t\leq T} |A_2(\Phi)(s,t_x,t)| + \sup_{(s,t_x)\in Q_T} \sup_{t_x\leq t\leq T} |\nabla A_2(\Phi)(s,t_x,t)| \\
&= \sup_{0\leq t\leq T} \sup_{Q_t} |A_2(\Phi)(s,t_x,t)| + \sup_{0\leq t\leq T} \sup_{Q_t} |\nabla A_2(\Phi)(s,t_x,t)| \\
&\leq 1 + \|\varphi\|_{0,I_{\Gamma_-}} + c(\operatorname{diam}(\Omega)) \max\left\{\|\rho_0\|_{0,\Omega_0}, |\rho_A|_{0,Q_T}\right\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\
&\quad + T(M+1) c(\alpha,\Omega) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned} \tag{4.88}$$

To obtain the α-Hölder norm, we first find the following pointwise estimate for all $(s,t_x) \in Q_T$ and $t_1, t_2 \in [t_x, T]$. By Lemma 4.15 and Lemma 4.34 follows

$$\begin{aligned}
&|A_2(\Phi)(s,t_x,t_1) - A_2(\Phi)(s,t_x,t_2)|_\infty \\
&= \left| \int_{t_1}^{t_2} \int_\Omega \nabla G(\Phi_2(s,t_x,\mu), y)\rho(y,\mu)\, dy + E_0(\Phi_2(s,t_x,\mu))\, d\mu \right|_\infty \\
&\leq |t_1-t_2| \sup_{t_x\leq t\leq T} \left| \int_\Omega \nabla G(\Phi_2(s,t_x,t), y)\rho(y,t)\, dy + E_0(\Phi_2(s,t_x,t)) \right|_\infty \\
&\leq |t_1-t_2| c_S(\Omega,\alpha) \sup_{t_x\leq t\leq T} \left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

We obtain for the Hölder norm by Lemma 4.34

$$\begin{aligned}
|A_2(\Phi)(s,t_x)|_{\alpha,[0,T]} &\leq \sup_{t_x\leq t_1,t_2\leq T} |t_1-t_2|^{1-\alpha} c_S(\Omega,\alpha) \sup_{t_x\leq t\leq T} \left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma}\right) \\
&\leq T^{1-\alpha} c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned} \tag{4.89}$$

Analogously, we obtain for ∇A_2 for all $(s,t_x) \in Q_T$ and $t_1, t_2 \in [t_x, T]$

$$\begin{aligned}
&|\nabla A_2(\Phi)(s,t_x,t_1) - \nabla A_2(\Phi)(s,t_x,t_2)|_\infty \\
&= \left| \int_{t_1}^{t_2} \left[\nabla \int_\Omega \nabla G(\Phi_2(s,t_x,\mu), y)\rho(y,\mu)\, dy + E_0(\Phi_2(s,t_x,\mu)) \right] \nabla \Phi_2(y,\mu)\, d\mu \right|_\infty \\
&\le |t_1 - t_2| \sup_{t_x \le t \le T} \left| \nabla \int_\Omega \nabla G(\Phi_2(s,t_x,t), y)\rho(y,t)\, dy \right|_\infty \sup_{t_x \le t \le T} |\nabla \Phi_2(s,t_x,t)|_\infty \\
&\quad + |t_1 - t_2| \, |E_0(\Phi_2(s,t_x,t))|_\infty \, |\nabla \Phi_2(s,t_x,t)|_\infty \\
&\le |t_1 - t_2| c_S(\Omega,\alpha) M \sup_{t_x \le t \le T} \left(\|\rho(t)\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha;\Gamma} \right).
\end{aligned}$$

We obtain for the Hölder norm by Lemma 4.34

$$\begin{aligned}
|\nabla A_2(\Phi)(s,t_x)|_{\alpha,[0,T]} &\le \sup_{t_x \le t_1, t_2 \le T} |t_1 - t_2|^{1-\alpha} M c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma} \right) \\
&\le T^{1-\alpha} M c_S(\Omega,\alpha) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma} \right).
\end{aligned} \tag{4.90}$$

Conclusively, we obtain by (4.88), (4.89) and (4.90) and since $T < 1$

$$\begin{aligned}
&\sup_{(s,t_x) \in Q_T} \|A_2(\Phi)(s,t_x)\|_{\alpha,[t_x,T]} + \sup_{(s,t_x) \in Q_T} \|\nabla A_2(\Phi)(s,t_x)\|_{\alpha,[t_x,T]} \\
&\quad \le 1 + |\varphi|_{0,Q_T} + c(\mathrm{diam}(\Omega)) \max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_t}\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\
&\qquad + 2T^{1-\alpha}(M+1) c(\alpha,\Omega) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma} \right).
\end{aligned}$$

□

We proceed to check the last restrictions in $W_1(M,T)$ and $W_2(M,T,K,\delta)$ that are connected to the invertibility of $A_1(\Phi)(x,t)$ and $A_2(\Phi)(s,t_x,t)$ with respect to the space variable.

Lemma 4.41. *Let Ω be a $C^{2,\alpha}$, ρ be defined in (4.31), $u_A \in C^{2,\alpha}(\Gamma)$ and let $\Phi \in W(M,T,K,\delta)$ and $\max\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\} \le \frac{\delta}{c(\mathrm{diam}(\Omega))}$. Then we have for $(s,t_x) \in Q_T$*

$$\left| \partial_{t_x} A_2(\Phi)(s,t_x,t=t_x) \cdot \begin{pmatrix} -\varphi_2'(s) \\ \varphi_1'(s) \end{pmatrix} \right| \ge \delta.$$

Moreover, the matrix $D_A(s,t_x)$ defined in (4.40) is invertible for every $(s,t_x) \in Q_T$.

Proof. First we note that by Lemma 4.34 follows $\rho(\cdot,t) \in C^\alpha(\bar{\Omega})$. We get by Lemma 4.22

$$\partial_{t_x} A_2(\Phi)(s,t_x,t=t_x) = \int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\, dy + E_0(\varphi(s)).$$

With the inverse triangle inequality follows

$$\begin{aligned}
&\left| \partial_{t_x} A_2(\Phi)(s,t_x,t=t_x) \cdot \left(-\varphi_2'(s), \varphi_1'(s) \right)^T \right| \\
&\quad = \left| \int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\, dy \cdot \varphi'(s)^\perp + E_0(\varphi(s)) \cdot \varphi'(s)^\perp \right| \\
&\quad \ge \left| E_0(\varphi(s)) \cdot \varphi'(s)^\perp \right|_\infty - \left| \int_\Omega \nabla G(\varphi(s),y)\rho(y,t_x)\, dy \cdot \varphi'(s)^\perp \right|.
\end{aligned} \tag{4.91}$$

The assumptions on E_0 given in Definition 4.23 yield

$$\left|E_0(\varphi(s)) \cdot \varphi'(s)^{\perp}\right| \geq 2\delta.$$

Due to the arc length parametrization of φ and the equivalence of the maximums and euclidean norm follows that $|\varphi'(s)|_\infty \leq \|\varphi'(s)\|_2 = 1$. Hence, we get for (4.91) with Lemma 4.16 and the sup-norm estimate of ρ in Lemma 4.34

$$\begin{aligned}
&\left|E_0(\varphi(s)) \cdot \varphi'(s)^{\perp}\right| - \left|\int_\Omega \nabla G(\varphi(s), y)\rho(y, t_x)\, dy \cdot \varphi'(s)^{\perp}\right| \\
&\geq 2\delta - c(\operatorname{diam}(\Omega)) \sup_{0\leq t\leq T} \|\rho(t)\|_{0,\Omega} |\varphi'(s)^{\perp}|_\infty \\
&\geq 2\delta - c(\operatorname{diam}(\Omega)) \max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\}.
\end{aligned}$$

As $\max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\} \leq \frac{\delta}{c(\operatorname{diam}(\Omega))}$, we eventually obtain

$$\left|\partial_{t_x} A_2(\Phi)(s, t_x, t = t_x) \cdot \begin{pmatrix} -\varphi_2'(s) \\ \varphi_1'(s) \end{pmatrix}\right| \geq \delta.$$

So far, we showed that $|\det D_A(s, t_x)| \geq \delta > 0$ for every $(s, t_x) \in Q_T$. It follows that the matrix $D_A(s, t_x)$ is invertible for every $(s, t_x) \in Q_T$. (See Lemma 4.28). $\square$

Lemma 4.42. *Let Ω be a $C^{2,\alpha}$ domain, ρ be defined in (4.31), $u_A \in C^{2,\alpha}(\Gamma)$ and $\Phi \in W(M, T, K, \delta)$. Then we have*
a) *for the operator A_1*

$$\sup_{0\leq t\leq T} \|I - \nabla A_1(\Phi)(t)\|_{0,\Omega_0} \leq T c_S(\alpha, \Omega) M \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right),$$

b) *and for the operator A_2*

$$\sup_{0\leq t\leq T} \|D_A - \nabla A_2(\Phi)(t)\|_{0,Q_t} \leq T c_S(\alpha, \Omega) M \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right)$$

with $D_A(s, t_x)$ defined in (4.43), $\kappa_\rho(\alpha)$ in (4.60) and $c_S(\Omega, \alpha)$ in Lemma 4.15.

Proof. First, we have by Lemma 4.34 that $\rho(\cdot, t) \in C^\alpha(\bar{\Omega})$.
a) By Lemma 4.22a) and Lemma 4.38a), we have

$$\begin{aligned}
&\|I - \nabla A_1(\Phi)(t)\|_{0,\Omega_0} \\
&= \sup_{x\in\Omega_0} \left|\int_0^t \nabla \int_\Omega \nabla G(\Phi_1(x, \mu), y)\rho(y, \mu)\, dy \nabla\Phi_1(x, \mu) + E_0(\Phi_1(x, \mu))\nabla\Phi_1(x, \mu)\, d\mu\right|_\infty \\
&\leq T M c_S(\alpha, \Omega) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

b) By Lemma 4.22b) and Lemma 4.38b), we have

$$\begin{aligned}
&\|D_A - \nabla A_2(\Phi)(t)\|_{0,Q_t} \\
&= \sup_{(s,t_x)\in Q_t} \left|\int_{t_x}^t \nabla \int_\Omega \nabla G(\Phi_2(s, t_x, \mu), y)\rho(y, \mu)\, dy \nabla\Phi_2(s, t_x, \mu) + E_0(\Phi_2(s, t_x, \mu))\nabla\Phi_2(s, t_x, \mu)\, d\mu\right|_\infty \\
&\leq T M c_S(\alpha, \Omega) \left(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}\right).
\end{aligned}$$

$\square$

We can now obtain the main result of this section. We show that A is a selfmap on the set $W(M,T,K,\delta)$.

Theorem 4.43. *Let Ω be a $C^{2,\alpha}$ domain with $0<\alpha<1$, ρ be defined in (4.31), $u_A \in C^{2,\alpha}(\Gamma)$ and $\Phi \in W(M,T,K,\delta)$ with*

$$
\begin{aligned}
T = \min \Bigg\{ & \left(\frac{1}{(3M + c_{mv}^{\alpha} M^{1+\alpha} + K + 3) c_S(\Omega,\alpha)(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})} \right)^{\alpha}, \frac{1}{2c_S(\alpha,\Omega) M\,(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})}, \\
& \frac{\delta}{4Mc_S(\alpha,\Omega)M\,(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})}, \frac{d}{c_S(\Omega,\alpha)(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})}, T_{E_0} \Bigg\}, \\
K = {} & |\varphi'|_{\alpha, I_{\Gamma_-}} + c_{mv} c_S(\Omega,\alpha)\,(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma})\, L_{\Gamma_-}^{1-\alpha} + 2\max\left\{ Mc_\rho, \|\rho_0\|^2_{0,\Omega_0}, \|\rho_A\|^2_{0,Q_t} \right\} + 1, \\
M = {} & 3 + 2\|x\|_{0,\Omega_0} + 2\delta + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}.
\end{aligned}
$$

with T_{E_0} defined in Lemma 4.31, c_ρ defined in (4.59), $\kappa_\rho(\alpha)$ defined in (4.60) and $c_S(\Omega,\alpha)$ defined in Lemma 4.15 and $d := \operatorname{dist}(\Gamma_+, \Omega_0)$. Further, let

$$
\max\left\{ \|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T} \right\} \leq \frac{\delta}{c(\operatorname{diam}\Omega)}.
$$

Then the operator A defines a selfmap on the set $W(M,T,K,\delta)$, i.e.

$$
A : W(M,T,K,\delta) \to W(M,T,K,\delta).
$$

Proof. First, we prove that the interface condition is fulfilled by $A_1(\Phi)(x,t)$ and $A_2(\Phi)(s,t_x,t)$. Choose x_0 and $(s_0,0)$ with $x_0 = \varphi(s_0)$. Then we obtain $\Phi_1(x_0,t) = \Phi_2(s_0,0,t)$ since $\Phi_1, \Phi_2 \in W(M,T,K,\delta)$. Hence,

$$
\begin{aligned}
A_1(\Phi)(x_0,t) &= x_0 - \int_0^t \int_\Omega \nabla G(\Phi_1(x_0,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_1(x_0,\mu))\,d\mu \\
&= \varphi(s_0) - \int_0^t \int_\Omega \nabla G(\Phi_2(s_0,0,\mu),y)\rho(y,\mu)\,dy - E_0(\Phi_2(s_0,0,\mu))\,d\mu \\
&= A_2(\Phi)(s_0,0,t).
\end{aligned}
$$

The inflow condition is fulfilled by $A_1(\Phi)$ since for $x \in \Omega_0$ holds

$$
A_1(\Phi)(x,0) = x. \tag{4.92}
$$

The question is whether $A_1(\Phi)$ maps the initial domain into Ω for every $t \in [0,T]$. It is not excluded that $A_1(\Phi)$ might map a point into the convex domain Ω_-. We therefore use now the interface condition and $A_2(\Phi)$ to show that $A_1(\Phi)$ expands into Ω.
$A_2(\Phi)$ is a streamline with streamline parameter t. The direction of the field is thus given by the derivative with respect to t. Hence,

$$
\frac{d}{dt} A_2(\Phi)(s,t_x,t) = -\int_\Omega \nabla G(\Phi_2(s,t_x,t),y)\rho(y,t)\,dy + E_0(\Phi_2(s,t_x,t)).
$$

The flow direction for every inflowing point is thus given for $t = t_x$ by (using the inflow condition for $\Phi_2 \in W_2(M, T, K, \delta)$)

$$\begin{aligned}\frac{d}{dt}A_2(\Phi)(s, t_x, t_x) &= -\int_\Omega \nabla G(\Phi_2(s, t_x, t_x), y)\rho(y, t)\, dy + E_0(\Phi_2(s, t_x, t_x)) \\ &= -\int_\Omega \nabla G(\varphi(s), y)\rho(y, t)\, dy + E_0(\varphi(s))\end{aligned}$$

We now show that with $\max\{\|\rho_0\|_{0,\Omega},\, \|\rho_A\|_{0,Q_t}\} \le \frac{\delta}{c(\operatorname{diam}(\Omega))}$, it is determined by E_0 whether Γ_- is the inflow or outflow boundary. By Definition 4.23, Lemma 4.16 and the sup-norm estimate of Lemma 4.34 follows for every $(s, t_x) \in Q_t$

$$\begin{aligned}&\left|\varphi^\perp(s) \cdot \left(\int_\Omega \nabla G(\varphi(s), y)\rho(y, t_x)\, dy + E_0(\varphi(s))\right)\right| \\ &\quad\ge \left|\varphi^\perp(s) \cdot E_0(\varphi(s)))\right| - \left|\varphi^\perp(s) \cdot \int_\Omega \nabla G(\varphi(s), y)\rho(y, t_x)\, dy\right| \\ &\quad\ge 2\delta - \left|\varphi^\perp(s)\right| \left|\int_\Omega \nabla G(\varphi(s), y)\rho(y, t_x)\, dy\right| \\ &\quad\ge 2\delta - c(\operatorname{diam}\Omega)\|\rho(t_x)\|_{0,\Omega} \\ &\quad\ge 2\delta - c(\operatorname{diam}\Omega)\max\{\|\rho_0\|_{0,\Omega_0},\, \|\rho_A\|_{0,Q_T}\} \\ &\quad\ge 2\delta - \delta = \delta > 0.\end{aligned}$$

Thus Γ_- is the inflow boundary with respect to $\frac{d}{dt}A_2(\Phi(s, t_x, t_x)$ for every $(s, t_x) \in Q_t$. As $A_1(\Phi)$ and $A_2(\Phi)$ fulfil the interface condition, it is only possible for $A_1(\Phi)$ to expand into Ω. It is left to determine an upper bound for T for which $A_1(\Phi)$ maps into Ω. Let therefore $d := \operatorname{dist}(\Gamma_+, \Omega_0)$. Then we have for every $z \in \Gamma_+$ and $x \in \Omega_0$ with alike computations to Lemma 4.37 and $T \le \frac{d}{2c_S(\Omega,\alpha)(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma})}$

$$\begin{aligned}|z - A_1(\Phi)(x, t)|_\infty &= \left|z - x + \int_0^t \int_\Omega \nabla G(\Phi_1(x, \mu), y)\rho(y, \mu)\, dy - E_0(\Phi_1(x, \mu))\, d\mu\right|_\infty \\ &\ge |z - x|_\infty - \left|\int_0^t \int_\Omega \nabla G(\Phi_1(x, \mu), y)\rho(y, \mu)\, dy - E_0(\Phi_1(x, \mu))\, d\mu\right|_\infty \\ &\ge |z - x|_\infty - Tc_S(\Omega, \alpha)(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}) \\ &\ge d - Tc_S(\Omega, \alpha)(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha;\Gamma}) \ge \frac{d}{2}.\end{aligned}$$

Thus the image of $A_1(\Phi)$ is a subset of Ω and $\Phi : \Sigma_t \to \Omega$.

Next, we collect the results of the previous Lemmas with the goal of determining the constants M, T and K such that $A(\Phi) \in W(M, T, K, \delta)$. We obtain with Lemmas 4.37a), 4.38a), 4.39a) and 4.40a) and since $T<1$

$$
\begin{aligned}
\|A_1(\Phi)\|_{1,\alpha,\Omega;\alpha,[0,T]} &\le \|x\|_{0,\Omega_0} + Tc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\quad + 1 + TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\quad + 2c_{mv}^{\alpha}Tc_S(\Omega,\alpha)M^{1+\alpha}\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\quad + 1 + \|x\|_{\Omega_0} + 2T^{1-\alpha}Mc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\le 2 + 2\|x\|_{0,\Omega_0} + T^{1-\alpha}c_S(\Omega,\alpha)\left(2c_{mv}^{\alpha}M^{1+\alpha}+3M+1\right)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right). \qquad (4.93)
\end{aligned}
$$

We obtain for the operator A_2 with Lemmas 4.37b), 4.38b) and 4.40b)

$$
\begin{aligned}
\|A_2(\Phi)\|_{1,Q_t;\alpha,[0,T]} &\le \|\varphi\|_{0,I_{\Gamma_-}} + Tc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\quad + 1 + c(\mathrm{diam}(\Omega))\max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\} + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}\\
&\quad + TMc_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\quad + 1 + \|\varphi\|_{0,I_{\Gamma_-}} + c(\mathrm{diam}(\Omega))\max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\}\\
&\quad + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} + 2T^{1-\alpha}(M+1)c(\alpha,\Omega)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)\\
&\le 2 + 2\|\varphi\|_{0,I_{\Gamma_-}} + 2c(\mathrm{diam}(\Omega))\max\left\{\|\rho_0\|_{0,\Omega_0}, \|\rho_A\|_{0,Q_T}\right\}\\
&\quad + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} + T^{1-\alpha}c_S(\Omega,\alpha)(3M+3)(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}). \qquad (4.94)
\end{aligned}
$$

For the Hölder constant in space of $\nabla A_2(\Phi)$, we get with 4.39b)

$$
\begin{aligned}
&\sup_{0\le t\le T}|A_2(\Phi)(t)|_{\alpha,Q_t}\\
&\le |\varphi'|_{\alpha,I_{\Gamma_-}} + c_{mv}c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)L_{\Gamma_-}^{1-\alpha} + \max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\}\\
&\quad + T^{1-\alpha}c_S(\Omega,\alpha)(K+M+M^{1+\alpha})(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}) \qquad (4.95)
\end{aligned}
$$

For the invertibility condition, we obtain with Lemma 4.42a)

$$
\sup_{0\le t\le T}\|I-\nabla A_1(\Phi)(t)\|_{0,\Omega_0} \le Tc_S(\alpha,\Omega)M\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right).
$$

and with Lemma 4.42b) for the function $A_2(\Phi)$

$$
\sup_{0\le t\le T}\|D_A-\nabla A_2(\Phi)(t)\|_{0,Q_t} \le Tc_S(\alpha,\Omega)M\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)
$$

Choice of T,M,K: Set

$$
\begin{aligned}
T = \min\Bigg\{&\left(\frac{1}{(3M+c_{mv}^{\alpha}M^{1+\alpha}+K+3)c_S(\Omega,\alpha)(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma})}\right)^{\alpha}, \frac{1}{2c_S(\alpha,\Omega)M\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)},\\
&\frac{\delta}{4Mc_S(\alpha,\Omega)M\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma})\right)}, \frac{d}{c_S(\Omega,\alpha)(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}}, T_{E_0}\Bigg\}
\end{aligned}
$$

$$
K = |\varphi'|_{\alpha,I_{\Gamma_-}} + c_{mv}c_S(\Omega,\alpha)\left(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha;\Gamma}\right)L_{\Gamma_-}^{1-\alpha} + 2\max\left\{Mc_\rho, \|\rho_0\|_{0,\Omega_0}^2, \|\rho_A\|_{0,Q_t}^2\right\} + 1,
$$

$$
M = 3 + 2\|x\|_{0,\Omega_0} + 2\delta + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}.
$$

with T_{E_0} defined in Lemma 4.31. Hence,

$$\begin{aligned}\|A_1(\Phi)\|_{1,\alpha,\Omega_0;\alpha,[0,T]} &\leq M\\ \|A_2(\Phi)\|_{1,Q_t;\alpha,[0,T]} &\leq M\\ \sup_{0\leq t\leq T}\|A_2(\Phi)(t)\|_{\alpha,Q_t} &\leq K\\ \sup_{0\leq t\leq T}\|D_A-\nabla A_2(\Phi)(t)\|_{0,Q_t} &\leq \frac{\delta}{4M}\\ \sup_{0\leq t\leq T}\|I-\nabla A_1(\Phi)(t)\|_{0,\Omega_0} &\leq \frac{1}{2}\end{aligned}$$

Furthermore, since $\max\left\{\|\rho_0\|_{0,\Omega_0},\|\rho_A\|_{0,Q_T}\right\}\leq \frac{\delta}{c(\mathrm{diam}(\Omega))}$, we obtain by Lemma 4.41

$$\left|\partial_{t_x}A_2(\Phi)(s,t_x,t=t_x)\cdot\varphi'(s)^{\perp}\right|\geq\delta.$$

Conclusively, $A: W(M,T,K,\delta)\to W(M,T,K,\delta)$. □

4.7 Continuity of A

Unfortunately, the operator A is not a contraction in $C^{1,\alpha,\alpha}(\bar{\Omega},[0,T])$. Instead, we will prove in this section that A is a contraction in the sup-norm. Due to the derivation of A, we will need to estimate the distance of ρ defined by two distinct streamline functions. We encounter the same technical difficulties as in section 4.5 caused by the lack of convexity of Ω and the piecewise definition of Φ. We therefore first focus on a bound for $\|\rho(t)-\tilde{\rho}(t)\|_{0,\Omega}$, where ρ and $\tilde{\rho}$ are given by (4.31) for two streamline functions $\Phi,\tilde{\Phi}\in W(M,T,K,\delta)$.

Lemma 4.44. *Let Ω be a $C^{2,\alpha}$ domain and $\Phi,\tilde{\Phi}\in W(M,T,K,\delta)$ and ρ, $\tilde{\rho}$ defined by (4.31). Then we have for $t\in[0,T]$*

$$\begin{aligned}&\|\rho(t)-\tilde{\rho}(t)\|_{0,\Omega}\\ &\quad\leq\max\left\{c_\rho, ML_{\rho_0}, M(L_{\rho_A}+\|\rho_A\|^2_{0,Q_T})\right\}\left(\|\tilde{\Phi}_1(t)-\Phi_1(t)\|_{0,\Omega_0}+\|\tilde{\Phi}_2(t)-\Phi_2(t)\|_{0,Q_t}\right)\end{aligned}$$

with c_ρ defined in Lemma 4.34 and L_{ρ_0},L_{ρ_A} being the Lipschitz constants of ρ_0 and ρ_A.

Proof. We conduct a case analysis in $y\in\Omega$. We denote the outer boundary part of Ω_t^1 by Γ_t^1. The interface is denoted as Γ_t^{IF}. Analogously, we denote the outer boundary part of $\tilde{\Omega}_t^1$ by $\tilde{\Gamma}_t^1$ and the interface by $\tilde{\Gamma}_t^{IF}$.

Case 1: $y\in\Omega_t^1\cap\tilde{\Omega}_t^1$.
Since $y\in\tilde{\Omega}_t^1$, the mapping $z=\Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)\in\Omega_t^1$ is well-defined. The line segment $\overrightarrow{yz}$ is not necessarily contained in Ω_t^1. Choose the point of intersection a_1 with any boundary such that $\overrightarrow{a_1z}\subset\Omega_t^1$. By the triangle inequality

$$\begin{aligned}|\rho(y,t)-\tilde{\rho}(y,t)|&=|\rho(y,t)-\rho(a_1,t)+\rho(a_1,t)-\tilde{\rho}(y,t)|\\ &\leq|\rho(y,t)-\rho(a_1,t)|+|\rho(a_1,t)-\tilde{\rho}(y,t)|\,. \qquad (4.96)\end{aligned}$$

Since y and a_1 are two points in Ω, the first term is bounded with Lemma 4.34

$$|\rho(y,t) - \rho(a_1,t)| \leq c_\rho |y - a_1|_\infty. \tag{4.97}$$

For the last term, we use the Lipschitz continuity of ρ_0. By (4.31) follows,

$$\begin{aligned}
|\rho(a_1,t) - \tilde{\rho}(y,t)| &= \left| \frac{\rho_0(\Phi_1^{-1}(a_1,t))}{1 + t\rho_0(\Phi_1^{-1}(a_1,t))} - \frac{\rho_0(\tilde{\Phi}_1^{-1}(y,t))}{1 + t\rho_0(\tilde{\Phi}_1^{-1}(y,t))} \right| \\
&= \left| \frac{\rho_0(\Phi_1^{-1}(a_1,t)) - \rho_0(\tilde{\Phi}_1^{-1}(y,t))}{\left(1 + t\rho_0(\Phi_1^{-1}(a_1,t))\right)\left(1 + t\rho_0(\tilde{\Phi}_1^{-1}(y,t))\right)} \right| \\
&\leq \left| \rho_0(\Phi_1^{-1}(a_1,t)) - \rho_0(\tilde{\Phi}_1^{-1}(y,t)) \right| \\
&\leq L_{\rho_0} \left| \Phi_1^{-1}(a_1,t) - \tilde{\Phi}_1^{-1}(y,t) \right|_\infty .
\end{aligned} \tag{4.98}$$

We will now use the identity $\tilde{\Phi}_1^{-1}(y,t) = \Phi_1^{-1}(\Phi_1(\tilde{\Phi}_1^{-1}(y,t)))$ which is valid as the domain of definition is the same for $\Phi_1(x,t)$ and $\tilde{\Phi}_1(x,t)$. Further follows for the line segment $\overrightarrow{a_1\Phi_1(\tilde{\Phi}_1^{-1}(y))} = \overrightarrow{a_1 z} \subset \Omega_t^1$ due to the choice of a_1. Thus we obtain with Theorem 2.15

$$\begin{aligned}
L_{\rho_0} \left| \Phi_1^{-1}(a_1,t) - \tilde{\Phi}_1^{-1}(y,t) \right|_\infty &= L_{\rho_0} \left| \Phi_1^{-1}(a_1,t) - \Phi_1^{-1}(\Phi_1(\tilde{\Phi}_1^{-1}(y,t))) \right|_\infty \\
&\leq L_{\rho_0} |\tilde{\Phi}_1^{-1}(a_1,t) - \Phi_1^{-1}(z,t)|_\infty \\
&\leq L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1} |a_1 - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty.
\end{aligned} \tag{4.99}$$

We will use the following ideas: Since the line segments $\overrightarrow{ya_1}$ and $\overrightarrow{a_1 z}$ are a disjoint decomposition of $\overrightarrow{yz}$ follows $|y - a_1|_\infty + |a_1 - z|_\infty = |y - z|_\infty$. Further since $y \in \tilde{\Omega}_t^1$, we use the identity $y = \tilde{\Phi}_1(\tilde{\Phi}_1^{-1}(y,t),t)$. We then get for (4.96) by (4.97) and (4.99)

$$\begin{aligned}
|\rho(y,t) - \tilde{\rho}(y,t)| &\leq |\rho(y,t) - \rho(a_1,t)| + |\rho(a_1,t) - \tilde{\rho}(y,t)|_\infty \\
&\leq \max\left\{c_\rho, L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} \left(|y - a_1|_\infty + |a_1 - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty\right) \\
&\leq \max\left\{c_\rho, L_{\rho_0} \|\nabla \Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\} \left(|y - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty\right) \\
&\leq \max\left\{c_\rho, L_{\rho_0} M\right\} |\tilde{\Phi}_1(\tilde{\Phi}_1^{-1}(t),t) - \Phi_1(\tilde{\Phi}_1^{-1}(t),t)|_{0,\Omega_t^1 \cap \tilde{\Omega}_t^1} \\
&\leq \max\left\{c_\rho, L_{\rho_0} M\right\} |\tilde{\Phi}_1(t) - \Phi_1(t)|_{0,\Omega_0}.
\end{aligned}$$

The last step followed as Φ_1 and $\tilde{\Phi}_1$ had the same argument.

Case 2: $y \in \Omega_t^2 \cap \tilde{\Omega}_t^2$

The idea is the same as in case 1. Let $z := \Phi_2(\tilde{\Phi}_2^{-1}(y,t),t) \in \Omega_t^2$. If the line segment $\overrightarrow{zy}$ intersects a boundary, then choose the intersection point a_1 such that $\overrightarrow{za_1} \subset \Omega_t^2$. Then follows pointwise by the triangle inequality

$$\begin{aligned}
|\rho(y,t) - \tilde{\rho}(y,t)| &= |\rho(y,t) - \rho(a_1,t) + \rho(a_1,t) - \tilde{\rho}(y,t)| \\
&\leq |\rho(y,t) - \rho(a_1,t)| + |\rho(a_1,t) - \tilde{\rho}(y,t)| .
\end{aligned} \tag{4.100}$$

Since y and a_1 are two points in Ω, the first term is bounded with Lemma 4.34

$$|\rho(y,t)-\rho(a_1,t)| \le c_\rho |y-a_1|_\infty. \tag{4.101}$$

For the second term of (4.100), we use the Lipschitz continuity of ρ_A. We obtain by (4.31) and with the same computations as in the proof of Lemma 4.33b)

$$\begin{aligned}|\rho(a_1,t)-\tilde{\rho}(y,t)| &= \left| \frac{\rho_A(\Phi_2^{-1}(a_1,t))}{1+(t-[\Phi_2^{-1}(a_1)]_2)\rho_A(\Phi_2^{-1}(a_1,t))} - \frac{\rho_A(\tilde{\Phi}_2^{-1}(y,t))}{1+(t-[\tilde{\Phi}_2^{-1}(y)]_2)\rho_A(\tilde{\Phi}_2^{-1}(y,t))} \right| \\ &\le (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t}) \left| \Phi_2^{-1}(a_1,t)-\tilde{\Phi}_2^{-1}(y,t) \right|_\infty .\end{aligned}$$

The identity $\Phi_2^{-1}(\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t),t) = \Phi_2^{-1}(z,t)$ is well defined, as Φ_2 and $\tilde{\Phi}_2$ have the same domain of definition. Further we know due to the choice of a_1 that $\overrightarrow{a_1 y} \subset \Omega_t^2$. We get for the previous term with Theorem 2.15

$$\begin{aligned}&(L_{\rho_A}+\|\rho_A\|^2_{0,Q_t}) \left| \Phi_2^{-1}(a_1,t)-\tilde{\Phi}_2^{-1}(y,t) \right|_\infty \\ &\quad = (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t}) \left| \Phi_2^{-1}(a_1,t)-\Phi_2^{-1}(\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t),t) \right|_\infty \\ &\quad \le (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t}) \|\nabla\Phi_2^{-1}(t)\|_{0,\Omega} |a_1-\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t)|_\infty. \end{aligned} \tag{4.102}$$

Since $y \in \tilde{\Omega}_t^2$, we use the identity $y = \tilde{\Phi}_2(\tilde{\Phi}_2^{-1}(y,t),t)$ and get for (4.100) by (4.101) and (4.102)

$$\begin{aligned}&|\rho(y,t)-\tilde{\rho}(y,t)| \\ &\le \max\left\{c_\rho, (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t})\|\nabla\Phi_2^{-1}(t)\|_{0,\Omega_t^2}\right\} \left(|y-a_1|_\infty + |a_1-\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t)|_\infty\right) \\ &\le \max\left\{c_\rho, (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t})\|\nabla\Phi_2^{-1}(t)\|_{0,\Omega_t^2}\right\} |y-\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t)|_\infty \\ &\le \max\left\{c_\rho, (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t})\|\nabla\Phi_2^{-1}(t)\|_{0,\Omega_t^2}\right\} |\tilde{\Phi}_2(\tilde{\Phi}_2^{-1}(y,t),t)-\Phi_2(\tilde{\Phi}_2^{-1}(y,t),t)|_\infty \\ &\le \max\left\{c_\rho, (L_{\rho_A}+\|\rho_A\|^2_{0,Q_t})M\right\} \|\tilde{\Phi}_2(t)-\Phi_2(t)\|_{0,\Omega_0}.\end{aligned}$$

Case 3: $y \in \Omega_t^2 \cap \tilde{\Omega}_t^1$

The mapping $z = \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t) \in \Omega_t^1$ is well defined as Φ_1 and $\tilde{\Phi}_1$ have the same domain of definition Ω_0. The line segment $\overrightarrow{zy}$ intersects the boundary Γ_t^1 at least once. Choose the point of intersection a_1 such that $\overrightarrow{a_1 z} \subset \Omega_t^1$. Using the triangle inequality we get

$$\begin{aligned}|\rho(y,t)-\tilde{\rho}(y,t)| &= |\rho(y,t)-\rho(a_1,t)+\rho(a_1,t)-\tilde{\rho}(y,t)| \\ &\le |\rho(y,t)-\rho(a_1,t)| + |\rho(a_1,t)-\tilde{\rho}(y,t)|.\end{aligned} \tag{4.103}$$

The first term of (4.103) is bounded by Lemma 4.34

$$|\rho(y,t)-\rho(a_1,t)| \le c_\rho |y-a_1|_\infty. \tag{4.104}$$

For the second term of (4.103), we conduct analogous computations to case 1.

$$\begin{aligned}|\rho(a_1,t)-\tilde{\rho}(y,t)| &= \left| \frac{\rho_0(\Phi_1^{-1}(a_1,t))}{1+t\rho_0(\Phi_1^{-1}(a_1,t))} - \frac{\rho_0(\tilde{\Phi}_1^{-1}(y,t))}{1+t\rho_0(\tilde{\Phi}_1^{-1}(y,t))} \right| \\ &\le L_{\rho_0} \left| \Phi_1^{-1}(a_1,t)-\tilde{\Phi}_1^{-1}(y,t) \right|_\infty .\end{aligned}$$

The identity $\tilde{\Phi}_1^{-1}(y,t) = \Phi_1^{-1}(\Phi_1(\tilde{\Phi}_1^{-1}(y,t)))$ is valid as both Φ_1 and $\tilde{\Phi}_1$ are defined on Ω_0. Further, the line segment $\overrightarrow{a_1\Phi_1(\tilde{\Phi}_1^{-1}(y))} = \overrightarrow{a_1 z} \subset \Omega_t^1$ due to the choice of a_1. Thus follows with Theorem 2.15

$$\begin{aligned} L_{\rho_0}\left|\Phi_1^{-1}(a_1,t) - \tilde{\Phi}_1^{-1}(y,t)\right| &= L_{\rho_0}\left|\Phi_1^{-1}(a_1,t) - \Phi_1^{-1}(\Phi_1(\tilde{\Phi}_1^{-1}(y,t),t),t)\right|_\infty \\ &\leq L_{\rho_0}\|\nabla\Phi_1^{-1}(t)\|_{0,\Omega_t^1}|a_1 - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty. \end{aligned} \tag{4.105}$$

Summing up, we get with the identity $y = \tilde{\Phi}_1(\tilde{\Phi}_1^{-1}(y,t),t)$ for (4.103) by (4.104) and (4.105)

$$\begin{aligned} |\rho(y,t) - \tilde{\rho}(y,t)| &\leq |\rho(y,t) - \rho(a_1,t)| + |\rho(a_1,t) - \tilde{\rho}(y,t)|_\infty \\ &\leq \max\left\{c_\rho, L_{\rho_0}\|\nabla\Phi_1^{-1}(t)\|_{0,\Omega_t^1}\right\}\left(|y - a_1|_\infty + |a_1 - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty\right) \\ &\leq \max\left\{c_\rho, L_{\rho_0}|\nabla\Phi_1^{-1}(t)|_{0,\Omega_t^1}\right\}\left(|y - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty\right) \\ &\leq \max\{c_\rho, L_{\rho_0}M\}\,|\tilde{\Phi}_1(\tilde{\Phi}_1^{-1}(y,t),t) - \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t)|_\infty \\ &\leq \max\{c_\rho, L_{\rho_0}M\}\,\|\tilde{\Phi}_1(t) - \Phi_1(t)\|_{0,\Omega_0}. \end{aligned}$$

Case 4 $y \in \Omega_t^1 \cap \tilde{\Omega}_t^2$
The computations are analogous to case 3 and are therefore omitted. It results

$$|\rho(y,t) - \tilde{\rho}(y,t)| \leq \max\{c_\rho, L_{\rho_0}M\}\,\|\tilde{\Phi}_1(t) - \Phi_1(t)\|_{0,\Omega_0}.$$

Case 5 $y \in \tilde{\Omega}_t^1 \cap \Omega\backslash\Omega_t$
Denote $z = \Phi_1(\tilde{\Phi}_1^{-1}(y,t),t) \in \Omega_t^1$. The line segment $\overrightarrow{zy}$ intersects the boundary of Γ_t^1 at least once. Choose the point of intersection a_1 such that $\overrightarrow{za_1} \subset \Omega_t^1$. As $a_1 \in \Gamma_t^1$ and by (4.31), we have $\rho(a_1,t) = 0$. We can thus add $\rho(a_1,t)$ without changing the value of the term.

$$\begin{aligned} |\rho(y,t) - \tilde{\rho}(y,t)| &\leq |0 - \tilde{\rho}(y,t)| \\ &= |\rho(a_1,t) - \tilde{\rho}(y,t)|\,. \end{aligned}$$

This is the setting of case 1 starting from step (4.98). We obtain

$$|\rho(a_1,t) - \tilde{\rho}(y,t)| \leq L_{\rho_0}M\|\Phi_1(t) - \tilde{\Phi}_1(t)\|_{0,\Omega_0}.$$

Case 6: $y \in \tilde{\Omega}_t^2 \cap \Omega\backslash\Omega_t$**,** $y \in \Omega_t^1 \cap \Omega\backslash\tilde{\Omega}_t$**,** $y \in \Omega_t^2 \cap \Omega\backslash\tilde{\Omega}_t$
These three cases are analogous to case 5 and are therefore omitted.

Summing up all six cases, we get

$$\begin{aligned} &\|\rho(t) - \tilde{\rho}(t)\|_{0,\Omega} \\ &\quad\leq \max\left\{c_\rho, ML_{\rho_0}, M(L_{\rho_A} + \|\rho_A\|_{0,Q_T}^2)\right\}\left(\|\Phi_1(t) - \tilde{\Phi}_1(t)\|_{0,\Omega_0} + \|\Phi_2(t) - \tilde{\Phi}_2(t)\|_{0,Q_t}\right). \end{aligned}$$

□

The next Lemma shows the continuity of A in the $C^0(\Sigma_t)$-norm. The Lipschitz constant depends on the time T. For small T, A is thus a contraction.

Lemma 4.45. *Let Ω be a $C^{2,\alpha}$ domain, ρ be defined by (4.31) and $u_A \in C^{2,\alpha}(\Gamma)$. It follows for $\Phi, \tilde{\Phi} \in W(M,T,K,\delta)$*

$$\sup_{0\leq t\leq T} \|A(\Phi)(t) - A(\tilde{\Phi})(t)\|_{0,\Sigma_t} \leq Tc_L \sup_{0\leq t\leq T} \left(\|\tilde{\Phi}_1(t) - \Phi_1(t)\|_{0,\Omega_0} + \|\tilde{\Phi}_2(t) - \Phi_2(t)\|_{0,Q_t} \right)$$

with

$$c_L = \max\left\{ c_{mv} c_S(\Omega,\alpha)(\kappa_\rho(\alpha) + \|u_A\|_{2,\alpha,\Gamma}), c_\rho, ML_{\rho_0}, M(L_{\rho_A} + \|\rho_A\|^2_{0,Q_t}) \right\}$$

with c_ρ and $\kappa_\rho(\alpha)$ defined in Lemma 4.34 and $c_S(\Omega,\alpha)$ defined in Lemma 4.15 and L_{ρ_0}, L_{ρ_A} being the Lipschitz constants for ρ_0 and ρ_A.

Proof. First, we note that by Lemma 4.34 follows $\rho(\cdot,t) \in C^\alpha(\bar{\Omega})$.
The norm is given due to the definition of the product space as

$$\|A(\Phi)(t) - A(\tilde{\Phi})(t)\|_{0,\Sigma_t} = \|A_1(\Phi)(t) - A_1(\tilde{\Phi})(t)\|_{0,\Omega_0} + \|A_2(\Phi)(t) - A_2(\tilde{\Phi})(t)\|_{0,Q_t}.$$

We will demonstrate the computations for $A_2(\Phi)$. We have by rearranging terms

$$\begin{aligned}
&\|A_2(\Phi)(t) - A_2(\tilde{\Phi})(t)\|_{0,Q_t} \\
&= \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t \int_\Omega \nabla G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - \int_\Omega \nabla G(\tilde{\Phi}_2(s,t_x,\mu),y)\tilde{\rho}(y,\mu)\,dy\,d\mu \right|_\infty \\
&\quad + \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t E_0(\Phi_2(s,t_x,\mu)) - E_0(\tilde{\Phi}_2(s,t_x,\mu))\,d\mu \right|_\infty \\
&\leq \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t \int_\Omega \nabla G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - \int_\Omega \nabla G(\tilde{\Phi}_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\,d\mu \right|_\infty \qquad (4.106) \\
&\quad + \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t \int_\Omega \nabla G(\tilde{\Phi}_2(s,t_x,\mu),y)\left[\rho(y,\mu) - \tilde{\rho}(y,\mu)\right]\,dy\,d\mu \right|_\infty \qquad (4.107) \\
&\quad + \sup_{(s,t_x)\in Q_t} \left| \int_{t_x}^t E_0(\Phi_2(s,t_x,\mu)) - E_0(\tilde{\Phi}_2(s,t_x,\mu))\,d\mu \right|_\infty. \qquad (4.108)
\end{aligned}$$

Due to 4.15, $(G_1\rho)(t) \in C^1(\Omega)$. We thus have by Lemma 2.20 and Lemma 4.15 for (4.106)

$$\begin{aligned}
&\left| \int_{t_x}^t \int_\Omega \nabla G(\Phi_2(s,t_x,\mu),y)\rho(y,\mu)\,dy - \int_\Omega \nabla G(\tilde{\Phi}_2(s,t_x,\mu),y)\rho(y,\mu)\,dy\,d\mu \right|_\infty \\
&\leq \int_0^t \left| (G_1\rho)(\Phi_2(s,t_x,\mu)) - (G_1\rho)(\tilde{\Phi}_2(s,t_x,\mu)) \right|_\infty d\mu \\
&\leq \int_0^t c_{mv} \|\nabla(G_1\rho)(\mu)\|_{0,\Omega} \left| \Phi_2(s,t_x,\mu) - \tilde{\Phi}_2(s,t_x,\mu) \right|_\infty d\mu \\
&\leq Tc_{mv}c_S(\Omega,\alpha) \sup_{0\leq t\leq T} \|\rho(t)\|_{\alpha,\Omega} \sup_{0\leq t\leq T} \left\| \Phi_2(t) - \tilde{\Phi}_2(t) \right\|_{0,Q_t}. \qquad (4.109)
\end{aligned}$$

We bound (4.108) by Lemma 2.20 and Lemma 4.15

$$\begin{aligned}\left|\int_{t_x}^{t} E_0(\Phi_2(s,t_x,\mu)) - E_0(\tilde{\Phi}_2(s,t_x,\mu))\,d\mu\right|_\infty &\le \int_0^t \left|E_0(\Phi_2(s,t_x,\mu)) - E_0(\tilde{\Phi}_2(s,t_x,\mu))\right|_\infty d\mu \\ &\le \int_0^t c_{mv}\,\|\nabla E_0\|_{0,\Omega}\,|\Phi_2(s,t_x,\mu) - \tilde{\Phi}_2(s,t_x,\mu)|_\infty\,d\mu \\ &\le T c_{mv} c_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha,\Gamma} \sup_{0\le t\le T}\left\|\Phi_2(t) - \tilde{\Phi}_2(t)\right\|_{0,Q_t}. \qquad (4.110)\end{aligned}$$

Last, (4.107) is bounded by Lemma 4.16 and 4.44

$$\begin{aligned}&\sup_{(s,t_x)\in Q_t}\left|\int_{t_x}^{t}\int_\Omega \nabla G(\tilde{\Phi}_2(s,t_x,t),y)\,[\rho(y,\mu)-\tilde{\rho}(y,\mu)]\;dy\,d\mu\right|_\infty \\ &\le \int_0^t \sup_{x\in\Omega}\left|\int_\Omega \nabla G(x,y)\,[\rho(y,\mu)-\tilde{\rho}(y,\mu)]\;dy\right|_\infty d\mu \\ &\le \int_0^t \sup_{x\in\Omega}\left|\int_\Omega |\nabla G(x,y)|_\infty\,|\rho(y,\mu)-\tilde{\rho}(y,\mu)|\;dy\right|_\infty d\mu \\ &\le \int_0^t c(\operatorname{diam}\Omega)\,\|\rho(\mu)-\tilde{\rho}(\mu)\|_{0,\Omega}\;d\mu \\ &\le Tc(\operatorname{diam}\Omega)\sup_{0\le t\le T}\|\rho(t)-\tilde{\rho}(t)\|_{0,\Omega} \\ &\le T\max\left\{c_\rho, ML_{\rho_0}, M(L_{\rho_A}+\|\rho_A\|_0^2)\right\}\sup_{0\le t\le T}\left(\left\|\tilde{\Phi}_1(t)-\Phi_1(t)\right\|_{0,\Omega_0}+\left\|\tilde{\Phi}_2(t)-\Phi_2(t)\right\|_{0,Q_t}\right). \\ &\hspace{30em}(4.111)\end{aligned}$$

Summing up, we get with (4.109), (4.110) and (4.111) and by Lemma 4.34

$$\begin{aligned}&\|A_2(\Phi)(t) - A_2(\tilde{\Phi})(t)\|_{0,Q_t} \le Tc_{mv}\kappa_\rho(\alpha)c_S(\Omega,\alpha)\sup_{0\le t\le T}\|\Phi_2(t)-\tilde{\Phi}_2(t)\|_{0,Q_t} \\ &\quad + Tc_{mv}c_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha,\Gamma}\sup_{0\le t\le T}\|\Phi_2(t)-\tilde{\Phi}_2(t)\|_{0,Q_t} \\ &\quad + T\max\left\{c_\rho, ML_{\rho_0}, M(L_{\rho_A}+\|\rho_A\|_0^2)\right\}\sup_{0\le t\le T}\left(\|\tilde{\Phi}_1(t)-\Phi_1(t)\|_{0,\Omega_0}+\|\tilde{\Phi}_2(t)-\Phi_2(t)\|_{0,Q_t}\right) \\ &\le Tc_L\left(\|\tilde{\Phi}_1-\Phi_1\|_{0,\Omega_0;0,[0,T]}+\|\tilde{\Phi}-\Phi\|_{0,Q_t,0,[0,T]}\right)\end{aligned}$$

with

$$c_L = \max\left\{c_{mv}c_S(\Omega,\alpha)(\kappa_\rho(\alpha)+\|u_A\|_{2,\alpha,\Gamma}), c_\rho, ML_{\rho_0}, M(L_{\rho_A}+\|\rho_A\|_{0,Q_T}^2)\right\}.$$

The computations are done analogously in case of the operator A_1. We obtain the bound

$$\sup_{0\le t\le T}\|A_1(\Phi)(t)-A_1(\tilde{\Phi})(t)\|_{0,\Omega_0} \le Tc_L\sup_{0\le t\le T}\left(\|\tilde{\Phi}_1(t)-\Phi_1(t)\|_{0,\Omega_0}+\|\tilde{\Phi}_2(t)-\Phi_2(t)\|_{0,Q_t}\right).$$

Conclusively, we obtain for the operator A

$$\sup_{0\le t\le T}\|A(\Phi)(t)-A(\tilde{\Phi})(t)\|_{0,\Sigma_t} \le Tc_L\left(\|\tilde{\Phi}_1-\Phi_1\|_{0,\Omega_0;0,[0,T]}+\|\tilde{\Phi}_2-\Phi_2\|_{0,Q_t;0,[0,T]}\right).$$

$\square$

4.8 Existence of a Fixed Point

It is not possible to apply the Banach fixed point theorem to show the existence of a fixed point to the operator A as A is not a contraction in $C^{1,\alpha,\alpha}(\bar{\Omega},[0,T])$. Instead, we follow the idea of Huang [44, 43] and show the existence with a compactness argument.

We start with collecting results from literature that will be needed in the following. First, we introduce the concept of a *precompact* set by the following Lemma.

Lemma 4.46. *[3, 2.5, p.100] For every subset A of a metric space (X,d) are equivalent*

1. *Every sequence in A contains a convergent subsequence with limit in A.*
2. *(A,d) is complete and A is precompact.*
3. *A is compact.*

The goal is to prove that $W_1(M,T)$ and $W_1(M,T,K,\delta)$ are precompact and additionally that the sets are complete. Therefore, we list two results on precompactness in Hölder spaces.

Lemma 4.47. *[3, p.136, U2.15] Let $\bar{\Omega}$ be a compact domain in $\mathbb{R}^n$ and let S be a bounded set in $C^{0,1}(\bar{\Omega})$. Then S is precompact in $C^{0,\beta}(\bar{\Omega})$ for $0<\beta<1$.*

Increasing the regularity of the domain and the bounded set S, we obtain

Lemma 4.48. *[34, Lemma 6.36] Let Ω be a $C^{1,\alpha}$ domain in $\mathbb{R}^2$ and let S be a bounded set in $C^{1,\alpha}(\bar{\Omega})$. Then S is precompact in $C^{j,\beta}(\bar{\Omega})$ if $j+\beta<1+\alpha$.*

We can conclude that the sets $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are precompact in $C^{1,\beta;\beta}(\bar{\Omega},[0,T])$ for $0\leq\beta\leq\alpha$.

Lemma 4.49. *The sets $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are precompact in $C^{1,\beta;\beta}(\bar{\Omega},[0,T])$.*

Proof. $\bar{\Omega}$ is a compact domain and $[0,T]$ is a compact interval. Further $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are bounded sets in $C^{1,\alpha;\alpha}(\bar{\Omega}_0,[0,T])$ and $C^{1,\alpha;\alpha}(Q_t,[0,T])$. Then $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are precompact in $C^{1,\beta;\beta}(\bar{\Omega}_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$ by Lemma 4.48 and 4.47. □

Second, we show that $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are closed w.r.t. to the $C^{1,\beta;\beta}$ norm for $0<\beta<\alpha$.

Lemma 4.50. *$W_1(M,T)$ and $W_2(M,T,K,\delta)$ are closed in $C^{1,\beta;\beta}(\bar{\Omega}_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$ for all $0<\beta<\alpha$.*

Proof. To show that $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are closed in $C^{1,\beta;\beta}(\bar{\Omega}_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$, we have to show that the limit of every convergent sequence in W_i is in W_i.
We begin with $W_1(M,T)$. Pick a convergent sequence $\{\Phi_1^n\}_n \in W_1(M,T)$ with limit function Φ_1, i.e.

$$\|\Phi_1^n - \Phi_1\|_{1,\beta,\Omega_0;\beta,[0,T]} \xrightarrow{n\to\infty} 0.$$

Conclusively, Φ_1^n and $\nabla\Phi_1^n$ converge uniformly to Φ_1 and $\nabla\Phi_1$. It follows that Φ_1^n and $\nabla\Phi_1^n$ also converge pointwise to Φ_1 and $\nabla\Phi_1$.
We have to show that the limit function $\Phi_1 \in W_1(M,T)$. For the initial condition, we get pointwise for every $x \in \Omega_0$

$$|\Phi_1(x,0) - x|_\infty = \lim_{n\to\infty} |\Phi_1(x,0) - \Phi_1^n(x,0)|_\infty = 0.$$

Hence, $\Phi_1(x,0) = x$.

We show that $\|\Phi_1\|_{1,\alpha,\Omega_0;\alpha,[0,T]} \le M$. Since $\Phi_1^n \in W_1(M,T)$, we have for every n a constant $M_1^n \le M$, such that for every $x, x_1, x_2 \in \Omega_0$ and $t, t_1, t_2 \in [0,T]$ follows

$$|\Phi_1^n(x_1,t)|_\infty + |\nabla\Phi_1^n(x_2,t)|_\infty + |\Phi_1^n(x,t_1)| + |\Phi_1^n(x,t_2)| \le M_1^n \tag{4.112}$$

Next, we show that $\nabla\Phi_1 \in C^\alpha(\Omega)$. Let therefore be $M_2^n \le M$, $x \ne y \in \Omega_0$ be fixed and $t \in [0,T]$. We have pointwise

$$|\nabla\Phi_1(x,t) - \nabla\Phi_1(y,t)|_\infty = \lim_{n\to\infty} |\nabla\Phi_1^n(x,t) - \nabla\Phi_1^n(y,t)|_\infty \le \lim_{n\to\infty} M_2^n |x-y|_\infty^\alpha. \tag{4.113}$$

It follows that $\nabla\Phi_1 \in C^\alpha(\Omega_0)$ Analogously, we obtain for $t_1 \ne t_2 \in [0,T]$ and $x \in \Omega_0$ and constants $M_2^n, M_3^n \le M$

$$|\Phi_1(x,t_1) - \Phi_1(x,t_2)|_\infty = \lim_{n\to\infty} |\Phi_1^n(x,t_1) - \Phi_1^n(x,t_2)|_\infty \le \lim_{n\to\infty} M_3^n |t_1 - t_2|_\infty^\alpha, \tag{4.114}$$

$$|\nabla\Phi_1(x,t_1) - \nabla\Phi_1(x,t_2)|_\infty = \lim_{n\to\infty} |\nabla\Phi_1^n(x,t_1) - \nabla\Phi_1^n(x,t_2)|_\infty \le \lim_{n\to\infty} M_4^n |t_1 - t_2|_\infty^\alpha. \tag{4.115}$$

We then get pointwise for all $x, x_i \in \Omega_0$ with $i = 1,..,4$ and $t, t_i \in [0,T]$ with $i = 1,..,6$ and since $\sum_{i=1}^4 M_i^n \le M$ for every n

$$\begin{aligned}
&|\Phi_1(x_1,t)|_\infty + |\nabla\Phi_1(x_2,t)|_\infty + \frac{|\nabla\Phi_1(x_3) - \nabla\Phi_1(x_4)|_\infty}{|x_3 - x_4|_\infty} + |\Phi_1(x,t_1)| + |\Phi_1(x,t_2)| \\
&\quad + \frac{|\Phi_1(x,t_3) - \Phi_1(x,t_4)|_\infty}{|t_3 - t_4|_\infty} + \frac{|\nabla\Phi_1(x,t_5) - \nabla\Phi_1(x,t_6)|_\infty}{|t_5 - t_6|_\infty} \\
&= \lim_{n\to\infty} \Bigg(|\Phi_1^n(x_1,t)| + |\nabla\Phi_1^n(x_2,t)|_\infty + \frac{|\nabla\Phi_1^n(x_3,t) - \nabla\Phi_1^n(x_4,t)|_\infty}{|x_3 - x_4|_\infty} + |\Phi_1^n(x,t_1)| + |\Phi_1^n(x,t_2)| \\
&\quad + \frac{|\Phi_1^n(x,t_3) - \Phi_1^n(x,t_4)|_\infty}{|t_3 - t_4|_\infty} + \frac{|\nabla\Phi_1^n(x,t_5) - \nabla\Phi_1^n(x,t_6)|_\infty}{|t_5 - t_6|_\infty} \Bigg) \\
&\le \lim_{n\to\infty} (M_1^n + M_2^n + M_3^n + M_4^n) \\
&\le \lim_{n\to\infty} M = M.
\end{aligned}$$

We thus obtain $\|\Phi_1\|_{1,\beta,\Omega_0;\beta,[0,T]} \leq M$.

Last, we check with the pointwise convergence of the sequence $\Phi_n(x,t)$

$$|I - \nabla\Phi_1(x,t)|_\infty = \lim_{n\to\infty} |I - \nabla\Phi_1^n(x,t)|_\infty \leq \frac{1}{2}. \tag{4.116}$$

Thus, for the limit Φ_1 of every sequence $\Phi_1^n \in W_1(M,T)$ with $\|\Phi_1 - \Phi_1^n\|_{1,\beta,\Omega_0;\beta,[0,T]} \xrightarrow{n\to\infty} 0$ follows $\Phi_1 \in W_1(M,T)$. Consequently, $W_1(M,T)$ is closed in $C^{1,\beta,\beta}(\bar{\Omega}_0,[0,T])$ for all $0<\beta<\alpha$.

We now pick a convergent sequence $\{\Phi_2^n\}_n \in W_2(M,T,K,\delta)$ with limit function Φ_2, i.e.

$$\|\Phi_2^n - \Phi_2\|_{1,\beta,Q_t;\beta,[0,T]} \xrightarrow{n\to\infty} 0.$$

Conclusively, Φ_2^n and $\nabla\Phi_2^n$ converge uniformly to Φ_2 and $\nabla\Phi_2$. It follows that Φ_2^n and $\nabla\Phi_2^n$ also converge pointwise to Φ_2 and $\nabla\Phi_2$. We can now check the initial condition and boundedness of Φ_2 in an analogously to Φ_1. Let us explicitly present the restrictions of $W_2(M,T,K,\delta)$ that are different to $W_1(M,T)$. For every Φ_2^n, we have the corresponding matrix D^n. It then follows

$$|D - \nabla\Phi_2(t)|_\infty = \lim_{n\to\infty} |D^n - \nabla\Phi_2^n(t)|_\infty \leq \frac{\delta}{4M}.$$

Last, we obtain

$$|\partial_{t_x}\Phi_2(s,t_x,t=t_x)\cdot\varphi'^{\perp}| = \lim_{n\to\infty} |\partial_{t_x}\Phi_2^n(s,t_x,t=t_x)\cdot\varphi'^{\perp}|_\infty \geq \delta > 0.$$

Thus, for the limit Φ_2 of every sequence $\Phi_2^n \in W_2(M,T,K,\delta)$ with $\|\Phi_2 - \Phi_2^n\|_{1,\beta,Q_t;\beta,[0,T]} \xrightarrow{n\to\infty} 0$ follows $\Phi_2 \in W_2(M,T,K,\delta)$. Consequently, $W_2(M,T,K,\delta)$ is closed in $C^{1,\beta,\beta}(Q_t,[0,T])$ for all $0<\beta<\alpha$. □

Lemma 4.51. *Every sequence in $W_1(M,T)$ and $W_2(M,T,K,\delta)$ contains a convergent subsequence in $C^{1,\beta;\beta}(\Omega_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$ whose limit is in $W_1(M,T)$ and $W_2(M,T,K,\delta)$. The sets $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are compact subsets of $C^{1,\beta;\beta}(\Omega_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$.*

Proof. Due to Lemma 4.50, the sets $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are closed in $C^{1,\beta;\beta}(\Omega_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$. By Lemma 4.5, the Hölder space $C^{1,\beta,\beta}(\Sigma_t,[0,T])$ is a Banach space. As closed subsets of a Banach space, $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are complete w.r.t. $C^{1,\beta,\beta}(\Sigma_t,[0,T])$. By Lemma 4.49, $W_1(M,T)$ and $W_2(M,T,K,\delta)$ are precompact in $C^{1,\beta;\beta}(\Omega_0,[0,T])$ and $C^{1,\beta;\beta}(Q_t,[0,T])$. The assertion follows with Lemma 4.46. □

We now show that the set $W(M,T,K,\alpha)$ is compact in the $C^{1,\beta;\beta}(\Sigma_t,[0,T])$-norm.

Theorem 4.52. *$W(M,T,K,\delta)$ is a compact subset of $C^{1,\beta;\beta}(\Sigma_t,[0,T])$.*

Proof. The product space $W_1(M,T)\times W_2(M,T,K,\delta)$ is compact in $C^{1,\beta;\beta}(\Sigma_t,[0,T])$ as each factor is compact [64, Theorem 17.8]. We will now prove that

$W(M,T,K,\delta) \subset W_1(M,T) \times W_2(M,T,K,\delta)$ is closed in $C^{1,\beta;\beta}(\Sigma_t,[0,T])$ and consequently is also compact.
Therefore, choose a convergent sequence $\Phi_n = (\Phi_1^n, \Phi_2^n) \in W(M,T,K,\delta)$ with

$$\|\Phi - \Phi_n\|_{1,\beta,\Sigma_t,\beta,[0,T]} \xrightarrow{n\to\infty} 0.$$

and show that the limit function Φ fulfils the interface condition. Φ_1^n and Φ_2^n converge uniformly and consequently pointwise to Φ_1 and Φ_2. Then we have pointwise for all x_0 and $(s_0, 0)$ with $x_0 = \varphi(s_0)$

$$|\Phi_1(x_0,t) - \Phi_2(s_0,0,t)|_\infty = \lim_{n\to\infty} |\Phi_1^n(x_0,t) - \Phi_2^n(s_0,0,t)|_\infty = 0. \tag{4.117}$$

It follows that the interface condition is inherited to the limit and that every limit function $\Phi \in W(M,T,K,\delta)$. We conclude that $W(M,T,K,\delta)$ is a closed subset of the compact product set $W_1(M,T) \times W_2(M,T,K,\delta)$ and thus also a compact subset of $C^{1,\beta,\beta}(\Sigma_t,[0,T])$. □

We arrive at the final result. We choose the following sequence

$$\Phi_n = A(\Phi_{n-1}), \qquad \Phi_0 \in W(M,T,M,\delta) \text{ arbitrary.} \tag{4.118}$$

and prove that a fixed point $\Phi \in W(M,T,K,\delta)$ exists. From now on let $\{\Phi_n\}_n$ be understood as defined in (4.118).

Theorem 4.53. *Let Ω be a $C^{2,\alpha}$ domain. Let T, M and K be defined in Theorem 4.43. Then the sequence Φ_n has a convergent subsequence in the $C^{1,\beta;\beta}(\Sigma_t,[0,T])$-norm with limit Φ in $W(M,T,K,\delta)$.*

Proof. As A is a selfmap due to Theorem 4.43, Φ_n is a sequence in $W(M,T,M,\delta)$. The assertion follows with Lemma 4.52 and Lemma 4.46. □

We can now show that there exists a unique fixed point Φ to the operator A.

Theorem 4.54 (Existence of a fixed point to A)**.** *Let Ω be a $C^{2,\alpha}$ domain. Let $\hat{T}$, M and K be defined as in Theorem 4.43. Let $T = \min\left\{\hat{T}, \frac{1}{2}c_L\right\}$ with c_L defined in Lemma 4.45 and* $\max\{\|\rho_A\|_{0,Q_T}, \|\rho_0\|_{0,\Omega_0}\} \leq \frac{\delta}{c(\mathrm{diam}(\Omega))}$.
Then the sequence $\{\Phi_n\}_n$ defined in (4.118) converges to $\Phi \in W(M,T,K,\delta)$. Further, the limit Φ is the unique fixed point for the operator A.

Proof. **Existence:**
By Lemma 4.45 and the choice of $T \leq \frac{1}{2c_L}$, we first obtain the existence of a limit function

$\Phi \in C^{0,0}(\Sigma_t, [0,T])$. It follows

$$\begin{aligned}
\|\Phi_{n+1} - \Phi_n\|_{0,\Sigma_t,0,[0,T]} &= \|A(\Phi_n) - A(\Phi_{n-1})\|_{0,\Sigma_t;0,[0,T]} \\
&\leq T c_L \|\Phi_n - \Phi_{n-1}\|_{0,\Sigma_t;0,[0,T]} \\
&\leq \frac{1}{2} \|\Phi_n - \Phi_{n-1}\|_{0,\Sigma_t;0,[0,T]} \\
&\leq \left(\frac{1}{2}\right)^n \|\Phi_1 - \Phi_0\|_{0,\Sigma_t,0,[0,T]} .
\end{aligned}$$

By Theorem 4.53, there exists a convergent subsequence $\Phi_{n_l} \in W(M,T,K,\delta)$ of Φ_n with limit $\tilde{\Phi} \in W(M,T,K,\delta)$. We will prove that $\tilde{\Phi}$ and Φ are identical by comparing Φ and $\tilde{\Phi}$ in the sup-norm. We have

$$\begin{aligned}
\|\Phi - \tilde{\Phi}\|_{0,\Sigma_t;0,[0,T]} &= \|\Phi - \Phi_n + \Phi_n - \Phi_{n_l} + \Phi_{n_l} - \tilde{\Phi}\|_{0,\Sigma_t;0,[0,T]} \\
&\leq \|\Phi - \Phi_n\|_{0,\Sigma_t;0,[0,T]} + \|\Phi_n - \Phi_{n_l}\|_{0,\Sigma_t;0,[0,T]} + \|\Phi_{n_l} - \tilde{\Phi}\|_{0,\Sigma_t;0,[0,T]}. \quad (4.119)
\end{aligned}$$

It is immediately clear that

$$\begin{aligned}
\|\Phi - \Phi_n\|_{0,\Sigma_t;0,[0,T]} &\xrightarrow{n\to\infty} 0 \\
\|\Phi_{n_l} - \tilde{\Phi}\|_{0,\Sigma_t;0,[0,T]} &\xrightarrow{l\to\infty} 0
\end{aligned}$$

as the sequences Φ_n and Φ_{n_l} converge to their respective limits. It is left to show that for $(n,l) \to (\infty, \infty)$ follows

$$\|\Phi_n - \Phi_{n_l}\|_{0,\Sigma_t;0,[0,T]} \to 0.$$

Φ_{n_l} is a subsequence of Φ_n. As Φ_n is converges to $\Phi \in C^{0,0}(\Sigma_t, [0,T])$, Φ_{n_l} must converge to the same limit as subsequence. We can conclude that $\|\tilde{\Phi} - \Phi\|_{0,\Sigma_t;0,[0,T]} \xrightarrow{n\to\infty} 0$ and thus $\Phi = \tilde{\Phi}$ due to the choice of the sup-norm.

We have shown that Φ_n converges to a limit function $\Phi \in W(M,T,K,\delta)$. Still, it is not proved whether Φ is a fixed point to A. Using again Lemma 4.45, we obtain

$$\begin{aligned}
\|\Phi - A(\Phi)\|_{0,\Sigma_t;0,[0,T]} &= \lim_{n\to\infty} \|\Phi_n - A(\Phi)\|_{0,\Sigma_t;0,[0,T]} \\
&\leq \lim_{n\to\infty} \|A(\Phi_{n-1}) - A(\Phi)\|_{0,\Sigma_t;0,[0,T]} \\
&\leq \frac{1}{2} \lim_{n\to\infty} \|\Phi_n - \Phi\|_{0,\Sigma_t;0,[0,T]} = 0.
\end{aligned}$$

Thus, $\Phi \in W(M,T,K,\delta)$ is a fixed point to the operator A.

Uniqueness: Assume that there is a second fixed point $\hat{\Phi}$ to A. We obtain with Lemma 4.45

$$\|\hat{\Phi} - \Phi\|_{0,\Sigma_t;0,[0,T]} = \|A(\hat{\Phi}) - A(\Phi)\|_{0,\Sigma_t;0,[0,T]} \leq \frac{1}{2} \|\hat{\Phi} - \Phi\|_{0,\Sigma_t;0,[0,T]}$$

This is only possible if $\Phi = \hat{\Phi}$. Thus the fixed point is unique. □

With the existence of a fixed point $\Phi \in W(M,T,K,\delta)$, we can also conclude that there exists a unique classical solution to (CP 4.2).

Theorem 4.55 (Existence and uniqueness of a classical solution (u,ρ))**.** *Let Ω be a $C^{2,\alpha}$ domain. Let M, T and K be as defined in Theorem 4.54 and let* $\max\{\|\rho_A\|_{0,Q_T}, \|\rho_0\|_{0,\Omega_0}\} \le \frac{\delta}{c(\operatorname{diam}\Omega)}$. *Then there exists a classical unique solution $(u,\rho) \in C^{2,\alpha;0}(\bar{\Omega},[0,T]) \times C^{0,1;\alpha}(\bar{\Omega},[0,T])$ to the coupled problem (CP 4.2) with streamline function $\Phi \in W(M,T,K,\delta)$.*

Proof. By Theorem 4.54, we have the existence of a fixed point $\Phi \in W(M,T,K,\delta)$ to the operator A. Due to the construction of A, the fixed point Φ is thus the streamline function corresponding to the solution ρ of (CP 4.2). As $\Phi \in W(M,T,K,\delta)$, the images of Φ_1 and Φ_2 are non-overlapping. Lemma 4.26 and 4.29 ensure the existence of the inverse functions $\Phi_1^{-1} \in C^1(\Omega)$ and $\Phi_2^{-1} \in C^1(\Omega)$. By Theorem 4.20, we have the solution $\rho \in C^{0,0}(\bar{\Omega},[0,T])$. Lemma 4.34 with $\beta = 1$ and Lemma 4.36 give the Lipschitz continuity in space and the α-Hölder continuity in time, i.e. $\rho \in C^{0,1;\alpha}(\Omega,[0,T])$. By Corollary 2.14 we obtain the extension up to the boundary $\rho \in C^{0,1;\alpha}(\bar{\Omega},[0,T])$.

As $\rho(\cdot,t) \in C^{0,1}(\bar{\Omega}) \subset C^{0,\alpha}(\bar{\Omega})$, we have by Lemma 4.11

$$u(x,t) = \int_{\Omega} G(x,y)\rho(y,t)\,dy + u_0(x)$$

and $u \in C^{2,\alpha;0}(\bar{\Omega},[0,T])$.

Thus we found the unique solution $(u,\rho) \in C^{2,\alpha;0}(\bar{\Omega},[0,T]) \times C^{0,1;\alpha}(\bar{\Omega},[0,T])$ to (CP 4.2) with streamline function $\Phi \in W(M,T,K,\delta)$. $\square$

4.9 Continuation of the Solution in Time

We proved the short time existence for a solution (u,ρ) to (CP 4.2) on a time interval $[0,T]$, where T depends on the choice of the boundary data u_A and ρ_A, the initial distribution ρ_0, the geometry of the domain Ω and α indicating the Hölder regularity of these quantities. We will now show that the solution can be continued in time. The idea is to use the solution (u,ρ) in $[0,T]$ and define a new initial distribution by $\rho_0(x) = \rho(x,T)$. Since $\rho \in C^{0,1;0,\alpha}(\Omega,[0,T])$, the new initial distribution inherits the regularity. The proof of short time existence can then be applied again on the new interval of existence $[T,T_2]$.

Theorem 4.56 (Continuation of the solution)**.** *Let Ω be a $C^{2,\alpha}$ domain, $u_A \in C^{2,\alpha}(\Omega)$ and $\rho_A \in C^{0,1}(I_{\Gamma_-} \times [0,\infty])$. Let $L_{\hat{\rho}_0} := c_\rho$ be the Lipschitz constant of $\rho(x,T)$ as defined in Lemma 4.34. If $(H_{convex}(\operatorname{supp}_{x\in\Omega}\{\rho(x,T)\}))\backslash\Omega_- \subset \Omega$, then the solution (u,ρ) obtained in Theorem 4.55 can be extended on a time interval $[T,T_2]$, where T_2 depends on α, the boundary data u_A and ρ_A, the geometry of the domain Ω and $L_{\hat{\rho}_0}$.*

Proof. By Theorem 4.55 follows $\rho \in C^{0,1;\alpha}(\bar{\Omega},[0,T])$. The charge distribution at time T is thus given by $\rho(x,T)$ which is a Lipschitz continuous function with respect to x. By Lemma 4.34, the Lipschitz constant is given by $L_{\hat{\rho}_0} = c_\rho$. We now define

$$\rho(x,T) =: \hat{\rho}_0(x) \tag{4.120}$$

and obtain the new initial domain as $\hat{\Omega}_0 = H_{convex}(\text{supp}_{x\in\Omega}\{\hat{\rho}_0(x)\})\backslash\Omega_- \subset \Omega$. We have to check if $\hat{\rho}_0(x) = \rho_A(x,T)$ for $x \in \Gamma_-$. This is fulfilled automatically due to the choice of $\rho_A \in C^{0,1}(I_{\Gamma_-}, [0,\infty])$ and since $\hat{\rho}_0$ consists of previously inflowing points in a neighbourhood of Γ_-.
We can now apply Theorem 4.55 to the new setting and obtain a solution
$(u,\rho) \in C^{2,\alpha;0}(\bar{\Omega},[T,T_2]) \times C^{0,1;\alpha}(\bar{\Omega},[T,T_2])$. The constant T_2 depends on the boundary data u_A and ρ_A, the geometry of the domain Ω, α and the Lipschitz constant of $\hat{\rho}_0$, i.e. $L_{\hat{\rho}_0} := c_\rho$. □

The last Theorem thus gives the continuation of the solution (u,ρ) in time. The time of existence decreases with every continuation step as the constant $L_{\hat{\rho}}$ and the size of Ω_0 increase. When the charge support increases, then the distance between Γ_+ and Ω_0 decreases which reflects into the time of existence T.

4.10 Remarks about the Chapter

In this Chapter, we presented an approach using a system of integro-differential operators to show the short time existence of the time dependent coupled problem (CP 4.2). We therefore generalized the works [44, 55] by the implementation of inflow boundary data ρ_A and Dirichlet boundary conditions u_A for the Poisson equation. Further, we had the additional difficulty of working on a nonconvex and not simply connected domain Ω.

To obtain the short time existence of a solution in Theorem 4.55, $\|\rho_A\|_{0,Q_t}$ and $\|\rho_0\|_{0,\Omega_0}$ have to obey certain restrictions. Both quantities are bounded by $\frac{\delta}{c(\operatorname{diam}\Omega)}$ where the constant δ is connected to the choice of boundary data u_A through the vector field E_0. There are two strategies to follow. For a given domain

1. fix u_A and choose ρ_A and ρ_0 small enough
2. fix ρ_0 and ρ_A and choose the potential difference $u_A|_{\Gamma_-} - u_A|_{\Gamma_+}$ big enough.

The time of existence T depends on the boundary data u_A, ρ_A, the initial distribution ρ_0, the domain Ω and α that defines the Hölder space in which the solution is sought in. Thus T is determined by only a priori given data. It is difficult to obtain a qualitative interpretation for the size of T with respect to the size of the domain, as we do not know precisely how the constant $c_S(\Omega,\alpha)$ is influenced by Ω. However, we can comment on a fixed geometry Ω and increasing boundary data u_A, ρ_0 and ρ_A: the greater $\|u_A\|_{2,\alpha,\Gamma}$, $\|\rho_0\|_{0,\Omega_0}$ and $\|\rho_A\|_{0,Q_t}$, the smaller T becomes.
Further, we obtained that the solution (u,ρ) is extendable into a next time interval $[T,T_1]$ as long as the support of $\rho(x,T)$ is contained in Ω. This restriction is based on two reasons. Recall, that Φ_1 is defined on Ω_0. In Lemma 4.39, we used Lemma 2.20 to obtain

$$|\Phi_1(x,t) - \Phi_1(z,t)|_\infty^\alpha \leq c_{mv}|\nabla\Phi_1|_{0,\Omega_0}^\alpha |x-z|_\infty^\alpha.$$

Lemma 2.20 requires the outer boundary to be convex which is the reason why we assume Ω_0 to be convex. Second, we did not model the outflow of charge and chose T such that $\Phi_1(\cdot, t) : \Omega_0 \to \Omega_t^1 \subset \Omega$. The solution can thus only be continued until the first charge particle reaches the outflow boundary.

The solution $(u, \rho) \in C^{2,\alpha;0}(\bar{\Omega}, [0, T]) \times C^{0,1;\alpha}(\bar{\Omega}, [0, T])$ of (CP 4.2) is not a classical solution to (CP 4.1), as ρ is neither differentiable in space nor in time. Let us choose the boundary data $\rho_0 \in C^1(\bar{\Omega})$ and $\rho_A \in C^1(Q_T)$ and recall that the fixed point $\Phi \in W(M, T, K, \delta)$ is of $C^{1,\alpha;\alpha}(\Sigma_t, [0, T])$ regularity. Then ρ defined in (4.31) is differentiable on each of the subdomains Ω_t^2 and $\Omega \backslash \Omega_t^2$ with respect to space. Due to the construction of the operator A, we know that the fixed point $\Phi = A(\Phi)$ is also differentiable with respect to t. Since

$$-\nabla\Phi^{-1}(\Phi(\tau, t), t)\frac{d}{dt}\Phi(\tau, t) = \partial_t\Phi^{-1}(\Phi(\tau, t), t),$$

we conclude that $\Phi^{-1}(y, t)$ is also differentiable with respect to t. We obtain by (4.31) that ρ is differentiable with respect to t and obtain a piecewise classical solution to (CP 4.1) on Ω_t^2 and $\Omega \backslash \Omega_t^2$. The solution can not be continued immediately, as the new initial charge distribution $\hat{\rho}_0(x) := \rho(x, T)$ is not globally differentiable on Ω. However, the argumentation of the presented proof can be applied by considering the additional interface in $\hat{\rho}_0(x)$. In time, we will thus obtain a solution ρ to (CP 4.1) that is a piecewise classical solution on each of a finite number of subdomains of Ω.

Chapter 5

Steady-State Radially Symmetric Setting

In the Chapters 5 and 6, we address the question of existence and uniqueness of a classical solution to the two dimensional steady state coupled problem.

Problem (CP 6.1). *Let Ω be a $C^{2,\alpha}$ domain with boundary $\Gamma = \Gamma_- \cup \Gamma_+$. Let $u_A \in C^{2,\alpha}(\Gamma)$. Find $(u,\rho) \in C^{2,\alpha}(\Omega) \times C^{1,\alpha}(\Omega)$ such that*

$$-\Delta u(x) = \rho(x) \qquad x \in \Omega \tag{5.1a}$$

$$u(x) = u_A(x) \qquad x \in \Gamma \tag{5.1b}$$

$$E(x) = -\nabla u(x) \qquad x \in \Omega \tag{5.1c}$$

$$\operatorname{div}(E\rho) = 0 \qquad x \in \Omega \tag{5.1d}$$

$$\rho(x) = \rho_A(x) \qquad x \in \Gamma_-, \tag{5.1e}$$

with $\rho_A \in C^{1,\alpha}(\Gamma_-)$.

The existence of steady state solutions (u,ρ) to a variant of the vortex patch problem was obtained by Styles et al. in [56] in the spaces $(H^1(\Omega), L^2(\Omega))$. However, we are interested in the existence of a classical solution as in the time dependent cases of Chapters 3 and 4. In this Chapter, we investigate the radially symmetric setting on an annular domain. The solution to the radially symmetric setting can be found by classical means (see 8.2.1). This Chapter is meant to develop a theoretical framework to prepare the proof of the general solution to (CP 6.1) in Chapter 6. We therefore reformulate the simplified one-dimensional problem as a fixed point problem. By means of Green's functions, we explicitly find the solution operator $L\rho = u'$ of the Dirichlet problem for the Poisson equation (5.1a)-(5.1b). For the transport problem (5.1d)-(5.1e), we will determine the solution operator $Tu' = \rho$ to an integrated formulation. We show by the Banach fixed point theorem, that there exists a unique fixed point $\rho \in C^0(I)$ to the operator $T \circ L$. The fixed point then leads to the existence of a solution $(u,\rho) \in C^2(I) \times C^1(I)$ of the radially symmetric coupled problem.

To the end of this chapter, Ω is assumed to be an annular domain with midpoint in the origin and the radii r_0 and r_1, i.e.

$$\Omega = \{x \,:\, r_0 \leq \|x\|_2 \leq r_1\}\,. \tag{5.2}$$

The inflow boundary is given by

$$\Gamma_- = \{x : \|x\|_2 = r_0\} \tag{5.3}$$

and the outflow boundary is given by

$$\Gamma_+ = \{x : \|x\|_2 = r_1\}\,. \tag{5.4}$$

Let u, E and ρ be radially symmetric, i.e. for $r \in [r_0, r_1]$ with $r = \sqrt{x^2 + y^2}$ holds

$$\begin{aligned} u(x, y) &= u(r), \\ E(x, y) &= E(r), \\ \rho(x, y) &= \rho(r). \end{aligned}$$

The quantities thus only depend on the distance from the inflow boundary Γ_- and are independent of a tangential component. Using polar coordinates, we obtain for the differential operators

$$\begin{aligned} \nabla_x u(r) &= e_r \partial_r u(r) + \frac{e_\varphi}{r} \partial_\varphi u(r) = e_r \partial_r u(r), \\ \Delta_x u(r) &= \frac{1}{r} \partial_r \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \partial_\varphi^2 u(r) = \frac{1}{r} \partial_r \left(r \frac{\partial u}{\partial r} \right), \\ \operatorname{div}(E(r)\rho(r)) &= \frac{1}{r} \partial_r (r e_r \cdot (E\rho)) + \frac{1}{r} \partial_\varphi (e_\varphi \cdot (E\rho)) = \frac{1}{r} \partial_r (r e_r \cdot (E\rho)) \end{aligned}$$

with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $e_r = [\cos\varphi\,, \sin\varphi]$ and $e_\varphi = [-\sin\varphi\,, \cos\varphi]$ as unit normal vectors in direction of r and φ. Eventually, we get for the electrical field

$$E(r) = -\nabla u(r) = -e_r \partial_r u(r) - \frac{e_\varphi}{r} \partial_\varphi u(r, \varphi) = -e_r \partial_r u(r).$$

Using polar coordinates, the two dimensional model problem (CP 6.1) reduces to a one-dimensional problem on the interval $[r_0, r_1]$. In the following, $'$ denotes differentiation with respect to r. This chapter is concerned with the following setting:

Problem (CP 5.1). *Let $I = [r_0,\, r_1]$. Find $(u, \rho) \in C^2(I) \times C^1(I)$, such that*

$$-\frac{1}{r} \partial_r (r u'(r)) = \rho(r) \tag{5.5a}$$

$$u(r_0) = u_{A_1} \tag{5.5b}$$

$$u(r_1) = u_{A_2} \tag{5.5c}$$

$$\frac{1}{r} \partial_r (r u' \rho(r)) = 0 \tag{5.5d}$$

$$\rho(r_0) = \rho_A \tag{5.5e}$$

with $u_{A_1} > u_{A_2}$ and $\rho_A > 0$ constants.

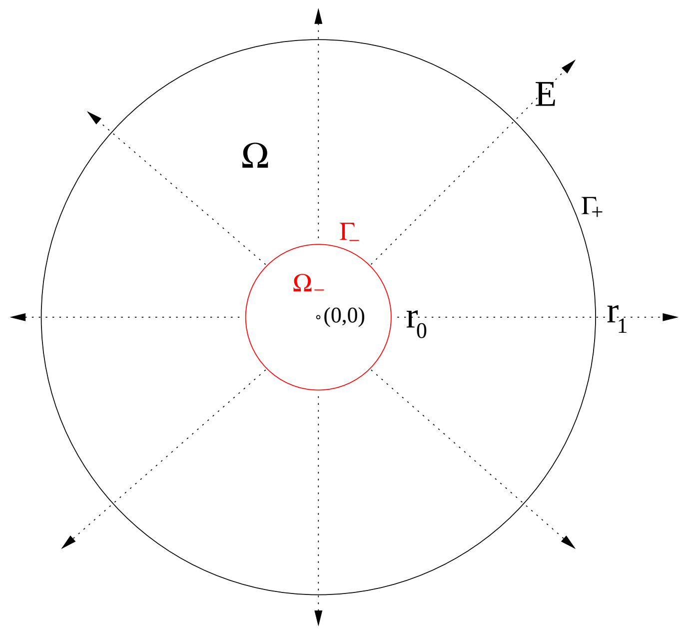


Figure 5.1: Annular domain Ω with electrical field E

The Poisson and transport equations reduce to one-dimensional differential equations with variable coefficients. For the boundary conditions u_{A_1}, u_{A_2} and ρ_A, the physical model gives hints on their size and signs. The potential is generated by a difference of voltages on the boundaries. The peak is assumed to be on the inflow boundary, which results in $u_{A_1} \gg u_{A_2}$. The charge density ρ is defined to be positive, conclusively $\rho_A > 0$.
As one-dimensional differential equations, the solutions u and ρ are sought in the spaces of continuous differentiable functions $C^2(\Omega)$ and $C^1(\Omega)$.

We proceed as follows. In section 5.1 we derive Green's function for the Poisson equation. We thus represent u as an integral equation that uses the solution ρ of the transport equation as argument. Because of the simple structure of (5.5d), we obtain the solution ρ by direct integration. Apparently, ρ depends on u' which expresses the coupling of the two problems. In section 5.3 we formulate the solution operators L for the Poisson equation and T for the transport equation. The composite operator $T \circ L$ is applied to a subset $R(M) \subset C^0([r_0, r_1])$ of all those $\rho \in C^0([r_0, r_1])$ for which the composition $T \circ L$ is well defined. To show existence and uniqueness of (CP 5.1), we apply the Banach fixed point theorem to $T \circ L$.

Theorem 5.1 (Banach Fixed Point Theorem)**.** *([65, p.19]) Let W be a closed non-empty subspace of a Banach space X and assume that the operator $A : W \to W$ is contractive, i.e.*

$$\|Au - Aw\| \leq q\|u - w\|, \quad \forall u, w \in W, \, 0 < q < 1.$$

Then a unique fixed point exists for A in the set W, i.e. there exists a unique $v \in W$, such that $Av = v$.

We will investigate how to choose the constant M in the set $R(M)$ for $T \circ L$ being a contraction and selfmap.

5.1 Partial Problems

In this section, we will find representations of the solutions u to the Dirichlet problem for the Poisson equation (5.5a)-(5.5c) and ρ of the transport problem (5.5d)-(5.5e).

5.1.1 The Poisson Equation

In the radially symmetric setting, the two-dimensional Poisson equation (5.1a) reduces to a one-dimensional second order differential equation with variable coefficients.

Problem (Po 5.2). *Let $I = [r_0, r_1]$. For a given right-hand side function $\rho \in C^0(I)$, find $u \in C^2(I)$, such that*

$$-u''(r) - \frac{1}{r}u'(r) = \rho(r) \tag{5.6a}$$

$$u(r_0) = u_{A_1} \tag{5.6b}$$

$$u(r_1) = u_{A_2}. \tag{5.6c}$$

The boundary values u_{A_1} and u_{A_2} are constant with $u_{A_1} > u_{A_2}$.

One technique to represent the solution of (Po 5.2) is given by the method of Green's function. Green's function enables us to express u as an integral equation applied to the right-hand side function ρ of (5.6a). In terms of the coupled problem (CP 5.1), this representation is of great advantage. As ρ is the argument of the integral equation, we immediately obtain the solution operator L of the Poisson equation of (Po 5.2). We will now begin to derive Green's function for (Po 5.2) and some related results that will be used in section 5.3 to show the requirements of the Banach fixed point theorem.

We will now present explicitly how to obtain the Green's function for (Po 5.2). As first step, we transform (5.6a)-(5.6c) into a boundary value problem with homogeneous boundary conditions. We then determine Green's function for the new problem and use it to derive the one for u.

The function

$$\Psi(r) = \frac{(r_1 - r)u_{A_1} + (r - r_0)u_{A_2}}{r_1 - r_0} \tag{5.7}$$

is a linear function with $\Psi(r_0) = u_{A_1}$ and $\Psi(r_1) = u_{A_2}$. Subtracting Ψ of u, we obtain

$$w(r) = u(r) - \Psi(r) \tag{5.8}$$

for $r \in I$. Then w solves the second order differential equation (5.6a) subject to modified right-hand side data and homogeneous boundary conditions.

Lemma 5.2. *Let $r \in I$. Then w solves the boundary value problem*

$$-w''(r) - \frac{1}{r}w'(r) = \rho(r) + \frac{1}{r}\left(\frac{u_{A_2} - u_{A_1}}{r_1 - r_0}\right) \tag{5.9a}$$

$$w(r_0) = 0 \tag{5.9b}$$

$$w(r_1) = 0. \tag{5.9c}$$

Proof. By (5.8), we compute

$$\begin{aligned} -w''(r) - \frac{1}{r}w'(r) &= -u''(r) - \frac{1}{r}u'(r) + \Psi''(r) + \frac{1}{r}\Psi'(r) \\ &= \rho(r) + \frac{1}{r}\left(\frac{u_{A_2} - u_{A_1}}{r_1 - r_0}\right). \end{aligned}$$

□

We now derive the Green's function for w. Afterwards, we easily obtain an integral representation for u by $u = w + \Psi$. For second order differential equations, Green's function is characterised by the following criteria.

Definition 5.3. *[35, p.14] We call a function $G(r,t)$ the* Green's function *for (Po 5.2) if it fulfils the following conditions*

1. *$G(r,t)$ fulfils as function of r with $r \neq t$ the homogeneous ODE, i.e. $-\partial_r^2 G(r,t) - \frac{1}{r}\partial_r G(r,t) = 0$.*
2. *$G(r,t)$ fulfils as function of r the boundary conditions, i.e. $G(r_0,t) = 0 = G(r_1,t)$.*
3. *$G(r,t)$ is continuous, $\frac{\partial G}{\partial r}$ jumps for $r = t$ with -1, i.e. $\lim_{r\to t_-} \partial_r G(r,t) - \lim_{r\to t_+} \partial_r G(r,t) = -1$.*

The definition suggests that $G(r,t)$ is a piecewise defined function for $x < t$ and $x > t$. In the following Lemma, we will compute Green's function for (5.9a)-(5.9c).

Lemma 5.4. *The solution w of (5.9a)-(5.9c) is given by*

$$w(r) = \int_{r_0}^{r} G_1(r,t)f(t)\,dt + \int_{r}^{r_1} G_2(r,t)f(t)\,dt \tag{5.10}$$

where $f(r)$ denotes the right-hand side of (5.9a)

$$f(r) = \rho(r) + \frac{1}{r}\left(\frac{u_{A_2} - u_{A_1}}{r_1 - r_0}\right).$$

$G_1(r,t)$ and $G_2(r,t)$ are the components of Green's function

$$G(r,t) = \begin{cases} (a_1 + b_1)\log(r) + a_2 + b_2\log(t) =: G_2(r,t), & r < t \\ (a_1 - b_1)\log(r) + a_2 + b_2\log(t) =: G_1(r,t), & r > t \end{cases} \tag{5.11}$$

with

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\log\left(\frac{r_0}{r_1}\right)} \begin{pmatrix} -b_1 \log(r_0 r_1) - 2b_2 \\ 2\log r_0 \log r_1 b_1 + b_2 \log(r_0 r_1) \end{pmatrix}$$

and

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{t}{2} \begin{pmatrix} 1 \\ -\log t \end{pmatrix}.$$

Proof. We follow the method described in [20, pp. 158-160].
We begin by finding the fundamental system for the homogeneous ordinary differential equation corresponding to (5.9a), i.e.

$$-w''(r) - \frac{1}{r} w'(r) = 0. \tag{5.12}$$

The fundamental system is given by $v = \{v_1, v_2\}$, $v_1, v_2 \in \mathbb{R}$, such that v_1 and v_2 are linearly independent and solve (5.12). Every solution to (5.12) is thus a linear combination
$w = v_1 + sv_2$ with $s \in \mathbb{R}$. A first solution to (5.12) is easily found as $v_1(r) = \log(r)$. Due to the absence of w in (5.12), the second solution is constant, say $v_2 = 1$. Then the fundamental system is given by

$$v = \{\log r, 1\}. \tag{5.13}$$

Green's function has to solve the ordinary differential equation on both sides of the diagonal $x = t$. A piecewise definition is thus reasonable. Further, due to condition (1) of Definition 5.3, $G(r,t)$ solves the homogeneous differential equation. We use as ansatz a linear combination of v_1 and v_2 with coefficient functions depending on t. This is clearly a solution to (5.12).

$$G(r,t) = \begin{cases} (a_1(t) + b_1(t))v_1(r) + (a_2(t) + b_2(t))v_2(r), & r < t \\ (a_1(t) - b_1(t))v_1(r) + (a_2(t) - b_2(t))v_2(r), & r > t \end{cases}.$$

At the diagonal $x = t$, the functions b_j, $j = 1, 2$ are decisive for $G(r,t)$ to obey condition (3) of Definition 5.3. We choose $b_j(t)$ such that the continuity and the jump for the derivative are fulfilled at $r = t$. With $G_1(t,t) - G_2(t,t) = 0$ and $\partial_r G_1(r = t, t) - \partial_r G_2(r = t, t) = -1$, we obtain the system

$$\begin{aligned} b_1(t)v_1(r) + b_2(t)v_2(r) &= 0 \\ b_1(t)v_1'(r) + b_2(t)v_2'(r) &= \frac{1}{2}. \end{aligned}$$

It results

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{t}{2} \begin{pmatrix} 1 \\ -\log t \end{pmatrix}.$$

a_1 and a_2 are now chosen to fulfil the boundary conditions for $G(r,t)$ and thus to satisfy condition (2) of Definition 5.3.

$$
\begin{aligned}
G(r_0,t) &= (a_1+b_1)v_1(r_0)+(a_2+b_2)v_2(r_0)=0\\
G(r_1,t) &= (a_1-b_1)v_1(r_1)+(a_2-b_2)v_2(r_1)=0.
\end{aligned}
$$

It results

$$
\begin{pmatrix} \log r_0 & 1\\ \log r_1 & 1\end{pmatrix}\begin{pmatrix} a_1\\ a_2\end{pmatrix}+\begin{pmatrix} \log r_0 & 1\\ -\log r_1 & -1\end{pmatrix}\begin{pmatrix} b_1\\ b_2\end{pmatrix}=0
$$

and we obtain

$$
\begin{pmatrix} a_1\\ a_2\end{pmatrix}=\frac{1}{\log r_0-\log r_1}\begin{pmatrix} -b_1(\log r_0+\log r_1)-2b_2\\ 2\log r_0\log r_1 b_1+b_2(\log r_0+\log r_1)\end{pmatrix}.
$$

Green's function is determined by

$$
G(r,t)=\begin{cases} (a_1+\frac{t}{2})\log(r)+(a_2-\frac{t}{2}\log(t)), & r<t\\ (a_1-\frac{t}{2})\log(r)+(a_2+\frac{t}{2}\log(t)), & r>t\end{cases}.
$$

□

With the result of Lemma 5.4, we have an integral representation for u.

Theorem 5.5. *The solution $u \in C^2(I)$ of (5.6a)-(5.6c) is given by*

$$
u(r)=\int_{r_0}^{r} G_1(r,t)\rho(t)\,dt+\int_{r}^{r_1} G_2(r,t)\rho(t)\,dt+\frac{\log(r)(u_{A_1}-u_{A_2})+\log r_0 u_{A_2}-\log r_1 u_{A_1}}{\log\left(\frac{r_0}{r_1}\right)} \tag{5.14}
$$

with G_1 and G_2 as defined in Lemma 5.4.

Proof. Recall the definition (5.8), i.e. $w=u-\Psi$. By (5.10) and (5.7), we obtain

$$
\begin{aligned}
u(r) &= \int_{r_0}^{r} G_1(r,t)f(t)\,dt+\int_{r}^{r_1} G_2(r,t)f(t)\,dt+\Psi(r)\\
&= \int_{r_0}^{r} G_1(r,t)f(t)\,dt+\int_{r}^{r_1} G_2(r,t)f(t)\,dt+\frac{(r_1-r)u_{A_1}+(r-r_0)u_{A_2}}{r_1-r_0}.
\end{aligned}
$$

As $f(r)=\rho(r)+\left(\frac{1}{r}\frac{u_{A_2}-u_{A_1}}{r_1-r_0}\right)$, we obtain the assumption by computing

$$
\begin{aligned}
&\int_{r_0}^{r} G_1(r,t)\frac{1}{t}\frac{u_{A_2}-u_{A_1}}{r_1-r_0}\,dt+\int_{r}^{r_1} G_2(r,t)\frac{1}{t}\frac{u_{A_2}-u_{A_1}}{r_1-r_0}\,dt+\frac{(r_1-r)u_{A_1}+(r-r_0)u_{A_2}}{r_1-r_0}\\
&\quad=\frac{r\log(r_1)+r_0\log(r)-\log(r_1)r_0-\log(r)r_1+\log(r_0)r_1-r\log(r_0)}{\log(\frac{r_0}{r_1})}\frac{u_{A_2}-u_{A_1}}{r_1-r_0}\\
&\quad\quad+\frac{(r_1-r)u_{A_1}+(r-r_0)u_{A_2}}{r_1-r_0}\\
&\quad=\frac{\log(r)(u_{A_1}-u_{A_2})+\log r_0 u_{A_2}-\log r_1 u_{A_1}}{\log\left(\frac{r_0}{r_1}\right)}.
\end{aligned}
$$

□

We will see in section 5.1.2 that the solution ρ of the Transport equation in (CP 5.1) depends only on u'. We state the integral equation for u' as it will be used frequently.

Lemma 5.6. *The integral equation for $u'(r)$ is given by*

$$u'(r) = \int_{r_0}^{r} \partial_r G_1(r,t)\rho(t)\,dt + \int_{r}^{r_1} \partial_r G_2(r,t)\rho(t)\,dt + \frac{u_{A_1} - u_{A_2}}{r\log\left(\frac{r_0}{r_1}\right)}.$$

Proof. We differentiate (5.14)

$$\begin{aligned} u'(r) = &\lim_{t\to r^-} G_1(r,t)\rho(r) + \int_{r_0}^{r} \partial_r G_1(r,t)\rho(t)\,dt \\ &- \lim_{t\to r^+} G_2(r,t)\rho(r) + \int_{r}^{r_1} \partial_r G_2(r,t)\rho(t)\,dt + \frac{u_{A_1} - u_{A_2}}{r\log\left(\frac{r_0}{r_1}\right)}. \end{aligned}$$

Condition (3) of Definition 5.3 claims the continuity of $G(r,t)$ at $t = r$, i.e.

$$\lim_{t\to r-} G_1(r,t)f(r) - \lim_{t\to r^+} G_2(r,t)\rho(r) = 0.$$

Hence,

$$u'(r) = \int_{r_0}^{r} \partial_r G_1(r,t)\rho(t)\,dt + \int_{r}^{r_1} \partial_r G_2(r,t)\rho(t)\,dt + \frac{u_{A_1} - u_{A_2}}{r\log\left(\frac{r_0}{r_1}\right)}.$$

□

The next two Lemmas are auxiliary results for showing the requirements of the Banach fixed point Theorem in section 5.2. To prove that the operator $T \circ L$ is a selfmap and contraction, the signs of the kernels $G_1(r,t)$ and $G_2(r,t)$ are of importance.

Lemma 5.7. *Let G_1 and G_2 be defined as in Lemma 5.4. Then holds for all $r, t \in [r_0,\, r_1]$*

$$\partial_r G_1(r,t) \le 0 \tag{5.15}$$

$$\partial_r G_2(r,t) \ge 0. \tag{5.16}$$

Proof. We check the signs of $\partial_r G_1(r,t)$ and $\partial_r G_2(r,t)$.

$$\partial_r G_1(r,t) = -\frac{t}{2r}\frac{\log\left(\frac{r_0^2}{t^2}\right)}{\log\left(\frac{r_0}{r_1}\right)} = \frac{t}{r}\frac{\overbrace{\log\left(\frac{t}{r_0}\right)}^{\ge 0}}{\underbrace{\log\left(\frac{r_0}{r_1}\right)}_{\le 0}} \le 0$$

and

$$\partial_r G_2(r,t) = \frac{t}{2r}\frac{\log\left(\frac{t^2}{r_1^2}\right)}{\log\left(\frac{r_0}{r_1}\right)} = \frac{t}{r}\frac{\overbrace{\log\left(\frac{t}{r_1}\right)}^{\leq 0}}{\underbrace{\log\left(\frac{r_0}{r_1}\right)}_{\leq 0}} \geq 0.$$

□

For the constant density $\rho(r) = 1$, we compute the integrals of Lemma 5.6 explicitly.

Lemma 5.8. *With $G_1(r,t)$ and $G_2(r,t)$ as defined in Lemma 5.4 holds*

$$\int_{r_0}^{r} |\partial_r G_1(r,t)|\, dt = \frac{-1}{r\log\left(\frac{r_0}{r_1}\right)}\left(\frac{r^2}{2}\log\left(\frac{r}{r_0}\right) - \frac{r^2 - r_0^2}{4}\right)$$

$$\int_{r}^{r_1} |\partial_r G_2(r,t)|\, dt = \frac{1}{r\log\left(\frac{r_0}{r_1}\right)}\left(\frac{r^2}{2}\log\left(\frac{r_1}{r}\right) - \frac{r_1^2 - r^2}{4}\right).$$

Proof. By direct computations, the representations of $\partial_r G_1(r,t)$ and $\partial_r G_2(r,t)$ in Lemma 5.6 and Lemma 5.7, we get

$$\begin{aligned}
\int_{r_0}^{r} |\partial_r G_1(r,t)|\, dt &= -\int_{r_0}^{r} \partial_r G_1(r,t)\, dt = -\int_{r_0}^{r} \frac{t}{r}\frac{\log\left(\frac{t}{r_0}\right)}{\log\left(\frac{r_0}{r_1}\right)}\, dt \\
&= \frac{-1}{r\log\left(\frac{r_0}{r_1}\right)}\left[\frac{t^2}{2}\log\left(\frac{t}{r_0}\right) - \frac{t^2}{4}\right]_{r_0}^{r} \\
&= \frac{-1}{r\log\left(\frac{r_0}{r_1}\right)}\left(\frac{r^2}{2}\log\left(\frac{r}{r_0}\right) - \frac{r^2 - r_0^2}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{r}^{r_1} |G_2'(r,t)|\, dt &= \int_{r}^{r_1} G_2'(r,t)\, dt = \int_{r}^{r_1} \frac{t}{r}\frac{\log\left(\frac{t}{r_1}\right)}{\log\left(\frac{r_0}{r_1}\right)}\, dt \\
&= \frac{1}{r\log\left(\frac{r_0}{r_1}\right)}\left[\frac{t^2}{2}\log\left(\frac{t}{r_1}\right) - \frac{t^2}{4}\right]_{r}^{r_1} \\
&= \frac{1}{r\log\left(\frac{r_0}{r_1}\right)}\left(\frac{-r^2}{2}\log\left(\frac{r}{r_1}\right) - \frac{r_1^2 - r^2}{4}\right) \\
&= \frac{1}{r\log\left(\frac{r_0}{r_1}\right)}\left(\frac{r^2}{2}\log\left(\frac{r_1}{r}\right) - \frac{r_1^2 - r^2}{4}\right).
\end{aligned}$$

□

5.1.2 Transport Equation

The second component of the coupled problem is the transport equation. The steady state transport equation reduces in the radially symmetric case to a one-dimensional equation with variable coefficients. We obtain the following one-dimensional setting

Problem (Tr 5.3). *Let $I = [r_0, r_1]$. For a given $u' \in C^1(I)$, find $\rho \in C^1(I)$, such that*

$$\frac{1}{r}\partial_r(ru'(r)\rho(r)) = 0 \tag{5.17a}$$

$$\rho(r_0) = \rho_A \tag{5.17b}$$

for all $r \in I$ and $\rho_A > 0$ constant.

For a given u', the solution ρ of (Tr 5.3) is easily obtained.

Lemma 5.9. *Let $I = [r_0, r_1]$, $u' \in C^1(I)$ with $u'(r) \neq 0$ for $r \in I$. Then the solution to the boundary value problem (Tr 5.3) is given by*

$$\rho(r) = \frac{\rho_A r_0 u'(r_0)}{ru'(r)}. \tag{5.18}$$

Proof. By integration of (5.17a), we obtain

$$\rho(r) = \frac{C}{ru'(r)}.$$

With the boundary conditions (5.17b), we obtain

$$C = \rho_A r_0 u'(r_0).$$

□

5.2 Formulation of Solution Operators L and T

We are now going back to the coupled radially symmetric problem (CP 5.1). This section carries out the formulation of the solution operators L of the Poisson and T of transport equation. Further, we will define a set of functions $R(M) \subset C^0(I)$ in which we will seek for a fixed point of the composite operator $T \circ L$ in section 5.3.

First, we define the solution operators T and L by using the results of section 5.1.

Lemma 5.10. *Let $I = [r_0, r_1]$ and $u' \in C^1(I)$ and $u'(r) \neq 0$ for $r \in I$. Then the solution operator T for the transport problem (Tr 5.3) is given by*

$$Tu'(r) = \frac{\rho_A r_0 u'(r_0)}{ru'(r)} \tag{5.19}$$

and $Tu' \in C^1(I)$.

Proof. By Lemma 5.9, we know that the solution ρ is given by

$$\rho(r) = \frac{\rho_A r_0 u'(r_0)}{r u'(r)} \tag{5.20}$$

and $Tu'(r)$ fulfils the boundary conditions (5.5e)

$$Tu'(r_0) = \rho_A. \tag{5.21}$$

Since $u' \in C^1(I)$, Tu' is once continuously differentiable. □

By Lemma 5.10, it is clear that for obtaining the solution ρ of (CP 5.1), it is not necessary to know u. For the Banach fixed point iterations in section 5.3, it is thus sufficient to iterate over u'. The solution operator L for the Poisson equation is hence understood as $L\rho = u'$ with u' being the derivative of the solution u to (Po 5.2).

Lemma 5.11. *Let $I = [r_0, r_1]$ and $\rho \in C^0(I)$. Then the solution operator L for the radially symmetric Poisson problem (Po 5.2) is given by*

$$L\rho(r) = \int_{r_0}^{r} \partial_r G_1(r,t)\rho(t)\,dt + \int_{r}^{r_1} \partial_r G_2(r,t)\rho(t)\,dt + \frac{u_{A_1} - u_{A_2}}{r \log\left(\frac{r_0}{r_1}\right)} \tag{5.22}$$

with $\partial_r G_1(r,t)$ and $\partial_r G_2(r,t)$ defined in Lemma 5.7. It holds $L\rho \in C^1(I)$.

Proof. By Lemma 5.6, we have the integral representation of u' in terms of Green's function. The assumption follows as the argument in the integral equation of u' is ρ. $L\rho$ is once continuously differentiable due to Theorem 5.5. □

To show existence and uniqueness of (CP 5.1), we prove the existence of a unique fixed point of the composite operator $T \circ L$. The approach is to prove that $T \circ L$ is a contraction and selfmap on a set $R(M) \subset C^0(I)$. This is equivalent to prove that there exists a solution to the coupling of the integrated transport equation (5.18) and (5.5a)-(5.5c). For the fixed point ρ of $T \circ L$ follows $L\rho \in C^1(I)$ due to Lemma 5.11. With Theorem 5.10, we conclude that the fixed point is also differentiable, i.e. $\rho \in C^1(I)$ and thus (u, ρ) is a classical solution to (CP 5.1).
Up to now, we have not yet constrained the set of functions in which it is reasonable to search for u' and ρ. The physical model implies restrictions on ρ such as it must stay positive for all iterations. We define the subset $R(M)$ of $C^0(I)$ by

$$R(M) = \left\{\rho \in C^0([r_0, r_1]) : 0 < \rho \leq \rho_A \leq M\right\} \tag{5.23}$$

with M being a constant to be defined later on. The boundedness of ρ by its inflow boundary data ρ_A is justified by the absence of sources in the interval. It is not immediately obvious that the boundedness of ρ also affects the sign of u'. By (5.19) follows that u' must not change signs on the interval $[r_0, r_1]$. As $u_{A_1} > u_{A_2}$, the model suggests $u' < 0$. With the following computation follows that $u'(r_0)$ is required to be negative since $r = r_0$ is the inflow boundary point of I. The outward normal vector in the radially symmetric setting is given by $\vec{n}(r_0) = -e_r(r_0)$. We obtain

$$0 \overset{!}{>} \vec{n}(r_0) \cdot E(r_0) = e_r(r_0) \cdot e_r(r_0) u'(r_0) = u'(r_0).$$

Yet, not every choice of M, and thus the upper bound of ρ_A, leads to $u'(r_0) < 0$. The next Lemma gives a first restriction for the size of the constant M.

Lemma 5.12. *Let $I = [r_0, r_1]$. Let $\rho \in R(c_R)$ with*

$$c_R := \frac{u_{A_1} - u_{A_2}}{\frac{1}{2} r_0^2 \log\left(\frac{r_0}{r_1}\right) + \frac{1}{4}\left(r_1^2 - r_0^2\right)} \tag{5.24}$$

Then holds

$$u'(r_0) < 0. \tag{5.25}$$

Proof. With Lemma 5.6 and 5.7, we get for $r = r_0$

$$u'(r_0) = \underbrace{\frac{1}{r_0 \log(\frac{r_0}{r_1})}}_{<0} \left(\int_{r_0}^{r_1} \underbrace{t \log\left(\frac{t}{r_1}\right) \rho(t)}_{\leq 0} \, dt + \underbrace{u_{A_1} - u_{A_2}}_{>0} \right).$$

Hence for $u'(r) < 0$

$$u_{A_1} - u_{A_2} > -\int_{r_0}^{r_1} t \log\left(\frac{t}{r_1}\right) \rho(t) \, dt$$

In particular, this must also be true for the upper bound of ρ

$$\begin{aligned} u_{A_1} - u_{A_2} &\geq -\int_{r_0}^{r_1} t \log\left(\frac{t}{r_1}\right) \rho_A \, dt \\ &= -\rho_A \left[\frac{1}{2} t^2 \log\left(\frac{t}{r_1}\right) - \frac{1}{4} t^2\right]_{r_0}^{r_1} \\ &= -\rho_A \left(-\frac{1}{2} r_0^2 \log\left(\frac{r_0}{r_1}\right) - \frac{1}{4}\left(r_1^2 - r_0^2\right)\right). \end{aligned}$$

Set

$$\rho_A < \frac{1}{2} \left(\frac{u_{A_1} - u_{A_2}}{\frac{1}{2} r_0^2 \log\left(\frac{r_0}{r_1}\right) + \frac{1}{4}\left(r_1^2 - r_0^2\right)} \right) =: c_R. \tag{5.26}$$

The assertion is proved. ☐

We thus found a first restriction for ρ_A. To show the contraction property of $T \circ L$ in Theorem 5.20, we will have to restrict ρ_A further. As an immediate consequence of Lemma 5.12 follows $u'(r) < 0$ for all $r \in [r_0, r_1]$.

Lemma 5.13. *Let $\rho \in R(c_R)$ with c_R defined in Lemma 5.12. Then holds for all $r \in [r_0, r_1]$*

$$u'(r) < 0. \tag{5.27}$$

Proof. First, by integration of (5.5a), we obtain an alternative representation of $u'(r)$.

$$u'(r) = -\frac{1}{r}\int_{r_0}^{r} t\rho(t)\,dt + \frac{c}{r}.$$

With $r = r_0$, the constant c is determined as $c = r_0 u'(r_0)$. Since $\rho \in R(c_R)$, Lemma 5.12 confirms that $u'(r_0) < 0$. The representation

$$u'(r) = -\frac{1}{r}\int_{r_0}^{r} \underbrace{t\rho(t)}_{>0}\,dt + \underbrace{\frac{r_0 u'(r_0)}{r}}_{<0} \tag{5.28}$$

proves the assertion (5.27). □

Before proving that $T \circ L$ satisfies the requirements of the Banach fixed point Theorem, we introduce a last result on the extrema of $ru'(r)$.

Lemma 5.14. *Let $I = [r_0, r_1]$ and $\rho \in R(c_R)$ with c_R defined as in Lemma 5.12. Then holds*

$$r_0 u'(r_0) > r_1 u'(r_1)$$

and

$$\min_{r\in I} |ru'(r)| = |r_0 u'(r_0)|,$$
$$\max_{r\in I} |ru'(r)| = |r_1 u'(r_1)|.$$

Proof. This lemma is an immediate conclusion of (5.28)

$$ru'(r) = \underbrace{-\int_{r_0}^{r} t\rho(t)\,dt}_{<0} + \underbrace{r_0 u'(r_0)}_{<0}.$$

It follows

$$0 > r_0 u'(r_0) > r_1 u'(r_1)$$

and conclusively the inequalities

$$|r_0 u'(r_0)| < |r_1 u'(r_1)|,$$
$$\min_{r\in I} |ru'(r)| = |r_0 u'(r_0)|,$$
$$\max_{r\in I} |ru'(r)| = |r_1 u'(r_1)|.$$

□

5.3 Existence of a Fixed Point

We are now ready to prove that $T \circ L$ satisfies the requirements of the Banach fixed point theorem, that is $T \circ L$ is a selfmap and a contraction on $R(M)$. We begin to show that $T \circ L$ maps $R(c_R)$ into itself.

Lemma 5.15. *Let $I = [r_0, r_1]$ and c_R defined as in Lemma 5.12. Then $T \circ L$ is a selfmap on the set $R(c_R)$.*

Proof. We show the positivity of $T \circ L\rho$. As $\rho \in R(c_R)$, we obtain with (5.28) for all $r \in [r_0, r_1]$

$$L\rho(r) = -\frac{1}{r}\int_{r_0}^{r} t\rho(t)\,dt + \frac{r_0 L\rho(r_0)}{r} < 0.$$

Denote $u'(r) = L\rho(r)$. Hence, by (5.19)

$$Tu'(r) = \frac{r_0\rho_A u'(r_0)}{ru'(r)} > 0.$$

Second, we verify that $T \circ L\rho(r)$ is bounded by ρ_A. By (5.19) and using (5.28) for u', we get

$$Tu'(r) = \frac{r_0\rho_A u'(r_0)}{ru'(r)} = \frac{-r_0\rho_A u'(r_0)}{r\left(\frac{1}{r}\int_{r_0}^{r} \underbrace{t\rho(t)}_{>0}\,dt - \underbrace{\frac{r_0}{r}u'(r_0)}_{<0}\right)} \leq \frac{-r_0\rho_A u'(r_0)}{-r\frac{r_0}{r}u'(r_0)} = \rho_A \leq c_R.$$

The boundedness of ρ_A by c_R is immediate due to set $R(c_R)$. $\square$

The second requirement for the Banach fixed point Theorem is that $T \circ L$ is a contraction. We are going to prove this in two steps. First, we show that $T \circ L$ is continuous on the set $R(c_R)$. With a further restriction on ρ_A, we reduce the size of the continuity constant and obtain the desired contraction property.

We begin with pointwise estimates of the difference of two elements ρ and $\tilde{\rho}$ in $R(c_R)$ in terms of $u'(r)$ and $\tilde{u}'(r)$.

Lemma 5.16. *Let $\rho, \tilde{\rho} \in R(c_R)$ and c_R defined as in Lemma 5.12. Then holds point wise for $r \in [r_0, r_1]$*

$$|\rho(r) - \tilde{\rho}(r)| \leq \frac{\rho_A}{\left|r_0\,\rho_A \int_{r_0}^{r_1} G_2'(r_0,t)\,dt + \frac{u_{A_1} - u_{A_2}}{\log(\frac{r_0}{r_1})}\right|}\left(r_0|u'(r_0) - \tilde{u}'(r_0)| + r|\tilde{u}'(r) - u'(r)|\right). \quad (5.29)$$

Proof. By (5.19) and adding $0 = \tilde{u}'(r)\tilde{u}'(r_0) - \tilde{u}'(r)\tilde{u}'(r_0)$, we obtain

$$\begin{aligned}
|\rho(r) - \tilde{\rho}(r)| &= \left| \frac{\rho_A r_0 u'(r_0)}{r u'(r)} - \frac{\rho_A r_0 \tilde{u}'(r_0)}{r \tilde{u}'(r)} \right| \\
&= \left| \frac{\rho_A r_0 u'(r_0)\tilde{u}'(r) - \rho_A r_0 \tilde{u}'(r_0) u'(r)}{r \tilde{u}'(r) u'(r)} \right| \\
&= \left| \frac{\rho_A r_0}{r} \frac{u'(r_0)\tilde{u}'(r) - \tilde{u}'(r)\tilde{u}'(r_0) + \tilde{u}'(r)\tilde{u}'(r_0) - \tilde{u}'(r_0)u'(r)}{\tilde{u}'(r)u'(r)} \right| \\
&= \left| \frac{\rho_A r_0}{r} \frac{\tilde{u}'(r)(u'(r_0) - \tilde{u}'(r_0)) + \tilde{u}'(r_0)(\tilde{u}'(r) - u'(r))}{\tilde{u}'(r)u'(r)} \right| \\
&\leq \left| \frac{\rho_A r_0}{r} \frac{u'(r_0) - \tilde{u}'(r_0)}{u'(r)} \right| + \left| \frac{\rho_A r_0}{r} \frac{\tilde{u}'(r_0)}{\tilde{u}'(r)} \frac{\tilde{u}'(r) - u'(r)}{u'(r)} \right| .
\end{aligned} \tag{5.30}$$

For the first term of (5.30), we use Lemma 5.14 and Lemma 5.6 and get

$$\begin{aligned}
\left| \frac{\rho_A r_0}{r} \frac{u'(r_0) - \tilde{u}'(r_0)}{u'(r)} \right| &= \rho_A \frac{|r_0(u'(r_0) - \tilde{u}'(r_0))|}{|r u'(r)|} \\
&\leq \rho_A \frac{|r_0(u'(r_0) - \tilde{u}'(r_0))|}{|r_0 u'(r_0)|} \\
&\leq \rho_A \frac{r_0 |u'(r_0) - \tilde{u}'(r_0)|}{\left| r_0\, \rho_A \int_{r_0}^{r_1} G_2'(r_0, t)\, dt + \frac{u_{A_1} - u_{A_2}}{\log(\frac{r_0}{r_1})} \right|} .
\end{aligned} \tag{5.31}$$

We turn our attention to the second term of (5.30). Lemma 5.14 implies

$$\frac{u'(r_0)}{u'(r)} \leq \frac{r}{r_0}.$$

Due to Lemmas 5.13 and 5.14, we obtain

$$\begin{aligned}
\left| \frac{\rho_A r_0}{r} \frac{\tilde{u}'(r_0)}{\tilde{u}'(r)} \frac{\tilde{u}'(r) - u'(r)}{u'(r)} \right| &\leq \rho_A \left| \frac{r_0 r}{r_0} \frac{\tilde{u}'(r) - u'(r)}{r u'(r)} \right| \\
&\leq \rho_A \left| \frac{r(\tilde{u}'(r) - u'(r))}{r_0 u'(r_0)} \right| \\
&\leq \rho_A \frac{r |\tilde{u}'(r) - u'(r)|}{\left| r_0\, \rho_A \int_{r_0}^{r_1} G_2'(r_0, t)\, dt + \frac{u_{A_1} - u_{A_2}}{\log(\frac{r_0}{r_1})} \right|} .
\end{aligned} \tag{5.32}$$

The assertion follows by combining (5.30)-(5.32).

□

The next Lemma shows the continuity of L for $\rho, \tilde{\rho} \in R(c_R)$ evaluated on the interval boundary $r = r_0$.

Lemma 5.17. *Let $\rho, \tilde{\rho} \in R(c_R)$ and c_R defined as in Lemma 5.12. Then holds point wise*

$$|L\rho(r_0) - L\tilde{\rho}(r_0)| \leq \left(-\frac{r_0}{2} - \frac{r_1^2 - r_0^2}{4r_0 \log(\frac{r_0}{r_1})} \right) \|\rho - \tilde{\rho}\|_{0,I}$$

Proof. By direct computations, we obtain from (5.22)

$$\begin{aligned}
|L\rho(r_0) - L\tilde{\rho}(r_0)| &= \left| \int_{r_0}^{r_1} \partial_r G_2(r_0, t)\rho(t)\, dt - \int_{r_0}^{r_1} \partial_r G_2(r_0, t)\tilde{\rho}(t)\, dt \right| \\
&= \left| \int_{r_0}^{r_1} \partial_r G_2(r_0, t) \left(\rho(t) - \tilde{\rho}(t)\right)\, dt \right| \\
&\leq \int_{r_0}^{r_1} |\partial_r G_2(r_0, t)|\, |\rho(t) - \tilde{\rho}(t)|\, dt \\
&\leq \|\rho - \tilde{\rho}\|_{0,I} \int_{r_0}^{r_1} |\partial_r G_2(r_0, t)|\, dt
\end{aligned}$$

With Lemma 5.8, we get explicitly

$$\int_{r_0}^{r_1} |\partial_r G_2(r_0, t)|\, dt = -\frac{r_0}{2} - \frac{r_1^2 - r_0^2}{4r_0 \log(\frac{r_0}{r_1})}.$$

□

The next Lemma is a generalisation of Lemma 5.17. We show that L is a continuous operator in ρ with continuity constant depending on the interval I.

Lemma 5.18. *Let $\rho, \tilde{\rho} \in R(c_R)$ with c_R defined as in Lemma 5.12. Then holds point wise for $r \in [r_0, r_1]$*

$$|L\rho(r) - L\tilde{\rho}(r)| \leq \frac{(r_1^2 - r_0^2)}{2r} \|\rho - \tilde{\rho}\|_{0,I}$$

Proof. By (5.22), we obtain

$$\begin{aligned}
&|L\tilde{\rho}(r) - L\rho(r)| \\
&\quad = \left| \int_{r_0}^{r} \partial_r G_1(r, t)\rho(t)\, dt + \int_{r}^{r_1} \partial_r G_2(r, t)\rho(t)\, dt - \int_{r_0}^{r} \partial_r G_1(r, t)\tilde{\rho}(t)\, dt - \int_{r}^{r_1} \partial_r G_2(r, t)\tilde{\rho}(t)\, dt \right| \\
&\quad = \left| \int_{r_0}^{r} \partial_r G_1(r, t)(\rho(t) - \tilde{\rho}(t))\, dt + \int_{r}^{r_1} \partial_r G_2(r, t)(\rho(t) - \tilde{\rho}(t))\, dt \right| \\
&\quad \leq \left| \int_{r_0}^{r} \partial_r G_1(r, t)(\rho(t) - \tilde{\rho}(t))\, dt \right| + \left| \int_{r}^{r_1} \partial_r G_2(r, t)(\rho(t) - \tilde{\rho}(t))\, dt \right| \\
&\quad \leq \int_{r_0}^{r} |\partial_r G_1(r, t)|\, |\rho(t) - \tilde{\rho}(t)|\, dt + \int_{r}^{r_1} |\partial_r G_2(r, t)|\, |\rho(t) - \tilde{\rho}(t)|\, dt \\
&\quad \leq \|\rho - \tilde{\rho}\|_{0,I} \left(\int_{r_0}^{r_1} |\partial_r G_1(r, t)|\, dt + \int_{r_0}^{r_1} |\partial_r G_2(r, t)|\, dt \right).
\end{aligned}$$

We compute the integrals over $\partial_r G_1$ and $\partial_r G_2$ and get with Lemma 5.8

$$\begin{aligned}
&\int_{r_0}^{r_1} |\partial_r G_1(r,t)|\, dt + \int_{r_0}^{r_1} |\partial_r G_2(r,t)|\, dt = -\int_{r_0}^{r_1} \partial_r G_1(r,t)\, dt + \int_{r_0}^{r_1} \partial_r G_2(r,t)\, dt \\
&= \frac{-1}{r\log\left(\frac{r_0}{r_1}\right)} \left(\frac{r_1^2}{2}\log\left(\frac{r_1}{r_0}\right) - \frac{r_1^2 - r_0^2}{4}\right) + \frac{1}{r\log\left(\frac{r_0}{r_1}\right)} \left(\frac{r_0^2}{2}\log\left(\frac{r_1}{r_0}\right) - \frac{r_1^2 - r_0^2}{4}\right) \\
&= \frac{-1}{r\log\left(\frac{r_0}{r_1}\right)} \left(\frac{r_1^2}{2}\log\left(\frac{r_1}{r_0}\right) - \frac{r_0^2}{2}\log\left(\frac{r_1}{r_0}\right)\right) \\
&= \frac{r_1^2 - r_0^2}{2r}.
\end{aligned}$$

□

We use the previous Lemmas to show that the operator $L \circ T$ is continuous.

Theorem 5.19. *Let $\rho, \tilde{\rho} \in R(c_R)$ with c_R defined as in Lemma 5.12. Then holds*

$$\|T \circ L\rho - T \circ L\tilde{\rho}\|_{0,I} \leq K\|\rho - \tilde{\rho}\|_{0,I}$$

with $K = \rho_A \frac{|b-2a+c|}{|\rho_A(a-c)+u_{A_1}-u_{A_2}|}$ where

$$\begin{aligned}
a &:= \frac{r_0^2}{2}\log\left(\frac{r_1}{r_0}\right), \\
b &:= \frac{r_1^2}{2}\log\left(\frac{r_1}{r_0}\right), \\
c &:= \frac{r_1^2 - r_0^2}{4}.
\end{aligned}$$

Proof. By Lemma 5.16, we obtain

$$\begin{aligned}
&\|T \circ L\rho - T \circ L\tilde{\rho}\|_{0,I} \\
&\leq \sup_{r_0 \leq r \leq r_1} \left| \frac{\rho_A}{\left| r_0\, \rho_A \int_{r_0}^{r_1} \partial_r G_2(r_0,t)\, dt + \frac{u_{A_1} - u_{A_2}}{\log(\frac{r_0}{r_1})} \right|} \left(r_0 |L\rho(r_0) - L\tilde{\rho}(r_0)| + r|L\tilde{\rho}(r) - L\rho(r)|\right) \right|.
\end{aligned}$$

By Lemma 5.17 and Lemma 5.18, we get

$$
\begin{aligned}
\|T\circ L\rho - T\circ L\tilde{\rho}\|_{0,I} &\le \sup_{r_0\le r\le r_1} \frac{\left|\rho_A\left(r_0\left(-\frac{r_0}{2}-\frac{r_1^2-r_0^2}{4r_0\log(\frac{r_0}{r_1})}\right)+r\frac{(r_1^2-r_0^2)}{2r}\right)\right|}{\left|r_0\,\rho_A\int_{r_0}^{r_1}\partial_r G_2(r_0,t)\,dt+\frac{u_{A_1}-u_{A_2}}{\log(\frac{r_0}{r_1})}\right|}\|\rho-\tilde{\rho}\|_{0,I}\\
&= \frac{\left|\rho_A\left(-\frac{r_0^2}{2}-\frac{r_1^2-r_0^2}{4\log(\frac{r_0}{r_1})}+\frac{r_1^2-r_0^2}{2}\right)\right|}{\left|r_0\,\rho_A\int_{r_0}^{r_1}\partial_r G_2(r_0,t)\,dt+\frac{u_{A_1}-u_{A_2}}{\log(\frac{r_0}{r_1})}\right|}\|\rho-\tilde{\rho}\|_{0,I}\\
&= \frac{\rho_A\left|-\frac{r_1^2-r_0^2}{4\log(\frac{r_0}{r_1})}+\frac{r_1^2-2r_0^2}{2}\right|}{\left|r_0\,\rho_A\int_{r_0}^{r_1}\partial_r G_2(r_0,t)\,dt+\frac{u_{A_1}-u_{A_2}}{\log(\frac{r_0}{r_1})}\right|}\|\rho-\tilde{\rho}\|_{0,I}\,.
\end{aligned}
$$

With Lemma 5.8, we compute the integral in the denominator.

$$
\begin{aligned}
\|T\circ L\rho - T\circ L\tilde{\rho}\|_{0,I} &\le \frac{\rho_A\left|-\frac{r_1^2-r_0^2}{4\log(\frac{r_0}{r_1})}+\frac{r_1^2-2r_0^2}{2}\right|}{\left|\frac{r_0\,\rho_A}{r_0\log\left(\frac{r_0}{r_1}\right)}\left(\frac{r_0^2}{2}\log(\frac{r_1}{r_0})-\frac{r_1^2-r_0^2}{4}\right)+\frac{u_{A_1}-u_{A_2}}{\log(\frac{r_0}{r_1})}\right|}\|\rho-\tilde{\rho}\|_{0,I}\\
&= \frac{\left|\frac{\rho_A}{\log(\frac{r_0}{r_1})}\right|\left|\frac{r_1^2-r_0^2}{4}-\frac{r_1^2-2r_0^2}{2}\log\left(\frac{r_0}{r_1}\right)\right|_0}{\left|\frac{1}{\log(\frac{r_0}{r_1})}\right|\left|\rho_A\left(\frac{r_0^2}{2}\log(\frac{r_1}{r_0})-\frac{r_1^2-r_0^2}{4}\right)+u_{A_1}-u_{A_2}\right|}\|\rho-\tilde{\rho}\|_{0,I}\\
&= \rho_A\frac{\left|\frac{r_1^2-r_0^2}{4}-\frac{r_1^2-2r_0^2}{2}\log\left(\frac{r_0}{r_1}\right)\right|}{\left|\rho_A\left(\frac{r_0^2}{2}\log(\frac{r_1}{r_0})-\frac{r_1^2-r_0^2}{4}\right)+u_{A_1}-u_{A_2}\right|}\|\rho-\tilde{\rho}\|_{0,I}\,.
\end{aligned}
$$

The constant K is thus given by

$$K=\rho_A\frac{|b-2a+c|}{|\rho_A\,(a-c)+u_{A_1}-u_{A_2}|}$$

□

The continuity constant for $T\circ L$ depends on the geometry of the domain, given here by the interval boundaries, and the boundary data u_{A_1}, u_{A_2} and ρ_A. It stands out that ρ_A is a multiplicative factor in K. By choosing ρ_A sufficiently small, it is possible to diminish the continuity constant such that $K<1$. We thus get a second condition on the constant M and have to reduce the set $R(M)$ further.

Theorem 5.20. *With the definitions of Theorem 5.19, set*

$$c_L := \frac{u_{A_1}-u_{A_2}}{|b-2a+c|+|a-c|}. \tag{5.33}$$

Then follows for the constant K of Theorem 5.19 for $\rho \in R(\min\{c_R, c_L\})$ that $K < 1$. Conclusively, the operator $T \circ L$ is a contraction on $R(\min\{c_R, c_L\})$, i.e.

$$\|T \circ L\rho - T \circ L\tilde{\rho}\|_{0,I} \leq K\|\rho - \tilde{\rho}\|_{0,I}$$

with $K < 1$.

Proof. To obtain a contraction, it must holds $K < 1$. From Theorem 5.19, we know that for $\rho, \tilde{\rho} \in R(c_R)$ holds

$$K = \rho_A \frac{|b - 2a + c|}{|\rho_A (a - c) + u_{A_1} - u_{A_2}|}$$

First choose ρ_A such that

$$\rho_A |a - c| \leq u_{A_1} - u_{A_2}.$$

Then holds by the inverse triangle inequality

$$\rho_A \frac{|b - 2a + c|}{|\rho_A (a - c) + u_{A_1} - u_{A_2}|} \leq \rho_A \frac{|b - 2a + c|}{|u_{A_1} - u_{A_2}| - \rho_A |a - c|}$$

To obtain $K < 1$, set

$$\rho_A \frac{|b - 2a + c|}{|u_{A_1} - u_{A_2}| - \rho_A |a - c|} \overset{!}{<} 1.$$

As the denominator is positive, it follows

$$\rho_A |b - 2a + c| < u_{A_1} - u_{A_2} - \rho_A |a - c|$$

and thus

$$\rho_A < \frac{u_{A_1} - u_{A_2}}{|b - 2a + c| + |a - c|}.$$

We thus set

$$c_L := \min\left\{\frac{u_{A_1} - u_{A_2}}{|b - 2a + c| + |a - c|}, \frac{u_{A_1} - u_{A_2}}{|a - c|}\right\} = \frac{u_{A_1} - u_{A_2}}{|b - 2a + c| + |a - c|}.$$

We now choose $\rho_A \leq \min\{c_L, c_R\}$ and obtain

$$\|T \circ L\rho - T \circ L\tilde{\rho}\|_{0,I} \leq K\|\rho - \tilde{\rho}\|_{0,I}$$

with $K < 1$. □

We are now arriving at the main result of the Chapter.

Theorem 5.21 (Existence and Uniqueness of a solution)**.** *Let $I = [r_0, r_1]$ and $M = \min\{c_L, c_R\}$. Then the operator*
$T \circ L$ has a unique fixed point $\rho \in R(M)$. Consequently, the coupled problem (CP 5.1) has a unique classical solution $(u, \rho) \in C^2(I) \times C^1(I)$ with $\rho \in R(M)$.

Proof. As shown in Lemma 5.15 and Theorem 5.20, $T \circ L$ is a selfmap and a contraction on the set $R(M)$. Due to the Banach fixed point theorem, there exists a unique fixed point ρ in $R(M)$. Consequently $\rho \in C^0(I)$ solves the integrated transport equation (5.18). The Poisson solution is given by Theorem 5.5. As Green's function is twice continuously differentiable, we obtain $u \in C^2(I)$. As ρ is the fixed point to $T \circ L$ holds $\rho = T \circ L\rho$. By Lemma 5.11 holds first $L\rho \in C^1(I)$. We apply Lemma 5.10 and obtain $\rho \in C^1(I)$. We found the classical solution $(u, \rho) \in C^2(I) \times C^1(I)$ to (CP 5.1) with $\rho \in R(c_R, c_L)$. □

With a transformation argument, we then obtain the radially symmetric solution
$(u, \rho) \in C^2(\Omega) \times C^0(\Omega)$.

Theorem 5.22 (Existence of a Radially Symmetric Solution on Ω)**.** *Let Ω be the annular domain as defined in (5.2). Let u, ρ and E be radially symmetric. Then the two dimensional coupled problem (CP 6.1) has a radially symmetric classical solution $(u, \rho) \in C^2(\Omega) \times C^0(\Omega)$.*

Proof. This result is an immediate consequence of Theorem 5.21 and the transformation of polar into Cartesian coordinates. □

5.4 Remarks about the Chapter

In this Chapter, we proved the existence and uniqueness of a radially symmetric solution to the two dimensional steady state coupled problem on an annular domain. We introduced the solution operators L of the Poisson equation and T of the transport equation. We used the Banach fixed point Theorem to prove the unique existence of a fixed point to the composite operator $T \circ L$ on a set $R(M)$ of continuous functions ρ that are bounded by its inflow boundary data.

We are free to choose the applied potential difference $u_{A_2} - u_{A_1}$. The inflow boundary data ρ_A, however, depends on the choice of r_0, r_1 and $u_{A_2} - u_{A_1}$. Having a fixed domain Ω, the greater the applied potential $u_{A_2} - u_{A_1}$, the greater the upper bound for ρ_A becomes. On the other hand, for a fixed potential difference, an increasing size of the domain will reduce the acceptable size of ρ_A.

To prove the existence of a fixed point, we applied the composite operator $T \circ L$ on the set $R(M)$ of functions ρ. The fixed point is the solution ρ of the transport equation in the coupled problem. The Banach fixed point iterations are thus an algorithm to solve (CP 5.1). Let the iterations be defined by $\rho_{n+1} = T \circ L\rho_n$ with $\rho_0 \in R(M)$. Then for every $\rho_A \in R(M)$, the algorithm converges to the solution ρ of (CP 5.1) and it holds the a priori estimate

$$\|\rho - \rho_n\|_{0,I} \leq \frac{K^n}{1-K}\|\rho_1 - \rho_0\|_{0,I} \leq 2M\frac{K^n}{1-K}. \tag{5.34}$$

With this error estimate, we obtain a maximum number of iterations that are necessary to approximate the exact solution by the iterations ρ_n for a given accuracy. This bound holds for every choice of inflow boundary data with $\rho_A \leq M$. In section 8, we will solve the discretized version of (CP 5.1) using a staggered algorithm. We will investigate numerically the dependence on ρ_A for a fixed choice of $u_{A_2} - u_{A_1}$ and r_0, r_1 and compare the results to the ones that we have obtained in this Chapter.

Chapter 6

Steady State Coupled Problem

In this chapter, we analyze the two-dimensional steady state coupled problem focussing on the existence of a continuous classical solution. The problem setting reads

Problem (CP 6.1). *Let Ω be an open bounded $C^{2,\alpha}$ domain with boundary $\Gamma = \Gamma_- \cup \Gamma_+$. Find $(u, \rho) \in C^{2,\alpha}(\bar{\Omega}) \times C^{\alpha}(\bar{\Omega})$ such that*

$$-\Delta u(x) = \rho(x) \qquad x \in \Omega \tag{6.1a}$$

$$u(x) = u_A(x) \qquad x \in \Gamma \tag{6.1b}$$

$$\rho^2(x) + E \cdot \nabla \rho = 0 \qquad x \in \Omega \tag{6.1c}$$

$$\rho(x) = \rho_A(x) \qquad x \in \Gamma_- \tag{6.1d}$$

$$E(x) = -\nabla u(x) \qquad x \in \Omega \tag{6.1e}$$

where $u_A|_{\Gamma_-} = u_{A_1}$ and $u_A|_{\Gamma_+} = u_{A_2}$, $u_{A_1} > u_{A_2}$ are constant and $\rho_A \geq 0$ with $\rho_A \in C^{1,\alpha}(\Gamma_-)$.

The method to show existence of a solution to (CP 6.1) is closely related to the one of Chapter 5. We introduce the solution operators L for the Poisson equation and T for the transport equation. The idea is to formulate the coupled problem as a fixed point problem. Starting from a vector field E_0, the Poisson and transport equations are solved alternating until convergence is obtained. Indeed, provided that $\|\rho_A\|_{1,\alpha,\Gamma}$ is sufficiently small, we will show the existence and uniqueness of a fixed point $E = L \circ TE$ in a set $W(M, \delta_1, \delta_2, \delta_3) \subset C^{1,\alpha}(\bar{\Omega})$ by the Banach fixed point theorem.

Let us first give a brief overview of this Chapter. In section 6.1, we will introduce the solution operator L for the Dirichlet problem for the Poisson equation (6.1a)-(6.1b) by $(L\rho)(x) = E(x) = -\nabla u(x)$. In contrast to Chapter 5, we do not search for an explicit representation of the operator L. To successfully apply the Banach fixed point theorem to $L \circ T$, it suffices to use standard existence results and a priori estimates for the Poisson solution in Hölder spaces as they can be found in [34].

The main focus in this Chapter is on the discussion of the Dirichlet problem for the nonlinear transport equation (6.1c)-(6.1d). In section 6.2, we introduce the streamline function $\Phi : Q \to \Omega$ defined as solution of

$$
\begin{aligned}
\frac{d\Phi(s,t)}{ds} &= E(\Phi(s,t)) && s \in [0,a], t \in [0, L_{\Gamma_-}] \\
\Phi(0,t) &= \varphi(t) && t \in [0, L_{\Gamma_-}]
\end{aligned}
$$

where $a \in \mathbb{R}$ and φ is, in agreement with Chapter 2, the parametrization of the inflow boundary Γ_- defined on the interval $I_{\Gamma_-} := [0, L_{\Gamma_-}]$. We will discuss restrictions on the right hand side vector field E to obtain a streamline function $\Phi \in C^{1,\alpha}(\bar{\Omega})$. Furthermore, we will investigate whether Φ is invertible. With the observation that on a streamline the nonlinear transport equation can be solved explicitly, we introduce the solution operator T for (6.1c)-(6.1d) for $x \in \Omega$ by

$$
TE(x) = \left. \frac{\rho_A(\varphi(t))}{1 + s \cdot \rho_A(\varphi(t))} \right|_{(s,t)=\Phi^{-1}(x)}. \tag{6.2}
$$

Clearly, a solution ρ only exists on Ω if the streamline function Φ is invertible. In section 6.3, we show that T is continuous on a set of functions $W(M, \delta_1, \delta_2, \delta_3)$. $W(M, \delta_1, \delta_2, \delta_3)$ contains all those vector fields E for which the corresponding streamline functions Φ exist and are invertible.

Section 6.4 deals with the main result of this chapter: the existence of a classical solution (u, ρ) to (CP 6.1). Provided that $\|\rho_A\|_{1,\alpha,\Omega}$ is sufficiently small, we prove that the composite operator $L \circ T$ is a selfmap and contraction on $W(M, \delta_1, \delta_2, \delta_3)$. By the Banach fixed point theorem, we conclude that there exists a unique fixed point $E = L \circ TE \in W(M, \delta_1, \delta_2, \delta_3)$. It then follows by the definition of the operator L and T that there exists a unique classical solution (u, ρ) to (CP 6.1) with $-\nabla u \in W(M, \delta_1, \delta_2, \delta_3)$.

6.1 Poisson Equation

The first subproblem of (CP 6.1) is the Dirichlet problem for the Poisson equation.

Problem (Po 6.2). *Let Ω be a bounded $C^{2,\alpha}$ domain. Given a right-hand side function $\rho \in C^{\alpha}(\bar{\Omega})$, find the solution $u \in C^{2,\alpha}(\bar{\Omega})$ to*

$$
\begin{aligned}
-\Delta u &= \rho && x \in \Omega && (6.3a) \\
u &= u_A && x \in \Gamma && (6.3b)
\end{aligned}
$$

where $u_A|_{\Gamma_-} = u_{A_1}$, $u_A|_{\Gamma_+} = u_{A_2}$ and $u_{A_1} > u_{A_2}$ are constants.

As standard example for elliptic partial differential equations, the classical analysis of the Poisson equation is found in a vast literature. The theory in the framework of Hölder spaces is presented in e.g. [34, 51]. We will list briefly some classical results that will be needed in the following. With the regularity assumptions on Ω, ρ and u_A in (Po 6.2), (6.3a)-(6.3b) has a unique solution.

Theorem 6.1. *[34, Theorem 6.14] Let Ω be an open bounded $C^{2,\alpha}$ domain and $\rho \in C^{\alpha}(\bar{\Omega})$. Further, let $u_A \in C^{2+\alpha}(\bar{\Omega})$. Then the Dirichlet problem*

$$\begin{aligned} -\Delta u &= \rho & x &\in \Omega, \\ u &= u_A & x &\in \Gamma \end{aligned}$$

has a unique solution lying in $C^{2+\alpha}(\bar{\Omega})$.

Similar to Chapter 5, we will now introduce the operator L that gives the gradient field of the solution of the Poisson equation (6.3a)-(6.3b). However, we do not need an explicit representation for L to prove the unique existence of a fixed point of the composite operator $L \circ T$. Instead, it is enough to work with an abstract formulation. Applied to the right hand side function ρ, L maps onto the gradient field of the solution u of (6.3a)-(6.3b).

Definition 6.2. *Let Ω be an open bounded $C^{2,\alpha}$ domain. We denote the solution operator $L : C^{\alpha}(\bar{\Omega}) \to C^{1,\alpha}(\bar{\Omega})$ to the Poisson problem (6.3a)-(6.3b) by*

$$E(x) = (L\rho)(x), \quad x \in \Omega, \tag{6.4}$$

with $E = -\nabla u$.

By Theorem 6.1, we know that $L\rho$ exists uniquely for every $\rho \in C^{\alpha}(\bar{\Omega})$. Although we do not need an explicit representation of L, we do need to know some of its properties. The listing of the following results might seem incoherent but the necessity will become clear in section 6.4. An important property used in the following sections is the linearity of the Laplace operator. An equivalent formulation of (Po 6.2) is given by the decomposition $u = u_0 + u_1$ with

$$-\Delta u_0 = 0 \qquad x \in \Omega \tag{6.5a}$$

$$u_0 = u_A \qquad x \in \Gamma \tag{6.5b}$$

and

$$-\Delta u_1 = \rho \qquad x \in \Omega \tag{6.6a}$$

$$u_1 = 0 \qquad x \in \Gamma. \tag{6.6b}$$

We obtain the following existence result for u_0 and u_1.

Lemma 6.3. *Let Ω be a $C^{2,\alpha}$ domain, $\rho \in C^{\alpha}(\bar{\Omega})$ and $u_A \in C^{2,\alpha}(\Gamma)$. Then there exist unique solutions $u_0 \in C^{2,\alpha}(\bar{\Omega})$ to (6.5a)-(6.5b) and $u_1 \in C^{2,\alpha}(\bar{\Omega})$ to (6.6a)-(6.6b).*

Proof. This is an immediate consequence of Theorem 6.1. □

The Laplace equation (6.5a)-(6.5b) only depends on the shape of the domain and the boundary data. In case of the Banach fixed point iterations, the solution u_0 is determined a priori and does not change with varying ρ.

Next, we will introduce a priori estimates for the solution u of (6.3a)-(6.3b). These will be used later on to bound the difference $\|L\rho - L\tilde{\rho}\|_{1,\alpha;\Omega}$ in section 6.4. The following theorem is the famous maximum principle for the Laplace equation.

Theorem 6.4. *[34, Theorem 3.5] (Strong maximum principle)*
Let $\Delta u = 0$ in an open bounded domain Ω and suppose there exists a point $y \in \Omega$ for which $u(y) = \sup_\Omega u$. Then u is constant. Consequently a harmonic function cannot assume an interior maximum or minimum value unless it is constant.

An immediate consequence is the weak maximum and minimum principle.

Theorem 6.5. *[34, Theorem 3.1] (Weak maximum and minimum principle)*
Let Ω be an open bounded domain. Let $u \in C^2(\Omega) \cup C^0(\bar{\Omega})$ with $-\Delta u(x) = 0$ in Ω. Then

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x), \qquad \inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega} u(x).$$

These theorems are used to give a first a priori bound on the sup-norm of u.

Theorem 6.6. *[34, Theorem 3.7]*
Let $-\Delta u = \rho$ in an open bounded domain Ω and $u \in C^0(\bar{\Omega}) \cup C^2(\Omega)$. Then

$$\sup_{x\in\Omega} |u(x)| \leq \sup_{x\in\partial\Omega} |u(x)| + c(\operatorname{diam}\Omega) \sup_{x\in\Omega} |\rho(x)|.$$

A priori bounds to the classical solution of the Poisson equation in the $C^{2,\alpha}(\Omega)$ norm are given by the Schauder a priori estimates. As we search for a solution on the bounded domain $\bar{\Omega}$, we need an estimate up to the boundary of Ω, see [51, p. 106 ff] or [34, Theorem 6.6].

Theorem 6.7. *(Schauder's Estimate)*
Let Ω be an open bounded $C^{2,\alpha}$ domain with $\alpha \in (0,1)$. Let $u \in C^{2,\alpha}(\bar{\Omega})$ be the solution of $-\Delta u = \rho, u|_\Gamma = u_A$ with $\rho \in C^\alpha(\bar{\Omega})$ and $u_A \in C^{2,\alpha}(\Gamma)$. Then holds

$$\|u\|_{2,\alpha,\Omega} \leq c_S(\Omega,\alpha)\left(\|u\|_{0,\Omega} + \|\rho\|_{\alpha,\Omega} + \|u_A\|_{2,\alpha,\Gamma}\right). \tag{6.7}$$

The solution u to (Po 6.2) is thus a priori bounded in terms of the right-hand side function ρ and boundary data. It will play an important role in section 6.4 where we show that $L \circ T$ is a self map and contraction on a set $W(M, \delta_1, \delta_2, \delta_3) \subset C^{1,\alpha}(\bar{\Omega})$.

6.2 Streamline Function

The second subproblem of (CP 6.1) is the nonlinear transport equation.

Problem (Tr 6.3). *Let Ω be an open bounded $C^{2,\alpha}$ domain. For a given vector field $E \in C^{1,\alpha}(\bar{\Omega})$, find $\rho \in C^{1,\alpha}(\bar{\Omega})$, such that*

$$\begin{aligned} \rho^2 + E \cdot \nabla\rho &= 0 & x &\in \Omega \\ \rho &= \rho_A & x &\in \Gamma_- \end{aligned}$$

with $\rho_A \in C^{1,\alpha}(\Gamma_-)$ and $\rho_A \geq 0$.

Before beginning with the analysis of the transport equation, we need to do some preliminary work. Hyperbolic partial differential equations such as the transport equation reduce to ordinary differential equations on the streamlines. We therefore begin our study with the streamline function Φ as it will be the key to derive the transport solution operator T. Let us define the parameter set

$$Q = \left\{(s,t) : s \in [0,a],\, t \in I_{\Gamma_-}\right\}$$

with $a \in \mathbb{R}$. For a given continuous vector field $E \in C^{0,1}(\Omega)$, the streamline function Φ to (Po 6.2) is defined as the solution of the autonomous ordinary differential equation

$$\frac{d\Phi(s,t)}{ds} = E(\Phi(s,t)) \qquad (s,t) \in Q \tag{6.9a}$$

$$\Phi(0,t) = \varphi(t) \qquad t \in I_{\Gamma_-}. \tag{6.9b}$$

First, regard $a \in \mathbb{R}$ as an arbitrary parameter. We will specify it more exactly later on.

A classical solution of the transport equation requires the existence of a $C^{1,\alpha}(Q)$ streamline function and as indicated by (6.2) the existence of its inverse function Φ^{-1}. This section contains a comprehensive analysis for the existence and uniqueness of a solution to (6.9a)- (6.9b). Moreover, we investigate restrictions on the vector field E to obtain an invertible streamline function Φ for the model problem. Φ has to be a bijective function from Q onto Ω because only then it is guaranteed that every point $x \in \Omega$ is covered by a streamline.

The right hand side function E of (6.9a) does not explicitly depend on the streamline parameter s which classifies the differential equation as autonomous. General existence theorems for streamline functions that are given in the literature (e.g. [41, Chapter 17]) use a different parametrization for the differential equation. The standard form is given by

$$\begin{aligned} \frac{d}{ds}\Psi(s,x_0) &= E(\Psi(s,x_0)) \\ \Psi(0,x_0) &= x_0 \end{aligned}$$

for $s \in [0,a]$ and $x_0 \in \Omega$. The streamlines Ψ thus begin in every point $x \in \Omega$. For completeness, we will present the proof for existence and uniqueness of $\Phi \in C^1(Q)$ for the chosen particular case (6.9a)-(6.9b) in which the streamline functions start only from the inflow boundary. We therefore generalize the Picard-Lindelöf Theorem as found in [39, Theorem 1.1].

Theorem 6.8. *Let $\Omega \subset \mathbb{R}^2$ be a C^1 domain and $E \in C^{0,1}(\Omega)$ with Lipschitz constant L_E. Then a unique global solution $\Phi : Q \to \Omega$ of the initial value problem*

$$\frac{d\Phi(s,t)}{ds} = E(\Phi(s,t)) \qquad s \in [0,a],\, t \in I_{\Gamma_-} \tag{6.10}$$
$$\Phi(0,t) = \varphi(t) \qquad t \in I_{\Gamma_-} \tag{6.11}$$

exists for every continuous φ. Further, $\Phi(s,t)$ is continuously differentiable with respect to s.

Proof. We use the Banach fixed point theorem to show existence and uniqueness of a solution $\Phi(s,t)$ to (6.10).

Given $\Phi \in C^0(Q)$, we define the operator $R(\Phi)$

$$R(\Phi)(s,t) = \varphi(t) + \int_0^s E(\Phi(\tau,t))\, d\tau.$$

R is a selfmap:
The mapping $(s,t) \mapsto E(\Phi(s,t))$ is continuous on Q for fixed $\Phi \in C(Q,\Omega)$. It holds $\|E(\Phi)\|_{0,Q} \le \|E\|_{0,\Omega} = M$ and thus the integral $\int_0^s E(\Phi(\tau,t))\, d\tau$ is well-defined. It holds

$$\begin{aligned}
|R(\Phi)(s_1,t_1) - R(\Phi)(s_2,t_2)|_\infty &\le |\varphi(t_1) - \varphi(t_2)|_\infty + \left| \int_0^{s_1} E(\Phi(\tau,t_1))\, d\tau - \int_0^{s_2} E(\Phi(\tau,t_2))\, d\tau \right|_\infty \\
&= |\varphi(t_1) - \varphi(t_2)|_\infty \\
&\quad + \left| \int_0^{s_1} E(\Phi(\tau,t_1))\, d\tau - \int_0^{s_2} E(\Phi(\tau,t_1)) - E(\Phi(\tau,t_1))\, d\tau - \int_0^{s_2} E(\Phi(\tau,t_2))\, d\tau \right|_\infty \\
&= |\varphi(t_1) - \varphi(t_2)|_\infty + \left| \int_{s_2}^{s_1} E(\Phi(\tau,t_1))\, d\tau \right|_\infty + \left| \int_0^{s_2} E(\Phi(\tau,t_1)) - E(\Phi(\tau,t_2))\, d\tau \right|_\infty \\
&\le |\varphi(t_1) - \varphi(t_2)|_\infty + \int_{s_2}^{s_1} |E(\Phi(\tau,t_1))|_\infty\, d\tau + \int_0^{s_2} |E(\Phi(\tau,t_1)) - E(\Phi(\tau,t_2))|_\infty\, d\tau \\
&\le \sup \left|\varphi'(t)\right|_\infty |t_1 - t_2|_\infty + M\,|s_2 - s_1|_\infty + L_E \int_0^{s_2} |\Phi(\tau,t_1) - \Phi(\tau,t_2)|_\infty\, d\tau.
\end{aligned}$$

Let $B_\epsilon((s_1,t_1))$ a ball around (s_1,t_2) with radius ϵ. Let $(s_2,t_2) \in B_\epsilon((s_1,t_1))$ for $\epsilon > 0$. Since Φ is continuous, it holds that $|\Phi(s,t_1) - \Phi(s,t_2)|_\infty \le c\epsilon$ and

$$|R(\Phi)(s_1,t_1) - R(\Phi)(s_2,t_2)|_\infty \le \sup|\varphi'(t)|_\infty \epsilon + M\epsilon + s_2 c L \epsilon = C\epsilon.$$

With $\epsilon \to 0$, it follows that $R(\Phi) \in C^0(Q)$ and thus the operator R is a self-map.

R is a contraction
In the next step, we introduce a weighted sup-norm for $C(Q)$

$$\|u\|_{\tilde{C}(Q)} := \sup_{(s,t)\in Q} \left[\exp(-2L_E s)|u(s,t)|_\infty\right].$$

This norm is equivalent on finite domains to the usual sup-norm. Thus $C(Q)$ forms a Banach space equipped with the norm above. It holds

$$
\begin{aligned}
&\sup_{(s,t)\in Q}\left[\exp(-2L_E s)|R(\Phi)(s,t) - R(\tilde{\Phi})(s,t)|_\infty\right] \\
&= \sup_{(s,t)\in Q}\left[\exp(-2L_E\mu)|\int_0^s E(\Phi(\mu,t)) - E(\tilde{\Phi}(\mu,t))\,d\mu|_\infty\right] \\
&\le \sup_{(s,t)\in Q}\left[\exp(-2L_E s)\int_0^s |E(\Phi(\mu,t)) - E(\tilde{\Phi}(\mu,t))|_\infty\,d\mu\right] \\
&\le \sup_{(s,t)\in Q}\left[\exp(-2L_E s)\int_0^s \exp(2L\mu)\exp(-2L_E\mu)L_E|\Phi(\mu,t) - \tilde{\Phi}(\mu,t)|_\infty\,d\mu\right] \\
&\le \sup_{(s,t)\in Q}\left[L_E\exp(-2L_E s)\int_0^s \exp(2L_E\mu)\|\Phi - \tilde{\Phi}\|_{\tilde{C}(Q)}\,d\mu\right] \\
&= \sup_{(s,t)\in Q}\left[L_E\exp(-2L_E s)\frac{1}{2L_E}(\exp(2L_E s) - 1)\|\Phi - \tilde{\Phi}\|_{\tilde{C}(Q)}\right] \\
&\le \frac{1}{2}\|\Phi - \tilde{\Phi}\|_{\tilde{C}(Q)}.
\end{aligned}
$$

The assumptions for Banach's fixed point theorem are fulfilled. It follows that there exists a unique fixed point Φ. With the fundamental theorem of calculus, Φ is continuously differentiable with respect to s. □

The existence and uniqueness of $(u,\rho) \in (C^{2,\alpha}(\bar{\Omega}), C^\alpha(\bar{\Omega}))$ requires $\Phi \in C^{1,\alpha}(Q)$. The next theorem shows that if $\nabla E \in C^\alpha(\bar{\Omega})$, then Φ is differentiable.

Theorem 6.9. *Let Ω be an open and bounded C^1 domain, $\nabla E \in C^\alpha(\Omega)$ with Hölder constant L_α. Then follows $\partial_t\Phi \in C^0(Q)$.*

Proof. First, we differentiate (6.10) with respect to t and change the order of differentiation.

$$\frac{d^2\Phi(s,t)}{ds\,dt} = \nabla E(\Phi(s,t))\frac{d\Phi(s,t)}{dt}.$$

Φ exists uniquely due to Theorem 6.8 and is considered as a known function. The differential equation is thus solved with respect to $\frac{d\Phi}{dt} =: \Phi_t$. Again, we use Banach's fixed point theorem.

Given $\Phi_t \in C^0(Q)$, define the operator $S(\Phi_t)$ by

$$S(\Phi_t)(s,t) = \varphi'(t) + \int_0^s \nabla E(\Phi(\tau,t))\Phi_t(\tau,t)\,d\tau.$$

Due to the continuity of Φ and Φ_t, we know that $\|\nabla E(\Phi)\|_{0,Q} \le \|\nabla E\|_{0,\Omega} =: M_1$ and $\|\Phi_t\|_{0,Q} =: M_2$ for constants $M_1, M_2 < \infty$. First we show that $S(\Phi_t)$ defines a selfmap.

S is a selfmap:

$$
\begin{aligned}
&|S(\Phi_t)(s_1,t_1) - S(\Phi_t)(s_2,t_2)|_\infty \\
&\quad \le |\varphi_t(t_1) - \varphi_t(t_2)|_\infty + \left| \int_0^{s_1} \nabla E(\Phi(\tau,t_1))\Phi_t(\tau,t_1)\,d\tau - \int_0^{s_2} \nabla E(\Phi(\tau,t_2))\Phi_t\,d\tau \right|_\infty \\
&\quad \le |\varphi_t(t_1) - \varphi_t(t_2)|_\infty + \left| \int_{s_2}^{s_1} \nabla E(\Phi)\Phi_t(\tau,t_2)\,d\tau \right|_\infty \\
&\qquad + \left| \int_0^{s_2} \nabla E(\Phi)(\Phi_t(\tau,t_1) - \Phi_t(\tau,t_2)) + (\nabla E(\Phi(\tau,t_1)) - \nabla E(\Phi(\tau,t_2)))\Phi_t(\tau,t_2)\,d\tau \right|_\infty \\
&\quad \le |\varphi_t(t_1) - \varphi_t(t_2)|_\infty + \int_0^{s_2} |\nabla E(\Phi)|_\infty \, |\Phi_t(\tau,t_1) - \Phi_t(\tau,t_2)|_\infty \,d\tau \\
&\qquad + \int_0^{s_2} |\nabla E(\Phi(\tau,t_1)) - \nabla E(\Phi(\tau,t_2))|_\infty |\Phi_t(\tau,t_2)|_\infty \,d\tau + \left| \int_{s_2}^{s_1} |\nabla E(\Phi)|_\infty |\Phi_t(\tau,t_2)|_\infty \,d\tau \right| \\
&\quad \le |\varphi_t(t_1) - \varphi_t(t_2)|_\infty + M_1 \int_0^{s_2} |\Phi_t(\tau,t_1) - \Phi_t(\tau,t_2)|_\infty \,d\tau \\
&\qquad + \int_0^{s_2} L_\alpha M_2 |\Phi(\tau,t_1)) - \Phi(\tau,t_2))|_\infty^\alpha \,d\tau + M_1 M_2 |s_2 - s_1|.
\end{aligned}
$$

Let $B_\epsilon((s_1,t_1))$ be a ball of radius ϵ around (s_1,s_2) and $(s_2,t_2) \in B_\epsilon((s_1,t_1))$ for an $\epsilon > 0$. Since Φ, φ' and Φ_t are continuous, it holds that $|\Phi_t(s_1,t_1) - \Phi_t(s_2,t_2)|_\infty \le c\epsilon$ and

$$
|S(\Phi_t)(s_1,t_1) - S(\Phi_t)(s_2,t_2)|_\infty \le c\epsilon + csM_1\epsilon + cs_2 L_\alpha M_2 \epsilon^\alpha + cM_1 M_2 \epsilon \le C\epsilon^\alpha.
$$

Since $\alpha \in (0,1)$, it follows that $S(\Phi_t) \in C(Q)$ for $\epsilon \to 0$. The operator S defines a self-map.

S is a contraction:
We introduce a weighted sup-norm

$$
\|u\|_{\tilde{C}(Q,\Omega)} := \sup_{(s,t)\in Q} \left[\exp(-2M_1 s)|u(s,t)|_\infty\right].
$$

This norm is equivalent to the usual sup-norm. Thus $C(Q)$ forms a Banach space equipped with the weighted norm. It holds

$$
\begin{aligned}
&\sup_{(s,t)\in Q}\left[\exp(-2M_1 s)\left|S(\Phi_t)(s,t)-S(\tilde{\Phi}_t)(s,t)\right|_\infty\right]\\
&=\sup_{(s,t)\in Q}\left[\exp(-2M_1 s)\left|\int_0^s \nabla E(\Phi(\tau,t))\Phi_t(\tau,t)-\nabla E((\Phi)(\tau,t))\tilde{\Phi}_t(\tau,t)\,d\tau\right|_\infty\right]\\
&\le\sup_{(s,t)\in Q}\left[\exp(-2M_1 s)\int_0^s \exp(2M_1\tau)\exp(-2M_1\tau)\left|\nabla E(\Phi)(\Phi_t(\tau,t)-\tilde{\Phi}_t(\tau,t))\right|_\infty\,d\tau\right]\\
&\le\sup_{(s,t)\in Q}\left[\exp(-2M_1 s)\int_0^s \exp(2M_1\tau)\exp(-2M_1\tau)\,|\nabla E(\Phi)|_{0,Q}\left|\Phi_t(\tau,t)-\tilde{\Phi}_t(\tau,t)\right|_\infty\,d\tau\right]\\
&\le\sup_{(s,t)\in Q}\left[M_1\exp(-2M_1 s)\int_0^s \exp(2M_1\tau)\|\Phi_t-\tilde{\Phi}_t\|_{\tilde{C}(Q)}\,d\tau\right]\\
&\le\sup_{(s,t)\in Q}\left[M_1\|\Phi_t-\tilde{\Phi}_t\|_{\tilde{C}(Q)}\exp(-2M_1 s)\frac{1}{2M_1}(\exp(2M_1 s)-1)\right]\\
&\le\frac{1}{2}\|\Phi_t-\tilde{\Phi}_t\|_{\tilde{C}(Q)}.
\end{aligned}
$$

By the Banach fixed point theorem, there exists a unique fixed point $S(\Phi_t)=\Phi_t$. Φ_t is continuously differentiable with respect to s due to the fundamental theorem of calculus. □

6.2.1 Properties of the Streamline Function

Not every streamline function Φ leads to a possible solution of the model problem (CP 6.1). For example, as $\rho(x)=TE(x)$ indicated in (6.2) is defined on Ω, we have to ensure that for every point $x\in\Omega$ there exists a $(s,t)\in Q$ such that $x=\Phi(s,t)$. We will now discuss restrictions on the vector field E in order to obtain streamlines that are feasible for the coupled problem.

Definition 6.10. *We call the streamline function* Φ feasible for the problem setting *if the following conditions are fulfilled*

1. *No two streamlines intersect on* Ω.
2. Φ *maps the parameter set* Q *surjectively on* Ω.
3. *There are no closed streamlines in* Ω.

By excluding the intersection of any two streamlines, we obtain that Φ is injective. Conditions (1) and (2) are then equivalent to the bijectivity of Φ.
Textbooks on dynamical systems [36, 38, 37, 41] contain the necessary information to ensure that Φ is feasible for the coupled problem. Recall that the initial condition (6.9b) indicates that every streamline starts on the inflow boundary.

Definition 6.11. *[38, p.176] For $t_0 \in [0, L_{\Gamma_-}]$, the curve in the three dimensional space with $(s, x) \in \mathbb{R} \times \mathbb{R}^2$ defined by*

$$\{(s, \Phi(s, t_0)) : s \in [0, a]\}$$

with starting point $(0, \varphi(t_0))$ is called the trajectory *through $\varphi(t_0)$.*

It follows that the streamlines are the projection of the trajectories onto Ω. We use this observation to show that no two streamlines intersect.

Lemma 6.12. *[38, p.176] The streamlines to the autonomous differential equation (6.9a)-(6.9b) do not intersect.*

Proof. The streamlines are the projection of the trajectories $(s, \Phi(s, p)) \subset \mathbb{R}^3$ onto Ω. The right-hand side E of (6.9a) is independent of s. Choose $x_0 \in \Omega$. For every (s, t) with $\Phi(s, t) = x_0$, $\Phi(s, t)$ has the same slope since $E(\Phi(s, t)) = E(x_0)$ is constant. Thus if the trajectories intersect the line $x = x_0$, they must always have the same slope. After projecting the trajectories onto Ω, it follows that no two streamlines can intersect. $\square$

With the previous Lemma, we are certain that no two streamlines intersect. However, there are other phenomena that impede feasible streamlines. We therefore further investigate the context of dynamical systems. First, let us generalize the initial value problem (6.9a)-(6.9b) to

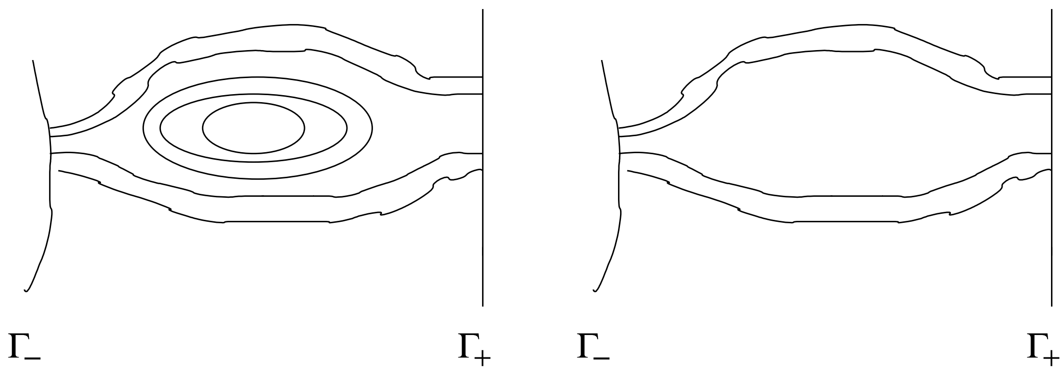
$$\begin{aligned} \frac{d\Psi(s, x_0)}{ds} &= E(\Psi(s, x_0)), && s \in [0, b],\ x_0 \in \Omega \\ \Psi(0, x_0) &= x_0 && x_0 \in \Omega. \end{aligned}$$

The difference to (6.9a)-(6.9b) is based on the initial condition. In case of $\Psi(s, x_0)$, every point $x_0 \in \Omega$ is a starting point for a streamline function. We will see that without any restrictions on E, the range of Φ might only be a subset of the range of Ψ.

Definition 6.13. *[37, p.38] An* equilibrium point *of the vector field $E \in C^{1,\alpha}(\Omega)$ is a point p such that $E(p) = 0$. If p is an equilibrium point, then the streamline starting from p is the point itself, i.e. $\Psi(s, p) = p, -\infty < s < \infty$. The trajectory of the critical point p is the line in $\mathbb{R}^3$ given by $x = p$, $-\infty < s < \infty$.*

Definition 6.14. *[36, p. 15] If there exists a $0 < T < \infty$ such that $\Psi(s + T, x_0) = \Psi(s, x_0)$ for all s, then we call Ψ* a periodic solution.

We will now focus again on (6.9a)-(6.9b) and explain why it is necessary to exclude equilibrium points of E and periodic solutions of Ψ. Having periodic orbits is equivalent to obtaining closed streamlines. Figures 6.1a and 6.1b illustrate the consequences for (6.9a)-(6.9b). Since every streamline $\Phi(\cdot, t)$ starts from the inflow boundary at $\varphi(t)$, $\Phi(\cdot, t)$ would never attain any point x belonging to a periodic streamline Ψ. Consequently, Φ does not map surjectively on Ω. Fortunately, the existence of period streamlines is excluded a priori for our model problem as E is a gradient field.



(a) Streamline Function Ψ (b) Streamline Function Φ

Figure 6.1: Phase Portrait with Periodical Streamlines

Theorem 6.15. *[38, p. 434] There do not exist periodic orbits for gradient systems.*

Theorem 6.15 excludes the possibility of having closed streamlines for the general streamline function Ψ. Conclusively, the situation illustrated in Figure 6.1b is impossible for Φ.

Nevertheless, equilibrium points give not only rise to periodic orbits.

Lemma 6.16. *[37, p.38] If p is an equilibrium point, then no streamline other than $\Psi(s,p) = p$ can reach the line $x = p, -\infty < s < \infty$. This implies: if p is a equilibrium point and $\Psi(s, x_0) \neq p$ tends to p, then either $s \to \infty$ or $s \to -\infty$.*

In Figure 6.2b, the previous Lemma is illustrated in a one-dimensional case. If an equilibrium point p exists, then Φ will converge to the first equilibrium point for $s \to \infty$. The remaining part of the interval is not contained in the range of Φ. It is thus necessary to exclude possible equilibrium points, i.e. $E(x) \neq 0, \forall x \in \Omega$.

(a) Streamline Function Ψ (b) Streamline Function Φ

Figure 6.2: Phase Portrait Containing Equilibrium Points

Lemma 6.17. *Let Ω be an open bounded domain. Let $E = -\nabla u \in C^0(\Omega)$ with $\inf_{x\in\Omega} |E(x)|_\infty \geq \delta_1 > 0$. Then a parameter set*

$$Q = \left\{(s,t) \,:\, 0 \leq s \leq l(t),\, t \in [0, L_{\Gamma_-}]\right\} \tag{6.12}$$

exists, such that Φ maps Q surjectively on Ω.

Proof. E is defined in $\bar{\Omega}$ and thus gives rise to streamlines through every point $x \in \Omega$. By Theorem 6.15 and the assumption $E(x) \neq 0$ for every $x \in \Omega$, neither closed streamlines nor equilibrium points exist. Thus, every point $x \in \Omega$ is in the range of Φ. The parameter s

depends on the starting point $\varphi(t)$ of the streamline and the local strength of the vector field E. Apparently, we obtain a different interval of definition for s for every fixed $t \in [0, L_{\Gamma_-}]$. Choose $l(t)$ such that $\Phi(l(t), t) \in \Gamma_+$ for every $t \in [0, L_{\Gamma_-}]$. Then Q defined in (6.12) proves the assertion. □

We collect the previous results for Φ and obtain the following theorem.

Theorem 6.18. *Let Ω be a $C^{2,\alpha}$ domain and Q as defined in (6.12). Let $E = -\nabla u \in C^0(\Omega)$ with $\inf_{x\in\bar{\Omega}} |E(x)|_\infty \geq \delta_1 > 0$ and Φ the solution of (6.9a)-(6.9b). Then*

$$\Phi : Q \to \Omega$$

is a bijection.

Proof. By Lemma 6.12 and Theorem 6.15 holds that Φ is injective. Moreover, Lemma 6.17 proves the surjectivity of Φ. It follows that Φ is bijective. □

So far, the maximum $l(t)$ of the streamline parameter s is only an abstract quantity. We know that it depends on the starting point φ and since $E(x) \neq 0$ for $x \in \bar{\Omega}$, we also know that $l(t)$ is finite for every t. In section 6.2.4, we will investigate an upper bound and show that $l(t)$ depends on the boundary data u_A and the minimal field strength $\inf_{x\in\bar{\Omega}} |E(x)|_\infty$.

6.2.2 Hölder Continuity of the Streamline Function

To the end of this Chapter, let Q be defined in (6.12). So far, we obtained $\Phi \in C^1(Q)$. To show that $\nabla\Phi$ is also Hölder continuous, we use a different representation of the streamline function. Integrating (6.9a) with respect to s and using the initial condition (6.9b) gives

$$\Phi(s,t) = \varphi(t) + \int_0^s E(\Phi(\mu, t))\, d\mu \tag{6.13}$$

Before beginning, we need to introduce *Grönwall's inequality* that we will use frequently. There exist multiple versions of this inequality in the literature. We pick the formulation of [55, Lemma 3.1, p. 89], as it is the most convenient one for our estimates in the following. The proof is given in the appendix, Lemma A.2.

Lemma 6.19 (Grönwall's inequality)**.** *Assume $I = [s_0, s_1]$ and $q \geq 0, u \geq 0 \in C^0(I)$ and $c \geq 0 \in C^1(I)$. If*

$$q(s) \leq c(s) + \int_{s_0}^s u(\tau) q(\tau)\, d\tau \tag{6.14}$$

then holds

$$q(s) \leq c(s_0) \exp\left(\int_{s_0}^s u(\tau)\, d\tau\right) + \int_{s_0}^s c'(\tau) \exp\left(\int_\tau^s u(\mu)\, d\mu\right) d\tau.$$

We are now ready to conclude that $\partial_s\Phi \in C^\alpha(Q)$.

Lemma 6.20. *Let Ω be a $C^{1,\alpha}$ domain and $E \in C^\alpha(\Omega)$. Then $\partial_s\Phi \in C^\alpha(Q)$.*

Proof. By the chain rule (2.9) holds for (6.9a)

$$|\partial_s\Phi|_{\alpha,Q} = |E(\Phi)|_{\alpha,Q} \leq |E|_{\alpha,\Omega}\|\nabla\Phi\|^\alpha_{0,Q} < \infty.$$

The Hölder coefficient $|\partial_s\Phi|_{\alpha,Q}$ is finite and thus $\partial_s\Phi \in C^\alpha(Q)$. □

Also, we show that $\partial_t\Phi \in C^\alpha(Q)$.

Lemma 6.21. *Let Ω be a $C^{1,\alpha}$ domain and $E \in C^{1,\alpha}(\Omega)$. Then $\partial_t\Phi \in C^\alpha(Q)$.*

Proof. We show that the Hölder coefficient of $\partial_t\Phi$ is bounded pointwise for all $(s_1, t_1) \neq (s_2, t_2) \in Q$. We only consider points with $s_1 \neq s_2$ and $t_1 \neq t_2$. The cases where $s_1 \neq s_2$ and $t_1 = t_2$ or $t_1 \neq t_2$ and $s_1 = s_2$ are simpler special cases. They can be derived easily from the following computations and are thus omitted.
Let now $s_1 \neq s_2$ and $t_1 \neq t_2$. The dependence of $[0, l(t)]$ on t makes it necessary to distinguish two cases. Two distinct points $(s_1, t_1) \neq (s_2, t_2) \in Q$ lead to two distinct intervals of existence for s_1 and s_2. It holds that $s_1 \in [0, l(t_1)]$ for a point (s_1, t_1) and $s_2 \in [0, l(t_2)]$ for (s_2, t_2). The minimum of $l(t_1)$ and $l(t_2)$ determines a triangle whose vertices $(s_1, t_1), (s_2, t_2)$ and either (s_1, t_2) or (s_2, t_1) are elements of Q.

Case 1: $l(t_1) \leq l(t_2)$, i.e. $[0,\, l(t_1)] \subset [0,\, l(t_2)]$. Consequently, $(s_1, t_2) \in Q$.
It holds pointwise

$$\begin{aligned}\frac{|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|^\alpha_\infty} &= \frac{|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_1,t_2) + \partial_t\Phi(s_1,t_2) - \partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|^\alpha_\infty} \\ &\leq \frac{|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_1,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|^\alpha_\infty} + \frac{|\partial_t\Phi(s_1,t_2) - \partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|^\alpha_\infty}. \end{aligned} \tag{6.15}$$

On the interval $[0, l(t_1)]$, we get for numerator of the second term of (6.15)

$$\begin{aligned}|\partial_t\Phi(s_1,t_2) - \partial_t\Phi(s_2,t_2)|_\infty &= \left|\int_{s_1}^{s_2} \nabla E(\Phi(\mu,t_2))\partial_t\Phi(\mu,t_2)\, d\mu\right|_\infty \\ &\leq |s_2 - s_1|\, \|\nabla E(\Phi)\|_{0,Q}\, \|\partial_t\Phi\|_{0,Q} \\ &\leq |s_2 - s_1|\, \|\nabla E\|_{0,\Omega}\, \|\partial_t\Phi\|_{0,Q}\,. \end{aligned} \tag{6.16}$$

For the first term of (6.15), we get for the numerator for every $s_1 \in [0,\, l(t_1)]$

$$
\begin{aligned}
&|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_1,t_2)|_\infty \\
&\quad\le |\varphi'(t_1) - \varphi'(t_2)|_\infty + \left| \int_0^{s_1} \nabla E(\Phi(\mu,t_1))\partial_t\Phi(\mu,t_1) - \nabla E(\Phi(\mu,t_2))\partial_t\Phi(\mu,t_2)\, d\mu \right|_\infty \\
&\quad\le \left|\varphi'(t_1) - \varphi'(t_2)\right|_\infty + \int_0^{s_1} |\nabla E(\Phi(\mu,t_1))\partial_t\Phi(\mu,t_1) - \nabla E(\Phi(\mu,t_2))\partial_t\Phi(\mu,t_2)|_\infty\, d\mu \\
&\quad\le \left|\varphi'(t_1) - \varphi'(t_2)\right|_\infty \\
&\quad+ \int_0^{s_1} |[\nabla E(\Phi(\mu,t_1)) - \nabla E(\Phi(\mu,t_2))]\, \partial_t\Phi(\mu,t_1) - \nabla E(\Phi(\mu,t_2))\, [\partial_t\Phi(\mu,t_1) - \partial_t\Phi(\mu,t_2)]|_\infty\, d\mu \\
&\quad\le \left|\varphi'(t_1) - \varphi'(t_2)\right|_\infty + \int_0^{s_1} |\nabla E(\Phi(\mu,t_1)) - \nabla E(\Phi(\mu,t_2))|_\infty\, |\partial_t\Phi(\mu,t_1)|_\infty\, d\mu \\
&\qquad+ \int_0^{s_1} |\nabla E(\Phi(\mu,t_2))|_\infty |\partial_t\Phi(\mu,t_1) - \partial_t\Phi(\mu,t_2)|_\infty\, d\mu.
\end{aligned}
$$

Grönwalls inequality (Lemma 6.19) is applicable with respect to s_1. We get

$$
\begin{aligned}
&|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_1,t_2)|_\infty \le |\varphi'(t_1) - \varphi'(t_2)|_\infty \exp\left(\int_0^{s_1} |\nabla E(\Phi(\mu,t_2))|_\infty d\mu \right) \\
&+ \int_0^{s_1} |\nabla E(\Phi(\mu,t_1)) - \nabla E(\Phi(\mu,t_2))|_\infty |\partial_t\Phi(\mu,t_1)|_\infty \exp\left(\int_\mu^{s_1} |\nabla E(\Phi(\tau,t_2))|_\infty d\tau \right) d\mu. \quad (6.17)
\end{aligned}
$$

We obtain for (6.15) with (6.16), (6.17) and since $|(s_1,t_1) - (s_2,t_2)|_\infty \ge |t_1 - t_2|$

$$
\begin{aligned}
&\frac{|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha} \\
&\quad\le \frac{|s_2 - s_1|}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha} \left\|\nabla E\right\|_{0,\Omega} \left\|\partial_t\Phi\right\|_{0,Q} + \frac{|\varphi'(t_1) - \varphi'(t_2)|_\infty}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha} \exp\left(\int_0^{s_1} \|\nabla E\|_{0,\Omega} d\mu \right) \\
&\qquad+ \int_0^{s_1} \frac{|\nabla E(\Phi(\mu,t_1)) - \nabla E(\Phi(\mu,t_2))|_\infty}{|t_1 - t_2|^\alpha} |\partial_t\Phi(\mu,t_1)|_\infty \exp\left(\int_\mu^{s_1} \|\nabla E\|_{0,\Omega} d\tau \right) d\mu.
\end{aligned}
$$

By the chain rule (2.9) and since $|(s_1,t_1)-(s_2,t_2)|_\infty \ge |s_1 - s_2|$ follows for the Hölder coefficient

$$
\begin{aligned}
&\frac{|\partial_t\Phi(s_1,t_1) - \partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha} \le \sup_{s_1,s_2 \in [0,L_{\Gamma\_}]} |s_2 - s_1|^{1-\alpha} \left\|\nabla E\right\|_{0,\Omega} \left\|\partial_t\Phi\right\|_{0,Q} \\
&\qquad+ |\varphi'|_{\alpha,I_{\Gamma\_}} \exp\left(\int_0^{s_1} \|\nabla E\|_{0,\Omega} d\mu \right) + \int_0^{s_1} |\nabla E|_{\alpha,\Omega} \left\|\partial_t\Phi\right\|_{0,Q}^{1+\alpha} \exp\left(\int_\mu^{s_1} \|\nabla E\|_{0,\Omega} d\tau \right) d\mu \\
&\quad\le \|l\|_{0,I_{\Gamma\_}}^{1-\alpha} \left\|\nabla E\right\|_{0,\Omega} \left\|\partial_t\Phi\right\|_{0,Q} + |\varphi'|_{\alpha,I_{\Gamma\_}} \exp\left(\|l\|_{0,I_{\Gamma\_}} \|\nabla E\|_{0,\Omega} \right) \\
&\qquad+ \|l\|_{0,I_{\Gamma\_}} |\nabla E|_{\alpha,\Omega} \|\partial_t\Phi\|_{0,Q}^{1+\alpha} \exp\left(\|l\|_{0,I_{\Gamma\_}} \|\nabla E\|_{0,\Omega} \right) \\
&\quad\le \|l\|_{0,I_{\Gamma\_}}^{1-\alpha} \left\|\nabla E\right\|_{0,\Omega} \left\|\partial_t\Phi\right\|_{0,Q} + \left(|\varphi'|_{\alpha,I_{\Gamma\_}} + \|l\|_{0,I_{\Gamma\_}} |\nabla E|_{\alpha,\Omega} \|\partial_t\Phi\|_{0,Q}^{1+\alpha} \right) \exp\left(\|l\|_{0,I_{\Gamma\_}} \|\nabla E\|_{0,\Omega} \right).
\end{aligned}
$$

Case 2: $l(t_2) < l(t_1)$. Then $[0,\, l(t_2)] \subset [0,\, l(t_1)]$ and $(s_2, t_1) \in Q$.

This case is only sketched as it works analogously to case 1.

$$\frac{|\partial_t\Phi(s_1,t_1)-\partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} = \frac{|\partial_t\Phi(s_1,t_1)-\partial_t\Phi(s_2,t_1)+\partial_t\Phi(s_2,t_1)-\partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_0^\alpha}$$
$$\leq \frac{|\partial_t\Phi(s_1,t_1)-\partial_t\Phi(s_2,t_1)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_0^\alpha} + \frac{|\partial_t\Phi(s_2,t_1)-\partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_0^\alpha}. \tag{6.18}$$

We get immediately for the first term of (6.18) for all $s_1, s_2 \in [0, l(t_2)]$

$$|\partial_t\Phi(s_1,t_2)-\partial_t\Phi(s_2,t_2)|_\infty \leq |s_1-s_2|\|\nabla E\|_{0,\Omega}\|\partial\Phi\|_{0,Q}. \tag{6.19}$$

For the second term of (6.18), we have for all $s_2 \in [0, l(t_2)]$

$$\begin{aligned}
&|\partial_t\Phi(s_2,t_1)-\partial_t\Phi(s_2,t_2)|_\infty \\
&\quad\leq |\varphi'(t_1)-\varphi'(t_2)|_\infty + \int_0^{s_2} |\nabla E(\Phi(\mu,t_1))-\nabla E(\Phi(\mu,t_2))|_\infty|\partial_t\Phi(\mu,t_1)|_\infty\,d\mu \\
&\qquad+ \int_0^{s_2} |\nabla E(\Phi(\mu,t_2))|_\infty|\partial_t\Phi(\mu,t_1)-\partial_t\Phi(\mu,t_2)|_\infty d\mu.
\end{aligned}$$

Grönwalls inequality (Lemma 6.19) is applicable with respect to s_2. We get

$$\begin{aligned}
&|\partial_t\Phi(s_2,t_1)-\partial_t\Phi(s_2,t_2)|_\infty \leq |\varphi'(t_1)-\varphi'(t_2)|_\infty \exp\left(\int_0^{s_2}|\nabla E(\Phi(\mu,t_2))|_\infty d\mu\right) \\
&+ \int_0^{s_2} |\nabla E(\Phi(\mu,t_1))-\nabla E(\Phi(\mu,t_2))|_\infty|\partial_t\Phi(\mu,t_1)|_\infty \exp\left(\int_\mu^{s_2}|\nabla E(\Phi(\tau,t_2))|_\infty d\tau\right)d\mu.
\end{aligned} \tag{6.20}$$

After bounding the previous equation, we obtain for (6.18) with (6.19), (6.20) and the chain rule (2.9)

$$\begin{aligned}
&\frac{|\partial_t\Phi(s_1,t_1)-\partial_t\Phi(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\
&\leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\|\nabla E\|_{0,\Omega}\|\partial_t\Phi\|_{0,Q} + \left(|\varphi'|_{\alpha,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\|\partial_t\Phi\|_{0,\Omega}^{1+\alpha}\right)\exp\left(\|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\right).
\end{aligned}$$

As both cases are bounded by the same global constant, we conclude

$$\begin{aligned}
&|\partial_t\Phi|_{\alpha,Q} \\
&\leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\|\nabla E\|_{0,\Omega}\|\Phi_t\|_{0,Q} + \left(|\varphi'|_{\alpha,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\|\partial_t\Phi\|_{0,\Omega}^{1+\alpha}\right)\exp\left(\|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\right) \\
&< \infty.
\end{aligned}$$

□

We have thus proved that $\Phi \in C^{1,\alpha}(Q)$.

6.2.3 Existence of the Inverse Streamline Function

Since Φ is a bijective function, it follows immediately that an inverse function $\Phi^{-1} : \bar{\Omega} \to Q$ exists. We will prove that also $\Phi^{-1} \in C^{1,\alpha}(\bar{\Omega})$. We therefore first define the Jacobian determinant of Φ.

Definition 6.22. *The determinant of the Jacobian matrix $\nabla\Phi(s,t)$ is denoted for $(s,t) \in Q$ by*

$$J(\Phi)(s,t) = \partial_s\Phi_1(s,t)\partial_t\Phi_2(s,t) - \partial_s\Phi_2(s,t)\partial_t\Phi_1(s,t). \tag{6.21}$$

We use a standard result on the differentiability of the inverse function.

Theorem 6.23. *[40, Theorem 171.2] Let Ω be an open domain and let the function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable. Let the Jacobian matrix $\nabla f(x)$ be invertible for every $x \in \Omega$. If f is injective, then the inverse function $f^{-1} : f(\Omega) \to \mathbb{R}^2$ is continuously differentiable.*

To ensure the existence of a differentiable function Φ^{-1}, the Jacobian matrix $\nabla\Phi(s,t)$ must be invertible for all $(s,t) \in Q$. Let $|J(\Phi)(s,t)| > 0$ for all $(s,t) \in Q$. Then the inverse matrix $(\nabla\Phi(s,t))^{-1}$ is given by

$$[\nabla\Phi(s,t)]^{-1} = \frac{1}{J(\Phi)(s,t)} \begin{pmatrix} \partial_t\Phi_2(s,t) & -\partial_t(\Phi_1(s,t)) \\ -[E(\Phi(s,t))]_2 & [E(\Phi(s,t))]_1 \end{pmatrix} \tag{6.22}$$

with $\vec{E}(x) = ([E(x)]_1, [E(x)]_2)$ and $\partial_t\vec{\Phi}(s,t) = (\partial_t\Phi_1(s,t), \partial_t\Phi_2(s,t))$.
It must hold $|J(\Phi)(s,t)| > 0$ for $\nabla\Phi$ to be invertible. We will now analyze if and how we can restrict E such that $J(\Phi)(s,t)$ does not vanish.

Lemma 6.24. *(see [10, p.169]). $J(\Phi)(s,t)$ is given for $(s,t) \in Q$ by*

$$J(\Phi)(s,t) = J(\Phi(0,t)) \exp\left(\int_0^s \operatorname{div} E(\Phi(\tau,t))\, d\tau\right).$$

Proof.

$$\begin{aligned}
\frac{d}{ds}J(\Phi)(s,t) &= \left(\frac{d}{ds}\frac{\partial\Phi_1}{\partial s}\right)\frac{\partial\Phi_2}{\partial t} + \frac{\partial\Phi_1}{\partial s}\left(\frac{d}{ds}\frac{\partial\Phi_2}{\partial t}\right) - \left(\frac{d}{ds}\frac{\partial\Phi_1}{\partial t}\right)\frac{\partial\Phi_2}{\partial s} - \frac{\partial\Phi_1}{\partial t}\left(\frac{d}{ds}\frac{\partial\Phi_2}{\partial s}\right) \\
&= \left(\frac{\partial[E]_1}{\partial\Phi_1}\frac{\partial\Phi_1}{\partial s} + \frac{\partial[E]_1}{\partial\Phi_2}\frac{\partial\Phi_2}{\partial s}\right)\frac{\partial\Phi_2}{\partial t} + \left(\frac{\partial[E]_2}{\partial\Phi_1}\frac{\partial\Phi_1}{\partial t} + \frac{\partial[E]_2}{\partial\Phi_2}\frac{\partial\Phi_2}{\partial t}\right)\frac{\partial\Phi_1}{\partial s} \\
&\quad - \left(\frac{\partial[E]_1}{\partial\Phi_1}\frac{\partial\Phi_1}{\partial t} + \frac{\partial[E]_1}{\partial\Phi_2}\frac{\partial\Phi_2}{\partial t}\right)\frac{\partial\Phi_2}{\partial s} - \left(\frac{\partial[E]_2}{\partial\Phi_1}\frac{\partial\Phi_1}{\partial s} + \frac{\partial[E]_2}{\partial\Phi_2}\frac{\partial\Phi_2}{\partial s}\right)\frac{\partial\Phi_1}{\partial t} \\
&= \frac{\partial[E]_1}{\partial\Phi_1}\left(\frac{\partial\Phi_1}{\partial s}\frac{\partial\Phi_2}{\partial t} - \frac{\partial\Phi_1}{\partial t}\frac{\partial\Phi_2}{\partial s}\right) + \frac{\partial[E]_2}{\partial\Phi_2}\left(\frac{\partial\Phi_1}{\partial s}\frac{\partial\Phi_2}{\partial t} - \frac{\partial\Phi_1}{\partial t}\frac{\partial\Phi_2}{\partial s}\right) \\
&= J(\Phi)\operatorname{div} E(x)|_{x=\Phi(s,t)}.
\end{aligned}$$

Solving the ordinary differential equation gives

$$\log J(\Phi)(s,t) = \int_0^s \operatorname{div} E(\Phi(\mu,t))\, d\mu + \log c(t).$$

With the initial conditions (6.9b), i.e. $\Phi(0,t) = \varphi(t)$, we get

$$J(\Phi)(s,t) = J(\Phi(0,t)) \exp\left(\int_0^s \operatorname{div} E\left(\Phi(\mu,t)\right)\, d\mu\right).$$

□

In fact, $J(\Phi)(0,t)$ is the inflow boundary condition for E. Thus, we can prove the positivity of $J(\Phi)(s,t)$ for all $(s,t) \in Q$ and conclusively the existence of an inverse streamline function for our problem setting.

Lemma 6.25. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $E(x) \cdot \vec{n}(x) \le -\delta_2 < 0$ for $\delta_2 > 0$ and $x \in \Gamma_-$. Then*

$$\inf_{t \in I_{\Gamma_-}} |J(\Phi(0,t))| \ge \delta_2 > 0.$$

Moreover holds $\Phi^{-1} \in C^1(\Omega)$.

Proof. Let $\vec{E}(x) = ([E(x)]_1, [E(x)]_2)$ and $\vec{\varphi}(t) = (\varphi_1(t), \varphi_2(t))$. By (6.21) follows

$$J(\Phi(0,t)) = [E(\varphi(t))]_1 \varphi_2'(t) - [E(\varphi(t))]_2 \varphi_1'(t). \tag{6.23}$$

As φ is the parametrization of the inflow boundary with respect to the arc length, the vector $(-\varphi_2'(t)\,,\, \varphi_1'(t))$ is the outward normal vector to $x = \varphi(t) \in \Gamma_-$.
We get with the assumption $E(x) \cdot \vec{n}(x) \le -\delta_2$

$$\inf_{t \in I_{\Gamma_-}} |J(\Phi)(0,t)| = \inf_{t \in I_{\Gamma_-}} |E(\varphi(t)) \cdot (\varphi'(t))^{\perp}| \ge \delta_2 > 0.$$

The matrix (6.22) is thus invertible for every $(s,t) \in Q$. By Theorem 6.23, we obtain that Φ^{-1} is differentiable with gradient

$$\begin{aligned} \nabla \Phi^{-1}(x) &= \left[\nabla_{(s,t)} \Phi(\Phi^{-1}(x))\right]^{-1} \\ &= \frac{1}{J(\Phi)(\Phi^{-1}(x))} \begin{pmatrix} \partial_t \Phi_2(\Phi^{-1}(x)) & -\partial_t(\Phi_1(\Phi^{-1}(x))) \\ -[E(x)]_2 & [E(x)]_1 \end{pmatrix}. \end{aligned} \tag{6.24}$$

□

We now easily obtain the Hölder continuity of $\nabla \Phi^{-1}$.

Lemma 6.26. *Let Ω be an open bounded $C^{2,\alpha}$ domain, $E = -\nabla u$ be a gradient field and $\inf_{x \in \bar{\Omega}} |E(x)|_\infty \ge \delta_1 > 0$. Let $E \cdot \vec{n} \le -\delta_2 < 0$ for $x \in \Gamma_-$. Then holds $\Phi^{-1} \in C^{1,\alpha}(\Omega)$.*

Proof. Lemma 6.25 yields that $\Phi^{-1} \in C^1(\Omega)$. All entries in the Jacobian matrix (6.24) of Φ^{-1} are first order derivatives of Φ. Using Lemma 2.11, Lemma 6.20 and 6.21, we obtain the boundedness of the α-Hölder coefficient of $\nabla \Phi^{-1}$ and thus the assumption. □

6.2.4 Boundedness of Φ and Φ^{-1}

In this last part of the section, we introduce upper bounds for $\|\Phi\|_{1,\alpha,Q}$ and $\|\Phi^{-1}\|_{1,\alpha,\Omega}$ in terms of E, Ω, δ_1, δ_2 and u_A. Of major interest is the range of $\Phi^{-1}(x)$, $x \in \Omega$, as it gives an insight into the size of the streamline parameter $l(t)$. The main intention of this section becomes evident by mentioning that $\|\Phi\|_{1,\alpha,Q}$ and $\|\Phi^{-1}\|_{1,\alpha,\Omega}$ are factors of the contraction constant of $L \circ T$. A bound in terms of known, bounded data is thus necessary.

Theorem 6.27. *Let Ω be an open bounded C^1 domain and $E \in C^1(\Omega)$ with $E = -\nabla u$ being a gradient field. u satisfies the boundary conditions $u|_{\Gamma_-} = u_{A_1}$ and $u|_{\Gamma_+} = u_{A_2}$ with $u_{A_2} < u_{A_1}$. Further assume $\inf_{x\in\Omega} |E(x)|_\infty \geq \delta_1 > 0$. Then holds*

$$\sup_{0\leq t\leq L_{\Gamma_-}} |l(t)| \leq \frac{u_{A_1} - u_{A_2}}{\delta_1^2} =: c_l.$$

Proof. The vector field E is a gradient field with potential function u. Let Φ be the streamline function to E. Then holds for $t \in [0, L_{\Gamma_-}]$

$$u(\Phi(l(t), t)) - u(\Phi(0, t)) = u_{A_2} - u_{A_1}. \tag{6.25}$$

u is a scalar function. By the mean value theorem for scalar functions exists a $\xi \in [0, l(t)]$ such that

$$u(\Phi(l(t), t)) - u(\Phi(0, t)) = \frac{d}{ds} u(\Phi(s, t))|_{s=\xi}(l(t) - 0). \tag{6.26}$$

We obtain for the derivative by (6.9a)

$$\begin{aligned}\frac{d}{ds} u(\Phi(s, t)) &= \nabla u(\Phi(s, t)) \cdot \frac{d}{ds}\Phi(s, t)\\ &= \nabla u(\Phi(s, t)) \cdot (-\nabla u(\Phi(s, t)))\\ &= -\|E(\Phi(s, t))\|_2^2 \end{aligned}\tag{6.27}$$

where $\|\cdot\|_2$ denotes the euclidean norm. We get for (6.25) with (6.26) and (6.27) and the equivalence of the euclidean and maximum norm

$$u_{A_1} - u_{A_2} = u(\Phi(0, t)) - u(\Phi(l(t), t)) = l(t)\|E(\Phi(\xi, t))\|_2^2 \geq l(t) \inf_{x\in\Omega} |E(x)|_\infty^2 \geq l(t)\delta_1^2$$

and conclusively

$$l(t) \leq \frac{u_{A_1} - u_{A_2}}{\delta_1^2} =: c_l.$$

□

To obtain an upper bound for $l(t)$, we exploited that E is a gradient field. We denote the upper bound of $l(t)$ by c_l as the constant is included in almost every upcoming estimate. We have to keep in mind that c_l depends on the boundary data u_A, in other words the applied potential, and the minimal field strength $\inf_{x\in\Omega} |E(x)|_\infty$. A dependence on the size of the domain Ω is implicitly given by $\inf_{x\in\Omega} |E(x)|_\infty$. For fixed boundary data u_A, c_l will increase with an increasing diameter of Ω, as then $\inf_{x\in\Omega} |E(x)|_\infty$ decreases.

A simple consequence is the upper bound for Φ^{-1}.

Lemma 6.28. *Let Ω be an open bounded C^1 domain and $-\nabla u = E \in C^1(\Omega)$. Further assume $\inf_{x\in\Omega} |E|_\infty \geq \delta_1 > 0$. Then holds*

$$\|\Phi^{-1}\|_{0,\Omega} \leq \max\left\{c_l,\, L_{\Gamma_-}\right\}$$

where $u|_{\Gamma_-} =: u_{A_1}$ and $u|_{\Gamma_-} =: u_{A_2}$

Proof. Since $\Phi : Q \to \Omega$ is a bijective function, Φ^{-1} maps into the parameter set

$$Q = \left\{(s,t) \,:\, 0 \le s \le l(t),\, t \in [0, L_{\Gamma_-}] =: I_{\Gamma_-}\right\}.$$

We have $\Phi_2^{-1}(x) \in [0, L_{\Gamma_-}]$ for $x \in \Omega$ and

$$\left\|\Phi_2^{-1}\right\|_{0,\Omega} \le L_{\Gamma_-}.$$

By Theorem 6.27, we obtain an upper bound for the first component Φ_1^{-1}. We have

$$\|\Phi^{-1}\|_{0,\Omega} \le \max\left\{c_l, L_{\Gamma_-}\right\}.$$

□

We proceed by finding upper bounds for the gradients $\nabla\Phi$ and $\nabla\Phi^{-1}$ where $\nabla\Phi$ is understood as $\nabla_{(s,t)}\Phi(s,t)$ and $\nabla\Phi^{-1}(x)$ as $\nabla_x\Phi^{-1}(x)$.

Lemma 6.29. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $-\nabla u = E \in C^{1,\alpha}(\Omega)$ with $\inf_{x\in\bar\Omega} |E(x)|_\infty \ge \delta_1 > 0$. Then holds*

$$\|\nabla\Phi\|_{0,Q} \le \|E\|_{0,\Omega} + \exp\left(c_l\|\nabla E\|_{0,\Omega}\right).$$

Proof. It is immediately clear by (6.9a) that

$$\|\partial_s\Phi\|_{0,Q} = \|E(\Phi)\|_{0,Q} = \|E\|_{0,\Omega}.$$

For the partial derivative $\partial_t\Phi(s,t)$ holds with the integral representation (6.13)

$$\partial_t\Phi(s,t) = \varphi'(t) + \int_0^s \nabla E(\Phi(\mu,t))\partial_t\Phi(\mu,t)\,d\mu.$$

We estimate pointwise for all $(s,t) \in Q$ and due to the arc length parametrization of φ

$$\begin{aligned}|\partial_t\Phi(s,t)|_\infty &\le \left|\varphi'(t)\right|_\infty + \int_0^s |\nabla E(\Phi(\mu,t))|_\infty \,|\partial_t\Phi(\mu,t)|_\infty\, d\mu\\ &\le 1 + \int_0^s |\nabla E(\Phi(\mu,t))|_\infty \,|\partial_t\Phi(\mu,t)|_\infty\, d\mu.\end{aligned}$$

By Grönwall's inequality (Lemma 6.19), we obtain pointwise for all $(s,t) \in Q$

$$|\partial_t\Phi(s,t)|_\infty \le \exp\left(\int_0^s |\nabla E(\Phi(\mu,t))|_\infty d\mu\right).$$

Applying the supremum over $(s,t) \in Q$ and by Theorem 6.27, we have

$$\begin{aligned}\|\partial_t\Phi\|_{0,Q} &\le \exp\left(\|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\right)\\ &\le \exp\left(c_l\|\nabla E\|_{0,\Omega}\right).\end{aligned}$$

The supremum of the Jacobian matrix is bounded by Lemma 2.12. We get

$$\|\nabla\Phi\|_{0,Q} \le \|\partial_s\Phi\|_{0,Q} + \|\partial_t\Phi\|_{0,Q} \le \|E\|_{0,\Omega} + \exp\left(c_l\|\nabla E\|_{0,\Omega}\right).$$

□

Next, we find a bound for the α Hölder coefficient of $\nabla\Phi$ in terms of E, Ω and δ_1 .

Lemma 6.30. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $-\nabla u = E \in C^{1,\alpha}(\Omega)$ with $\inf_{x\in\bar{\Omega}} |E(x)|_\infty \geq \delta_1 > 0$. Then holds*

$$|\nabla\Phi|_{\alpha,Q} \leq |E|_{\alpha,\Omega}\|\nabla\Phi\|_{0,Q}^{\alpha} + c_l^{1-\alpha}\,\|\nabla E\|_{0,\Omega}\,\|\partial_t\Phi\|_{0,Q} \\ + \left(|\varphi'|_{\alpha,I_{\Gamma_-}} + c_l|\nabla E|_{\alpha,\Omega}\|\partial_t\Phi\|_{0,Q}^{1+\alpha}\right)\exp\left(c_l\|\nabla E\|_{0,\Omega}\right)$$

where $u|_{\Gamma_-} =: u_{A_1}$ and $u|_{\Gamma_-} =: u_{A_2}$ and c_l defined in Theorem 6.27

Proof. The Hölder semi-norm of the Jacobian matrix is bounded by Lemma 2.12. We get

$$|\nabla\Phi|_{\alpha,Q} \leq |\partial_s\Phi|_{\alpha,Q} + |\partial_t\Phi|_{\alpha,Q}. \tag{6.28}$$

The necessary computations to bound $|\nabla\Phi|_{\alpha,Q}$ are done in the proofs of Lemma 6.20 and 6.21. It holds by the chain rule (2.9)

$$|\partial_s\Phi|_{\alpha,Q} = |E(\Phi)|_{\alpha,Q} \leq |E|_{\alpha,\Omega}\|\nabla\Phi\|_{0,Q}^{\alpha}. \tag{6.29}$$

By Lemma 6.21, we also obtain

$$|\partial_t\Phi|_{\alpha,Q} \\ \leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\,\|\nabla E\|_{0,\Omega}\,\|\partial_t\Phi\|_{0,Q} + \left(|\varphi'|_{\alpha,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}|\nabla E|_{\alpha,\Omega}\|\partial_t\Phi\|_{0,Q}^{1+\alpha}\right)\exp\left(\|l\|_{0,I_{\Gamma_-}}\|\nabla E\|_{0,\Omega}\right).$$

Due to Theorem 6.27,

$$|\partial_t\Phi|_{\alpha,Q} \\ \leq c_l^{1-\alpha}\,\|\nabla E\|_{0,\Omega}\,\|\partial_t\Phi\|_{0,Q} + \left(|\varphi'|_{\alpha,I_{\Gamma_-}} + c_l|\nabla E|_{\alpha,\Omega}\|\partial_t\Phi\|_{0,Q}^{1+\alpha}\right)\exp\left(c_l\|\nabla E\|_{0,\Omega}\right). \tag{6.30}$$

The assertion follows by substituting (6.29) and (6.30) into (6.28). □

We will now estimate $\nabla\Phi^{-1}$ in the sup-norm.

Lemma 6.31. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $-\nabla u = E \in C^{1,\alpha}(\bar{\Omega})$ with* $\inf|E(x)|_\infty \geq \delta_1 > 0$. *Further assume for the Jacobian determinant* $\inf|J(\Phi(0,t))| \geq \delta_2 > 0$. *Then holds*

$$\|\nabla\Phi^{-1}\|_{0,\Omega} \leq \frac{2}{\delta_2}\exp\left(c_l\,\|\operatorname{div} E\|_{0,\Omega}\right)\max\left\{\|\partial_t\Phi\|_{0,Q}\,,\,\|\partial_s\Phi\|_{0,Q}\right\}.$$

with c_l defined in Theorem 6.27.

Proof. We obtain by (6.24), Lemma 6.24 and Theorem 6.27

$$\begin{aligned}
\|\nabla\Phi^{-1}\|_{0,\Omega} &= \sup_{(s,t)\in Q} \left|(\nabla\Phi(s,t))^{-1}\right|_\infty \\
&= \sup_{(s,t)\in Q} \left| \frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))} \begin{pmatrix} \partial_t\Phi_2(s,t) & -\partial_t\Phi_1(s,t) \\ -\partial_s\Phi_2(s,t) & \partial_s\Phi_1(s,t) \end{pmatrix} \right|_\infty \\
&\le \frac{\sup_{(s,t)\in Q} \left|\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)\right|_\infty}{\inf_{t\in I_{\Gamma_-}} |J(\Phi(0,t))|} \left\| \begin{pmatrix} \partial_t\Phi_2 & -\partial_t\Phi_1 \\ -\partial_s\Phi_2 & \partial_s\Phi_1 \end{pmatrix} \right\|_{0,Q} \\
&\le \frac{\exp\left(\|l\|_{0,I_{\Gamma_-}} \|\operatorname{div} E(\Phi)\|_{0,Q}\right)}{\inf_t |J(\Phi(0,t))|} \max\left\{2\|\partial_t\Phi\|_{0,Q}\,,\, 2\|\partial_s\Phi\|_{0,Q}\right\} \\
&\le \frac{2}{\delta_2} \exp\left(c_l \|\operatorname{div} E\|_{0,\Omega}\right) \max\left\{\|\partial_t\Phi\|_{0,Q}\,,\, \|\partial_s\Phi\|_{0,Q}\right\}.
\end{aligned}$$

□

Eventually, we find the Hölder coefficient of $\nabla\Phi^{-1}$.

Lemma 6.32. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $-\nabla u = E \in C^{1,\alpha}(\Omega)$. Let $\inf_{x\in\Omega} |E(x)|_\infty \ge \delta_1 > 0$ and $\inf_{t\in I_{\Gamma_-}} |J(\Phi(0,t))| \ge \delta_2 > 0$. Then holds*

$$\begin{aligned}
\|\nabla\Phi^{-1}\|_{\alpha,\Omega} \le C_1(\delta_1,\delta_2,\Omega)\|\nabla\Phi^{-1}\|_{0,\Omega}^\alpha \max\left\{\|\partial_t\Phi\|_{0,Q},\, \|\partial_s\Phi\|_{0,Q}\right\} \\
+ C_2(\delta_1,\delta_2,\Omega)\|\nabla\Phi^{-1}\|_{0,\Omega}^\alpha \max\left\{|\partial_t\Phi|_{\alpha,Q},\, |\partial_s\Phi|_{\alpha,Q}\right\}
\end{aligned}$$

with

$$C_1(c_l,\delta_2,\Omega) = \frac{2}{\delta_2} \exp\left(c_l \|\nabla E\|_{0,\Omega}\right)$$

and

$$\begin{aligned}
C_2(c_l,\delta_2,\Omega) = \frac{1}{\delta_2^2}\left(|E|_{\alpha,\Omega} + \|E\|_{0,\Omega}|\varphi'|_{\alpha,I_{\Gamma_-}}\right) \exp\left(c_l \|\nabla E\|_{0,\Omega}\right) \\
+ \frac{1}{\delta_2} \exp\left(c_l \|\operatorname{div} E\|_{0,\Omega}\right) \left(c_l |\nabla E|_{\alpha,\Omega} |\partial_t\Phi|_\infty^\alpha + c_l^{1-\alpha} \|\nabla E\|_{0,\Omega}\right)
\end{aligned}$$

with c_l defined in Theorem 6.27.

Proof. It holds by (6.24) and Lemma 6.24

$$\nabla\Phi^{-1}(x) = \frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))} \left.\underbrace{\begin{pmatrix} \partial_t\Phi_2(s,t) & -\partial_t\Phi_1(s,t) \\ -E_2(\Phi(s,t)) & E_1(\Phi(s,t)) \end{pmatrix}}_{=:A(s,t)}\right|_{(s,t)=\Phi^{-1}(x)}.$$

By the product and chain rule of Lemma 2.11

$$\begin{aligned}\left|\nabla\Phi^{-1}\right|_{\alpha,\Omega} &= \left|\left[\nabla_{(s,t)}\Phi(\Phi^{-1}(x))\right]^{-1}\right|_{\alpha,\Omega} \\ &\leq \|\nabla\Phi^{-1}\|_{0,\Omega}^{\alpha} \sup_{(s,t)\in Q}\left|\frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))}\right| |A|_{\alpha,Q} \\ &+ \|\nabla\Phi^{-1}\|_{0,\Omega}^{\alpha}\left|\frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))}\right|_{\alpha,Q} \|A\|_{0,Q}. \end{aligned} \tag{6.31}$$

The matrix A is bounded in terms of $\partial_s\Phi$ and $\partial_t\Phi$ for both the sup-norm and the Hölder semi norm

$$\|A\|_{0,Q} \leq 2\max\left\{\|\partial_t\Phi\|_{0,Q},\ \|E\|_{0,\Omega}\right\} \tag{6.32}$$

$$|A|_{\alpha,Q} \leq 2\max\left\{|\partial_t\Phi|_{\alpha,Q},\ |E|_{\alpha,\Omega}\right\}. \tag{6.33}$$

The sup-norm of $1/\det(\nabla\Phi(s,t))$ has already been bounded in Lemma 6.31 as

$$\left\|\frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))}\right\|_{0,Q} \leq \frac{1}{\delta_2}\exp\left(c_l\,\|\nabla E\|_{0,\Omega}\right). \tag{6.34}$$

The last step is to bound the α-Hölder norm of the Jacobian determinant.

$$\begin{aligned}&\left|\frac{\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)}{J(\Phi(0,t))}\right|_{\alpha,\Omega} \\ &\leq \frac{\left|\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)\right|_{\alpha,Q}}{\|J(\Phi(0,t))\|_{0,I_{\Gamma_-}}} + \frac{\left\|\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)\right\|_{0,Q}}{|J(\Phi(0,t))|_{\alpha,I_{\Gamma_-}}}.\end{aligned}$$

We begin with $J(\Phi(0,t))$. By the assumption $\inf_{0\leq t\leq L_{\Gamma_-}}|J(\Phi(0,t))|\geq\delta_2$, (6.23) and (2.9), we get

$$\begin{aligned}\left|\frac{1}{J(\Phi(0,t))}\right|_{\alpha,I_{\Gamma_-}} &\leq \sup_{t_1,t_2\in I_{\Gamma_-}} \frac{\left|\frac{1}{J(\Phi(0,t_1))}-\frac{1}{J(\Phi(0,t_2))}\right|_{\infty}}{|t_1-t_2|^{\alpha}} \\ &\leq \sup_{t_1,t_2\in I_{\Gamma_-}} \frac{|J(\Phi(0,t_2))-J(\Phi(0,t_1))|_{\infty}}{|J(\Phi(0,t_2))J(\Phi(0,t_1))|_{\infty}|t_1-t_2|^{\alpha}} \\ &\leq \frac{1}{\delta_2^2}\sup_{t_1,t_2\in I_{\Gamma_-}} \frac{|J(\Phi(0,t_2))-J(\Phi(0,t_1))|_{\infty}}{|t_1-t_2|^{\alpha}} \\ &\leq \frac{1}{\delta_2^2}\left(|E|_{\alpha,\Omega}\|\varphi'\|_{0,I_{\Gamma_-}}^{1+\alpha} + \|E\|_{0,\Omega}|\varphi'|_{\alpha,I_{\Gamma_-}}\right) \\ &\leq \frac{1}{\delta_2^2}\left(|E|_{\alpha,\Omega} + \|E\|_{0,\Omega}|\varphi'|_{\alpha,I_{\Gamma_-}}\right). \end{aligned} \tag{6.35}$$

The last inequality followed due to the arc length parametrization of φ.

To bound the exponential function, define $p(s,t) = -\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu$. It holds that $p \in C^\alpha(Q)$ and by the chain rule (2.9)

$$\begin{aligned} |\exp(p)|_{\alpha,Q} &= \sup_{(s_1,t_1)\neq(s_2,t_2)\in Q} \frac{|\exp(p(s_1,t_1)) - \exp(p(s_2,t_2)|}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\ &\leq \|\exp(p)\|_{0,Q}\,|p|_{\alpha,Q}. \end{aligned} \tag{6.36}$$

Since the exponential function is monotone increasing, it holds

$$|\exp(p)| \leq \exp(\|p\|_{0,Q}).$$

To bound $|p|_{\alpha,Q}$, we need to distinguish two cases as in Lemma 6.21.

Case 1: (s_1,t_2), $(s_2,t_2) \in Q$ and $l(t_1) < l(t_2)$. Then $(s_1,t_2) \in Q$.
We obtain

$$\frac{|p(s_1,t_1)-p(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \leq \frac{|p(s_1,t_1)-p(s_1,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} + \frac{|p(s_1,t_2)-p(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}. \tag{6.37}$$

We get for the first term of (6.37) by the chain rule (2.9) for all $(s_1,t_1) \neq (s_2,t_2) \in Q$

$$\begin{aligned} \frac{|p(s_1,t_1)-p(s_1,t_2)|}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} &\leq \frac{1}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}\int_0^{s_1} |\operatorname{div} E(\Phi(\mu,t_1)) - \operatorname{div} E(\Phi(\mu,t_2))|_\infty \,d\mu \\ &\leq \int_0^{s_1} \frac{|\operatorname{div} E(\Phi(\mu,t_1)) - \operatorname{div} E(\Phi(\mu,t_2))|_\infty}{|t_1-t_2|^\alpha}\,d\mu \\ &\leq \int_0^{s_1} |\operatorname{div} E|_{\alpha,\Omega}\,|\partial_t\Phi|_{0,Q}^\alpha\,d\mu \\ &\leq \|l\|_{0,I_{\Gamma_-}}\,|\nabla E|_{\alpha,\Omega}\,\|\partial_t\Phi\|_{0,Q}^\alpha. \end{aligned} \tag{6.38}$$

We obtain easily for the second term of (6.37)

$$\begin{aligned} \frac{|p(s_1,t_2)-p(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} &= \frac{1}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}\left|\int_{s_2}^{s_1} \operatorname{div} E(\Phi(\mu,t))\,d\mu\right|_\infty \\ &\leq \frac{|s_2-s_1|}{|s_1-s_2|^\alpha}\,\|\operatorname{div} E\|_{0,\Omega} \\ &\leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\,\|\nabla E\|_{0,\Omega}. \end{aligned} \tag{6.39}$$

Case 2: (s_1,t_2), $(s_2,t_2) \in Q$ and $l(t_1) < l(t_2)$. Then $(s_1,t_2) \in Q$.
The computations are done analogously and we obtain the same result

$$\frac{|p(s_1,t_1)-p(s_2,t_2)|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \leq \|l\|_{0,I_{\Gamma_-}}\,|\nabla E|_{\alpha,\Omega}\,\|\partial_t\Phi\|_{0,Q}^\alpha + \|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\,\|\nabla E\|_{0,\Omega}. \tag{6.40}$$

Eventually, we obtain for (6.36) with (6.37)-(6.40) and by Theorem 6.27

$$\begin{aligned}
&\left|\exp\left(-\int_0^s \operatorname{div} E(\Phi(\mu,t))\,d\mu\right)\right|_{\alpha,Q} \\
&\le \exp\left(\|l\|_{I_{\Gamma_-},0}\|\operatorname{div} E\|_{0,\Omega}\right)\left(\|l\|_{0,I_{\Gamma_-}}|\nabla E|_{\alpha,\Omega}\|\partial_t\Phi\|_{0,Q}^{\alpha}+\|l\|_{0,I_{\Gamma_-}}^{1-\alpha}\|\nabla E\|_{0,\Omega}\right) \\
&\le \exp\left(c_l\|\operatorname{div} E\|_{0,\Omega}\right)\left(c_l|\nabla E|_{\alpha,\Omega}\|\partial_t\Phi\|_{0,Q}^{\alpha}+c_l^{1-\alpha}\|\nabla E\|_{0,\Omega}\right).
\end{aligned} \tag{6.41}$$

The assertion follows by substituting (6.32)-(6.34) and (6.41) into (6.31). □

The inverse function is defined only on the open domain Ω due to Theorem 6.23. However, since we have proved the boundedness of $\|\Phi^{-1}\|_{1,\alpha,\Omega}$, we can extend Φ^{-1} up to the boundary.

Lemma 6.33. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $-\nabla u = E \in C^{1,\alpha}(\bar{\Omega})$. Further let $\|E\|_{1,\alpha,\Omega} \le M$, $\inf_{x\in\bar{\Omega}}|E(x)|_\infty \ge \delta_1 > 0$ and $\inf_{t\in I_{\Gamma_-}}|J(\Phi(0,t))| \ge \delta_2 > 0$. Then the inverse streamline function $\nabla\Phi^{-1}: \Omega \to \mathbb{R}^2$ is continuously extendable up to the boundary, i.e.*

$$\Phi^{-1} \in C^{1,\alpha}(\bar{\Omega}).$$

Proof. By Lemma 2.20 and Lemma 6.31 holds

$$|\Phi^{-1}(x)-\Phi^{-1}(y)|_\infty \le c_{mv}|\nabla\Phi^{-1}|_{0,\Omega}|x-y|_\infty$$

$\nabla\Phi^{-1}$ is bounded with Lemma 6.31 and 6.29 by given constants

$$|\Phi^{-1}(x)-\Phi^{-1}(y)|_\infty \le c_{mv}c(M,\delta_2,\Omega,c_l)|x-y|_\infty.$$

Φ^{-1} is uniformly continuous and thus by Lemma 2.13 extendable up to the boundary.
We use the same argumentation for $\nabla\Phi^{-1}$. The Hölder coefficient is bounded with Lemma 6.32 and Lemma 6.29 by given constants. Hence,

$$|\nabla\Phi^{-1}|_{\alpha,\Omega} \le c(M,\delta_2,\Omega,c_l)|x-y|^\alpha.$$

By Lemma 2.13 follows that $\nabla\Phi^{-1} \in C^\alpha(\bar{\Omega})$ with Hölder constant $c(M,\delta_2,\Omega,c_l)$. □

6.3 Transport Solution Operator

With the knowledge on the streamline function, we now proceed and investigate the Dirichlet problem for the transport equation

Problem (Tr 6.3). *Let Ω be an open bounded $C^{2,\alpha}$ domain. For a given vector field $E \in C^{1,\alpha}(\bar{\Omega})$, find $\rho \in C^{1,\alpha}(\bar{\Omega})$, such that*

$$\rho^2 + E\cdot\nabla\rho = 0 \qquad x \in \Omega \tag{6.42a}$$

$$\rho = \rho_A \qquad x \in \Gamma_- \tag{6.42b}$$

with $\rho_A \in C^{1,\alpha}(\Gamma_-)$ and $\rho_A \ge 0$.

By means of the streamline function Φ, we are able to find the solution $\rho(\Phi(s,t))$ of (6.42a)-(6.42b) on a streamline. This representation is used to define the solution operator T for (Tr 6.3). The goal in this section is to prove the continuity of the operator T in C^α with respect to E. We will encounter several difficulties caused by the variability of the domain of definition Q for the streamline functions. Let us begin with determining the solution ρ on the streamlines. For simplicity, we denote the transformed boundary condition on I_{Γ_-} by $\rho_A(t) := \rho_A(\varphi(t))$. Whenever ρ_A is used, the argument indicates the domain of definition of ρ_A.

Theorem 6.34. *Let Ω be a $C^{2,\alpha}$ domain, $E \in C^{1,\alpha}(\bar{\Omega})$ and $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$. Then the solution to (6.42a)-(6.42b) is given on a streamline by*

$$\rho(\Phi(s,t)) = \hat{\rho}(s,t) := \frac{\rho_A(t)}{1 + s \cdot \rho_A(t)} \tag{6.43}$$

for $(s,t) \in Q$ and $\rho_A(t) := \rho_A(\varphi(t))$. It holds that $\hat{\rho} \in C^{1,\alpha}(\mathbb{R}_+ \times I_{\Gamma_-})$.

Proof. With $x = \Phi(s,t)$, we get for (6.42a) by (6.9a) and the chain rule

$$\begin{aligned} \rho^2(\Phi(s,t)) + E(\Phi(s,t)) \cdot \nabla\rho(\Phi(s,t)) &= \rho^2(\Phi(s,t)) + \frac{d}{ds}\Phi(s,t) \cdot \nabla\rho(\Phi(s,t)) \\ &= \rho^2(\Phi(s,t)) + \frac{d}{ds}\rho(\Phi(s,t)) = 0. \end{aligned} \tag{6.44}$$

(6.44) is an ordinary differential equation. Separation of variables leads to

$$-\int 1\, ds = \int \frac{d\rho(\Phi(s,t))}{\rho^2(\Phi(s,t))}$$

and by integration, we have

$$-s = -\frac{1}{\rho(\Phi(s,t))} + c(t).$$

Using the boundary conditions (6.9b) for Φ and (6.42b) for ρ, we obtain

$$\rho(\Phi(0,t)) = \rho_A(t) = -\frac{1}{c(t)}.$$

It follows

$$\rho(\Phi(s,t)) = \frac{\rho_A(t)}{1 + s\rho_A(t)} =: \hat{\rho}(s,t).$$

$\hat{\rho}$ is defined for every $s \in \mathbb{R}_+$. Since $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$, it holds due to the chain rule that $\hat{\rho} \in C^{1,\alpha}(\mathbb{R}_+ \times I_{\Gamma_-})$. $\square$

To formulate the solution operator T, we first introduce the set of vector fields for which we obtain feasible streamlines. We therefore define the following set W depending on constants $M, \delta_1, \delta_2, \delta_3 > 0$.

Definition 6.35. *Let Ω be a $C^{2,\alpha}$ domain and $M, \delta_1, \delta_2, \delta_3 > 0$ constants. Then we define the set $W(M, \delta_1, \delta_2, \delta_3) \subset C^{1,\alpha}(\Omega)$ by*

$$\begin{aligned} W(M, \delta_1, \delta_2, \delta_3) = \big\{ & E = E_0 + E_1 \in C^{1,\alpha}(\Omega) \,:\, E_0, E_1 \in C^{1,\alpha}(\Omega), -\nabla u = E \text{ gradient field}, \\ & u|_{\Gamma_-} = u_{A_1}, u|_{\Gamma_+} = u_{A_2}, \|E\|_{1,\alpha;\Omega} \leq M, \inf_{x \in \bar{\Omega}} |E(x)|_\infty \geq \delta_1, \\ & \|E_1\|_{0,\Omega} \leq \delta_3, \vec{n} \cdot E \leq -\delta_2 < 0 \text{ on } \Gamma_-, \vec{n} \cdot E \geq \delta_2 > 0 \text{ on } \Gamma_+ \} \end{aligned} \tag{6.45}$$

with $2\delta_1 = \inf_{x \in \bar{\Omega}} |E_0(x)|_\infty$ and $E_0 = -\nabla u_0$ given as solution of the Laplace equation

$$\begin{aligned} -\Delta u_0 &= 0 & x &\in \Omega \\ u_0 &= u_A & x &\in \Gamma. \end{aligned}$$

The set $W(M, \delta_1, \delta_2, \delta_3)$ contains all restrictions on E that have been shown in section 6.2 to be necessary to obtain feasible streamlines. We herein substitute $|J(\Phi)(0,t)| \geq \delta_2$ by the equivalent restriction on the direction of the flow field $\vec{n} \cdot E \leq -\delta_2$ on Γ_-. Also, we claim $\vec{n} \cdot E \geq \delta_2$ on Γ_+ which ensures that for all $E \in W(M, \delta_1, \delta_2, \delta_3)$, Γ_- is the inflow and Γ_+ is the outflow boundary. The constants M and δ_3 depend on boundary data and the geometry of Ω as we will understand later on. In section 6.4, we will make an explicit choice for M and δ_3 to show that the composite operator $L \circ T$ is a selfmap on the set $W(M, \delta_1, \delta_2, \delta_3)$.

Lemma 6.36. *Let $M \geq c_S(\Omega, \alpha)\|u_A\|_{2,\alpha;\Gamma}$, $0 \leq \delta_2 \leq 2\delta_1$ and $\delta_3 \geq 0$. Then the set $W(M, \delta_1, \delta_2, \delta_3)$ is nonempty.*

Proof. Set $E = E_0$ with $E_0 = -\nabla u_0$ being the solution of the Laplace equation with boundary data $u|_{\Gamma_-} = u_{A_1}, u|_{\Gamma_+} = u_{A_2}$. The boundary conditions u_{A_1} and u_{A_2} are constant and form equipotential curves. Thus $E_0(x)$ with $x \in \Gamma$ is perpendicular to Γ and points into the direction or exactly opposite to the normal vector $\vec{n}$. Hence holds $E_0(x) = -c_1(x)\vec{n}(x)$ for $x \in \Gamma_-$ and $c_1 > 0$ and $E_0(x) = c_2(x)\vec{n}(x)$ for $x \in \Gamma_+$ and $c_2 > 0$. We have with the equivalence of the euclidean and the maximums norm

$$c_1(x) = \|c_1(x)\vec{n}(x)\|_2 = \|E_0(x)\|_2 \geq |E_0(x)|_\infty \geq 2\delta_1.$$

We obtain for $x \in \Gamma_-$

$$\vec{n}(x)E_0(x) = -\vec{n}(x) \cdot \vec{n}(x)c_1 = -c_1 \leq -2\delta_1.$$

Analogously we obtain for $x \in \Gamma_+$

$$\vec{n}(x)E_0(x) = \vec{n}(x) \cdot \vec{n}(x)c_2(x) = c_2(x) \geq 2\delta_1.$$

Further, we have with Schauder's estimates

$$\|E_0\|_{1,\alpha;\Omega} \leq c_S(\Omega, \alpha)\|u_A\|_{2,\alpha;\Gamma} \tag{6.46}$$

Choose $\delta_2 := 2\delta_1$ and $M = c_S(\Omega, \alpha)\|u_A\|_{2,\alpha;\Gamma}$. Then $E \in W(M, \delta_1, \delta_2, \delta_3)$ and $W(M, \delta_1, \delta_2, \delta_3)$ is nonempty. $\square$

Further, the set $W(M, \delta_1, \delta_2, \delta_3)$ is also convex.

Lemma 6.37. *Choose $\delta_3 \le \delta_1$. Then the set $W(M, \delta_1, \delta_2, \delta_3)$ is convex.*

Proof. We pick two arbitrary elements E and $\tilde{E}$ of $W(M, \delta_1, \delta_2, \delta_3)$ and show that $\hat{E} = t\tilde{E} + (1-t)E \in W(M, \delta_1, \delta_2, \delta_3)$. As $E = -\nabla u$ and $\tilde{E} = -\nabla \tilde{u}$, it holds that $\hat{E}$ is a gradient field with u satisfying the boundary conditions.

$$\|\hat{E}\|_{1,\alpha,\Omega} = \|t\tilde{E} + (1-t)E\|_{1,\alpha,\Omega} \le \|t\tilde{E}\|_{1,\alpha,\Omega} + \|(1-t)E\|_{1,\alpha,\Omega} \le tM + (1-t)M \le M.$$

As the normal vector n depends only on the boundary, it holds for the inflow boundary condition

$$\vec{n} \cdot \hat{E} = \vec{n} \cdot (t\tilde{E} + (1-t)E) = t\,\vec{n} \cdot \tilde{E} + (1-t)\vec{n} \cdot E \le t\,\delta_2 + (1-t)\delta_2 \le \delta_2.$$

Analogously,

$$\vec{n} \cdot \hat{E} = \vec{n} \cdot (t\tilde{E} + (1-t)E) = t\,\vec{n} \cdot \tilde{E} + (1-t)\vec{n} \cdot E > \delta_2.$$

Last, we show that the minimum of $\hat{E}$ remains greater than δ_1. Choose $\delta_3 := \delta_1$.

$$\begin{aligned}
\|\hat{E}\|_0 &= \|t\tilde{E} + (1-t)E\|_0 \\
&= \|t(E_0 + \tilde{E}_1) + (1-t)(E_0 + E_1)\|_0 \\
&\ge \|E_0\| - \|t\tilde{E}_1 + (1-t)E_1\|_0 \\
&\ge \inf \|E_0\|_0 - (t\|\tilde{E}_1\|_0 + (1-t)\|E_1\|_0) \\
&\ge 2\delta_1 - \delta_1 = \delta_1.
\end{aligned}$$

□

Before defining the solution operator T, we first denote Φ as operator applied to the vector field E.

Definition 6.38. *Let Ω be a $C^{2,\alpha}$ domain. Then the streamline operator Φ with argument $E \in W(M, \delta_1, \delta_2, \delta_3)$ is denoted*

$$\Phi(E, s, t) = \varphi(t) + \int_0^s E(\Phi(E, \mu, t))\,d\mu. \tag{6.47}$$

With this definition and Theorem 6.34, we immediately find the solution operator for (6.42a)-(6.42b).

Theorem 6.39. *Let Ω be an open bounded $C^{2,\alpha}$-domain and $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$. Let further $E \in W(M, \delta_1, \delta_2, \delta_3)$. Then the solution operator T to (Tr 6.3) is given by*

$$TE(x) = \hat{\rho}(\Phi^{-1}(E, x)). \tag{6.48}$$

It holds that $TE \in C^{1,\alpha}(\bar{\Omega})$.

Proof. First, we show that $TE \in C^0(\bar{\Omega})$. Second, we prove that $\nabla TE \in C^\alpha(\bar{\Omega})$.
Since $E \in W(M, \delta_1, \delta_2, \delta_3)$ and by section 6.2, the streamline function $\Phi(E, s, t)$ exists uniquely and has an inverse function $\Phi^{-1}(E) \in C^{1,\alpha}(\bar{\Omega})$. Theorem 6.34 then yields (6.48). Since $\Phi^{-1} \in C^{1,\alpha}(\bar{\Omega})$ and $\hat{\rho} \in C^\alpha(\mathbb{R} \times I_{\Gamma_-})$, TE is continuous up to the boundary. It follows $TE \in C^0(\bar{\Omega})$.

With $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$ follows $\hat{\rho} \in C^{1,\alpha}(Q)$. Moreover, as $E \in C^{1,\alpha}(\bar{\Omega})$, we know due to Lemma 6.26 that $\nabla\Phi^{-1} \in C^{0,\alpha}(\bar{\Omega})$. We show that the gradient ∇TE is bounded. It holds by Theorem 6.34

$$\nabla TE(x) = \nabla_x \hat{\rho}(\Phi^{-1}(x)) = \nabla_{(s,t)}\hat{\rho}(\Phi^{-1}(x)) \cdot \nabla\Phi^{-1}(x).$$

Hence,

$$\|\nabla TE\|_{0,\Omega} \leq \|\nabla_{(s,t)}\hat{\rho}(\Phi^{-1}) \cdot \nabla\Phi^{-1}\|_{0,\Omega} \leq \|\nabla_{(s,t)}\hat{\rho}\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega} < \infty.$$

We get for the Hölder coefficient

$$\begin{aligned} |TE|_{\alpha,\Omega} &\leq |\nabla\hat{\rho}(\Phi^{-1}) \cdot \nabla\Phi^{-1}|_{\alpha,\Omega} \\ &\leq \|\nabla\hat{\rho}\|_{0,Q}|\nabla\Phi^{-1}|_{\alpha,\Omega} + |\nabla\hat{\rho}|_{\alpha,Q}\|\nabla\Phi^{-1}\|_{0,\Omega}^{1+\alpha} < \infty. \end{aligned}$$

Conclusively, we obtain $\nabla TE \in C^\alpha(\bar{\Omega})$. $\square$

By Theorem 6.1, it suffices to have $TE \in C^\alpha(\bar{\Omega})$ to obtain a new iterate $L \circ TE \in C^{2,\alpha}(\bar{\Omega})$. The previous Theorem shows that in the fixed point iterations, TE is even of $C^{1,\alpha}(\bar{\Omega})$ regularity assuming that $E \in C^{1,\alpha}(\bar{\Omega})$. In fact, to show that $L \circ T$ is a contraction and most importantly for TE to be a classical solution to (CP 6.1), TE has to be a $C^{1,\alpha}(\bar{\Omega})$ function.

Recall that we want to prove the existence of a fixed point to the operator $L \circ T$ by the Banach fixed point theorem. The essential steps are to show that $L \circ T$ is a selfmap and a contraction on the set $W(M, \delta_1, \delta_2, \delta_3)$. The continuity of L with respect to TE is immediately given due to the Schauder estimates (Theorem 6.7) while the continuity of the transport solution operator is not obvious. We now begin to show the continuity of the operator T with respect to E. We will need several steps before obtaining the final result. Let us simplify the notations and use

$$\begin{aligned} \Phi(E, s, t) &= \Phi(s, t), & \Phi^{-1}(E, x) &= \Phi^{-1}(x) \\ \Phi(\tilde{E}, s, t) &= \tilde{\Phi}(s, t), & \Phi^{-1}(\tilde{E}, x) &= \tilde{\Phi}^{-1}(x) \end{aligned}$$

In the following Q is always understood as the parameter set of Φ and $\tilde{Q}$ as the parameter set of $\tilde{\Phi}$.

Lemma 6.40. *Let Ω be a $C^{2,\alpha}$ domain. Let $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ and $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$. Then holds for the transport solution operator*

$$\left\|TE - T\tilde{E}\right\|_{\alpha,\Omega} \leq \|\nabla\hat{\rho}\|_{\alpha,Q\cup\tilde{Q}} \left(1 + (c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega})^\alpha\right) \left\|\tilde{\Phi}^{-1} - \Phi^{-1}\right\|_{\alpha,\Omega}$$

with

$$Q \cup \tilde{Q} = \left\{(s,t) : s \in [0, \max\left\{l(t), \tilde{l}(t)\right\}],\ t \in I_{\Gamma_-}\right\}.$$

Proof. The Hölder norm splits up into

$$\begin{aligned}\left\|TE - T\tilde{E}\right\|_{\alpha,\Omega} &= \left\|\hat{\rho}(\Phi^{-1}) - \hat{\rho}(\tilde{\Phi}^{-1})\right\|_{\alpha,\Omega} \\ &= \left\|\hat{\rho}(\Phi^{-1}) - \hat{\rho}(\tilde{\Phi}^{-1})\right\|_{0,\Omega} + \left|\hat{\rho}(\Phi^{-1}) - \hat{\rho}(\tilde{\Phi}^{-1})\right|_{\alpha,\Omega}. \end{aligned} \tag{6.49}$$

We begin with a pointwise estimate for the first term of (6.49) for all $x \in \Omega$. As the inflow boundary data $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$, it holds also $\hat{\rho} \in C^{1,\alpha}(Q)$. Moreover, $\hat{\rho}$ is defined on $\mathbb{R}^+ \times [0, L_{\Gamma_-}]$ which is a convex set. We therefore may apply the Mean Value Theorem 2.15.

$$\left|\hat{\rho}(\Phi^{-1}(x)) - \hat{\rho}(\tilde{\Phi}^{-1}(x))\right|_\infty \le \|\nabla\hat{\rho}\|_{0,Q\cup\tilde{Q}} \left|\Phi^{-1}(x) - \tilde{\Phi}^{-1}(x)\right|_\infty. \tag{6.50}$$

We proceed to evaluate the second term of (6.49)

$$\left|\hat{\rho}(\Phi^{-1}) - \hat{\rho}(\tilde{\Phi}^{-1})\right|_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{\left|\hat{\rho}(\Phi^{-1}(x)) - \hat{\rho}(\tilde{\Phi}^{-1}(x)) - \hat{\rho}(\Phi^{-1}(y)) + \hat{\rho}(\tilde{\Phi}^{-1}(y))\right|_\infty}{|x-y|_\infty^\alpha}. \tag{6.51}$$

Set $z_1(\tau) = \Phi^{-1}(x) + \tau(\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x))$ and $z_2(\tau) = \Phi^{-1}(y) + \tau(\tilde{\Phi}^{-1}(y) - \Phi^{-1}(y))$ with $\tau \in [0,1]$. Both line segments $z_1(\tau)$ and $z_2(\tau)$ are contained in $\mathbb{R}_+ \times I_{\Gamma_-}$, i.e. in the domain of definition of $\hat{\rho}$. It holds with Theorem 2.15

$$\begin{aligned}&\left|\hat{\rho}(\Phi^{-1}(x)) - \hat{\rho}(\tilde{\Phi}^{-1}(x)) - \hat{\rho}(\Phi^{-1}(y)) + \hat{\rho}(\tilde{\Phi}^{-1}(y))\right|_\infty \\ &= \left|\int_0^1 \nabla\hat{\rho}(z_1(\tau))\,d\tau \left[\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x)\right] - \int_0^1 \nabla\hat{\rho}(z_2(\tau))\,d\tau \left[\tilde{\Phi}^{-1}(y) - \Phi^{-1}(y)\right]\right|_\infty \\ &\le \left|\int_0^1 \left[\nabla\hat{\rho}(z_1(\tau)) - \nabla\hat{\rho}(z_2(\tau))\right] d\tau \left[\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x)\right]\right|_\infty \\ &\quad + \left|\int_0^1 \nabla\hat{\rho}(z_2(\tau))\,d\tau \left[\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x) - \tilde{\Phi}^{-1}(y) + \Phi^{-1}(y)\right]\right|_\infty \\ &\le \int_0^1 \left|\nabla\hat{\rho}(z_1(\tau)) - \nabla\hat{\rho}(z_2(\tau))\right|_\infty d\tau \left|\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x)\right|_\infty \qquad (6.52) \\ &\quad + \int_0^1 \left|\nabla\hat{\rho}(z_2(\tau))\right|_\infty d\tau \left|\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x) - \tilde{\Phi}^{-1}(y) + \Phi^{-1}(y)\right|_\infty. \qquad (6.53)\end{aligned}$$

As $\hat{\rho} \in C^{1,\alpha}(Q \cup \tilde{Q})$ we obtain by Lemma 2.16 and Lemma 2.20 for the difference in terms of $\nabla\hat{\rho}$ in (6.52)

$$\int_0^1 \left|\nabla\hat{\rho}(z_1(\tau)) - \nabla\hat{\rho}(z_2(\tau))\right|_\infty d\tau \le |\nabla\hat{\rho}|_{\alpha,Q\cup\tilde{Q}} \left(c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega}\right)^\alpha |x-y|_\infty^\alpha. \tag{6.54}$$

It then follows for the α-Hölder coefficient (6.51) by (6.52), (6.53) and (6.54)

$$\begin{aligned}|\hat{\rho}(\Phi^{-1}) - \hat{\rho}(\tilde{\Phi}^{-1})|_{\alpha,\Omega} \le{}& |\nabla\hat{\rho}|_{\alpha,Q\cup\tilde{Q}} \left(c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega}\right)^\alpha \|\tilde{\Phi}^{-1} - \Phi^{-1}\|_{0,\Omega} \\ &+ \|\nabla\hat{\rho}\|_{0,Q\cup\tilde{Q}} |\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega}.\end{aligned} \tag{6.55}$$

Hence, it holds for (6.49) by (6.50) and (6.55)

$$\begin{aligned}\|TE - T\tilde{E}\|_{\alpha,\Omega} &\le \|\nabla\hat{\rho}\|_{0,Q\cup\tilde{Q}}\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{0,\Omega} + \|\nabla\hat{\rho}\|_{0,Q\cup\tilde{Q}}|\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega} \\ &\quad + |\nabla\hat{\rho}|_{\alpha,Q\cup\tilde{Q}}(c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega})^{\alpha}\|\tilde{\Phi}^{-1} - \Phi^{-1}\|_{0,\Omega} \\ &\le \|\nabla\hat{\rho}\|_{\alpha,Q\cup\tilde{Q}}\left(1 + (c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega})^{\alpha}\right)\|\tilde{\Phi}^{-1} - \Phi^{-1}\|_{\alpha,\Omega}.\end{aligned}$$

□

So far, we have found a bound to $\|TE - T\tilde{E}\|_{\alpha,\Omega}$ in terms of the difference of two inverse streamline functions $\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{\alpha,\Omega}$. One factor of the preceding constant contains $\|\nabla\Phi^{-1}\|_{0,\Omega}$ which has been bounded in section 6.2.4 by a number depending on the geometry of the domain Ω, the boundary data u_A and the constants $M, \delta_1, \delta_2, \delta_3$ defining the set W. The second factor is $\|\nabla\hat{\rho}\|_{\alpha,Q\cup\tilde{Q}}$. In the following Lemmas, we will bound $\|\nabla\hat{\rho}\|_{\alpha,Q\cup\tilde{Q}}$ by the inflow boundary data $\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$. Later on, the method will be to choose $\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$ small enough such that the continuity constant of T diminishes in order to obtain a contraction and selfmap T on the set $W(M, \delta_1, \delta_2, \delta_3)$.

Lemma 6.41. *Let $\rho_A \in C^1(I_{\Gamma_-})$ and $\|\rho_A\|_{0,I_{\Gamma_-}} < 1$. Then we obtain for $\hat{\rho}$ defined in Theorem 6.34*

$$\|\nabla\hat{\rho}\|_{0,Q} \le \|\rho_A\|_{1,I_{\Gamma_-}}.$$

Proof. Since $s \in [0,\, l(t)]$ with $t \in I_{\Gamma_-}$ and $1 > \rho_A(t) > 0$, we get pointwise

$$\begin{aligned}|\nabla\hat{\rho}(s,t)|_{\infty} &= \left|\frac{1}{(1 + s\rho_A(t))^2}\begin{pmatrix} -\rho_A^2(t) \\ \rho_A'(t)\end{pmatrix}\right|_{\infty} \le \left|\begin{pmatrix} -\rho_A^2(t) \\ \rho_A'(t)\end{pmatrix}\right|_{\infty} \\ &= \max\left\{|\rho_A^2(t)|\,,\, |\rho_A'(t)|\right\} \\ &\le \|\rho_A\|_{1,I_{\Gamma_-}}.\end{aligned}$$

Applying the sup-norm gives the assertion. □

Next, we will bound the Hölder coefficient of $\nabla\rho$.

Lemma 6.42. *Let the inflow boundary data $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$ and $\|\rho_A\|_{1,I_{\Gamma_-}} < 1$. Then follows for $\hat{\rho} \in C^{1,\alpha}(Q)$ defined in Theorem 6.34 that*

$$|\nabla\hat{\rho}|_{\alpha,Q} \le \left(2c_l + 2c_l L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\}\right)\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$$

with c_l defined in Theorem 6.27.

Proof. It holds

$$\begin{aligned}|\nabla\hat{\rho}|_{\alpha,Q} &= \left|\frac{1}{(1 + s\rho_A(t))^2}\begin{pmatrix} -\rho_A^2(t) \\ \rho_A'(t)\end{pmatrix}\right|_{\alpha,Q} \\ &\le \max\left\{\left|\frac{\rho_A^2(t)}{(1 + s\rho_A(t))^2}\right|_{\alpha,Q}, \left|\frac{\rho_A'(t)}{(1 + s\rho_A(t))^2}\right|_{\alpha,Q}\right\}. \qquad (6.56)\end{aligned}$$

Let us focus on the first term of (6.56). With the product rule (2.8) holds

$$\left|\frac{\rho_A^2(t)}{(1+s\rho_A(t))^2}\right|_{\alpha,Q} \leq \left|(1+s\rho_A(t))^{-2}\right|_{\alpha,Q} \left\|\rho_A^2\right\|_{0,I_{\Gamma_-}} + \left\|(1+s\rho_A(t))^{-2}\right\|_{0,Q} \left|\rho_A^2\right|_{\alpha,I_{\Gamma_-}}$$
$$\leq \left|(1+s\rho_A(t))^{-2}\right|_{\alpha,Q} \left\|\rho_A^2\right\|_{0,I_{\Gamma_-}} + \left|\rho_A^2\right|_{\alpha,I_{\Gamma_-}}. \tag{6.57}$$

To obtain the α-semi norm of the quadratic function in the denominator, we apply again the product rule (2.8).

$$\left|(1+s\rho_A(t))^{-2}\right|_{\alpha,Q} \leq 2\left\|(1+s\rho_A(t))^{-1}\right\|_{0,Q} \left|(1+s\rho_A(t))^{-1}\right|_{\alpha,Q}$$
$$\leq 2\left|(1+s\rho_A(t))^{-1}\right|_{\alpha,Q} \tag{6.58}$$

and analogously for $|\rho_A^2|_{\alpha,I_{\Gamma_-}}$

$$\left|\rho_A^2\right|_{\alpha,I_{\Gamma_-}} \leq 2\left\|\rho_A\right\|_{0,I_{\Gamma_-}} \left|\rho_A\right|_{\alpha,I_{\Gamma_-}}. \tag{6.59}$$

It is left to find an upper bound for (6.58).

$$\begin{aligned}
\left|\frac{1}{1+s\rho_A(t)}\right|_{\alpha,Q} &= \sup_{(s_1,t_1)\neq(s_2,t_2)\in Q} \frac{\left|\frac{1}{1+s_1\rho_A(t_1)} - \frac{1}{1+s_2\rho_A(t_2)}\right|_\infty}{|(s_1,\,t_1)-(s_2,\,t_2)|_\infty^\alpha} \\
&= \sup_{(s_1,t_1)\neq(s_2,t_2)} \frac{\left|\frac{s_2\rho_A(t_2)-s_1\rho_A(t_1)}{(1+s_1\rho_A(t_1))(1+s_2\rho_A(t_2))}\right|_\infty}{|(s_1,\,t_1)-(s_2,\,t_2)|_\infty^\alpha} \\
&\leq \sup_{(s_1,t_1),(s_2,t_2)} \frac{|s_2\rho_A(t_2)-s_1\rho_A(t_1)|_\infty}{|(s_1,\,t_1)-(s_2,\,t_2)|_\infty^\alpha} \\
&= \sup_{(s_1,t_1)\neq(s_2,t_2)} \frac{|(s_2-s_1)\rho_A(t_2)+s_1(\rho_A(t_2)-\rho_A(t_1))|_\infty}{|(s_1,\,t_1)-(s_2,\,t_2)|_\infty^\alpha} \\
&\leq \sup_{(s_1,t_1)\neq(s_2,t_2)} \left(\frac{|s_2-s_1|_\infty|\rho_A(t_2)|_\infty}{|s_1-s_2|_\infty^\alpha} + \frac{|s_1|_\infty|\rho_A(t_2)-\rho_A(t_1)|_\infty}{|t_1-t_2|_\infty^\alpha}\right) \\
&\leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} \left|\rho_A\right|_{\alpha,I_{\Gamma_-}}.
\end{aligned} \tag{6.60}$$

Hence, we obtain for (6.57) with (6.58), (6.59) and (6.60) and since $|\rho_A|_{1,I_{\Gamma_-}} \leq 1$

$$\begin{aligned}
\left|\frac{\rho_A^2(t)}{(1+s\rho_A(t))^2}\right|_{\alpha,Q} &\leq 2\left|(1+s\rho_A(t))^{-1}\right|_{\alpha,Q} \left\|\rho_A\right\|_{0,Q}^2 + 2\left|\rho_A\right|_{\alpha,I_{\Gamma_-}} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} \\
&\leq 2\left(\|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} \left|\rho_A\right|_{\alpha,I_{\Gamma_-}}\right) \left\|\rho_A\right\|_{0,I_{\Gamma_-}}^2 + 2\left|\rho_A\right|_{\alpha,I_{\Gamma_-}} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} \\
&\leq 2\|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} + 2\|l\|_{0,I_{\Gamma_-}} \left|\rho_A\right|_{\alpha,I_{\Gamma_-}} + 2\left|\rho_A\right|_{\alpha,I_{\Gamma_-}}.
\end{aligned} \tag{6.61}$$

Analogously, we obtain for the second term of (6.56) with (6.57) and (6.60)

$$\begin{aligned}
\left|\frac{\rho_A'(t)}{(1+s\rho_A(t))^2}\right|_{\alpha,Q} &\leq \|\rho_A'\|_{0,Q} \left|(1+s\rho_A(t))^{-2}\right|_{\alpha,Q} + 2|\rho_A'|_{\alpha,I_{\Gamma_-}} \\
&\leq 2\|\rho_A'\|_{0,I_{\Gamma_-}} \left|(1+s\rho_A(t))^{-1}\right|_{\alpha,Q} + 2|\rho_A'|_{\alpha,I_{\Gamma_-}} \\
&\leq 2\|\rho_A'\|_{0,I_{\Gamma_-}} \left(\|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \left\|\rho_A\right\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} \left|\rho_A\right|_{\alpha,I_{\Gamma_-}}\right) + 2|\rho_A'|_{\alpha,I_{\Gamma_-}}.
\end{aligned} \tag{6.62}$$

Since $|\rho_A|_{\alpha,I_{\Gamma_-}} \leq |t_1 - t_2|^{1-\alpha} \|\rho_A'\|_{0,I_{\Gamma_-}} \leq L_{\Gamma_-}^{1-\alpha} \|\rho_A'\|_{0,I_{\Gamma_-}}$, we get for (6.56) by (6.61) and (6.62) and Theorem 6.27

$$\begin{aligned}|\nabla \hat{\rho}|_{\alpha,Q} &\leq \left(2\|l\|_0^{1-\alpha} + 2\|l\|_0 L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\}\right) \|\rho_A\|_{1,\alpha,I_{\Gamma_-}} \\ &\leq \left(2c_l + 2c_l L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\}\right) \|\rho_A\|_{1,\alpha,I_{\Gamma_-}}.\end{aligned}$$

□

The remaining part of this section deals with finding an upper bound of $\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{\alpha,\Omega}$ in terms of $\|E - \tilde{E}\|_{1,\alpha,\Omega}$. As we do not have much information on the inverse streamline function Φ^{-1}, we first bound $\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{\alpha,\Omega}$ in terms of $\|\Phi - \tilde{\Phi}\|_{\alpha,\Omega}$. Unfortunately to bound the latter term, we will encounter several difficulties due to the variability of the parameter set Q in E. Two streamline functions defined on vector fields E and $\tilde{E}$ lead to distinct parameter sets Q and $\tilde{Q}$. The comparison of the distance of Φ and $\tilde{\Phi}$ in a point (s,t) is only possible on the intersection of Q and $\tilde{Q}$. As Φ does not in general map $Q \cap \tilde{Q}$ surjectively on Ω, this would restrict the goal of bounding $|TE - T\tilde{E}|_{\alpha,\Omega}$. To overcome this problem, we extend the vector field E into a sufficiently large domain Ω^+. The corresponding extended streamline function is then defined on a parameter set $Q^+ \supset Q$. Therefore, we can compare the streamline functions to any two extended field E^+ and $\tilde{E}^+$ on the fixed set Q. In 6.3.1 we define the space $W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ in which the extension of every $E \in W(M, \delta_1, \delta_2, \delta_3)$ is contained.

6.3.1 Extension of the Vector Field E

The streamline functions for two vector fields E and $\tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ are defined on different parameter sets Q and $\tilde{Q}$, say, in order to map surjectively onto Ω (see Lemma 6.17). We now change the point of view: Having one fixed parameter set Q and a sufficiently large domain $\Omega^+ \supset \Omega$, the task is to define an extended vector field $\tilde{E}^+$ defined on Ω^+ such that $\tilde{E}^+|_\Omega = \tilde{E}$ and $\Phi(\tilde{E}^+, Q) \subset \Omega^+$. The extended vector field shall lead to invertible streamlines. TE^+ thus has to obey analogous restrictions to E as illustrated in section 6.2.1.

Definition 6.43. *Let Ω^+ be an open bounded $C^{2,\alpha}$ domain with boundary $\Gamma^{ext} = \Gamma_-^{ext} \cup \Gamma_+^{ext}$ and Ω as defined in Chapter 2 with boundary $\Gamma = \Gamma_- \cup \Gamma_+$. Ω^+ is called the* extended domain of Ω *if*

$$\Omega^+ \supset \Omega$$

and Ω^+ shares the inflow boundary with Ω, i.e. $\Gamma_-^{ext} = \Gamma_-$. The outer boundary is called Γ_+^{ext}. We will denote the ϵ-neighborhood of Γ_+ by

$$U_\epsilon(\Gamma_+) = \left\{x \in \Omega^+ \backslash \Omega : \mathrm{dist}(x, \Gamma^+) \leq \epsilon\right\}.$$

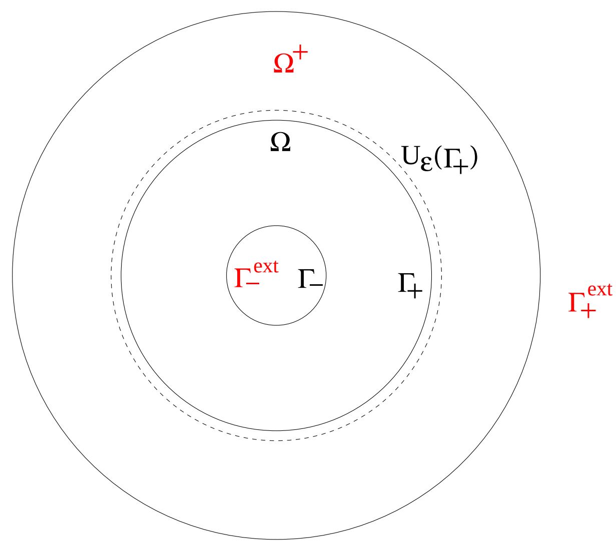


Figure 6.3: Extended domain Ω^+

Definition 6.44. *Let Ω be a $C^{2,\alpha}$ domain and $\Omega^+ \supset \Omega$ as in Definition 6.43. Let $M, \delta_1, \delta_2, \epsilon, c^+ > 0$. Then the set of* extended vector fields *on Ω^+ is given by*

$$W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+) = \big\{E^+ \in C^{1,\alpha}(\Omega^+) \,:\, E^+|_\Omega = E \in W(M,\delta_1,\delta_2,\delta_3),\ E^+ \textit{gradient field},$$
$$\|E^+\|_{1,\alpha,\Omega^+} \le c_1(M + \|u_A\|_{0,\Gamma}),\ \inf_{x\in\Omega^+} |E^+(x)|_\infty \ge c^+\delta_1,$$
$$\vec{n}\cdot E^+ \le -\delta_2 < 0 \ \textit{on}\ \Gamma_-,\ \vec{n}\cdot E^+ > 0 \ \textit{on}\ \Gamma_+^+\big\}\,.$$

where c^+ is a constant depending on the extended domain Ω^+ and c_1 is a constant depending on given quantities and the constants defining the set W^+.

The method of extending the vector field to obtain streamline functions defined on a larger parameter set is a known practice. Alber applied in [2] Calderon's extension Theorem to the vector field E. He obtained an extension of E from the Sobolev space $H^3(\bar{\Omega})$ into $H^3(\mathbb{R}^3)$, with $\bar{\Omega} \subset \mathbb{R}^3$ bounded domain. In our situation, we have to carefully construct the vector field E^+ for not losing the properties to obtain an invertible streamline function $\Phi(E^+, s, t) \in C^{1,\alpha}(\Omega^+)$. Instead of directly deriving E^+ from $E \in W(M,\delta_1,\delta_2,\delta_3)$, we extend the corresponding potential function u to u^+. The first condition that E^+ is a gradient field is thus automatically fulfilled. The main tools we use are the $C^{2,\alpha}(\bar{\Omega})$-continuity of u and a general extension lemma for $C^{2,\alpha}(\bar{\Omega})$ functions on $C^{2,\alpha}$ domains [34, Lemma 6.37].

Theorem 6.45. *[34, Lemma 6.37] Let $\Omega \subset \mathbb{R}^2$ be an open bounded $C^{2,\alpha}$ domain and let Ω' be an open set containing $\bar{\Omega}$. Suppose $u \in C^{2,\alpha}(\bar{\Omega})$. Then there exists a function $w \in C_0^{2,\alpha}(\Omega')$ such that $w = u$ in Ω and*

$$\|w\|_{2,\alpha;\Omega'} \le c(\Omega,\Omega')\|u\|_{2,\alpha;\Omega}$$

where $c(\Omega,\Omega')$ is a constant depending only on Ω and Ω'.

The previous Lemma ensures the existence of a $C^{2,\alpha}$-extension of a function $u \in C^{2,\alpha}(\bar{\Omega})$ into a larger domain Ω'. However, it lacks any qualitative information about w. The restrictions to obtain valid streamlines are not necessarily fulfilled.

Before starting with the actual construction of the extended function, we need a result on the distance function $\operatorname{dist}(x, \Gamma_+)$. If the domain Ω is a $C^{2,\alpha}$ domain, then the distance function itself is of $C^{2,\alpha}(U_\epsilon(\Gamma_+))$ regularity for an $\epsilon > 0$.

Theorem 6.46. *[48, p.231] Let Ω be a $C^{2,\alpha}$ domain. Then there exists an $\epsilon > 0$ such that every point $x \in U_\epsilon(\Gamma_+)$ has a unique nearest point on $\partial\Omega$. The function*

$$\operatorname{dist}(x, \Gamma_+) = \min_{x_0 \in \partial\Omega} |x - x_0|_\infty$$

is a $C^{2,\alpha}(U_\epsilon(\Gamma_+))$ function. Further holds

$$\|\nabla \operatorname{dist}(x, \Gamma_+)\|_2 = 1$$

with $\|\cdot\|_2$ being the euclidean norm. For $x \in U_\epsilon(\Gamma_+)$, the vector $\nabla \operatorname{dist}(x, \Gamma_+)$ points in the direction of the outward normal vector $\vec{n}(x_0)$ to Γ_+ evaluated at the unique nearest point x_0.

We proceed to derive an extension u^+ to u. First, we show that $u^+ \in C^{2,\alpha}(\Omega^+)$. Second, we verify that $E^+ = -\nabla u^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$.

Define the one-dimensional cut-off function $\chi \in C^3([-\infty, \infty])$ for some $\epsilon > 0$ as

$$\chi(x) = \begin{cases} 1, & x < \epsilon \\ \gamma(x), & \epsilon \leq |x| \leq 2\epsilon \\ 0, & x > 2\epsilon \end{cases}$$

where γ is defined in appendix A.2 and $0 \leq \chi(x) \leq 1$ for all x.

Theorem 6.47. *Let Ω be an open bounded $C^{2,\alpha}$ domain and let $\Omega^+ \supset \Omega$ be as defined in Definition 6.43. Let $-\nabla w$ be a $C^{1,\alpha}$ extension of $E = -\nabla u \in W(M, \delta_1, \delta_2, \delta_3)$. Let v be the solution of the Laplace equation*

$$\begin{aligned} -\Delta v(x) &= 0 && x \in \Omega^+ \backslash \Omega \\ v(x) &= c_v(x) && x \in \Gamma_+ \cup \Gamma_+^{ext} \end{aligned}$$

with $c_v|_{\Gamma_+} =: c_{v_1}$ and $c_v|_{\Gamma_+^{ext}} =: c_{v_2}$ constant and $c_{v_1} > c_{v_2}$.
Then there exists an $\epsilon > 0$, such that the function

$$u^+(x) = \begin{cases} u(x) & x \in \Omega \\ \chi(\operatorname{dist}(x, \Gamma^+))w(x) + (1 - \chi(\operatorname{dist}(x, \Gamma^+)))v(x) & x \in U_{2\epsilon}(\Gamma_+) \\ v(x) & x \in \Omega^+ \backslash U_{2\epsilon} \cup \Omega \end{cases}$$

is a $C^{2,\alpha}(\Omega^+)$ extension for every u with $-\nabla u = E \in W(M, \delta_1, \delta_2, \delta_3)$.

Proof. Let Ω' be an open domain such that $\bar{\Omega} \subset \Omega' \subset \Omega^+$. Then by Theorem 6.45, we obtain an extension $w \in C_0^{2,\alpha}(\Omega')$ of u. Since w has compact support in Ω', it is extended by 0 into Ω^+ and therefore $w \in C^{2,\alpha}(\Omega^+)$.
It holds that $w \in C^{2,\alpha}(\Omega^+)$, $v \in C^{2,\alpha}(\Omega^+\backslash\Omega)$, $\chi \in C^\infty([-\infty,\infty])$ and Theorem 6.46 yields $\operatorname{dist}(x,\Gamma_+) \in C^{2,\alpha}(U_{2\epsilon}(\Gamma_+))$. Therefore, we have $u^+ \in C^{2,\alpha}(U_{2\epsilon}(\Gamma^+))$. On the other hand is w the $C^{2,\alpha}(\Omega^+)$-extension to u and for $x \in U_\epsilon(\Gamma_+)$ holds $1 - \chi(\operatorname{dist}(x,\Gamma^+)) = 0$. Conclusively, $u^+ \in C^{2,\alpha}(\Omega \cup U_{2\epsilon}(\Gamma_+))$. The same argument holds for all $x \in \Omega^+\backslash(U_\epsilon(\Gamma_+) \cup \Omega)$. Since $\chi(\operatorname{dist}(x,\Gamma^+)) = 0$ for $x \in U_{2\epsilon}(\Gamma_+)\backslash U_\epsilon(\Gamma_+)$, v is a $C^{2,\alpha}$ extension into $\Omega^+\backslash(U_{2\epsilon} \cup \Omega)$. Thus $u^+ \in C^{2,\alpha}(\Omega)$. □

In the following Lemma, we will prove that $-\nabla u^+$ fulfils the restrictions in $W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$.

Lemma 6.48. *Let Ω be an open bounded $C^{2,\alpha}$ domain and $\Omega^+ \supset \Omega$ as defined in Definition 6.43. Set $c_{v_2} = -\min\left\{\frac{1}{8}, M\right\} \frac{\delta_1}{c_S(\Omega^+\backslash\Omega,\alpha)}$ on Γ_+^{ext} and $c_{v_1} = -\frac{1}{2}\min\left\{\frac{1}{8}, M\right\} \frac{\delta_1}{c_S(\Omega^+\backslash\Omega,\alpha)}$ on Γ_+. Set $c^+ = \min\left\{\frac{1}{8}, M\right\} \frac{c_p(\Omega^+)}{c_S(\Omega^+\backslash\Omega,\alpha)}$ with $c_S(\Omega^+\backslash\Omega)$ defined in Theorem 6.7 and $c_p(\Omega^+)$ being a constant depending on the domain Ω^+.*
Then there exists an $\epsilon > 0$, such that the extension $E^+ = -\nabla u^+$ to any $E \in W(M,\delta_1,\delta_2,\delta_3)$ is element of $W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$.

Proof. By choosing an appropriate constant $\epsilon > 0$, we will prove that all the restrictions in W^+ are fulfilled by $E^+ = -\nabla u^+$. The crucial point in the argumentation is to choose a global constant ϵ, i.e. it holds for the extension E^+ to any $E \in W(M,\delta_1,\delta_2,\delta_3)$. To choose ϵ, we use the following properties.

Claim 1:
We use the $C^{2,\alpha}(\Omega^+)$ continuity of w to define $\epsilon > 0$. For $x \in \Gamma_+$ holds $-\nabla w = E$. Since the extension $-\nabla w$ to any E is continuous in Ω^+, it holds due to $E \in W(M,\delta_1,\delta_2,\delta_3)$ that $\inf_{x\in\Gamma_+} |\nabla w(x)|_\infty \geq \delta_1 > 0$. Again by the $C^{2,\alpha}$ continuity of w, there exists an $\epsilon > 0$, such that for the extension $-\nabla w$ to any $E \in W(M,\delta_1,\delta_2,\delta_3)$ holds

$$\inf_{x\in U_\epsilon} |\nabla w(x)| \geq \frac{\delta_1}{2}$$

and

$$\inf_{x\in U_{2\epsilon}} |\nabla w(x)| \geq \frac{\delta_1}{4}.$$

Claim 2:
Next, we will compare the angle between the gradients $-\nabla w$, $-\nabla v$ and $-\nabla\chi(\operatorname{dist}(x,\Gamma^+))$. The boundary conditions for u are constant on Γ_+ and thus form an equipotential curve. For $x \in \Gamma_+$, $-\nabla w(x)$ is thus perpendicular to Γ_+ and points into the direction of the outward normal vector due to the outflow condition in $W(M,\delta_1,\delta_2,\delta_3)$. The analogous argumentation holds for v. Due to the constant boundary value $v|_{\Gamma_+} = c_v$, the gradient field $-\nabla v$ points in the direction of the normal outward vector for $x \in \Gamma_+$. By

$$-\nabla\chi(\operatorname{dist}(x,\Gamma_+)) = -\underbrace{\chi'(\operatorname{dist}(x,\Gamma_+))}_{\leq 0}\nabla\operatorname{dist}(x,\Gamma_+) \tag{6.63}$$

and Theorem 6.46, we obtain that $-\nabla\chi(\text{dist}(x,\Gamma_+))$ also points in the direction of the outward normal vector $\vec{n}(x)$.
Due to their continuity, ∇v, ∇w and $\nabla\chi(\text{dist}(x,\Gamma_+))$ cannot change their direction extensively in a small neighborhood of Γ_+. There exists an $\epsilon > 0$, such that the cosine of the pairwise enclosed angles of ∇v, $\nabla w - \nabla v$ and $\nabla\chi(\text{dist}(x,\Gamma_+))$ at $x \in U_{2\epsilon}(\Gamma_+)$ is greater than 0.9 for the extension $-\nabla w(x)$ to any vector field $E \in W(M,\delta_1,\delta_2,\delta_3)$.

Claim 3:
The boundary condition c_v is given by $c_v|_{\Gamma_+} > c_v|_{\Gamma_+^{ext}}$. Since $c_v|_{\Gamma_+} < 0$ it certainly holds $v|_{\Gamma_+} < w|_{\Gamma_+}$. Due to the continuity there exists an $\epsilon > 0$ such that $w(x) \geq v(x)$ for $x \in U_{2\epsilon}(\Gamma^+)$.

Choice of ϵ:
Let γ be an upper bound for the pairwise enclosed angles of ∇v, $\nabla\chi(\text{dist}(x,\Gamma_+))$ and $\nabla w - \nabla v$. Choose $\epsilon > 0$ such that for the extension $-\nabla w = E$ to any $E \in W(M,\delta_1,\delta_2,\delta_3)$ holds

1. $w(x) \geq v(x)$ for $x \in U_{2\epsilon}$,
2. $\inf_{x\in U_\epsilon} |\nabla w| \geq \frac{\delta_1}{2}$,
3. $\inf_{x\in U_{2\epsilon}} |\nabla w| \geq \frac{\delta_1}{4}$,
4. $\cos(\gamma) \geq 0.9$.

It is reasonable that $\nabla w(x) - \nabla v(x)$ points outward for $x \in U_{2\epsilon}(\Gamma^+)$: By the inverse triangle inequality, Schauder's estimate (Theorem 6.7) and the choice of the boundary conditions c_v holds

$$\begin{aligned}
|\nabla w(x) - \nabla v(x)|_\infty &\geq |\nabla w(x)|_\infty - |\nabla v(x)|_\infty \\
&\geq |\nabla w(x)|_\infty - c_S(\Omega^+\backslash\Omega,\alpha)\|c_v\|_{2,\alpha,\Gamma} \\
&\geq \frac{\delta_1}{4} - \frac{\delta_1}{8} = \frac{\delta_1}{8}.
\end{aligned}$$

The strength of the vector field $-\nabla v$ is smaller than the one of $-\nabla w$ and thus the sum is mainly governed by $-\nabla w$.

Verification of the conditions on E^+:
Due to the construction of u^+ it is immediately clear, that $E^+|_\Omega = E$. Moreover,
$E^+ = -\nabla u^+$ forms a gradient field and the inflow condition is fulfilled, as E does. The outflow condition is fulfilled, due to $u^+(x) = v(x)$ for $x \in \Gamma_+^+$ and the properties of the Laplace equation.

Boundedness from below:
To show that $\inf_{x\in\Omega^+} |-\nabla u^+(x)|_\infty \geq c^+\delta_1$, we have to distinguish three cases.

Case 1: $x \in U_\epsilon$

Then

$$u^+ = w.$$

It holds due to the choice of ϵ

$$\inf_{x \in U_\epsilon} | - \nabla u^+(x)|_\infty = \inf_{x \in U_\epsilon} | - \nabla w(x)| \geq \frac{\delta_1}{2} > 0.$$

Case 2: $x \in \Omega^+ \backslash (U_{2\epsilon} \cup \Omega)$
Then

$$u^+ = v.$$

To find a lower bound, consider the solution of the Laplace equation

$$-\Delta p = 0 \qquad x \in \Omega^+ \tag{6.64a}$$

$$p = -\frac{1}{2} \qquad x \in \Gamma_+ \tag{6.64b}$$

$$p = -1 \qquad x \in \Gamma_+^{ext}. \tag{6.64c}$$

The solution v is then given by $v = c_v p$ with the gradient $\nabla v = c_v \nabla p$. As p is the solution of a harmonic function, it holds by the strong maximum principle (Theorem 6.4) that there exist no inner extrema. Conclusively, the gradient $\nabla p(x) \neq 0$ and there exists a lower bound $\inf_{x \in \Omega^+ \backslash \Omega} |\nabla p(x)|_\infty =: c_p$. It follows

$$\inf_{x \in \Omega^+ \backslash \Omega} |\nabla v(x)|_\infty = \inf_{x \in \Omega^+ \backslash \Omega} |c_v \nabla p(x)|_\infty \geq \min \left\{ \frac{1}{8}, M \right\} \frac{c_p(\Omega^+)}{c_S(\Omega^+ \backslash \Omega, \alpha)} \delta_1.$$

The actual size of the constant $c_p(\Omega^+)$ depends on the size and shape of the extended domain but is always greater than 0.

Case 3: $x \in U_{2\epsilon} \backslash U_\epsilon$
Then

$$u^+(x) = \chi(\mathrm{dist}(x, \Gamma)) w(x) + (1 - \chi(\mathrm{dist}(x, \Gamma))) v(x).$$

We have to prove that

$$\begin{aligned} &|\nabla \left[\chi(\mathrm{dist}(x, \Gamma^+)) w(x) \right] + \nabla \left[(1 - \chi(\mathrm{dist}(x, \Gamma^+))) v(x) \right] | \\ &\quad = |\nabla \chi(\mathrm{dist}(x, \Gamma^+))(w(x) - v(x)) + \chi(\mathrm{dist}(x, \Gamma^+))(\nabla w(x) - \nabla v(x)) + \nabla v(x)| \geq c\delta_1 \end{aligned}$$

for some constant $c > 0$. Due to the choice of ϵ holds

$$|\nabla \chi(\mathrm{dist}(x, \Gamma^+)) \underbrace{(w(x) - v(x))}_{\geq 0} + \underbrace{\chi(\mathrm{dist}(x, \Gamma^+))}_{\geq 0} (\nabla w(x) - \nabla v(x)) + \nabla v(x)|.$$

By condition (4), all gradients are contained in an angle for which the cosine is 0.9 or larger, i.e. the angles are smaller than 90° for $x \in U_{2\epsilon} \backslash U_\epsilon$. Conclusively, the length of the sum of the gradients increases and is larger than the length of any single gradient, in particular greater than $\inf_{x\in U_{2\epsilon}\setminus U_\epsilon} |\nabla v(x)|_\infty$. We confirm that this is a lower bound for $x \in U_{2\epsilon}$, as for all $x_0 \in \Omega^+$ with $\operatorname{dist}(x_0, \Gamma^+) = 2\epsilon$ holds $\chi(2\epsilon) = 0$ and $\chi'(2\epsilon) = 0$. Hence,

$$
\begin{aligned}
&|\nabla\chi(\operatorname{dist}(x_0,\Gamma^+))(w(x_0) - v(x_0)) + \chi(\operatorname{dist}(x_0,\Gamma^+))(\nabla w(x_0) - \nabla v(x_0)) + \nabla v(x_0)|_\infty \\
&\quad = |\nabla v(x_0)|_\infty \\
&\quad \geq c_p(\Omega^+) \min\left\{\frac{1}{8}, M\right\} \frac{\delta_1}{c_S(\Omega^+\setminus\Omega, \alpha)}.
\end{aligned}
$$

To sum up, it holds for u^+:

$$
\inf_{x\in\Omega^+} |-\nabla u^+(x)|_\infty \geq \min\left\{\frac{1}{2}, \min\left\{\frac{1}{8}, M\right\} \frac{c_p(\Omega^+)}{c_S(\Omega^+\setminus\Omega,\alpha)}\right\} \delta_1 > 0.
$$

Boundedness from above:
By Lemma 6.45 holds

$$
\|-\nabla w\|_{1,\alpha,\Omega^+} \leq c(\Omega, \Omega')\|u\|_{2,\alpha,\Omega}. \tag{6.65}
$$

For every $x \in \Omega$, there exists a point $(z_1, x_2) \in \Gamma$, such that the line segment $\overrightarrow{(z_1,x_2)(x_1,x_2)}$ is fully contained in Ω. We get pointwise for every $x \in \Omega$

$$
\begin{aligned}
|u(x)|_\infty &= \left|\int_{z_1}^{x_1} \partial_y u(y, x_2)\, dy - u(z_1, x_2)\right|_\infty \\
&\leq \left|\int_{z_1}^{x_1} \partial_y u(y, x_2)\, dy\right|_\infty + |u_A(z_1, x_2)|_\infty \\
&\leq |z_1 - x_1|\, |\partial_{x_1} u|_{0,\Omega} + |u_A|_{0,\Gamma} \\
&\leq \operatorname{diam}(\Omega)\, |\nabla u|_{0,\Omega} + |u_A|_{0,\Gamma}
\end{aligned}
$$

We thus get for (6.65)

$$
\|-\nabla w\|_{1,\alpha,\Omega^+} \leq c(\Omega', \Omega)(\operatorname{diam}\Omega |E|_{0,\Omega} + \|u_A\|_{0,\Gamma}) \leq c(\Omega, \Omega')\left(M + \|u_A\|_{0,\Gamma}\right). \tag{6.66}
$$

We obtain with Schauder's estimates for v

$$
\begin{aligned}
\|v\|_{2,\alpha;\Omega^+\setminus\Omega} &\leq c_S(\Omega^+\setminus\Omega, \alpha)\|v\|_{2,\alpha;\Gamma_+\cup\Gamma_+^{ext}} \\
&\leq c_S(\Omega^+\setminus\Omega, \alpha)\, \|c_v\|_{2,\alpha;\Gamma_+\cup\Gamma_+^{ext}} \\
&= c_s(\Omega^+\setminus\Omega, \alpha) \min\left\{\frac{1}{8}, M\right\} \frac{\delta_1}{c_S(\Omega^+\setminus\Omega, \alpha)} \\
&\leq M\delta_1.
\end{aligned} \tag{6.67}
$$

With Lemma A.6, we have $\|\chi(\operatorname{dist}(x, \Gamma_+))\|_{2,\alpha,\Omega^+\setminus\Omega} \leq \frac{c(U_{2\epsilon}(\Gamma_+))}{\epsilon^{2+\alpha}}$. We obtain with (6.66) and (6.67)

$$
\|-\nabla u^+\|_{1,\alpha,\Omega^+} \leq \max\left\{1, \frac{c(\Omega, \Omega', U_{2\epsilon}(\Gamma_+), \delta_1)}{\epsilon^{2+\alpha}}\right\} (M + \|u_A\|_{0,\Gamma}). \tag{6.68}
$$

□

For every Ω^+ we obtain a different set $W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$, as c^+ depends on Ω^+. The question is whether it is possible to find a finite domain Ω^+ such that the streamlines to any $E \in W(M,\delta_1,\delta_2,\delta_3)$ are satisfactorily extendable. The next Lemma shows that there exists such a domain with $\Omega^+ \supset \Phi(E^+,s,t)$ for any parameter set Q to $E \in W(M,\delta_1,\delta_2,\delta_3)$.

Lemma 6.49. *Let Ω be a $C^{2,\alpha}$ domain and Ω^+ be as defined in Definition 6.43. Let the outer boundary Γ_+^{ext} be a ball of radius R with midpoint in the centre of mass of Ω. Choose R such that*

$$\operatorname{dist}(\Gamma_-,\Gamma_+^{ext}) \geq \frac{c(U_{2\epsilon}(\Gamma_+),\Omega,\Omega',\delta_1)(M+\|u_A\|_{0,\Gamma})c_l}{\epsilon}.$$

with $c(U_{2\epsilon}(\Gamma_+),\Omega,\Omega',\delta_1)$ defined in (6.68) and c_l defined in Theorem 6.27.
Set $c^+ = \min\left\{\frac{1}{8},M\right\}\frac{c_p(\Omega^+)}{c_S(\Omega^+\setminus\Omega,\alpha)}$. Then holds for the streamline function to any $E^+ \in W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$ that

$$\Phi(E^+,Q) \subset \Omega^+$$

for the parameter set Q to the streamline function $\Phi(E,s,t)$ to any $E \in W(M,\delta_1,\delta_2,\delta_3)$.

Proof. Let $E \in W(M,\delta_1,\delta_2,\delta_3)$. We obtain a lower bound for $l(t)$ by

$$\begin{aligned}
|\Phi(l(t),t)-\varphi(t)|_\infty &= \left|\int_0^{l(t)} E(\Phi(\mu,t))\,d\mu\right|_\infty \\
&\leq l(t)\sup_{s\in[0,l(t)]}|E(\Phi(s,t))|_\infty \\
&\leq l(t)\sup_{x\in\Omega}|E(x)|_\infty.
\end{aligned}$$

Hence,

$$l(t) \geq \frac{\inf_{0\leq t\leq L_{\Gamma_-}}|\Phi(l(t),t)-\varphi(t)|_\infty}{\sup_{x\in\Omega}|E(x)|_\infty} \geq \frac{\operatorname{dist}(\Gamma_-,\Gamma_+)}{\sup_{x\in\Omega}|E(x)|_\infty}.$$

Let now $E^+ \in W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$. By using (6.68) for an upper bound on $\|E^+\|_{0,\Omega^+}$, we obtain likewise a lower bound for $l^+(t)$ for every $E^+ \in W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$ and $0 \leq t \leq L_{\Gamma_-}$ by

$$\begin{aligned}
\inf_{0\leq t\leq L_{\Gamma_-}}|l^+(t)| &\geq \frac{\inf_t|\Phi(E^+,l^+(t),t)-\varphi(t)|_\infty}{\sup_{x\in\Omega^+}|E^+(x)|_\infty} \\
&\geq \frac{\epsilon\inf_t|\Phi(E^+,l^+(t),t)-\varphi(t)|_\infty}{c(U_{2\epsilon}(\Gamma_+),\Omega,\Omega',\delta_1)(M+\|u_A\|_{0,\Gamma})} \\
&\geq \frac{\epsilon\operatorname{dist}(\Gamma_-,\Gamma_+^{ext})}{c(U_{2\epsilon}(\Gamma_+),\Omega,\Omega',\delta_1)(M+\|u_A\|_{0,\Gamma})} \qquad (6.69)
\end{aligned}$$

By carefully tracking the bound of $\|E^+\|_{0,\Omega^+}$ in the proof of Lemma 6.48, we notice the following: $\|E^+\|_{0,\Omega^+}$ is bounded by a constant depending on the compact support Ω' for w and a further constant given by $\chi(\operatorname{dist}(x,\Gamma_+))$. However, $\chi(\operatorname{dist}(x,\Gamma_+))$ is constant except on $U_{2\epsilon}(\Gamma^+)$. By increasing the size of the domain Ω^+, neither the compact support Ω' nor the 2ϵ-neighborhood of Γ_+ are affected. The bound of $|-\nabla v|_{0,\Omega^+\setminus\Omega}$ is independent of any constant depending on the domain Ω^+.

The goal is to choose R big enough, such that $\Phi(E^+, Q) \subset \Omega^+$ for every choice of $E \in W(M, \delta_1, \delta_2, \delta_3)$. This is obtained if

$$\inf_{0\le t\le L_{\Gamma_-}} |l^+(t)| \ge \sup_{0\le t\le L_{\Gamma_-}} |l(t)|$$

and by (6.69) and Theorem 6.27 in particular if

$$\frac{\epsilon \operatorname{dist}(\Gamma_-, \Gamma_+^{ext})}{c(U_{2\epsilon}(\Gamma_+), \Omega, \Omega', \delta_1)(M + \|u_A\|_{0,\Gamma})} \overset{!}{\ge} c_l$$

Γ_+^{ext} is chosen as a ball with midpoint in the centre of mass of Ω. By increasing R, we automatically increase the distance between the inflow boundary Γ_- and the outer boundary Γ_+^{ext}. Choose R such that

$$\operatorname{dist}(\Gamma_-, \Gamma_+^{ext}) \ge \frac{c(U_{2\epsilon}(\Gamma_+), \Omega, \Omega', \delta_1)(M + \|u_A\|_{0,\Gamma})c_l}{\epsilon}.$$

It follows the assertion. □

6.3.2 Distance of two Streamline Functions

We are now ready to bound the difference of two streamline functions $\|\Phi^{-1}(E) - \Phi^{-1}(\tilde{E})\|_{\alpha,\Omega}$ in terms of $\|E - \tilde{E}\|_{1,\alpha,\Omega}$. The essential tool is the extension of the field E to obtain streamline functions defined on a common parameter set. To simplify notations, we set again

$$\begin{aligned} \Phi(E, s, t) &=: \Phi(s, t), & \Phi^{-1}(E, x) &=: \Phi^{-1}(x) \\ \Phi(\tilde{E}, s, t) &=: \tilde{\Phi}(s, t) & \Phi^{-1}(\tilde{E}, x) &=: \tilde{\Phi}^{-1}(x) \\ \Phi(\tilde{E}^+, s, t) &=: \tilde{\Phi}^+(s, t) & \Phi^{-1}(\tilde{E}^+, x) &=: \tilde{\Phi}_+^{-1}(x). \end{aligned}$$

We always identify the parameter set Q with $\Phi(E, s, t)$ and $\tilde{Q}$ with $\Phi(\tilde{E}, s, t)$.

We start proving a bound for the sup-norm $\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{0,\Omega}$ in terms of $\|\Phi - \tilde{\Phi}^+\|_{0,Q}$.

Lemma 6.50. *Let Ω be a $C^{2,\alpha}$ domain and $\Omega^+ \supset \Omega$ as in Lemma 6.49. Let $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ be vector fields with extension $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ to $\tilde{E}$. Then holds*

$$\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{0,\Omega} \le c_{mv} \|\nabla \tilde{\Phi}_+^{-1}\|_{0,\Omega^+} \|\tilde{\Phi}^+ - \Phi\|_{0,Q}.$$

Proof. The streamline functions Φ^{-1} and $\tilde{\Phi}^{-1}$ map Ω into distinct parameter sets Q and $\tilde{Q}$. Using the extension $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ to $\tilde{E}$, we get an extended streamline function

$\tilde{\Phi}^+$ with inverse function $\tilde{\Phi}_+^{-1}$. Then holds the identity $\tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) = \Phi^{-1}(x)$. We obtain pointwise for all $x \in \Omega$ with Lemma 2.20 and the identity $x = \Phi^{-1}(\Phi(x))$

$$\begin{aligned}
\left|\Phi^{-1}(x) - \tilde{\Phi}^{-1}(x)\right|_\infty &= \left|\tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) - \tilde{\Phi}^{-1}(x)\right|_\infty \\
&= \left|\tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) - \tilde{\Phi}_+^{-1}(x)\right|_\infty \\
&\leq c_{mv} \left\|\nabla\tilde{\Phi}_+^{-1}\right\|_{0,\Omega^+} \left|\tilde{\Phi}^+(\Phi^{-1}(x)) - x\right|_\infty \\
&= c_{mv} \left\|\nabla\tilde{\Phi}_+^{-1}\right\|_{0,\Omega^+} \left|\tilde{\Phi}^+(\Phi^{-1}(x)) - \Phi(\Phi^{-1}(x))\right|_\infty . \qquad (6.70)
\end{aligned}$$

We then obtain for the sup-norm of (6.70)

$$\left\|\Phi^{-1} - \tilde{\Phi}^{-1}\right\|_{0,\Omega} \leq c_{mv} \left\|\nabla\tilde{\Phi}_+^{-1}\right\|_{0,\Omega^+} \left\|\tilde{\Phi}^+ - \Phi\right\|_{0,Q} .$$

□

The next lemma states the pointwise distance for two streamline functions on a parameter set Q. The estimate is based on [10, Lemma 3.2].

Lemma 6.51. *Let Ω be a $C^{2,\alpha}$ domain and Ω^+ as in Lemma 6.49. Let $E \in W(M, \delta_1, \delta_2, \delta_3)$ be a vector field with streamline function $\Phi : Q \to \Omega$. Let $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ be the extension to $\tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$. The pointwise distance of the streamline functions Φ and $\tilde{\Phi}^+$ is given for all $(s, t) \in Q$ by*

$$|\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty \leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}(\Phi(\mu,t))|_\infty \exp\left(c_{mv} c_l \|\nabla\tilde{E}^+\|_{0,\Omega^+}\right) d\mu$$

with c_l defined in Theorem 6.27.

Proof. Φ maps the parameter set Q bijectively on Ω. Hence, every point $\Phi(s,t)$ with $(s,t) \in Q$ is in the domain of definition of $\tilde{E}^+$. We get by the triangle inequality

$$\begin{aligned}
|\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty &= \left|\int_0^s E(\Phi(\mu,t)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t))\, d\mu\right|_\infty \\
&\leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t))|_\infty \, d\mu \\
&= \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t)) + \tilde{E}^+(\Phi(\mu,t)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t))|_\infty \, d\mu \\
&\leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t))|_\infty + |\tilde{E}^+(\Phi(\mu,t)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t))|_\infty \, d\mu. \qquad (6.71)
\end{aligned}$$

By Lemma 2.20 holds

$$|\tilde{E}^+(\Phi(s,t)) - \tilde{E}^+(\tilde{\Phi}^+(s,t))|_\infty \leq c_{mv} \|\nabla\tilde{E}^+\|_{0,\Omega^+} |\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty.$$

Substituting the latter equation into (6.71) gives

$$\begin{aligned}
&|\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty \\
&\quad\leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t))|_\infty + c_{mv} \|\nabla\tilde{E}^+\|_{0,\Omega^+} |\Phi(\mu,t) - \tilde{\Phi}^+(\mu,t)|_\infty \, d\mu.
\end{aligned}$$

As $|\Phi(s,t) - \tilde{\Phi}(s,t)|_\infty$ is a function of s, we can apply Grönwall's inequality (Lemma 6.19)

$$|\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty \leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t))|_\infty \exp\left(c_{mv} \int_\mu^s \|\nabla \tilde{E}^+\|_{0,\Omega^+} d\tau\right) d\mu.$$

Since this estimate is valid for all $(s,t) \in Q$, we know that $|s| \leq \|l\|_{0,I_{\Gamma_-}}$. Further, since $\Phi : Q \to \Omega$ holds $|E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t))|_\infty = |E(\Phi(\mu,t)) - \tilde{E}(\Phi(\mu,t))|_\infty$. By Theorem 6.27 follows

$$\begin{aligned}|\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty &\leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}^+(\Phi(\mu,t))|_\infty \exp\left(c_{mv}\|l\|_{0,Q}\|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) d\mu \\ &\leq \int_0^s |E(\Phi(\mu,t)) - \tilde{E}(\Phi(\mu,t))|_\infty \exp\left(c_{mv}c_l\|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) d\mu.\end{aligned}$$

□

The previous Lemma is important for the upcoming estimates. The lack of convexity of the domain Ω generally excludes the applicability of the mean value theorem. In Lemma 2.20, we adjusted the mean value theorem to our geometry with an additional constant c_{mv}. Yet, in case of the Hölder coefficient $|\Phi^{-1} - \tilde{\Phi}^{-1}|_{\alpha,\Omega}$, the four point difference makes an analogous argumentation of 2.20 not feasible. We use the parameter δ_3 in $W(M, \delta_1, \delta_2, \delta_3)$ to constrain the distance of two streamlines starting at the same point $\varphi(t)$ along $s \in [0, l(t)]$.

According to the notations in section 6.1, let E_0 be the gradient field of the solution of the Laplace equation (6.5a)-(6.5b). E_0 is element of $W(M, \delta_1, \delta_2, \delta_3)$. Due to the constant boundary conditions $u_A|_{\Gamma_-}$ and $u_A|_{\Gamma_+}$, the field E_0 is perpendicular to Γ_+ and Γ_-. As a direct consequence, the streamlines Φ_0 are also perpendicular thereon. Since E_0 is a gradient field, the streamlines always follow the direction of steepest descent [38, p.190]. Furthermore, there do not exist any inner extrema by Theorem 6.4. Recall that we chose $u_A|_{\Gamma_-} > u_A|_{\Gamma_+}$, the supremum is thus attained at the inflow boundary Γ_-. Conclusively, the streamlines Φ_0 can not approach Γ_- arbitrarily close again after the initial point at $s = 0$. Let now be E_0^+ the extension to the vector field E_0 with the corresponding streamline function Φ_0^+. Once Φ_0^+ has left the domain Ω, the distance of every point $\Phi_0^+(s,t)$ for $l_0(t) \leq s \leq l(t)$ to Γ_- is at least $\operatorname{dist}(\Gamma_-, \Gamma_+)$. Let us now define

$$d(s,t_0) = \operatorname{dist}(\Phi_0^+(s,t_0), \Gamma_-).$$

Since Φ_0^+ is the streamline function to the gradient field of the Laplace equation (6.5a)-(6.5b) and due to its continuity and the reasoning above, we know that $d(s,t_0) \neq 0$ for all $0 < s \leq l^+(t)$ and all $0 \leq t \leq L_{\Gamma_-}$. Further, it does not exist a point $s_0 \in (0, l^+(t)]$ with $\lim_{s \to s_0} d(s,t_0) = 0$, as u_0 attains its maximum at the inflow boundary and there do not exist any inner extrema. Next, we define

$$c_{\Phi_0} := \inf_{0 \leq t \leq L_{\Gamma_-}} \inf_{0 < s \leq l^+(t)} \left\{\frac{d(s,t)}{s}\right\} \tag{6.72}$$

which gives the minimal speed for a particle travelling along the streamlines. Since $d(s,t) > 0$ for all $0 < s \leq l(t)$ and $0 \leq t \leq L_{\Gamma_-}$, the constant c_{Φ_0} is positive, i.e. $c_{\Phi_0} > 0$. Again, we emphasize that due to the continuity of E_0 such a constant c_{Φ_0} exists. The vector field E_0 is given as soon as the problem is defined and depends on Ω and the potential difference $u_{A_1} - u_{A_2}$. We therefore regard c_{Φ_0} as a constant depending on the geometry and the inflow boundary data u_A.

We now use the constant c_{Φ_0} to show that the point $\Phi(s,t)$ is in a small neighborhood of the reference point $\Phi_0^+(s,t)$. It is important to use the reference point $\Phi_0^+(s,t)$ as it corresponds to the solution of the Laplace equation which is determined as soon as the problem is defined. Using this observation, we restrict the distance of any two streamline functions with vector fields $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ that are evaluated at the same point. Most importantly, we find a qualitative result for the line segment connecting $\Phi(s,t)$ and $\tilde{\Phi}(s,t)$. The tool to obtain the following result is to choose the parameter δ_3 in $W(M, \delta_1, \delta_2, \delta_3)$ appropriately. In the choice, we will also include the assumption of Lemma 6.37 that δ_3 has to be smaller than δ_1 in order to obtain the convexity of the set $W(M, \delta_1, \delta_2, \delta_3)$. δ_1 is not relevant for the following proof but it will be used in Theorem 6.57 to show that $L \circ T$ is a selfmap on $W(M, \delta_1, \delta_2, \delta_3)$.

Lemma 6.52. *Let Ω be a $C^{2,\alpha}$ domain and Ω^+ as in Lemma 6.49. Let $E_0 \in W(M, \delta_1, \delta_2, \delta_3)$ be the gradient of the solution to the Laplace equation (6.5a)-(6.5b). Let $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ and $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ the extension to $\tilde{E}$. Moreover, set*

$$\delta_3 = \min\left\{\delta_1, \frac{c_{\Phi_0}}{4\exp\left(c_{mv} c_l \frac{c(\Omega,\Omega',U_{2\epsilon}(\Gamma^+),\delta_1)(M+\|u_A\|_{0,\Gamma})}{\epsilon^{2+\alpha}}\right)}\right\}. \tag{6.73}$$

with $c(\Omega, \Omega', U_{2\epsilon}(\Gamma^+), \delta_1)$ defined in (6.68) and c_l defined in Theorem 6.27. Then follows for the line segment

$$z(\tau) := \Phi(s,t) + \tau(\tilde{\Phi}^+(s,t) - \Phi(s,t)), \qquad \tau \in [0,1], (s,t) \in Q \tag{6.74}$$

that

$$z(\tau) \subset \Omega^+ \tag{6.75}$$

for $\tau \in [0,1]$ and all $(s,t) \in Q$.

Proof. Due to the properties of the set $W(M, \delta_1, \delta_2, \delta_3)$, both vector fields E and $\tilde{E}$ are decomposed into $E = E_0 + E_1$ and $\tilde{E} = E_0 + \tilde{E}_1$. We show that the perturbations by E_1 and $\tilde{E}_1$ are small and that every two points $\Phi(s,t)$ and $\tilde{\Phi}^+(s,t)$ are found in a ball of radius $\frac{\operatorname{dist}(\Phi_0^+(s,t),\Gamma_-)}{2}$. First, however, we will show that every point $\Phi(s,t)$ is found in a ball of radius $\frac{\operatorname{dist}(\Phi_0^+(s,t),\Gamma_-)}{4}$ around $\Phi_0^+(s,t)$. This also indicates that the shortest distance of a point $\Phi(s,t)$ to the inflow boundary is $\frac{3\operatorname{dist}(\Phi_0^+(s,t),\Gamma_-)}{4}$.

Let us begin with the mutual distance of $\Phi(s,t)$ and $\Phi_0^+(s,t)$. Extend E_0 to

$E_0^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$. We obtain with Lemma 6.51

$$\begin{aligned}
|\Phi(s,t) - \Phi_0^+(s,t)|_\infty &\le \int_0^s |E(\Phi(\mu,t)) - E_0(\Phi(\mu,t))|_\infty \exp\left(c_l c_{mv} \|\nabla E_0^+\|_{0,\Omega^+}\right) \, d\mu \\
&\le s\|E(\Phi) - E_0(\Phi)\|_{0,Q} \exp\left(c_l \, c_{mv} \|\nabla E_0^+\|_{0,\Omega^+}\right) \\
&\le s\|E - E_0\|_{0,\Omega} \exp\left(c_l \, c_{mv} \|\nabla E_0^+\|_{0,\Omega^+}\right) \\
&\le s\|E_1\|_{0,\Omega} \exp\left(c_l \, c_{mv} \|\nabla E^+\|_{0,\Omega^+}\right).
\end{aligned}$$

By the choice of δ_3 and since $E_0^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$, we get pointwise for every $(s,t) \in Q$

$$\begin{aligned}
|\Phi(s,t) - \Phi_0^+(s,t)|_\infty &\le s\,\delta_3 \exp\left(c_{mv} c_l \frac{c(\Omega, \Omega', U_{2\epsilon}(\Gamma^+), \delta_1)(M + \|u_A\|_{0,\Gamma})}{\epsilon^{2+\alpha}}\right) \\
&\le \frac{s c_{\Phi_0}}{4} \\
&\le \frac{\mathrm{dist}(\Phi_0^+(s,t), \Gamma_-)}{4}.
\end{aligned} \tag{6.76}$$

For $s = 0$, the distance between $\Phi(0,t)$ and $\Phi_0^+(0,t)$ is thus 0. The distance between two points $\Phi(s,t)$ and $\Phi_0(s,t)$ increases in s. However, the point $\Phi(s,t)$ is always contained in a ball of radius $\frac{\mathrm{dist}\left\{\Phi_0^+(s,t),\Gamma_-\right\}}{4}$ around $\Phi_0^+(s,t)$ which, due to its size, can not intersect Ω_-. Conclusively, the inflow boundary can not be intersected by the line segment connectig $\Phi(s,t)$ and $\Phi_0^+(s,t)$, i.e. $\overrightarrow{\Phi(s,t)\Phi_0^+(s,t)} \subset \Omega$.

Now, we will find the distance of two points Φ and $\tilde{\Phi}^+$. We compare the streamline functions Φ and $\tilde{\Phi}^+$ directly by Lemma 6.51 and obtain for $(s,t) \in Q$

$$\begin{aligned}
|\tilde{\Phi}^+(s,t) - \Phi(s,t)|_\infty &\le \int_0^s |\tilde{E}^+(\Phi(\mu,t)) - E(\Phi(\mu,t))|_\infty \exp\left(c_l c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \, d\mu \\
&\le s\|\tilde{E} - E\|_{0,\Omega} \exp\left(c_l \; c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \\
&\le 2s\delta_3 \exp\left(c_l c_{mv} \frac{c(\Omega, \Omega', U_{2\epsilon}(\Gamma^+), \delta_1)(M + \|u_A\|_{0,\Gamma})}{\epsilon^{2+\alpha}}\right) \\
&\le s\frac{c_{\Phi_0}}{2} \\
&\le \frac{\mathrm{dist}(\Phi_0^+(s,t), \Gamma_-)}{2}.
\end{aligned} \tag{6.77}$$

By (6.76), the point $\Phi(s,t)$ lays in a ball of radius $\frac{\mathrm{dist}(\Phi_0(s,t),\Gamma_-)}{4}$ around $\Phi_0^+(s,t)$. The shortest distance of $\Phi(s,t)$ to the inflow boundary is thus $\frac{3\,\mathrm{dist}(\Phi_0(s,t),\Gamma_-)}{4}$. By (6.77), $\tilde{\Phi}^+(s,t)$ lays in a ball of radius $\frac{\mathrm{dist}(\Phi_0(s,t),\Gamma_-)}{2}$ around $\Phi(s,t)$. Thus the inflow boundary can not be intersected by the line segment $\overrightarrow{\Phi(s,t), \tilde{\Phi}(s,t)}$, i.e. $\overrightarrow{\Phi(s,t), \tilde{\Phi}(s,t)} \subset \Omega^+$ The assertion is proved.

$\square$

The relevance of the previous result will get clear in the following Lemma. We proceed with bounding $|\Phi^{-1} - \tilde{\Phi}^{-1}|_{\alpha,\Omega}$.

Lemma 6.53. *Let Ω be a $C^{2,\alpha}$ domain and $\Omega^+ \supset \Omega$ as defined in Definition 6.43. Let $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ and $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ the extension to $\tilde{E}$. Let δ_3 be as in Lemma 6.52. Then holds for the corresponding streamline functions*

$$|\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega} \leq |\nabla\tilde{\Phi}_+^{-1}|_{\alpha,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha\|\tilde{\Phi}^+ - \Phi\|_{0,Q} \\ + c_{mv}\left\|\nabla\tilde{\Phi}_+^{-1}\right\|_{0,\Omega^+}\|\nabla\Phi^{-1}\|_{0,\Omega}|\tilde{\Phi}^+ - \Phi|_{\alpha,Q}. \tag{6.78}$$

Proof. The Hölder semi-norm is given by

$$|\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x) - \tilde{\Phi}^{-1}(y) + \Phi^{-1}(y)|_\infty}{|x-y|_\infty^\alpha}. \tag{6.79}$$

The streamline functions Φ^{-1} and $\tilde{\Phi}^{-1}$ map Ω into the parameter sets Q and $\tilde{Q}$. Extend the vector field $\tilde{E}$ to $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ and obtain the extended streamlines $\tilde{\Phi}^+(s,t)$ with inverse mapping $\tilde{\Phi}_+^{-1}$. Then holds the identity $\tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) = \Phi^{-1}(x)$ for every $x \in \Omega$. For every $x, y \in \Omega$, we get for the numerator of (6.79)

$$\begin{aligned} &\left|\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x) - \tilde{\Phi}^{-1}(y) + \Phi^{-1}(y)\right|_\infty \\ &= \left|\tilde{\Phi}^{-1}(x) - \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) - \tilde{\Phi}^{-1}(y) + \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(y)))\right|_\infty \\ &= \left|\tilde{\Phi}_+^{-1}(x) - \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) - \tilde{\Phi}_+^{-1}(y) + \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(y)))\right|_\infty. \end{aligned} \tag{6.80}$$

The line segments $z_1(\tau) = x + \tau(\tilde{\Phi}^+(\Phi^{-1}(x)) - x)$ and $z_2(\tau) = y + \tau(\tilde{\Phi}^+(\Phi^{-1}(y)) - y)$, $\tau \in [0,1]$ are fully contained in Ω^+ due to the choice of δ_3 and Lemma 6.52. We may apply Theorem 2.15 and get

$$\begin{aligned} &\left|\tilde{\Phi}_+^{-1}(x) - \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(x))) - \tilde{\Phi}_+^{-1}(y) + \tilde{\Phi}_+^{-1}(\tilde{\Phi}^+(\Phi^{-1}(y)))\right|_\infty \\ &= \left|\int_0^1 \nabla\tilde{\Phi}_+^{-1}(z_1(\tau))\,d\tau\left[\tilde{\Phi}^+(\Phi^{-1}(x)) - x\right] - \int_0^1 \nabla\tilde{\Phi}^{-1}(z_2(\tau))\,d\tau\left[\tilde{\Phi}^+(\Phi^{-1}(y)) - y\right]\right|_\infty \\ &\leq \left|\int_0^1 \nabla\tilde{\Phi}_+^{-1}(z_1(\tau)) - \nabla\tilde{\Phi}_+^{-1}(z_2(\tau))\,d\tau\left[\tilde{\Phi}^+(\Phi^{-1}(x)) - x\right]\right|_\infty \\ &\quad + \left|\int_0^1 \nabla\tilde{\Phi}_+^{-1}(z_1(\tau))\,dt\left[\tilde{\Phi}^+(\Phi^{-1}(x)) - x - \tilde{\Phi}^+(\Phi^{-1}(y)) + y\right]\right|_\infty \\ &\leq \int_0^1 \left|\nabla\tilde{\Phi}_+^{-1}(z_1(\tau)) - \nabla\tilde{\Phi}_+^{-1}(z_2(\tau))\right|_\infty d\tau\left|\tilde{\Phi}^+(\Phi^{-1}(x)) - x\right|_\infty \\ &\quad + \int_0^1 \left|\nabla\tilde{\Phi}_+^{-1}(z_1(\tau))\right|_\infty d\tau\left|\tilde{\Phi}^+(\Phi^{-1}(x)) - x - \tilde{\Phi}^+(\Phi^{-1}(y)) + y\right|_\infty. \end{aligned} \tag{6.81}$$

Since $\nabla\tilde{\Phi}_+^{-1} \in C^\alpha(\Omega^+)$, it holds with Lemma 2.16

$$\int_0^1 \left|\nabla\tilde{\Phi}_+^{-1}(z_1(t)) - \nabla\tilde{\Phi}_+^{-1}(z_2(t))\right| dt \leq |\nabla\tilde{\Phi}_+^{-1}|_{\alpha,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha\,|x-y|_\infty^\alpha.$$

Substituting the latter equation into (6.81) and using the identity $x = \Phi(\Phi^{-1}(x))$, we obtain for (6.80)

$$\begin{aligned}
&\left|\tilde{\Phi}^{-1}(x) - \Phi^{-1}(x) - \tilde{\Phi}^{-1}(y) + \Phi^{-1}(y)\right|_\infty \\
&\quad \leq |\nabla\tilde{\Phi}_+^{-1}|_{\alpha,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha\,|x-y|_\infty^\alpha\left|\tilde{\Phi}^+(\Phi^{-1}(x)) - \Phi(\Phi^{-1}(x))\right|_\infty \\
&\qquad + \left\|\nabla\tilde{\Phi}_+^{-1}\right\|_{0,\Omega^+}\left|\tilde{\Phi}^+(\Phi^{-1}(y)) - \Phi(\Phi^{-1}(y)) - \tilde{\Phi}^+(\Phi^{-1}(x)) + \Phi(\Phi^{-1}(x))\right|_\infty.
\end{aligned}$$

Eventually, we obtain for (6.79) by the chain rule (2.9) and Lemma 2.20

$$\begin{aligned}
|\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega} &\leq |\nabla\tilde{\Phi}_+^{-1}|_{\alpha,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha\|\tilde{\Phi}^+(\Phi^{-1}) - \Phi(\Phi^{-1})\|_{0,\Omega} \\
&\quad + \|\nabla\tilde{\Phi}_+^{-1}\|_{0,\Omega^+}\left|\tilde{\Phi}^+ - \Phi\right|_{\alpha,Q}\sup_{x,y\in\Omega}\frac{|\Phi^{-1}(x) - \Phi^{-1}(y)|^\alpha}{|x-y|^\alpha} \\
&\leq |\nabla\tilde{\Phi}_+^{-1}|_{\alpha,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha\|\tilde{\Phi}^+ - \Phi\|_{0,Q} \\
&\quad + c_{mv}\left\|\nabla\Phi_+^{-1}\right\|_{0,\Omega^+}\|\nabla\Phi^{-1}\|_{0,\Omega}|\tilde{\Phi}^+ - \Phi|_{\alpha,Q}.
\end{aligned}$$

□

With the same technique as in the previous Lemma, we now bound the Hölder coefficient of $|\Phi - \tilde{\Phi}|_{\alpha,Q}$

Lemma 6.54. *Let Ω be a $C^{2,\alpha}$ domain. Let E and $\tilde{E} \in W(M,\delta_1,\delta_2,\delta_3)$ and $\tilde{E}^+ \in W^+(M,\delta_1,\delta_2,\delta_3,\epsilon,c^+)$ the extension to $\tilde{E}$. Let δ_3 be chosen as in Lemma 6.52. Then holds for the corresponding streamline functions*

$$|\Phi - \tilde{\Phi}^+|_{\alpha,Q} \leq (c_1 + c_2)\|\nabla E - \nabla\tilde{E}\|_{1,\Omega}$$

with

$$c_1 = c_{mv}c_l L_{\Gamma_-}^{1-\alpha}\exp\left(c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)|\partial_t\Phi|_{0,Q}$$

and

$$\begin{aligned}
c_2 = {} & |\nabla\tilde{E}^+|_{\alpha,\Omega^+}\,c_l^2(\|\nabla\Phi\|_{0,Q} + \|\nabla\tilde{\Phi}^+\|_{0,Q})^\alpha\exp\left((1+c_{mv})c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right) \\
& + c_l^{1-\alpha}\left(1 + c_{mv}\|\nabla\tilde{E}^+\|_{0,\Omega^+}\ c_l\exp\left(c_{mv}\ c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)\right)
\end{aligned}$$

with c_l defined in Theorem 6.27.

Proof. The α- semi norm is given by

$$|\Phi - \tilde{\Phi}^+|_{\alpha,Q} = \sup_{(s_1,t_1),(s_2,t_2)}\frac{|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)|_\infty}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha}. \tag{6.82}$$

In the following, we show first a pointwise bound for the right-hand side of (6.82). We need to distinguish two cases.

Case 1: $l(t_1) < l(t_2)$

Then $(s_1, t_2) \in Q$. It holds for every $(s_1, t_1), (s_2, t_2) \in Q$ with $l(t_1) < l(t_2)$

$$\begin{aligned}
&\frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\
&= \frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_1,t_2) + \tilde{\Phi}^+(s_1,t_2) + \Phi(s_1,t_2) - \tilde{\Phi}^+(s_1,t_2) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\
&\leq \frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_1,t_2) + \tilde{\Phi}^+(s_1,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \qquad (6.83) \\
&\quad + \frac{\left|\Phi(s_1,t_2) - \tilde{\Phi}^+(s_1,t_2) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}. \qquad (6.84)
\end{aligned}$$

Let us first examine the numerator of (6.83). Since $\Phi : Q \to \Omega$ and $\Omega \subset \Omega^+$, $\tilde{E}^+(\Phi(s,t))$ is defined for $(s,t) \in Q$ into the domain of definition of $\tilde{E}^+$. We obtain by the triangle inequality

$$\begin{aligned}
&\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_1,t_2) + \tilde{\Phi}^+(s_1,t_2)\right|_\infty \qquad (6.85) \\
&= \left|\int_0^{s_1} E(\Phi(\mu,t_1)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t_1)) - E(\Phi(\mu,t_2)) + \tilde{E}^+(\tilde{\Phi}^+(\mu,t_2))\,d\mu\right|_\infty \\
&\leq \int_0^{s_1} \Big|E(\Phi(\mu,t_1)) - \tilde{E}^+(\Phi(\mu,t_1)) + \tilde{E}^+(\Phi(\mu,t_1)) - \tilde{E}(\tilde{\Phi}(\mu,t_1)) \\
&\qquad - E(\Phi(\mu,t_2)) - \tilde{E}^+(\Phi(\mu,t_2)) + \tilde{E}^+(\Phi(\mu,t_2)) + \tilde{E}(\tilde{\Phi}(\mu,t_2))\Big|_\infty\,d\mu \\
&\leq \int_0^{s_1} \left|E(\Phi(\mu,t_1)) - \tilde{E}^+(\Phi(\mu,t_1)) - E(\Phi(\mu,t_2)) + \tilde{E}^+(\Phi(\mu,t_2))\right|_\infty\,d\mu \qquad (6.86) \\
&\quad + \int_0^{s_1} \left|\tilde{E}^+(\Phi(\mu,t_1)) - \tilde{E}^+(\tilde{\Phi}^+(\mu,t_1)) - \tilde{E}^+(\Phi(\mu,t_2)) + \tilde{E}^+(\tilde{\Phi}^+(\mu,t_2))\right|_\infty\,d\mu. \qquad (6.87)
\end{aligned}$$

Due to the choice of δ_3 and Lemma 6.52 holds for the line segment
$z(\tau) = \Phi(s,t) + \tau(\tilde{\Phi}^+(s,t) - \Phi(s,t)) \subset \Omega^+$, $\tau \in [0,1]$. Theorem 2.15 may thus be applied. Define the matrix

$$B(\mu,t_1) = \int_0^1 \nabla\tilde{E}^+(\Phi(\mu,t_1) + \tau(\tilde{\Phi}^+(\mu,t_1) - \Phi(\mu,t_1))\,d\tau.$$

Let us omit the variable μ of integration for a better overview. We get with Theorem 2.15 for (6.87)

$$\begin{aligned}
&\int_0^{s_1} \left|\tilde{E}^+(\Phi(t_1)) - \tilde{E}^+(\tilde{\Phi}^+(t_1)) - \tilde{E}^+(\Phi(t_2)) + \tilde{E}^+(\tilde{\Phi}^+(t_2))\right|_\infty\,d\mu \\
&= \int_0^{s_1} \left|B(t_1)\cdot\left[\Phi(t_1) - \tilde{\Phi}^+(t_1)\right] - B(t_2)\cdot\left[\Phi(t_2) - \tilde{\Phi}^+(t_2)\right]\right|_\infty\,d\mu \\
&\leq \int_0^{s_1} \left|B(t_1)\cdot\left[\Phi(t_1) - \tilde{\Phi}^+(t_1) - \Phi(t_2) + \tilde{\Phi}^+(t_2)\right]\right|_\infty + \left|[B(t_1) - B(t_2)]\cdot\left[\Phi(t_2) - \tilde{\Phi}^+(t_2)\right]\right|_\infty\,d\mu \\
&\leq \int_0^{s_1} |B(t_1)|_\infty \left|\Phi(t_1) - \tilde{\Phi}^+(t_1) - \Phi(t_2) + \tilde{\Phi}^+(t_2)\right|_\infty + |B(t_1) - B(t_2)|_\infty \left|\Phi(t_2) - \tilde{\Phi}^+(t_2)\right|_\infty\,d\mu.
\end{aligned}$$

Substituting the previous equation into (6.87), we have for (6.85)

$$\begin{aligned}
&\left|\Phi(s_1,t_1)-\tilde{\Phi}^+(s_1,t_1)-\Phi(s_1,t_2)+\tilde{\Phi}^+(s_1,t_2)\right|_\infty \\
&\quad\le \int_0^{s_1} |B(t_1)|_\infty \left|\Phi(t_1)-\tilde{\Phi}^+(t_1)-\Phi(t_2)+\tilde{\Phi}^+(t_2)\right|_\infty + |B(t_1)-B(t_2)|_\infty \left|\Phi(t_2)-\tilde{\Phi}^+(t_2)\right|_\infty d\mu \\
&\qquad + \int_0^{s_1} \left|E(\Phi(\mu,t_1))-\tilde{E}^+(\Phi(\mu,t_1))-E(\Phi(\mu,t_2))+\tilde{E}^+(\Phi(\mu,t_2))\right|_\infty d\mu.
\end{aligned}$$

With Grönwall's inequality (Lemma 6.19), we get

$$\begin{aligned}
&\left|\Phi(s_1,t_1)-\tilde{\Phi}^+(s_1,t_1)-\Phi(s_1,t_2)+\tilde{\Phi}^+(s_1,t_2)\right|_\infty \\
&\quad\le \int_0^{s_1} \left|E(\Phi(\mu,t_1))-\tilde{E}^+(\Phi(\mu,t_1))-E(\Phi(\mu,t_2))+\tilde{E}^+(\Phi(\mu,t_2))\right|_\infty \exp\left(\int_\mu^{s_1}|B(r,t_1)|_\infty\,dr\right) d\mu \\
&\qquad + \int_0^{s_1} |B(t_1)-B(t_2)|_\infty|\Phi(t_2)-\tilde{\Phi}^+(t_2)|_\infty \exp\left(\int_\mu^{s_1}|B(r,t_1)|_\infty\,dr\right) d\mu.
\end{aligned}$$

Since $|(s_1,t_1)-(s_2,t_2)|_\infty \ge |t_1-t_2|_\infty$, we get for (6.83) with the previous equation

$$\begin{aligned}
&\frac{\left|\Phi(s_1,t_1)-\tilde{\Phi}^+(s_1,t_1)-\Phi(s_1,t_2)+\tilde{\Phi}^+(s_1,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha} \\
&\le \frac{1}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}\int_0^{s_1} \left|E(\Phi(t_1))-\tilde{E}^+(\Phi(t_1))-E(\Phi(t_2))+\tilde{E}^+(\Phi(t_2))\right|_\infty \exp\left(\int_\mu^{s_1}|B(t_1)|_\infty\,dr\right) d\mu \\
&\quad + \frac{1}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}\int_0^{s_1} |B(t_1)-B(t_2)|_\infty|\Phi(t_2)-\tilde{\Phi}^+(t_2)|_\infty \exp\left(\int_\mu^{s_1}|B(r,t_1)|_\infty\,dr\right) d\mu \\
&\le \int_0^{s_1} \frac{\left|E(\Phi(t_1))-\tilde{E}^+(\Phi(t_1))-E(\Phi(t_2))+\tilde{E}^+(\Phi(t_2))\right|_\infty}{|t_1-t_2|_\infty^\alpha} \exp\left(\int_\mu^{s_1}|B(r)|_\infty\,dr\right) d\mu \qquad (6.88)\\
&\quad + \int_0^{s_1} \frac{|B(t_1)-B(t_2)|_\infty}{|t_1-t_2|_\infty}|\Phi(t_2)-\tilde{\Phi}(t_2)|_\infty \exp\left(\int_\mu^{s_1}|B(r,t_1)|_\infty\,dr\right) d\mu. \qquad (6.89)
\end{aligned}$$

It holds for (6.88) by the chain rule (2.9), Lemma 2.20 and Theorem 6.27

$$\begin{aligned}
&\int_0^{s_1} \frac{\left|E(\Phi(t_1))-\tilde{E}^+(\Phi(t_1))-E(\Phi(t_2))+\tilde{E}^+(\Phi(t_2))\right|_\infty}{|t_1-t_2|_\infty^\alpha} \exp\left(\int_\mu^{s_1}|B(r)|_\infty\,dr\right) d\mu \\
&\quad\le c_{mv}\int_0^{s_1} \left\|\nabla E-\nabla\tilde{E}^+\right\|_{0,\Omega} \frac{|\Phi(t_1)-\Phi(t_2)|_\infty}{|t_1-t_2|_\infty^\alpha}\cdot\exp\left(\int_\mu^{s_1}|B(r)|_\infty\;dr\right) d\mu \\
&\quad\le c_{mv}c_l\left\|\partial_t\Phi\right\|_{0,Q}|t_2-t_1|^{1-\alpha}\exp\left(c_l\left\|\nabla\tilde{E}^+\right\|_{0,\Omega^+}\right)\left\|\nabla E-\nabla\tilde{E}\right\|_{0,\Omega}. \qquad (6.90)
\end{aligned}$$

We use Lemma 2.16 to bound $|B(\mu,t_1)-B(\mu,t_2)|$ in (6.89). It holds

$$|B(\mu,t_1)-B(\mu,t_2)|_\infty \le |\nabla\tilde{E}^+|_{\alpha,\Omega^+}\left(\|\nabla\Phi\|_{0,Q}+\|\nabla\tilde{\Phi}^+\|_{0,Q}\right)^\alpha|t_1-t_2|_\infty^\alpha.$$

Hence we get for (6.89) by Theorem 6.27

$$\begin{aligned}
&\int_0^{s_1} \frac{|B(t_1)-B(t_2)|_\infty}{|t_1-t_2|_\infty^\alpha}|\Phi(t_2)-\tilde{\Phi}^+(t_2)|_\infty \exp\left(\int_\mu^{s_1}|B(r,t_1)|_\infty\,dr\right) d\mu \\
&\quad\le c_l|\nabla\tilde{E}^+|_{\alpha,\Omega^+}\left(\|\nabla\Phi\|_{0,Q}+\|\nabla\tilde{\Phi}^+\|_{0,Q}\right)^\alpha \exp\left(c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)\|\Phi-\tilde{\Phi}^+\|_{0,Q}.
\end{aligned}$$

We bound $\|\Phi - \tilde{\Phi}^+\|_{0,Q}$ by Lemma 6.51 and obtain for (6.89)

$$
\begin{aligned}
&\int_0^{s_1} \frac{|B(t_1) - B(t_2)|_\infty}{|t_1 - t_2|_\infty} |\Phi(t_2) - \tilde{\Phi}(t_2)|_\infty \exp\left(\int_\mu^{s_1} |B(r,t_1)|_\infty \, dr\right) d\mu \\
&\leq |\nabla \tilde{E}^+|_{\alpha,\Omega^+} c_l (\|\nabla \Phi\|_{0,Q} + \|\nabla \tilde{\Phi}^+\|_{0,Q})^\alpha \exp\left(c_{mv} c_l \|\nabla E^+\|_{0,\Omega^+}\right) \\
&\qquad \cdot c_l \exp\left(c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \|E - \tilde{E}\|_{0,\Omega} \\
&\leq |\nabla \tilde{E}^+|_{\alpha,\Omega^+} \, c_l^2 (\|\nabla \Phi\|_{0,Q} + \|\nabla \tilde{\Phi}^+\|_{0,Q})^\alpha \exp\left((1 + c_{mv}) c_l \|\nabla E^+\|_{0,\Omega^+}\right) \|E - \tilde{E}\|_{0,\Omega}. \qquad (6.91)
\end{aligned}
$$

Let us proceed with (6.84). We may apply the mean value theorem in s, since $[s_1, s_2] \subset [0, l(t_2)]$. We have for the numerator by (6.9a) and the triangle inequality

$$
\begin{aligned}
&\left|\Phi(s_1,t_2) - \tilde{\Phi}^+(s_1,t_2) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty \\
&\quad \leq \sup_{s\in[s_1,s_2]} \left|\partial_s \Phi(s,t_2) - \partial_s \tilde{\Phi}^+(s,t_2)\right|_\infty |s_1 - s_2| \\
&\quad \leq \sup_{s\in[s_1,s_2]} \left|E(\Phi(s,t_2)) - \tilde{E}^+(\tilde{\Phi}^+(s,t_2))\right|_\infty |s_1 - s_2| \\
&\quad \leq \sup_{s\in[s_1,s_2]} \left(\left|E(\Phi(s,t_2)) - \tilde{E}^+(\Phi(s,t_2))\right|_\infty + \left|\tilde{E}^+(\Phi(s,t_2)) - \tilde{E}^+(\tilde{\Phi}^+(s,t_2))\right|_\infty\right) |s_1 - s_2| \,.
\end{aligned}
$$

With Lemma 2.20 and Lemma 6.51, we get

$$
\begin{aligned}
&|\Phi(s_1,t_2) - \tilde{\Phi}^+(s_1,t_2) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)|_\infty \\
&\quad \leq \left[\|E - \tilde{E}^+\|_{0,\Omega} + c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+} |\Phi(s,t) - \tilde{\Phi}^+(s,t)|_\infty\right] |s_1 - s_2| \\
&\quad \leq \left[\|E - \tilde{E}^+\|_{0,\Omega} + c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \|E - \tilde{E}\|_{0,\Omega}\right] |s_1 - s_2| \\
&\quad \leq \left[1 + c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right)\right] \|E - \tilde{E}\|_{0,\Omega} \, |s_1 - s_2| \,.
\end{aligned}
$$

Hence, (6.84) is bounded pointwise for every $(s_1,t_1), (s_2,t_2) \in Q$

$$
\begin{aligned}
&\frac{\left|\Phi(s_1,t_2) - \tilde{\Phi}^+(s_1,t_2) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1) - (s_2,t_2)|_\infty^\alpha} \\
&\quad \leq \left(1 + c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla \tilde{E}^+(x)\|_{0,\Omega^+}\right)\right) \|E - \tilde{E}\|_{0,\Omega} \, |s_1 - s_2|^{1-\alpha} \\
&\quad \leq c_l^{1-\alpha} \left(1 + c_{mv} \|\nabla \tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla \tilde{E}^+(x)\|_{0,\Omega^+}\right)\right) \|E - \tilde{E}\|_{0,\Omega}. \qquad (6.92)
\end{aligned}
$$

Case 2: $l(t_2) < l(t_2)$

Then the point $(s_2, t_1) \in Q$ and it holds for all $(s_1, t_2), (s_2, t_2)$ with $l(t_2) < l(t_1)$

$$\frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}$$

$$= \frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_2,t_1) + \tilde{\Phi}^+(s_2,t_1) + \Phi(s_1,t_2) - \tilde{\Phi}^+(s_2,t_1) - \Phi(s_2,t_1) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}$$

$$\le \frac{\left|\Phi(s_1,t_1) - \tilde{\Phi}^+(s_1,t_1) - \Phi(s_2,t_1) + \tilde{\Phi}^+(s_2,t_1)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}$$

$$+ \frac{\left|\Phi(s_2,t_1) - \tilde{\Phi}^+(s_2,t_1) - \Phi(s_2,t_2) + \tilde{\Phi}^+(s_2,t_2)\right|_\infty}{|(s_1,t_1)-(s_2,t_2)|_\infty^\alpha}.$$

The calculations are analogous to the ones in case 1 and lead to the same result.

To conclude, we get for (6.82) with (6.90), (6.91) and (6.92)

$$\begin{aligned}|\Phi - \tilde{\Phi}^+|_{\alpha,Q} &\le c_{mv} c_l \|\partial_t \Phi\|_{0,Q} L_{\Gamma_-}^{1-\alpha} \exp\left(c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \|\nabla E - \nabla \tilde{E}\|_{0,\Omega} \\ &+ |\nabla \tilde{E}^+|_{\alpha,\Omega^+}\, c_l^2 (2\|\nabla\Phi\|_{0,Q} + \|\nabla \tilde{\Phi}^+\|_{0,Q})^\alpha \exp\left((1+c_{mv}) c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right) \|E - \tilde{E}\|_{0,\Omega} \\ &+ c_l^{1-\alpha}\left[1 + c_{mv}\|\nabla \tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right)\right] \|E - \tilde{E}\|_{0,\Omega}.\end{aligned}$$

□

6.3.3 Continuity of T

We are now collecting the previous results to prove that $T : W(M, \delta_1, \delta_2, \delta_3) \to C^{1,\alpha}(\bar{\Omega})$ is continuous in E.

Theorem 6.55. *Let Ω be a $C^{2,\alpha}$ domain and Ω^+ chosen as in Lemma 6.49. Let $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$ with $\|\rho_A\|_{1,I_{\Gamma_-}} < 1$ and $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$ with extension $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ to $\tilde{E}$. Let δ_3 be chosen as in Lemma 6.52. Then holds for the operator $T : W(M, \delta_1, \delta_2, \delta_3) \to C^{1,\alpha}(\bar{\Omega})$*

$$\|TE - T\tilde{E}\|_{\alpha,\Omega} \le C_T \|\rho_A\|_{1,\alpha,I_{\Gamma_-}} \|E - \tilde{E}\|_{1,\Omega}.$$

with

$$C_T = \left(2c_l + 2c_l L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\} + 1\right) \max\{c_1, c_2\}, \tag{6.93}$$

and

$$c_1 = c_{mv} c_l \|\nabla\Phi\|_{0,Q} L_{\Gamma_-}^{1-\alpha} \exp\left(c_l \|\nabla \tilde{E}^+\|_{0,\Omega^+}\right)$$

and

$$\begin{aligned}c_2 = c_{mv} &\left\|\nabla\Phi_+^{-1}\right\|_{0,\Omega^+} \|\nabla\Phi^{-1}\|_{0,\Omega} \left[c_l^{1-\alpha}\left(1 + c_{mv}\|\nabla\tilde{E}^+\|_{0,\Omega^+} c_l \exp\left(c_{mv} c_l \|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)\right)\right. \\ &\left.+ |\nabla\tilde{E}^+|_{\alpha,\Omega^+}\, c_l^2 (2\|\nabla\Phi\|_{0,Q} + \|\nabla\tilde{\Phi}^+\|_{0,Q})^\alpha \exp\left((1+c_{mv}) c_l \|\nabla E^+\|_{0,\Omega^+}\right)\right] \\ &+ \|\nabla\Phi_+^{-1}\|_{0,\Omega^+} (2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^\alpha c_l \exp\left(c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right).\end{aligned}$$

Proof. By Lemma 6.40, we have

$$\|TE - T\tilde{E}\|_{\alpha,\Omega} \leq \|\nabla\hat{\rho}\|_{\alpha,Q\cup\tilde{Q}} \left(1 + (2c_{mv}\|\nabla\Phi^{-1}\|_{0,\Omega} + c_{mv}\|\nabla\tilde{\Phi}^{-1}\|_{0,\Omega})^{\alpha}\right) \left\|\tilde{\Phi}^{-1} - \Phi^{-1}\right\|_{\alpha,\Omega}.$$

By Lemma 6.41 and Lemma 6.42, we have

$$\begin{aligned}\|\nabla\hat{\rho}\|_{\alpha,\Omega} &\leq \|\rho_A\|_{1,I_{\Gamma_-}} + \left(2c_l + 2c_l L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\}\right)\|\rho_A\|_{1,\alpha,I_{\Gamma_-}} \\ &\leq \left(2c_l + 2c_l L_{\Gamma_-}^{1-\alpha} + 2\max\left\{L_{\Gamma_-}^{1-\alpha}, 1\right\} + 1\right)\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}.\end{aligned}$$

Lemmas 6.50 and 6.51 give

$$\|\Phi^{-1} - \tilde{\Phi}^{-1}\|_{0,\Omega} \leq c_{mv}\|\nabla\tilde{\Phi}_+^{-1}\|_{0,\Omega^+} c_l \exp\left(c_{mv}c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)\|E - \tilde{E}\|_{0,\Omega}.$$

Eventually, Lemma 6.53 and Lemma 6.54 lead to

$$|\tilde{\Phi}^{-1} - \Phi^{-1}|_{\alpha,\Omega} \leq c_1\|\nabla E - \nabla\tilde{E}\|_{0,\Omega} + c_2\|E - \tilde{E}\|_{0,\Omega}$$

with

$$c_1 = c_{mv}^2 \left\|\nabla\Phi_+^{-1}\right\|_{0,\Omega^+} \|\nabla\Phi^{-1}\|_{0,\Omega} c_l \left\|\partial_t\Phi\right\|_{0,Q} L_{\Gamma_-}^{1-\alpha}\exp\left(c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)$$

and

$$\begin{aligned}c_2 = c_{mv}\left\|\nabla\Phi_+^{-1}\right\|_{0,\Omega^+}\|\nabla\Phi^{-1}\|_{0,\Omega}\cdot &\left[c_l^{1-\alpha}\left(1 + c_{mv}\|\nabla\tilde{E}^+\|_{0,\Omega^+} c_l\exp\left(c_{mv}c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right)\right)\right. \\ &\left. + |\nabla\tilde{E}^+|_{\alpha,\Omega^+} c_l^2(2\|\nabla\Phi\|_{0,Q} + \|\nabla\tilde{\Phi}^+\|_{0,Q})^{\alpha}\exp\left((1 + c_{mv})c_l\|\nabla E^+\|_{0,\Omega^+}\right)\right] \\ &+ \|\nabla\Phi_+^{-1}\|_{0,\Omega^+}(2 + c_{mv}\|\nabla\tilde{\Phi}^+\|_{0,Q}\|\nabla\Phi^{-1}\|_{0,\Omega})^{\alpha} c_l\exp\left(c_l\|\nabla\tilde{E}^+\|_{0,\Omega^+}\right).\end{aligned}$$

□

C_T is bounded by a constant depending on the boundary data u_A, the streamline functions Φ and Φ^+ and the vector fields E and E^+. We will justify that this unclear constant is bounded.

Lemma 6.56. *Let Ω be a $C^{2,\alpha}$ domain and Ω^+ chosen as in Lemma 6.49. Let $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$ with $\|\rho_A\|_{1,Q} < 1$ and $E, \tilde{E} \in W(M, \delta_1, \delta_2, \delta_3)$. Further let $\tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ be the extension of $\tilde{E}$. Let δ_3 chosen as in Lemma 6.52. Then the constant C_T in Theorem 6.55 is bounded and depends only on the geometry of Ω and Ω^+, the constants $\delta_1, \delta_2, M, \epsilon, c^+$ and the boundary data u_A and c_v for the extended domain.*

Proof. By section 6.2.4, we know that $\|\nabla\Phi\|_{0,Q}$, $\|\nabla\Phi^{-1}\|_{0,\Omega}$ and c_l are bounded in terms of M, δ_1 and δ_2. Furthermore, c_{mv} and L_{Γ_-} depend on the geometry of Ω_-. Since $E^+, \tilde{E}^+ \in W^+(M, \delta_1, \delta_2, \delta_3, \epsilon, c^+)$ follows immediately

$$\|\nabla E^+\|_{\alpha,\Omega^+} \leq \max\left\{1, \frac{c(\Omega, \Omega', U_{2\epsilon}(\Gamma_+), \delta_1)}{\epsilon^{2,\alpha}}\right\}(M + \|u_A\|_{0,\Gamma}).$$

It is left to bound the extended streamline function. Herein, we will encounter the maximal streamline parameter l^+ for the extended vector sets E^+. We get by Lemma 6.27

$$\|l^+\|_{0,[0,L_{\Gamma\_}]} \leq \frac{|u_{A_1} - c_{v_2}|_\infty}{\min\left\{\frac{1}{2}, \min\left\{\frac{1}{8}, M\right\} \frac{c_p(\Omega^+)}{c_S(\Omega^+ \backslash \Omega, \alpha)}\right\} \delta_1^2}$$

where c_{v_2} are the constant boundary data on Γ_+^{ext} for the function v in the extension of u in Theorem 6.47.

By Lemma 6.31 follows

$$\|\nabla \tilde{\Phi}_+^{-1}\|_{0,\Omega^+} \leq C_1(\Omega, \Omega^+, M, \delta_1, \delta_2, \epsilon, \alpha, c^+, |u_{A_1} - c_{v_2}|)$$

and by Lemma 6.32 holds

$$|\nabla \Phi^{-1}|_{\alpha,\Omega} \leq C_2(\Omega, \Omega^+, M, \delta_1, \delta_2, \epsilon, \alpha, c^+, |u_{A_1} - c_{v_2}|)$$

All occurring constants are bounded. Thus the two constants c_1 and c_2 in Theorem 6.55 are bounded and conclusively so is the Lipschitz constant c_T.

□

6.4 Existence of a Solution

We will now consider the composite operator $L \circ T$ and show with an appropriate choice of the constant M and the inflow boundary function ρ_A that $L \circ T$ is a self map and contraction on the set $W(M, \delta_1, \delta_2, \delta_3)$. With the Banach fixed point Theorem, we then prove the existence and uniqueness of a fixed point of the composite operator $L \circ T$. Conclusively, there exists a classical solution to (CP 6.1).

Define the sequence

$$E^n(x) = L \circ T E^{n-1}(x), \quad n = 1, ...$$

with E^0 being an arbitrary element of $W(M, \delta_1, \delta_2, \delta_3)$.

Theorem 6.57. *Let Ω be a $C^{2,\alpha}$ domain and $\rho_A \in C^{1,\alpha}(I_{\Gamma\_})$. Let $M = c_S(\Omega, \alpha)(1 + \|u_A\|_{2,\alpha,\Gamma})$ with $c_S(\Omega, \alpha)$ defined in Theorem 6.7. Let*

$$\|\rho_A\|_{1,I_{\Gamma\_}} = \min\left\{\frac{1}{c_4(\Omega, c_l, \delta_2, M, \alpha)}, \frac{\delta_3}{c(\Omega)}\right\} \tag{6.94}$$

with

$$c_4(\Omega, c_l, \delta_2, \Omega, M, \alpha) := c(\Omega) \frac{\exp\left(c_l M\right)}{\delta_2} \max\left\{\exp\left(c_l M\right), M\right\} \max\left\{L_{\Gamma\_}^{1-\alpha}, c_l, L_{\Gamma\_}^{1-\alpha} c_l^{1-\alpha}\right\}$$

and $c(\Omega)$ being a constant depending on the domain.

Then $L \circ T$ defines a selfmap on the set $W(M, \delta_1, \delta_2, \delta_3)$.

Proof. By Theorem 6.39, we know that $TE \in C^{1,\alpha}(\bar{\Omega})$ and conclusively also in $C^{\alpha}(\bar{\Omega})$. Since Ω is a $C^{2,\alpha}$ domain and $u_A \in C^{2,\alpha}(\Gamma)$, Theorem 6.1 states that $L \circ TE_{n-1} \in C^{1,\alpha}(\bar{\Omega})$. The composite operator thus maps into function spaces of desired regularity.

We will now show that $L \circ TE^{n-1}$ fulfils the restrictions in $W(M, \delta_1, \delta_2, \delta_3)$. By the definition of L follows that $L \circ TE^{n-1} = E^n = -\nabla u^n$ is a gradient field. Let $\rho^n = TE^{n-1}$.

As presented in section 6.1, the Laplace operator Δ is linear and the solution to the Poisson equation (6.1a)-(6.1b) is decomposed into $u^n = u_0 + u_1^n$. u_0 is the solution of the Laplace equation

$$-\Delta u_0 = 0 \qquad x \in \Omega \tag{6.95a}$$

$$u_0 = u_A \qquad x \in \Gamma. \tag{6.95b}$$

u_0 is the same for every iteration, as it is independent of ρ^n and is only determined through an a priori given right hand side function u_A and the shape of Ω. u_1^n is the solution of the Poisson equation

$$-\Delta u_1^n = \rho^n \qquad x \in \Omega \tag{6.96a}$$

$$u_1^n = 0 \qquad x \in \Gamma \tag{6.96b}$$

For every iteration u_1^n changes until convergence is obtained. Let us denote L_1 as solution operator to (6.96a)-(6.96b). We thus obtain $-\nabla u^n = L \circ TE^{n-1} = E_0 + L_1 \circ TE^{n-1}$. By Lemma 6.3 follows that $E^0 \in C^{1,\alpha}(\bar{\Omega})$ and $L_1 \circ TE^{n-1} \in C^{1,\alpha}(\bar{\Omega})$.

Boundedness of E^n

Schauder's a priori estimate (Theorem 6.7) and Theorem 6.6 give

$$\begin{aligned} \|L \circ TE^{n-1}\|_{1,\alpha;\Omega} &\le \|u^n\|_{2,\alpha,\Omega} \\ &\le c_S(\Omega,\alpha)\left(\|u_A\|_{2,\alpha;\Gamma} + \|u^n\|_{0,\Omega} + \|TE^{n-1}\|_{\alpha,\Omega}\right) \\ &\le c_S(\Omega,\alpha)\left(\|u_A\|_{2,\alpha;\Gamma} + c_2(\Omega)\|TE^{n-1}\|_{\alpha,\Omega}\right). \end{aligned} \tag{6.97}$$

It holds for the Hölder norm

$$\|TE^{n-1}\|_{\alpha,\Omega} = \|TE^{n-1}\|_{0,\Omega} + |TE^{n-1}|_{\alpha,\Omega}. \tag{6.98}$$

The sup-norm of TE^{n-1} is bounded by the inflow boundary data. We have with Theorem 6.39

$$\begin{aligned} \left\|TE^{n-1}\right\|_{0,\Omega} &= \sup_{x\in\Omega}\left|\hat{\rho}(\Phi^{-1}(E^{n-1},x))\right|_{\infty} \\ &= \sup_{(s,t)\in Q}\left|\frac{\rho_A(t)}{1+s\rho_A(t)}\right|_{\infty} \le \|\rho_A\|_{0,I_{\Gamma_-}}. \end{aligned} \tag{6.99}$$

For the Hölder seminorm holds by the chain rule (2.9)

$$|TE^{n-1}|_{\alpha,\Omega} = |\hat{\rho}(\Phi^{-1}(E^{n-1},x))|_{\alpha,\Omega} \le |\hat{\rho}|_{\alpha,Q}\|\nabla\Phi^{-1}(E^{n-1})\|_{0,\Omega}^{\alpha}. \tag{6.100}$$

We have for the Hölder coefficient of $\hat{\rho}$

$$|\hat{\rho}|_{\alpha,Q} = \left|\frac{\rho_A(t)}{1+s\rho_A(t)}\right|_{\alpha,Q} \leq |\rho_A|_{\alpha,I_{\Gamma_-}} \left\|\frac{1}{1+s\rho_A(t)}\right\|_{0,Q} + \|\rho_A\|_{0,I_{\Gamma_-}} \left|\frac{1}{1+s\rho_A(t)}\right|_{\alpha,Q}$$
$$\leq |\rho_A|_{\alpha,I_{\Gamma_-}} + \|\rho_A\|_{0,I_{\Gamma_-}} \left|\frac{1}{1+s\rho_A(t)}\right|_{\alpha,Q}.$$

It is left to bound $\left|\frac{1}{1+s\rho_A(t)}\right|_{\alpha,Q}$. Since $s > 0$ and $\rho_A > 0$ follows

$$\left|\frac{1}{1+s_1\rho_A(t_1)} - \frac{1}{1+s_2\rho_A(t_2)}\right| = \left|\frac{s_2\rho_A(t_2) - s_1\rho_A(t_1)}{(1+s_1\rho_A(t_1))(1+s_2\rho_A(t_2))}\right|$$
$$\leq |s_2 - s_1||\rho_A(t_2)| + |s_1||\rho_A(t_2) - \rho_A(t_1)| \tag{6.101}$$

holds

$$\left|\frac{1}{1+s\rho_A(t)}\right|_{\alpha,Q} = \sup_{\substack{(s_1,t_1),(s_2,t_2)\in Q \\ (s_1,t_1)\neq(s_2,t_2)}} \frac{\left|\frac{1}{1+s_1\rho_A(t_1)} - \frac{1}{1+s_2\rho_A(t_2)}\right|}{|(s_1,\, t_1) - (s_2,\, t_2)|_\infty^\alpha}$$
$$\leq \|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \|\rho_A\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} |\rho_A|_{\alpha,I_{\Gamma_-}}. \tag{6.102}$$

To complete the bound for (6.100), we use Lemma 6.31 and bound $\nabla\Phi^{-1}(E^{n-1}, x)$ in terms of constants used in the definition of $W(M, \delta_1, \delta_2, \delta_3)$.

$$\|\nabla\Phi^{-1}(E^{n-1})\|_{0,\Omega} \leq \frac{2}{\delta_2} \exp\left(c_l \left\|\operatorname{div} E^{n-1}\right\|_{0,\Omega}\right) \max\left\{\|\partial_t\Phi(E^{n-1})\|_{0,Q},\, \|\partial_s\Phi(E^{n-1})\|_{0,Q}\right\}$$
$$\leq \frac{2\exp(c_l M)}{\delta_2} \max\left\{\exp(c_l M),\, M\right\}$$
$$=: c_3(c_l, \delta_2, M) \tag{6.103}$$

Assume that $\|\rho_A\|_{0,Q} < 1$. It holds further that $|\rho_A|_{\alpha,I_{\Gamma_-}} \leq L_{\Gamma_-}^{1-\alpha}|\rho_A'|_{0,I_{\Gamma_-}}$. Collecting the terms, we have for (6.98) by (6.99)-(6.103) and Theorem 6.27

$$\begin{aligned}
&\|TE^{n-1}\|_{\alpha,\Omega} \\
&\leq \|\rho_A\|_{0,I_{\Gamma_-}} + c_3(\delta_1, \delta_2, u_A, M)\left(|\rho_A|_{\alpha,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \|\rho_A\|_{0,I_{\Gamma_-}}^2 + \|l\|_{0,I_{\Gamma_-}} \|\rho_A\|_{0,I_{\Gamma_-}} |\rho_A|_{\alpha,I_{\Gamma_-}}\right) \\
&\leq \|\rho_A\|_{0,I_{\Gamma_-}} + c_3(\delta_1, \delta_2, u_A, M)\left(|\rho_A|_{\alpha,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \|\rho_A\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} |\rho_A|_{\alpha,I_{\Gamma_-}}\right) \\
&\leq \|\rho_A\|_{0,I_{\Gamma_-}} + c_3(\delta_1, \delta_2, u_A, M)\left(L_{\Gamma_-}^{1-\alpha}\|\rho_A'\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}}^{1-\alpha} \|\rho_A\|_{0,I_{\Gamma_-}} + \|l\|_{0,I_{\Gamma_-}} L_{\Gamma_-}^{1-\alpha}\|\rho_A'\|_{0,I_{\Gamma_-}}\right) \\
&\leq c_3(\delta_1, \delta_2, u_A, M) \max\left\{L_{\Gamma_-}^{1-\alpha}, L_{\Gamma_-}^{1-\alpha}\|l\|_{0,I_{\Gamma_-}}^{1-\alpha}, \|l\|_{0,I_{\Gamma_-}}\right\} \|\rho_A\|_{1,I_{\Gamma_-}} \\
&\leq c_3(\delta_1, \delta_2, u_A, M) \max\left\{L_{\Gamma_-}^{1-\alpha}, c_l, L_{\Gamma_-}^{1-\alpha} c_l^{1-\alpha}\right\} \|\rho_A\|_{1,I_{\Gamma_-}}.
\end{aligned}$$

The previous bound is independent of n. Hence, we get for (6.97)

$$\|L \circ TE^{n-1}\|_{\alpha,\Omega} \leq c_S(\Omega, \alpha)\left(\|u_A\|_{2,\alpha;\Gamma} + c_4(\delta_1, \delta_2, \Omega, M, \alpha)\|\rho_A\|_{1,I_{\Gamma_-}}\right)$$

with

$$c_4(\Omega, \delta_2, c_l, M, \alpha) = c_2(\Omega) c_3(c_l, \delta_2, M) \max\left\{L_{\Gamma_-}^{1-\alpha}, c_l, L_{\Gamma_-}^{1-\alpha} c_l^{1-\alpha}\right\}.$$

Boundedness of $L_1 \circ TE^{n-1}$:
We need to find an upper bound for $\|L_1 \circ TE^{n-1}\|_{0,\Omega}$. We use Lemma 4.16 of Chapter 4 and obtain

$$\begin{aligned} \|L_1 \circ TE^{n-1}\|_{0,\Omega} &\leq c_5(\operatorname{diam}\Omega)\|TE^{n-1}\|_{0,\Omega} \\ &\leq c_5(\operatorname{diam}\Omega)\|\rho_A\|_{0,I_{\Gamma_-}}. \end{aligned} \tag{6.104}$$

Choice of ρ_A and M:
Choose

$$\|\rho_A\|_{\alpha,I_{\Gamma_-}} = \min\left\{\frac{1}{c_4(\Omega, c_l, \delta_2, M, \alpha)}, \frac{\delta_3}{c_5(\Omega)}\right\} \tag{6.105}$$

and

$$M = c_S(\Omega, \alpha)(1 + \|u_A\|_{2,\alpha,\Gamma}).$$

The choice of M and ρ_A is reasonable, because they only depend on constants given in the definition of the set $W(M, \delta_1, \delta_2, \delta_3)$, the given boundary function u_A and the geometry of the domain. All these quantities are given in the problem definition.

We can now show that the boundedness restrictions on $L \circ TE^{n-1}$ are fullfilled

$$\|L_1 \circ TE^{n-1}\|_{0,\Omega} \leq \delta_3$$

and

$$\|L \circ TE^{n-1}\|_{1,\alpha,\Omega} \leq c_S(\Omega, \alpha)(1 + \|u_A\|_{2,\alpha,\Gamma}) = M.$$

Boundedness from below and inflow/outflow conditions:
As the vector field E_0 is given by the solution of the Laplace equation, it follows $\inf_{x\in\bar\Omega} |E_0(x)|_\infty = 2\delta_1$. The lower bound on $|E^n(x)|_\infty$ is now easily found with the inverse triangle inequality and the choice of ρ_A in (6.105). By (6.104), we get

$$\begin{aligned} \inf_{x\in\bar\Omega} |L \circ TE^{n-1}(x)|_\infty &= \inf_{x\in\bar\Omega} |E_0 + L_1 \circ TE^{n-1}(x)|_\infty \\ &\geq \inf_{x\in\bar\Omega} |E_0|_\infty - \sup_{x\in\bar\Omega} |L_1 \circ TE^{n-1}(x)|_\infty \\ &\geq 2\delta_1 - c_5(\operatorname{diam}\Omega)\|\rho_A\|_{\alpha,I_{\Gamma_-}} \\ &\geq 2\delta_1 - \delta_3 \\ &\geq 2\delta_1 - \delta_1 = \delta_1. \end{aligned}$$

Due to the constant boundary conditions, the field E^0 is perpendicular to the boundary. Further, since L_1 is the solution operator of the Poisson equation with homogeneous boundary conditions, the field $L_1 \circ TE^{n-1}$ is also perpendicular to the boundary Γ. It is left to show that the field strength at $x \in \Gamma_-$ is bounded by δ_1. The iterated field $L \circ TE^{n-1}$ points into the same direction as the outward normal vector at the outflow boundary and in the opposite direction at the inflow boundary. It holds for a function $c(x) > 0$ and $x \in \Gamma_-$

$$L \circ TE^{n-1}(x) = -c(x)\vec{n}(x).$$

Since $\inf_{x\in\bar{\Omega}} |L \circ TE^{n-1}(x)| \geq \delta_1$ holds with the equivalence of the euclidean and the maximum norm

$$c(x) = \|c(x)\vec{n}(x)\|_2 = \|L \circ TE^{n-1}(x)\|_2 \geq |L \circ TE^{n-1}(x)|_\infty \geq \delta_1.$$

It follows for $x \in \Gamma_-$

$$\vec{n}(x) \cdot (L \circ TE^{n-1}(x)) = -c(x)\vec{n}(x) \cdot \vec{n}(x) = -c(x) \leq -\delta_1$$

and analogously for $x \in \Gamma_+$

$$\vec{n}(x) \cdot (L \circ TE^{n-1}(x)) = c(x)\vec{n}(x) \cdot \vec{n}(x) = c \geq \delta_1.$$

By choosing $\delta_2 := \delta_1$ follows the assertion.

□

We arrive at the main results of this Chapter. First, we show that a fixed point $E \in W(M, \delta_1, \delta_2, \delta_3)$ exists to the operator $L \circ T$.

Theorem 6.58 (Existence and Uniqueness of a Fixed Point)**.** *Let Ω be a $C^{2,\alpha}$ domain, $u_A \in C^{2,\alpha}(\Gamma)$, $\rho_A \in C^{1,\alpha}(I_{\Gamma_-})$ with*

$$\|\rho_A\|_{1,\alpha,I_{\Gamma_-}} = \min\left\{\frac{1}{c_4(\Omega, c_l, \delta_2, M, \alpha)}, \frac{\delta_3}{c_5(\operatorname{diam}\Omega)}, \frac{1}{2c_T c_S(\Omega,\alpha)c_2(\Omega)}\right\} \tag{6.106}$$

with $c_4(\Omega, c_l, \delta_2, M, \alpha)$ and $c_5(\operatorname{diam}\Omega)$ defined in Theorem 6.57, c_T defined in Theorem 6.55 and $c_S(\Omega, \alpha)$ defined in Theorem 6.7. Let M and δ_2 be defined as in Theorem 6.57 and δ_3 be chosen as in Lemma 6.52. Then the operator $L \circ T$ has a unique fixed point in the set $W(M, \delta_1, \delta_2, \delta_3)$.

Proof. We use the Banach fixed point theorem. By Lemma 6.57, $L \circ T$ is a selfmap on $W(M, \delta_1, \delta_2, \delta_3)$. It is left to show that $L \circ T$ is a contraction.

With Schauder's a priori estimate (Theorem 6.7) and Theorem 6.6, we get

$$\begin{aligned}
&\|L \circ TE^{n+1} - L \circ TE^n\|_{1,\alpha,\Omega} = \|L \circ (TE^{n+1} - TE^n)\|_{1,\alpha,\Omega} \\
&\quad \leq c_S(\alpha,\Omega)\left(\|TE^{n+1} - TE^n\|_{\alpha,\Omega} + \|u^{n+2} - u^{n+1}\|_{0,\Omega} + \|u_A - u_A\|_{2,\alpha,\Gamma}\right) \\
&\quad \leq c_S(\alpha,\Omega)c_2(\Omega)\|TE^{n+1} - TE^n\|_{\alpha,\Omega}.
\end{aligned}$$

By Theorem 6.55, we obtain

$$\|L \circ TE^{n+1} - L \circ TE^n\|_{1,\alpha,\Omega} \leq c_S(\alpha,\Omega)c_2(\Omega)C_T\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}\|E^n - E^{n-1}\|_{1,\Omega}.$$

Due to the assumption on $\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$ holds

$$\begin{aligned}\|L \circ TE^{n+1} - L \circ TE^n\|_{1,\alpha,\Omega} &\leq \frac{1}{2}\|E^n - E^{n-1}\|_{1,\Omega}\\ &\leq \frac{1}{2}\|E^n - E^{n-1}\|_{1,\alpha,\Omega}.\end{aligned}$$

Hence, $L \circ T$ is a contraction on the set $W(M,\delta_1,\delta_2,\delta_3)$. By the Banach fixed point Theorem, there exists a unique fixed point $E \in W(M,\delta_1,\delta_2,\delta_3)$ to $L \circ T$. □

We can conclude that there exists a unique classical solution to (CP 6.1).

Theorem 6.59 (Existence and Uniqueness of a Classical Solution to (CP 6.1))**.** *Let Ω be an open bounded $C^{2,\alpha}$ domain, $u_A \in C^{2,\alpha}(\Gamma)$, $\rho_A \in C^{1,\alpha}(\Gamma)$. Let $\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$, M and δ_2 be chosen as in Theorem 6.58. Moreover, let δ_3 be chosen as in Lemma 6.52. Then there exists a classical solution $(u,\rho) \in C^{2,\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$ to (CP 6.1) with $-\nabla u \in W(M,\delta_1,\delta_2,\delta_3)$.*

Proof. By Theorem 6.58, there exists a unique fixed point $E \in W(M,\delta_1,\delta_2,\delta_3)$ to $L \circ T$, i.e.

$$L \circ TE = E.$$

The fixed point $E \in W(M,\delta_1,\delta_2,\delta_3)$ is a gradient field $E = -\nabla u$. We therefore obtain the unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ to (CP 6.1) with $-\nabla u \in W(M,\delta_1,\delta_2,\delta_3)$. Since T is the solution operator to the transport problem (6.1c)-(6.1d), we obtain the solution $\rho = TE \in C^{1,\alpha}(\bar{\Omega})$ by Theorem 6.39. We found the unique solution $(u,\rho) \in C^{2,\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$ to (CP 6.1) with $-\nabla u \in W(M,\delta_1,\delta_2,\delta_3)$.

□

6.5 Remarks about the Chapter

In this Chapter, we have proved the existence and uniqueness of a classical solution (u,ρ) to the steady state two-dimensional problem (CP 6.1) with $-\nabla u$ in a set of vector fields $W(M,\delta_1,\delta_2,\delta_3) \subset C^{1,\alpha}(\bar{\Omega})$. We defined the solution operator L for the Poisson and T for the transport problem. By the Banach fixed point theorem, we proved that a unique fixed point $E = L \circ TE$ exists provided that the inflow boundary data $\|\rho_A\|_{1,I_{\Gamma_-}}$ is sufficiently small.

In the argumentation for the existence of the streamline functions and especially to obtain upper bounds, we exploited one crucial fact in the problem definition: The convective field E of the transport equation is a gradient field. Due to this knowledge, we were able to determine an upper bound for the streamline parameter s.

The fundamental idea in this Chapter is to formulate the coupled problem as a fixed point problem. In the continuous case that we presented, we can conclude that the Banach fixed point iterations are the staggered algorithm. In the following Chapters we will use this result to introduce a discretization method for the coupled problem. Furthermore, we will illustrate the advantage of following the presented approach in comparison to a compactness argument applied in Chapter 4. With the error estimate for the Banach fixed point iterations and standard approximation results, we will show in Chapter 7 that we immediately obtain an error estimate for the staggered algorithm.

Let us comment on the choice of the vector field E. Vector fields $E = E_0 + E_1$ in the set $W(M, \delta_1, \delta_2, \delta_3)$ lead to invertible streamline functions of $C^{1,\alpha}$ regularity. The constants M, δ_1 and δ_2 depend on the choice of the boundary data u_A and the domain Ω. The vector field E_0 determined by the solution of the Laplace equation was considered as the dominant field. E_0 only depends on the boundary data u_A and the geometry of the domain Ω. As solution to a harmonic equation, E_0 does not have any inner extrema and thus $|E(x)|_\infty$ is naturally bounded below by a constant δ_1. We chose the second component E_1 as a small perturbation to E, as the supremum of $|E_1(x)|_\infty$ shall be bounded by this very δ_1. Indeed, this choice of vector fields is realistic for the underlying physical model. To obtain corona discharge, we need a strong electrical field E_0 such that ions are emitted into the system. The ions then give rise to the vector field E_1.
With the previous observations, we justify the method to bound the Hölder coefficient of $TE - T\tilde{E}$ in Lemma 6.53 and 6.54. Herein we first restricted the vector field E_1 further, by claiming that it is smaller than the minimum of δ_1 and δ_3 where δ_3 was a constant determined by the streamline function Φ_0 of E_0. As E_1 is small, it is realistic to assume that the streamline Φ corresponding to $E_0 + E_1$ is only a small perturbation of the streamline function Φ_0. We then used that two points $\Phi(s,t)$ and $\tilde{\Phi}(s,t)$ can be connected by a line segment that is contained in Ω as the graphs of both streamline functions lay in a small neighborhood of Φ_0.

The inflow boundary function ρ_A was the key to prove existence and uniqueness of a classical solution. We chose $\|\rho_A\|_{1,\alpha,I_{\Gamma_-}}$ sufficiently small to obtain a selfmap and contraction $L \circ T$ on $W(M, \delta_1, \delta_2, \delta_3)$. The actual size of ρ_A depends on several quantities which all depend on the geometry of the domain Ω, the extended domain Ω^+ and the boundary data u_A and c_v in case of the extended domain. It is difficult to give a qualitative interpretation for the dependence of ρ_A on the size of the domain and the boundary data, as for example holds for the maximum of the streamline parameter

$$\|l\|_{0,I_{\Gamma_-}} \leq \frac{u_{A_1} - u_{A_2}}{\inf_{x\in\bar{\Omega}} |E(x)|_\infty^2}.$$

For an increasing potential difference, the infimum for the field strength will surely also increase. However, for fixed boundary conditions u_A and an increasing diameter of Ω, the field strength will decrease. Conclusively, in this case ρ_A will decrease too.

Chapter 7

Discretization Methods

In this chapter, we focus on discretization methods for the time independent coupled problem (CP 6.1). Discretization methods are best analyzed in Sobolev spaces where error estimates and stability properties are available. We therefore restrain the analysis to Sobolev spaces. First, we list briefly error and stability estimates for the Poisson and linear transport equations. In section 7.4, we investigate the nonlinear transport equation under a new point of view. We formulate (CP 6.1) as variational inequality and prove the unique existence of a continuous solution in a space V_A. Further we are interested in the Galerkin discretization of the variational inequality. We derive an error estimate for a discrete bilinear solution on a quadrilateral mesh and show that it converges to the continuous solution. We proceed to investigate whether there is a connection between the solutions of the variational inequality and (CP 6.1). If we further restrict the space $W(M, \delta_1, \delta_2, \delta_3)$ defined in Chapter 6, then the classical solution ρ is also in the space V_A. Due to uniqueness, the solution of the variational inequality must be equal to the classical solution ρ. Conclusively, the discrete solution of the variational inequality also converges to the solution ρ of (Tr 6.3).

In section 7.5, we introduce the so-called *staggered algorithm* which is an algorithm to solve the coupled problem [1, 59]. The idea is simple, after initializing a first vector field E, the transport and Poisson problem are solved alternating until convergence is obtained. For the continuous problem, the staggered algorithm are the Banach fixed point iterations as used in the Chapters 5 and 6. Especially regarding numerical results, it is of great interest if the discretized staggered algorithm converges. Having the contraction property in the continuous case and stability and error estimates for the Poisson and transport problem, it is possible to prove an error estimate for the discrete staggered algorithm. This part is understood as an outline for future work and does not claim completeness.

7.1 Notations

Let $\Omega \subset \mathbb{R}^2$ be an open $C^{2,\alpha}$ domain. Then we define the space of *Lesbesgue square-integrable functions* by

$$L^2(\Omega) := \left\{ v : \int_\Omega v^2 \, dx < \infty \right\}.$$

Equipped with the norm

$$\|v\|^2_{L^2(\Omega)} := \int_\Omega v^2 \, dx, \tag{7.1}$$

$L^2(\Omega)$ forms a Banach space. We will denote

$$(v, w)_{L^2(\Omega)} := \int_\Omega vw \, dx \tag{7.2}$$

as the $L^2(\Omega)$ *scalar product*.

Let k be an integer and m be a multi-index with $|m| \leq k$. The *Sobolev space* of order k is denoted by

$$H^k(\Omega) := \left\{ v : \|\partial^m v\|_{L^2(\Omega)} < \infty, |m| \leq k \right\}. \tag{7.3}$$

Equipped with the norm

$$\|v\|^2_{H^k(\Omega)} := \sum_{|m| \leq k} \|\partial^m v\|^2_{L_2(\Omega)}, \tag{7.4}$$

and inner product

$$(v, w)_{H^k(\Omega)} := \sum_{|m| \leq k} (\partial^m v, \partial^m w)_{L^2(\Omega)} \tag{7.5}$$

$H^k(\Omega)$ is a Hilbert space.

For the analysis of the transport equation, we define additional boundary norms. Let $E \in (L^2(\Omega))^2$ be a convective field and $\rho_A \in L^2(\Gamma)$. Then we denote

$$|\rho_A|^2_\Gamma := \int_\Gamma \vec{n} \cdot E \; \rho_A^2 \, dx \tag{7.6}$$

with $\vec{n}$ being the outward pointing normal vector to Γ. Analogously, we define the norms on boundary parts Γ_- and Γ_+ by

$$|\rho_A|^2_{\Gamma_-} := \int_{\Gamma_-} \vec{n} \cdot E \; \rho_A^2 \, dx, \tag{7.7}$$

$$|\rho_A|^2_{\Gamma_+} := \int_{\Gamma_+} \vec{n} \cdot E \; \rho_A^2 \, dx. \tag{7.8}$$

We will also need the *dual space* of the usual Sobolev spaces.

Definition 7.1. *[13, p.123] Let $m \geq 1$. Given $u \in L^2(\Omega)$, define the norm*

$$\|u\|_{\tilde{H}^{-m}(\Omega)} := \sup_{v \in H^m(\Omega)} \frac{(u,v)_{L^2(\Omega)}}{\|v\|_{m,\Omega}}.$$

We define $\tilde{H}^{-1}(\Omega)$ to be the completion of $L^2(\Omega)$ w.r.t. $\|\cdot\|_{\tilde{H}^{-m}(\Omega)}$. For the Sobolev space built on $L^2(\Omega)$, we identify the dual space of $H^m(\Omega)$ with $\tilde{H}^{-m}$. Moreover, by the definition of $\tilde{H}^{-m}$, there is a dual pairing $\langle u, v\rangle$ for all $u \in \tilde{H}^{-m}, v \in H^m$, i.e. $\langle u, v\rangle$ is a bilinear form, and

$$\langle u, v\rangle := (u,v)_{L^2(\Omega)}. \tag{7.9}$$

7.2 Discretization of the Poisson Equation

The discretization of the Poisson equation is a well-studied problem. We briefly list some properties for completeness. Let

$$V^P = \big\{v \in H^1(\Omega) \,:\, v = u_A \text{ on } \Gamma\big\}$$

and

$$V_0^P = \big\{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma\big\} = H_0^1(\Omega)$$

where P indicates that the sets correspond to the **P**oisson equation. The variational formulation for (Po 6.2) is given by:

Find $u \in V^P$, such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \rho v \, dx, \qquad \forall v \in V_0^P. \tag{7.10}$$

Let $\mathcal{T}^h$ be a partition of Ω into quadrilateral elements $\tau \in \mathcal{T}^h$ and $n = |\mathcal{T}^h|$ be the number of elements. Let h be the mesh size with $h = \max_{\tau \in \mathcal{T}^h} \{ \text{ longest side of } \tau\}$. Let $V^{P,h}$ and $V_0^{P,h}$ be the discretization into finite element spaces of V^P and V_0^P with basis functions of degree p. Then the discrete variational problem reads:

Find $u^h \in V^{P,h}$, such that

$$\int_\Omega \nabla u^h \cdot \nabla v \, dx = \int_\Omega \rho v \, dx, \qquad \forall v \in V_0^{P,h}. \tag{7.11}$$

We will now list well-known results that we will need in the following.

Regularity of u: [45, Section 4.5]
The regularity of the solution u depends on the regularity of the right hand side data ρ. Since Ω is smooth, the following estimate follows

$$\|u\|_{H^2(\Omega)} \leq c\|\rho\|_{L^2(\Omega)} \tag{7.12}$$

with c being a constant independent of ρ.

Error estimate: [13, Table 3, p. 82]
Let $u \in H^2(\Omega)$ and u^h piecewise linear. Then we have the following error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leq ch\|u\|_{H^2(\Omega)} \tag{7.13}$$

with c being a constant independent of h.

Stability: [45, Section 2.1]
Let α be the V_0^P-ellipticity constant of the bilinear form $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, i.e.

$$a(v,v) \geq \alpha \|v\|_{V^P}^2 \quad \forall v \in V_0^P.$$

Then follows the stability estimate for the solution u to (7.10)

$$\alpha \|u\|_V \leq \|\rho\|_{L^2(\Omega)}. \tag{7.14}$$

7.3 Discretization of the Linear Transport Equation

The steady state linear transport problem is given by

Problem (Tr 7.1). *Let Ω be a $C^{2,\alpha}$ domain. For a given $E \in (C^1(\Omega))^2$, find $\rho \in C^1(\Omega)$ such that*

$$\operatorname{div}(E\rho) = 0 \qquad x \in \Omega \tag{7.15a}$$

$$\rho(x) = \rho_A(x) \qquad x \in \Gamma_-. \tag{7.15b}$$

As in the case of the Poisson problem, discretization methods for the linear transport equation are well studied. For hyperbolic partial differential equation, it is a known phenomenon to obtain spuriously oscillating numerical solutions as soon as the inflow boundary data are not globally smooth. Many authors have published methods to stabilize numerical solutions, for example the streamline upwind Petrov-Galerkin (SUPG) method. Information about the SUPG method are found among others in the works [15, 16]. In [12], the authors compare the application of the standard Galerkin, SUPG and Least squares methods for (Tr 7.1). In the following, we introduce the standard Galerkin method to discretize the linear transport equation. Let therefore first be

$$V^T = \left\{ v \in H^1(\Omega) \,:\, v = \rho_A \text{ on } \Gamma_- \right\}$$

and

$$V_0^T = \left\{ v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_- \right\}.$$

where T indicates that the sets correspond to the **T**ransport equation. The boundary conditions are employed strongly into the space. The variational formulation is given by:

Find $\rho \in V^T$, such that

$$\int_\Omega \operatorname{div}(E\rho)v\,dx = 0 \qquad \forall v \in V_0^T.$$

Let $\mathcal{T}^h$ be a partition of Ω into curved quadrilateral elements $\tau \in \mathcal{T}^h$ with $n = |\mathcal{T}^h|$ as number of elements. Let $V^{T,h}$ and $V_0^{T,h}$ be the finite element discretization of V^T and V_0^T. We use bilinear basis functions.

The *standard Galerkin* method is then given by:

Find $\rho^h \in V^{T,h}$, such that

$$\int_\Omega \operatorname{div}(E\rho^h)v\,dx = 0 \qquad \forall v \in V_0^{T,h}. \tag{7.16}$$

The existence and uniqueness of the Galerkin solution is proved by means of the following Galerkin graph norm.

Definition 7.2. *[50, Lemma 3] Let $\frac{1}{2}\operatorname{div} E \geq \sigma > 0$. Then we denote the graph norm for the Galerkin method by*

$$|||\rho|||_G^2 = |\rho|_{\Gamma_-}^2 + \frac{\sigma}{2}\|\rho\|_{L^2(\Omega)}^2. \tag{7.17}$$

It holds the following error estimate.

Lemma 7.3. *[12, 50] Let $\rho \in H^2(\Omega)$. Then holds for the Galerkin approximate solution ρ^h*

$$|||\rho - \rho^h|||_G \leq ch\|\rho\|_{H^2(\Omega)} \tag{7.18}$$

with a constant c independent of h and ρ.

7.4 Discretization of the Nonlinear Transport Equation

In contrast to the Poisson and linear transport problem, the variational theory for the nonlinear transport problem is not a standard result. Recall, that we want to solve the nonlinear transport problem (Tr 6.3)

$$E \cdot \nabla\rho + \rho^2 = 0 \qquad x \text{ in } \Omega \tag{7.19a}$$

$$\rho = \rho_A \qquad x \text{ on } \Gamma_- \tag{7.19b}$$

with $E \in (C^1(\Omega))^2$ being a given bounded vector field. To the end of this section, let $0 < \rho_A$ be constant.

One method to solve and discretize the nonlinear transport problem is the Method of Characteristics (MOC). However, one encounters difficulties in the context of the coupled problem.

The Poisson equation is discretized by the Galerkin finite element method and is therefore based on a quadrilateral mesh. The MOC on the other hand solves the transport equation on the streamlines and is therefore line based. The streamlines start from the boundary and thereon not only on the nodes of the quadrilateral mesh. Hence it is difficult to extract the relevant information for ρ and the method becomes numerically unstable. We are therefore interested to use a Galerkin discretization. Standard methods to prove existence and uniqueness are not applicable, as the nonlinear form for (7.19a) is not coercive.

In the following approach, we formulate (Tr 6.3) as variational inequality. Let us first find the ansatz and test spaces. Apparently, the solution ρ to (Tr 6.3) must be differentiable which implies that the ansatz space V_A is a subset of $H^1(\Omega)$. We define the operator $A : V_A \to L^2(\Omega)$ according to the form

$$A\rho = E \cdot \nabla\rho + \rho^2 \tag{7.20}$$

with

$$V_A(\sigma_u, \sigma_p, \sigma_A) = \left\{\rho \in H^1 \,:\, \rho|_{\Gamma_-} = \rho_A,\, \sigma_u \geq \rho \geq \sigma_A > 0 \text{ } a.e.,\, |\nabla\rho| \leq \sigma_p \text{ } a.e.\right\}$$

and σ_u, σ_p , $\sigma_A > 0$ are constant parameter. We will now prove that for every choice of finite σ_u, σ_p and σ_A, a solution exists to the variational inequality (Tr 7.2) below. Further, we will prove that a solution exists to the discretization of (Tr 7.2). In section 7.4.4, we then choose the parameter σ_u, σ_p , $\sigma_A > 0$ such that the solution to the variational inequality (Tr 7.2) is also the solution of the classical transport problem (Tr 6.3). The discretization of the variational inequality is thus a discretization for the classical transport problem. In the following, for the convenience of notation, we will write V_A instead of $V_A(\sigma_u, \sigma_p, \sigma_A)$.

Lemma 7.4. *Let $E \in (L^2(\Omega))^2$ and $\rho \in V_A$. Then $A\rho \in L^2(\Omega)$.*

Proof. With the Cauchy Schwarz inequality, the boundedness conditions in V_A and since $E \in (L^2(\Omega))^2$ holds

$$\begin{aligned}\int_\Omega (A\rho)^2\, dx &= \int_\Omega (E \cdot \nabla\rho + \rho^2)^2\, dx \\ &\leq 2\int_\Omega (E \cdot \nabla\rho)^2\, dx + 2\int_\Omega (\rho^2)^2\, dx \\ &\leq 2\int_\Omega E^2\, dx \int_\Omega |\nabla\rho|^2\, dx + 2\int_\Omega \rho^4\, dx \\ &\leq 2\|E\|^2_{L^2(\Omega)} \mathrm{vol}(\Omega)\sigma_p^2 + 2\sigma_u^4 \mathrm{vol}(\Omega) < \infty.\end{aligned}$$

□

It holds that $A\rho \in L^2(\Omega)$ for $\rho \in V_A$. Consequently, it would be sufficient to use a test function space $V \subset L^2(\Omega)$. However, since $\tilde{H}^{-1}(\Omega) \supset L^2(\Omega)$, let us choose the test space $V_A \subset H^1(\Omega)$.

The dual pairing $\langle A\rho, v\rangle$ is defined and we have a symmetrical formulation for the test and ansatz spaces.

The variational inequality for (Tr 6.3) reads:

Problem (Tr 7.2). *Let Ω be a Lipschitz domain and let $E \in (H^1(\Omega))^2$ with $|E|_{L^\infty(\Omega)} \leq M$ and $\sigma_A \geq \frac{1}{2} \operatorname{div} E \geq 0$ with σ_A defined in V_A. Find $\rho \in V_A$, such that*

$$\langle A\rho \, , \, v - \rho\rangle \geq 0 \qquad \forall v \in V_A. \tag{7.21}$$

7.4.1 Existence and uniqueness of a solution to (Tr 7.2)

A concept used in connection with variational inequalities is the theory of monotone operators as it is presented in [26, 49]. If the form associated to the variational inequality is monotone, then the condition of coerciveness to prove existence and uniqueness may be relaxed.

Definition 7.5. *[49, III Definition 1.1] Let K be a closed convex subset of a reflexive Banach space X with dual X'. A mapping $S : K \to X'$ is called* monotone*, if and only if*

$$\langle Su - Sv, \, u - v\rangle \geq 0 \qquad \text{for all } u, v \in K.$$

The monotone mapping S is called strictly monotone *if*

$$\langle Su - Sv, \, u - v\rangle = 0 \qquad \text{implies } u = v.$$

Definition 7.6. *[49, III Definition 1.2] Let K be a closed convex set of a reflexive Banach space X with dual X'. The mapping $S : K \to X'$ is called* weakly continuous *if there holds*

$$\langle Sx_n, v\rangle \xrightarrow{n\to\infty} \langle Sx, v\rangle \quad \forall v \in X$$

whenever

$$\langle x_n, v\rangle \xrightarrow{n\to\infty} \langle x, v\rangle. \quad \forall v \in X$$

The mapping $S : K \to X'$ is continuous on finite dimensional subspaces *if for any finite dimensional subspace $M \subset X$ the restriction of S to $K \cap M$ is weakly continuous, namely if*

$$S : K \cap M \to X'$$

is weakly continuous.

The following theorem proves the existence and uniqueness of a solution to a variational inequality.

Theorem 7.7. *[49, III Theorem 1.4] Let K be a closed bounded convex subset of X $(\neq \emptyset)$ and let $S : K \to X'$ be monotone and continuous on finite dimensional subspaces. Then there exists a $u \in K$ such that*

$$\langle Su, \, v - u\rangle \geq 0 \qquad \text{for all } v \in K.$$

If S is strictly monotone, then the solution u to the variational inequality is unique.

In the following Lemmas, we will verify the conditions needed to apply Theorem 7.7 to (Tr 7.2). First, we show that V_A fulfils the assumptions of Theorem 7.7.

Lemma 7.8. *The set V_A is convex, bounded and closed.*

Proof. Convexity:
The space V_A is clearly convex. Choose $\rho_1, \rho_2 \in V_A$, then

$$\begin{aligned} \lambda\rho_1 + (1-\lambda)\rho_2 &\geq \lambda\sigma_A + (1-\lambda)\sigma_A = \sigma_A \\ \lambda\rho_1 + (1-\lambda)\rho_2 &\leq \lambda\sigma_u + (1-\lambda)\sigma_u = \sigma_u \\ \lambda\nabla\rho_1 + (1-\lambda)\nabla\rho_2 &\leq \lambda\sigma_p + (1-\lambda)\sigma_p = \sigma_p \\ \lambda\nabla\rho_1 + (1-\lambda)\nabla\rho_2 &\geq -\lambda\sigma_p - (1-\lambda)\sigma_p = -\sigma_p. \end{aligned}$$

Boundedness:
For every $\rho \in V_A$ holds due to the boundedness conditions in the space

$$\|\rho\|^2_{H^1(\Omega)} = \|\rho\|^2_{L^2(\Omega)} + \|\nabla\rho\|^2_{L^2(\Omega)} \leq \operatorname{vol}(\Omega)(\sigma_u^2 + \sigma_p^2) < \infty. \tag{7.22}$$

Consequently, V_A is bounded with respect to the H^1-norm.

Closed Set:
Cases $\langle 4\rangle$ and $\langle 5\rangle$ of [3, 6.18] show that the limit ρ of a convergent sequence ρ_n fulfils the boundary conditions on Γ_- and the boundedness of ρ. For a strongly convergent sequence with $\|\nabla\rho_n - \nabla\rho\|_{L^2(\Omega)} \to 0$ follows $\nabla\rho_n \xrightarrow{n\to\infty} \nabla\rho$ almost everywhere. Since $|\nabla\rho_n|$ is bounded a.e., it also holds that the limit function $|\nabla\rho| \leq \sigma_p$ a.e.

$\square$

Next, we prove that the operator A given by (7.20) is strictly monotone.

Lemma 7.9. *Let $0 \leq \frac{1}{2}\operatorname{div} E \leq \sigma_A$. Then the mapping $A : V_A \to \tilde{H}^{-1}$ is strictly monotone with*

$$\langle A\rho_1 - A\rho_2, \rho_1 - \rho_2\rangle \geq \frac{1}{2}|||\rho_1 - \rho_2|||^2_G. \tag{7.23}$$

Proof. Let $\rho_1, \rho_2 \in V_A$. We show that the following dual pairing is bounded from below. By the chain rule, the Gauss's divergence theorem and since $0 < \sigma_A \leq \rho \leq \sigma_u$ and $\frac{1}{2}\operatorname{div} E \leq \sigma_A$, we obtain

$$
\begin{aligned}
\langle E\cdot\nabla\rho_1+\rho_1^2-E\cdot\nabla\rho_2-\rho_2^2,\,\rho_1-\rho_2\rangle &= \int_\Omega (E\cdot\nabla\rho_1+\rho_1^2-E\cdot\nabla\rho_2-\rho_2^2)(\rho_1-\rho_2)\,dx\\
&= \int_\Omega \frac{1}{2}E\cdot\nabla(\rho_1-\rho_2)^2+(\rho_1+\rho_2)(\rho_1-\rho_2)^2\,dx\\
&= \int_\Omega \frac{1}{2}\operatorname{div}(E(\rho_1-\rho_2)^2)\,dx-\int_\Omega \frac{1}{2}\operatorname{div}E(\rho_1-\rho_2)^2\,dx+\int_\Omega(\rho_1+\rho_2)(\rho_1-\rho_2)^2\,dx\\
&\geq \int_\Omega \frac{1}{2}\operatorname{div}(E(\rho_1-\rho_2)^2)-\frac{1}{2}\sup|\operatorname{div}E|\int_\Omega(\rho_1-\rho_2)^2\,dx+\inf_{x\in\Omega}|\rho_1+\rho_2|\int_\Omega(\rho_1-\rho_2)^2\,dx\\
&\geq \frac{1}{2}\int_\Gamma n\cdot E(\rho_1-\rho_2)^2 ds_x+\sigma_A\int_\Omega(\rho_1-\rho_2)^2\,dx\\
&= \frac{1}{2}\int_{\Gamma_-} n\cdot E(\rho_A-\rho_A)^2\,ds_x+\frac{1}{2}\int_{\Gamma_+} n\cdot E(\rho_1-\rho_2)^2\,ds_x+\sigma_A\int_\Omega(\rho_1-\rho_2)^2\,dx\\
&\geq \frac{1}{2}|\rho_1-\rho_2|_+^2+\sigma_A\|\rho_1-\rho_2\|^2_{L^2(\Omega)}=\frac{1}{2}|||\rho_1-\rho_2|||_G^2\geq 0.
\end{aligned}
$$

If $\rho_1\neq\rho_2$, then $\langle A\rho_1-A\rho_2,\,\rho_1-\rho_2\rangle>0$. Conclusively, whenever $\langle A\rho_1-A\rho_2,\,\rho_1-\rho_2\rangle=0$ then $\rho_1=\rho_2$. A is strictly monotone on V_A. ☐

Remark 7.10. *The condition $\frac{1}{2}\operatorname{div}E\leq\sigma_A$ is needed to prove the monotonicity of the operator A. In the space V_A we claim additionally that $\rho_A\geq\sigma_A$. At a first glance, those two conditions might seem contradictory in case of the coupled problem. For the solution (u,ρ) of (CP 6.1) holds*

$$
\operatorname{div}E=\rho. \tag{7.24}
$$

Thus, to retain applicability to the coupled problem, the constant σ_A must be chosen such that

$$
\rho\geq\sigma_A\geq\frac{1}{2}\rho.
$$

The crucial point in the following argumentation is that ρ_A is constant. We show in section 7.4.4 that with $\sigma_A=\frac{1}{2}\rho_A$ and ρ_A sufficiently small, the coupled problem is still well-defined.

To apply Theorem 7.7, we need to show that the operator A is weakly continuous on finite dimensional subspaces. We are able to show the strong continuity for A.

Lemma 7.11. *Let E be a bounded vector field. Then the operator $A: V_A\to\tilde{H}^{-1}$ is strongly continuous.*

Proof. For two elements $v,w\in V_A$ holds due to the embedding $L^2(\Omega)\subset\tilde{H}^{-1}(\Omega)$

$$
\|Av-Aw\|_{\tilde{H}^{-1}}=\sup_{z\in H^1}\frac{(Av-Aw,\,z)_{L^2(\Omega)}}{\|z\|_{H^1}}\leq\|Av-Aw\|_{L^2(\Omega)}.
$$

With the restrictions for $v, w \in V_A$, we bound the last term by

$$\begin{aligned}
\|Av - Aw\|^2_{L^2(\Omega)} &= \|E \cdot \nabla(v-w) + v^2 - w^2\|^2_{L^2(\Omega)} \\
&= \|E \cdot \nabla(v-w) + (v-w)(v+w)\|^2_{L^2(\Omega)} \\
&\le 2\int_\Omega (E \cdot \nabla(v-w))^2\,dx + 2\int_\Omega (v-w)^2(v+w)^2\,dx \\
&\le 2\int_\Omega (\sup|E|)^2 |\nabla v - \nabla w|^2\,dx + 8\sigma_u^2 \int_\Omega (v-w)^2\,dx \\
&\le \max\left\{2\sup|E|^2, 8\sigma_u^2\right\} \|v-w\|^2_{H^1}.
\end{aligned}$$

If $\|v - w\|^2_{H^1(\Omega)} \to 0$, then $\|Av - Aw\|^2_{H^1(\Omega)} \to 0$. Conclusively, the operator A is strongly continuous on V_A. $\square$

With the previous results, we show that (Tr 7.2) is uniquely solvable in the set V_A.

Theorem 7.12. *Let $E \in H^1(\Omega)$ with $|E|_{L^\infty(\Omega)} \le M$ and $\sigma_A \ge \frac{1}{2}\operatorname{div} E \ge 0$. Then there exists a unique solution $\rho \in V_A$ for (Tr 7.2).*

Proof. By Lemma 7.8, the set V_A is convex, closed and bounded with respect to the $H^1(\Omega)$ norm. Lemmas 7.9 and 7.11 give the monotonicity and continuity of A on V_A. It is left to show that A is continuous on finite dimensional subspaces. By [31, 2.3], a strongly continuous nonlinear operator is weakly continuous. By [31, Remark 1.3], weak continuity on the set V_A, implies weak continuity on finite dimensional subspaces. The assertion is proved. $\square$

7.4.2 Discretization

We use a Galerkin discretization following the approach of [17, 28]. Let $\mathcal{T}^h$ be a partition of Ω into curved quadrilateral elements $\tau \in \mathcal{T}^h$ and $n = |\mathcal{T}^h|$ be the number of elements. The curved elements are chosen to exactly fit the geometry at the boundary. Thus, for the elements at the boundaries, the mapping from the standard quadrilateral to the actual element involves functions which are not polynomial. On the standard quadrilateral element bilinear functions are used and V^h denotes the finite element discretization of $V = H^1(\Omega)$ with bilinear basis functions.

The discrete convex set $V_A^h \subset V^h$ should fulfill two conditions [28, p.94]

1. V_A^h should reduce to a finite number of constraints.
2. V_A^h should be a "good" approximation to V_A.

The actual meaning and benefit of the second condition becomes clear in the proof of Theorem 7.16. So far, we choose V_A^h by

$$V_A^h = \left\{\rho^h \in V^h \,:\, \rho^h|_{\Gamma_-} = \rho_A,\, \sigma_u \geq \rho^h(y_j) \geq \sigma_A > 0, |\nabla \rho^h| \leq \sigma_p, j = 1,2,3,\, i = 1,..,N\right\}.$$

The following Lemma shows that $V_A^h \subset V_A$ which is in fact a good approximation.

Lemma 7.13. *Let ρ_A be constant. Then follows $V_A^h \subset V_A$.*

Proof. The inflow boundary condition ρ_A is chosen constant. Due to the curved elements, we have $\rho^h|_{\Gamma_-} = \rho|_{\Gamma_-}$. For a bilinear function on a quadrilateral element holds that the maximum and minimum are obtained on the vertices. Thus $\rho^h(x) \leq \rho^h(y_j) \leq \sigma_u$ and $\rho^h(x) \geq \rho^h(y_j) \geq \sigma_A > 0$, $j = 1,..,4$ for every element $\tau \in \mathcal{T}^h$. For every function in $\rho^h \in V_A^h$ follows $|\nabla \rho^h| \leq \sigma_p$. Thus every function $\rho^h \in V_A^h$ is also a function of V_A. □

7.4.3 A Priori Estimate

We want to determine an error estimate for the discrete solution ρ^h. As first step, we determine an a priori estimate for the Galerkin discretization.

Lemma 7.14. *Let $\rho \in V_A$ be the solution to (Tr 7.2) and $\rho^h \in V_A^h$ be the Galerkin solution. Then holds for all $v \in V_A$ and $v^h \in V_A^h$*

$$\begin{aligned}|||\rho - \rho^h|||_G^2 \leq{}& 4c_1\|v - \rho^h\|_{H^1(\Omega)} + 4c_1\|v^h - \rho\|_{H^1(\Omega)} + 4|\rho - v^h|_\Gamma^2 \\ &+ \frac{16}{\sigma_A}\|E \cdot \nabla\left(\rho - v^h\right)\|_{L^2(\Omega)}^2 + \frac{32}{\sigma_A}(2\sigma_u - \sigma_A)^2\|\rho - v^h\|_{L^2(\Omega)}^2\end{aligned}$$

with $c_1 = c(\Omega)\left(\sup|E|\sigma_p + \sigma_u^2\right)$.

Proof. We follow the method in [28, Theorem 1]. By the definition of ρ and ρ^h holds

$$\begin{aligned}\langle A\rho,\, v - \rho\rangle &\geq 0 && \forall v \in V_A,\\ \langle A\rho^h,\, v^h - \rho^h\rangle &\geq 0 && \forall v^h \in V_A^h.\end{aligned}$$

Adding the previous inequalities gives

$$\langle A\rho,\, \rho\rangle + \langle A\rho^h,\, \rho^h\rangle \leq \langle A\rho,\, v\rangle + \langle A\rho^h,\, v^h\rangle.$$

Subtracting $\langle A\rho,\, \rho^h\rangle + \langle A\rho^h,\, \rho\rangle$ from both sides gives

$$\langle A\rho,\, \rho - \rho^h\rangle + \langle A\rho^h,\, \rho^h - \rho\rangle \leq \langle A\rho,\, v - \rho^h\rangle + \langle A\rho^h,\, v^h - \rho\rangle.$$

Grouping terms and adding $-\langle A\rho,\, \rho - v^h\rangle + \langle A\rho,\, \rho - v^h\rangle$ to the right-hand side leads to

$$\begin{aligned}\langle A\rho - A\rho^h,\, \rho - \rho^h\rangle &\leq \langle A\rho,\, v - \rho^h\rangle - \langle A\rho,\, \rho - v^h\rangle + \langle A\rho,\, \rho - v^h\rangle + \langle A\rho^h,\, v^h - \rho\rangle\\ &= \langle A\rho,\, v - \rho^h\rangle + \langle A\rho,\, v^h - \rho\rangle + \langle A\rho - A\rho^h,\, \rho - v^h\rangle.\end{aligned}$$

By Lemma 7.9 and the Cauchy Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{2}|||\rho-\rho^h|||_G^2 &\le \langle A\rho - A\rho^h,\, \rho-\rho^h\rangle \\ &\le \langle A\rho,\, v-\rho^h\rangle + \langle A\rho,\, v^h-\rho\rangle + \langle A\rho - A\rho^h,\, \rho - v^h\rangle \\ &\le \|A\rho\|_{\tilde H^{-1}}\|v-\rho^h\|_{H^1} + \|A\rho\|_{\tilde H^{-1}}\|v^h-\rho\|_{H^1} + \langle A\rho - A\rho^h,\, \rho - v^h\rangle. \qquad (7.25)\end{aligned}$$

We now limit the three terms of (7.25). For the first term of (7.25) holds due to the inclusion $\tilde H^{-1}(\Omega) \supset L^2(\Omega)$

$$\|A\rho\|_{\tilde H^{-1}} = \sup_{z\in K}\frac{(A\rho, z)_{0,\Omega}}{\|z\|_{H^1(\Omega)}} \le \|A\rho\|_{L^2(\Omega)}.$$

We get for the L^2-norm of $A\rho$

$$\begin{aligned}\|A\rho\|_{L^2(\Omega)} &= \|E\cdot\nabla\rho - \rho^2\|_{L^2} \le \|E\cdot\nabla\rho\|_{L^2} + \|\rho^2\|_{L^2} \\ &= \left(\int_\Omega (E\cdot\nabla\rho)^2\; dx\right)^{\frac12} + \left(\int_\Omega \rho^4\, dx\right)^{\frac12} \\ &= \left(\mathrm{vol}(\Omega)\,(E\cdot\nabla\rho)^2\; dx\right)^{\frac12} + \left(\mathrm{vol}(\Omega)\int_\Omega \rho^4\, dx\right)^{\frac12} \qquad (7.26)\\ &\le c(\Omega)\left(\sup|E|\sigma_p + \sigma_u^2\right) =: c_1. \qquad (7.27)\end{aligned}$$

Next, we bound the term $\langle A\rho - A\rho^h, \rho - v^h\rangle$ of (7.25). Green's formula reads [46, Equation (1.4)]

$$(E\cdot\nabla v,\, w)_{L^2(\Omega)} = (\vec n\cdot E v,\, w)_{L^2(\Gamma)} - (v,\, E\cdot\nabla w)_{L^2(\Omega)} - (v,\, w\,\mathrm{div}\,E)_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned}&\langle A\rho - A\rho^h, \rho - v^h\rangle = \int_\Omega \Big(E\cdot\nabla(\rho-\rho^h)\Big)(\rho - v^h) + (\rho^2-\rho_h^2)(\rho-v^h)\,dx \\ &= \int_\Gamma \vec n\cdot E(\rho-\rho^h)(\rho-v^h)\,ds_x + \int_\Omega (\rho^h-\rho)E\cdot\nabla(\rho-v^h) + (\rho+\rho_h-\mathrm{div}\,E)(\rho-\rho_h)(\rho-v^h)\,dx \\ &\le |\rho-\rho^h|_\Gamma|\rho-v^h|_\Gamma + \|\rho-\rho^h\|_{L^2(\Omega)}\|E\cdot\nabla\left(\rho-v^h\right)\|_{L^2(\Omega)} \\ &\quad + \|\rho-\rho_h\|_{L^2(\Omega)}\|\underbrace{(\rho+\rho_h-\mathrm{div}\,E)}_{\le 2\sigma_u-\sigma_A}(\rho-v^h)\|_{L^2(\Omega)}. \qquad (7.28)\end{aligned}$$

With the Cauchy-Schwarz inequality, we have with the method of [46, Theorem 1.1]

$$\begin{aligned}|\rho-\rho^h|_\Gamma\,|\rho-v^h|_\Gamma &= \sqrt{\frac12}|\rho-\rho^h|_\Gamma\,\sqrt{2}|\rho-v^h|_\Gamma \\ &\le \frac14|\rho-\rho^h|_\Gamma^2 + |\rho-v^h|_\Gamma^2 \\ \|\rho-\rho^h\|_{L^2(\Omega)}\,\|E\cdot\nabla\left(\rho-v^h\right)\|_{L^2(\Omega)} &= \sqrt{\frac{\sigma_A}{8}}\|\rho-\rho^h\|_{L^2(\Omega)}\sqrt{\frac{8}{\sigma_A}}\|E\cdot\nabla\left(\rho-v^h\right)\|_{L^2(\Omega)} \\ &\le \frac{\sigma_A}{16}\|\rho-\rho^h\|_{L^2(\Omega)}^2 + \frac{4}{\sigma_A}\|E\cdot\nabla\left(\rho-v^h\right)\|_{L^2(\Omega)}^2 \\ (2\sigma_u-\sigma_A)\|\rho-\rho^h\|\|\rho-v^h\|_{L^2(\Omega)} &= \sqrt{\frac{\sigma_A}{8}}\|\rho-\rho^h\|_{L^2(\Omega)}\sqrt{\frac{8}{\sigma_A}}(2\sigma_u-\sigma_A)\|\rho-v^h\|_{L^2(\Omega)} \\ &\le \frac{\sigma_A}{16}\|\rho-\rho^h\|_{L^2(\Omega)}^2 + \frac{8}{\sigma_A}(2\sigma_u-\sigma_A)^2\|\rho-v^h\|_{L^2(\Omega)}^2.\end{aligned}$$

Hence, it follows for (7.28)

$$
\begin{aligned}
\langle A\rho - A\rho^h, \rho - v^h \rangle &\leq \frac{1}{4}|\rho - \rho^h|_\Gamma^2 + \frac{\sigma_A}{8}\|\rho - \rho^h\|_{L^2(\Omega)}^2 \\
&+ |\rho - v^h|_\Gamma^2 + \frac{4}{\sigma_A}\|E \cdot \nabla \left(\rho - v^h\right)\|_{L^2(\Omega)}^2 + \frac{8(2\sigma_u - \sigma_A)^2}{\sigma_A}\|\rho - v^h\|_{L^2(\Omega)}^2 \\
&= \frac{1}{4}|||\rho - \rho^h|||_G^2 + |\rho - v^h|_\Gamma + \frac{4}{\sigma_A}\|E\nabla(\rho - v^h)\|_{L^2(\Omega)}^2 + \frac{8(2\sigma_u - \sigma_A)^2}{\sigma_A}\|\rho - v^h\|_{L^2(\Omega)}^2. \qquad (7.29)
\end{aligned}
$$

Substituting (7.27) and (7.29) into (7.25), we have finally

$$
\begin{aligned}
\frac{1}{4}|||\rho - \rho^h|||_G^2 &\leq c_1\|v - \rho^h\|_{H^1} + c_1\|v^h - \rho\|_{H^1(\Omega)} + |\rho - v^h|_\Gamma^2 \\
&+ \frac{4}{\sigma_A}\|E \cdot \nabla \left(\rho - v^h\right)\|_{L^2(\Omega)}^2 + \frac{8}{\sigma_A}(2\sigma_u - \sigma_A)^2\|\rho - v^h\|_{L^2(\Omega)}^2.
\end{aligned}
$$

□

By renaming the constants follows the next Corollary.

Corollary 7.15. *Let $\rho \in V_A$ be the solution to (Tr 7.2) and $\rho^h \in V_A^h$ be the Galerkin solution. Then holds for all $v \in V_A$ and $v^h \in V_A^h$*

$$
|||\rho - \rho^h|||_G^2 \leq c_3(\|v - \rho^h\|_{H^1(\Omega)} + \|v^h - \rho\|_{H^1(\Omega)}) + c_4\|v^h - \rho\|_{H^1(\Omega)}^2
$$

with $c_3 := 4c(\Omega)\left(\sup|E|\sigma_p + \sigma_u^2\right)$ and $c_4 := 4c_b \sup_\Omega\{|E|\} + \frac{16}{\sigma_A}\sup_\Omega\{|E|\} + \frac{32}{\sigma_A}(2\sigma_u - \sigma_A)^2$ where σ_p, σ_u and σ_A are defined in the set V_A.

Proof. We bound the terms in Lemma 7.14. Let us first reduce

$$
\|E \cdot \nabla(\rho - v^h)\|_{L^2(\Omega)}^2 \leq \sup_\Omega\{|E|\}\|\nabla\rho - \nabla v^h\|_{L^2(\Omega)}^2.
$$

By the trace theorem [13, Theorem 3.1], it holds for the boundary norm

$$
\begin{aligned}
|\rho - v^h|_\Gamma^2 &\leq \sup\{|n \cdot E|\}\|\rho - v^h\|_{L^2(\Gamma)}^2 \\
&\leq c_b \sup\{E|\}\|\rho - v^h\|_{H^1(\Omega)}^2.
\end{aligned}
$$

The assertion follows with Lemma 7.14 and choosing

$$
c_3 := 4c(\Omega)\left(\sup|E|\sigma_p + \sigma_u^2\right)
$$

and

$$
c_4 := 4c\sup_\Omega|E| + \frac{16}{\sigma_A}\sup_\Omega\{|E|\} + \frac{32}{\sigma_A}(2\sigma_u - \sigma_A)^2
$$

with c being a constant and σ_u, σ_A and σ_p defined in the space V_A. □

In the next Lemma, we will obtain an error estimate for (Tr 7.2) and prove the convergence of the discrete solution to ρ. Here it will become decisive to have a "good" approximation V_A^h to V_A such that we will finally clarify the meaning.

Theorem 7.16. *Let $\rho \in H^2(\Omega)$ be the solution to (Tr 7.2). For the Galerkin solution $\rho^h \in V_A^h$ holds*

$$|||\rho - \rho^h|||_G^2 \leq \max\{c_3, c_4\}\, c(\Omega) h(\|\rho\|_{H^2(\Omega)} + h\|\rho\|_{H^2(\Omega)}^2)$$

with c_3 and c_4 defined in Corollary 7.15.

Proof. Corollary 7.15 yields

$$|||\rho - \rho^h|||_G^2 \leq c_3(\|v - \rho^h\|_{H^1(\Omega)} + \|v^h - \rho\|_{H^1(\Omega)}) + c_4\|v^h - \rho\|_{H^1(\Omega)}^2 \tag{7.30}$$

for all $v \in V_A$ and $v^h \in V_A^h$.

We begin by estimating the term $\|v - \rho^h\|_{H^1}$. Due to Lemma 7.13 holds $V_A^h \subset V_A$. Since (7.30) is valid for all $v \in V_A$ and $\rho^h \in V_A^h$, we may choose a particular $v \in V_A$. We choose $v = \rho^h \in V_A^h \subset V_A$. The term reduces to

$$\|v - \rho^h\|_{H^1} = \|\rho^h - \rho^h\|_{H^1} = 0.$$

It remains to bound $\|\rho - v^h\|_{H^1(\Omega)}$. By [13, Theorem 6.7, p.82] holds

$$\|\rho - v^h\|_{H^1(\Omega)}^2 \leq c(\Omega)h^2\|\rho\|_{H^2(\Omega)}^2.$$

It results for (7.30)

$$\begin{aligned}
|||\rho - \rho^h|||_G^2 &\leq c_3 c(\Omega) h\|\rho\|_{H^2(\Omega)} + c_4 c(\Omega)h^2\|\rho\|_{H^2(\Omega)}^2 \\
&\leq \max\{c_3,\ c_4\}\, c(\Omega) h(\|\rho\|_{H^2(\Omega)} + h\|\rho\|_{H^2(\Omega)}^2).
\end{aligned}$$

□

We thus obtained an error estimate for the discrete solution ρ^h. For $h \to 0$ follows that ρ^h converges to ρ.

7.4.4 Connection between (Tr 6.3) and (Tr 7.2)

In the previous section, we investigated the variational inequality (Tr 7.2) associated to the nonlinear transport problem and proved that a continuous solution exists in the set V_A. Further, we have shown that the discrete solution ρ^h converges to the continuous solution ρ. Recall that T was the solution operator of the transport equation defined in Theorem 6.39. To distinguish the two problem settings, let T_A be the solution operator for (Tr 7.2) given by

$$T_A E = \rho \in V_A. \tag{7.31}$$

It is immediately clear that the two solutions $T_A E$ and TE are not elements of the same space and thus do not need necessarily to be the same. While TE is the classical solution to (Tr 6.3), $T_A E$ is the solution to the variational inequality and is thus included in the Sobolev space

$H^1(\Omega)$. Moreover, (Tr 7.2) restricts the solution T_AE by incorporating pointwise bounds for T_AE and ∇T_AE into the ansatz space V_A. The goal is to choose the constants σ_A, σ_p and σ_u such that the classical solution $TE \in V_A$. As we have proved existence and uniqueness of both solutions TE to (Tr 7.2) and T_AE to (CP 6.1), we could then conclude that $T_AE \equiv TE$ and the problems (CP 6.1) and (Tr 7.2) would be equivalent. Furthermore, we obtain that the discrete solution ρ^h of section 7.4.2 also converges to TE and arrive at the main purpose of this section: We have found a way to discretize the nonlinear transport problem (CP 6.1).

Let us start to examine the constants σ_A, σ_p and σ_u. With the notations of Chapter 6 and Theorem 6.39, the solution operator T is given by

$$TE(x) = \hat{\rho}(\Phi^{-1}(E,x)) := \frac{\rho_A}{1+s\rho_A}\Big|_{(s,t)=\Phi^{-1}(E,x)} \tag{7.32}$$

with $\rho_A > 0$ constant. Recall the set $W(M,\delta_1,\delta_2,\delta_3)$ defined in Chapter 6 by

$$\begin{aligned} W(M,\delta_1,\delta_2,\delta_3) = \big\{E = E_0 + E_1 \in C^{1,\alpha}(\Omega)\,:\, & E_0, E_1 \in C^{1,\alpha}(\Omega), E = -\nabla u \text{ gradient field}, \\ & u|_{\Gamma_-} = u_{A_1}, u|_{\Gamma_+} = u_{A_2},\ \|E\|_{1,\alpha;\Omega} \le M,\ \inf_{x\in\bar{\Omega}} |E(x)|_\infty \ge \delta_1, \\ & \|E_1\|_{0,Q} \le \delta_3,\ \vec{n}\cdot E \le \delta_2 < 0 \text{ on } \Gamma_-,\ \vec{n}\cdot E > 0 \text{ on } \Gamma_+\big\}. \end{aligned}$$

To show the equivalence of the solutions TE and T_AE, we need to restrict the set $W(M,\delta_1,\delta_2,\delta_3)$ further. Define

$$\begin{aligned} W(M,\delta_1,\delta_2,\delta_3,\sigma_A) = \big\{E = E_0 + E_1 \in C^{1,\alpha}(\Omega)\,:\, & E_0, E_1 \in C^{1,\alpha}(\Omega), E = -\nabla u \text{ gradient field}, \\ & u|_{\Gamma_-} = u_{A_1}, u|_{\Gamma_+} = u_{A_2},\ \|E\|_{1,\alpha;\Omega} \le M,\ \inf_{x\in\bar{\Omega}} |E(x)|_\infty \ge \delta_1, \\ & \|E_1\|_{0,Q} \le \delta_3,\ \vec{n}\cdot E \le \delta_2 < 0 \text{ on } \Gamma_-,\ \vec{n}\cdot E > 0 \text{ on } \Gamma_+, \frac{1}{2}\operatorname{div} E(x) \le \sigma_A\Big\}. \end{aligned}$$

Given $E \in W(M,\delta_1,\delta_2,\delta_3,\sigma_A)$ and ρ_A sufficiently small, we are able to prove that $TE \in V_A$.

Theorem 7.17. *Let Ω be a $C^{2,\alpha}$ domain and $E \in W(M,\delta_1,\delta_2,\delta_3,\sigma_A)$. Choose $\sigma_A := \frac{\rho_A}{2}$, $\sigma_u := \rho_A$ and*

$$\sigma_p := \rho_A^2 \frac{2}{\delta_2} \exp\Big(c_l \,\|\operatorname{div} E\|_{0,\Omega}\Big)\left(\|E\|_{0,\Omega} + \exp\left(c_l \|\nabla E\|_{0,\Omega}\right)\right).$$

with c_l defined in Theorem 6.27. Moreover, let $\rho_A \le \frac{1}{c_l}$. Then holds $TE \in V_A$.

Proof. Let $(s,t) = \vec{\Phi}^{-1}(x) = (\Phi_1^{-1}(x), \Phi_2^{-1}(x))$. Then TE is bounded pointwise for every $x \in \Omega$ by

$$|TE(x)| \le \sup_{x\in\Omega} \left|\frac{\rho_A}{1+\Phi_1^{-1}(E,x)\rho_A}\right| \le \rho_A \underbrace{\left\|\frac{1}{1+\Phi_1^{-1}(E)\rho_A}\right\|_{0,\Omega}}_{\le 1} \le \rho_A. \tag{7.33}$$

Second, we bound TE pointwise from below. By Theorem 6.27 and $\rho_A \leq \frac{1}{c_l}$ holds for every $x \in \Omega$

$$
\begin{aligned}
|TE(x)| &\geq \inf_{x\in\Omega} \left| \frac{\rho_A}{1+\Phi_1^{-1}(E,x)\rho_A} \right| \\
&\geq \frac{\rho_A}{\sup_{x\in\Omega} |1+\Phi_1^{-1}(E,x)\rho_A|} \\
&\geq \frac{\rho_A}{1+c_l\rho_A} \\
&\geq \frac{\rho_A}{2}.
\end{aligned} \tag{7.34}
$$

It is left to prove a pointwise estimate for the gradient. Since ρ_A is constant, it follows that $\rho_A' = 0$. By Lemma 6.41 and Lemma 6.31, we have

$$
\begin{aligned}
|\nabla TE(x)|_\infty &= \left|\nabla_{(s,t)}\hat{\rho}(\Phi^{-1}(x))\nabla\Phi^{-1}(x)\right|_\infty \\
&\leq \left\|\nabla_{(s,t)}\hat{\rho}(\Phi^{-1})\right\|_{0,\Omega} \left|\nabla_x\Phi^{-1}(x)\right|_\infty \\
&\leq \max\left\{\|\rho_A\|_{0,\Omega}^2,\, \|\rho_A'\|_{0,Q}\right\} \frac{2}{\delta_2} \exp\left(c_l \left\|\operatorname{div} E\right\|_{0,\Omega}\right) \max\left\{\|\partial_t\Phi\|_{0,Q}\,,\, \|\partial_s\Phi\|_{0,Q}\right\} \\
&\leq \rho_A^2 \frac{2}{\delta_2} \exp\left(c_l \left\|\operatorname{div} E\right\|_{0,\Omega}\right) \max\left\{\|\partial_t\Phi\|_{0,Q}\,,\, \|\partial_s\Phi\|_{0,Q}\right\}
\end{aligned} \tag{7.35}
$$

$\nabla\Phi$ is bounded by Lemma 6.29

$$
\|\nabla\Phi\|_{0,Q} \leq \|E\|_{0,\Omega} + \exp\left(c_l\|\nabla E\|_{0,\Omega}\right).
$$

It follows for (7.35)

$$
|\nabla TE(x)|_\infty \leq \frac{2\rho_A^2}{\delta_2} \exp\left(c_l \left\|\operatorname{div} E\right\|_{0,\Omega}\right) \left(\|E\|_{0,\Omega} + \exp\left(c_l\|\nabla E\|_{0,\Omega}\right)\right). \tag{7.36}
$$

Choose the constants in the space V_A by $\sigma_u := \rho_A$, $\sigma_A := \frac{1}{2}\rho_A$ and

$$
\sigma_p := \rho_A \frac{2}{\delta_2} \exp\left(c_l \left\|\operatorname{div} E\right\|_{0,\Omega}\right) \left(\|E\|_{0,\Omega} + \exp\left(c_l\|\nabla E\|_{0,\Omega}\right)\right).
$$

Then the classical solution $TE \in V_A$. $\square$

Next, we prove that TE and T_AE are equal.

Lemma 7.18. *Let Ω be a $C^{2,\alpha}$ domain and $E \in W(M, \delta_1, \delta_2, \delta_3, \sigma_\rho)$. Let the constants σ_A, σ_u and σ_p of the space V_A be chosen as in Theorem 7.17. Then holds that $TE \equiv T_AE$.*

Proof. Since $E \in W(M, \delta_1, \delta_2, \delta_3, \sigma_\rho)$ and by Theorem 6.39, $TE \in C^{1,\alpha}(\Omega)$ is the unique solution to (Tr 6.3). Due to Theorem 7.12, there exists a unique solution $T_AE \in V_A$. With the choice of the constants in Theorem 7.17, it holds that $TE \in V_A$. Since we have proved uniqueness of the solution in both cases, it holds that $TE \equiv T_AE$. $\square$

Last, we show that the discrete solution T_A^h converges to the classical solution TE for $h \to 0$.

Lemma 7.19. *Let $TE \in C^{1,\alpha}(\bar{\Omega})$ the solution of (CP 6.1). Let further $T_AE \in H^2(\Omega)$ be the exact solution and $T_A^hE \in V_A^h$ the Galerkin solution of (Tr 7.2) in section 7.4.2. Then yields*

$$|||TE - T_A^hE|||_G^2 \leq \max\{c_3, c_4\}\, c(\Omega)h(\|T_AE\|_{H^2(\Omega)} + h\|T_AE\|_{H^2(\Omega)}^2) \tag{7.37}$$

with c_3 and c_4 defined in Corollary 7.15.

Proof. Since $TE \in C^{1,\alpha}(\bar{\Omega})$ as classical solution of (CP 6.1), it is also element of $L^2(\Omega)$ and $L^2(\Gamma)$. By Theorem 7.16 follows

$$\begin{aligned}
|||TE - T_A^hE|||_G^2 &= |||TE - T_AE + T_AE - T_A^hE|||_G^2 \\
&\leq |||TE - T_AE|||_G^2 + |||T_AE - T_A^hE|||_G^2 \\
&\leq 0 + |||T_AE - T_A^hE|||_G^2 \\
&\leq \max\{c_3, c_4\}\, c(\Omega)h(\|T_AE\|_{H^2(\Omega)} + h\|T_AE\|_{H^2(\Omega)}^2)
\end{aligned}$$

□

Remark 7.20. *To prove that $TE \equiv T_AE$, we had to add an additional restriction into the space $W(M, \delta_1, \delta_2, \delta_3)$ by $\frac{1}{2}\operatorname{div} E \leq \sigma_A$. Since all our results should be considered in the framework of the coupled problem, the question is whether the existence result in Chapter 6 is still valid for the new space $W(M, \delta_1, \delta_2, \delta_3, \sigma_A)$. The contraction property of the composition $L \circ T$ holds unchanged, as it depends only on the choice of ρ_A and restrictions on E that are still available in $W(M, \delta_1, \delta_2, \delta_3, \sigma_A)$. We thus only need to verify that $L \circ T$ is a selfmap on $W(M, \delta_1, \delta_2, \delta_3, \sigma_A)$. First of all, we show that $\frac{1}{2}\operatorname{div} E \leq \sigma_A$ is a sensible restriction in case of the coupled problem. For the solution (u, ρ) to (CP 6.1) holds $\rho = \operatorname{div} E$. Then follows by (7.33) and (7.34) and since $\frac{1}{2}\rho_A = \sigma_A$ for all $x \in \Omega$*

$$\rho_A \geq \rho(x) \geq \frac{1}{2}\rho_A = \sigma_A \geq \frac{1}{2}\operatorname{div} E(x) = \frac{1}{2}\rho(x). \tag{7.38}$$

The assumptions are thus not contradictory for the solution (u, ρ). We will now show that $L \circ T$ is a selfmap on $W(M, \delta_1, \delta_2, \delta_3, \sigma_A)$. Since $L\rho = -\nabla u$ and u being the solution of the Poisson equation, it follows

$$\operatorname{div}(L \circ TE^{n-1}) = TE^{n-1}.$$

Hence, by (7.33)

$$\operatorname{div}(L \circ TE^{n-1}) \leq \rho_A$$

and

$$\frac{1}{2}\operatorname{div}(L \circ TE^{n-1}) \leq \frac{1}{2}\rho_A = \sigma_A.$$

Conclusively, a classical solution also exists to (CP 6.1) in $W(M, \delta_1, \delta_2, \delta_3, \sigma_A)$. The discretization method introduced in the previous section is therefore applicable to discretize the nonlinear transport problem in (CP 6.1).

7.5 The Staggered Algorithm

In the previous sections, we obtained approximation results for the Poisson equation as well as linear and nonlinear transport equation. In this section, we introduce the *staggered algorithm* which is an algorithm used to solve the discrete coupled problem. This section should be understood as an outline for future work and we do not claim completeness.

The staggered algorithm is an iterative method to solve the coupled problem, initially applied by [1, 59]. The underlying idea is simple: Starting off by initializing a first vector field E, e.g. by solving the Laplace equation, the transport and Poisson equations are solved alternating until convergence is obtained. Recall that in Chapter 6 we used the solution operator L of the Poisson equation and the solution operator T of the transport equation. The composition $L \circ T$ is a self-map and contraction on the set $W(M, \delta_1, \delta_2, \delta_3)$. Conclusively, Banach's fixed point iterations are the staggered algorithm.

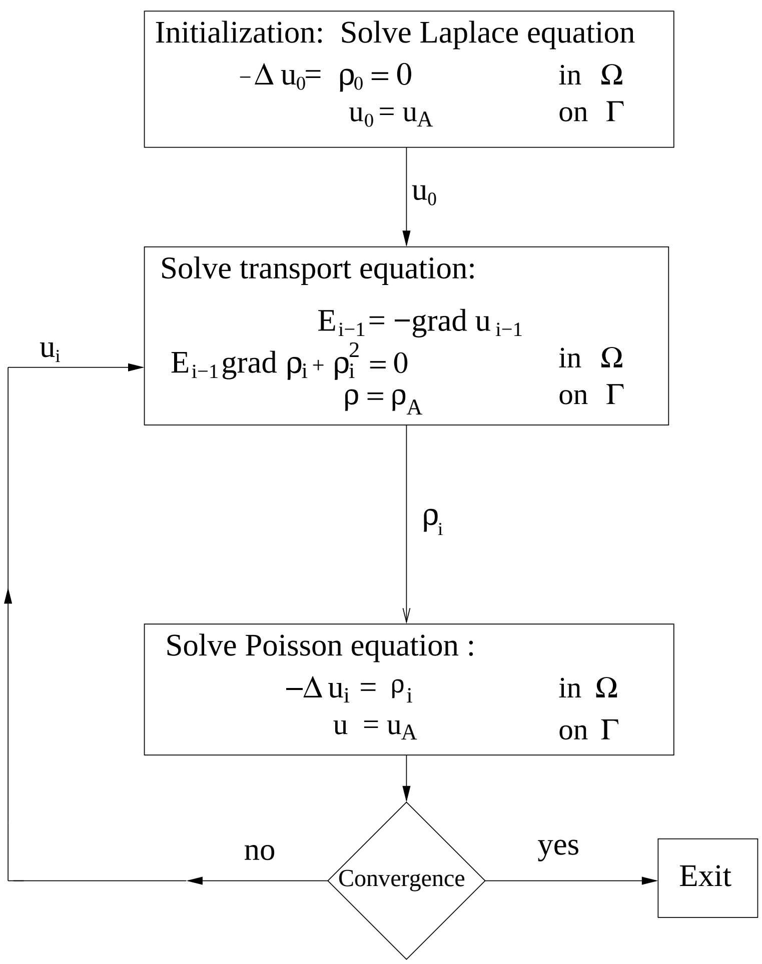


Figure 7.1: Staggered Algorithm

In this section only, we will apply the transport solution operator to the Poisson solution u, i.e.

$$Tu = \rho \tag{7.39}$$

$$L\rho = u. \tag{7.40}$$

For a a finite element implementation, we have to discretize the coupled problem. Let therefore L^h be the solution operator of the discrete Poisson equation and T^h the solution operator of the discrete transport equation. i.e.

$$T^h u^h = \rho^h \tag{7.41}$$

$$L^h \rho^h = u^h. \tag{7.42}$$

In this chapter, we will again conduct our analysis in Sobolev spaces. Based on the results of Chapter 6 and due to numerical evidence, we assume that analogous results of Chapter 6 hold in Sobolev spaces. The error estimate of the Banach fixed point iterations together with approximation and stability estimates of the Poisson and transport problem lead to an immediate error bound of the discrete staggered algorithm. We thus eventually clarify the advantage of using the fixed point approach of Chapter 6 in comparison of applying a compactness argument similar to Chapter 4 or as it is done in [5, 56].

Let us define the following sequences

$$\begin{aligned} u_n &= L \circ T u_{n-1}, \\ u_n^h &= L^h \circ T^h u_{n-1}^h, \\ \tilde{u}_n^h &= L \circ T u_n^h. \end{aligned}$$

To prove that the staggered algorithm converges, we want to obtain a bound of the kind

$$\|u - u_n^h\|_{H^1(\Omega)} = \|L \circ Tu - L^h \circ T^h u_n^h\|_{H^1(\Omega)} \leq c_1^n |u|_{H^2(\Omega)} + c_2 h |u|_{H^2(\Omega)}$$

with $c_1 < 1$. We now make the following assumptions:

(1) $L \circ T$ is a contraction: Let $L \circ T$ be a contraction on $H^1(\Omega)$, that is for $u_1, u_2 \in H^1(\Omega)$ holds

$$\|L \circ Tu_1 - L \circ Tu_2\|_{H^1(\Omega)} \leq K \|u_1 - u_2\|_{H^1(\Omega)} \tag{7.43}$$

with $K < 1$.

Furthermore, use a discretization method such that the following two assumptions are fulfilled.
(2) Error estimate: Let $Tu^h \in H^1(\Omega)$ and $0 < \beta \leq 1$. For $u^h \in V^h$ holds

$$\|Tu^h - T_h u^h\|_{L^2(\Omega)} \leq c_E^T h^\beta \|Tu^h\|_{H^1(\Omega)}. \tag{7.44}$$

for a constant c_E^T independent of h and Tu^h.
(3) Boundedness: Let u_0^h be the solution of the discrete Poisson equation.
For $u_n^h = L^h \circ T^h u_{n-1}^h \in V_h^p$, $n > 1$, follows

$$\|Tu_n^h\|_{H^1(\Omega)} \leq c_T \tag{7.45}$$

for a constant c_T independent of h.

Remark 7.21. *One possible discretization technique that might fulfil condition (2) is the dicontinuous Galerkin method. In case of the linear transport equation, we have the following error estimate [45, p. 194].*

$$\|\rho - \rho^h\|_{L^2} \leq Ch^{\frac{1}{2}} \|\rho\|_{H^1(\Omega)}.$$

Since Tu_n^h is bounded by the inflow boundary data, we know that $\|Tu_n^h\|_{L^2(\Omega)} \leq \rho_A$. Numerical experiments backed up assumption (2) where we clearly obtained that $\|Tu_n^h\|_{H^1(\Omega)}$ is bounded by a constant for all $h = \frac{1}{2^i}$ with $i = 0, .., 8$ and n until convergence is obtained.

The contraction property of the operator $L \circ T$ is the crucial point in the following argumentation. Let now u be the exact Poisson solution of the coupled problem. Then holds $u = L \circ Tu$. We begin with the first step to prove the convergence of the discrete staggered algorithm.

$$\begin{aligned} \|u - u_n^h\|_{H^1(\Omega)} &= \|u - u_n + u_n - u_n^h\|_{H^1(\Omega)} \\ &\leq \|u - u_n\|_{H^1(\Omega)} + \|u_n - u_n^h\|_{H^1(\Omega)}. \end{aligned} \tag{7.46}$$

By (7.43), the first term of (7.46) is bounded by

$$\|u - u_n\|_{H^1(\Omega)} = \|L \circ Tu - L \circ Tu_{n-1}\|_{H^1(\Omega)} \leq K\|u - u_{n-1}\|_{H^1(\Omega)}.$$

With the error estimate of the Banach fixed point iterations, we obtain

$$\|u_1 - u_n\|_{H^1(\Omega)} \leq \frac{K^n}{1-K}\|u - u_0\|_{H^1(\Omega)}.$$

In the second term of (7.46), u_n and u_n^h are not immediately comparable since neither stability nor approximation results are applicable. We perturb the difference by adding the auxiliary function $\pm\tilde{u}_n^h$. The new occurring combinations of terms can be estimated.

$$\begin{aligned} \|u_n - u_n^h\|_{H^1(\Omega)} &= \|u_n - \tilde{u}_n^h + \tilde{u}_n^h - u_n^h\|_{H^1(\Omega)} \\ &\leq \|u_n - \tilde{u}_n^h\|_{H^1(\Omega)} + \|\tilde{u}_n^h - u_n^h\|_{H^1(\Omega)} \\ &= \|L \circ Tu_{n-1} - L \circ Tu_{n-1}^h\|_{H^1(\Omega)} + \|L \circ Tu_{n-1}^h - L_h \circ T_h u_{n-1}^h\|_{H^1(\Omega)}. \end{aligned} \tag{7.47}$$

The first term of (7.47) is bounded by the contraction property (7.43)

$$\|L \circ Tu_{n-1} - L \circ Tu_{n-1}^h\|_{H^1(\Omega)} \leq K\|u_{n-1} - u_{n-1}^h\|_{H^1(\Omega)} \tag{7.48}$$

with $K < 1$.

We proceed to the second term of (7.47). Adding $\pm L_h \circ Tu_{n-1}^h$ and the triangle inequality gives

$$\begin{aligned} &\|L \circ Tu_{n-1}^h - L_h \circ T_h u_{n-1}^h\|_{H^1(\Omega)} \\ &= \|L \circ Tu_{n-1}^h - L_h \circ Tu_{n-1}^h + L_h \circ Tu_{n-1}^h - L_h \circ T_h u_{n-1}^h\|_{H^1(\Omega)} \\ &\leq \|L \circ Tu_{n-1}^h - L_h \circ Tu_{n-1}^h\|_{H^1(\Omega)} + \|L_h \circ Tu_{n-1}^h - L_h \circ T_h u_{n-1}^h\|_{H^1(\Omega)}. \end{aligned} \tag{7.49}$$

The first term of (7.49) is bounded with the error estimate (7.13) of the Poisson equation

$$\|L \circ T u_{n-1}^h - L_h \circ T u_{n-1}^h\|_{H^1(\Omega)} \leq c_E^L h \|L \circ T u_{n-1}^h\|_{H^2(\Omega)}.$$

With the stability estimate (7.14) follows

$$\|L \circ T u_{n-1}^h\|_{H^2(\Omega)} \leq \frac{1}{\alpha} \|T u_{n-1}^h\|_{L^2(\Omega)}. \tag{7.50}$$

For the second term of (7.49), we use the continuity of L_h with continuity constant c_{L_h} and (7.44) and get

$$\begin{aligned} \|L_h \circ T u_{n-1}^h - L_h \circ T_h u_{n-1}^h\|_{H^1(\Omega)} &\leq c_{L_h} \|T u_{n-1}^h - T_h u_{n-1}^h\|_{L^2(\Omega)} \\ &\leq c_E^T c_{L_h} h^\beta \|T u_{n-1}^h\|_{H^1(\Omega)}. \end{aligned} \tag{7.51}$$

We get for (7.47) with (7.48), (7.50), (7.51) and (7.45)

$$\begin{aligned} \|u_n - u_n^h\|_{H^1(\Omega)} &\leq K \|u_{n-1} - u_{n-1}^h\|_{H^1(\Omega)} + \frac{c_E^L}{\alpha} h \|T u_{n-1}^h\|_{L^2} + c_E^T c_{L_h} h^\beta \|T u_{n-1}^h\|_{H^1(\Omega)} \\ &\leq K \|u_{n-1} - u_{n-1}^h\|_{H^1(\Omega)} + (\frac{c_E^L}{\alpha} + c_E^T c_{L_h}) h^\beta c_T \\ &=: K \|u_{n-1} - u_{n-1}^h\|_{H^1(\Omega)} + \delta h^\beta. \end{aligned}$$

The first term comes from the contraction property of $L \circ T$ with a constant $K < 1$. Further, we have a second term δh. Thus with successive iterations, the first term will become smaller. By reducing the mesh size, the second term will diminish. Let us denote $\epsilon_n := \|u_n - u_n^h\|_{H^1(\Omega)}$. Then we have

$$\epsilon_n \leq K \epsilon_{n-1} + \delta h^\beta.$$

Then we have for ϵ_0 with the knowledge that u_0 is the solution of the Laplace equation

$$\epsilon_0 = \|u_0 - u_0^h\|_{H^1(\Omega)} \leq h \|u_0\|_{H^2(\Omega)}.$$

Let us now write down the first iterations

$$\begin{aligned} &\epsilon_1 \leq K \epsilon_0 + \delta h^\beta \\ &\epsilon_2 \leq K \epsilon_1 + \delta h^\beta = K^2 \epsilon_0 + K \delta h^\beta + \delta h^\beta. \end{aligned}$$

We obtain

$$\epsilon_n \leq \epsilon_0 K^n + \delta \ h^\beta \sum_{i=0}^{n} K^i.$$

For $n \to \infty$, we obtain

$$\epsilon_n \xrightarrow{n \to \infty} \epsilon_\infty := \delta \ h^\beta \sum_{i=0}^{\infty} K^i = \frac{\delta \ h^\beta}{1 - K}.$$

It follows that

$$\|u_n - u_n^h\|_{H^1(\Omega)} \leq \epsilon_\infty \leq \frac{\delta h^\beta}{1-K}$$

with $\delta = (\frac{c_E^L}{\alpha} + c_E^T c_{L_h}) c_L^T$ being independent of h.

We can now formulate an approximation result for the staggered algorithm.

Theorem 7.22. *With the assumptions (1)-(3), the staggered algorithm converges to the continuous solution $u \in H^2(\Omega)$. The error estimates is given by*

$$\|u - L^h \circ T^h u_n^h\|_{H^2(\Omega)} \leq \frac{K^n}{1-K}\|u_1 - u_0\|_{H^1(\Omega)} + \frac{\delta}{1-K} h^\beta$$

with $\delta = (\frac{c_E^L}{\alpha} + c_E^T c_{L_h}) c_L^T$.

7.6 Remarks about the Chapter

In this chapter, we presented discretization methods for the Poisson and transport problem. The main intention was to develop a discretization technique for the nonlinear transport equation. Therefore, we formulated (Tr 6.3) as variational inequality. By the theory of monotone operators, we proved the unique existence of a continuous solution in a set V_A. Further, we developed an error estimate for the Galerkin solution to the corresponding discrete variational problem and thus proved that the discrete solution converges. By restricting the set $W(M, \delta_1, \delta_2, \delta_3)$ used in Chapter 6, we were able to show the equivalence of the classical solution of (Tr 6.3) and the one of the variational inequality (Tr 7.2). Eventually, we presented that the discretization method for (Tr 7.2) is feasible to discretize the nonlinear transport problem of (Tr 6.3).

The second intention in this Chapter was to underline the usage of the Banach fixed point iterations to prove existence and uniqueness of a solution to (CP 6.1). Regarding numerical results, it is of great interest if the discrete staggered algorithm converges to the exact solution. On the basis of the results of Chapter 6, we assumed that the staggered algorithm is also a contraction in the Sobolev space $H^1(\Omega)$. Beside the contraction property, we had to assume stability and error results for the continuous transport solution operator applied to a discrete coefficient function u^h that we have not studied so far. These open problems give an immediate suggestion for future works. In fact, we were able to prove the convergence of the discrete staggered algorithm which followed immediately of the contraction property of the continuous operator $L \circ T$ and error and stability estimates.
In comparison to the compactness arguments presented in Chapter 4, the approach of Chapter 6 proved advantageous if one aims for discretization results.

In the next Chapter, we will conduct numerical experiments for the time independent steady state case. We will investigate the radial symmetric coupled problem of Chapter 5. It is shown numerically that the staggered algorithm quickly converges. The numbers of iterations that are needed numerically to obtain a given accuracy are even below the theoretical number given by the a priori estimate (5.34) for the Banach fixed point iterations. We see herein another confirmation that the assumptions (1)-(3) are likely to be proved.

Chapter 8

Numerical Experiments

In this chapter, we will underline the theoretical results of the Chapters 5 to 7 by numerical experiments. All experiments are performed using the research code *MaiProgs* [54, 4, 53]. *MaiProgs* is a Fortran based programme package that has been initially developed by M. Maischak. During this PhD research, the routines have been extended by the finite element implementation of the transport equation on curved quadrilateral elements. Furthermore, we implemented different algorithms to solve the coupled problem. Considering the continuous coupled problem, the formulations using the nonlinear or linear transport equation are equivalent. In terms of the discretization, it becomes important to distinguish between the formulations. In this chapter, we introduce four algorithms and investigate their convergence behaviour. Next to the staggered algorithm that is used if both the Poisson and transport equations are linear, we will use Newton's method to deal with the nonlinearity of the problem. Here, we compare three different approaches

1. The coupled problem is solved in an outer loop by the staggered algorithm. The nonlinear transport equation is solved in an inner loop with Newton's iteration scheme.
2. The coupled problem is solved by Newton's iteration scheme in an outer loop. The linearized problem is solved by the staggered algorithm in each Newton iteration in an inner loop.
3. The coupled problem is solved by Newton's iteration scheme. The linearized problem is solved as a block-matrix system.

In section 8.1, we will define the algorithms accurately. Let in the following V^P, V_0^P and V^T, V_0^T denote the spaces defined in sections 7.2 and 7.3. The continuous variational formulation of the coupled problem with the linear transport equation is given by

Find $(u, \rho) \in V^P \times V^T$, such that

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega \rho v \, dx \qquad v \in V_0^P \tag{8.1}$$

$$\int_\Omega \operatorname{div}(\nabla u \rho) w \, dx = 0 \qquad w \in V_0^T. \tag{8.2}$$

For the nonlinear transport equation, we replace (8.2) by

$$\int_\Omega -\rho^2 w + \nabla u \cdot \nabla \rho w \, dx = 0 \qquad w \in V_0^T. \tag{8.3}$$

8.1 Description of the Used Algorithms

We will introduce four algorithms to solve the discrete coupled problem. The first method is the staggered algorithm which we use to discretize (8.1)-(8.2).

8.1.1 The Staggered Algorithm

We will briefly describe the iteration scheme given by the staggered algorithm. The discrete variational problem reads:
Find $(u^h, \rho^h) \in V^{P,h} \times V^{T,h}$ such that

$$\int_\Omega \nabla u^h \nabla v \, dx = \int_\Omega \rho^h v \, dx \qquad \forall v \in V_0^{P,h} \tag{8.4}$$

$$\int_\Omega \operatorname{div}(\nabla u^h \rho^h) w \, dx = 0 \qquad \forall w \in V_0^{T,h}. \tag{8.5}$$

The staggered algorithm is then given by

Algorithm 1 Staggered algorithm

Input: Initial guess $\rho_0^h := 0$, accuracy ϵ
Output: Approximation of solution (u, ρ) to (CP 6.1)
Initialize u_0**:** Solve (8.4) with $\rho^h := \rho_0^h$
$n := 0$
while $\|u_n^h - u_{n-1}^h\|_{V^P} + \|\rho_n^h - \rho_{n-1}^h\|_{V^T} \geq \epsilon$ **do**
  Solve (8.5) with $u^h := u_n^h$ and obtain ρ_{n+1}^h
  Solve (8.4) with $\rho^h := \rho_{n+1}^h$ and obtain u_{n+1}^h
  $n := n + 1$
end while

If we want to use the staggered algorithm to solve (8.1)-(8.3), we have to deal with the nonlinear transport equation. Furthermore, the coupled problem itself is nonlinear. A commonly used method for linearizing nonlinear problems is Newton's iteration scheme. Let therefore U and V be two Banach spaces and $F : U \to V$ a nonlinear operator. We then search for the solution u of $F(u) = 0$. We will need the *Gateaux derivative* of F.

Definition 8.1. *[6, Definition 4.3.2] The operator F is* Gateaux differentiable *at u_0 if and only if there exists $A \in \mathcal{L}(V, W)$ such that*

$$\lim_{\epsilon \to 0} \frac{F(u_0 + \epsilon\delta) - F(u_0)}{\epsilon} = A\delta, \quad \forall \delta \in V.$$

We denote $DF(u_0, \delta) := A\delta$.

Newton's method reads as follows: For an initial guess $u_0 \in U$, compute for $n = 0, 1, ..$

$$DF(u_n; \delta) = -F(u_n) \tag{8.6}$$

with respect to δ and update $u_{n+1} = u_n + \delta$. It is expressed in the following algorithm:

Algorithm 2 Newton's iteration method for Banach spaces [6, Section 4.4.1]

Input: Initial guess $u_0 \in U$, accuracy ϵ
Output: Approximation of $u \in U$ such that $F(u) = 0$
$k := 0$
while $\|\delta\|_V \geq \epsilon$ **do**
  Solve $DF(u_k, \delta) = -F(u_k)$ for δ
  Update $u_{k+1} := u_k + \delta$
  k:=k+1
end while

8.1.2 Linearization of Nonlinear Transport Equation

As first possibility, we use the staggered algorithm and solve the nonlinear transport equation by Newton's method. The operator F according to the nonlinear variational transport problem (8.3) is given by

$$F(\rho, w) = \int_\Omega \rho^2 w + E \cdot \nabla \rho w \, dx.$$

We want to solve $F(\rho, w) = 0$. The Gateaux derivative with respect to ρ is given by

$$\begin{aligned}
DF(\rho; \delta_\rho, w) &= \lim_{\epsilon \to 0} \frac{\int_\Omega (\rho + \epsilon\delta_\rho)^2 w + E \cdot \nabla(\rho + \epsilon\delta_\rho) w - \rho^2 w + E \cdot \nabla \rho \ w \, dx}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\int_\Omega \rho^2 w + 2\epsilon\rho\delta_\rho w + \epsilon^2 \delta_\rho^2 w + E \cdot \nabla \rho w + E \cdot \nabla \epsilon \delta_\rho w - \rho^2 w + E \cdot \nabla \rho w \, dx}{\epsilon} \\
&= \int 2\rho\delta_\rho w + E \cdot \nabla \delta_\rho w \, dx.
\end{aligned} \tag{8.7}$$

By (8.6) we have to solve the following discrete problem:
Find $\delta_\rho^h \in V_0^{T,h}$, such that

$$\int_\Omega 2\rho^h \delta_\rho^h w - \nabla u^h \cdot \nabla \delta_\rho^h w \, dx = - \int_\Omega \rho_h^2 w - \nabla u^h \cdot \nabla \rho^h w \, dx \quad \forall w \in V_0^{T,h}. \tag{8.8}$$

Applying Newton's iteration scheme, we solve the previous equation iteratively with respect to δ_ρ^h and obtain the solution ρ^h up to a given accuracy. For the coupled problem, we obtain the algorithm

Algorithm 3 Nonlinear Transport Equation

Input: Initial guess $\rho_0^h = 0$, accuracies ϵ_S and ϵ_N
Initialize u_0**:** Solve (8.4) with $\rho^h := \rho_0^h$
$n := 0$
while $\|\rho_n^h - \rho_{n-1}^h\|_{V^P} + \|u_n^h - u_{n-1}^h\|_{V^T} \geq \epsilon_S$ **do**
  Solve (8.8) by Alg. 2 with $u^h := u_n^h$, accuracy ϵ_N and obtain ρ_{n+1}^h
  Solve (8.4) for u_{n+1} with $\rho^h := \rho_{n+1}^h$
  n:=n+1
end while

8.1.3 Linearization of the Coupled Problem by Newton's Method

Another possibility is to linearize the overall coupled problem. The nonlinear operator F is defined according to (8.1)-(8.3) by

$$F\left(\begin{bmatrix} u \\ \rho \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix}\right) = \begin{pmatrix} \int_\Omega \nabla u \cdot \nabla v - \rho v \, dx \\ \int_\Omega -\rho^2 w + \nabla u \cdot \nabla \rho w \, dx \end{pmatrix} \tag{8.9}$$

with $v \in V_0^P$ and $w \in V_0^T$. We first compute the Gateaux derivative of F with respect to (u, ρ). Let $(\delta_u, \delta_\rho) \in V_0^P \times V_0^T$. Then follows

$$\begin{aligned}
&\lim_{\alpha \to 0} \frac{1}{\alpha} \left(\int_\Omega \nabla(u + \alpha\delta_u) \cdot \nabla v - (\rho + \alpha\delta_\rho) v \, dx - \int_\Omega \nabla u \cdot \nabla v - \rho v \, dx \right) \\
&\quad = \lim_{\alpha \to 0} \frac{1}{\alpha} \int_\Omega \alpha \nabla \delta_u \cdot \nabla v - \alpha \delta_\rho v \, dx \\
&\quad = \int_\Omega \nabla \delta_u \cdot \nabla v - \delta_\rho v \, dx
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\alpha \to 0} \frac{1}{\alpha} \left(\int_\Omega -(\rho + \alpha\delta_\rho)^2 w + \nabla(u + \alpha\delta_u) \cdot \nabla(\rho + \alpha\delta_\rho) w - \int_\Omega -\rho^2 w + \nabla u \cdot \nabla \rho w \, dx \right) \\
&= \lim_{\alpha \to 0} \frac{1}{\alpha} \left(\int_\Omega -(\rho^2 + 2\alpha\rho\delta_\rho + \alpha^2\delta_\rho^2) w + \nabla u \cdot \nabla \rho w + \alpha \nabla \delta_u \cdot \nabla \rho w + \alpha \nabla u \cdot \nabla \delta_\rho w + \alpha^2 \nabla \delta_u \cdot \nabla \delta_\rho w \right. \\
&\qquad \left. - \int_\Omega -\rho^2 w + \nabla u \cdot \nabla \rho w \, dx \right) \\
&= \lim_{\alpha \to 0} \frac{1}{\alpha} \left(\int_\Omega -2\alpha\rho\delta_\rho w - \alpha^2 \delta_\rho^2 w + \alpha \nabla \delta_u \cdot \nabla \rho w + \alpha \nabla u \cdot \nabla \delta_\rho w + \alpha^2 \nabla \delta_u \cdot \nabla \delta_\rho w \, dx \right) \\
&= \int_\Omega -2\rho\delta_\rho w + \nabla \delta_u \cdot \nabla \rho w + \nabla u \cdot \nabla \delta_\rho w \, dx.
\end{aligned}$$

We obtain the Gateaux derivative

$$\begin{aligned}
DF([u,\rho]; [\delta_u, \delta_\rho], [v, w]) &= \lim_{\alpha \to 0} \frac{F([u + \alpha\delta_u, \rho + \alpha\delta_\rho], [v, w]) - F([u, \rho], [v, w])}{\alpha} \\
&= \begin{pmatrix} \int_\Omega \nabla \delta_u \cdot \nabla v - \delta_\rho v \, dx \\ \int_\Omega -2\rho\delta_\rho w + \nabla \delta_u \cdot \nabla \rho w + \nabla u \cdot \nabla \delta_\rho w \, dx \end{pmatrix}.
\end{aligned} \tag{8.10}$$

By (8.6) we have to solve the following discrete problem:

Find $(\delta_u^h, \delta_\rho^h) \in V_0^{P,h} \times V_0^{T,h}$, such that

$$(\nabla \delta_u^h, \nabla v) - (\delta_\rho^h, v) = -(\nabla u^h, \nabla v) + (\rho^h, v) \qquad \forall v \in V_0^{P,h} \tag{8.11}$$

$$(\nabla \delta_u^h \cdot \nabla \rho^h, w) + (\nabla u^h \cdot \nabla \delta_\rho^h, w) - 2(\rho^h \delta_\rho^h, w) = (\rho_h^2 - \nabla u^h \cdot \nabla \rho^h, w) \qquad \forall w \in V_0^{T,h}. \tag{8.12}$$

(8.11) -(8.12) is now a linear problem in $(\delta_u^h, \delta_\rho^h)$ and can thus be solved by the staggered algorithm. We obtain the following algorithm for the coupled problem

Algorithm 4 Newton iterations using staggered algorithm

Input: $\rho_0^h := 0$, accuracies ϵ_S, ϵ_N .
Output: Approximate solution to (u, ρ) of (CP 6.1)
Initialize u_0^h: (8.4) with $\rho^h := \rho_0^h$
Initialize $(\delta_u^h, \delta_\rho^h)$: Solve (8.11) -(8.12) for $(\delta_u^h, \delta_\rho^h)$ by Alg. 1 with accuracy ϵ_S
$n := 0$
while $\|\delta_u^h\|_{V^P} + \|\delta_\rho^h\|_{V^T} \geq \epsilon_N$ **do**
  Solve (8.11) -(8.12) for $(\delta_u^h, \delta_\rho^h)$ by Alg. 1 with accuracy ϵ_S.
  Update the solution: $(u_{n+1}^h, \rho_{n+1}^h) = (u_n^h + \delta_u^h, \rho_n^h + \delta_\rho^h)$.
  $n := n + 1$
end while

8.1.4 Newton's Iteration Scheme Using a Block Matrix System

The last possibility uses the same linearization of the coupled problem as in the previous section. In comparison to section 8.1.3, the linearized problem is regarded as a block matrix system. Given $(u^h, \rho^h) \in V^{P,h} \times V^{T,h}$, we first describe the block matrix system for (8.11)-(8.12). Let therefore be $\{\Phi_i\}_{i=1}^n$ a basis for the space $V^{P,h}$ and $\{\Psi_i\}_{i=1}^m$ a basis for the space $V^{T,h}$. We can expand the finite element solutions as $\delta_u^h = \sum_{i=1}^n \alpha_i^u \Phi_i$ and $\delta_\rho^h = \sum_{i=1}^n \alpha_i^\rho \Psi_i$ where α_i^u and α_i^ρ are unknown real numbers to be determined.

Given $(u^h, \rho^h) \in V^{P,h} \times V^{T,h}$, we thus obtain the discrete solution by solving the following linear system for $(\vec{\alpha}^u, \vec{\alpha}^\rho)$

$$\begin{pmatrix} A & B_1 \\ B_2 & C \end{pmatrix} \begin{pmatrix} \vec{\alpha}^u \\ \vec{\alpha}^\rho \end{pmatrix} = \begin{pmatrix} \vec{L}^1_{(u^h,\rho^h)} \\ \vec{L}^2_{(u^h,\rho^h)} \end{pmatrix} \tag{8.13}$$

with

$$\begin{aligned} A_{ki} &= (\nabla \Phi_i, \nabla \Phi_k), \\ (B_1)_{kj} &= -(\Psi_j, \Phi_k), \\ (B_2)_{ji} &= (\nabla \Phi_i \cdot \nabla \rho^h, \Psi_j), \\ C_{jl} &= (\nabla u^h \cdot \nabla \Psi_l, \Psi_j) - 2(\rho^h \Psi_l, \Psi_j), \end{aligned}$$

and the right-hand side vectors

$$(L^1_{(u^h,\rho^h)})_i = -(\nabla u^h, \nabla \Phi_i) + (\rho^h, \Phi_i),$$
$$(L^2_{(u^h,\rho^h)})_j = (\rho_h^2 - \nabla u^h \cdot \nabla \rho^h, \Psi_j)$$

for $i, k = 1, ..n$ and $j, l = 1, ..m$.

Algorithm 5 Newton iterations using block matrix

Input: $\rho_0^h := 0$, accuracy ϵ .
Output: Approximate solution of (u, ρ) to (CP 6.1)
Initialize u_0^h: Solve (8.4) with $\rho^h := \rho_0^h$
Initialize $(\delta_u^h, \delta_\rho^h)$: Solve (8.13) with $(u^h, \rho^h) = (u_0^h, \rho_0^h)$ w.r.t. $(\delta_u^h, \delta_\rho^h)$.
$n := 0$
while $\|\delta_u^h\|_{V^P} + \|\delta_\rho^h\|_{V^T} \geq \epsilon$ **do**
  Solve (8.13) for $(\delta_u^h, \delta_\rho^h)$ with $(u^h, \rho^h) = (u_n^h, \rho_n^h)$
  Update the solution: $(u_{n+1}^h, \rho_{n+1}^h) = (u_n^h + \delta_u^h, \rho_n^h + \delta_\rho^h)$.
  $n := n + 1$
end while

8.2 Example 1: Steady State Radially Symmetric Coupled Problem

The first example deals with the radially symmetric steady state problem of Chapter 5. Let Ω be an annular domain with center of gravity in the origin, i.e.

$$\Omega = \left\{x \in \mathbb{R}^2 : r_0 \leq \|x\|_2 \leq r_1,\, r_0, r_1 > 0\right\}. \tag{8.14}$$

Recall the radially symmetric problem

Problem (CP 5.1). *Let $I = [r_0,\, r_1]$ with $r_1 \geq r_0$. Find $(u, \rho) \in C^2(I) \times C^0(I)$, such that*

$$-\frac{1}{r}\partial_r(ru'(r)) = \rho(r) \tag{8.15a}$$

$$u(r_0) = u_{A_1} \tag{8.15b}$$

$$u(r_1) = u_{A_2} \tag{8.15c}$$

$$\frac{1}{r}\partial_r(r\partial_r u(r)\rho(r)) = 0 \tag{8.15d}$$

$$\rho(r_0) = \rho_A \tag{8.15e}$$

with $u_{A_1} > u_{A_2} > 0$ and $\rho_A > 0$ constants.

8.2.1 Derivation of the Solution in Closed Form

In the radially symmetric case, we are able to determine a solution in closed form. We get by integration of the transport equation (8.15d)

$$ru'(r)\rho(r) = C.$$

With the boundary condition (8.15e) follows for the constant

$$C = r_0 u'(r_0)\rho(r_0) = r_0\rho_A u'(r_0).$$

While $r_0 > 0$ and $\rho_A > 0$ are known a priori, the value for the derivative $u'(r_0)$ is unknown. However, we can determine its sign and thus the sign of C. By the definition of the inflow boundary and the outward normal vector $\vec{n}(r_0) = -e_r(r_0)$, we obtain

$$0 > \vec{n}(r_0) \cdot E(r_0) = e_r(r_0) \cdot e_r(r_0)u'(r_0) = u'(r_0).$$

Conclusively, $C < 0$. Assume that $u'(r) \neq 0$ for all $r \in [r_0, r_1]$. We obtain for $\rho(r)$

$$\rho(r) = \frac{C}{ru'(r)}. \tag{8.16}$$

Next we focus on the Poisson equation (8.15a). Substituting (8.16) into (8.15a), we get

$$-\partial_r(ru'(r)) = \frac{C}{u'(r)}. \tag{8.17}$$

Multiplying (8.17) by $ru'(r)$, we have

$$ru'(r)\partial_r(ru'(r)) = \frac{1}{2}\partial_r(ru'(r))^2 = -Cr. \tag{8.18}$$

By integration of the previous equation follows

$$\frac{1}{2}r^2u'(r)^2 = -\frac{1}{2}Cr^2 + D.$$

Set $r = r_0$. We obtain

$$D = \frac{1}{2}\left(r_0^2u'(r_0)^2 + Cr_0^2\right) = \frac{C}{2}\left(r_0^2 + \frac{C}{\rho_A^2}\right).$$

We can now determine $u'(r)$ by

$$\begin{aligned} u'(r) &= \pm\frac{1}{r}\sqrt{-r^2C + 2D} \\ &= \pm\frac{\sqrt{-C}}{r}\sqrt{r^2 - \frac{2D}{C}}. \end{aligned}$$

We assumed that $u'(r) \neq 0$ for all $r \in [r_0, r_1]$. Since $u'(r_0) < 0$, it follows that $u'(r) < 0$ for all $r \in [r_0, r_1]$. Thus $u'(r)$ is given by

$$u'(r) = -\frac{\sqrt{-C}}{r}\sqrt{r^2 - \frac{2D}{C}}. \tag{8.19}$$

In fact, this representation for $u'(r)$ is valid without any further restrictions on the constants C and D, since

$$r^2 - \frac{2D}{C} = r^2 - r_0^2 - \frac{r_0u'(r_0)}{\rho_A} > 0$$

for every r_0, $r_1 > 0$ and $\rho_A > 0$.
We obtain the final representations of $\rho(r)$ by substituting (8.19) into (8.16)

$$\rho(r) = -\frac{\sqrt{-C}}{\sqrt{r^2 - \frac{2D}{C}}}. \tag{8.20}$$

To obtain $u(r)$, we integrate (8.19). We have to distinguish two cases.

Case 1: $\frac{2D}{C} \leq 0$
Set $a^2 = -\frac{2D}{C}$. Integrating $u'(r)$ by [14, p. 1086, No.189] gives

$$u(r) = -\sqrt{-C}\left(\sqrt{r^2 + a^2} - a\log\frac{a + \sqrt{r^2 + a^2}}{r}\right) + B. \tag{8.21}$$

Using the boundary conditions (8.15b)-(8.15c) we eliminate the constant B

$$u_{A_1} - u_{A_2} = -\sqrt{-C}\left(\sqrt{r_0^2 + a^2} - a\log\frac{a + \sqrt{r_0^2 + a^2}}{r_0}\right) + \sqrt{-C}\left(\sqrt{r_1^2 + a^2} - a\log\frac{a + \sqrt{r_1^2 + a^2}}{r_1}\right).$$

In the following experiments, C is determined numerically by the bisection method with an accuracy of $\epsilon = 10^{-12}$ for given boundary data ρ_A, u_{A_1} and u_{A_2},.

Case 2: $\frac{2D}{C} > 0$
Set $a^2 = \frac{2D}{C}$. Integrating $u'(r)$ by [14, p. 1087, No.217] gives

$$u(r) = -\sqrt{-C}\left(\sqrt{r^2 - a^2} - a \arccos \frac{a}{r}\right) + B \tag{8.22}$$

Using now the boundary conditions (8.15b)-(8.15c) we eliminate the constant B

$$u_{A_1} - u_{A_2} = -\sqrt{-C}\left(\sqrt{r_0^2 - a^2} - a \arccos \frac{a}{r_0}\right) + \sqrt{-C}\left(\sqrt{r_1^2 - a^2} - a \arccos \frac{a}{r_1}\right).$$

Again, in the numerical experiments, we determine C numerically by the bisection method with an accuracy of $\epsilon = 10^{-12}$ for given boundary data ρ_A, u_{A_1} and u_{A_2}.

We will now present the numerical results. The parameters of (CP 5.1) are chosen as

$$r_0 = 1, \quad r_1 = 2, \quad u_{A_1} = 4, \quad u_{A_2} = 1.$$

We use two criteria to investigate the approximation properties of (CP 5.1). Using a fixed inflow boundary condition $\rho_A = 0.5$, we investigate the convergence of the coupled problem with the linear or nonlinear transport equation by comparing the convergence of the Algorithms 1, 3 - 5. In a second approach, we use the formulation of the nonlinear transport equation and the Algorithm 3. We investigate whether changing inflow boundary data affects the convergence of the algorithm.

Although the radially symmetric problem is one-dimensional, the computations are done on a partition of the two-dimensional domain Ω. The computations for the Poisson equation are done on a quadrilateral mesh. The space $V^{P,h}$ uses quadratic basis functions. The linear system is solved by the CG algorithm with an accuracy of $\epsilon = 10^{-12}$. Ω is not a convex domain, thus while computing the right hand side or the error, evaluation points are found in Ω_-. We use the following approximation: Whenever $r = \sqrt{x_1^2 + x_2^2} \leq r_0$, the evaluation is moved onto the boundary, i.e. if $r \leq r_0$ then set $r = r_0$.

The computations for the transport equation are done on a curved quadrilateral mesh, thus the problem of having evaluation points outside of Ω is avoided. The space $V^{T,h}$ uses bilinear basis functions. The linear system is solved using the GMRES algorithm with an accuracy of $\epsilon = 10^{-12}$. Right hand sides and errors are computed in both cases using a 16×16-point Gaussian quadrature rule.

The block matrix system of Algorithm 5 is solved by the CGNE solver. Whenever the staggered algorithm is used, the stopping criterion is given by

$$\sqrt{\|u_n - u_{n-1}\|^2_{H^1(\Omega)} + \|\rho_n - \rho_{n-1}\|^2_{L^2(\Omega)}} \leq 10^{-8}.$$

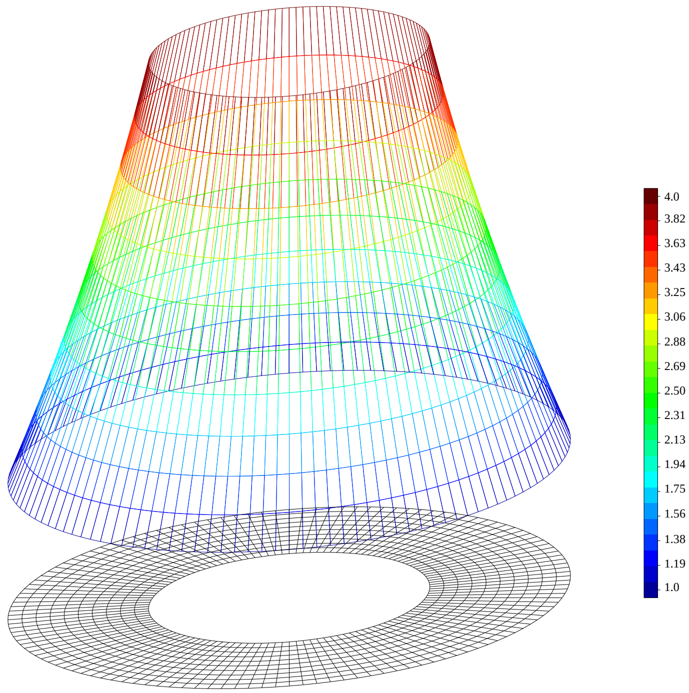


Figure 8.1: Example 1: u^h for $\rho_A = 0.5$

For the Newton schemes, we use the accuracy of 10^{-12}. In the tables shown below, α_u gives the convergence rate for the staggered algorithm computed for u whereas α_ρ gives the convergence rate for the staggered algorithm computed for ρ. The columns labeled *Staggered Iterations* and *Newton Iterations* show the number of iterations that are needed for the respective algorithm to converge. *CPU Time* states the computation time for the algorithm to obtain convergence. The computation time for the errors $\|u^h - u\|_{H^1(\Omega)}$ or $|||\rho^h - \rho|||_G$ is not included.

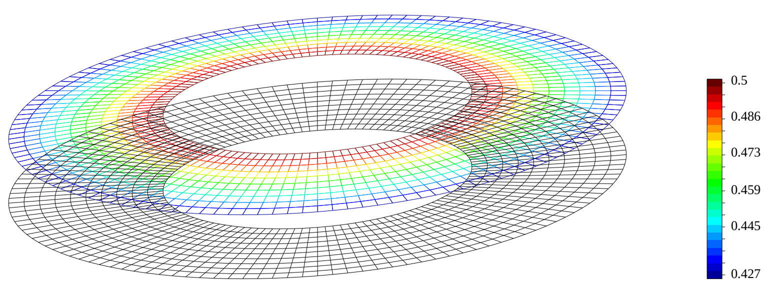


Figure 8.2: Example 1: Charge distribution ρ^h for $\rho_A = 0.5$

8.2.2 Comparison of Algorithms

We compare Algorithms 1 and 3-5 that were introduced in section 8.1. The inflow boundary data is chosen as $\rho_A = 0.5$. We obtain the constants

$$C = -2.025673, \quad D = 7.193863, \quad B = 1.577040.$$

The exact solution is found in section 8.2.1 in case 1.

Table 8.1: Ex 1: Algorithm 1 with $\rho_A = 0.5$

DOF	Poisson $\|u - u_h\|_{H^1}$	α_u	Transport $\|\|\|\rho - \rho_h\|\|\|_G$	α_ρ	Staggered Iterations	CPU Time
24	1.1002568		0.0043292		10	0.18
48	0.4931866	2.315	0.0080227	-1.78	10	0.30
416	0.1340417	1.207	0.0009267	1.999	9	0.34
1472	0.0494057	1.580	0.0003787	1.416	9	1.00
6144	0.0170219	1.491	0.0001155	1.662	8	2.80
25088	0.0059319	1.499	.2897E-04	1.966	8	11.81
103936	0.0020605	1.488	.5685E-05	2.291	8	52.23
412672	0.0007293	1.506	.1466E-05	1.966	8	268.35
1665024	0.0002567	1.497	.3613E-06	2.008	8	5474.38

The numerical results show that all four discretizations of the coupled problem converge to the exact solution. Although using different implementations of the transport equation, Algorithm 1 and 3 demonstrate the same approximation error. However, the linear transport formulation requires more iterations to obtain convergence: 8 versus 5 iterations. Going back to the analysis of Chapter 5, we can compute $M = \min\{c_R, c_L\}$. Let $a = \frac{r_0^2}{2}\log(\frac{r_1}{r_0})$, $b = \frac{r_1^2}{2}\log(\frac{r_1}{r_0})$ and $c = \frac{r_1^2 - r_0^2}{4}$ and with (5.24) and (5.33) we have

$$\begin{aligned} M &= \min\left\{\frac{u_{A_1} - u_{A_2}}{r_0^2\log\left(\frac{r_0}{r_1}\right) + \frac{1}{2}\left(r_1^2 - r_0^2\right)}, \frac{u_{A_1} - u_{A_2}}{|b - 2a + c| + |a - c|}\right\} \\ &= \min\{3.71815, 1.62463\} \\ &= 1.62463. \end{aligned} \tag{8.23}$$

We also compute the contraction constant using Theorem 5.20 as

$$K = \rho_A \frac{b - 2a + c}{\rho_A(a - c) + u_{A_1} - u_{A_2}} = 0.25786.$$

For an accuracy of $\epsilon = 10^{-8}$, we obtain an upper bound by the error formula (5.34) for the number of continuous Banach fixed point iterations by $n = 13$ iterations. This number is valid

Table 8.2: Ex 1: Algorithm 3 with $\rho_A = 0.5$

DOF	Poisson $\|u-u_h\|_{H_1}$	α_u	Transport $\|\|\|\rho-\rho_h\|\|\|_G$	α_ρ	Staggered Iterations	Newton Iterations	CPU Time
24	1.1004885		0.0290814		5	5	0.23
48	0.4932129	2.316	0.0140211	2.105	5	5	0.33
416	0.1340489	1.207	0.0025408	1.582	5	5	1.09
1472	0.0494069	1.580	0.0007803	1.868	5	5	2.69
6144	0.0170221	1.492	0.0002161	1.797	5	5	9.65
25088	0.0059319	1.499	.5410E-04	1.969	5	5	38.59
103936	0.0020605	1.488	.1197E-04	2.122	5	5	204.04
412672	0.0007293	1.506	.3036E-05	1.990	5	5	1099.50
1665024	0.0002567	1.497	.7540E-06	1.997	5	5	10250.00

Table 8.3: Ex 1: Algorithm 4 with $\rho_A = 0.5$

DOF	Poisson $\|u-u_h\|_{H_1}$	α_u	Transport $\|\|\|\rho-\rho_h\|\|\|_G$	α_ρ	Staggered Iterations	Newton Iterations	CPU Time
24	1.1004885		0.0290814		8	4	0.37
48	0.4932129	2.316	0.0140211	2.105	8	4	0.55
416	0.1340489	1.207	0.0025408	1.582	8	5	2.32
1472	0.0494069	1.580	0.0007803	1.868	8	5	6.36
6144	0.0170221	1.492	0.0002161	1.797	8	5	24.71
25088	0.0059319	1.499	.5410E-04	1.969	9	5	101.00
103936	0.0020605	1.488	.1197E-04	2.122	9	5	434.81
412672	0.0007293	1.506	.3036E-05	1.990	9	5	2085.10
1665024	0.0002567	1.497	.7540E-06	1.997	9	5	18420.00

for the continuous solution (u, ρ) in the sup-norm. Nevertheless, both Algorithms 1 and 3 stay below the maximal number of iterations in the continuous case. The Algorithms 4 and 5 converge to the exact solution with different approximation errors which is due to the different ways of discretization. In case of Algorithm 4, we stated the number of Newton iterations and the maximum number of inner staggered iterations for every degree of freedom. Using the block matrix system, 6 Newton iterations are needed to obtain the desired accuracy whereas in the case of the inner staggered algorithm, only 5 outer Newton iterations are observed. The computation time for Algorithm 5 is higher than for Algorithm 4, which is due to the computations of the matrices. Algorithm 4 is thus much more efficient and preferable.

We observe the same convergence rates for all the methods, that is 1.5 for the Poisson equation and 2 for the transport equation. We thus obtain better convergence rates than predicted in section 7 which should be 1 for the Poisson and 1 for the transport equation. The gap in

Table 8.4: Ex 1: Algorithm 5 with $\rho_A = 0.5$

	Poisson		Transport		Newton	CPU
DOF	$\lVert u - u_h \rVert_{H_1}$	α_u	$\lvert\lvert\lvert \rho - \rho_h \rvert\rvert\rvert_G$	α_ρ	Iterations	Time
24	1.0283330		0.0407271		6	0.14
48	0.4466469	2.406	0.0174493	2.446	6	0.22
416	0.1248048	1.181	0.0038807	1.392	6	1.68
1472	0.0459739	1.581	0.0010259	2.106	6	11.19
6144	0.0159006	1.486	0.0006584	0.621	6	173.63
25088	0.0055523	1.496	0.0001651	1.966	7	4156.93

convergence of the linear transport equation between numerical and theoretical results is well known, see e.g. [50, 12]. Further, the implemented problem is smooth and Ω does not contain any cusps or reentrant corners. We know that the exact solution (8.20)-(8.22) is the classical solution to (CP 5.1). Conclusively, the solution $(u, \rho) \in H^2(\Omega) \times H^1(\Omega)$ and does not contain any singularities which explains the good approximation behaviour.

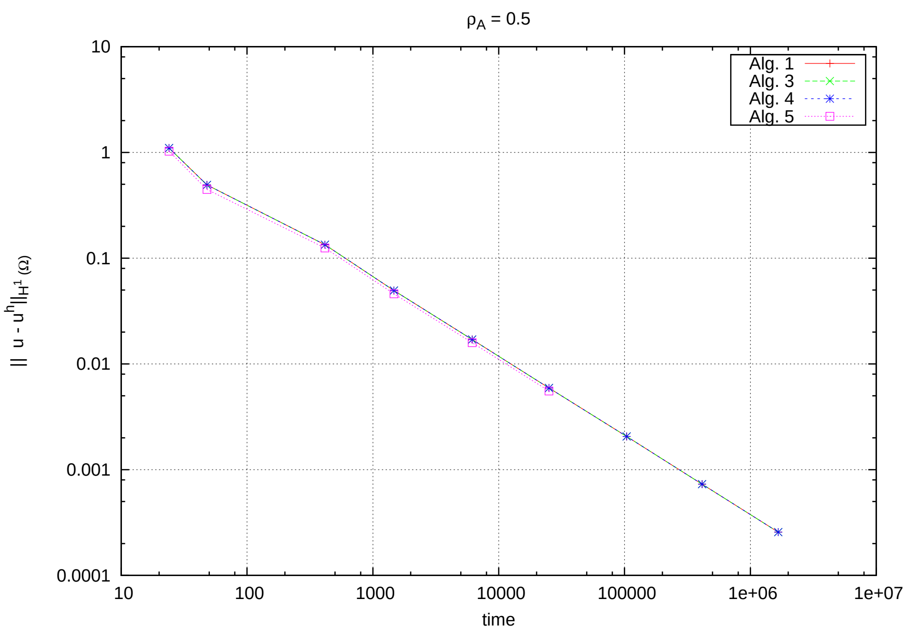


Figure 8.3: Example 1: $\lVert u - u^h \rVert_{H^1(\Omega)}$ for $\rho_A = 0.5$

8.2.3 Comparison of Inflow Boundary Data ρ_A

In this section, we examine the influence of the inflow boundary data ρ_A. We choose the problem formulation (CP 6.1) and use Algorithm 3. Since the convergence of Algorithm 3 has been shown

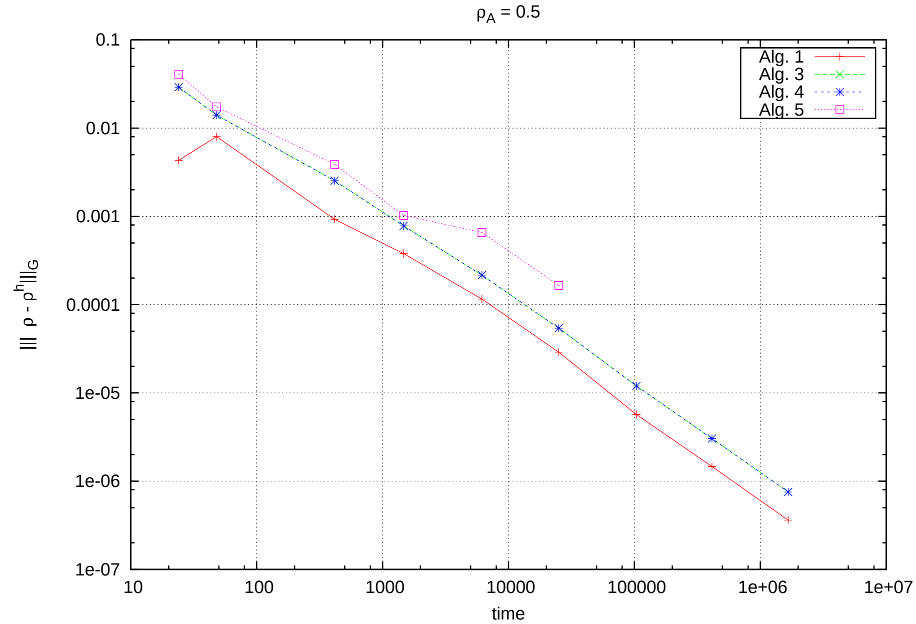


Figure 8.4: Example 1: $\|\rho - \rho^h\|_G$ for $\rho_A = 0.5$

in the previous example, we will now follow a different approach. In Chapter 5, we obtained an upper bound for the inflow data ρ_A to prove existence and uniqueness for a classical solution to (CP 5.1). ρ_A must be chosen, such that

$$\rho_A \leq M.$$

With the chosen parameters follows by (8.23)

$$\rho_A \leq \min\{3.71815, 1.62463\} = 1.62463.$$

Let us choose $\rho_A \in \{0.5, 1.6, 5\}$ and investigate whether the convergence of the staggered algorithm is influenced. Approximation errors and convergence rates for $\rho_A = 0.5$ are given in the previous section. For $\rho_A = 1.6$, we obtain the constants

$$C = -5.646765, \qquad D = 3.404342, \qquad B = 5.052825.$$

The exact solution can be found in section 8.2.1 in case 1.

As third choice, we choose $\rho_A = 5.0$ and obtain the constants

$$C = -12.170821, \qquad D = -3.122833, \qquad B = 4.504260.$$

The exact solution is given in section 8.2.1 in case 2. The first observation is that the greater ρ_A, the greater the number of staggered iterations to obtain convergence. While for $\rho_A = 0.5$ the staggered algorithm only needs five iterations, it is increasing to seven iterations for $\rho_A = 1.6$ and to twelve for $\rho_A = 5$. The three values for ρ_A are chosen according to the theory proven in Chapter 5. For the chosen geometry and potential difference, it is only possible to prove

Table 8.5: Ex 1: Algorithm 3 with $\rho_A = 1.6$

DOF	Poisson $\|u - u_h\|_{H_1}$	α_u	Transport $\|\|\|\rho - \rho_h\|\|\|_G$	α_ρ	Staggered Iterations	Newton Iterations	CPU Time
24	0.9500793		0.0905598		6	5	0.23
48	0.3964061	2.522	0.1495869	-1.45	7	5	0.37
416	0.1146870	1.149	0.0092399	2.579	7	5	1.04
1472	0.0422150	1.582	0.0035722	1.504	7	5	2.85
6144	0.0146799	1.479	0.0008128	2.072	7	5	11.43
25088	0.0051404	1.492	0.0002044	1.962	7	5	48.97
103936	0.0017904	1.484	.5070E-04	1.962	7	5	240.97
412672	0.0006345	1.505	.1308E-04	1.965	7	5	1275.99
1665024	0.0002235	1.496	.3217E-05	2.011	7	5	11660.00

Table 8.6: Ex 1: Algorithm 3 with $\rho_A = 5$

DOF	Poisson $\|u - u_h\|_{H_1}$	α_u	Transport $\|\|\|\rho - \rho_h\|\|\|_G$	α_ρ	Staggered Iterations	Newton Iterations	CPU Time
24	0.80417		2.09391		8	5	0.61
48	0.72365	0.304	2.43573	-0.44	8	5	0.65
416	0.12185	1.650	0.43675	1.592	10	6	2.51
1472	0.04862	1.454	0.18590	1.352	11	6	5.86
6144	0.01422	1.721	0.04920	1.861	11	6	27.40
25088	0.00439	1.670	0.01286	1.908	12	6	109.91
103936	0.00140	1.612	0.00325	1.936	12	6	537.18
412672	0.00047	1.584	0.00084	1.969	12	6	3136.21
1665024	0.00016	1.545	0.00021	2.005	12	6	23350.00

existence and uniqueness of a solution for $\rho_A = 0.5$ and $\rho_A = 1.6$. The Banach fixed point theorem is not applicable to $\rho_A = 5$ anymore, since $T \circ L$ is neither a selfmap nor a contraction. The staggered algorithm in the continuous case does not necessarily converge according to the developed theory. Yet, as we have the solution in closed form for this model problem, we know that a solution to (CP 5.1) exists for $\rho_A = 5$. Moreover, the numerical staggered algorithm converges for $\rho_A = 5$. Surely, the estimates of Chapter 5 are not sharp which explains part of the discrepancy. The estimates for the continuous solution are obtained in a different function space in comparison to the one in which the numerical solution is found in. In Sobolev spaces, Banach's fixed point theorem might prove the existence of a solution up to a larger value of ρ_A and thus explains the convergence of the algorithm for $\rho_A = 5$.

In the previous example, we computed the maximum number of iterations for an accuracy of 10^{-8} and $\rho_A = 0.5$ as 13. With increasing ρ_A, the maximum number of iterations also increases.

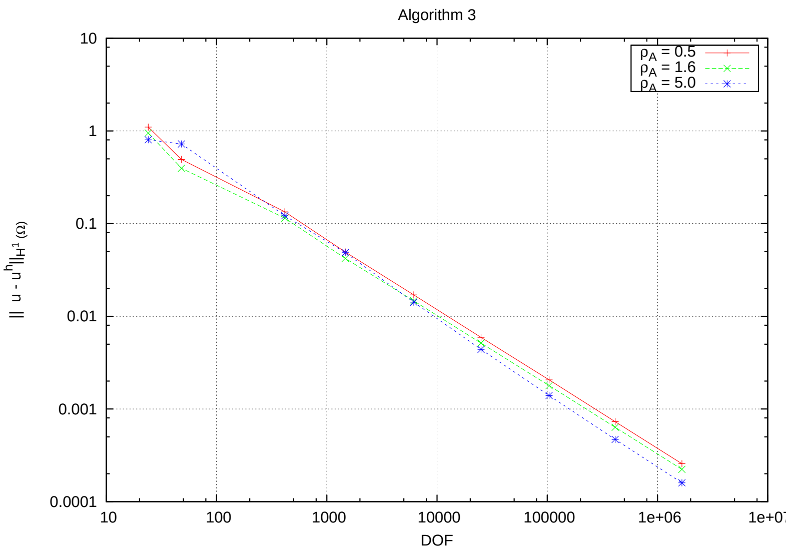


Figure 8.5: Example 1: $\|u - u^h\|_{H^1(\Omega)}$ for Algorithm 3

For $\rho_A = 5$, no prediction is possible with the methods of Chapter 5. However, the number of iterations for all three values of ρ_A is smaller. The staggered algorithm seems quickly convergent. The convergence rates are asymptotically the same for all three values of ρ_A that is 1.5 for the Poisson equation and 2 for the transport equation. This phenomenon has been discussed already in the previous example.

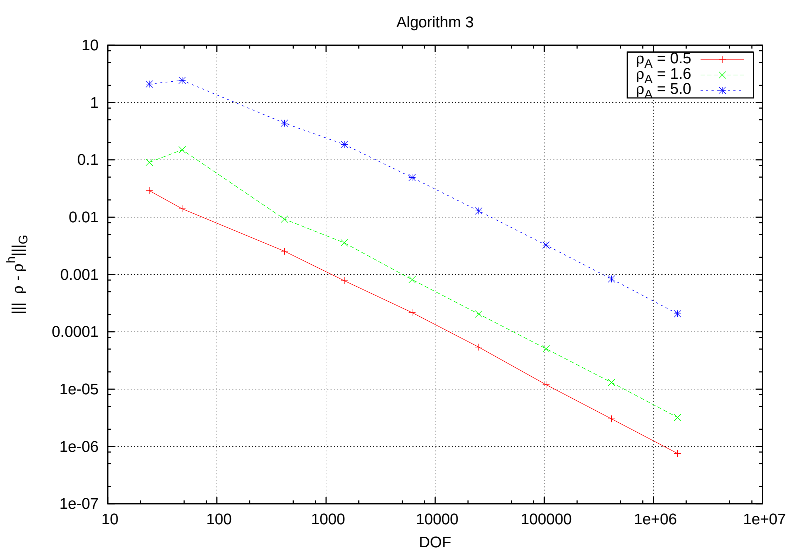


Figure 8.6: Example 1: $|||\rho - \rho^h|||_G$ for Algorithm 3

8.3 Example 2: Steady State Problem in 3d

Next, we present a three-dimensional example in which we proceed analogously to Example 1. Let Ω be a hollow sphere with

$$\Omega := \left\{x \in \mathbb{R}^3 : r_0 \leq \|x\|_2 \leq r_1\right\} \tag{8.24}$$

with inflow boundary $\Gamma_- := \{x : \|x\|_2 = r_0\}$ and outflow boundary $\Gamma_+ := \{x : \|x\|_2 = r_1\}$. We consider the steady state radially symmetric coupled problem which reduces to a one-dimensional problem with variable coefficients.

Problem (CP 8.1). *Let $I = [r_0, r_1]$. Find $(u, \rho) \in C^2(I) \times C^0(I)$, such that*

$$-\frac{1}{r^2}\partial_r \left(r^2 \partial_r u(r)\right) = \rho(r) \tag{8.25a}$$

$$u(r_0) = u_{A_1} \tag{8.25b}$$

$$u(r_1) = u_{A_2} \tag{8.25c}$$

$$-\frac{1}{r^2}\partial_r \left(r^2 \partial_r u(r)\rho(r)\right) = 0 \tag{8.25d}$$

$$\rho(r_0) = \rho_A \tag{8.25e}$$

with constants $u_{A_1} > u_{A_2}$ and $\rho_A > 0$.

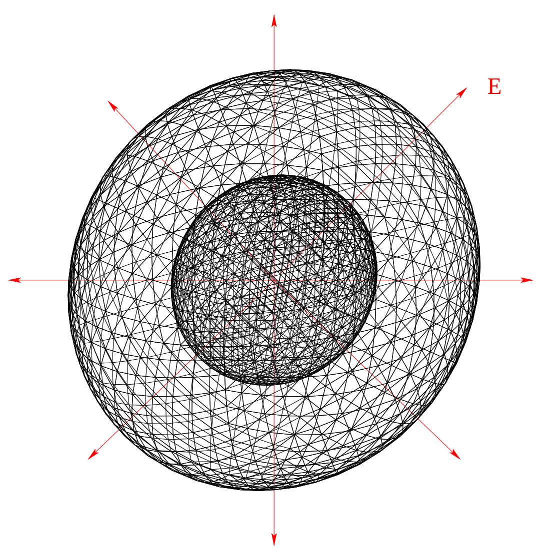


Figure 8.7: Hollow sphere Ω

The details in Chapter 5 describing the radially symmetric problem in $\mathbb{R}^2$ could similarly be done in $\mathbb{R}^3$. After determining a solution in closed form for (u', ρ) analogously to the two-dimensional case, we will investigate the three-dimensional problem numerically.

8.3.1 Solution in Closed Form

In three dimensions, we find a solution in closed form for u' and ρ.

By integration of (8.25d), we have

$$r^2\partial_r u(r)\rho(r) = C.$$

The constant C is determined with the boundary condition (8.25e) as $C = r_0^2\partial_r u(r_0)\rho(r_0)$. By the definition of the inflow boundary, we determine the sign of C as in two dimensions. Let $\vec{n}(r) = -e_r(r)$ be the normal outward vector and $E(r) = -e_r(r)\partial_r u(r)$. Then

$$0 < \vec{n}(r_0) \cdot E(r_0) = e_r(r_0)e_r(r_0)\partial_r u(r_0) = \partial_r u(r_0)$$

and consequently $C < 0$.

Let us assume that $u(r) \neq 0$ for $r \in [r_0, r_1]$. Then $\rho(r)$ is given for $r \in [r_0, r_1]$ by

$$\rho(r) = \frac{C}{r^2\partial_r u(r)}. \tag{8.26}$$

Substituting (8.26) into (8.25a) and multiplying by $r^4\partial_r u(r)$ gives

$$-r^2\partial_r u(r)\partial_r\left(r^2\partial_r u(r)\right) = -\frac{1}{2}\partial_r(r^2\partial_r u(r))^2 = r^2 C. \tag{8.27}$$

By integration, we obtain

$$\frac{1}{2}(r^2\partial_r u(r))^2 = -\frac{C}{3}r^3 + D. \tag{8.28}$$

where the constant D is determined for $r = r_0$ by

$$D = \frac{Cr_0}{2\rho_A^2} + \frac{1}{3}r_0^3 C.$$

Since $u'(r) \neq 0$ for all $r \in [r_0, r_1]$ and $u'(r_0) < 0$, it follows that $u'(r) < 0$. We obtain

$$r^2\partial_r u(r) = -\sqrt{-\frac{C}{2}r^3 + D}. \tag{8.29}$$

The solutions $u'(r)$ and $\rho(r)$ for $r \in [r_0, r_1]$ are given by

$$u'(r) = -\frac{\sqrt{-C}}{r^2}\sqrt{\frac{2}{3}}\sqrt{r^3 - \frac{3D}{C}} \tag{8.30}$$

and

$$\rho(r) = \sqrt{\frac{3}{2}}\frac{\sqrt{-C}}{\sqrt{r^3 - \frac{3D}{C}}}$$

To obtain the constants C and thus D, integrate 8.30 over $[r_0, r]$

$$u(r) = u_{A_1} + \int_{r_0}^{r} u'(t)\,dt. \tag{8.31}$$

For $r = r_1$, we obtain

$$u_{A_2} = u_{A_1} + \int_{r_0}^{r_1} u'(t)\, dt. \tag{8.32}$$

In the implementation, we apply the following method

- Outer loop: Apply Newton's method to (8.32) to obtain C.
- Inner loop: Use Gaussian quadrature to evaluate the integral in (8.31).

The constants are determined once in the beginning of the algorithm. We will focus on precision and use a 32-point Gaussian quadrature rule to compute the integral and an accuracy of $\epsilon = 10^{-10}$ for Newton's method.

8.3.2 Numerical Results

We discretize the formulation of the coupled problem using the linear transport equation and therefore apply Algorithm 1. Let $\mathcal{T}^h$ be a partition of the hollow sphere into prisms $\tau \in \mathcal{T}^h$.

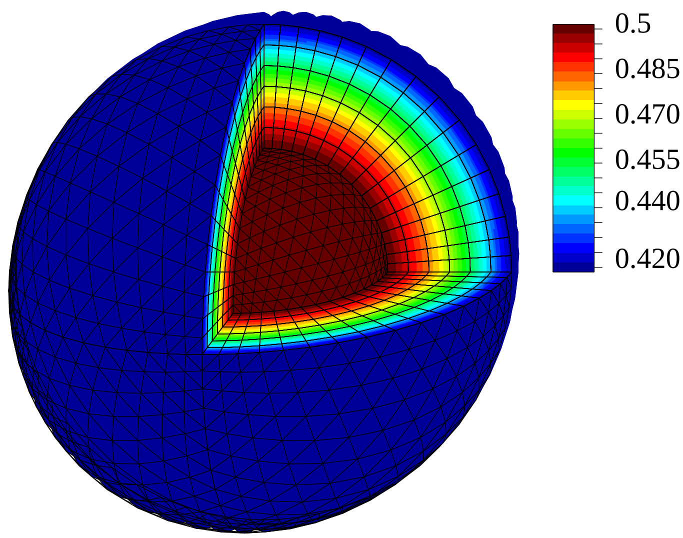


Figure 8.8: Example 2: ρ^h for $\rho_A = 0.5$ and DOF=11286

We use linear basis functions for $V^{P,h}$ and $V^{T,h}$. Since Ω is not a convex domain, there are evaluation points in the computations that are not in Ω. We use the following approximation: If $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq r_0$, then move the evaluation onto the boundary and set $r := r_0$. To solve the linear systems, we use iterative solvers. The linear system corresponding to the Poisson equation is solved with the CG solver and the linear system for the transport equation is solved by the CGNE solver, both with an accuracy of 10^{-12}. The integrals of the right hand side and errors are computed with a $4 \times 4 \times 4$ point Gaussian quadrature rule. We choose the parameter as

$$u_0 = 4, \quad u_1 = 1, \quad r_0 = 1, \quad r_1 = 2$$

The aim is to investigate whether the staggered algorithm converges in three dimensions and

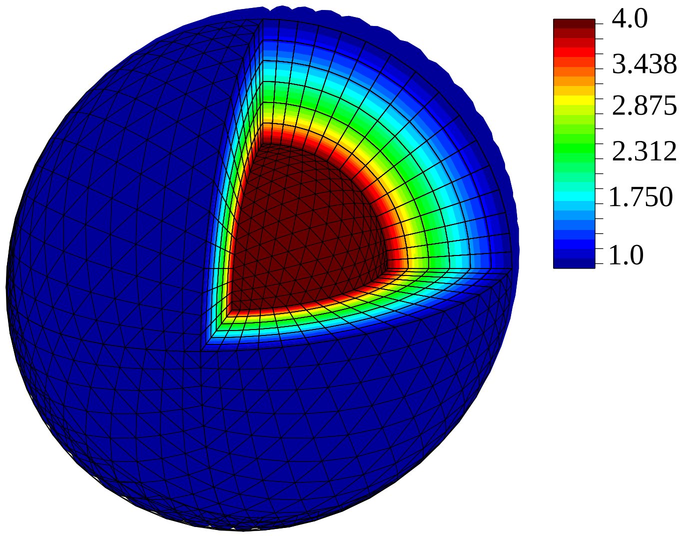


Figure 8.9: Example 2: u^h for $\rho_A = 0.5$ and DOF=11286

if yes, whether the convergence behaviour changes with different values of ρ_A. First at all, we can confirm that the staggered algorithm converges in three dimensions for the formulation of the coupled problem that uses the linear transport equation. The tables 8.7 and 8.8 contain the approximation error for the Poisson and transport equation and the corresponding convergence rates. The staggered algorithm converges quickly for both values of ρ_A. As in two dimensions,

Table 8.7: Ex 2: Algorithm 1 with $\rho_A = 0.5$

	Poisson		Transport		Staggered
DOF	$\|u - u_h\|_{H_1}$	α_u	$\|\rho - \rho_h\|_{L^2(\Omega)}$	α_ρ	Iterations
18	6.9027142		0.0294647		9
66	6.0057577	0.214	0.0242367	0.301	10
1290	2.2429156	0.663	0.0033514	1.331	9
11286	1.1276881	0.634	0.0011699	0.970	8
86058	0.6040090	0.615	0.0002416	1.553	8
737370	0.2918365	0.677	.5646E-04	1.354	8

we observe that with increasing ρ_A, the number of iterations also increases. While the algorithm needs about 8 iterations to converge for $\rho_A = 0.5$, it needs in average 13 iterations to converge for $\rho_A = 1.6$. Comparing Table 8.1 and 8.7, the staggered algorithm needs the same number of iterations for $\rho_A = 0.5$ in two and three dimensions. This is interesting not only because of the change of dimensions but even because the right hand side in the three-dimensional case is computed with less accuracy. The integrals are computed with a $4 \times 4 \times 4$ quadrature rule, while in two dimensions 16×16 quadrature points are used.

Table 8.8: Ex 2: Algorithm 3 with $\rho_A = 1.6$

	Poisson		Transport		Staggered
DOF	$\|u - u_h\|_{H_1}$	α_u	$\|\rho - \rho_h\|_{L^2(\Omega)}$	α_ρ	Iterations
18	6.3155316		0.2019701		15
66	4.8771643	0.398	0.2334961	-0.22	15
1290	1.8802057	0.641	0.0283733	1.418	13
11286	0.9431810	0.636	0.0077508	1.197	12
86058	0.5002869	0.624	0.0021641	1.256	12
737370	0.2428009	0.673	0.0004977	1.368	11

The convergence rate is not stable yet. In both cases it seems that the rate of the Poisson equation tends to 0.7. The convergence rate of the transport equation is not predictable yet. The approximation error for the Poisson equation is remarkably large. This might be caused by the linear basis function and the way maiprogs internally generates the mesh. In the first two steps for a low number of degrees of freedom, the mesh does not contain inner nodes. The derivative of the approximate solution is constant in r between the inflow and outflow boundary and thus naturally generates a big error.

8.4 Example 3: Electrostatic Spray Painting Process

As last example, we want to go back to the motivation of this work: The electrostatic spray painting process. So far, the theoretical and numerical results in this Thesis were done on $C^{2,\alpha}$ domains. In particular, these domains do not contain cusps or reentrant corners. The gun which is used to paint a workpiece contains an electrode having a high curvature. As a starting point, this electrode can be approximated by a point singularity. We now present a setup on a rectangular domain having a reentrant corner. We model the electrode, i.e. the inflow boundary, by the boundary part given by the reentrant corner and the outflow, i.e. the workpiece, is supposed to be the opposite boundary part (Figure 8.10). The tip of the corner is

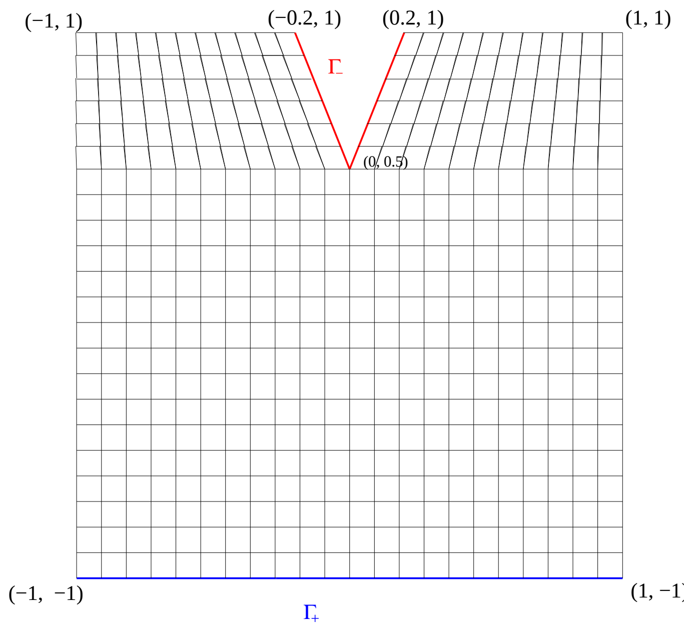


Figure 8.10: Mesh for Example 3

a point singularity. We investigate the formulation of the coupled problem that uses the linear transport equation and therefore Algorithm 1. We choose the parameter as

$$u_{A_1} = 4, \quad u_{A_2} = 1, \quad \rho_A = 0.5.$$

It is not possible to find a solution in closed form for this particular setting but we determine the $H^1(\Omega)$-norm of u^h and the L^2-norm of ρ^h. Using these sequences of numbers, we determine a good approximation for the exact solution by Aitken's extrapolation technique.

8.4.1 Numerical Results

The domain Ω is partitioned into quadrilateral elements. We use quadratic basis functions for the space $V^{P,h}$ and linear basis functions for the space $V^{T,h}$. The linear system of the Poisson equations is solved by the CG solver with an accuracy of $\epsilon = 10^{-12}$. The linear system for the

transport equation is solved by the CGNE solver with an accuracy of $\epsilon = 10^{-12}$. The right hand sides are computed with a 16×16 Gaussian quadrature rule. The accuracy for the staggered algorithm is set to 10^{-8}.
Table 8.9 confirms that the staggered algorithm converges although the domain is nonsmooth.

Table 8.9: Ex 3: Algorithm 1 with $\rho_A = 1.6$

	Poisson		Transport		Staggered
DOF	$\|u\|_{H_1} - \|u_h\|_{H_1}$	α_u	$\|\rho\|_{L^2(\Omega)} - \|\rho_h\|_{L^2(\Omega)}$	α_ρ	Iterations
37	0.1794444		0.1757060		23
167	0.1166833	0.571	0.0829415	0.996	21
583	0.0828583	0.548	0.0591498	0.541	20
2489	0.0549072	0.567	0.0357015	0.696	19
9167	0.0374420	0.587	0.0242033	0.596	19
37687	0.0249339	0.575	0.0162388	0.565	18
148453	0.0167392	0.581	0.0111624	0.547	18
596189	0.0112116	0.577	0.0073553	0.600	18

The number of iterations is significantly higher than in the comparable case on a smooth domain. On the smooth annular domain, the staggered algorithm needs 8 iterations (Table 8.1) in comparison to 19 iterations in the current example. The convergence rates are 0.9 for the Poisson equation and 0.5 for the transport equation. Again, due to the singularity of the domain, the convergence rates are smaller than in Example 1. In Figure 8.12, we present the electrical field $-\nabla u^h$ for 5199 degrees of freedom. It is clearly visible that the electrical field is huge around

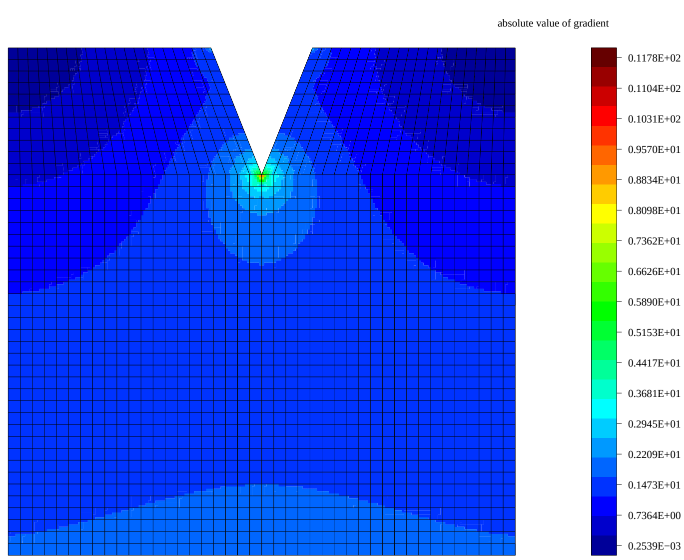


Figure 8.11: Example 3: Electrical field $-\nabla u^h$

the tip of the electrode which is caused by the singularity in the geometry. In Figure 8.12, we present the charge distribution ρ^h. The charge flows in at Γ_- and moves toward the outflow

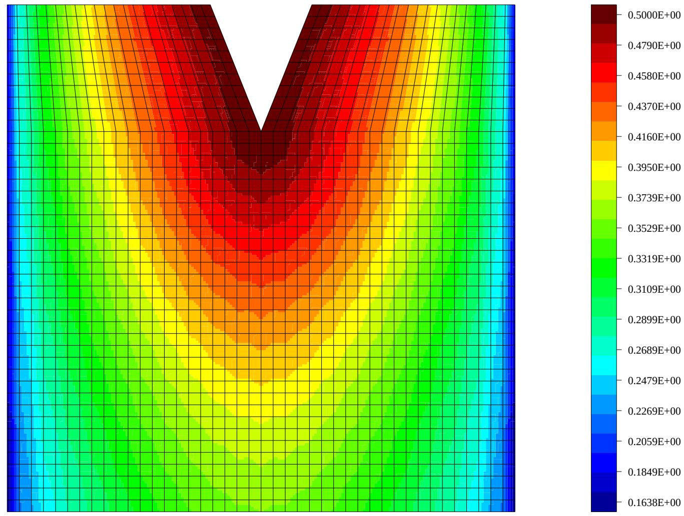


Figure 8.12: Example 3: Charge distribution ρ^h

boundary. We can see in Figure 8.12, that the charge concentration is high around the electrode and spreads into the domain. As we have proved in Chapter 6, the highest value of ρ is at the electrode and thus ρ is bounded by ρ_A.

This example opens the door for ideas for future work. The singularity at the tip demands a finer mesh structure than the used uniform one. One possibility is to use a graded mesh as it has been presented for the transport equation in [50]. It is high likely that grading toward the point singularity increases the speed of convergence. So far, numerical examples have been obtained for the electrostatic spray painting process, see e.g. [1]. To the best of our knowledge, a rigorous approximation theory for the staggered algorithm as indicated in 7.5 does not exist. The presented example suggests the convergence of the staggered algorithm and thus might be a starting point to develop an approximation theory for the coupled problem on a nonsmooth domain.

8.5 Remarks about the Chapter

In this chapter, we introduced four discretization methods for the two-dimensional steady state coupled problem. We distinguished between the approximation of the the coupled problem using the linear and nonlinear transport equations. To deal with the nonlinearity caused by the overall problem or, in terms of the staggered algorithm by the nonlinear transport equation, we introduced Newton's iteration scheme. It turned out that all the algorithms 1 and 3-5 converged to the exact solution. Further, we showed that the number of iterations to obtain convergence depends on the size of the inflow boundary function ρ_A. The larger ρ_A, the more iterations are needed. We could show that the staggered algorithm even converges for larger values of ρ_A than predicted by the theoretical results of Chapter 5. We then proceeded to the radially symmetric coupled problem in three dimensions. Again, the staggered algorithm performed satisfactorily. As last example, we investigated a problem setting to simulate the electrostatic spray painting process. Here, we approximated the electrode by a point singularity and thus obtained a non smooth domain. The staggered algorithm converged although the number of iterations is higher than for the example on a smooth domain.

Chapter 9

Conclusions and Further Research

9.1 Conclusions

In this thesis, we analysed a time dependent and time independent hyperbolic-elliptic coupled problem with respect to existence and uniqueness of a solution and its approximation properties.

In the Chapters 3 and 4, we addressed the question whether a classical solution exists for the continuous time dependent coupled problem. In [44, 55], the authors presented a method to formulate a similar problem as an integro-differential operator by means of the streamline function. By a compactness argument, they showed the existence of a unique fixed point and conclusively a classical solution to the coupled problem. This method was not immediately transferable as the two model problems differed in the choice of boundary conditions. The challenge in the analysis of the underlying problem of the corona discharge was to model a continuous inflow of charge into the domain and additionally to have nonhomogeneous boundary conditions for the Poisson problem. Chapter 3 outlined the methods used in Chapter 4 in a one dimensional setting. Therein, we presented the analysis for a homogeneous initial charge distribution mostly focusing on modeling the charge inflow into the domain. The idea was to introduce an inflow set to describe the time a charge particle flew into the domain. On this set, we defined a streamline function following the path of a particle emitted at the inflow boundary. In line with [44, 55], we obtained an integro-differential operator and proved the existence of a unique fixed point for a small time interval $[0, T]$ by the Banach fixed point theorem. As a consequence, we also obtained the short time existence of a classical solution. The presented two dimensional approach in Chapter 4 generalised the one of [44, 55] not only in terms of the boundary data but also in the choice of the geometry. We chose a domain that was homeomorph to an annular domain and thus contained the additional difficulty of being not simply connected and nonconvex. Chapter 4 used the same idea for modeling the charge inflow as Chapter 3 but dealt with an additional challenge. To be able to continue the solution in time, it is necessary to have a non homogeneous initial charge distribution. Therefore, we had to ensure the continuous transition of the mapped initial charge distribution with the inflowing charge. We introduced two streamline functions Φ_1 and Φ_2 for the mapping of the initial distribution and the inflowing charge and derived two integro-differential operators A_1 and A_2. Assuming that the boundary conditions ρ_0 and ρ_A were sufficiently small, we eventually could prove the unique existence of a fixed point (Φ_1, Φ_2), i.e. $(\Phi_1, \Phi_2) = (A_1(\Phi_1, \Phi_2), A_2(\Phi_1, \Phi_2))$ on a set W containing all those streamline functions leading to a well defined problem. Consequently, we could show the short time existence of a classical solution (u, ρ) to the model problem. Moreover, we proved that this solution is extendable up to the time until the support of the charge distribution reaches the outflow boundary. In both the one and two dimensional cases, the interval of existence $[0, T]$ of the solution depends on the chosen boundary data and the geometry of the domain.

Starting from Chapter 5, we focused on the steady state coupled problem. Chapters 5 and 6 followed the approach of using the solution operators L of the Poisson problem, i.e. $L\rho = -\nabla u = E$ and T of the transport problem, i.e. $TE = \rho$ to prove the existence of a solution. While L was applied to the right hand side function ρ of the Poisson equation, the operator T used the coefficient function E of the transport equation as its argument and was nonlinear. In Chapter 5, we investigated the radial symmetric coupled problem on an annular domain for which it reduced to a one dimensional problem. We were able to obtain explicit representations for the solution operators and defined the composite operator $T \circ L$. We could show by the Banach fixed point theorem that the composite operator $T \circ L$ had a unique fixed point provided ρ_A was sufficiently small. By the definition of the solution operators, we also obtained the unique existence of a classical solution. In Chapter 6, we investigated the general two dimensional case on a domain Ω that was chosen homeomorph to an annular domain. We followed the same methods as Chapter 5 and used the solution operators L and T. Again, we could prove by the Banach fixed point theorem and with ρ_A small enough, that a fixed point and thus a unique solution to the steady state two dimensional coupled problem exists. The choice of ρ_A was depending in both the one and two dimensional cases on the choice of the boundary data and the geometry which are all quantities that are known as soon as the problem is defined. The size of ρ_A might appear very restrictive and small. However, the method we have presented is useful and advantageous in comparison to the compactness argument illustrated in Chapter 4. As first consequence, we naturally defined an algorithm to solve the continuous coupled problem: The Banach fixed point iterations. By the well known error estimate, an upper bound for the number of iterations is immediately given. This iterative method is a staggered algorithm that solves the two problems alternating until convergence is obtained. The staggered algorithm is also known to be a way to solve the coupled problem numerically. In Chapter 7, we investiged discretization methods for the coupled problem. In particular, we assumed based on the results of Chapter 6 and numerical simulations that the operator $L \circ T$ is also a contraction in Sobolev spaces. The usefulness of this assumption became obvious: To show the convergence of the discrete staggered algorithm, we used the contraction property of the continuous operator $L \circ T$ together with stability and error estimates for the Poisson and transport problems and proved an error estimate for $\|u - L^h \circ T^h u^h\|_{H^1(\Omega)}$.

Chapter 8 eventually underlined the theoretical results for the steady state coupled problem by numerical experiments. While it is equivalent for the continuous coupled problem to use the linear or nonlinear transport equation, it is important to distinguish these two formulations in the discrete case. Beside the staggered algorithm, we presented three more algorithms that used Newton's method to linearize the problem. We could find a solution in closed form for the radial symmetric coupled problem of Chapter 5 and thus used it as example. As first experiment, we compared the convergence properties of the four algorithms. Indeed, all algorithms converged to the exact solution with the same rate of convergence. As second experiment, we focused on the formulation of the coupled problem with the nonlinear transport equation and used the staggered algorithm to investigate the influence of the inflow boundary condition ρ_A. It turned out that the larger ρ_A, the more iterations are necessary to obtain convergence. Compared to the theoretical results in Chapter 5, the staggered algorithm did converge even for a value of ρ_A for which $L \circ T$ is not a contraction anymore. To obtain an idea about future research, we presented a three dimensional radial symmetric example. The staggered algorithm converged satisfactorily. As last example, we presented a setting on a nonsmooth domain that is supposed to represent the electrostatic spray painting process. Again, we observed convergence although the number of iterations was significantly higher than in the corresponding continuous case.

9.2 Further Research

We will now suggest some ideas how this thesis might be extended and include some unanswered questions that appeared during its preparation.

Recall that the motivation of this work is the corona discharge in the electrostatic spray painting process. Since a work piece is painted in an open space, the problem could be seen as an exterior problem in $\mathbb{R}^2 \backslash \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 being two compact domains representing the gun and the target. During our literature review, we were not able to find Schauder estimates or continuity estimates for the Green's function for exterior problems in Hölder spaces. These results however would be necessary to reformulate the analysis of the Chapters 4 and 6 to an exterior setting. In Chapter 4, we could prove the unique existence of a solution for short time T and were able to extend it in time until the support of the charge distribution reached the outflow boundary. The painting process, however, does not stop as soon as the first colour particle arrives at the work piece. An immediate extension would be to incorporate the outflow of charge.

All our existence proofs in the Chapters 3 - 6 were obtained in classical spaces. In view of the variational theory, an immediate suggestion for further research is the development of analogous results in Sobolev spaces. With the methods of [5], existence and uniqueness might be proved easily. However, with this method future results for the discrete staggered algorithm do not immediately follow. As we explained above, it is advantageous to examine whether $L \circ T$ is a contraction on Sobolev spaces. Also, it is necessary to further examine stability properties of the continuous solution operator T applied to discrete vector fields E^h, in other words to examine the continuous transport equation with a discrete convective field. If both results are obtained, then 7.5 gives immediately an error estimate for the discrete staggered algorithm. This would open the door for several new aspects of investigation. The analysis in this thesis has been done on smooth domains. In case of the electrostatic spray painting model and indicated in section

8.4, the domain is nonsmooth due to the high curvature of the electrode. As starting point, the electrode can be modelled as a point singularity. An immediate extension is to pay tribute to the singularity and develop approximation results on anisotropic meshes.

Appendix A

Additional Results

A.1 Nonemptiness of W

Let the notations be as in Chapter 4. We show that the defined sets W_1 and W_2 are nonempty. Therefore, we use the electrical field $-\nabla u_0 = E_0$ with u_0 being the solution of the Laplace problem and show that the streamline functions Φ_1 and Φ_2 given through E_0 are in W_1 and W_2. We therefore need Grönwalls inequality which is an equality to bound the solution of an integral equation.

Lemma A.1 (Grönwall's inequality, version 1)**.** *[19, p. 37] Assume $I = [t_0, t_1]$ and $q, c, u \geq 0 \in C^0$. If*

$$q(t) \leq c(t) + \int_{t_0}^{t} u(s)q(s)\,ds \tag{A.1}$$

then holds

$$q(t) \leq c(t) + \int_{t_0}^{t} u(s)c(s)\exp\left(\int_{s}^{t} u(\tau)\,d\tau\right)\,ds$$

Lemma A.2 (Grönwall's inequality, version 2)**.** *[55, Lemma 3.1, p. 89] Assume $I = [t_0, t_1]$ and $q \geq 0, u \geq 0 \in C^0$ and $c \geq 0 \in C^1$. If*

$$q(t) \leq c(t) + \int_{t_0}^{t} u(s)q(s)\,ds \tag{A.2}$$

then follows

$$q(t) \leq c(t_0)\exp\left(\int_{t_0}^{t} u(s)\,ds\right) + \int_{t_0}^{t} c'(s)\exp\left(\int_{s}^{t} u(\tau)\,d\tau\right)\,ds$$

Proof. Set $\gamma(s) = \exp\left(-\int_{s_0}^{s} u(\mu)\, d\mu\right)$. Then,

$$
\begin{aligned}
\frac{d}{ds}\left(\gamma(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau\right) &= u(s)q(s)\gamma(s) + \gamma'(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau \\
&= u(s)q(s)\gamma(s) - \gamma(s)u(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau \\
&= u(s)q(s)\left(\gamma(s) - \int_{s_0}^{s} u(\tau)q(\tau)\, d\tau\right) \\
&\leq u(s)\gamma(s)c(s).
\end{aligned}
$$

Integrating both sides and using the initial condition gives

$$
\gamma(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau \leq \int_{s_0}^{s} u(\tau)\gamma(\tau)c(\tau)\, d\tau. \tag{A.3}
$$

With integration by parts, we have

$$
\int_{s_0}^{s} c'(\tau)\gamma(\tau) = c(s)\gamma(s) - c(s_0)\gamma(s_0) + \int_{s_0}^{s} c(\tau)u(\tau)\gamma(\tau)\, d\tau
$$

and thus (A.3) turns into

$$
\gamma(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau \leq -c(s)\gamma(s) + c(s_0) + \int_{s_0}^{s} c'(\tau)\gamma(\tau)\, d\tau.
$$

Multiplying (A.2) by $\gamma \geq 0$, we have the following inequality

$$
\begin{aligned}
q(s)\gamma(s) &\leq c(s)\gamma(s) + \gamma(s)\int_{s_0}^{s} u(\tau)q(\tau)\, d\tau \\
&\leq c(s)\gamma(s) - c(s)\gamma(s) + c(s_0) + \int_{0}^{s} c'(\tau)\gamma(\tau)\, d\tau \\
&\leq c(s_0) + \int_{s_0}^{s} c'(\tau)\gamma(\tau)\, d\tau.
\end{aligned}
$$

Multiplying by $1/\gamma$ gives the assertion. $\square$

Let us start with the streamline function Φ_1 which is due to (4.21a)-(4.21b) the solution of

$$
\frac{d}{dt}\Phi_1(x,t) = E_0(\Phi_1(x,t)) \qquad (x,t) \in \Omega \times [0,T] \tag{A.4}
$$

$$
\Phi_1(x,0) = x \qquad x \in \Omega. \tag{A.5}
$$

Integrating with respect to t, we obtain the integral equation

$$
\Phi_1(x,t) = x + \int_{0}^{t} E_0(\Phi_1(x,\mu))\, d\mu. \tag{A.6}
$$

We will now prove that $\Phi_1 \in W_1$.

Lemma A.3. *Let Ω be a $C^{2,\alpha}$ domain, $u_A \in C^{2,\alpha}(\Omega)$. Let $M \geq 3 + 2\|x\|_{0,\Omega}$ and $T \leq \min\left\{\frac{1}{2c_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}M}, \left(\frac{1}{2c_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}(2+M+M^{1+\alpha})}\right)^{\alpha}\right\}$. Then holds for Φ_1 defined in (A.6) that*

$$\Phi_1 \in W_1(M,T).$$

Proof. We show that Φ_1 fulfills the restrictions of W_1. First, it holds $\Phi_1(x,0) = x$. Second, we prove the boundedness of Φ_1. It holds for the sup-norm by Theorem 4.14

$$\begin{aligned}
\|\Phi_1\|_{0,\Omega_0} &\leq \|x\|_{0,\Omega} + \sup_{x\in\Omega_0}\left|\int_0^t E_0(\Phi_1(x,\mu))\,d\mu\right|_\infty \\
&\leq \|x\|_{0,\Omega_0} + \int_0^t \sup_{x\in\Omega_0}|E_0(\Phi_1(x,\mu))|_\infty\,d\mu \\
&\leq \|x\|_{0,\Omega_0} + \int_0^t \|E_0\|_{0,\Omega}\,d\mu \\
&\leq \|x\|_{0,\Omega_0} + Tc_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}.
\end{aligned}$$

It is more complicated to bound the gradient

$$\nabla\Phi_1(x,t) = I + \int_0^t \nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu.$$

We obtain pointwise

$$|\nabla\Phi_1(x,t)|_\infty \leq |I|_\infty + \int_0^t |\nabla E_0(\Phi_1(x,\mu))|_\infty\,|\nabla\Phi_1(x,\mu)|_\infty\,d\mu.$$

By Lemma A.1 and Theorem 4.14, we get

$$\begin{aligned}
|\nabla\Phi_1(x,t)|_\infty &\leq 1 + \int_0^t |\nabla E_0(\Phi_1(x,\mu))|_\infty \exp\left(\int_\mu^t |\nabla E_0(\Phi_1(x,\tau))|_\infty\,d\tau\right)d\mu \\
&\leq 1 + \int_0^t \|\nabla E_0(\Phi_1(\mu))\|_{0,\Omega_0} \exp\left(\int_\mu^t \|\nabla E_0(\Phi_1(\tau))\|_{0,\Omega_0}\,d\tau\right)d\mu \\
&\leq 1 + T\,\|\nabla E_0\|_{0,\Omega} \exp\left(T\|\nabla E_0\|_{0,\Omega}\right) \\
&\leq 1 + Tc_S(\alpha,\Omega)\,\|u_A\|_{2,\alpha,\Gamma} \exp\left(Tc_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}\right).
\end{aligned}$$

Next, we investigate the α- semi norm of the gradient. We start with the pointwise estimate

$$\begin{aligned}
&|\nabla\Phi_1(x_1,t) - \nabla\Phi_1(x_2,t)|_\infty \\
&= \left|\int_0^t \left[E_0(\Phi_1(x_1,\mu)) - E_0(\Phi_1(x_2,\mu))\right]\nabla\Phi_1(x_2,\mu) + \left[\nabla\Phi_1(x_1,\mu) - \nabla\Phi_1(x_2,\mu)\right]E_0(\Phi_1(x_2,\mu))\,d\mu\right|_\infty \\
&\leq \int_0^t |E_0(\Phi_1(x_1,\mu) - E_0(\Phi_1(x_2,\mu))|_\infty\,|\nabla\Phi_1(x_2,\mu)|_\infty\,d\mu \\
&\quad + \int_0^t |\nabla\Phi_1(x_1,\mu) - \nabla\Phi_1(x_2,\mu)|_\infty\,|E_0(\Phi_1(x_2,\mu))|_\infty\,d\mu.
\end{aligned}$$

By Lemma A.2, we obtain

$$
\begin{aligned}
&|\nabla\Phi_1(x_1,t)-\nabla\Phi_1(x_2,t)|_\infty \\
&\quad\le \int_0^t |E_0(\Phi_1(x_1,\mu)-E_0(\Phi_1(x_2,\mu))|_\infty\,|\nabla\Phi_1(x_2,\mu)|_\infty \exp\left(\int_s^t |\nabla E_0(\Phi_1(x,t))|_\infty\right)
\end{aligned}
$$

It follows for the Hölder semi norm by (2.9) and by Theorem 4.14 for every $t\in[0,T]$

$$
\begin{aligned}
|\nabla\Phi_1(t)|_{\alpha,\Omega} &\le \int_0^t |E_0(\Phi_1(\mu)|_\alpha\,\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\exp\left(\int_\mu^t \|\nabla E_0(\Phi_1(\mu))\|_{0,\Omega_0}\right)d\mu \\
&\le \int_0^t |E_0|_{\alpha,\Omega}\,\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}^\alpha\,\|\nabla\Phi_1(\mu)\|_{0,\Omega_0}\exp\left(T\|\nabla E_0\|_{0,\Omega}\right)d\mu \\
&\le Tc_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha,\Gamma}\sup_{0\le t\le T}\|\nabla\Phi_1(t)\|_{0,\Omega}^{1+\alpha}\exp\left(Tc_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}\right).
\end{aligned}
$$

Analogously, we bound the Hölder norm in time for Φ_1 and $\nabla\Phi_1$. It holds

$$
\begin{aligned}
\sup_{x\in\Omega_0}\|\Phi_1(x)\|_{\alpha,[0,T]} &\le \|x\|_{0,\Omega_0}+T^{1-\alpha}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} \\
\sup_{x\in\Omega_0}\|\nabla\Phi_1(x)\|_{\alpha,[0,T]} &\le 1+T^{1-\alpha}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}\sup_{0\le t\le T}\|\nabla\Phi_1(t)\|_{0,\Omega_0}.
\end{aligned}
$$

As last step, we have to investigate whether Φ_1 is invertible. We therefore estimate

$$
\begin{aligned}
\|I-\nabla\Phi_1(t)\|_{0,\Omega} &= \sup_{x\in\Omega_0}\left|I-I-\int_0^t \nabla E_0(\Phi_1(x,\mu))\nabla\Phi_1(x,\mu)\,d\mu\right|_\infty \\
&\le \int_0^t \|\nabla E_0(\Phi_1(\mu))\nabla\Phi_1(\mu)\|_{0,\Omega_0}\,d\mu \\
&\le T\,\|\nabla E_0\|_{0,\Omega}\,\|\nabla\Phi_1(t)\|_{0,\Omega_0}.
\end{aligned}
$$

By Theorem 4.14, we obtain

$$
\|I-\nabla\Phi_1\|_{0,\Omega}\le Tc_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}\,\|\nabla\Phi_1\|_{0,\Omega}.
$$

Choose T small enough, we have

$$
\begin{aligned}
\|\Phi_1\|_{1,\alpha,\Omega_0;\alpha,[0,T]} &\le 2\|x\|_{0,\Omega}+3 \\
\|I-\nabla\Phi_1\|_{0,\Omega} &\le \frac{1}{2}.
\end{aligned}
$$

□

Next, we show analogously that Φ_2 defined by

$$
\Phi_2(s,t_x,t)=\varphi(s)+\int_{t_x}^t E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu \tag{A.7}
$$

is element of the set $W_2(M,T,K,\delta)$. This proof is however more complicated then the previous one due to the time dependence of Q_t.

Lemma A.4. *Let Ω be a $C^{2,\alpha}$ domain, $u_A \in C^{2,\alpha}(\Omega)$.*
Let $M \geq 2\|\varphi\|_{0,I_{\Gamma_-}} + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma} + 3$, $K \geq c_{mv}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}L_{\Gamma_-}^{1-\alpha} + 1$ and let

$$T \leq \min\Big\{\frac{1}{2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}}, \frac{\delta}{4M^2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}},$$
$$\left(\frac{1}{c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}(3 + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma} + M + M^{1+\alpha} + 2K)}\right)^{\alpha}\Big\}$$

with $c_S(\Omega,\alpha)$ defined in Theorem 4.14. Then follows for Φ_2 defined in (A.7) that

$$\Phi_2 \in W_2(M,T,K,\delta).$$

Proof. We show that Φ_2 fulfills the restrictions in W_2. First, it holds $\Phi_2(s,t_x,t_x) = \varphi(s)$. Second, we prove the boundedness of Φ_2. It holds for the sup-norm by Theorem 4.14

$$\begin{aligned}
\|\Phi_2(t)\|_{0,Q_t} &\leq \|\varphi\|_{0,I_{\Gamma_-}} + \sup_{(s,t_x)\in Q_t}\left|\int_{t_x}^t E_0(\Phi_2(s,t_x,\mu))\,d\mu\right|_\infty \\
&\leq \|\varphi\|_{0,I_{\Gamma_-}} + \int_0^t \sup_{(s,t_x)\in Q_t} |E_0(\Phi_2(s,t_x,\mu))\,|_\infty\,d\mu \\
&\leq \|\varphi\|_{0,I_{\Gamma_-}} + Tc_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}.
\end{aligned}$$

The gradient of Φ_2 is given by

$$\begin{aligned}
\nabla\Phi_2(s,t_x,t) &= \begin{pmatrix} \varphi_1'(s) & -[E_0(\varphi(s))]_1 \\ \varphi_2'(s) & -[E_0(\varphi(s))]_2 \end{pmatrix} + \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu \\
&=: D(s) + \int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu.
\end{aligned}$$

We now bound the gradient pointwise

$$|\nabla\Phi_2(s,t_x,t)|_\infty \leq |D(s)|_\infty + \int_{t_x}^t |\nabla E_0(\Phi_2(s,t_x,\mu))|_\infty\,|\nabla\Phi_2(s,t_x,\mu)|_\infty\,d\mu.$$

By Lemma A.1 and Theorem 4.14, we get

$$\begin{aligned}
|\nabla\Phi_2(s,t_x,t)|_\infty &\leq |D(s)|_\infty + \int_{t_x}^t |D(s)|_\infty\,|\nabla E_0(\Phi_2(x,\mu))|_\infty \exp\left(\int_\mu^t |\nabla E_0(\Phi_2(x,\tau))\,d\tau\right)d\mu \\
&\leq \|D\|_{0,Q_t} + \int_0^t \|D\|_{0,Q_t}\,\|\nabla E_0(\Phi_2(\mu))\|_{0,\Omega_0} \exp\left(\int_\mu^t \|\nabla E_0(\Phi_2(\tau))\|_{0,\Omega_0}\,d\tau\right)d\mu \\
&\leq \|D\|_{0,Q_t} + T\,\|D\|_{0,Q_T}\,\|\nabla E_0\|_{0,\Omega}\exp\left(T\|\nabla E_0\|_{0,\Omega}\right) \\
&\leq \|D\|_{0,Q_t} + T\,\|D\|_{0,Q_T}\,c_S(\alpha,\Omega)\,\|u_A\|_{2,\alpha,\Gamma}\exp\left(Tc_S(\alpha,\Omega)\|u_A\|_{2,\alpha,\Gamma}\right).
\end{aligned}$$

We have for the sup-norm of D by Theorem 4.14 and the arc length parametrization of φ

$$\|D\|_{0,Q_T} \leq \|\varphi'\|_{0,I_{\Gamma_-}} + \|E_0(\varphi)\|_{0,Q_t} \leq 1 + c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}.$$

It is left to bound the $C^\alpha(Q_t)$ semi norm of the gradient. We start with a pointwise estimate.

$$
\begin{aligned}
&|\nabla\Phi_2(s_1,t_1,t)-\nabla\Phi_2(s_2,t_2,t)|_\infty\\
&\quad\le |D(s_1)-D(s_2)|_\infty+\int_{t_1}^{t_2}|\nabla E_0(\Phi_2(s_1,t_1,\mu))|_\infty\,|\nabla\Phi_2(s_1,t_1,\mu)|_\infty\,d\mu\\
&\qquad+\int_{t_2}^{t}|\nabla E_0(\Phi_2(s_1,t_1,\mu))-\nabla E_0(\Phi_2(s_2,t_2,\mu))|_\infty\,|\nabla\Phi_2(s_2,t_2,\mu)|_\infty\,d\mu\\
&\qquad+\int_{t_2}^{t}|\nabla\Phi_2(s_1,t_1,\mu)-\nabla\Phi_2(s_2,t_2,\mu)|_\infty\,|\nabla E_0(\Phi_2(s_2,t_2,\mu))|_\infty\,d\mu. \qquad (A.8)
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
g(s_1,s_2,t_1,t_2,t):=|D(s_1)-D(s_2)|_\infty&+\int_{t_1}^{t_2}|\nabla E_0(\Phi_2(s_1,t_1,\mu))|_\infty\,|\nabla\Phi_2(s_1,t_1,\mu)|_\infty\,d\mu\\
&+\int_{t_2}^{t}|\nabla E_0(\Phi_2(s_1,t_1,\mu))-\nabla E_0(\Phi_2(s_2,t_2,\mu))|_\infty\,|\nabla\Phi_2(s_2,t_2,\mu)|_\infty\,d\mu.
\end{aligned}
$$

By Lemma A.1, we obtain for (A.8)

$$
\begin{aligned}
&|\nabla\Phi_2(x_1,t)-\nabla\Phi_2(x_2,t)|_\infty\le g(s_1,s_2,t_1,t_2,t)\\
&\quad+\int_{t_x}^{t}g(s_1,s_2,t_1,t_2,\mu)|\nabla E_0(\Phi_2(s_1,t_1,\mu)|_\infty\exp\left(\int_\tau^t|\nabla E_0(\Phi_2(s_1,t_2,\tau)|_\infty\,d\tau\right)d\mu. \qquad (A.9)
\end{aligned}
$$

To bound $\|\nabla\Phi_2(t)\|_{\alpha,Q_t}$, let us first bound g. By (2.9) and Theorem 4.14, we obtain

$$
\|g\|_{\alpha,\Omega}\le\|D\|_{\alpha,Q_t}+T^{1-\alpha}c_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha,\Gamma}\,\|\nabla\Phi_2(t)\|_{0,Q_t}+Tc_S(\Omega,\alpha)\,\|u_A\|_{2,\alpha,\Gamma}\,\|\nabla\Phi_2(t)\|_{0,Q_t}^{1+\alpha}.
$$

D is bounded by (2.9) and Theorem 4.14

$$
\begin{aligned}
\|D\|_{\alpha,Q_t}&\le c_{mv}\|\nabla E_0\|_{0,\Omega}\|\varphi'\|_{0,[0,L_{\Gamma_-}]}\sup_{s_1,s_2\in I_{\Gamma_-}}|s_1-s_2|^{1-\alpha}\\
&\le c_{mv}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}L_{\Gamma_-}^{1-\alpha} \qquad (A.10)
\end{aligned}
$$

It follows for (A.9) by Theorem 4.14

$$
\begin{aligned}
\|\nabla\Phi_2\|_{\alpha,Q_t}&\le\|g\|_{\alpha,Q_t}+\int_{t_x}^{t}\|g\|_{\alpha,Q_t}\|\nabla E_0(\Phi_2(\mu))\|_{0,Q_t}\exp\left(\int_\tau^t\|\nabla E_0(\Phi_2(\mu))\|_{0,Q_t}\,d\tau\right)d\mu\\
&\le\|g\|_{\alpha,Q_t}+T\|g\|_{\alpha,Q_t}\|\nabla E_0\|_{0,\Omega}\exp\left(T\|\nabla E_0\|_{0,\Omega}\right)\\
&\le\|g\|_{\alpha,Q_t}+T\|g\|_{\alpha,Q_t}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}\exp\left(Tc_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}\right)
\end{aligned}
$$

Analogously, we obtain for the Hölder norms with respect to time

$$
\begin{aligned}
\sup_{(s,t_x)\in Q_t}\|\Phi_2(s,t_x)\|_{\alpha,[t_x,T]}&\le\|\varphi\|_{0,I_{\Gamma_-}}+2T^{1-\alpha}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}\\
\sup_{(s,t_x)\in Q_t}|\nabla\Phi_2(s,t_x)|_{\alpha,[t_x,T]}&\le T^{1-\alpha}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}\sup_{0\le t\le T}\|\nabla\Phi_2(t)\|_{0,Q_t}.
\end{aligned}
$$

We continue to investigate the remaining two conditions. Since E_0 is the solution of the Laplace problem, we know immediately

$$|\partial_{t_x}\Phi_2(s,t_x,t=t_x)\cdot\varphi'(s)^\perp| = |E_0(\varphi(s))\cdot\varphi'(s)^\perp| \geq \delta.$$

As last condition, we examine the invertibility of Φ. We get

$$\begin{aligned}
\sup_{s,t_x\in Q_t}|D(s)-\nabla\Phi_2(s,t_x,t)|_\infty &= \sup_{s,t_x\in Q_t}|D(s)-D(s)-\int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu|_\infty \\
&\leq \sup_{s,t_x\in Q_t}\left|\int_{t_x}^t \nabla E_0(\Phi_2(s,t_x,\mu))\nabla\Phi_2(s,t_x,\mu)\,d\mu\right|_\infty \\
&\leq T\left\|\nabla E_0\right\|_{0,\Omega}\left\|\nabla\Phi_2\right\|_{0,Q_t} \\
&\leq Tc_S(\alpha,\Omega)\left\|u_A\right\|_{2,\alpha,\Gamma}\left\|\nabla\Phi_2\right\|_{0,Q_t}.
\end{aligned}$$

If T is chosen small enough, then holds

$$\begin{aligned}
\|\Phi_2\|_{1,Q_t,\alpha,[0,T]} &\leq 2\|\varphi\|_{0,I_{\Gamma_-}} + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma} + 3 \\
\sup_{0\leq t\leq T}|\nabla\Phi_2(t)|_{\alpha,Q_t} &\leq c_{mv}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}L_{\Gamma_-}^{1-\alpha} + 1 \\
\sup_{s,t_x\in Q_t}|D(s)-\nabla\Phi_2(s,t_x,t)|_\infty &\leq \frac{\delta}{4M}.
\end{aligned}$$

□

As last condition, we need to ensure that Φ_1 and Φ_2 fulfil the transition condition on Γ_t^T.

Theorem A.5. *Let Ω be a $C^{2,\alpha}$ domain and $u_A \in C^{2,\alpha}(\Gamma)$.*
Let $M \geq 2\|x\|_{0,\Omega_0} + 2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma} + 3$, $K \geq c_{mv}c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}L_{\Gamma_-}^{1-\alpha} + 1$ and

$$\begin{aligned}
T \leq \min\Bigg\{&\frac{1}{2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}}, \frac{\delta}{4M^2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}}, \\
&\left(\frac{1}{2c_S(\Omega,\alpha)\|u_A\|_{2,\alpha,\Gamma}(3+c_S(\Omega,\alpha)\|u_A\|_{2,\alpha;\Gamma}+M+M^{1+\alpha}+2K)}\right)^\alpha\Bigg\}
\end{aligned}$$

with $c_S(\Omega,\alpha)$ defined in Theorem 4.14. Then holds for $\Phi = (\Phi_1,\Phi_2)$ with $\Phi_1 \in W_1(M,T)$ defined in (A.6) and $\Phi_2 \in W_2(M,T,K,\delta)$ defined in (A.7) that

$$\Phi \in W(M,T,K,\delta). \tag{A.11}$$

Proof. We have to show that Φ_1 and Φ_2 fulfil the transition condition i.e. for every $x_0 \in \Omega_0$ and $(s_0,0) \in Q_t$ with $x_0 = (x_1,x_2) = \varphi(s_0)$ shall hold

$$\Phi_1(x_0,t) = \Phi_2(s_0,0,t). \tag{A.12}$$

By differentiating (A.6) and (A.7), we have the differential equation of Φ_1 and Φ_2 going through x_0 and $(s_0,0)$

$$\begin{aligned}
\frac{d}{dt}\Phi_1(x_0,t) &= E_0(\Phi_1(x_0,t)) \\
\Phi_1(x_0,0) &= x_0
\end{aligned}$$

and

$$\frac{d}{dt}\Phi_2(s_0,0,t) = E_0(\Phi_2(s_0,0,t))$$
$$\Phi_2(s_0,0,0) = \varphi(s_0).$$

Both differential equations have the same right hand side function. Further, both have the same initial condition, as $\varphi(s_0) = x_0$. Since the differential equation is solved with respect to t, the solution is not affected by the different parametrization of Φ_1 and Φ_2. Due to the uniqueness theory of ordinary differential equations, both streamline functions must describe the same trajectory. It is thus demonstrated that for every t holds

$$\Phi_1(x_0,t) = \Phi_2(s_0,0,t).$$

Conclusively, Φ_1 and Φ_2 fulfil the transition condition and $\Phi = (\Phi_1, \Phi_2) \in W$. □

A.2 Bound for Cut-off Function

Observe that with the substitution $\mu = 2t - 1$ and symmetry about $s = 0$ we have

$$140\int_0^1 t^3(1-t)^3\,dt = \frac{140}{64}\int_0^1 (1-\mu^2)^3\,d\mu = \frac{140}{64}\left(1 - \frac{3}{3} + \frac{3}{5} - \frac{1}{7}\right) = 1. \tag{A.13}$$

Hence define for $x \in [0,1]$

$$\begin{aligned}\gamma(x) &= 1 - 140\int_0^x t^3(1-t)^3\,dt = 140\int_x^1 t^3(1-t)^3\,dt \\ &= 20x^7 - 70x^6 + 84x^5 - 35x^4 + 1.\end{aligned}$$

Define the transformation

$$F : \begin{cases} [0,1] & \to [\epsilon, 2\epsilon] \\ x & \mapsto (1+x)\epsilon \end{cases}.$$

Then the cut-off function $\chi \in C^3(\mathbb{R})$ is given by

$$\chi(s) = \begin{cases} 1, & s < \epsilon \\ \gamma(F^{-1}(s)), & \epsilon \le s \le 2\epsilon \\ 0, & s > 2\epsilon \end{cases}$$

with

$$F^{-1}(s) = \frac{s}{\epsilon} - 1$$

Lemma A.6. *It holds*

$$\|\chi\|_{2,\alpha,\mathbb{R}} \le \frac{52.5}{\epsilon^{2+\alpha}}.$$

Proof. We get for the sup-norm

$$\|\partial_s \chi\|_{0,\mathbb{R}} = 1.$$

Since $\partial_s F^{-1}(s) = \frac{1}{\epsilon}$ and $\partial_s^2 F^{-1}(s) = 0$ holds

$$\partial_s \chi(s) = \partial_x \hat{\chi}(F^{-1}(s))\partial_s F^{-1}(s) = \frac{\partial_x \hat{\chi}(F^{-1}(s))}{\epsilon}$$
$$\partial_s^2 \chi(s) = \partial_x^2 \hat{\chi}(F^{-1}(s))(\partial_s F^{-1}(s))^2 + \partial \hat{\chi}(F^{-1}(s))\partial_s^2 F^{-1}(s) = \frac{\partial_x^2 \hat{\chi}(F^{-1}(s)}{\epsilon^2}$$

We obtain

$$\|\partial_s \chi\|_{0,[\epsilon,2\epsilon]} = \frac{1}{\epsilon}\left\|\partial_x \hat{\chi}(F^{-1})\right\|_{0,[\epsilon,2\epsilon]} = \frac{1}{\epsilon}\left\|\partial_x \hat{\chi}\right\|_{0,[0,1]} = \frac{2.1875}{\epsilon}$$
$$\|\partial_s^2 \chi\|_{0,[\epsilon,2\epsilon]} = \frac{1}{\epsilon}\left\|\partial_x^2 \hat{\chi}(F^{-1})\right\|_{0,[\epsilon,2\epsilon]} = \frac{1}{\epsilon^2}\left\|\partial_x^2 \hat{\chi}\right\|_{0,[0,1]} = \frac{7.5131}{\epsilon}$$

and

$$\begin{aligned}\left\|\partial_s^2 \chi\right\|_{\alpha,[\epsilon,2\epsilon]} &= \frac{1}{\epsilon^2}\left\|\partial_x^2 \hat{\chi}(F^{-1})\right\|_{\alpha,[\epsilon,2\epsilon]} \leq \frac{1}{\epsilon^2}\left\|\partial_x^3 \hat{\chi}\right\|_{0,Q}\left\|F^{-1}\right\|_{\alpha,[\epsilon,2\epsilon]} \\ &= \frac{1}{\epsilon^3}\left\|\partial_x^3 \hat{\chi}\right\|_{0,[\epsilon,2\epsilon]} \frac{|s_1 - s_2|}{|s_1 - s_2|^\alpha} \\ &\leq \frac{1}{\epsilon^{2+\alpha}}\left\|\partial_x^3 \hat{\chi}\right\|_{0,[0,1]} = \frac{52.5}{\epsilon^{2+\alpha}}.\end{aligned}$$

Conclusively,

$$\left|\partial_s^2 \chi\right|_{\alpha,\mathbb{R}} \leq \frac{52.5}{\epsilon^{2+\alpha}}.$$

□

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