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Abstract

For this research project, we have been doing research on the shallow water equations: a set of hyperbolic partial differential equations. These equations exist as a set of three primary equations [2]. However, there is another version of the shallow water equations called the Saint Venant's equations. These equations are similar to the standard shallow water equations, but these equations have been reduced to one-dimension. The primary goal of our research has been to investigate the behavior and mathematical construction of the Saint Venant's equations and model these equations using COMSOL. Regardless of the equation type, standard or Saint Venant's, it is useful to note that these equations are only applicable under some restrictions such as hydrostatic balance and distance from one crest to another, on any two waves, must be greater than the distance from the free surface to the sea floor (bottom topography). These restrictions, along with initial conditions, are also a target in this research, and these conditions and equations can help with flood predictions and regulations not only now, but also in the future.

Keywords: shallow water equations, hyperbolic partial differential equations

1 Introduction and Motivation

The shallow water equations are hyperbolic or parabolic partial differential equations that describe fluid flow, especially around areas close to the oceans. These equations are derived from the Navier Stokes equations often used in studying partial differential equations. In recent history, these equations have been used in the field of meteorology to predict storm surge in coastal regions. Often times, the equations are used to model Kelvin waves in the

lakes and rivers, but in order for the equations to be valid, the wavelength must be larger than the basin holding the water. [1]
The reason we became interested in these equations in particular is because we want to understand more about meteorology. Coupling information between mathematics and meteorology is not only interesting, but useful. Also, since Louisiana's coast is among the fastest to disappear in the world, we want to learn more about waves eroding the coast away. [6]

1.1 Basics

The shallow water equations can only be used under very specific conditions. For example, the shallow water equations neglect surface tension and theoretically describe shallow water regions where there exists hydrostatic balance. In other words all particles and water particles through a particular parcel of water must all be flowing at the same rate or velocity; otherwise, these equations would not necessarily be useful. [4] We will analyze wave behavior by inputting initial boundary conditions in COMSOL to monitor wave behavior on the surface and spilling break.

2 Background Literature and Related Studies

In order to understand the shallow water equations, we must first be able to identify what given variables in the equations represent. Each of the three primary shallow water equations include the variables listed in the following sections. However, more variables are included to acknowledge those variables that play a role in the derivation in these equations. Lots of the literary background focuses on the variables used to define the primary Shallow Water Equations and the Saint Venant's Equations. A huge portion of our variable definitions come from Dawson's article on the Shallow Water Equations. [1] The variables and attributes characterizing the Saint Venant's Equations come from the following citation: [7]

2.1 Defining Variables

We will define the following variables as follows:

- $\dot{\vec{u}} := (u, v, w)$, where u,v, and w are essentially the x,y, and z components respectively. This vector is the three-dimensional velocity representation in the arbitrary body of water.
- x is the length component of the water (in meters)
- y is the width component of the water (in meters)
- t is the time parameter being used (in seconds)

- $g := -9.8m/s^2$ is accleration due to gravity
- $z = -h(x, y)$ is the surface defined to be the floor topography
- $z = n(x, y, t)$ is the vertical displacement of the free surface

  Without understanding these fundamental definitions, comprehending these equations becomes exceedingly more difficult, so that is why they are allotted to this section of the paper.

2.2 Shallow Water Equations

There are three primary shallow water equations, and these equations are derived from the continuity equation which is defined to be

$$\nabla \cdot \dot{\vec{u}} = 0 \tag{1}$$

For now, the continuity equation can be thought of as the gradient of the three-dimensional velocity. This equation is used to obtain the three shallow water equations. These equations can be denoted:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial n}{\partial x} = 0 \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial n}{\partial y} = 0 \tag{3}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}[(n+h)u] + \frac{\partial}{\partial y}[(n+h)v] = 0 \tag{4}$$

However, the focus of our research rests in the one-dimensional version of the shallow water equations: the Saint Venant's Equations. [4]

2.3 Saint Venant's Equations

There are two primary variations of the Saint Venant's Equations: Continuity and Momentum equations. For the research conducted with Dr. Dai, we use a variation of the Momentum Equation as recommended by COMSOL, and this equation with its components will be dissected in the 'Methods' section. The equations can be represented as follows:

- Continuity Equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{5}$$

- Momentum Equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial H}{\partial x} + gAS_f + gAh_t = 0 \tag{6}$$

- Primary Equation of Study:

$$e_a \frac{\partial^2 \dot{\vec{u}}}{\partial t^2} + d_a \frac{\partial \dot{\vec{u}}}{\partial t} + \nabla \cdot \Gamma = \dot{\vec{f}} \tag{7}$$

It should be noted that some more of those variables should be defined.
A:=cross-sectional area
Q:=discharge of water
H:=hydraulic head of water [7]

However, the equation we have been investigating in COMSOL has several components that we must discuss, and those transformations and equivocations will be shown in the Methods section. It should be noted that e_a is a basis vector and d_a is the damping coefficient. Γ is the conservative flux, and $\dot{\vec{f}}$ is external force due to water in our system.

3 Methods

The primary goal for this research is to see how initial boundary conditions can impact the formation and behavior of a free surface, denoted Z_s. One of the biggest methods in conducting this research was dissecting our equation of interest,7. By pulling this equation apart, we will be able to input our values as we wish into our Multiphysics package. In COMSOL, we worked in the Model Wizard window and selected the 1-dimensional PDE interface, which is where we begin inputting our designated values into our system.

3.1 COMSOL Inputs and Initial Boundary Conditions

When regarding our equation of interest,7, it is useful to note that we are required the insertion of two initial conditions; for this equation and/or wave, we have two initial conditions due to our non-linear PDE, and these conditions are listed below.

- Z_0=the depth of the water layer, where $Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$ [8]
- Z_f=sea floor topography, where $Z_f = ae^{-(x-x_0)^2} + k_1 x$ [3]

  The sea floor, Z_f, is defined in this manner as it offers a singular local extrema (maximum), and I wanted to see how the sea-surface profile, Z_s, is affected by the location of this extrema. This sea-floor general equation allowed me to manage the PDE a little better, and it was a suggested variation of the sea-floor. We were able to readjust the location of the extrema by altering x_0. The rest of the properties of the sea floor will be discussed in Section 4.1.

3.2 Equation Expansion and Coupling

Consider the original equation:

$$e_a \frac{\partial^2 \dot{\vec{u}}}{\partial t^2} + d_a \frac{\partial \dot{\vec{u}}}{\partial t} + \nabla \cdot \Gamma = \dot{\vec{f}}$$

$$\dot{\vec{f}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} Z \\ V \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} Z \\ V \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} VZ \\ -nu * V_x \end{bmatrix} = \begin{bmatrix} 0 \\ -g(Z_x + \frac{dZ_f}{dx} - V * V_x + nuV_x \frac{Z_x}{Z}) \end{bmatrix}$$

$$\Longrightarrow$$

$$\begin{bmatrix} \frac{\partial^2 Z}{\partial t^2} \\ \frac{\partial^2 V}{\partial t^2} \end{bmatrix} + \begin{bmatrix} \frac{\partial Z}{\partial t} \\ \frac{\partial V}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial x}(VZ) \\ -nu * \frac{\partial V_x}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ -g(Z_x + \frac{dZ_f}{dx}) - V * V_x + nuV_x \frac{Z_x}{Z} \end{bmatrix}$$

After we pull these equations out of their matrix forms we obtain the following system (due to coupling), we can rewrite these equations as

$$\frac{\partial^2 Z}{\partial t^2} + \frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}(VZ) = 0$$
$$\frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} - nu * \frac{\partial V_x}{\partial x} = -g(Z_x + \frac{dZ_f}{dx}) - V * V_x + nuV_x \frac{Z_x}{Z}$$

At this point, it should be noted that our vector, $\dot{\vec{u}}$, can actually be represented as the dependent variable matrix transposed and denoted

$$\dot{\vec{u}} = \begin{bmatrix} Z & V \end{bmatrix}^T$$

One more note to consider is that the conservative flux, denoted Γ, was represented as the 2 by 1 matrix

$$\begin{bmatrix} VZ \\ -nu * V_x \end{bmatrix} = \Gamma$$

[9]

3.3 COMSOL

COMSOL is a multiphysics package often used to represent and store physical systems that describe phenomena [5]. COMSOL is a useful package when working with the Saint Venant's Equations because it helps with visualization as a physical aquatic system can be viewed in one-dimension. While using COMSOL, we have decided to maintain the depth of the water layer, $Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$, as almost a constant. However, one of the primary modifications we make to the water layer is the location of the local extrema-or in our case-the global extrema. This can be changed by shifting the x_0 value (denoted as 3 in our Z_0 equation.)

4 Data and Models

Before offering visualization of our actual models within COMSOL, we will discuss some of the inputs we maintain in COMSOL. We have already discussed the depth of the water layer in Section 3.3 where we retain the same general equation throughout the research. Although manipulating the equation to create two or more extrema (maxima) values could be interesting, the difficulty with storing that data in COMSOL comes when we must keep crest-to-crest distance greater than the distance from Z_s to Z_f. This proves to be a daunting task because our water layer depth is not a consistent scalar value. Due to this, we keep a general equation for the sea floor, Z_f, as well:

$$Z_f = ae^{-(x-x_0)^2} + k_1 x \tag{8}$$

4.1 Floor Topography

The floor topography signifies one of our initial boundary conditions. For Z_f, we have three cases we consider, and two of those cases have two subcases. Before discussing these cases, variables used in the general sea-floor equation must be defined:

- a:=amplitude of the sea ridge
- x_0:=location of the sea ridge (local and global extrema of Z_f)
- k_1:=slope of the linear portions of sea floor (before and after sea ridge)

Each of the cases are described in Table 1 below:

TABLE 1	Amplitude of Sea Ridge (a)	location of sea ridge (x_0)	Linearity of Sea-Floor (k_1)
Case1a	0.003m	3	0.0015
Case1b	0.009m	3	0.0015
Case2a	0.003m	6	0.0015
Case2b	0.009m	6	0.0015
Case3a	0.007m	2	-0.0015

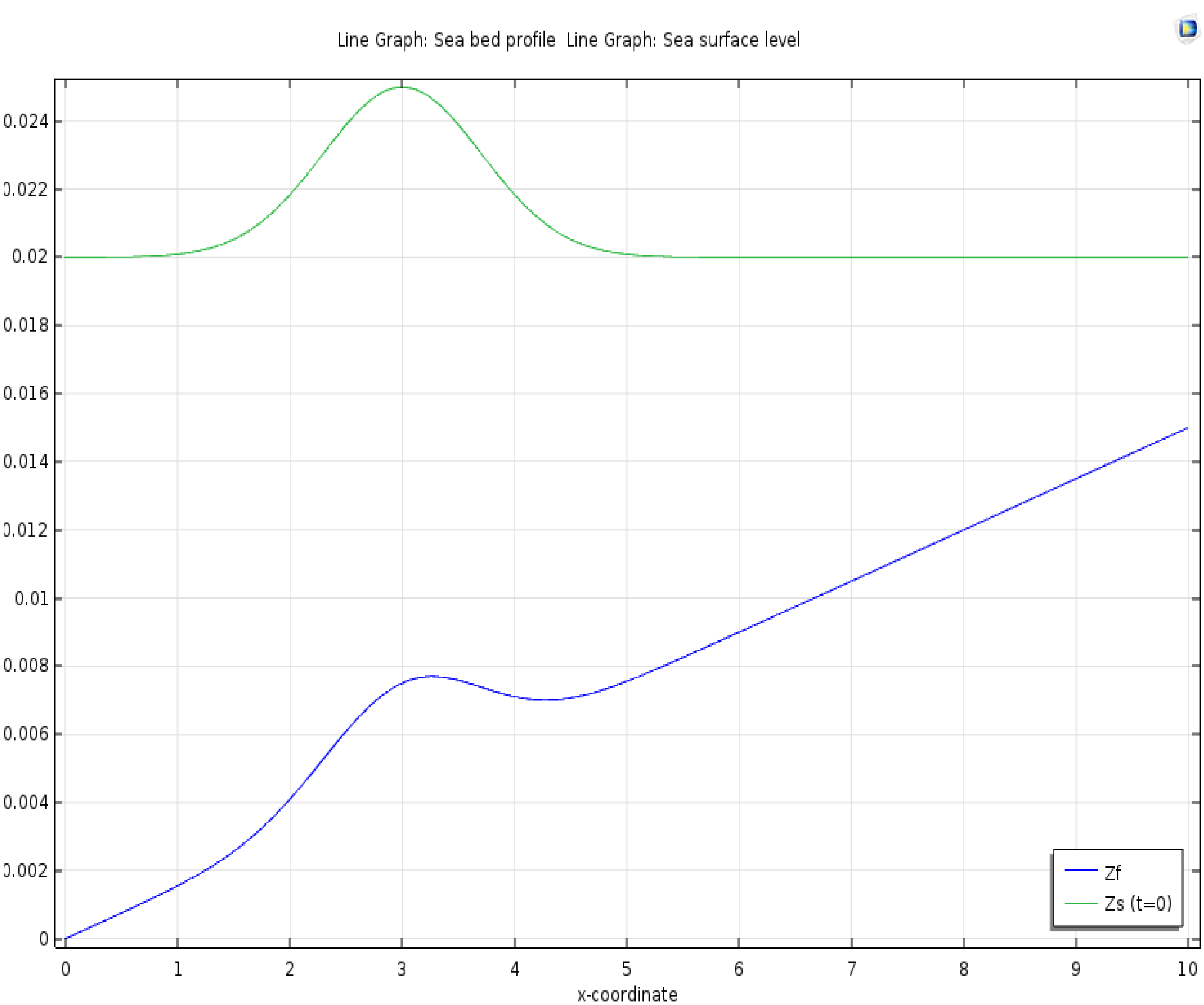
Values of Variables for Initial Boundary Conditions

The last thing that should be noted is that Case 1 considers a system in which a wave is propagated at the location of the local or global extrema of the sea floor. Case 2 considers a system in which a wave is propagated before the location of the local or global extrema of the sea floor, and Case 3 demonstrates how a wave behaves when the sea floor has a negative gradient and slope where the wave travels into a region of ascending water depth values.

4.2 Models

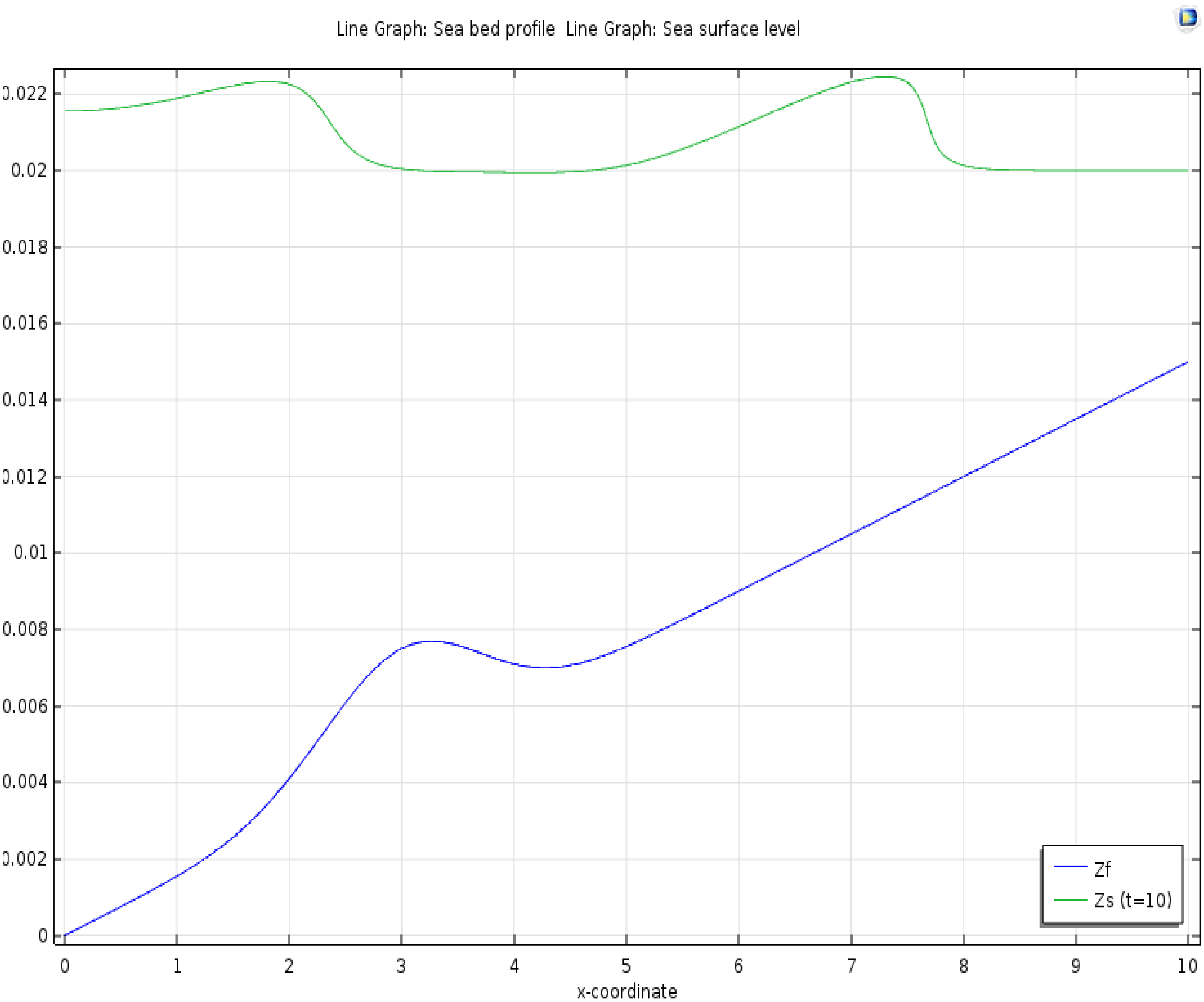
We will now show physical representations of our models for Cases 1,2, and 3 respectively. Each subcase will have exactly three models that show the behavior, motion, and spindrift of the wave from time $t = 0$ seconds to $t = 20$ seconds; however, Case 3a shows times $t = 0$ seconds, $t = 6$ seconds, and $t = 14$ seconds as we are monitoring a different system environment type.

Case 1a



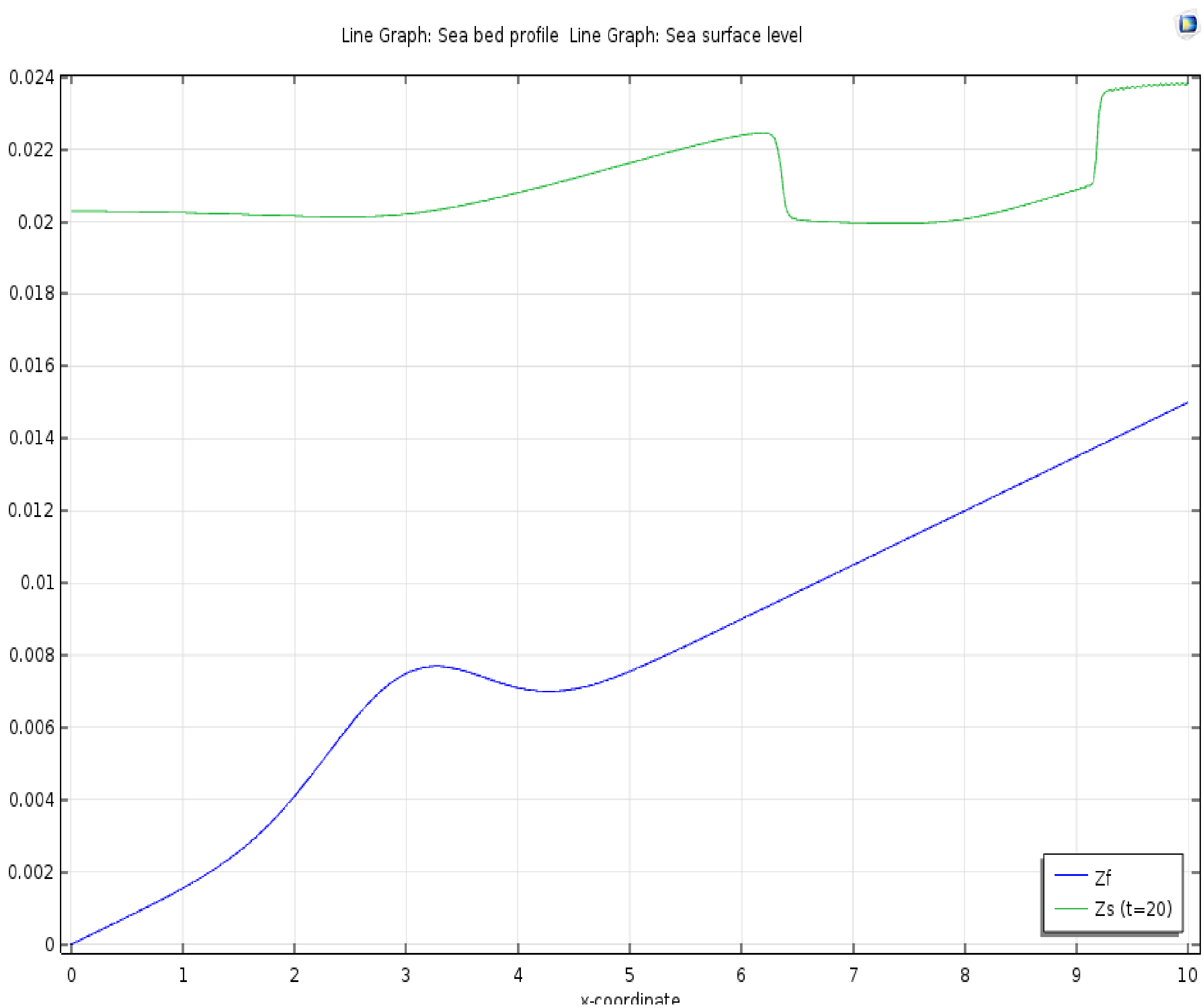
$$Z_f = 0.003e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 1a



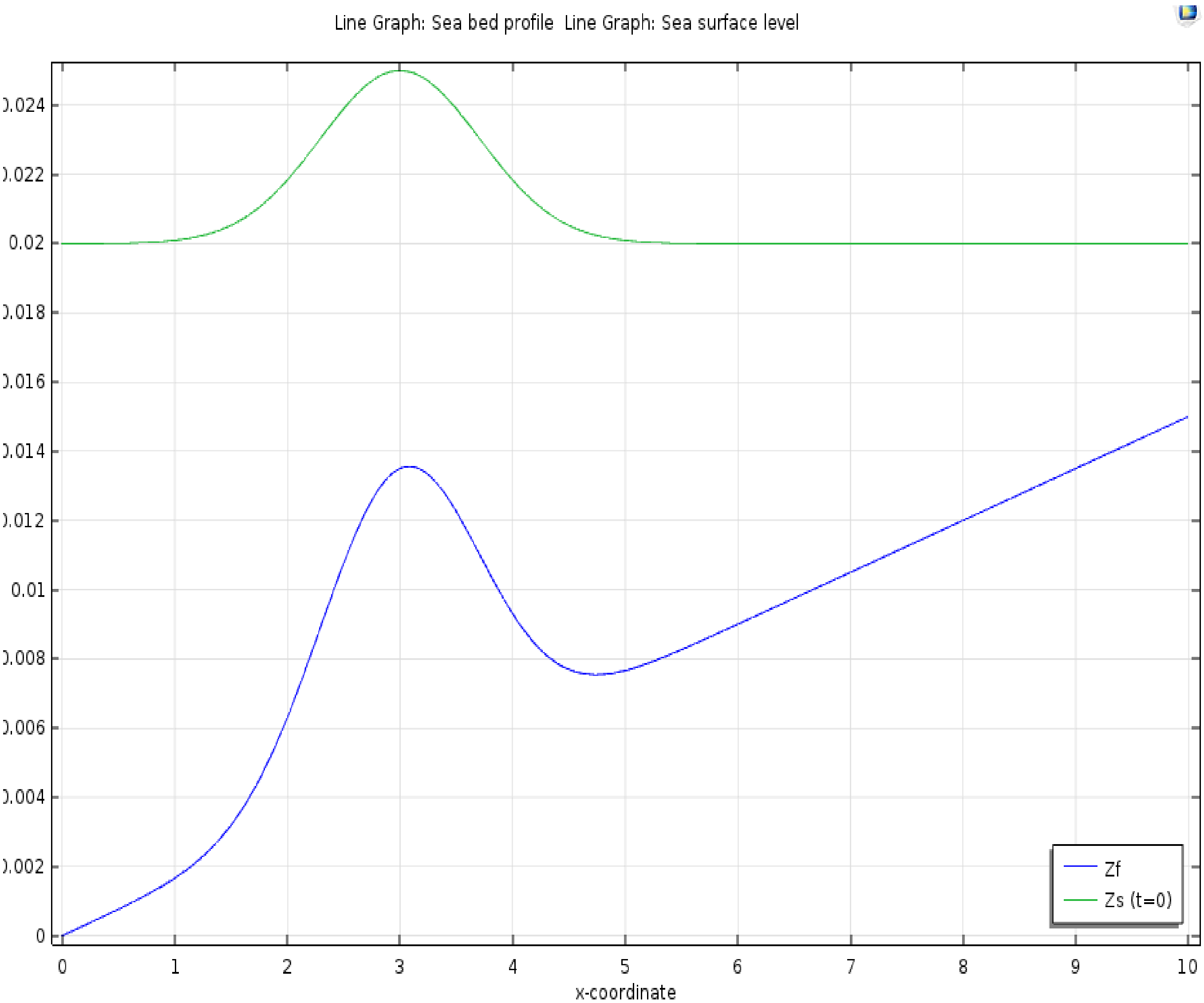
$$Z_f = 0.003e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 1a



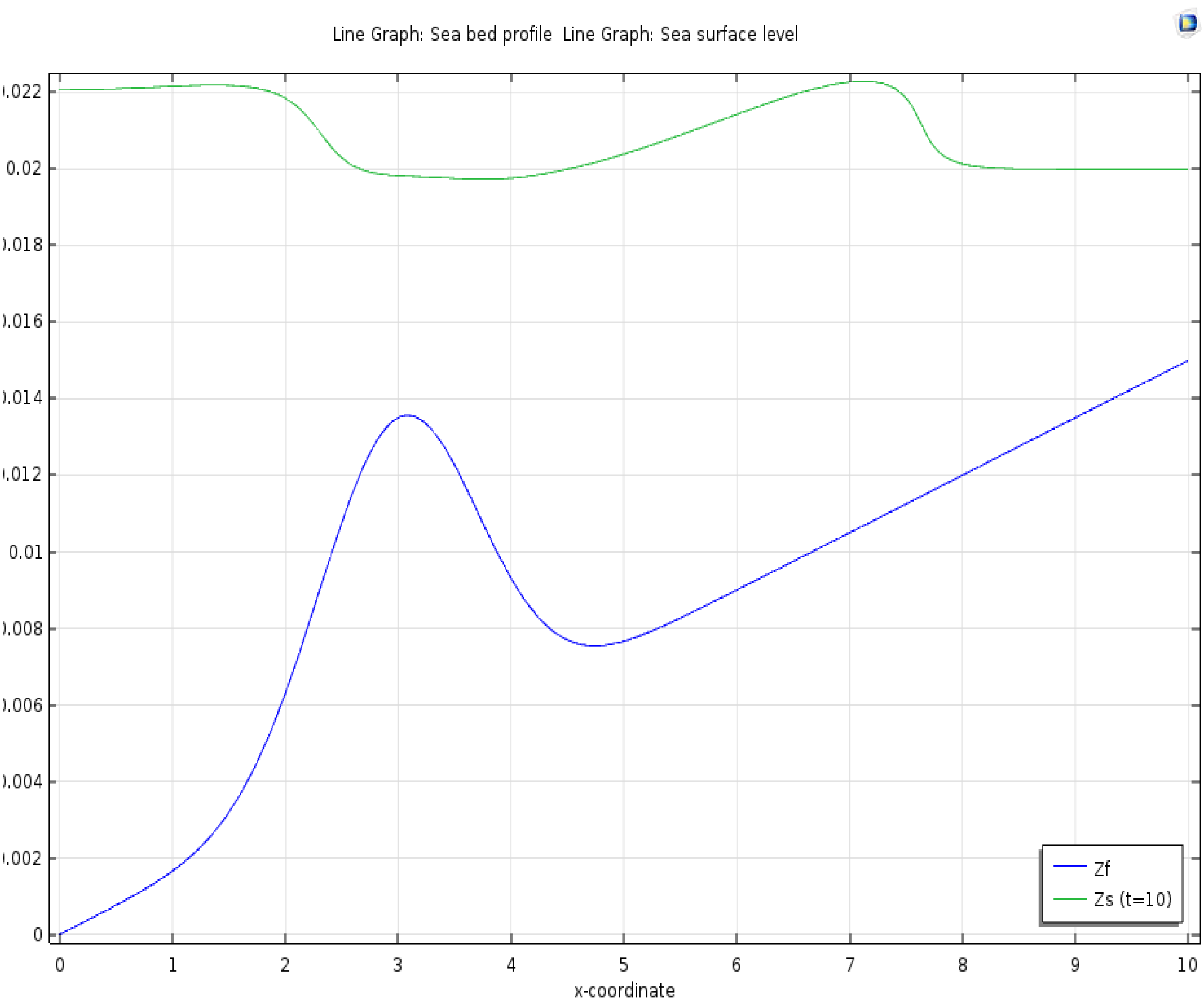
$$Z_f = 0.003e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 1b



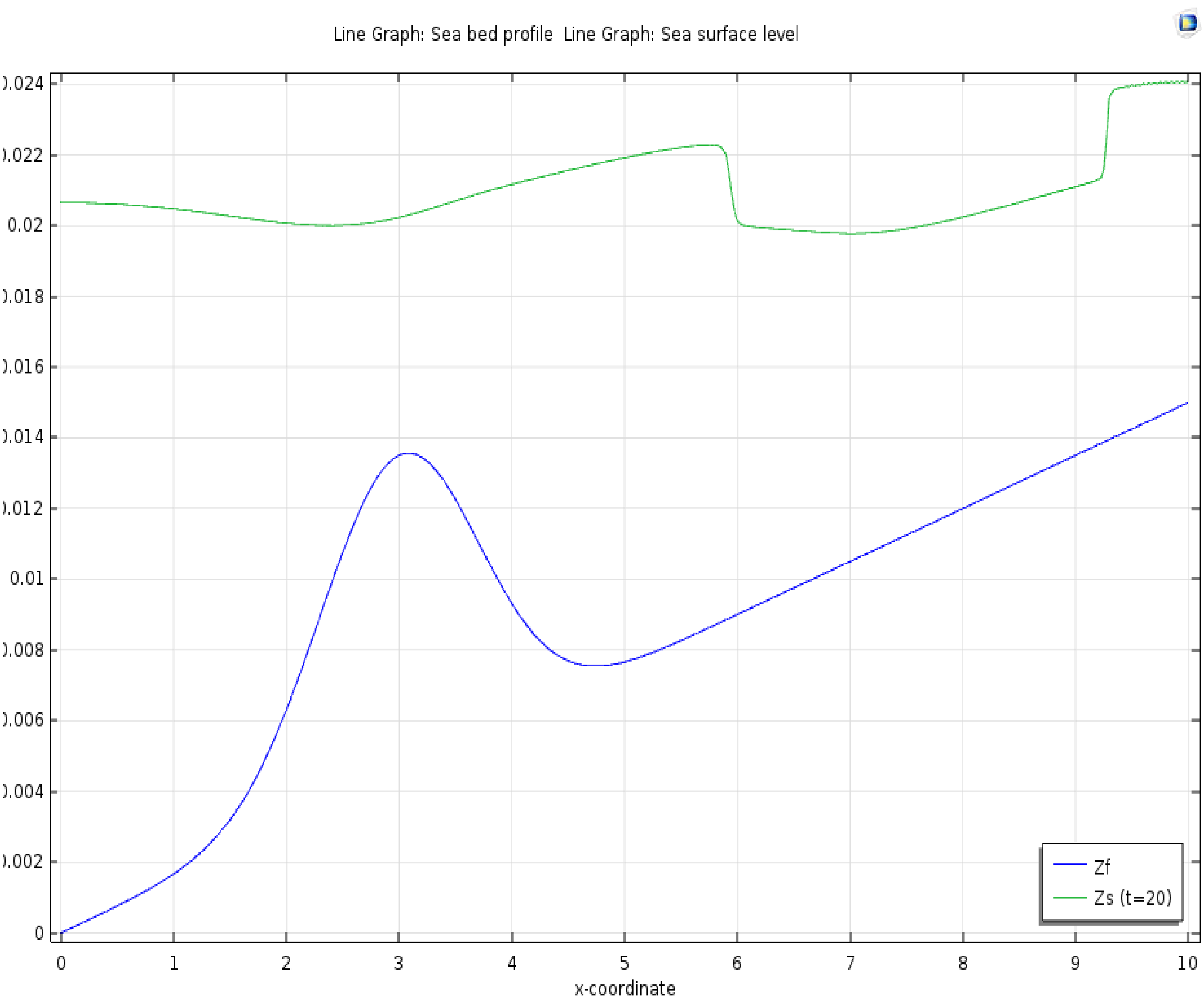
$$Z_f = 0.009e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 1b



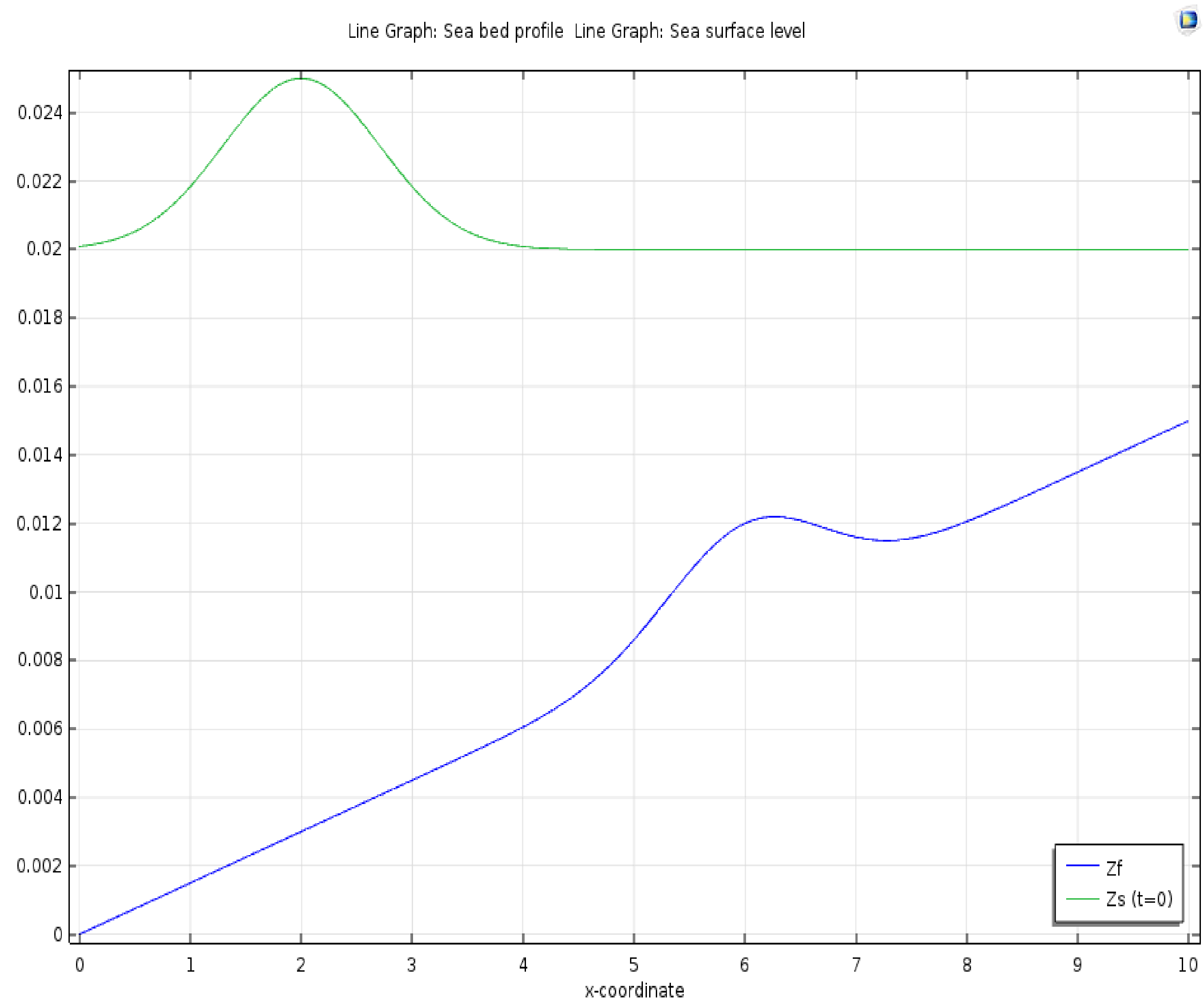
$$Z_f = 0.009e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 1b



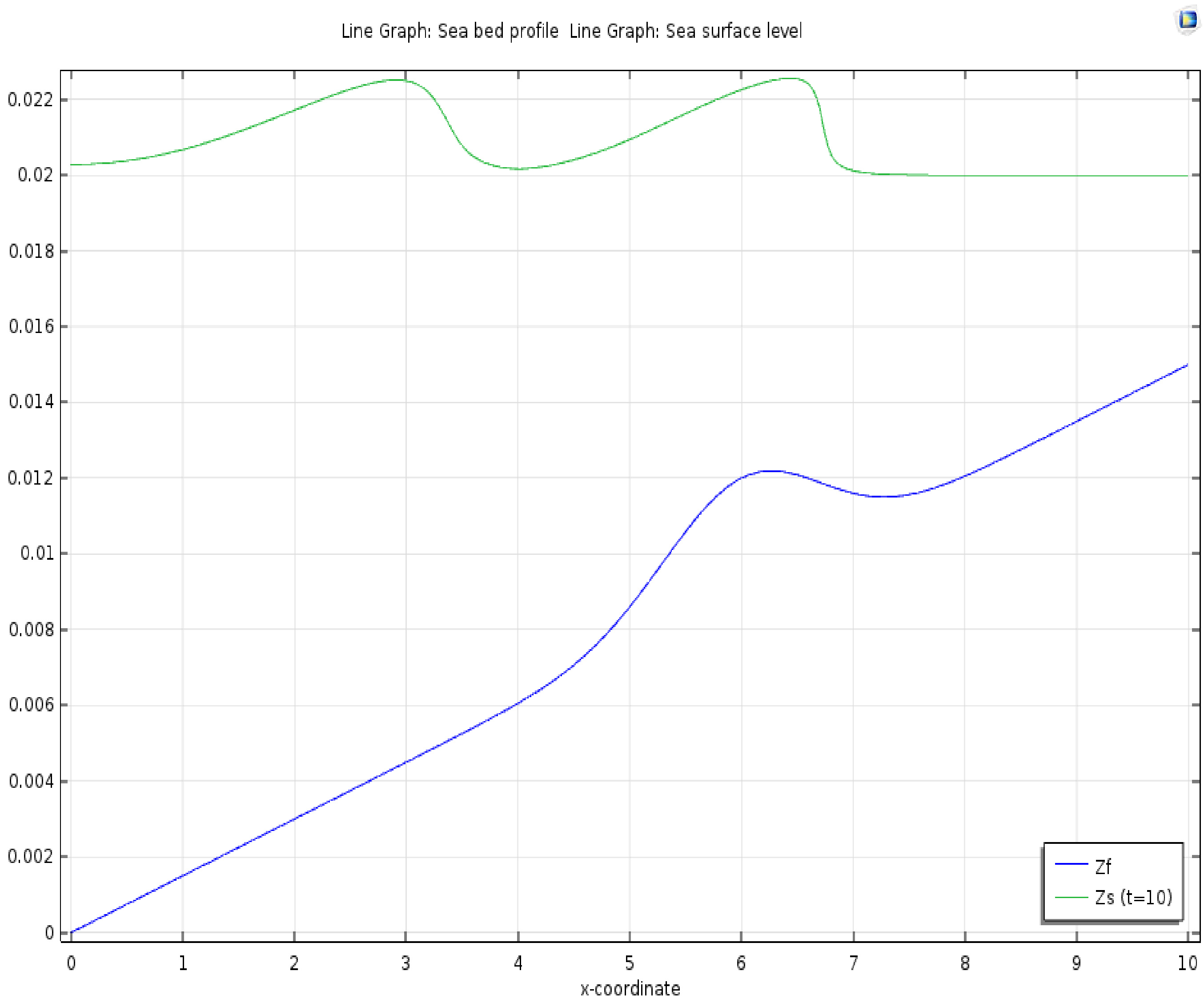
$$Z_f = 0.009e^{-(x-3)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-3)^2)}$$

Case 2a



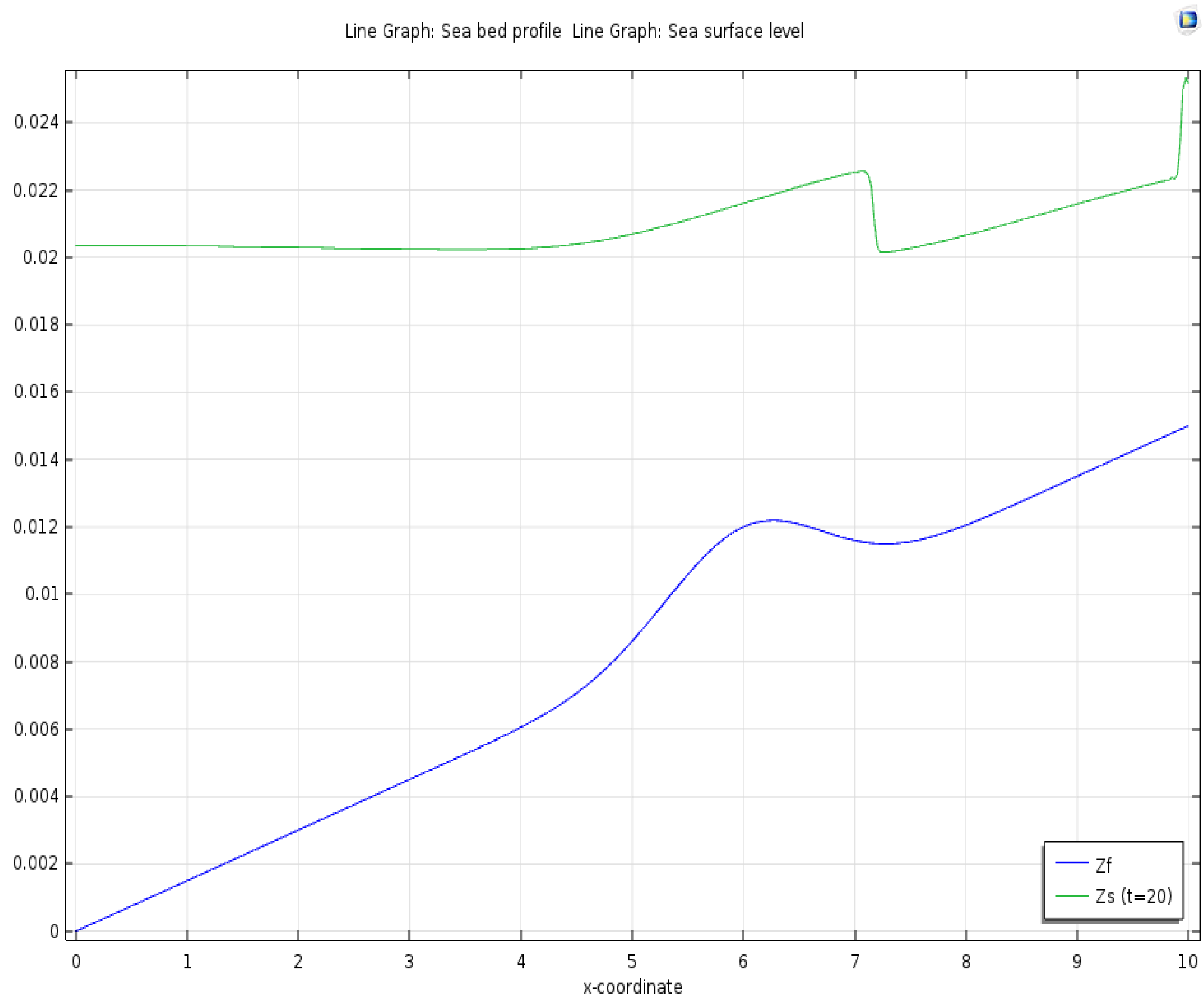
$$Z_f = 0.003e^{-(x-6)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 2a



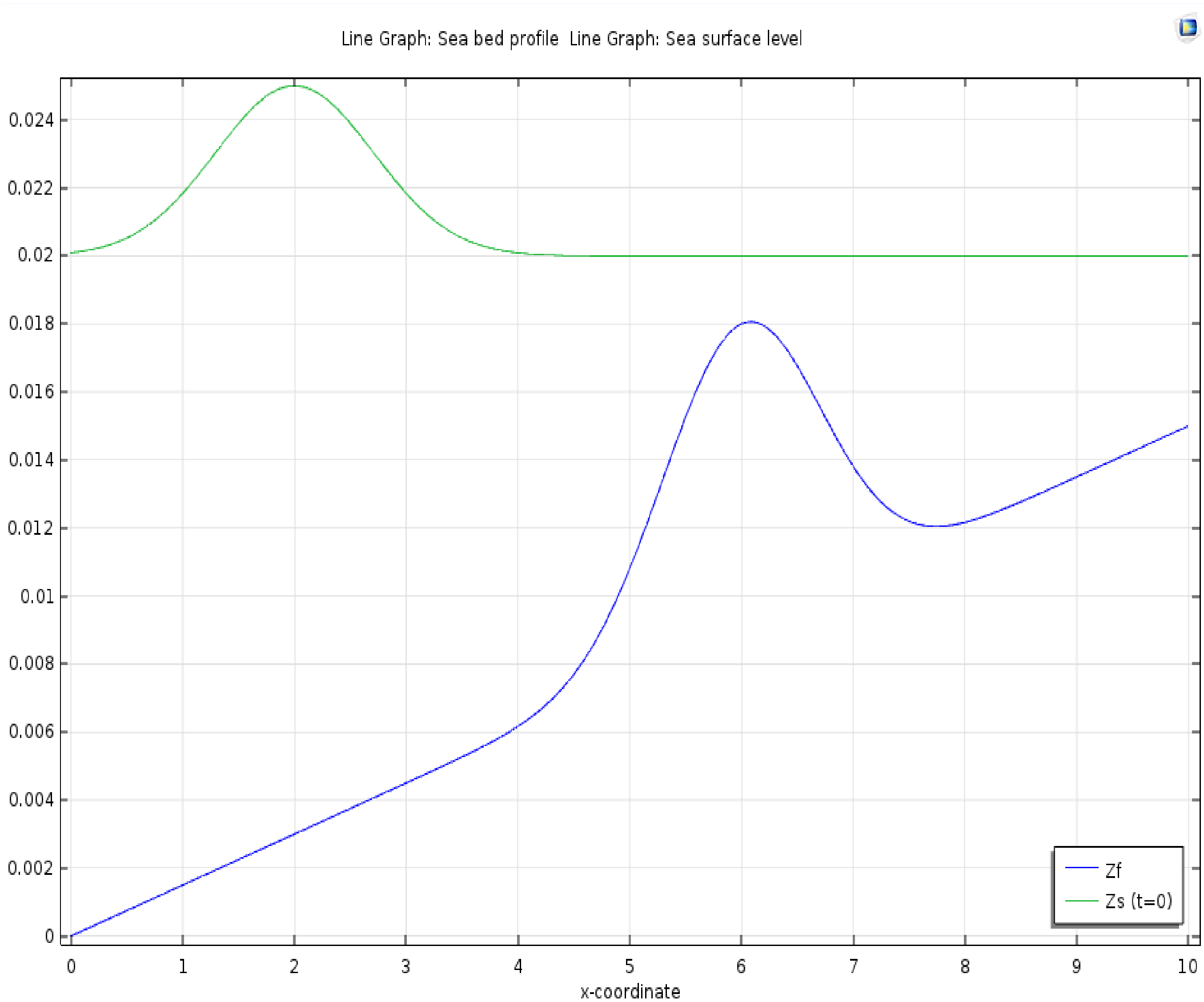
$Z_f = 0.003e^{-(x-6)^2} + 0.0015x$ and $Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$

Case 2a



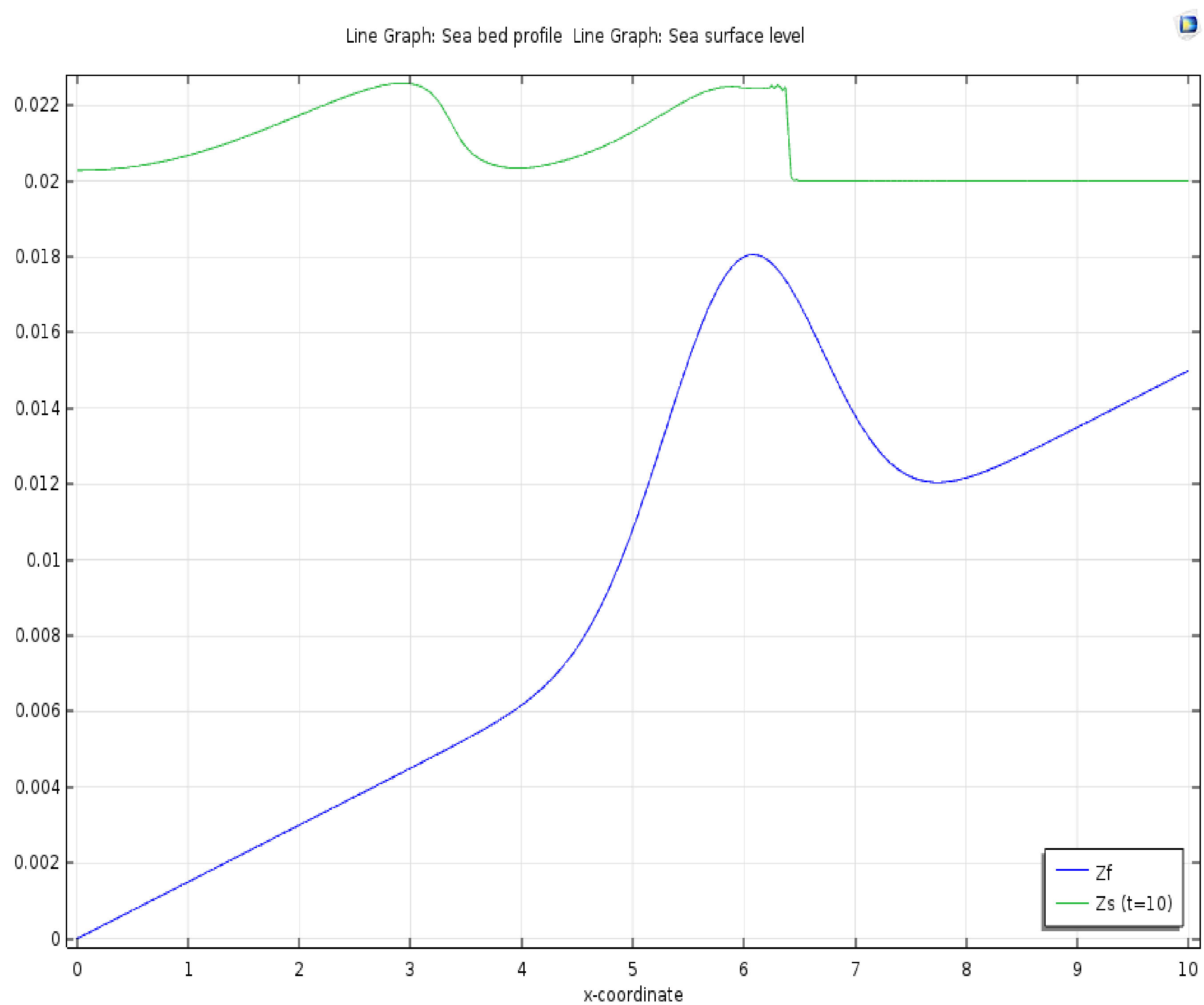
$$Z_f = 0.003e^{-(x-6)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 2b



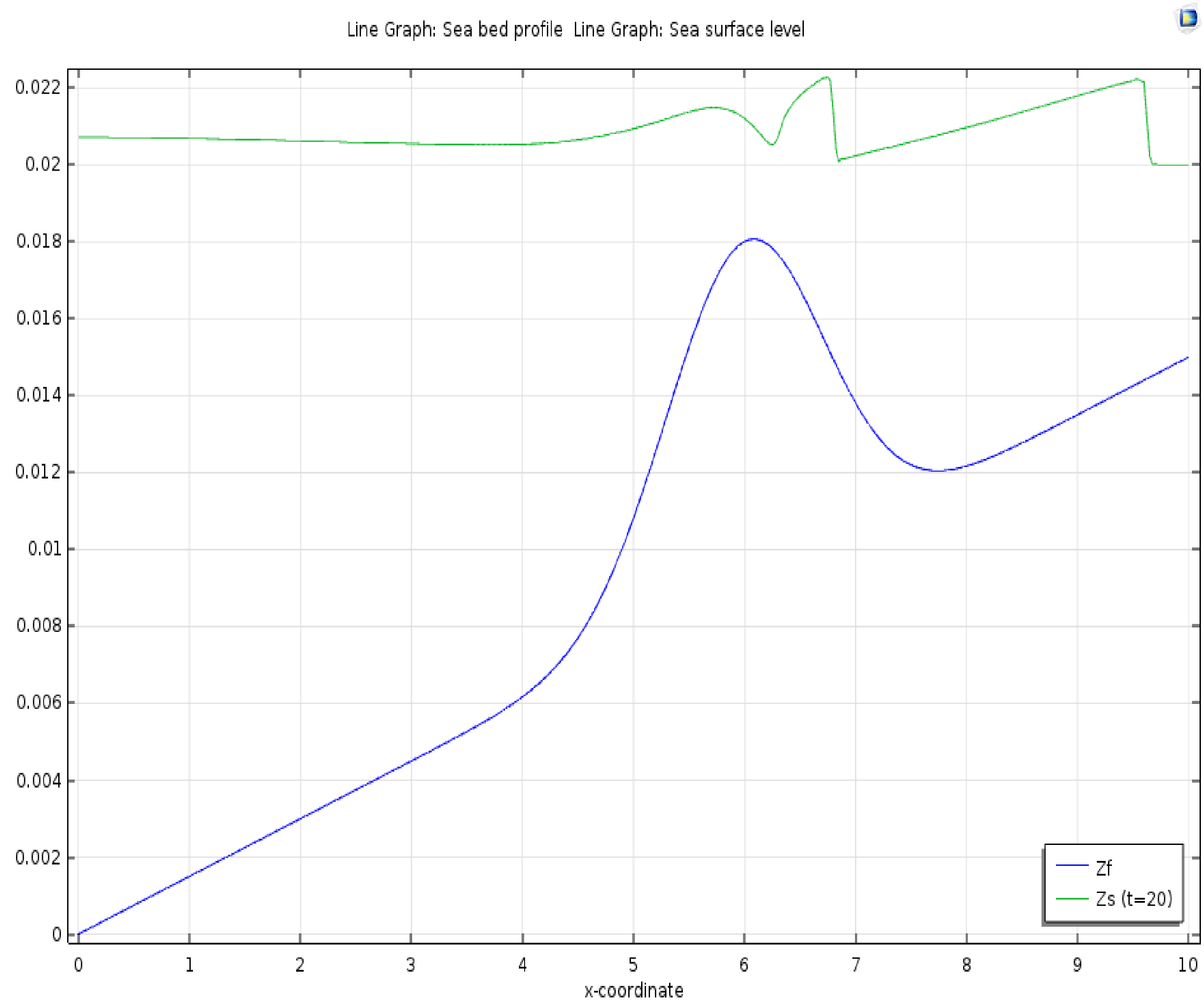
$$Z_f = 0.009e^{-(x-6)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 2b



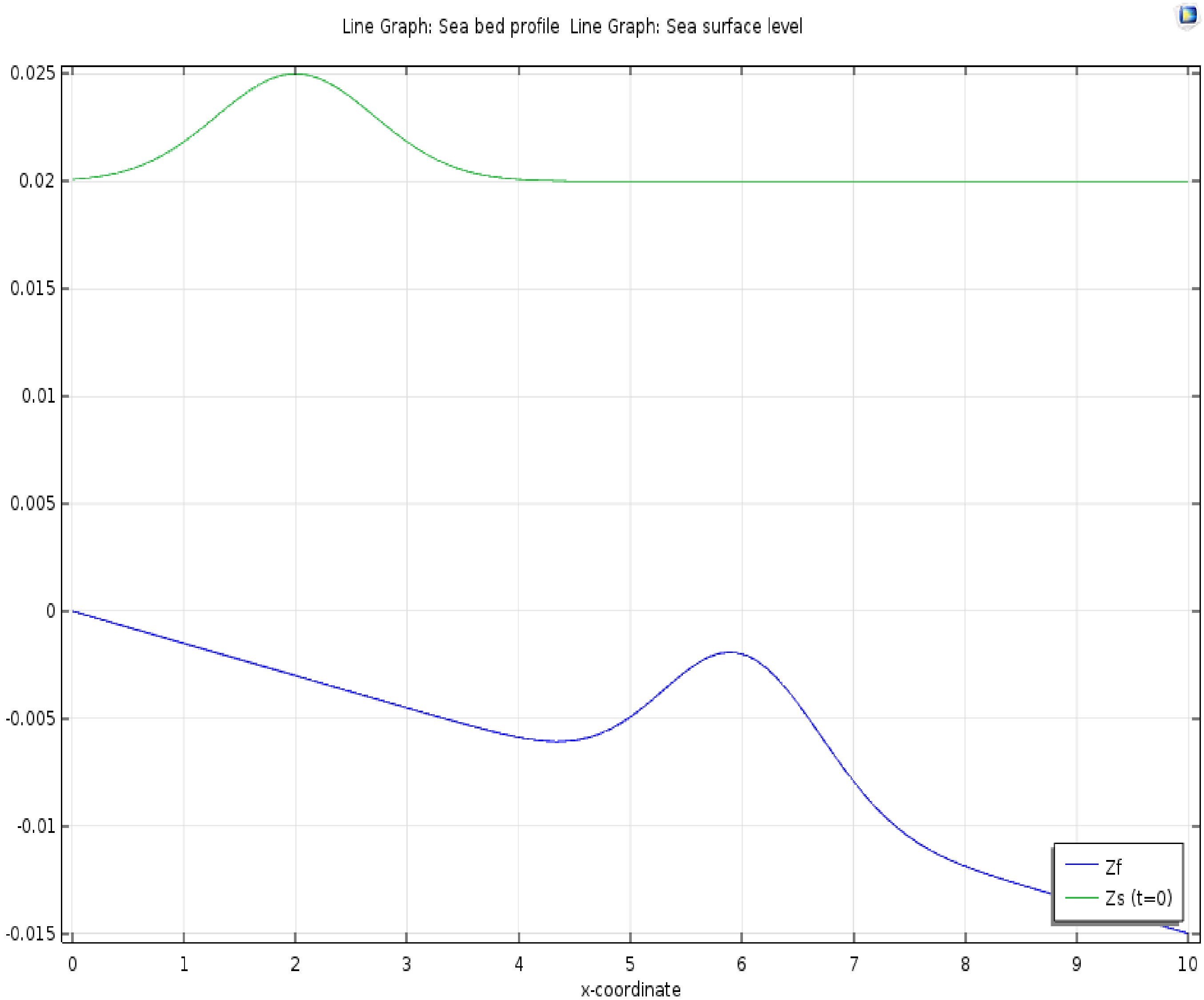
$$Z_f = 0.009e^{-(x-6)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 2b



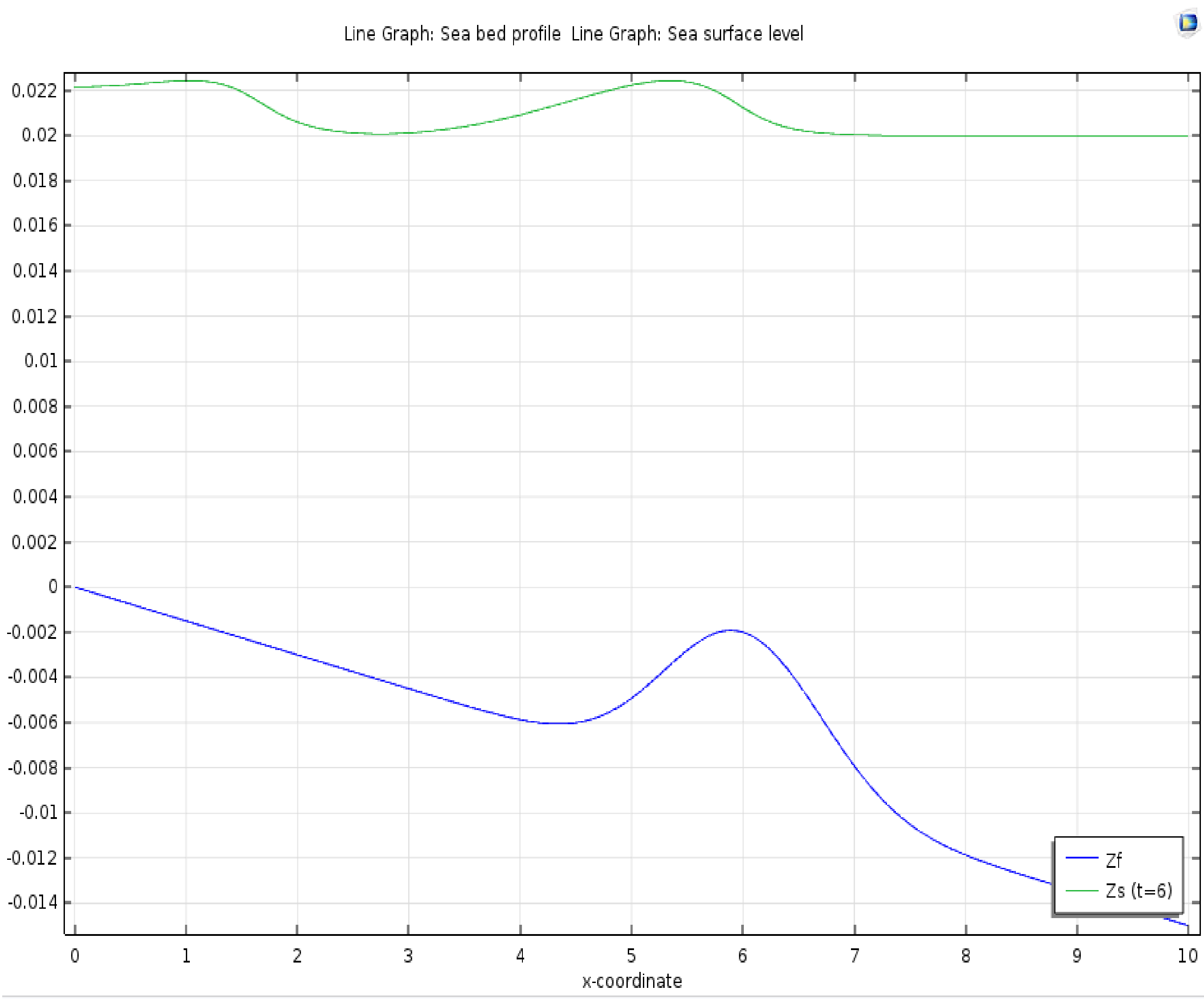
$$Z_f = 0.009e^{-(x-6)^2} + 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 3a



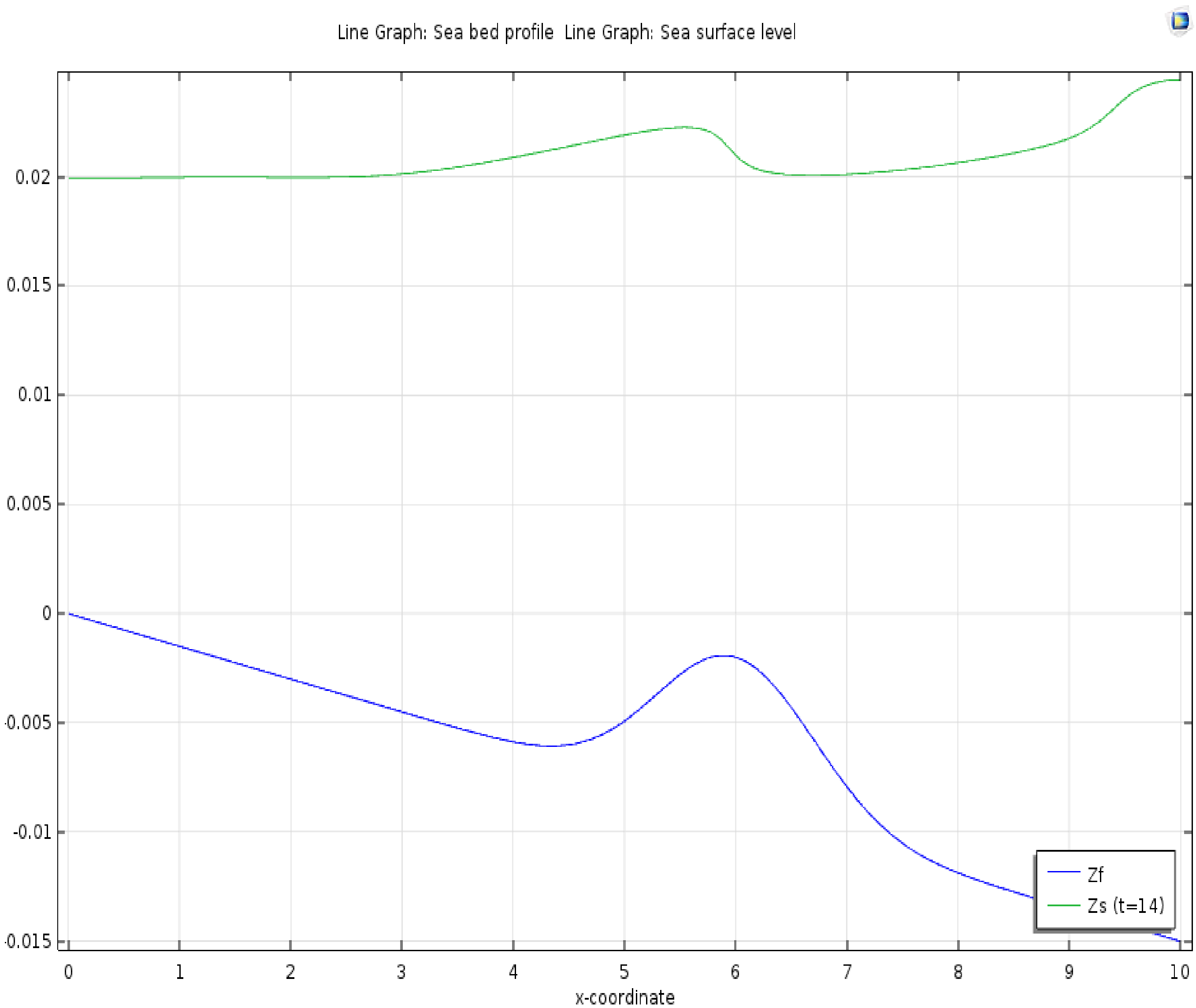
$$Z_f = 0.007e^{-(x-6)^2} - 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 3a



$$Z_f = 0.007e^{-(x-6)^2} - 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

Case 3a



$$Z_f = 0.007e^{-(x-6)^2} - 0.0015x \text{ and } Z_0 = 0.02 - Z_f + 0.005e^{(-(x-2)^2)}$$

5 Results

The results from our models depict a very calculated phenomenon involving wave behavior and spilling breakers. We refer to spilling breakers as waves that break as a result of some external influence caused by a bottom topography. [2]

Based off the models, it seems as though wave behavior and spilling breakers are minimally dependent on the height or amplitude of extrema along the sea floor when we narrow down our cased to when there is only one sea ridge or one local extrema for the bottom topography in our system. However, the wave behavior and spilling breakers are highly dependent on the location at which the wave is propagated and the location of the sea ridge or local extrema of the floor.

If we model a physical system (like we have done in our previous models), and let the x-axis be the horizontal direction or distance (in meters) with the y-axis representing vertical height (in meters), we note that if a wave is propagated at the location of the sea ridge, then when the wave breaks and spilling breakers occur, the final outcome is a fairly long wavelength versus when the wave is propagated before the location of the sea ridge. [10]

If a wave is propagated before the location of the global or local extrema of the sea floor, then the result is shorter wavelengths when spilling breakers occur. In other words, the distance between the crests of the two new waves, derived from the collapse of the original wave, are much closer than they would have been had that same wave (represented by the same arbitrary equation for any system) been produced at the same x location as the sea ridge. This event can be depicted by analyzing Case 1b at $t = 10$ seconds and Case 2b also at $t = 10$ seconds. Case 1b shows a wavelength of nearly 5 meters, while Case 2b showcases a wavelength of about 3 meters.

Next, we analyze the exemplification of how location of the sea ridge or global extrema of sea floor marginally affects the wave behavior and spilling breakers. It should be noted that we only consider a system with initially only one wave and one local (also global) extrema. Wavelength, spilling breakers, and wave behavior in this particular system are only loosely dependent on height of the sea ridge. Evidence for this phenomenon is demonstrated in Case 1a from $t = 0$ to $t = 20$ seconds compared to Case 1b from the same time interval. The amplitude of the sea ridge was increased from 0.003 meters to 0.009 meters, however the resultant waves have roughly the same behavior. Even the wavelengths at $t = 10$ seconds for the respective cases are only about 0.0025 different in length. This same circumstance occurs in Case 2a and Case 2b. In these cases both waves have roughly the same behavior over the allotted 20 second interval and wavelength differences are marginal.

6 Conclusion

Although the shallow water equations seem very theoretical and are applied under very tight restrictions involving hydrostatic balance and wavelength requirements, these hyperbolic nonlinear partial differential equations are very useful in modern meteorology and engineering as they help us project storm surge, water and wave celerity, flooding, and tsunamis. To farther analyze wave shape and propagation, we consider the original shallow water equations in one-dimension representing a special set of equations: the Saint Venant's equations. We can use packages such as COMSOL to input initial boundary conditions (water layer depth and sea floor topography) to generate physical systems to represent wave production and behavior over a designated time interval. When doing so, we obtain useful conclusions where it is discovered that wave behavior and spilling breakers are only loosely dependent on amplitude of a sea ridge in a system in which one wave is propagated and one sea ridge occurs. Also by using these equations we ascertain that wave behavior and spilling breakers are highly dependent on location of propagation. Waves produced at the location of the extrema of the sea floor produce longer wavelengths than waves produced before the location of the local extrema. Waves that are produced before the sea ridge and then travel over it over time produced choppy, rough water with very short crest-to-crest distances. We conclude that the shallow water equations are useful sets of mathematical tools to model global climatic phenomena and should not be overlooked.

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