Periodic Points and Sharkovsky's Theorem

Luke J Seaton

May 12, 2020

Project Advisor: Dr. John Doyle

Abstract

The number of periodic points of a function depends on the context. The number of complex periodic points and rational periodic points have been shown to be infinite and finite, respectively, if f is a polynomial of degree at least 2. However, the number of real periodic points can be either finite or infinite. Sharkovskys Theorem states that if p is left of q in the "Sharkovsky ordering" and the continuous function f has a point of period p, then f also has a point of period q. This statement becomes very powerful when considering a function that has points of period 3, all the way to the left side of the Sharkovsky ordering, since having a point of period 3 implies the existence of points of all periods. We explore a continuous function with points of period 3 where the function can be restricted to an interval containing points of period all other natural numbers.

1 Introduction

A dynamical system represents how the state of a system changes over time. This change can be captured as ongoing time or in intervals; that is, a dynamical system can be represented as either continuous or discrete. A continuous representation can be modeled through calculus and differential equations; however, a discrete representation involves iterative mappings where the output of the previous state is used as the input to the next iteration.

For example, given the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x + 5$ and $x_0 = 3$ is our starting point, we have $x_0 = 3, x_1 = 8, x_2 = 13, \ldots$ by using the output of one step as the input for the next, so that $f(x_n) = x_{n+1}$. When this sequence repeats itself from the beginning, we say the point x_0 is periodic. Formally, a point p is a **periodic point** of the function f with period m where m is the least number satisfying $f^m(p) = p$. The notation f^m represents m iterations of the function f [4].

In general, when this sequence eventually contains repetition, we say the point is **preperiodic** [4]. By the previous definitions, we have that all periodic points can be described as preperiodic. When $f(p) = p$ (the sequence is p, p, p, $\ldots$), we say that p is a fixed point. Fixed points are points of period 1 (let m=1).

Current work in number theory aims to answer where these periodic points occur and what those points look like. We look first at polynomials of degree ≥ 2. There are always infinitely many complex periodic points for polynomials of degree ≥ 2. [1]. By Northcott in 1949, we see that only finitely many periodic points can be rational numbers for polynomials of degree at least 2. [8]. However, the number of real periodic points can be either finite or infinite depending on the periods of orbits of f.

This paper uses iterations of a function f to describe the state of a dynamical system through the lens of a specific example. First, the iterative nature of f motivates a question about Sharkovsky's Theorem for restricted functions. Next, the iterations of the critical point are explored. Finally, a question is posed about the relationship between the iterations of f and the iterations of its critical point.

Acknowledgements

I thank Dr. John Doyle of Louisiana Tech University for his mentorship and conversation. I also thank Dr. Dorothy Dodge Robbins for comments that greatly improved the paper. The graphs were created using Desmos Graphing Calculator, and the analysis was completed using MathCAD.

2 Background

In 1975 James Yorke and Tien-Yien Li published a paper called *Period Three Implies Chaos* [3] where function f has an orbit of period three, then f also has orbits of every period n where $n \in \mathbb{N}$. This ordering that starts with three was further completed by Sharkovsky:

$$\begin{gathered}
3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \\
2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd 2 \cdot 9 \rhd \cdots \\
2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd 2^2 \cdot 9 \rhd \cdots \\
\cdots \\
\cdots \rhd 2^5 \rhd 2^4 \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.
\end{gathered}$$

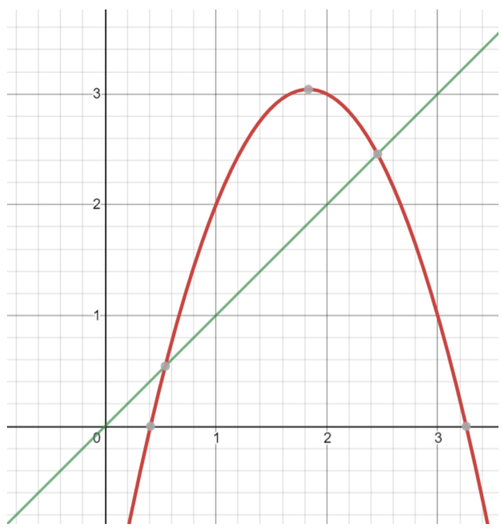
We see that a function with an orbit of period three has profound implications, and we now explore a particular function with period three.

2.1 Example Function with Period 3 Orbit

Consider the following function and its graph:

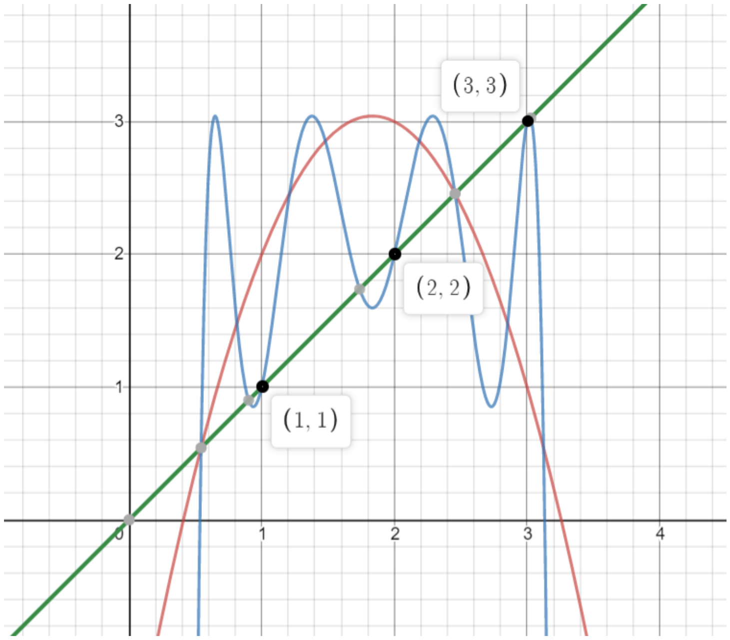
$$f(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2$$

This function contains an orbit of period three as seen by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$. This example shows that the point 1 is a point of period three.



The graph of function f is seen above in red along with the line $y = x$. This line shows fixed points of the function where the function f intercepts the line $y = x$ at exactly two points. By solving $f(x) = x$, we see the values of these points are $\frac{3}{2} + \frac{\sqrt{33}}{6}$ and $\frac{3}{2} - \frac{\sqrt{33}}{6}$.

To find the points of period three, we solve the equation $f^3(x) = x$. The solutions will intercept the line $y = x$. Fixed points will also intercept the graph as points of period three because $f(p) = p$ implies $f^3(p) = p$ for a fixed point p. The function f^3 is added in blue in the graph below.



This graphing shows the blue graph of f^3 intercepting the line $y = x$ at points 1, 2, and 3. These points are points of period 3, meaning if the function f is iterated three times starting with the points intercepting $y = x$, we can see $f^3(1) = 1$, $f^3(2) = 2$, and $f^3(3) = 3$.

While Yorke and Li's result was significant, Sharkovsky had already proven a more general result on the other side of the Iron Curtain [9].

Theorem 1. (Sharkovsky [7]) *If $f : I \to I$ is a continuous map that has a cycle of period m, then f has cycles of every period m' such that $m \triangleright m'$ (read: "m is left of m prime"), where*

$$\begin{gathered} 3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \\ 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright \cdots \\ 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright \cdots \\ \cdots \\ \cdots \triangleright 2^5 \triangleright 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1. \end{gathered}$$

This ordering of the natural numbers can describe the periodic points for every polynomial with degree ≥ 2. The odd numbers starting at 3 come first, followed by the odd numbers times two and then the odd numbers times 2^2. The pattern continues, and the ordering ends with decreasing powers of 2. This ordering includes all of the natural numbers.

Looking at our previous example through the lens of Sharkovsky, we see that our cycle of period three implies that there are cycles of period five, period seven, and so on, until we have a cycle of period one (a fixed point). Our main conjecture questions whether or not these cycles are found within nested intervals as the ordering is traversed. We restrict the function to these intervals as we traverse the ordering and use the following notation for restricted functions:

Definition 1. **([6])** *Let D, R, and S be sets with $R \subseteq D$ and let $f : D \to S$ be a function. The restriction of the function f to R, denoted $f|_R$, is defined by $f|_R(x) := f(x)$ for all $x \in R$.*

3 Main Result

The converse of Sharkovsky's theorem (also attributed to Sharkovsky) says that a function exists at every cut in this ordering. That is, given the conditions on the function, there exists a function with periodic points of period 5, but no points of period 3. Sharkovsky's theorem considers functions that map into the same interval. Shrinking that interval over iterations leads us to a question.

Motivating Question 1. *For all continuous polynomial function with degree ≤ 2, suppose function f has points of period three. Then there exists an interval I such that:*

- *$f|_I$ has period five*
- *$f|_I$ does not have period three*
- *$f(x) \in I$ for all $x \in I$.*

The final condition comes from Definition 1 and is necessary when we consider iterating the function f. The output of f must be used as the input for the next iteration; thus, the function f must map into itself.

We start with an interval I_3 that contains the orbit of period 3. From our previous example, where

$$f(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2,$$

we can choose $I_3 := [1, 3]$. We want to show that there exists a function $f|_{I_5}$ that when restricted to the interval $I_5 \subset I_3$, the interval I_5 contains points of period 5, but does not contain points of period 3.

Let $I_5 := [1.09, 2.98]$. This interval satisfies the conditions as the only points of period three are 1, 2, and 3. The numerical analysis below shows an orbit within the interval I_5.

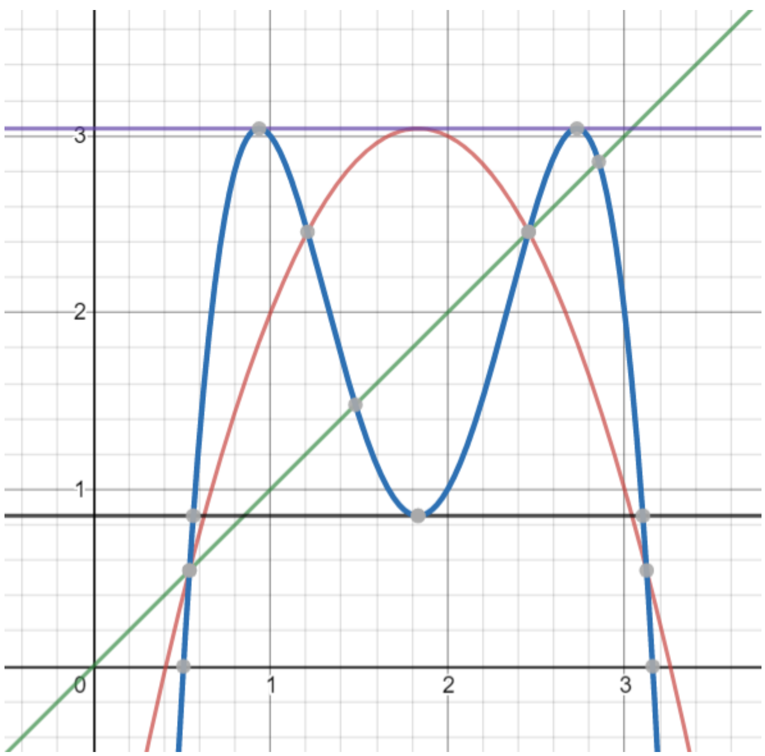
$$\begin{aligned}
f(1.61775667794253139 68\ldots) &= 2.97195672514243980 62\ldots \\
f(2.97195672514243980 62\ldots) &= 1.09697182410435567 67\ldots \\
f(1.09697182410435567 67\ldots) &= 2.22832425825570004 68\ldots \\
f(2.22832425825570004 68\ldots) &= 2.80763992051012656 02\ldots \\
f(2.80763992051012656 02\ldots) &= 1.61775667794253139 68\ldots
\end{aligned}$$

Each of the five inputs above are points of period 5 that are contained in the interval I_5. This pattern continues to follow the Sharkovsky Ordering for cycles of period seven, six, four, two, and one. However, this interval does not satisfy the final condition of our conjecture.

Counterexample: Consider the point at $x = \frac{11}{6}$. This point is the critical point of the function f, and $f(\frac{11}{6})$ describes the highest value of the curve f. However, $x = \frac{11}{6}$ exists inside of both I_3 and I_5 where $f(\frac{11}{6}) \notin I_3$ and $f(\frac{11}{6}) \notin I_5$. Therefore, any interval I must have an upper bound of at least $f(\frac{11}{6})$ to map f back into itself for the next iteration. ■

Iterations of the Critical Point

Through the example above, we note a significance of the critical point in relation to the position of the periodic points. The periodic points are bounded above by $f(\frac{11}{6})$, but the existence of a lower bound is uncertain. We see below how $f(c)$ becomes a critical point for the function f^2.



The original function f (red) and the line $y = x$ (green) are illustrated above. The second iteration of f is seen above in blue. The horizontal lines show the iterations of the critical point where $y = f(\frac{11}{6})$ (purple) and $y = f(f(\frac{11}{6}))$ (black).

The iterations of the critical point continue to bound the function in higher iterations as well. The fourth iteration of f can be seen in blue below.

4 Concluding Remarks

This work disproved a conjecture made about Sharkovsky's Theorem and restricted functions. The disproof consisted of a single counterexample that showed where the restricted function would not map back into the interval where it is defined. Our proof illustrated the importance of the critical point, and further research is needed to define the significance of the iterations of the critical point.

References

[1] I. N. Baker. Fixpoints of polynomials and rational functions. *Journal of the London Mathematical Society*, s1-39:615–622, 1964.

[2] Bau-Sen Du. An interesting application of the intermediate value theorem: A simple proof of sharkovsky's theorem and the towers of periodic points, 2015.

[3] Boris Hasselblatt Keith Burns. The sharkovsky theorem: A natural direct proof. page 15, 2008.

[4] Bjorn Poonen. The classification of rational preperiodic points of quadratic polynomials over $\mathbb{Q}$ a refined conjecture. *Mathematische Zeitschrift*, 228(1):11, 1987.

[5] Sylvie Ruette. Dense chaos for continuous interval maps. *Nonlinearity*, 18(4):16911698, May 2005.

[6] Bernd Schroder. *Mathematical Analysis: A Concise Introduction*. John Wiley and Sons, Inc., Hoboken, New Jersey, 2008.

[7] A. N. SHARKOVSKI. Coexistence of cycles of a continuous map of the line into itself. *International Journal of Bifurcation and Chaos*, 05(05):1263–1273, 1995.

[8] Joseph Silverman. Number theory and dynamical systems, 2012. MAA Invited Paper Session on the Beauty and Power of Number Theory.

[9] Veniamin L. Smirnov and Juan J. Tolosa. The sharkovsky theorem, 2017.