The max-flow min-cut property and ± 1-resistant sets

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Abstract

A subset of the unit hypercube $\{0,1\}^n$ is *cube-ideal* if its convex hull is described by hypercube and generalized set covering inequalities. In this paper, we provide a structure theorem for cube-ideal sets $S \subseteq \{0,1\}^n$ such that, for any point $x \in \{0,1\}^n$, $S - \{x\}$ and $S \cup \{x\}$ are cube-ideal. As a consequence of the structure theorem, we see that cuboids of such sets have the max-flow min-cut property.

1 Introduction

Take an integer $n \geq 1$. A *cuboid* is a family $\mathcal{C}$ of subsets of $[2n] := \{1,\ldots,2n\}$ such that

$$|C \cap \{1,2\}| = |C \cap \{3,4\}| = \cdots = |C \cap \{2n-1,2n\}| = 1 \quad \forall C \in \mathcal{C}.$$

$\mathcal{C}$ has a compact representation: There exists a unique $S \subseteq \{0,1\}^n$ such that

$$\{\chi_C : C \in \mathcal{C}\} = \{(z_1, 1-z_1, z_2, 1-z_2, \ldots, z_n, 1-z_n) : z \in S\}.$$

We will write $\mathcal{C} = \mathrm{cuboid}(S)$. Introduced in [5] and studied in [3], cuboids form an important class of clutters, to the extent that the major conjectures on clutters, namely the *Replication Conjecture* [6] and the $\tau = 2$ *Conjecture* [8] and the *f-Flowing Conjecture* [13], can be phrased equivalently in terms of cuboids. In fact, for the second and third conjectures, the equivalence goes beyond just a simple rephrasing and gets to the heart of the conjectures; we refer the interested reader to [3].

Consider the following primal-dual pair of linear programs for $w \in \mathbb{Z}^{2n}_+$:

$$(P) \quad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & x(C) \geq 1 \quad C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : i \in C \in \mathcal{C}) \leq w_i \quad i \in [2n] \\ & y \geq \mathbf{0}. \end{array}$$

$\mathcal{C}$ is *ideal* if (P) has an integral optimal solution for all $w \in \mathbb{Z}^{2n}_+$ [9], while $\mathcal{C}$ has the *max-flow min-cut property* if (D) has an integral optimal solution for all $w \in \mathbb{Z}^{2n}_+$ [7]. A classic result of Edmonds and Giles [10] and Hoffman [11] tells us that the max-flow min-cut property implies idealness. The converse however does not hold. In fact, understanding when the converse *does* hold is what the Replication and $\tau = 2$ Conjectures address. This is also what we address.

Looking at the known examples of cuboids that are ideal and do not have the max-flow min-cut property, we have noticed something curious:

Conjecture 1.1. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ such that* $\operatorname{cuboid}(S)$ *is ideal and does not have the max-flow min-cut property. Then there exists a point $x \in S$ such that* $\operatorname{cuboid}(S - \{x\})$ *is nonideal.*

For instance, consider the two sets

$$A := \{000, 101, 110, 011\} \subseteq \{0,1\}^3$$
$$B := \{1010, 0110, 0001, 1101, 0011, 1011, 0111, 1111\} \subseteq \{0,1\}^4.$$

These two examples are taken from a library of 745 *strictly non-polar* sets provided in [3]; they are sets number 1 and 9, respectively. Both of these sets have an ideal cuboid that does not have the max-flow min-cut property. In the first example, $\operatorname{cuboid}(A - \{x\})$ is nonideal for any point $x \in A$. In the second example, $\operatorname{cuboid}(B - \{x\})$ is nonideal for any point $x \in \{1010, 0110, 0001, 1101\}$, while $\operatorname{cuboid}(B - \{x\})$ is ideal for any point $x \in \{0011, 1011, 0111, 1111\}$.

In this paper, we prove the following weakening of Conjecture 1.1:

Theorem 1.2. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ such that* $\operatorname{cuboid}(S)$ *is ideal and does not have the max-flow min-cut property. Then there exists a point $x \in S$ such that* $\operatorname{cuboid}(S - \{x\})$ *is nonideal, or there exists a point $x \in \{0,1\}^n - S$ such that* $\operatorname{cuboid}(S \cup \{x\})$ *is nonideal.*

Our proof relies on a structure theorem, an intriguing fact in its own right. We need to set some notation and make some definitions first.

Take an integer $n \geq 1$. A *sub-hypercube of* $\{0,1\}^n$ is a subset of the form

$$\left\{x \in \{0,1\}^n : x_i = 0 \;\; i \in I, x_j = 1 \;\; j \in J\right\} \quad I, J \subseteq [n], I \cap J = \emptyset;$$

its *rank* is $n - |I| - |J|$. Given $a, b \in \{0,1\}^n$, the *distance* between a and b, denoted $\operatorname{dist}(a,b)$, is the number of coordinates a and b differ on. Denote by G_n the *skeleton graph* of $[0,1]^n$, whose vertices are the points in $\{0,1\}^n$, where $a, b \in \{0,1\}^n$ are adjacent if $\operatorname{dist}(a,b) = 1$. For a subset $X \subseteq \{0,1\}^n$, denote by $G_n[X]$ the subgraph of G_n induced on vertices X, and we say that X is *connected* if $G_n[X]$ is connected.

Take a set $S \subseteq \{0,1\}^n$. We refer to n as the *dimension of* S, to the points in S as *feasible*, and to the points in $\overline{S} := \{0,1\}^n - S$ as *infeasible*. The connected components of $G_n[S]$ are *feasible components*, while the components of $G_n[\overline{S}]$ are *infeasible components*. Given $x, y \in \{0,1\}^n$, denote by $x \triangle y$ the coordinate-wise sum of x, y modulo 2, and define $S \triangle y := \{x \triangle y : x \in S\}$. Take $i \in [n]$. Denote by e_i the i^{th} unit vector. To *twist coordinate* i is to replace S by $S \triangle e_i$. A set $S' \subseteq \{0,1\}^n$ is *isomorphic* to S, displayed as $S' \cong S$, if S' is obtained from S after relabeling and twisting some coordinates.

Given integers $n_1, n_2 \geq 0$ and $S_1 \subseteq \{0,1\}^{n_1}, S_2 \subseteq \{0,1\}^{n_2}$, the *product of* S_1 *and* S_2 is

$$S_1 \times S_2 := \{(x,y) : x \in S_1, y \in S_2\} \subseteq \{0,1\}^{n_1+n_2}.$$

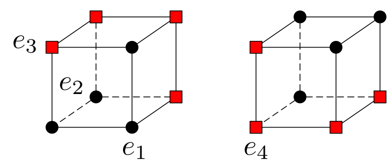


Figure 1: An illustration of C_8. Round points are feasible while square points are infeasible.

Denote by $\mathbf{0}, \mathbf{1}$ the all-zeros and all-ones vectors of appropriate dimensions, respectively; the dimension of the vectors will be clear from the context.

We are now ready to state our structure theorem:

Theorem 1.3. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ such that* $\mathrm{cuboid}(S)$ *is ideal,* $\mathrm{cuboid}(S - \{x\})$ *is ideal for each $x \in S$, and* $\mathrm{cuboid}(S \cup \{x\})$ *is ideal for each $x \in \overline{S}$. Then one of the following statements holds:*

(i) $S \cong A_k \times \{0,1\}^{n-k}$ *for some $k \in \{2, \dots, n\}$, where $A_k = \{\mathbf{0}, \mathbf{1}\} \subseteq \{0,1\}^k$,*

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ *for some $k \in \{3, \dots, n\}$, where $B_k = \{\mathbf{0}, e_1, \mathbf{1}\} \subseteq \{0,1\}^k$,*

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$*, where $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\} \subseteq \{0,1\}^4$,*

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ *for some $k \in \{3, \dots, n\}$, where $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$,*

(v) S is a sub-hypercube, or

(vi) every infeasible component of S is a sub-hypercube, and every feasible point has at most two infeasible neighbors.

Next we explain the main idea behind the proofs of our two theorems.

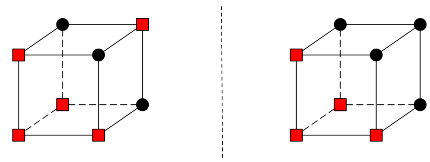
1.1 The notion of ± 1-resistance

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. We say that S is *cube-ideal* if $\mathrm{cuboid}(S)$ is ideal. We have the following nice characterization of cube-ideal sets:[1]

Theorem 1.4 ([3]). *Take an integer $n \geq 1$. Then a subset of $\{0,1\}^n$ is cube-ideal if, and only if, its convex hull is described by inequalities of the form*

$$x_i \geq 0 \text{ and } x_i \leq 1 \quad i \in [n] := \{1, \dots, n\}$$
$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [n], I \cap J = \emptyset.$$

[1]Just like idealness, it would be very useful to reformulate in terms of S what it means for $\mathrm{cuboid}(S)$ to have the max-flow min-cut property. Unfortunately, we are not yet aware of such a characterization.



P_3 S_3

The second type of constraints above are called *generalized set covering inequalities* [7]. Jon Lee refers to generalized set covering inequalities as *cropping inequalities*, and he has shown that if every infeasible component is a sub-hypercube, then S is cube-ideal [12].[2] Notice that generalized set covering inequalities are precisely the inequalities that cut off sub-hypercubes of $\{0,1\}^n$, providing yet another reason why cube-idealness is such a natural geometric concept. To prove Theorem 1.2 and Theorem 1.3, however, we need to replace cube-idealness by a weaker yet more tangible property. Let us elaborate.

Take $i \in [n]$. The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-*restriction of S over coordinate i*, and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-*restriction of S over coordinate i*. A *restriction* of S is a set obtained after a series of 0- and 1-restrictions. The *projection of S over coordinate i* is the set obtained from S after dropping coordinate i. A *minor of S* is what is obtained after a series of restrictions and projections. A minor is *proper* if at least one operation is applied.

Remark 1.5 ([3])**.** *If a set is cube-ideal, then so is every isomorphic minor of it.*

Hereinafter, the prefix "isomorphic" will be omitted from "isomorphic restriction" and "isomorphic minor".

Let $P_3 := \{110, 101, 011\} \subseteq \{0,1\}^3$ and $S_3 := \{110, 101, 011, 111\} \subseteq \{0,1\}^3$. These two sets are not cube-ideal because

$$\begin{aligned}\mathrm{conv}(P_3) &= \left\{x \in [0,1]^3 : (1-x_1) + (1-x_2) + (1-x_3) = 1\right\}\\ \mathrm{conv}(S_3) &= \left\{x \in [0,1]^3 : (1-x_1) + (1-x_2) + (1-x_3) \le 1\right\}.\end{aligned}$$

In fact, up to isomorphism, P_3 and S_3 are the only non-cube-ideal sets of dimension at most 3. As an immediate consequence of Remark 1.5, a cube-ideal set has none of P_3, S_3 as a minor. We will replace cube-idealness by the weaker yet more tangible property of having no P_3, S_3 minor. (This idea only works for tackling Theorem 1.2 and Theorem 1.3, and will not suffice for tackling Conjecture 1.1, because of B.)

We say that S is ± 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S - X$ and $S \cup X$ have no P_3, S_3 minor. Notice that the set in Theorem 1.3 is ± 1-resistant. This observation is key as we will prove stronger analogues of Theorem 1.2 and Theorem 1.3 for ± 1-resistant sets. In §2 we prove that ± 1-resistance is a minor-closed property and provide an excluded minor characterization for it. In §3 we state and outline the proof of an exact structure theorem for ± 1-resistance, which will imply Theorem 1.3. The proof of the structure theorem spans §4-§7. By using the structure theorem, we will prove in §8 that the cuboid of a ± 1-resistant set has the max-flow min-cut property, thereby implying Theorem 1.2.

[2] In fact, he proves that if every infeasible component is a sub-hypercube, then the canonical convex hull description of S is totally dual integral [12]. This does not mean that $\mathrm{cuboid}(S)$ has the max-flow min-cut property, but that the blocker does.

Our proofs heavily rely on another notion. S is 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor. Notice that a ± 1-resistant set is 1-resistant. The notion of 1-resistance was studied in [4] by us together with another author, though the prefix 1- was omitted there. It is worth pointing out that 1-resistance was come across much later than and as a result of ± 1-resistance, for the latter is a more natural notion to define; we only arrived at it after studying ± 1-resistance. Nevertheless, the following lemma for 1-resistant sets will be used frequently throughout the paper:

Lemma 1.6 ([4]). *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then S is a sub-hypercube.*

The paper is notationally heavy. To help the reader we have summarized what the symbols refer to in the last two pages of the manuscript.

1.2 Related notions and work

In [4] we showed that 1-resistance is a rich and multifaceted notion. We proved the Replication Conjecture and the $\tau = 2$ Conjecture for 1-resistant sets, but our attempts fell short of providing a structure theorem for these sets. We showed that there are infinitely many 1-resistant sets with an *ideal minimally non-packing* cuboid, and argued that in order to fully characterize the ideal minimally non-packing cuboids this class leads to, a structure theorem is likely needed. This paper achieves just that for the subclass of ± 1-resistant sets. There is another subclass for which this is achieved. Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. For an integer $k \geq 2$, S is k-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most k, $S \cup X$ has no P_3, S_3 minor. In [2] we described the structure of k-resistant sets for $k \geq 2$, and showed as a result that there are exactly three 2-resistant sets with an ideal minimally non-packing cuboid.

For an integer $k \geq 2$, we may say that S is $\pm k$-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most k, the symmetric difference $S \triangle X$ has no P_3, S_3 minor. Our structure theorem for ± 1-resistant sets, or the one for 2-resistant sets, can be used to recover the structure of $\pm k$-resistant sets; we leave that to the interested reader.

The notion of resistance came into being as we were looking for a counterexample among cuboids to the $\tau = 2$ Conjecture. We wrote a computer program to do this work for us, but to no avail. The theory of resistance explains why our attempts failed; see [1], Chapter 6 for more detail.

2 An excluded minor characterization for ± 1-resistant sets

Let us start with the following easy remark:

Remark 2.1 ([4]). *If a set is 1-resistant, then so is every minor of it.*

Take a set $F \subseteq \{0,1\}^3$ such that $\{101, 011\} \subseteq F \subseteq \{101, 011, 110, 111\}$. We refer to F, and any set isomorphic to it, as *fragile*. Notice that $F \cup \{110\}$ is either P_3 or S_3, so F is not 1-resistant. We have the

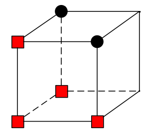
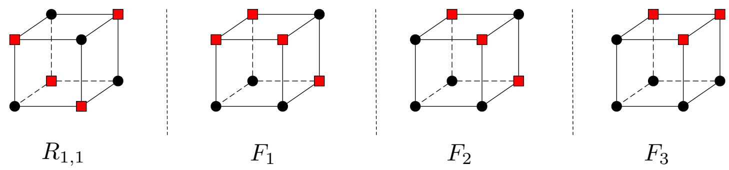


Figure 2: An illustration of a fragile set.



following excluded minor characterization of 1-resistance:

Theorem 2.2 ([4]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 1-resistant,

(ii) S has no fragile restriction and no $\left\{\mathbf{0}, \mathbf{1} - e_1\right\} \subseteq \{0,1\}^k, k \geq 4$ restriction,

(iii) S has no fragile minor.

Let us now turn to ± 1-resistant sets. We have the following easy remark:

Remark 2.3. *If a set is ± 1-resistant, then so is every restriction of it.*

The class of ± 1-resistant sets turns out to be closed under projections as well, but the reason is not as straightforward. Let

$$\begin{aligned} R_{1,1} &:= \{000, 110, 101, 011\} \subseteq \{0,1\}^3 \\ F_1 &:= \{000, 100, 010, 111\} \subseteq \{0,1\}^3 \\ F_2 &:= \{000, 100, 010, 001, 111\} \subseteq \{0,1\}^3 \\ F_k &:= \{\mathbf{0}, e_1, e_2, e_1 + e_2, \mathbf{1} - e_1 - e_2\} \subseteq \{0,1\}^k \quad k \geq 3 \end{aligned}$$

Notice that for each $k \geq 4$, F_k has an F_3 projection obtained after projecting away coordinates $4, \ldots, k$.

Remark 2.4. *The sets $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ± 1-resistant.*

Proof. Notice that $R_{1,1} - \{000\} = P_3$, $F_1 - \{000\} \cong P_3$, $F_2 - \{111\} \cong S_3$, and that for each $k \geq 3$, $F_k - \{e_1 + e_2\}$ has an S_3 projection obtained after projecting away coordinates $4, \ldots, k$. As a result, $\{R_{1,1}\} \cup \{F_k : k \geq 1\}$ are not ± 1-resistant. $\square$

We are now ready to prove the following:

Theorem 2.5. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S *is* ± 1*-resistant,*

(ii) S *has none of* $\{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}$ *as a restriction,*

(iii) S *has none of* $\{R_{1,1}, F_1, F_2, F_3\}$ *as a minor.*

Proof. By Remark 2.1, every minor of S is 1-resistant. We will use this throughout the proof.

(i) $\Rightarrow$ **(ii)** follows from Remark 2.3 and Remark 2.4.

(ii) $\Rightarrow$ **(iii)**: We will need the following three claims:

Claim 1. *Let* $R \subseteq \{0,1\}^4$ *be* 1*-resistant. Let* $N \subseteq \{0,1\}^3$ *be the projection of* R *over coordinate* 4*. Then the following statements hold:*

(1) if $N = R_{1,1}$*, then* R *has an* $R_{1,1}$ *restriction,*

(2) if $N = F_1$*, then* R *has an* F_1 *restriction,*

(3) if $N = F_2$*, then* R *has one of* F_1, F_2 *as a restriction.*

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the i-restriction of R over coordinate 4. Notice that $R_0 \cup R_1 = N$. **(1)** In this case, $R_0, R_1 \subseteq R_{1,1} = \{000, 110, 101, 011\}$. As $R_0, R_1 \ncong P_3$, it follows that $|R_0|, |R_1| \neq 3$. In fact, since R_0, R_1 are 1-resistant, we must have that $|R_0| \in \{0,1,4\}$ and $|R_1| \in \{0,1,4\}$. Since $|R_0| + |R_1| \geq 4$, it follows that one of R_0, R_1 is $R_{1,1}$, so R has an $R_{1,1}$ restriction. **(2)** We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since the 0-restriction of R over coordinate 1 is 1-resistant, it follows that $010 \in R_0$. Similarly, as the 0-restriction of R over coordinate 2 is 1-resistant, $100 \in R_0$. Because R_0 is 1-resistant, we have that $R_0 = F_1$, so R has an F_1 restriction. **(3)** We may assume, after possibly twisting coordinate 4, that $000 \in R_0$. Since R has no P_3, S_3 restriction, at least two of $100, 010, 001$ must belong to R_0. Without loss of generality, $100, 010 \in R_0$. As R_0 is 1-resistant, $111 \in R_0$, so R_0 is either F_1 or F_2, implying in turn that R has one of F_1, F_2 as a restriction. ◊

Claim 2. *Let* $R \subseteq \{0,1\}^4$ *be* 1*-resistant and have no* F_1, F_3 *restriction. If the projection of* R *over coordinate* 4 *is* F_3*, then* $R \cong F_4$*.*

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^3$ be the i-restriction of R over coordinate 4. Notice that $R_0 \cup R_1 = F_3$. Assume in the first case that $110 \in R_0 \cap R_1$. Since $100 \in R_0 \cup R_1$ and the 1-restriction of R over coordinate 1 is 1-resistant, it follows that $100 \in R_0 \cap R_1$. Similarly, $010 \in R_0 \cap R_1$. After possibly twisting coordinate 4 of R, we may assume that $001 \in R_0$. This implies that R_0 is isomorphic to either F_1 or F_3, which is not the case as R has no F_1, F_3 restriction. Assume in the remaining case that $110 \notin R_0 \cap R_1$. After possibly twisting coordinate 4 of R, we may assume that $110 \in R_0$ and $110 \notin R_1$. As $100 \in R_0 \cup R_1$ and the 1-restriction of R over coordinate 1 is 1-resistant, it follows that $100 \in R_0$ and $100 \notin R_1$. Similarly, $010 \in R_0$ and $010 \notin R_1$. Since $R_0 \ncong F_1, F_3$, it follows that $001 \notin R_0$ and so $001 \in R_1$. As R_0 is 1-resistant, $000 \in R_0$. Since R has no F_3 restriction, it follows that $000 \notin R_1$, implying in turn that $R \cong F_4$, as required. ◊

Claim 3. *Take an integer $k \geq 4$ and a 1-resistant set $R \subseteq \{0,1\}^{k+1}$ that has no F_3, F_k restriction. If the projection of R over coordinate $k+1$ is F_k, then $R \cong F_{k+1}$.*

Proof of Claim. For $i \in \{0,1\}$, let $R_i \subseteq \{0,1\}^k$ be the i-restriction of R over coordinate $k+1$. Then $R_0 \cup R_1 = F_k$. For $i \in \{0,1\}$, since R_i is 1-resistant, it follows that $|R_i \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| \neq 3$, and if $|R_i \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 2$ then the two points in $R_i \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}$ are adjacent. Since the restriction of R obtained after 0-restricting coordinates $3, \ldots, k$ is not isomorphic to F_3, one of the following holds:

- $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 2$: in this case, the 0-restriction of R over coordinates $[k+1] - \{1,2,3,k+1\}$ is not 1-resistant,
- $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}| = 4$: in this case, one of R_0, R_1 is F_k,
- one of $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|, |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|$ is 2 and the other one is 4: in this case, the 0-restriction of R over coordinates $[k+1] - \{1,2,3,k+1\}$ is not 1-resistant,
- one of $|R_0 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|, |R_1 \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\}|$ is 0 and the other one is 4.

Thus, the last case is the only possibility. In this case, since R has no F_k restriction, it follows that $R \cong F_{k+1}$, as required. $\diamondsuit$

Assume that S has an $N \in \{R_{1,1}, F_1, F_2, F_3\}$ minor, obtained after applying ℓ single projections and $n-3-\ell$ single restrictions, for some $\ell \in \{0, \ldots, n-3\}$. We need to show that S has one of $R_{1,1}, \{F_k : 1 \leq k \leq n\}$ as a restriction. A repeated application of Claim 1 implies that if $N \in \{R_{1,1}, F_1, F_2\}$, then S has one of $\{R_{1,1}, F_1, F_2\}$ as a restriction. We may therefore assume that $N = F_3$, and that S has no $\{R_{1,1}, F_1, F_2\}$ restriction. If $\ell = 0$, then S has an F_3 restriction, so we are done. We may therefore assume that $\ell \geq 1$ and S has no F_3 restriction. If $\ell = 1$, then by Claim 2, S has an F_4 restriction and we are done. We may therefore assume that $\ell \geq 2$ and S has no F_3, F_4 restriction. By repeatedly applying Claim 3, we see that S has one of $F_5, \ldots, F_n$ as a restriction, as required.

(iii) $\Rightarrow$ **(i)**: Assume that S is not ± 1-resistant. Since S is 1-resistant, there exists an $x \in S$ such that $S - \{x\}$ has an N minor for some $N \in \{P_3, S_3\}$. Thus, for some point $y \in \{0,1\}^3$, S has an $N \cup \{y\}$ minor. Since S is 1-resistant, $N \cup \{y\}$ is 1-resistant, so $N \cup \{y\}$ must be isomorphic to one of $R_{1,1}, F_1, F_2, F_3$. Thus, S has one of $\{R_{1,1}, F_1, F_2, F_3\}$ as a minor. $\square$

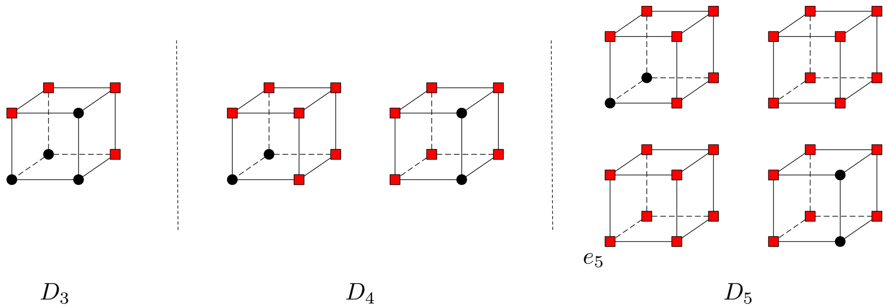
As a consequence,

Corollary 2.6. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is ± 1-resistant,

(ii) S has none of the following sets as a restriction:

$$\{F : F \text{ is fragile}\} \cup \left\{\left\{\mathbf{0}, \mathbf{1} - e_1\right\} \subseteq \{0,1\}^k : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\},$$



(iii) S has none of the following sets as a minor: $\{F : F \text{ is fragile}\} \cup \{R_{1,1}, F_1, F_2, F_3\}$.

In particular, ± 1-resistance is a minor-closed property.

Proof. This is an immediate consequence of Theorem 2.2 and Theorem 2.5. □

3 A structure theorem for ± 1-resistant sets

In this section, we state and outline the proof of an exact structure theorem for ± 1-resistant sets. We will need the following three ingredients:

Theorem 3.1. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected, then either*

- *$S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$,*
- *$S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, or*
- *S has a D_3 minor.*

Theorem 3.2. *Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S has a D_3 minor, then either*

- *$S \cong C_8 \times \{0,1\}^{n-4}$, or*
- *$S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$.*

Theorem 3.3. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is connected and has no D_3 minor, then either*

- *S is a sub-hypercube, or*
- *every infeasible component of S is a sub-hypercube, and every feasible point has at most two infeasible neighbors.*

Assuming the correctness of these three results, let us state and prove an exact structure theorem for ± 1-resistant sets:

Theorem 3.4. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is ± 1-resistant if, and only if, one of the following statements holds:*

(i) $S \cong A_k \times \{0,1\}^{n-k}$ *for some* $k \in \{2,\dots,n\}$*, where* $A_k = \{\mathbf{0},\mathbf{1}\} \subseteq \{0,1\}^k$*,*

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3,\dots,n\}$*, where* $B_k = \{\mathbf{0},e_1,\mathbf{1}\} \subseteq \{0,1\}^k$*,*

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$*, where* $C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\}$*,*

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3,\dots,n\}$*, where* $D_k = \{\mathbf{0}, e_2, \mathbf{1}-e_2, \mathbf{1}-e_2-e_3\} \subseteq \{0,1\}^k$*,*

(v) *S is a sub-hypercube, or*

(vi) *every infeasible component of S is a sub-hypercube, and every feasible point has at most two infeasible neighbors.*

Proof of Theorem 3.4, assuming Theorem 3.1, Theorem 3.2 and Theorem 3.3. $(\Rightarrow)$ Clearly, S is 1-resistant, so by Theorem 2.5 (iii), S has no $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected and has no D_3 minor, then (i) or (ii) holds by Theorem 3.1. If S has a D_3 minor, then (iii) or (iv) holds by Theorem 3.2. Otherwise, S is connected and has no D_3 minor, so (v) or (vi) holds by Theorem 3.3, as required. $(\Leftarrow)$ We will need the following claim:

Claim. *If S is ± 1-resistant, then so is $S \times \{0,1\}$.*

Proof of Claim. By Corollary 2.6 (ii), the excluded restrictions defining ± 1-resistance are

$$\{F : F \text{ is fragile}\} \cup \left\{\left\{\mathbf{0}, \mathbf{1}-e_1\right\} \subseteq \{0,1\}^k : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.$$

In particular, every excluded restriction of ± 1-resistance is not isomorphic to $F \times \{0,1\}$ for any set F. This proves the claim. $\Diamond$

It can be readily checked that the sets $\{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\}$ and C_8 are ± 1-resistant. Thus, after repeatedly applying the claim above, we see that the four classes (i)-(iv) are ± 1-resistant. It can also be readily checked that (v) gives a ± 1-resistant class. It remains to show that the restriction-closed class (vi) gives is ± 1-resistant. To this end, pick a set S satisfying (vi). Suppose for a contradiction that S is not ± 1-resistant. By Corollary 2.6 (ii), S has one of the following restrictions:

$$\{F : F \text{ is fragile}\} \cup \left\{\left\{\mathbf{0}, \mathbf{1}-e_1\right\} \subseteq \{0,1\}^k : k \geq 4\right\} \cup \{R_{1,1}\} \cup \{F_k : 1 \leq k \leq n\}.$$

Out of these sets, $R_{1,1}$ is the only set whose infeasible components are sub-hypercubes. Thus S must have an $R_{1,1}$ restriction. However, $R_{1,1}$ has a feasible point with three infeasible neighbors, implying in turn that S has a feasible point with three infeasible neighbors, a contradiction. $\square$

To complete the proof of Theorem 3.4, it remains to prove Theorem 3.1, Theorem 3.2 and Theorem 3.3; they are proved in §5, §6 and §7.1, respectively. Notice that Theorem 1.3 is an immediate consequence of Theorem 3.4.

4 Bridges

In this section we provide an ingredient needed for the proof of Theorem 3.1.

Take an integer $n \geq 2$. For a point $x \in \{0,1\}^n$ and distinct coordinates $i, j \in [n]$ such that $x_i = x_j = 0$, we refer to $\{x, x+e_i, x+e_j, x+e_i+e_j\}$ as a *square* that *initiates from* x and is *active in directions* e_i, e_j. Two squares are *parallel* if they are active in the same pair of directions. Two parallel squares are *neighbors* if the points they initiate from are neighbors.

Take a set $S \subseteq \{0,1\}^n$. A *bridge* is a square that contains feasible points from different feasible components. Notice that a bridge contains exactly two feasible points, which are non-adjacent and belong to different feasible components. In this section, we will prove the following statement:

> *Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.*

We will need three lemmas to prove this statement.

Lemma 4.1. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$, where direction e_n is not active in any bridge. If S' is obtained from S after projecting away coordinate n, then the feasible components of S project onto different feasible components of S'.*

Proof. For a point $x \in \{0,1\}^n$, denote by $x' \in \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. To prove the lemma, we may assume that S is not connected. It suffices to show that if K is a feasible component of S and $x \in S - K$, then $\operatorname{dist}(x', y') \geq 2$ for all $y \in K$. Well, since x does not belong to the component K, $\operatorname{dist}(x, y) \geq 2$ for all $y \in K$, implying in turn that

$$\operatorname{dist}(x', y') \geq \operatorname{dist}(x, y) - 1 \geq 1 \quad \forall\, y \in K.$$

In particular, $x' \notin \{y' : y \in K\}$. Suppose for a contradiction that $\operatorname{dist}(x', y') = 1$ for some $y \in K$. As the inequalities above are held at equality, there must be a coordinate $i \in [n-1]$ such that $y = x \triangle e_i \triangle e_n$. But then $\{x, x\triangle e_i, x\triangle e_n, x\triangle e_i\triangle e_n\}$ would be a bridge that is active in direction e_n, contrary to our assumption. Hence, $\operatorname{dist}(x', y') \geq 2$ for all $y \in K$, as required. □

Lemma 4.2. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2$ restriction. Take a point $x \in \{0,1\}^n$ and distinct coordinates $i, j, k \in [n]$. Then the following statements hold:*

(i) If $x\triangle e_i, x\triangle e_j, x\triangle e_k \in \overline{S}$, then $\left|\{x\triangle e_i\triangle e_j, x\triangle e_j\triangle e_k, x\triangle e_k\triangle e_i\} \cap S\right| \leq 1$.

(ii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\{x\triangle e_i\triangle e_k, x\triangle e_j\triangle e_k\} \cap S = \emptyset$.

(iii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\left|\{x\triangle e_k, x\triangle e_i\triangle e_j\triangle e_k\} \cap S\right| \geq 1$.

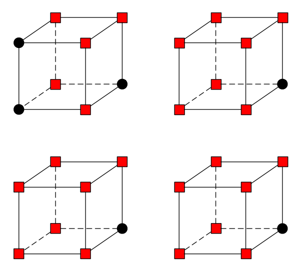
Proof. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and $i = 1, j = 2, k = 3$. Let $S' \subseteq \{0,1\}^3$ be the restriction of S obtained after 0-restricting coordinates $4, \ldots, n$.

(i): Suppose that $e_1, e_2, e_3 \in \overline{S}$. Assume for a contradiction that two of $e_1+e_2, e_2+e_3, e_3+e_1$, say e_1+e_2 and e_2+e_3, belong to S. If $e_1+e_3 \in S$, then S' is isomorphic to one of $P_3, S_3, R_{1,1}, F_2$, which cannot occur as S is 1-resistant and has no $R_{1,1}, F_2$ restriction. Otherwise, $e_1+e_3 \in \overline{S}$. Since $S' \ncong P_3$ and S is 1-resistant, it follows that $\mathbf{0}, e_1+e_2+e_3 \in S$, implying in turn that $S' \cong F_1$, a contradiction as S has no F_1 restriction.

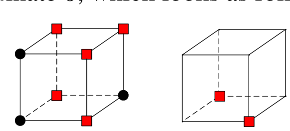
(ii): Suppose that $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1+e_2\}$ is a bridge. Then $e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. We need to prove that $\{e_1+e_3, e_2+e_3\} \cap S = \emptyset$. Suppose otherwise. After possibly relabeling coordinates $1,2$, we may assume that $e_1+e_3 \in S$. Thus, since $\mathbf{0}, e_1+e_2$ are in different feasible components, we must have that $|\{e_3, e_1+e_2+e_3\} \cap S| \leq 1$. After possibly twisting coordinates $1,2$, we may assume that $e_3 \in \overline{S}$. Since $e_1, e_2, e_3 \in \overline{S}$, we get from (i) that $|\{e_1+e_2, e_2+e_3, e_3+e_1\} \cap S| \leq 1$, a contradiction. Thus, $\{e_1+e_3, e_2+e_3\} \cap S = \emptyset$, so (ii) holds.

(iii) Suppose that $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1+e_2\}$ is a bridge. Then $S \cap \{\mathbf{0}, e_1, e_2, e_1+e_2\} = \{\mathbf{0}, e_1+e_2\}$, and $\{e_1+e_3, e_2+e_3\} \cap S = \emptyset$ by (ii). Since S is 1-resistant, it follows immediately that $\{e_3, e_1+e_2+e_3\} \cap S \neq \emptyset$, so (iii) holds. □

Lemma 4.3. *Take a set $S \subseteq \{0,1\}^5$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor, and in every minor, including S itself, every pair of bridges are parallel. If $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1+e_2\}$ is a bridge without neighboring bridges, then after possibly twisting coordinates 1 and 2, we have that $S = \{\mathbf{0}, e_3, e_1+e_2, e_1+e_2+e_4, e_1+e_2+e_5, e_1+e_2+e_4+e_5\}$:*

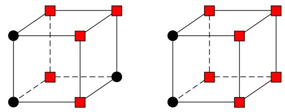


Proof. Let $B := \{\mathbf{0}, e_1, e_2, e_1+e_2\}$. As B is a bridge and $\mathbf{0} \in S$, $e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. It follows from Lemma 4.2 (ii) that $e_1+e_3, e_2+e_3 \in \overline{S}$. By Lemma 4.2 (iii) and the fact that B has no neighboring bridge, we get that exactly one of $e_3, e_1+e_2+e_3$ belongs to S. After twisting coordinates 1 and 2, if necessary, we may assume that $e_3 \in S$ and $e_1+e_2+e_3 \in \overline{S}$. Moreover, by Lemma 4.2 (ii), we have that $\{e_1+e_4, e_2+e_4\} \subseteq \overline{S}$. Let S' be the 0-restriction of S over coordinate 5, which looks as follows:



Claim 1. *$e_4 \in \overline{S}$ and $e_1+e_2+e_4 \in S$.*

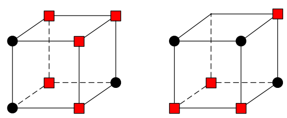
Proof of Claim. Suppose otherwise. Since B has no neighboring bridge in S, it follows from Lemma 4.2 (iii) that $e_4 \in S$ and $e_1+e_2+e_4 \in \overline{S}$. If $e_2+e_3+e_4 \in S$, then the 0-restriction of S' over coordinate 1 is either F_1 or F_3, which is not the case. Thus, $e_2+e_3+e_4 \in \overline{S}$. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in S$. As the 0-restriction of S' over coordinate 2 is not F_3, we have $e_1 + e_3 + e_4 \in \overline{S}$. Since the 1-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:



Observe however now that F_3 is obtained from S' after projecting away coordinate 1, a contradiction. $\lozenge$

Claim 2. $\{e_1 + e_3 + e_4, e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Suppose otherwise. After interchanging the roles of $1, 2$, if necessary, we may assume that $e_1 + e_3 + e_4 \in S$. If $e_3 + e_4 \in \overline{S}$, then $\{\mathbf{0}, e_3\}$ is a feasible component of S', so the square initiating from e_3 and active in directions e_1, e_4 is a bridge of S' that is not parallel to B, which is contrary to our assumption. Thus, $e_3 + e_4 \in S$. Since $\mathbf{0}, e_1 + e_2$ belong to different feasible components of S, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:

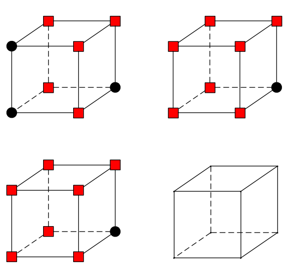


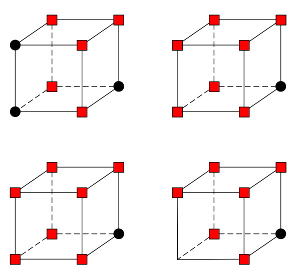
Observe however that S' has two non-parallel bridges, namely B and the square that initiates from $e_1 + e_4$ and is active in directions e_2, e_3, a contradiction. $\lozenge$

Claim 3. $\{e_3 + e_4, e_1 + e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in \overline{S}$. Since the 1-restriction of S' over coordinate 1 is also 1-resistant, we see that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, as required. $\lozenge$

We just determined the status of all the points in $\{x : x_5 = 0\}$. A similar argument applied to $\{x : x_4 = 0\}$ gives us the left figure below:





Consider the set obtained from S after 1-restricting over coordinate 1 and 0-restricting over coordinate 3; since this set is 1-resistant and not isomorphic to F_1, F_3, we get that $e_1 + e_4 + e_5 \in \overline{S}$ and $e_1 + e_2 + e_4 + e_5 \in S$. As the 1-restriction of S over coordinates $1, 2$ is not F_3, we get that $\mathbf{1} \in \overline{S}$. Now consider the set obtained from S after 1-restricting coordinate 2 and 0-restricting over coordinate 3; since this set is not F_3, we get that

$e_2+e_4+e_5 \in \overline{S}$. Note that $\{e_1+e_2, e_1+e_2+e_4, e_1+e_2+e_5, e_1+e_2+e_4+e_5\}$ forms a feasible component of S. Hence, as S does not have non-parallel bridges, it follows that $e_2+e_3+e_4+e_5, e_1+e_3+e_4+e_5 \in \overline{S}$, and also that $e_3+e_4+e_5 \in \overline{S}$. (See the right figure above.) Once again, as S does not have non-parallel bridges, it follows that $e_4+e_5 \in \overline{S}$, thereby finishing the proof. □

We are now ready to prove the main result of this section:

Proposition 4.4. *Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.*

Proof. Suppose for a contradiction that S has a pair of non-parallel bridges. (In particular, S is not connected.) We may assume that in every proper minor of S, every pair of bridges, if any, are parallel.

Claim 1. *Every direction is active in some bridge.*

Proof of Claim. Suppose for a contradiction that direction e_n is not active in any bridge. For a point $x \in \{0,1\}^n$, denote by $x' \subseteq \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. Notice first that by Lemma 4.1, the feasible components of S project onto different feasible components of S', the subset of $\{0,1\}^{n-1}$ obtained from S after projecting away coordinate n. We will derive a contradiction to the minimality of S by showing that S' has non-parallel bridges.

We will show that if B is a bridge of S, then $B' := \{x' : x \in B\}$ is still a bridge of S' that is active in the same directions as before. Since e_n is not active in any bridge of S, we may assume that $n \geq 3$ and $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$ where $\mathbf{0}, e_1+e_2$ belong to different feasible components of S, and $e_1, e_2 \in \overline{S}$. It follows from Lemma 4.2 (ii) that $\mathbf{0}, e_1+e_2 \in S'$ and $e_1, e_2 \in \overline{S'}$. Moreover, since the feasible components of S project onto different feasible components of S', we see that $\mathbf{0}, e_1+e_2$ belong to different feasible components of S'. Thus, B' is still a bridge of S' that is active in the same directions as before.

As a corollary, S' still has non-parallel bridges, thereby contradicting the minimality of S. ◊

Claim 2. *The following statements hold:*

(i) *if B, B' are non-parallel bridges that are not active in direction e_i, then $\{x : x_i = 0\}$ contains one of the bridges and $\{x : x_i = 1\}$ contains the other one,*

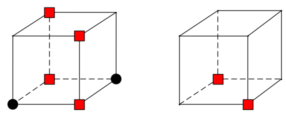
(ii) *if B, B', B'' are pairwise non-parallel bridges, then every direction is active in one of the bridges, and*

(iii) $n \in \{4, 5, 6\}$.

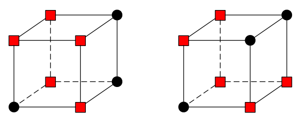
Proof of Claim. **(i)** For if not, then one of the restrictions of S over coordinate i contains B and B', thereby contradicting the minimality of S. **(ii)** Suppose for a contradiction that e_i is not active in any of B, B', B''. Then one of the hyperplanes $\{x : x_i = 0\}, \{x : x_i = 1\}$ contains at least two of B, B', B'', thereby contradicting (i). **(iii)** Let B, B' be non-parallel bridges. It follows from Lemma 4.2 (ii) that $n \geq 4$. If every direction is active in one of B, B', we get that $n = 4$. Otherwise, there is a direction e_i inactive in both B, B'. By Claim 1, there is a bridge B'' active in e_i. Clearly, B, B', B'' are pairwise non-parallel bridges. It now follows from (ii) that $n \le 6$, as required. $\Diamond$

Claim 3. $n \neq 4$.

Proof of Claim. Suppose for a contradiction that $n = 4$. Let B, B' be non-parallel bridges of S. We may assume that $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$, $\mathbf{0}, e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. By Lemma 4.2 (ii), $e_1+e_3, e_2+e_3, e_1+e_4, e_2+e_4 \in \overline{S}$:

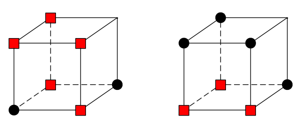


Assume in the first case that B' shares an active direction with B. After possibly relabeling coordinates $1, 2$, we may assume that B' is active in directions e_1, e_3. It follows from Claim 2 (i) that B' is contained in $\{x : x_4 = 1\}$. After possibly twisting coordinates $1, 2$, we may assume that $B' = \{e_4, e_1+e_4, e_3+e_4, e_1+e_3+e_4\}$. Since $e_1 + e_4 \in \overline{S}$, it follows that $e_4, e_1 + e_3 + e_4 \in S$ and $e_3 + e_4 \in \overline{S}$. Applying Lemma 4.2 (ii), we get that $e_3, e_2 + e_3 + e_4, e_1 + e_2 + e_4 \in \overline{S}$. Since the two restrictions of S over coordinate 4 are 1-resistant, it follows that $e_1 + e_2 + e_3, \mathbf{1} \in S$:



Observe, however, that 1-restricting S over coordinate 3 yields a set that is not 1-resistant, a contradiction.

Assume in the remaining case that B' is active in directions e_3, e_4. Observe that B' is not contained in $\{x : x_1 + x_2 = 1\}$. After possibly twisting coordinates $1, 2$, we may assume that B' initiates from $\mathbf{0}$. This means that $e_3, e_4 \in \overline{S}$ and $e_3 + e_4 \in S$. Applying Lemma 4.2 (iii), we get that $e_1 + e_2 + e_4 \in S$ and $e_1 + e_3 + e_4, e_2 + e_3 + e_4 \in S$:



The 1-restriction of S over coordinate 4, however, is isomorphic to either F_1 or F_3, a contradiction. $\Diamond$

Thus, we have that $n \in \{5, 6\}$. It follows from Claim 1 that there are $\lceil \frac{n}{2} \rceil = 3$ pairwise non-parallel bridges B_1, B_2, B_3. We get from Claim 2 (ii) that, after a possible relabeling, B_1 is active in e_1, e_2, B_2 is active in e_3, e_4, and

- if $n = 5$, then B_3 is active in e_3, e_5,
- if $n = 6$, then B_3 is active in e_5, e_6.

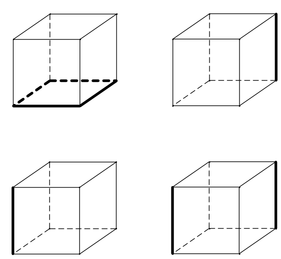
We can further say that,

Claim 4. *If B is a bridge different from B_1, B_2, B_3, then $n = 5$.*

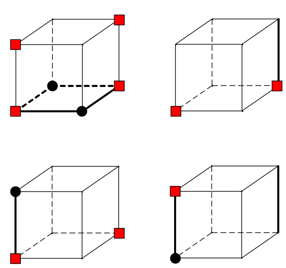
Proof of Claim. Suppose for a contradiction that $n = 6$. It follows from Claim 2 (ii) that B is parallel to one of B_1, B_2, B_3. Consider the bridge B_2. Since B_2, B_3 are inactive in e_1, e_2, it follows from Claim 2 (i) that the hyperplanes $\{x : x_1 = 0\}, \{x : x_2 = 0\}$ split B_2, B_3. Moreover, since B_2, B_1 are inactive in e_5, the hyperplane $\{x : x_5 = 0\}$ splits B_2, B_1. Hence, the square of B_2 – and of any bridge parallel to it – is uniquely determined once B_1 and B_3 are given, implying that B is not parallel to B_2. By the symmetry between B_1, B_2, B_3, we get that B is not parallel to B_1, B_3 either, a contradiction. $\lozenge$

Claim 5. $n \neq 5$.

Proof of Claim. Suppose for a contradiction that $n = 5$. After twisting coordinates $3, 4, 5$, if necessary, we may assume that B_1 initiates from $\mathbf{0}$. By Claim 2 (i), and after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates from e_5. Another application of Claim 2 (i) tells us that B_3 initiates from $e_1 + e_2 + e_4$:

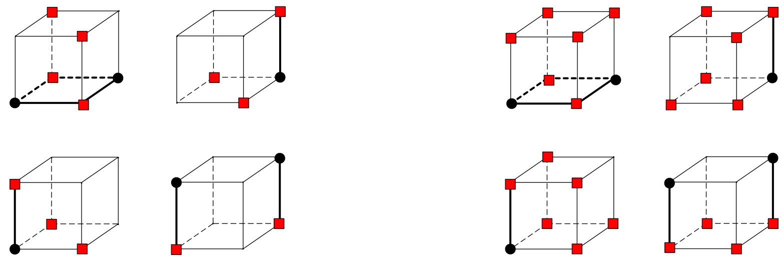


Assume in the first case that $\mathbf{0}, e_1 + e_2 \in \overline{S}$ and $e_1, e_2 \in S$. Then a repeated application of Lemma 4.2 (ii) tells us that $e_3, e_1 + e_2 + e_3, e_5, e_1 + e_2 + e_5, e_4, e_1 + e_2 + e_4 \in \overline{S}$. As a result, in the bridge B_2, we have that $e_3 + e_5, e_4 + e_5 \in S$:



Observe now that the restriction of S obtained after 0-restricting coordinates 1 and 2 is not 1-resistant, a contradiction.

Assume in the remaining case that $\mathbf{0}, e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. A repeated application of Lemma 4.2 (ii) to B_1, followed by an application of it to B_2, B_3 gives us the left figure below:



Applying Lemma 4.2 (ii) to B_2, B_3 gives us the right figure above, thereby yielding a contradiction as 0-restricting coordinates $4, 5$ of S yields a set that is not 1-resistant. This finishes the proof of the claim. $\diamondsuit$

Thus $n = 6$. After twisting coordinates $3, 4, 5, 6$, if necessary, we may assume that B_1 initiates from $\mathbf{0}$. Applying Claim 2 (i), we see that after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates from $e_5 + e_6$. Using Claim 2 (i), we see that B_3 must initiate from $e_1 + e_2 + e_3 + e_4$. (See Figure 3.)

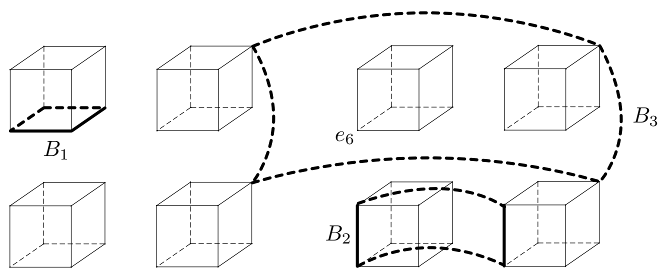


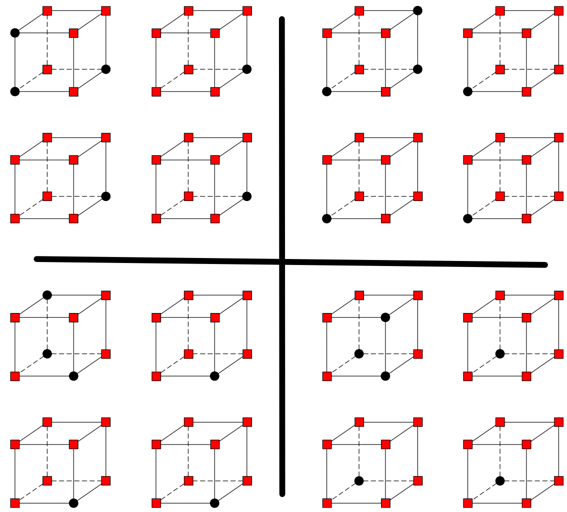
Figure 3: The locations of the bridges B_1, B_2, B_3 in $\{0, 1\}^6$

Recall from Claim 4 that B_1, B_2, B_3 are the only bridges of S. Let $S' \subseteq \{0, 1\}^5$ be the restriction of S obtained after 0-restricting coordinate 6. By assumption, every minor of S' has only parallel bridges. As a bridge in S' is not necessarily a bridge in S, we see that S' may have bridges other than B_1 (that will necessarily be parallel to it).

Claim 6. *B_1 does not have a neighboring bridge in S'.*

Proof of Claim. Suppose for a contradiction that B_1 has a neighboring bridge B in S'. Since B is not a bridge of S by Claim 4, it follows that the points in $B \cap S'$ are in the same feasible component of S. After applying Lemma 4.2 (ii) to B_1, we see that the points in $B_1 \cap S'$ also lie in this feasible component of S, a contradiction. $\diamondsuit$

We may now apply Lemma 4.3 to the bridge B_1 of S'. Depending on which points of B_1 are in S', and how coordinates $1, 2$ are twisted, we get that S' takes on one of the four possibilities shown below.



Consider the 3-dimensional restriction F of S containing B_2 and $B_2 \triangle e_6$. If S' takes one of the top-left, bottom-left or bottom-right possibilities, then F is not 1-resistant, which is not possible. Otherwise, S' takes the top-right possibility, in which case $F \cong F_1$, a contradiction. This finally finishes the proof of Proposition 4.4. □

5 Proof of Theorem 3.1

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. S is *polar* if either there are antipodal feasible points, or the feasible points agree on a coordinate:

$$\{x, \mathbf{1} - x\} \subseteq S \quad \text{for some } x \in \{0,1\}^n \qquad \text{or} \qquad S \subseteq \{x : x_i = a\} \quad \text{for some } i \in [n] \text{ and } a \in \{0,1\}.$$

This notion was introduced and studied in [3] and will be needed in this section.

We say that S is *separable* if there exist a partition of S into nonempty parts S_1, S_2 and distinct coordinates $i, j \in [n]$ such that either $S_1 \subseteq \{x : x_i = 0, x_j = 1\}$ and $S_2 \subseteq \{x : x_i = 1, x_j = 0\}$, or $S_1 \subseteq \{x : x_i = x_j = 0\}$ and $S_2 \subseteq \{x : x_i = x_j = 1\}$. Notice that if S is separable, then it is not connected.

Remark 5.1. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If a projection of S is separable, then so is S.*

Proposition 5.2. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Suppose there is a partition of S into nonempty parts S_1, S_2 such that $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. Then S_1 and S_2 are sub-hypercubes. In particular, S is polar.*

Proof. The sub-hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is infeasible. As S is 1-resistant, Lemma 1.6 implies that in each of the parallel sub-hypercubes $\{x : x_{n-1} = x_n = 0\}$ and $\{x : x_{n-1} = x_n = 1\}$, the feasible points form a sub-hypercube. That is, the two sets $S_1 = S \cap \{x : x_{n-1} = x_n = 0\}$ and $S_2 = S \cap \{x : x_{n-1} = x_n = 1\}$ are sub-hypercubes. We leave it as an easy exercise for the reader to check that S is polar. □

We are now ready to prove Theorem 3.1:

Proof of Theorem 3.1. Take an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and is not connected. We will describe S exactly unless it has a D_3 minor. Let us start with the following claim:

Claim 1. *S is separable.*

Proof of Claim. Let $k \geq 2$ be the number of feasible components of S. Let $S' \subseteq \{0,1\}^m$ be a projection of S of smallest dimension with exactly k feasible components. It then follows from Lemma 4.1 that every direction of $\{0,1\}^m$ is active in a bridge of S'. However, as S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, Proposition 4.4 implies that every pair of bridges of S' are parallel. As a result, $m = k = 2$ and S' is either $\{00, 11\}$ or $\{10, 01\}$. In particular, S' is separable, so S is separable by Remark 5.1. $\Diamond$

Thus, there is a partition of S into nonempty parts S_1, S_2 such that, after a possible twisting and relabeling, $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. As S is 1-resistant, Proposition 5.2 implies that S_1 and S_2 are sub-hypercubes, and that S is polar. In particular, since S is not a sub-hypercube, Lemma 1.6 implies that the points in S do not agree on a coordinate; notice that this property is preserved in every projection of dimension at least one.

Claim 2. *Either S has a D_3 minor, or one of S_1, S_2 is contained in the antipode of the other.*

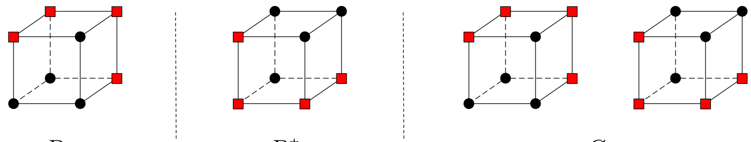
Proof of Claim. Suppose neither of S_1, S_2 is contained in the antipode of the other. As the points in the polar set S do not agree on a coordinate, there exists a point $x \in S_1$ such that $\mathbf{1} - x \in S_2$. As neither of S_1, S_2 is contained in the antipode of the other, there exist distinct coordinates $i, j \in [n-2]$ such that $x \triangle e_i \in S_1, x \triangle e_j \notin S_1, \mathbf{1} \triangle x \triangle e_i \notin S_2$ and $\mathbf{1} \triangle x \triangle e_j \in S_2$. Let S' be the minor of S obtained after projecting away coordinates $[n] - \{i, j, n-1, n\}$. It can be readily checked that $S' \cong \{0000, 1000, 1011, 1111\}$ (note that S_1, S_2 are sub-hypercubes). Clearly, S' has a D_3 projection, thereby proving the claim. $\Diamond$

If S has a D_3 minor, then we are done. Otherwise, one of S_1, S_2 is contained in the antipode of the other. After possibly relabeling S_1, S_2, we may assume that S_2 is contained in the antipode of S_1.

Claim 3. $2|S_2| \geq |S_1| \geq |S_2|$.

Proof of Claim. Clearly, $|S_1| \geq |S_2|$. Suppose for a contradiction that $|S_1| \geq 4|S_2|$. Since S_2 is contained in the antipode of S_1, it can be readily checked that S has an F_3 minor, a contradiction. $\Diamond$

As a result, either $|S_1| = |S_2|$ or $|S_1| = 2|S_2|$. It can now be readily checked that either $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, or $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, thereby finishing the proof of Theorem 3.1. $\square$



D_3 $D_3^\star$ C_8

6 D_3 minors and proof of Theorem 3.2

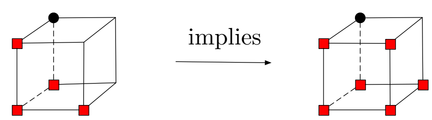
We will need four lemmas, the first of which is from another paper:

Lemma 6.1 ([4])**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$, where for all $x \in \{0,1\}^n$ and distinct $i, j \in [n]$, the following statement holds:*

if $x, x\triangle e_i, x\triangle e_j \in S$ then $x\triangle e_i\triangle e_j \in S$.

Then every feasible component of S is a sub-hypercube.

Let $D_3^\star := \{010, 011, 111, 101\} \subseteq \{0,1\}^3$. Observe that $D_3^\star$ is a twisting of $D_3 = \{000, 100, 010, 101\}$, and $C_8 = (D_3 \times \{0\}) \cup (D_3^\star \times \{1\})$. In the following lemma, we will use the following implication of Lemma 4.2 (i):

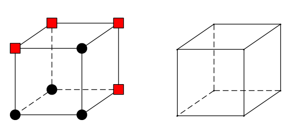


Lemma 6.2. *Let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, where the 0-restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$. Then,*

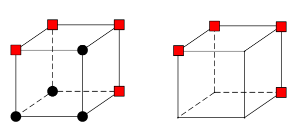
(i) every restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$, and

(ii) either $S \cong D_3 \times \{0,1\}^{n-3}$ or $S \cong C_8 \times \{0,1\}^{n-4}$.

Proof. **(i)** By a recursive argument, it suffices to show that each 3-dimensional restriction of S neighboring a $D_3, D_3^\star$ restriction is also a D_3 or a $D_3^\star$. Thus, we may assume that $n = 4$. After twisting coordinates $1, 2, 3$, if necessary, we may assume that the 0-restriction of S over coordinate 4 is D_3. So $S \cap \{x : x_4 = 0\} = \{0000, 1000, 0100, 1010\}$:

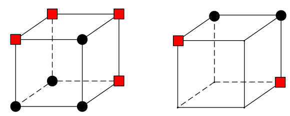


Assume in the first case that $\{0111, 1111\} \cap \overline{S} \neq \emptyset$. After applying Lemma 4.2 (i) twice, we see that $\{0111, 1111, 0011, 1101\} \subseteq \overline{S}$:



Since the two restrictions over coordinate 1 are 1-resistant, $|\{0101, 0001\} \cap S| \neq 1$ and $|\{1001, 1011\} \cap S| \neq 1$. In fact, as S has no F_3 minor, $\{0101, 0001\} \subseteq S$ if and only if $\{1001, 1011\} \subseteq S$. Moreover, as the 0-restriction of S over coordinate 3 is 1-resistant, it follows that $\{0101, 0001, 1001, 1011\} \cap S \neq \emptyset$. As a result, $\{0101, 0001, 1001, 1011\} \subseteq S$, implying in turn that 1-restricting S over coordinate 4 yields D_3.

Assume in the remaining case that $\{0111, 1111\} \cap \overline{S} = \emptyset$. As the 1-restriction of S over coordinate 3 (resp. coordinate 2) is not isomorphic to either of F_1, F_3, we get that $0011 \in \overline{S}$ (resp. $1101 \in \overline{S}$).

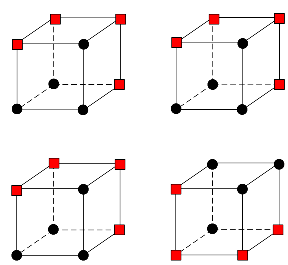


Since S has no F_1, F_3, S_3 restrictions, it follows that $0001, 1001 \in \overline{S}$. Since the 0-restriction of S over coordinate 2 (resp. coordinate 3) is 1-resistant, $1011 \in S$ (resp. $0101 \in S$), implying in turn that 1-restricting S over coordinate 4 yields $D_3^\star$.

(ii) It follows from (i) that $S = \bigcup_{y \in \{0,1\}^{n-3}} \left(F \times \{y\} : F \in \{D_3, D_3^\star\}\right)$. Let $R \subseteq \{0,1\}^{n-3}$ be the set of points y such that $S \cap \{x : x_i = y_{i-3} \quad 4 \leq i \leq n\} = D_3 \times \{y\}$.

Claim 1. *Every feasible component of R is a sub-hypercube. Similarly, every infeasible component of R is a sub-hypercube.*

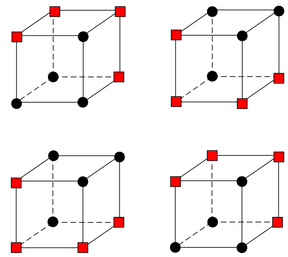
Proof of Claim. By Lemma 6.1, it suffices to prove that for each $y \in R$ and distinct coordinates $i, j \in [n-3]$, if $y, y\triangle e_i, y\triangle e_j \in R$ then $y\triangle e_i\triangle e_j \in R$. Suppose otherwise. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe that the 0-restriction of S' over coordinates $1, 2$ is fragile and therefore not 1-resistant, a contradiction. $\Diamond$

Claim 2. *R is connected. Similarly, $\overline{R}$ is connected.*

Proof of Claim. Suppose for a contradiction that $R \subseteq \{0,1\}^{n-3}$ is not connected. By Claim 1, every feasible component of R is a sub-hypercube, each of which must have rank at most $(n-3) - 2 = n-5$. Thus, there exist $y \in \{0,1\}^{n-3}$ and distinct coordinates $i, j \in [n-3]$ such that $y \in R$ and $y\triangle e_i, y\triangle e_j \in \overline{R}$. Since every infeasible component of R is also a sub-hypercube by Claim 1, it follows that $y\triangle e_i\triangle e_j \in R$. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe however that the 0-restriction of S' over coordinates $1, 2$ is fragile and therefore not 1-resistant, a contradiction. $\Diamond$

As a result, both $R, \overline{R}$ are sub-hypercubes, implying in turn that $R \cong \emptyset, \{0,1\}^{n-4} \times \{0\}, \{0,1\}^{n-3}$. If $R \cong \emptyset, \{0,1\}^{n-3}$ then $S \cong D_3 \times \{0,1\}^{n-3}$, and if $R \cong \{0,1\}^{n-4} \times \{0\}$ then $S \cong C_8 \times \{0,1\}^{n-4}$, thereby finishing the proof. $\square$

For each $k \geq 4$, recall that $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$, and let $D_k^\star := D_k \triangle e_k$.

Lemma 6.3. *Take integers $n \geq 3$ and $k \in \{3, \ldots, n\}$. Let $S \subseteq \{0,1\}^{n+1}$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then the following statements hold:*

(i) if the projection of S over coordinate $n+1$ is D_n, then S is either $D_{n+1}, D_{n+1}^\star$ or $D_n \times \{0,1\}$,

(ii) if the projection of S over coordinate $k+1$ is $D_k \times \{0,1\}^{n-k}$, then S is either $D_{k+1} \times \{0,1\}^{n-k}$, $D_{k+1}^\star \times \{0,1\}^{n-k}$ or $D_k \times \{0,1\}^{n-k+1}$.

Proof. In this proof, we use $\mathbf{1}$ to refer to the $(n+1)$-dimensional vector of all-ones, and use $\mathbf{1}'$ to refer to the n-dimensional vector of all-ones. **(i)** Assume that the projection of S over coordinate $n+1$ is D_n. Let

$$S_0 := S \cap \{x : x_i = 0, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1},$$
$$S_1 := S \cap \{x : x_i = 1, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1}.$$

Then

- $S = S_0 \cup S_1$,
- $S_0 \subseteq \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, and the projection of S_0 over coordinate $n+1$ is $\{\mathbf{0}, e_2\}$, and
- $S_1 \subseteq \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, and the projection of S_1 over coordinate $n+1$ is $\{\mathbf{1}' - e_2, \mathbf{1}' - e_2 - e_3\}$.

After twisting coordinate $n+1$, if necessary, we may assume that $\mathbf{0} \in S_0$. Then, since S_0 and S_1 are 1-resistant, we get that

$$S_0 = \{\mathbf{0}, e_2\} \quad \text{or} \quad \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}, \quad \text{and}$$
$$S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\} \quad \text{or} \quad \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \quad \text{or}$$
$$\{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}.$$

Claim 1. *If $S_0 = \{\mathbf{0}, e_2\}$, then $S = D_{n+1}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2\}$.

Assume in the first case that $n = 3$. If $S_1 = \{\mathbf{1}-e_2-e_4, \mathbf{1}-e_2-e_3-e_4\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 3 is not 1-resistant, which is not the case. If $S_1 = \{\mathbf{1}-e_2-e_4, \mathbf{1}-e_2-e_3-e_4, \mathbf{1}-e_2, \mathbf{1}-e_2-e_3\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 2 is isomorphic to F_3, which is again not the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_4$, as claimed.

Assume in the remaining case that $n \geq 4$. If $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\}$, then the points in $S = S_0 \cup S_1$ all agree on coordinate $n + 1$, so by Lemma 1.6, S is a sub-hypercube, which is not the case. If $S_1 = \{\mathbf{1}-e_2-e_{n+1}, \mathbf{1}-e_2-e_3-e_{n+1}, \mathbf{1}-e_2, \mathbf{1}-e_2-e_3\}$, then the projection of $S = S_0 \cup S_1$ over coordinates $[n + 1] - \{2, 3, n + 1\}$ is isomorphic to F_3, which cannot be the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_{n+1}$, as claimed. ♢

Claim 2. *If $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, then $S = D_n \times \{0, 1\}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$. As the projection of $S = S_0 \cup S_1$ over coordinates $[n + 1] - \{2, 3, n + 1\}$ is not isomorphic to F_3, it follows that $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = D_n \times \{0, 1\}$, as required. ♢

Thus, after twisting coordinate $n + 1$, if necessary, S is either D_{n+1} or $D_n \times \{0, 1\}$, so (i) holds.

(ii) Assume that the projection of S over coordinate $k + 1$ is $D_k \times \{0, 1\}^{n-k}$. For each point $y \in \{0, 1\}^{n-k}$, let $S_y := S \cap \{x : x_{i+k+1} = y_i, i \in [n - k]\} \subseteq \{0, 1\}^{n+1}$. Notice that $S = \bigcup_{y \in \{0,1\}^{n-k}} S_y$. For each $y \in \{0, 1\}^{n-k}$, pick an appropriate $S'_y \subseteq \{0, 1\}^{k+1}$ such that $S_y = S'_y \times \{y\}$. Notice that the projection of each S'_y over coordinate $k + 1$ is D_k. We therefore get from (i) that each S'_y is either D_{k+1}, $D^\star_{k+1}$ or $D_k \times \{0, 1\}$.

Claim 3. *All of $(S'_y : y \in \{0, 1\}^{n-k})$ are equal to one another.*

Proof of Claim. Suppose otherwise. Then there exists $y_1, y_2 \in \{0, 1\}^{n-k}$ such that $\mathrm{dist}(y_1, y_2) = 1$ and $S'_{y_1} \neq S'_{y_2}$. In particular, S has either $S' := (D_{k+1} \times \{0\}) \cup (D_k \times \{01, 11\})$ or $S'' := (D_{k+1} \times \{0\}) \cup (D^\star_{k+1} \times \{1\})$ as a restriction. However, the restriction of S' (resp. S'') obtained after 0-restricting coordinates $[n+1]-\{3, k+1, k + 2\}$ is not 1-resistant, so S cannot have either of S', S'' as a restriction, a contradiction. ♢

As a consequence, $S = D_{k+1} \times \{0, 1\}^{n-k}, D^\star_{k+1} \times \{0, 1\}^{n-k}$ or $D_k \times \{0, 1\}^{n-k+1}$, so (ii) holds. □

Lemma 6.4. *Take an integer $n \geq 5$ and a set $S \subseteq \{0, 1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. If the projection of S over coordinate n is $C_8 \times \{0, 1\}^{n-5}$, then $S = C_8 \times \{0, 1\}^{n-4}$.*

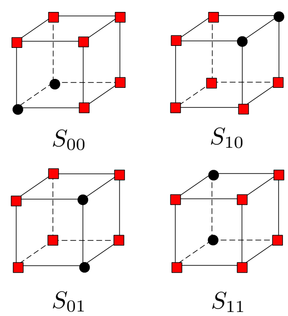
Proof. It suffices to prove this for $n = 5$. Assume that the projection of S over coordinate 5 is $C_8 = (D_3 \times \{0\}) \cup (D^\star_3 \times \{1\})$. For $i, j \in \{0, 1\}$, let $S_{ij} \subseteq \{0, 1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. After twisting coordinate 5, if necessary, we may assume that $\mathbf{0} \in S$.

Claim. *S has a D_3 restriction.*

Proof of Claim. Suppose for a contradiction that S does not have a D_3 restriction. In particular, $S_{00}, S_{01} \neq D_3$ and $S_{10}, S_{11} \neq D_3^\star$. Thus by Lemma 6.3 (i),

$$(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4 \quad \text{or} \quad D_4^\star,$$
$$(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D_4' \quad \text{or} \quad D_4' \triangle e_4,$$

where $D_4' = \{0100, 0110, 1011, 1111\} \subseteq \{0,1\}^4$. Since $\mathbf{0} \in S$, we must have that $(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4$. Thus, $S_{00} = \{000, 010\}$ and $S_{01} = \{100, 101\}$. Since the restriction of S obtained after 0-restricting coordinates 1 and 5 is not isomorphic to D_3, it follows that $(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D_4' \triangle e_4$. So, $S_{10} = \{101, 111\}$ and $S_{11} = \{010, 011\}$:



Observe however that the 1-restriction of S over coordinates $2, 3$ is not 1-resistant, a contradiction. ◊

Thus, $S \cong D_3 \times \{0,1\}^2$ or $C_8 \times \{0,1\}$ by Lemma 6.2 (ii). It can be readily checked that S must be in fact equal to $C_8 \times \{0,1\}$, as required. □

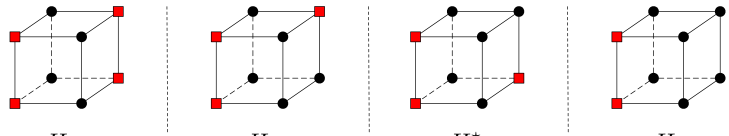
We are now ready to prove Theorem 3.2:

Proof of Theorem 3.2. Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. Assume that S has a D_3 minor. We will describe S exactly. Among all projections of S with a D_3 restriction, pick one $S' \subseteq \{0,1\}^\ell$ of largest dimension $\ell \in \{3, \ldots, n\}$. We may assume, after a possible relabeling, that S' is obtained from S after projecting away coordinates $[n] - [\ell]$. It follows from Lemma 6.2 (ii) that, after a possible twisting and relabeling, $S' = C_8 \times \{0,1\}^{\ell-4}$ or $S' = D_3 \times \{0,1\}^{\ell-3}$.

Claim. *If $S' = C_8 \times \{0,1\}^{\ell-4}$, then $\ell = n$.*

Proof of Claim. This follows immediately from Lemma 6.4 and the maximal choice of S'. ◊

Thus, if $S' = C_8 \times \{0,1\}^{\ell-4}$, then $S \cong C_8 \times \{0,1\}^{n-4}$. Otherwise, $S' = D_3 \times \{0,1\}^{\ell-3}$. In this case, a repeated application of Lemma 6.3 (ii) implies that $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{\ell, \ldots, n\}$ (note that $D_k, D_k^\star$ are isomorphic), thereby finishing the proof of Theorem 3.2. □



H_1 H_2 $H_2^\star$ H_3

7 Infeasible sub-hypercubes and Theorem 3.3

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. In this section, we will prove the following statement:

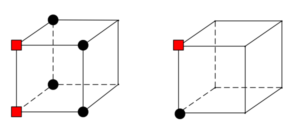
> *Assume that S is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a sub-hypercube.*

Proving this statement requires three technical lemmas. Given $i \in \{0,1\}$, denote by $S_i \subseteq \{0,1\}^{n-1}$ the i-restriction of S over coordinate n. Let

$$\begin{aligned} H_1 &:= \{100, 010, 101, 011\} \subseteq \{0,1\}^3 \\ H_2 &:= \{100, 010, 101, 011, 110\} \subseteq \{0,1\}^3 \\ H_2^\star &:= \{100, 010, 101, 011, 111\} \subseteq \{0,1\}^3 \\ H_3 &:= \{100, 010, 101, 011, 110, 111\} \subseteq \{0,1\}^3 \end{aligned}$$

Lemma 7.1. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. If $S_0 \in \{H_1, H_2, H_2^\star, H_3\}$, then $|\{000, 001\} \cap S_1| \neq 1$.*

Proof. Suppose, for a contradiction, that $H_1 \subseteq S_0 \subseteq H_3$ and $|\{000, 001\} \cap S_1| = 1$. After twisting coordinate 3, if necessary, we may assume that $000 \in S_1$ and $001 \in \overline{S_1}$. So S may be displayed as below:



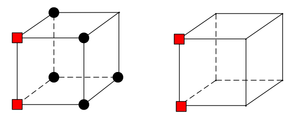
Since the 0-restriction of S over coordinate 1 is not isomorphic to either F_1 or F_3, we get that $011 \in \overline{S_1}$, and since this restriction is not isomorphic to D_3, we get that $010 \in \overline{S_1}$. By the symmetry between coordinates $1, 2$, we get that $\{100, 101\} \subseteq \overline{S_1}$. But then the 0-restriction of S over coordinate 3 is isomorphic to either $P_3, R_{1,1}, F_1$ or F_2, a contradiction. $\square$

Lemma 7.2. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_0 \in \{H_2, H_2^\star, H_3\}$ and $\{000, 001\} \cap S_1 = \emptyset$. Then the following statements hold:*

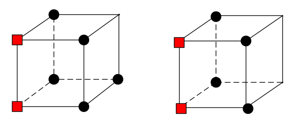
(i) *$S_1 \in \{H_1, H_2, H_2^\star, H_3\}$, and*

(ii) *if $S_1 = H_1$, then $S_0 = H_3$.*

Proof. **(i)** After twisting coordinate 3, if necessary, we may assume that $S_0 \in \{H_2, H_3\}$. We may therefore display S as:



Since the 0-restriction of S over coordinate 1 is 1-resistant, it follows that $|\{010, 011\} \cap S_1| \neq 1$, and since the 0-restriction of S over coordinate 2 is 1-resistant, it follows that $|\{100, 101\} \cap S_1| \neq 1$. Thus, as the 0-restriction of S over coordinate 3 is 1-resistant, either $\{010, 011\} \subseteq S_1$ or $\{100, 101\} \subseteq S_1$. After relabeling coordinates $1, 2$, if necessary, $\{010, 011\} \subseteq S_1$. Since the 0-restriction of S over coordinate 3 is not isomorphic to D_3 or F_3, it follows that $\{100, 101\} \subseteq S_1$ also:



Hence, $S_1 \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_1 = H_1$, then as the 1-restriction of S over coordinate 1 is not isomorphic to F_3, it follows that $111 \in S_0$, so $S_0 = H_3$, as required. □

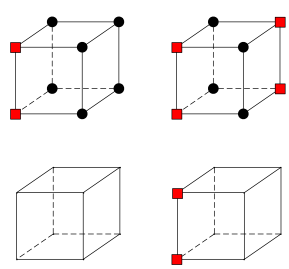
Given that $n \geq 2$ and $i, j \in \{0, 1\}$, denote by $S_{ij} \subseteq \{0, 1\}^{n-2}$ the restriction of S obtained after i-restricting coordinate $n - 1$ and j-restricting coordinate n.

Lemma 7.3. *Let $S \subseteq \{0, 1\}^5$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_{00} = H_3, S_{10} = H_1$ and $\{000, 001\} \cap S_{11} = \emptyset$. Then the following statements hold:*

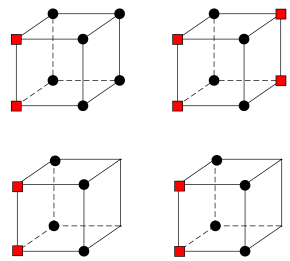
(i) $S_{01}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$, and

(ii) if $S_{11} = H_1$ then $S_{01} = H_3$, and therefore $S_1 = S_0$.

Proof. **(i)** For $i, j \in \{0, 1\}$, denote by $R_{ij} \subseteq \{0, 1\}^5$ the restriction of S obtained after i-restricting coordinate 3 and j-restricting coordinate 5.



Notice that $R_{00} = R_{10} = H_2$ and $001 \notin R_{01} \cup R_{11}$. It therefore follows from Lemma 7.1 that $000 \notin R_{01} \cup R_{11}$. We get from Lemma 7.2 (i)-(ii) that $R_{01}, R_{11} \in \{H_2, H_2^\star, H_3\}$:



As a result, $S_{00}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_{11} = H_1$, then R_{01} and R_{11} must be equal to H_2, implying in turn that $S_{01} = H_3$, as required. $\square$

We are now ready to prove the first main result of this section:

Proposition 7.4. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that has no $R_{1,1}, F_1, F_2, F_3$ and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a sub-hypercube.*

Proof. We prove this by induction on $n \geq 2$. The base case $n = 2$ holds trivially. For the induction step, assume that $n \geq 3$. Let K be the infeasible component of S containing x. If every neighbor of x belongs to S, then $K = \{x\}$ and we are done. Otherwise, we may assume that $x \in \{\mathbf{0}, e_3\} \subseteq K$ and $i = 1, j = 2$. For each $y \in \{0,1\}^{n-3}$, let $S_y := S \cap \big\{x : x_{3+i} = y_i, i \in [n-3]\big\}$ and choose an appropriate $R_y \subseteq \{0,1\}^3$ such that $S_y = R_y \times \{y\}$.

Claim 1. $R_{\mathbf{0}} \in \{H_2, H_2^\star, H_3\}$.

Proof of Claim. We know by assumption $\{000, 001\} \subseteq \overline{R_{\mathbf{0}}}$. Assume in the first case $x = \mathbf{0}$. Then $\{100, 010, 110\} \subseteq R_{\mathbf{0}}$. Since $R_{\mathbf{0}}$ is 1-resistant, $R_{\mathbf{0}} \cap \{101, 011\} \neq \emptyset$. In fact, $\{101, 011\} \subseteq R_{\mathbf{0}}$ because $R_{\mathbf{0}} \not\cong D_3, F_3$. Subsequently, $R_{\mathbf{0}} \in \{H_2, H_3\}$. Assume in the remaining case $x = e_3$. Then $\{101, 011, 111\} \subseteq R_{\mathbf{0}}$. Similar to the first case, since $R_{\mathbf{0}}$ is 1-resistant, $R_{\mathbf{0}} \cap \{100, 010\} \neq \emptyset$. In fact, $\{100, 010\} \subseteq R_{\mathbf{0}}$ because $R_{\mathbf{0}} \not\cong D_3, F_3$. Subsequently, $R_{\mathbf{0}} \in \{H_2^\star, H_3\}$, as required. $\lozenge$

If $n = 3$, then $K = \{\mathbf{0}, e_3\}$ by Claim 1, and the induction step is complete.

We may therefore assume that $n \geq 4$. Let S' be the projection of S over coordinate 3. Then S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. Hence, since $\mathbf{0} \in \overline{S'}$ and $\{e_1, e_2, e_1 + e_2\} \subseteq S'$, the induction hypothesis implies that the infeasible component of S' containing $\mathbf{0}$ – call it K' – is a sub-hypercube. For the next claim, call a point $y \in \{0,1\}^{n-3}$ *involved* if $R_y \in \{H_2, H_2^\star, H_3\}$ and $00y \in K'$. Notice that $\mathbf{0} \in \{0,1\}^{n-3}$ is involved.

Claim 2. *K consists precisely of the points in $\{0,1\}^n$ projecting onto K'.*

Proof of Claim. ($\supseteq$) The set of points in $\{0,1\}^n$ projecting onto a point in K' clearly belong to K and form a sub-hypercube. ($\subseteq$) Suppose, for a contradiction, the reverse inclusion does not hold. Then there must exist points $z, z + e_3 \in \{0,1\}^n$ satisfying $|\{z, z + e_3\} \cap S| = 1$ which project onto a point $z' \in \{0,1\}^{n-1}$ such that z' belongs to S' and is adjacent to a point in K'. Notice that $|\{z, z + e_3\} \cap K| = 1$.

Pick a point $t' \in \{0,1\}^{n-1}$ such that

i. $t' \in K'$,

ii. there exists an involved $y^\star \in \{0,1\}^{n-3}$ such that $t' = 00y$, and

iii. $\mathrm{dist}(t', z')$ is minimized subject to i-ii, in this order of priority.

After a relabeling of the coordinates, if necessary, we may assume that $t' = \mathbf{0} \in \{0,1\}^{n-1}$. Since $z' \notin K'$, we get that $\mathrm{dist}(\mathbf{0}, z') \geq 1$. It follows from Lemma 7.1 that $\mathrm{dist}(\mathbf{0}, z') \geq 2$.

Since K' is a sub-hypercube, there exist an integer $d \geq 2$ and distinct coordinates $j_1, j_2, \ldots, j_d \in [n] - \{3\}$ such that $z' = \sum_{i=1}^d e_{j_i}$ and

$$\sum_{i=1}^k e_{j_i} \in K' \qquad k = 1, \ldots, d-1.$$

In words, there exists an infeasible path in $\overline{S'}$ starting from $\mathbf{0}$, ending at z', and of (the shortest possible) length $\mathrm{dist}(\mathbf{0}, z') = d$. In what follows, we essentially take a walk on this path starting from $\mathbf{0}$, repeatedly apply Lemma 7.1, Lemma 7.2 and Lemma 7.3, prove that each R_y encountered on the path (i.e. $00y$ is a vertex on the path) is one of $H_1, H_2, H_2^\star, H_3$, thereby reaching a contradiction because this cannot be the case for the last vertex z'.

Notice that

$$\sum_{i=1}^k e_{j_i} \in K \quad \text{and} \quad e_3 + \sum_{i=1}^k e_{j_i} \in K \qquad k = 1, \ldots, d-1.$$

Thus, since $R_{\mathbf{0}} \in \{H_2, H_2^\star, H_3\}$, we have $j_1 \in [n] - \{1,2,3\}$. After relabeling the coordinates, if necessary, we may assume that $j_1 = 4$. Since $R_{\mathbf{0}} \in \{H_2, H_3\}$ and $\{000, 001\} \cap R_{e_1} = \emptyset$, it follows from Lemma 7.2 (i) that $R_{e_1} \in \{H_1, H_2, H_2^\star, H_3\}$. Our minimal choice of $t' = \mathbf{0}$ implies that $R_{e_1} = H_1$ (otherwise, e_4 would satisfy i-ii and $\mathrm{dist}(e_4, z') < \mathrm{dist}(t', z')$, thereby contradicting the minimality of $t' = \mathbf{0}$).

We now get from Lemma 7.2 (ii) that $R_{\mathbf{0}} = H_3$, and from Lemma 7.1 that $d \geq 3$. Since $j_2 \in [n] - \{1,2,3,4\}$, we may assume that $j_2 = 5$. So $e_4 + e_5 \in K'$. As $\mathbf{0}, e_4, e_4 + e_5 \in K'$ and K' is a sub-hypercube, it follows that $e_5 \in K'$. Since $\{000, 001\} \cap R_{e_1+e_2} = \emptyset$, we get from Lemma 7.3 that either

- $R_{e_1+e_2} \in \{H_2, H_2^\star, H_3\}$, or
- $R_{e_2} = H_3$ and $R_{e_1+e_2} = H_1$.

The first case is not possible as it contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_4 + e_5$ would be a better choice. However, the second case is not possible either as it also contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_5$ would be a better choice. In both cases, we reached the desired contradiction, thereby finishing the proof of Claim 2. ◊

Claim 2 completes the induction step as the set of points in $\{0,1\}^n$ projecting onto a point in K' also forms a sub-hypercube (whose rank is larger than K' by one). This finishes the proof of Proposition 7.4. □

7.1 Proof of Theorem 3.3

Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. Assume that S is connected and has no D_3 minor.

Assume in the first case that there is an infeasible component K that is not a sub-hypercube. We will prove that S is a sub-hypercube.

Claim 3. *Take a point x and distinct coordinates $i,j \in [n]$ such that $x \in K$ and $x \triangle e_i \in S$. If $x \triangle e_i \triangle e_j \in S$, then $x \triangle e_j \in K$.*

Proof of Claim. For if not, $x \triangle e_j \in S$, so by Proposition 7.4, the infeasible component of S containing x, which is K, is a sub-hypercube, a contradiction. $\Diamond$

This claim has the following subtle implication:

Claim 4. *The points in S agree on a coordinate.*

Proof of Claim. Take a point $y \in K$ and a direction $i \in [n]$ such that $y \triangle e_i \in S$. We may assume that $y = \mathbf{0}$ and $i = 1$. As S is connected, it follows from Claim 1 that $S \subseteq \{x : x_1 = 1\}$, as required. $\Diamond$

As S is 1-resistant, it follows from Lemma 1.6 that S is a sub-hypercube, as required.

Assume in the remaining case that every infeasible component of S is a sub-hypercube. We claim that every feasible point has at most two infeasible neighbors, thereby finishing the proof of Theorem 3.3. Suppose otherwise. Then there is a feasible point x with three infeasible neighbors $x \triangle e_i, x \triangle e_j, x \triangle e_k$, for distinct $i,j,k \in [n]$. Since every infeasible component is a sub-hypercube, it follows that $x \triangle e_i \triangle e_j, x \triangle e_j \triangle e_k, x \triangle e_k \triangle e_i$ are feasible. But then the 3-dimensional restriction of S containing $x \triangle e_i, x \triangle e_j, x \triangle e_k$ is isomorphic to either $R_{1,1}$ or F_2, a contradiction. $\Box$

8 The cuboid of a ± 1-resistant set has the max-flow min-cut property.

Let us start with the following fascinating result:

Theorem 8.1 ([4])**.** *A* 1*-resistant set is cube-ideal.*

That is, the cuboid of a 1-resistant set is ideal. When does the cuboid have the max-flow min-cut property? This has been answered partially. To elaborate, take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall from §5 that S is polar if either there are antipodal feasible points, or the feasible points agree on a coordinate. S is *strictly polar* if every restriction of it, including S itself, is polar [3]. The following highly nontrivial result was proved in [4]:

Theorem 8.2 ([4])**.** *Let S be a* 1*-resistant set. Then* $\text{cuboid}(S)$ *has the max-flow min-cut property if, and only if, S is strictly polar.*

Hence, to prove the title statement of this section, it suffices to prove that a ± 1-resistant is strictly polar. We need the following immediate remark:

Remark 8.3. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If S is strictly polar, then so is $S \times \{0,1\}$.*

Lemma 8.4. *Take an integer $n \geq 1$ and a nonempty set $S \subseteq \{0,1\}^n$ without an $R_{1,1}$ restriction, where every infeasible component is a sub-hypercube. Then*

- *$|S| \geq 2^{n-1}$, and*
- *if $|S| = 2^{n-1}$, then S is either a sub-hypercube of rank $n-1$ or the union of antipodal sub-hypercubes of rank $n-2$.*

In particular, S is strictly polar.

Proof. We prove this by induction on $n \geq 1$. The base cases $n \in \{1,2\}$ are clear as $S \neq \emptyset$. For the induction step, assume that $n \geq 3$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a sub-hypercube of rank $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty. Since every infeasible component of both S_0, S_1 is a sub-hypercube, we may apply the induction hypothesis. Thus, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$, so by the induction hypothesis, each S_i is either a sub-hypercube of rank $n-2$ or the union of antipodal sub-hypercubes of rank $n-3$. If one of S_0, S_1 is a sub-hypercube, then as every infeasible component of S is a sub-hypercube, S is either a sub-hypercube of rank $n-1$ or the union of antipodal sub-hypercubes of rank $n-2$. Otherwise, each one of S_0, S_1 is the union of two antipodal sub-hypercubes of rank $n-3$. As S has no $R_{1,1}$ restriction, it must be that $S_0 = S_1$, implying in turn that S is the union of antipodal sub-hypercubes of rank $n-2$, thereby completing the induction step. □

We are now able to prove Theorem 8.5, stating that every ± 1-resistant set is strictly polar:

Theorem 8.5. *A ± 1-resistant set is strictly polar.*

Proof. Take an integer $n \geq 1$ and a ± 1-resistant set $S \subseteq \{0,1\}^n$. Then by Theorem 3.4, either

(i) $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2,\dots,n\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3,\dots,n\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3,\dots,n\}$,

(v) S is a sub-hypercube, or

(vi) every infeasible component of S is a sub-hypercube, and every feasible point has at most two infeasible neighbors.

Observe that $\{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\}$ and C_8 are strictly polar sets. As a result, in cases (i)-(iv), the set S is strictly polar by Remark 8.3. A sub-hypercube is strictly polar, so in case (v), S is strictly polar. For the last case (vi), as S is ± 1-resistant it has no $R_{1,1}$ restriction by Remark 2.3, so Lemma 8.4 implies that S is strictly polar, as required. □

As a consequence,

Corollary 8.6. *The cuboid of a* ± 1*-resistant set has the max-flow min-cut property.*

Proof. This follows from Theorem 8.2 and Theorem 8.5. □

Theorem 1.2 follows immediately.

Acknowledgements

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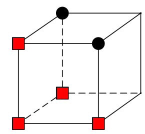
Special operations

$$S \triangle y = \{x \triangle y : x \in S\}$$
$$S_1 \times S_2 = \{(x, y) : x \in S_1, y \in S_2\}$$

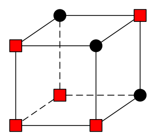
Special sets

$$R_{1,1} = \{000, 101, 110, 011\}$$
$$P_3 = \{110, 101, 011\}$$
$$S_3 = \{110, 101, 011, 111\}$$
$$C_8 = \{0000, 1000, 0100, 1010, 0101, 0111, 1111, 1011\}$$
$$F_1 = \{000, 100, 010, 111\}$$
$$F_2 = \{000, 100, 010, 001, 111\}$$
$$F_3 = \{000, 100, 010, 001, 110\}$$
$$D_3 = \{000, 100, 010, 101\}$$
$$D_3^\star = \{010, 011, 111, 101\}$$
$$D_4' = \{0100, 0110, 1011, 1111\}$$
$$H_1 = \{100, 010, 101, 011\}$$
$$H_2 = \{100, 010, 101, 011, 110\}$$
$$H_2^\star = \{100, 010, 101, 011, 111\}$$
$$H_3 = \{100, 010, 101, 011, 110, 111\}$$
$$A_k = \{\mathbf{0}, \mathbf{1}\} \subseteq \{0,1\}^k \quad k \geq 2$$
$$B_k = \{\mathbf{0}, e_1, \mathbf{1}\} \subseteq \{0,1\}^k \quad k \geq 3$$
$$D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k \quad k \geq 4$$
$$D_k^\star = D_k \triangle e_k \quad k \geq 4$$
$$F_k = \{\mathbf{0}, e_1, e_2, e_1 + e_2, \mathbf{1} - e_1 - e_2\} \subseteq \{0,1\}^k \quad k \geq 4$$

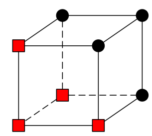
Figures of special sets



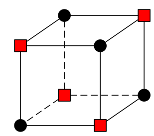
A fragile set



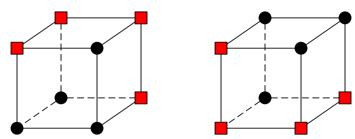
P_3



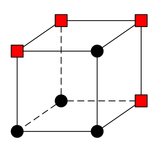
S_3



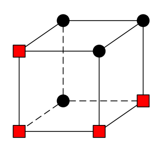
$R_{1,1}$



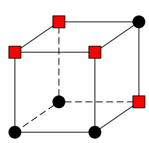
C_8



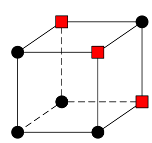
D_3



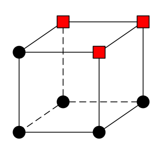
$D_3^\star$



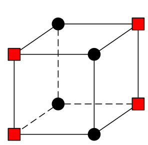
F_1



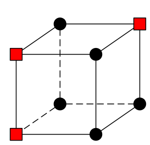
F_2



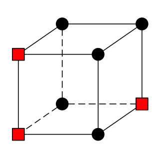
F_3



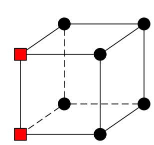
H_1



H_2



$H_2^\star$



H_3