Science Front Publishers *Journal for Foundations and Applications of Physics*, vol. 7, No. 1 (2020)
(sciencefront.org) ISSN 2394-3688

A Nonlinear Approach for Quantum Mechanics

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(Received 24 July 2020, Accepted 19 September 2020, Published 29 September 2020)

Abstract

This work represents a possible way to achieve the Einstein-de Broglie soliton-particle concept. The weakly nonlinear Klein-Gordon equation (nonlinear quantum mechanics) is investigated by the asymptotic perturbation (AP) method for a particle confined in a box. The quantization of the energy with a slight difference with respect to the standard (linear) quantum mechanics is obtained. Both relativistic and non-relativistic cases are considered and the transition frequencies are slightly different for the linear and nonlinear quantum mechanics. Experimental verification is needed to choose between the two theories.

Keywords: Nonlinear field theory, Soliton, Quantum mechanics

1. Introduction

Solitons, i.e. exponentially localised nonlinear waves, are exact solutions for many nonlinear partial differential equations (NPDEs) in 1+1 dimensions [1-3]. However, it has been demonstrated by the asymptotic perturbation (AP) method (a perturbation method based on Fourier expansion and spatio-temporal rescalings) [4-6] that solitons and dromions, their counterpart in 2+1 dimensions, appear as approximate solutions in physically relevant systems [7-9]. The AP method can be applied to the description of soliton interactions in nonlinear dispersive media without using the complexity of the inverse scattering method.

In the Einstein-de Broglie soliton-particle theory [10], localized nondispersive waves with finite energy can be then interpreted as images of extended particles with relativistic energy and momentum satisfying quantum wave-particle relations

$$E = \hbar\omega, \quad \lambda = h/p \tag{1}$$

between particle and wave properties. Energy E and momentum p of the soliton are then connected by the quantum relations (1) to the (circular) frequency ω and wavelength λ of the carrier wave.

According to this concept, a particle corresponds to a localized regular solution of a nonlinear field equation. It is well known that de Broglie with his famous "theory of double solution" made an attempt to give a causal interpretation of the quantum mechanics and to represent the electron as a localized solution, which obeys a nonlinear equation. Far from the particle, the nonlinear equation must reduce to the linear equations of quantum mechanics, for example to the Klein-Gordon equation if the spin is neglected. This opinion about quantum mechanics is very similar to Einstein's ideas according to which particles are clots of some material fields satisfying the nonlinear field equations. The wave-particle dualism is then interpreted in the framework of a classical nonlinear field model.

Obviously much work is needed to develop the Einstein-de Broglie soliton-particle theory in the framework of the approximate solutions of nonlinear field models. However, the role of nonlinearity in the formation of extended particles is essential. Moreover, it is interesting to observe that during the period 1955-1963 Skyrme proposed a model of baryons as topological solitons. His model could describe a stable extended particle with a unit topological charge and all finite dynamical characteristics. [11]

In previous works we have demonstrated that envelope solitons in the weakly nonlinear Klein-Gordon equation (nonlinear quantum mechanics) propagate with the group velocity of the carrier wave [12]. Subsequently we have investigated nonlinear vectorial relativistic fields [13], symmetry hidden models [15] and the nonlinear Dirac equation [14]. Also in these cases dromions correspond to particles and moreover chaotic and fractal solutions are possible.

In this paper we seek approximate analytic solutions for the weakly nonlinear Klein-Gordon equation in a box (a particle in a infinitely deep potential) and then the relevant Lorentz invariant partial differential equation is

$$\frac{\partial^2 u(\underline{X},t)}{c^2 \partial t^2} - \Delta u(\underline{X},t) + a u(\underline{X},t) + \lambda u^3(\underline{X},t) = 0, \qquad (2)$$

$u(\underline{X},t)=0$ for $0 \le x, y, z \le L$ and where a is the coefficient of the dispersive term and λ is the parameter of cubic nonlinearity. The particle is constrained to move in a box of side L. Weak nonlinearity induces a slow variation of the amplitudes of the linear dispersive waves and our aim is to derive the nonlinear equations that describe such evolution, obviously in appropriate "slow" and "coarse-grained" variables defined by equations (3). Since the amplitude of Fourier modes are not constant, higher order harmonics appear and Δ stands for the Laplacian. The linear part of the equation (2), i.e. $\varepsilon = 0$, is satisfied by a linear superposition of plane waves Fourier modes

$$A \exp\left[i(\underline{K}\,\underline{X} - \omega t)\right], \qquad (3)$$

with dispersion relation $\omega = \omega(K)$, group velocity V_G and phase velocity V_F given by

$$\omega = c\sqrt{K^2 + a}, \quad V_G = \frac{\underline{K} c^2}{\omega}, \; V_F = \frac{\omega}{\underline{K}}. \qquad (4)$$

The AP method provides a systematic means to obtain increasingly accurate solution by increasing the order of approximation in terms of a small parameter ε. This is accomplished by three processes: obtaining the form of solution in term of harmonic components, introducing a slow time scale and solving directly for the various harmonic components via harmonic balance.

In Section 2 we calculate the lowest order non linear approximate solution for the non relativistic case and demonstrate that the amplitude modulation of the nonlinear waves is described by the nonlinear Schrodinger (NLS) equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + \frac{\varepsilon\hbar^2}{2m}|\psi|^2\psi = 0,$$

$\psi(\underline{X},t)=0$ for $0\le x, y, z\le L$ where ψ is the amplitude of the Fourier mode (3). The approximate solution for the nonlinear equation is given by the product of two parts, the carrier wave ($\exp[i(\underline{K}\cdot\underline{X}-\omega t)]$) and the localized particle-like function ψ . In this way we have obtained a classical solution to the wave-particle dualism. According to the Einstein-de Broglie soliton-particle concept, we must find a link between particle and wave properties.

We find the energy levels and transition frequencies that present slight differences with the standard (linear) quantum mechanics. In Sec. 3 we consider the relativistic case and discuss possible experimental verification. Finally, in the last section the most important findings are recapitulated and some possible extensions of the present work indicated.

2. The non relativistic case

We express the solution $u(x,t)$ of equation (2) in the following form

$$u = \psi \exp(-i\frac{mc^2}{\hbar}t), \tag{6}$$

where m is the particle mass and $\hbar$ the Planck constant. Inserting the expression (6) into equation (2) yields in the non relativistic limit

$$-2i\frac{m}{\hbar}\psi_t - \psi_{xx} + \lambda|\psi|^2\psi + (a - \frac{m^2c^2}{\hbar})\psi = 0 \tag{7}$$

and if we choose

$$a = \frac{m^2c^2}{\hbar} \tag{8}$$

we arrive at the nonlinear Schrodinger equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + \lambda\frac{\hbar^2}{2m}|\psi|^2\psi \tag{9}$$

If we set

$$\psi'(\underline{X},t) = \psi(\underline{X})\exp(-i\frac{E}{\hbar}t), \quad K^2 = \frac{2mE}{\hbar^2}, \tag{10}$$

and for simplicity drop the apex, we obtain

$$E\psi = -\frac{\hbar^2}{2m}\Delta\psi + \lambda\frac{\hbar^2}{2m}|\psi|^2\psi. \tag{11}$$

Now we use the asymptotic perturbation (AP) method and with the substitutions $\psi\to\varepsilon\psi$ transform the equation (11) into

$$E\psi = -\frac{\hbar^2}{2m}\Delta\psi + \lambda\varepsilon^2 \frac{\hbar^2}{2m}|\psi|^2\psi \, , \tag{12}$$

where ε is a bookkeeping device that will be set to unity in the final analysis.
In the following the calculations are in one spatial dimension. At the end of the Section we show the results for the general case in 3+1 dimensions.
We introduce the spatial variable change

$$\xi = \varepsilon X \, . \tag{13}$$

Weak nonlinearity induces a slow variation of the amplitudes of the linear dispersive waves and our aim is to derive the nonlinear equations that describe such evolution, obviously in appropriate "slow" and "coarse-grained" variable defined by equations (13). The modulation of the amplitudes is best described in terms of the rescaled variable ξ , that account for the need to look on larger space and time scales, in order to obtain a significant contribution from the nonlinear term.

Since the amplitude of Fourier modes are not constant, higher order harmonics appear in the higher order perturbative solution.

We set

$$\psi(\underline{X}) = \sum_{n=-\infty}^{+\infty} \varepsilon^{r_n} \psi_n(\xi,\varepsilon) \exp\left(inKx\right), \tag{14}$$

where $r_n = |n| - 1$ for $n \neq 0$, $r_0 = 1$. For simplicity $\psi_1 = \varphi$ and $\psi_{-1} = \chi$ .The functions $\psi_n(\xi,\varepsilon)$ depend parametrically on □ and we assume that their limit for $\varepsilon \to 0$ exists and is finite.
Equation (11) can be written more explicitly in the following form

$$\psi = \varphi \exp(iKx) + \chi \exp(-iKx) + h.o.t. \tag{15}$$

where *h.o.t.*=higher order terms.
Note that the variable change (13) implies that differentiation with respect to the fast variable x must be substituted in the following way

$$\partial_x \to \varepsilon\partial_\xi + inK \, , \tag{16}$$

The variable change (13) permits to determine the asymptotic behavior of the solution, when the nonlinear effects can supply a not negligible contribution. The expansion of the solution (14) is used for the elimination of the predominant linear part of the equation (12) and it allows to calculate the possible interactions among the different harmonics, created by the nonlinear terms. Substituting (14) in equation (12) and considering the different equations obtained for every harmonic at the lowest order of approximation in ε, we obtain for $n=1$

$$2iK\varphi_\xi - 3\lambda\varphi^2\chi = 0 \, , \tag{17}$$

$$2iK\chi_\xi + 3\lambda\chi^2\varphi = 0 \, . \tag{18}$$

The validity of the approximate solution should be expected to be restricted on bounded intervals of the ξ -variable and on time-scale $X = O(\frac{1}{\varepsilon})$. If one wishes to study solutions on intervals such that $\xi = O(\frac{1}{\varepsilon})$ then the higher terms must be included in the system (17-18).

Using the *ansatz*

$$\varphi = \rho \exp(i\vartheta) \,,\; \chi = \mu \exp(i\alpha) \,, \tag{19}$$

equations (17-18) yield

$$2K\rho_\xi + 3\lambda \sin(\vartheta + \alpha)\rho^2 \mu = 0 \,, \tag{20}$$

$$2K\mu_\xi - 3\lambda \sin(\vartheta + \alpha)\rho\mu^2 = 0 \,, \tag{21}$$

$$2K\rho\vartheta_\xi + 3\lambda \cos(\vartheta + \alpha)\rho^2 \mu = 0 \,, \tag{22}$$

$$2K\mu\alpha_\xi - 3\lambda \cos(\vartheta + \alpha)\rho\mu^2 = 0 \,, \tag{23}$$

Equation (20-21) and equations (22-23) imply respectively that

$$(\rho\mu) = constant. \quad (\vartheta + \alpha) = constant. \tag{24}$$

To avoid unbounded solutions, we set

$$(\vartheta + \alpha) = 0 \text{ or } \pi \text{ and } \rho = constant, \mu = constant \tag{25}$$

and then

$$\vartheta = BX, \quad \alpha = -Bx + \delta, \quad \delta = \vartheta + \alpha, \quad B = -\frac{3\lambda\rho\mu}{2K}\cos(\vartheta + \alpha) \tag{26}$$

where δ is the phase difference between ϑ and α.

From the solution (15), we can write the approximate solution

$$\psi = \rho \exp(i(KX + BX)) + \chi \exp(-i(KX + BX - \delta)) \,. \tag{27}$$

Now we apply the contour condition and

$$\psi(0) = 0 \,,\; \psi(L) = 0 \,, \tag{28}$$

and obtain

$$2(B + K)L = 2n\pi \,,\; \rho = \chi, \quad \delta = \pi \,, \tag{29}$$

where *n* is the quantum number. From the normalization condition

$$\int_0^L |\psi|^2 dx = 1 \,,\; |\psi|^2 = \rho^2 + \chi^2 + 2\rho\chi \cos(2(Kx + Bx) - \delta) \,, \tag{30}$$

we get

$$B = \frac{3\lambda}{4(n\pi - BL)} \approx \frac{3\lambda}{4n\pi} \,, \quad \rho^2 = \frac{1}{2L} \,. \tag{31}$$

The first order approximate solution (27) becomes

$$\psi = i\sqrt{\frac{2}{L}}\sin(KX + BX)\,. \tag{32}$$

For the validity of the AP method (the spatial dimension of the box must be very large with respect the wavelength number K), it is requested that

$$L >> \frac{1}{K} = \frac{\hbar}{\sqrt{2mE}}\ , \qquad n >> \frac{1}{\pi}\,. \tag{33}$$

The energy levels are

$$E = \frac{\hbar^2}{2m}(\frac{n\pi}{L} - B)^2\,. \tag{34}$$

and the transition frequency between the level *n+1* and *n* is

$$\nu = \frac{h}{8\pi mL}(\frac{\pi}{L}(2n+1) - 2B) \tag{35}$$

For *B=0* we obtain the prevision of the standard quantum mechanics. The difference between the two frequencies (standard value minus the nonlinear prevision) is

$$\Delta\nu = \frac{hB}{4\pi mL} = \frac{3\Lambda c}{16n\pi^2 L}, \tag{36}$$

with a relative shift

$$\frac{\Delta\nu}{\nu} = \Lambda\frac{3L}{2n(2n+1)\lambda_C\pi^2}, \tag{37}$$

where

$$\lambda_C = \frac{h}{mc} \tag{38}$$

is the Compton wavelength and we have set for dimensional reasons

$$\lambda = \Lambda\frac{mc}{h} \tag{39}$$

and Λ is an a dimensional parameter of order one ($\Lambda \approx 1$). The frequency (35) can be also written as

$$\nu = \frac{c}{\lambda_T}, \qquad \lambda_T = \frac{8l^2\lambda_C}{2n+1-(\frac{3l\Lambda}{2\pi^2 n})}. \tag{40}$$

where λ_T is the transition frequency (not to confuse with λ the nonlinear parameter).

For the validity of the AP method is requested that

$$K >> B, \quad n^2 >> \frac{3l}{4\pi^2}, \quad l = L/\lambda_C\,. \tag{41}$$

For fixed *n* we get that the maximum value of the ratio *l* in (41) is

$$l_{MAX} = \frac{4n^2}{3} \tag{42}$$

and for a large value of relative shift (37) we insert (42) in (37) in order to obtain a maximum value for the relative shift and obtain

$$\frac{\Delta \nu}{\nu} = \frac{2n}{\pi^2(2n+1)} \tag{43}$$

The maximum value of the relative shift is *0.1* for a very large quantum number.

For example we consider an electron for *n=10* and $L=133\,\lambda_C$ (see (41)) , then the frequency (40) is $1.7\ 10^{16}$ *Hz (*wavelength $18\ 10^{-9}$ *m=18 nm* and photon energy near *0.069 Kev))* and the relative shift (43) is *0.097*.

The non relativistic approach is justified because for the velocity of the particle, we get

$$\beta = \frac{V}{c} = \frac{\sqrt{2E}}{c\sqrt{m}} = \frac{n\pi\hbar}{mcL} \approx 0.04 << 1, \tag{44}$$

for *n=10* and *l=133*. Now we return to the general case in 3+1 dimensions. Following the same reasoning, we find for the energy levels (particle in a cubic box of side *L*)

$$E = \frac{\hbar^2 K^2}{2m} = \frac{\hbar^2}{2m}\left[(\frac{n_y\pi}{L} - B_y)^2 + (\frac{n_y\pi}{L} - B_y)^2 + (\frac{n_y\pi}{L} - B_y)^2\right], \tag{45}$$

where n_x, n_y, n_z are three quantum numbers. With $\underline{B}$ =(B_x, B_y, B_z) we obtain

$$E = \frac{\hbar^2}{2m}\left[(n_x^2 + n_y^2 + n_z^2)\frac{\pi^2}{L^2} - 2\underline{K}\cdot\underline{B} + B^2\right] \tag{46}$$

where $\underline{K} = (K_x, K_y, K_z)$ and the dot stands for the scalar product. In the three dimensional case we find that

$$\lambda = \Lambda\frac{h}{mc}. \tag{47}$$

3. The relativistic case

In the relativistic case we start from equation (2) and set

$$u(\underline{X}, t) = \psi(\underline{X})\exp(-i\frac{E}{\hbar}t), \tag{48}$$

where

$$E = \hbar\omega = \sqrt{(pc)^2 + (mc^2)^2} \tag{49}$$

Equation (2) yields

$$a = \frac{E^2}{\hbar^2 c^2}, \tag{50}$$

and

$$E = c\sqrt{m^2c^2 + \hbar^2\left[(\frac{n_x\pi}{L} - B_x)^2 + (\frac{n_y\pi}{L} - B_y)^2 + (\frac{n_z\pi}{L} - B_z)^2\right]}, \tag{51}$$

or with $\underline{B}$ =(B_x , B_y , B_z)

$$E = c\sqrt{m^2c^2 + \hbar^2\left[(n_x^2 + n_y^2 + n_z^2)\frac{\pi^2}{L^2} - 2\underline{K}\cdot\underline{B} + B^2\right]} \quad , \tag{52}$$

and the transition frequencies are easily derived.

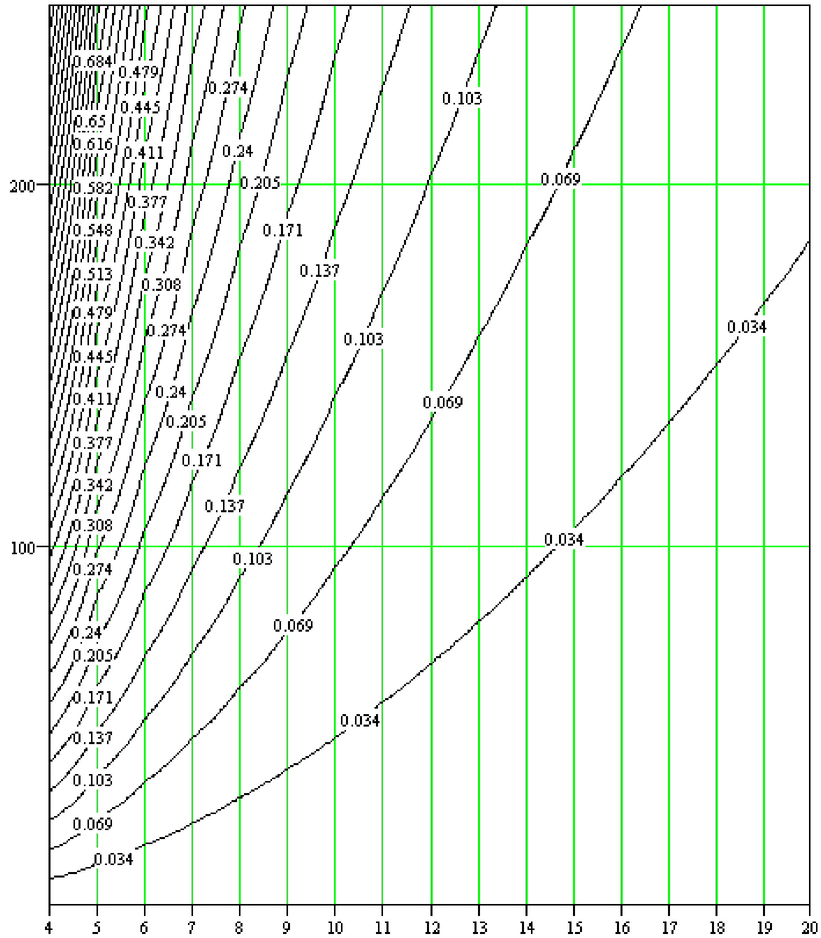

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Fig. 1: Contour plot for the relative shift (37) with X=quantum number *n* (from *4* to *20*)and Y=the ratio *l* (41) (from *1* to *500*) between the side of the box and the Compton wavelength. The numbers on the plot stand for relative shift (37) and we have set $\Lambda = 1$. In the regions where the value (37) exceeds *0.1* the AP method is not applicable, because the nonlinearity is too large (the ratio *l* exceeds the value (42)).

From (43) and (44) we see that the relativistic correction and nonlinear term are of the same order (about *1-10 %*) and we must include in (35) the relativistic correction that is

$$\Delta E = -\frac{p^4}{8m^3c^2} \, . \tag{53}$$

We easily find for the transition frequency between the level *n+1* and *n*

$$\nu_{QM} = \frac{c}{8\pi\lambda_C l}(\frac{\pi}{l}(2n+1)) - \frac{c}{128 l^4 \lambda_C}(4n^3+6n^2+4n+1), \qquad (54)$$

for the standard quantum mechanics with the relativistic correction and

$$\nu_{NLQM} = \frac{c}{8\pi\lambda_C l}(\frac{\pi}{l}(2n+1) - \frac{3\Lambda}{2\pi n}) - \frac{c}{128 l^4 \lambda_C}(4n^3+6n^2+4n+1) \qquad (55)$$

for the nonlinear quantum mechanics prevision, where $\Lambda \approx 1$. The difference between the two frequencies (standard value with the relativistic correction minus the nonlinear prevision) is always (36) with a relative shift given at the first order by (37).

In Fig. 1 we show a contour plot for the relative shift (37) with X=quantum number *n* (from *4* to *20*) and Y=the ratio *l* (41) (from *1* to *250*) between the side of the box and the Compton wavelength. The numbers on the plot stand for relative shift (37). In the regions where the value (37) exceeds *0.1,* the nonlinearity is too large (the ratio *l* exceeds the value (42)) and the AP method is not valid. For $l<l_{MAX}$ the previsions are more reliable, but the relative shift decreases.

4. Conclusion

We have then demonstrated that the Einstein-de Broglie particle-soliton concept can be derived from a perturbation analysis of a relativistic nonlinear field theory in 3+1 dimensions. We have used the AP method, based on spatial rescaling and harmonic balance, in order to construct approximate solutions for a weakly nonlinear Lorentz invariant complex scalar field theory in 3+1 dimensions. For the case of a particle confined in a box the approximate solution is given by a Fourier expansion in which the coefficients are power series of a small parameter and vary slowly in space. Substituting the expression of the solution into the original equation and projecting onto each Fourier mode, we have derived a nonlinear system of partial differential equations which describes the spatial evolution of the oscillation amplitudes of Fourier modes at the lowest order of approximation. We have obtained the quantization of the energy with a slight difference respect to the standard (linear) quantum mechanics both in the relativistic and non relativistic case. The transition frequencies are slightly different for the linear and nonlinear quantum mechanics. In this way we can experimentally check the validity of the nonlinear quantum mechanics observing transition frequencies with quantum number around *n=10* (box side around $L=125\lambda_C$).

The present work can be extended in various different ways:

i) higher order perturbation analysis in order to derive a more refined approximate solution and possible deviations from the evolution laws of quantum mechanics.

ii) inclusion of an external field to describe the interaction of the particle-like solutions, for example with the electromagnetic field.

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