



*Research article*

## Effective governing equations for heterogenous porous media subject to inhomogeneous body forces<sup>†</sup>

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**Abstract:** We derive a new homogenized model for heterogeneous porous media driven by inhomogeneous body forces. We assume that the *fine scale*, characterizing the heterogeneities in the medium, is larger than the *pore scale*, but nonetheless much smaller than the size of the material (*the coarse scale*). We decouple spatial variations and assume periodicity on the fine scale. Fine scale variations are formally reflected in a *locally unbounded* source for the arising system of partial differential equations. We apply the asymptotic homogenization technique to obtain a well-defined coarse scale Darcy-type model. The resulting problem is driven by an effective source which comprises both the coarse scale divergence of the average body force, and additional contributions which are to be computed solving a well-defined diffusion-type cell problem which is driven solely by fine scale variations of the given force. The present model can be used to predict the effect of externally applied magnetic (or electric) fields on ferrofluids (or electrolytes) flowing in porous media. This work can, in perspective, pave the way for investigations of the effect of applied forces on complex and heterogeneous hierarchical materials, such as systems of fractures or cancerous biological tissues.

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**Keywords:** heterogeneous porous media; asymptotic homogenization; locally unbounded source; ferrofluids; hierarchical materials

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## 1. Introduction

The flow of Newtonian fluids slowly percolating through a rigid porous matrix can be macroscopically described by the Darcy law, which linearly relates the fluid discharge to the pressure gradient (via the hydraulic conductivity tensor) and externally applied volume loads. Darcy's law can be derived by means of suitable upscaling techniques. It can be viewed as an approximation of the balance equations of linear momentum for a viscous fluid flowing in the pores in the context of the mixture theory, see, e.g., [29], as well as the two-scale asymptotic limit (when the average radius of the pores approaches zero sufficiently faster with respect to the whole domain's size) of the Stokes' problem for low Reynolds number fluids. The latter result can be shown both via formal asymptotic expansions of the fields, see, e.g., [11], and by a rigorous convergence approach as in [1, 12]. The homogenized differential problem for intrinsically incompressible fluids reads formally as a standard elliptic problem in terms of the fluid pressure, where the hydraulic conductivity plays the role of diffusion and the external fluid sources, together with the divergence of the average body forces, appear collectively as a driving volume source.

Although such a problem describes the *homogenized* fluid mechanics, i.e., where pore-scale inhomogeneities are smoothed out, the hydraulic conductivity and the effective body force can in turn exhibit heterogeneities on well-separated *fine* and *coarse* spatial scales, the former being much larger than the pore scale and, at the same time, much smaller than the coarse one characterizing the size of the whole domain. Relevant examples involving hierarchical porous materials and thus upscaling of the Darcy law itself can be found in the context of biological tissues, such as vascularized (growing) tumors (see, e.g., [13, 21, 22, 24, 37]), as well as fractured rocks exhibiting different levels of porosity, as in [3].

Here we embrace the asymptotic homogenization technique to derive the effective governing equations for a heterogeneous porous media governed by the Darcy law and subject to an inhomogeneous body force, ignoring external mass supply and assuming incompressibility of the fluid phase. Therefore, the system of partial differential equations (PDEs) under consideration is a diffusion-type problem in terms of the pressure driven by an inhomogeneous volume source given by the divergence of the body force. Thus, whenever both fine and coarse scale variations of the fields are considered, the latter source is *locally unbounded* when the ratio between scales approaches zero. We decouple spatial scales and formally proceed employing power series expansions of the pressure and velocity fields in terms of the ratio between the two length scales, and further assume fine scale periodicity. We derive a well-defined effective model which describes the coarse scale behavior of the medium.

The new feature of this work resides in the volume load which drives the coarse scale fluid flow, whose coarse scale divergence appears as a heterogeneous source for the homogenized system of elliptic PDEs.

The above mentioned source involves both the cell average of the body force, and an additional

contribution that is related to the solution of a diffusion-type periodic cell problem driven solely by fine scale variations of the given force. Whenever the latter does not depend on the fine scale, the additional contribution vanishes and the resulting formulation coincides with the homogenized problem for diffusion, see, e.g., [4, 7, 19].

We illustrate the potential of the presented results in investigating the action of inhomogeneous body forces, such as those arising from the application of electromagnetic fields, on magnetorheological fluids and electrolytes percolating heterogeneous tissues. The remainder of the work is organized as follows. In Section 2 we introduce the Darcy equation for the medium, assuming that both the body force and the hydraulic conductivities are spatially heterogeneous. In Section 3 we introduce the asymptotic homogenization technique and decouple spatial variations into the fine scale, which characterizes the material heterogeneities, and the coarse scale, which represents the average size of the domain. In Section 4 we present the new coarse scale model. In Section 5 we discuss the results and highlight relevant physical scenarios of interest. In Section 6 we present our conclusions and discuss further development of the work.

## 2. Darcy's law for inhomogeneous body forces

We consider a heterogeneous porous medium identified with an open and bounded domain  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ , made of  $N$  non-intersecting inclusions\* such that

$$\bar{\Omega} = \bigcup_{\alpha=0}^N \bar{\Omega}_\alpha = \bar{\Omega}_0 \cup \bar{\Omega}_I; \quad \Omega_I = \bigcup_{\beta=1}^N \Omega_\beta, \quad (2.1)$$

where  $\alpha \in \{0, 1, 2, \dots, N\}$ ,  $\beta \in \{1, 2, \dots, N\}$ . The domain  $\Omega_0$  represents the host (matrix) phase, while  $\Omega_I$  represents the union of the  $N$  inclusions. From now on, relationships where the indices  $\alpha$  and  $\beta$  appear are meant to hold for every  $\alpha \in \{0, 1, 2, \dots, N\}$  and  $\beta \in \{1, 2, \dots, N\}$ , respectively. We assume that each inclusion of the medium is characterized by a symmetric and positive definite hydraulic conductivity tensor  $\mathbf{K}_\alpha(\mathbf{x})$ ,  $\mathbf{x} \in \Omega_\alpha$ , that is

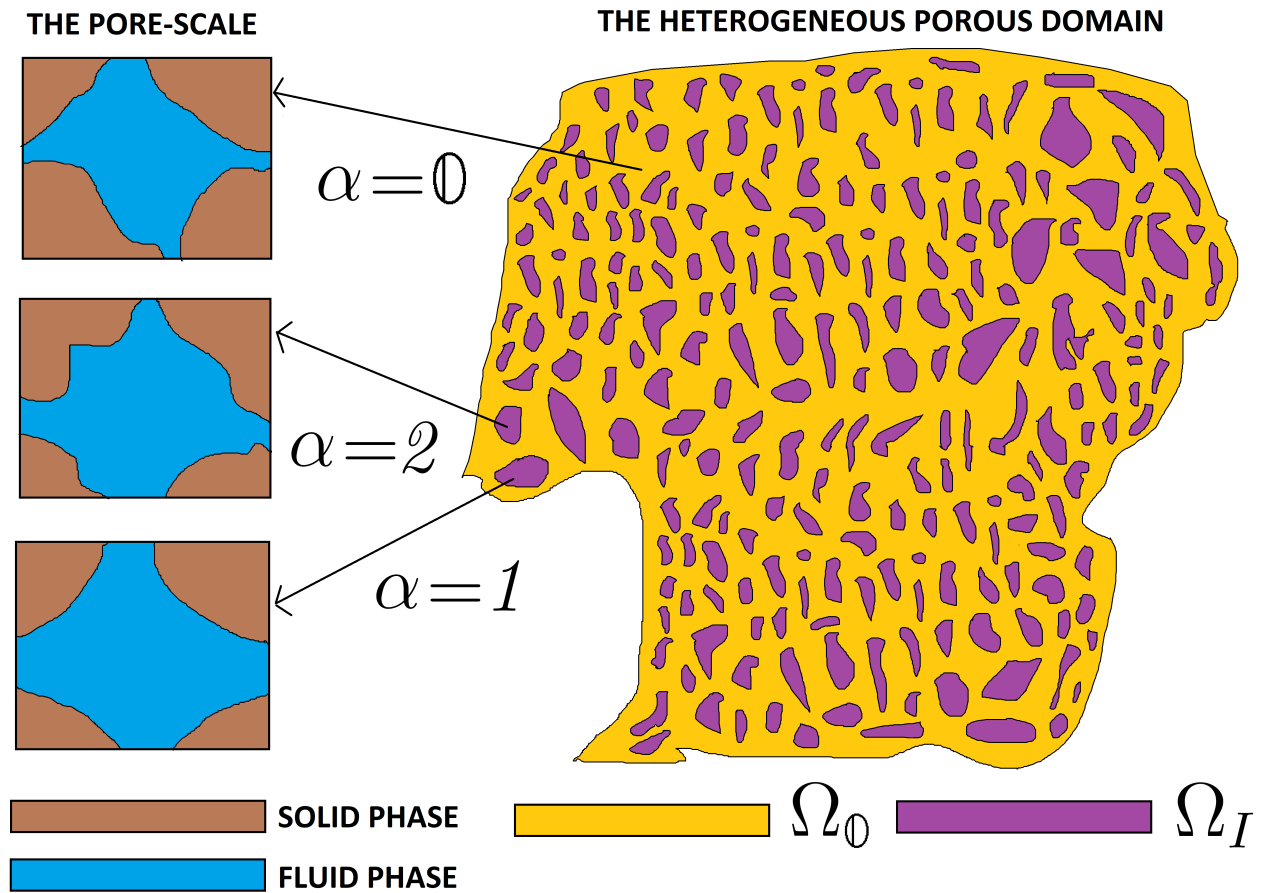
$$\mathbf{K}_\alpha = \mathbf{K}_\alpha^\top \quad \text{and} \quad \mathbf{a} \mathbf{K}_\alpha \mathbf{a} > 0 \quad \forall \mathbf{a} \neq \mathbf{0}, \quad (2.2)$$

where  $\mathbf{a}$  is a constant vector and that the fluid flow is governed by Darcy's law in each  $\Omega_\alpha$ , i.e.,

$$\mathbf{u}_\alpha = -\mathbf{K}_\alpha(\mathbf{x}) \left( \nabla p_\alpha(\mathbf{x}) + \hat{\mathbf{f}}_\alpha(\mathbf{x}) \right). \quad (2.3)$$

It is worth remarking that, as our starting point is a Darcy's representation (2.3), the pore structure (made of a fluid and a solid phase) is considered already smoothed out, and details concerning pore-scale geometries and porosities of each phase are effectively encoded in the hydraulic conductivity tensors  $\mathbf{K}_\alpha$ . A sketch of the heterogeneous porous domain in two dimensions is provided in Figure 1, where it is also emphasized that each inclusion and the matrix are in principle characterized by different pore-structures, and consequently, by a different hydraulic conductivity tensor.

\*The whole formulation actually holds for a generic combination of inclusions, fibers, and strata. Nonetheless, we refer to inclusions only here and thereafter for the sake of clarity of exposition.



**Figure 1.** A 2D sketch of the heterogeneous porous domain (right hand side), vs sample pore-scale structures characterising the matrix and the inclusions (left hand side). As we embrace Darcy's law as a starting point, heterogeneities at the pore-scale are smoothed out, and their properties are encoded in the heterogeneous hydraulic conductivities  $K_\alpha(\mathbf{x})$ . However, the difference in hydraulic properties between different inclusions and the matrix is still clearly resolved at this stage.

Here, for every inclusion  $\Omega_\alpha$ ,  $\mathbf{u}_\alpha$  represents the fluid velocity<sup>†</sup>,  $p_\alpha$  the interstitial pressure and  $\hat{\mathbf{f}}_\alpha$  formally plays the role of the body force effectively acting on the fluid flowing in the pores characterizing each individual inclusion. We further assume that the fluid flow is intrinsically incompressible, so that ignoring external volume sources, we have

$$\nabla \cdot \mathbf{u}_\alpha = 0, \quad \mathbf{x} \in \Omega_\alpha. \quad (2.4)$$

Substituting Darcy's law (2.3) in the divergence-free constraint (2.4), we obtain the differential problem in terms of the pressure in each inclusion and the matrix, i.e.,

$$\nabla \cdot (K_\alpha(\mathbf{x}) \nabla p_\alpha(\mathbf{x})) = \nabla \cdot \mathbf{f}_\alpha(\mathbf{x}), \quad \mathbf{x} \in \Omega_\alpha, \quad (2.5)$$

where we set

$$\mathbf{f}_\alpha = -K \hat{\mathbf{f}}_\alpha. \quad (2.6)$$

<sup>†</sup>It should be in principle multiplied by the porosity, however, the latter can be incorporated in the definition of each  $K_\alpha$ .

The differential problem (2.5), once equipped with appropriate external boundary conditions on  $\partial\Omega$ , as well as interface conditions across the boundary between phases, represents a classical diffusion-type elliptic system of PDEs in the unknowns  $p_\alpha$ . The right hand sides formally play the role of given sources in terms of the divergence of the body force  $\mathbf{f}_\alpha$ . Furthermore, the interface conditions for every  $\beta = 1, 2 \dots N$ , are prescribed as follows:

$$p_\beta = p_0 \quad \text{on } \Gamma_\beta, \quad (2.7)$$

$$\mathbf{u}_\beta \cdot \mathbf{n}_\beta = \mathbf{u}_0 \cdot \mathbf{n}_\beta \quad \text{on } \Gamma_\beta, \quad (2.8)$$

where we set  $\Gamma_\beta := \partial\Omega_0 \cap \partial\Omega_\beta$ , and  $\mathbf{n}_\beta$  is the unit outward vector normal to the interface  $\Gamma_\beta$ . Relationships (2.7) and (2.8) are standard interface conditions in the context of porous media flow, and represent the continuity of pressures and the (local) continuity of fluid fluxes.

The latter conditions can be rewritten in terms of the pressures only by exploiting Darcy's law (2.3) and (2.6), i.e.,

$$\left(-\mathbf{K}_\beta \nabla p_\beta + \mathbf{f}_\beta\right) \cdot \mathbf{n}_\beta = \left(-\mathbf{K}_0 \nabla p_0 + \mathbf{f}_0\right) \cdot \mathbf{n}_\beta \quad \text{on } \Gamma_\beta. \quad (2.9)$$

In the next section, we apply the asymptotic homogenization technique to equations (2.5), (2.7), and (2.9) to obtain the effective system of PDEs for a heterogeneous porous medium subject to an inhomogeneous body force.

### 3. The asymptotic homogenization technique

We assume that the scale at which the inclusions are clearly resolved (which could be identified as the maximum of the distances between adjacent inclusions),  $d$ , that is the *fine scale*, and the *coarse scale*  $L$ , which here represents the size of the whole domain  $\Omega$ , are well-separated. We then define their ratio  $\epsilon$  as follows:

$$\frac{d}{L} = \epsilon \ll 1. \quad (3.1)$$

**Remark 1** (A diffusion problem driven by a locally unbounded source). *We are interested in emphasizing the role of fine scale spatial heterogeneities that are resolved on  $\mathbf{x}/\epsilon$ . In particular, both hydraulic conductivities and body forces are heterogeneous and therefore we consider a spatial dependence as given below*

$$\mathbf{K}_\alpha(\mathbf{x}) = \mathbf{K}_\alpha(\mathbf{x}, \mathbf{x}/\epsilon), \quad \mathbf{f}_\alpha(\mathbf{x}) = \mathbf{f}_\alpha(\mathbf{x}, \mathbf{x}/\epsilon). \quad (3.2)$$

*The differential problem given by equations (2.5), (2.7), and (2.9) formally reads as a standard, well-posed, diffusion-type problem driven by heterogeneous sources given by the divergence of regular body forces  $\mathbf{f}_\alpha$ , supplemented by the continuity of pressures and local fluxes conditions at the interface between phases. However, the source term comprises spatial derivatives of the body force and therefore is  $O(1/\epsilon)$  whenever spatial variations on  $\mathbf{x}/\epsilon$  are taken into account. In this sense, the problem (2.5) can be formally viewed as diffusion problem driven by a locally unbounded source, which is well-defined for every  $\mathbf{x} \in \Omega$ , but asymptotically behaving as  $1/\epsilon$  when assuming a representation of the type (3.2). The systems of PDEs given by (2.5) happens nonetheless to be homogenizable when assuming fine scale periodicity, as shown in the following sections.*

Enforcing condition (3.1), we decouple spatial scales by introducing a new local variable as follows:

$$\mathbf{y} := \frac{\mathbf{x}}{\epsilon}, \quad (3.3)$$

where from now on  $\mathbf{x}$  and  $\mathbf{y}$  represent the coarse and fine scale spatial coordinates, respectively, and are to be formally considered independent variables. We note that the fields  $p_\alpha$  and  $\mathbf{u}_\alpha$ , as well as the hydraulic conductivities  $K_\alpha$  and the inhomogeneous body forces  $\mathbf{f}_\alpha$ , are assumed to be functions of both  $\mathbf{x}$  and  $\mathbf{y}$ .

We now employ a power series representation of the variables  $\mathbf{u}_\alpha$  and  $p_\alpha$ , as well as the given forces  $\mathbf{f}_\alpha$ , in terms of the small parameter  $\epsilon$ , namely

$$\mathbf{u}_\alpha(\mathbf{x}, \mathbf{y}) = \mathbf{u}_\alpha^\epsilon(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \mathbf{u}_\alpha^{(l)}(\mathbf{x}, \mathbf{y}) \epsilon^l, \quad (3.4)$$

$$p_\alpha(\mathbf{x}, \mathbf{y}) = p_\alpha^\epsilon(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} p_\alpha^{(l)}(\mathbf{x}, \mathbf{y}) \epsilon^l, \quad (3.5)$$

$$\mathbf{f}_\alpha(\mathbf{x}, \mathbf{y}) = \mathbf{f}_\alpha^\epsilon(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \mathbf{f}_\alpha^{(l)}(\mathbf{x}, \mathbf{y}) \epsilon^l. \quad (3.6)$$

We emphasize that, in general, even though the forces  $\mathbf{f}_\alpha$  are supposed to be given and not dependent on pressures and velocities, they could in principle be prescribed as being related to the solution of another multiscale problem where power series representations are exploited. Therefore, in the most general case we assume that (3.6) holds, and whenever the forces are not related to a multiscale problem and are simply prescribed, then we can simply set  $\mathbf{f}_\alpha^\epsilon = \mathbf{f}_\alpha^{(0)}$ .

Substituting the power series representations (3.4–3.6) into the system of PDEs given by (2.5), (2.7), and (2.9), we obtain the following multiscale system of PDEs in terms of  $\mathbf{u}_\alpha^\epsilon$  and  $p_\alpha^\epsilon$ , driven by the given body forces  $\mathbf{f}_\alpha$ .

$$\begin{aligned} & \epsilon^2 \nabla_{\mathbf{x}} \cdot (K_\alpha(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{x}} p_\alpha^\epsilon(\mathbf{x}, \mathbf{y})) + \epsilon \nabla_{\mathbf{x}} \cdot (K_\alpha(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} p_\alpha^\epsilon(\mathbf{x}, \mathbf{y})) + \\ & \epsilon \nabla_{\mathbf{y}} \cdot (K_\alpha(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{x}} p_\alpha^\epsilon(\mathbf{x}, \mathbf{y})) + \nabla_{\mathbf{y}} \cdot (K_\alpha(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} p_\alpha^\epsilon(\mathbf{x}, \mathbf{y})) = \\ & \epsilon^2 \nabla_{\mathbf{x}} \cdot \mathbf{f}_\alpha^\epsilon(\mathbf{x}, \mathbf{y}) + \epsilon \nabla_{\mathbf{y}} \cdot \mathbf{f}_\alpha^\epsilon(\mathbf{x}, \mathbf{y}) \quad \text{in } \Omega_\alpha, \end{aligned} \quad (3.7)$$

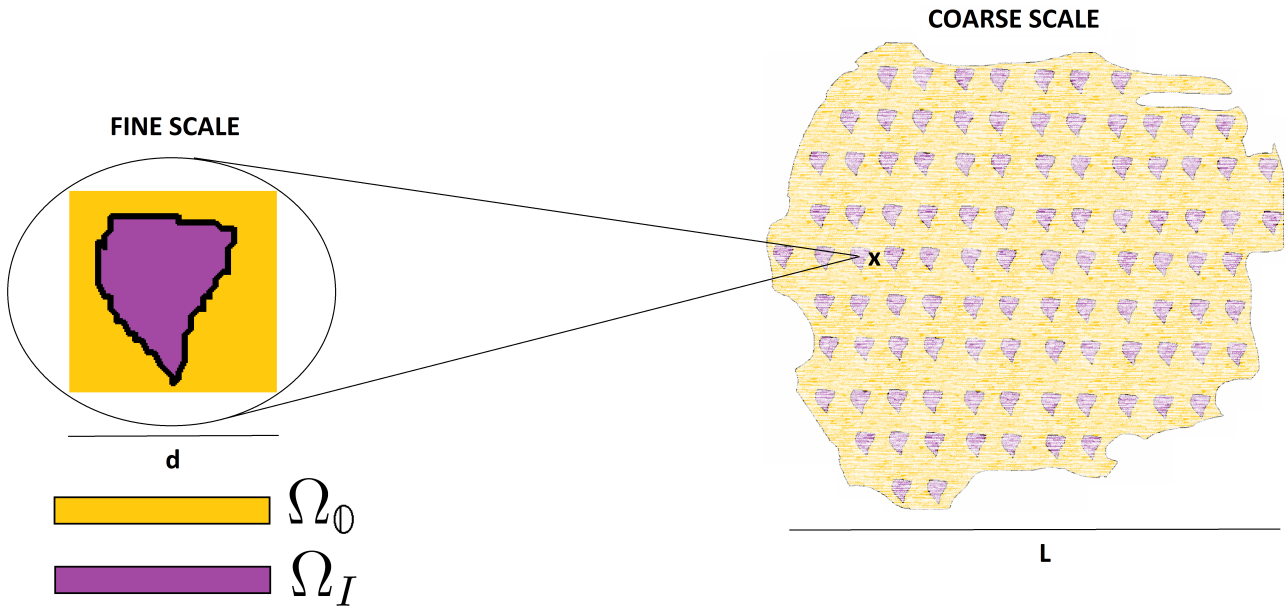
$$\left( -K_\beta \nabla_{\mathbf{y}} p_\beta^\epsilon - \epsilon K_\beta \nabla_{\mathbf{x}} p_\beta^\epsilon + \epsilon \mathbf{f}_\beta^\epsilon \right) \cdot \mathbf{n}_\beta = \left( -K_0 \nabla_{\mathbf{y}} p_0^\epsilon - \epsilon K_0 \nabla_{\mathbf{x}} p_0^\epsilon + \epsilon \mathbf{f}_0^\epsilon \right) \cdot \mathbf{n}_0 \quad \text{on } \Gamma_\beta, \quad (3.8)$$

$$p_\beta^\epsilon = p_0^\epsilon \quad \text{on } \Gamma_\beta, \quad (3.9)$$

where we have applied the chain rule according to the spatial scale decoupling (3.3), i.e.,

$$\nabla \rightarrow \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{y}}. \quad (3.10)$$

The operators  $\nabla_{\mathbf{x}}$  and  $\nabla_{\mathbf{y}}$  represent the gradient with respect to the coarse scale and fine scale variables  $\mathbf{x}$  and  $\mathbf{y}$ , respectively.



**Figure 2.** A 2D representation of the fine and coarse scale geometries. On the right hand side, the coarse scale domain where the matrix and the inclusions cannot be distinguished is shown. On the left hand side, the periodic unit which is representative of the fine scale is shown.

**Remark 2** (Assumptions concerning the geometry).

#### A- Local periodicity

We assume that the field  $p_\alpha$  and  $\mathbf{u}_\alpha$ , as well as the hydraulic conductivity tensor  $\mathbf{K}_\alpha$  and the body force  $\mathbf{f}_\alpha$  are  $\mathbf{y}$ -periodic. This assumption permits us to study fine scale variations of the fields on a restricted portion of the domain. In particular, we identify  $\Omega$  with the corresponding periodic cell.

#### B- Macroscopic uniformity

Furthermore, we also account for macroscopic uniformity, that is, we neglect geometrical variations of the cell and inclusions with respect to the coarse scale variable  $\mathbf{x}$  (see, e.g., [6, 11, 21, 22]) for the sake of simplicity. As such, it is sufficient to consider only one periodic cell for every macroscale point  $\mathbf{x}$  to fully represent the fine scale structure, and, since  $\Omega_\alpha$  does not depend on the macroscale  $\mathbf{x}$ , we have

$$\nabla_{\mathbf{x}} \cdot \int_{\Omega_\alpha} (\bullet) \, d\mathbf{y} = \int_{\Omega_\alpha} \nabla_{\mathbf{x}} \cdot (\bullet) \, d\mathbf{y}. \quad (3.11)$$

#### C- A two-phase problem

Finally, without loss of generality and for the sake of ease of presentation only, we present the derivation that follows by focusing on one inclusion only within the cell, i.e.,  $\alpha = 0, 1, \beta = 1$ . As such, there is only one inclusion in periodic cell  $\Omega$ , so that

$$\Omega_I = \Omega_I, \quad \text{and} \quad \bar{\Omega} = \bar{\Omega}_0 \cup \bar{\Omega}_I, \quad (3.12)$$

where  $\Omega_0$  and  $\Omega_I$  represent the host matrix and inclusion cell portions, respectively, see Figure 2 for a two-dimensional sketch of the fine scale vs the coarse scale when assuming assumptions (A – C). We also set  $\Gamma := \Gamma_I$ ,  $\mathbf{n} := \mathbf{n}_I$ , and use the index  $I$  instead of  $I$  for quantities defined in the inclusion within the cell, cf. (3.12).

In the following section we embrace the assumptions (A – C) illustrated in the above Remark 2, and derive the leading order coarse-scale governing equations in terms of the zero-th order pressures and velocities via the upscaling of the multiscale system of PDEs (3.7–3.9). This is done by exploiting the differential equations arising from equating the same powers of  $\epsilon^l$  for  $l = 0, 1, 2$ . The dependency of pressures, velocities, hydraulic conductivities, and body forces on both  $\mathbf{x}$  and  $\mathbf{y}$  is to be understood unless otherwise specified.

#### 4. The coarse-scale Darcy's-type model

We now equate the same powers of  $\epsilon$  in ascending order, starting from  $\epsilon^0$ .

$\epsilon^0$  Coarse scale leading order pressure

Equating the same powers of  $\epsilon^0$  in (3.7–3.9) yields

$$\nabla_{\mathbf{y}} \cdot (\mathbf{K}_0 \nabla_{\mathbf{y}} p_0^{(0)}) = 0 \text{ in } \Omega_0, \quad (4.1)$$

$$\nabla_{\mathbf{y}} \cdot (\mathbf{K}_I \nabla_{\mathbf{y}} p_I^{(0)}) = 0 \text{ in } \Omega_I, \quad (4.2)$$

$$\mathbf{K}_I \nabla_{\mathbf{y}} p_I^{(0)} \cdot \mathbf{n} = \mathbf{K}_0 \nabla_{\mathbf{y}} p_0^{(0)} \cdot \mathbf{n} \text{ on } \Gamma, \quad (4.3)$$

$$p_I^{(0)} = p_0^{(0)} \text{ on } \Gamma. \quad (4.4)$$

Relationships (4.1–4.4), equipped with periodicity conditions on  $\partial\Omega$ , form a standard diffusion-type cell problem that admits a unique solution up to a  $\mathbf{y}$ -constant function. In particular, as any continuous (by means of (4.4))  $\mathbf{y}$ - constant function is also periodic and solves (4.1–4.4), we deduce that

$$p_I^{(0)} = p_0^{(0)} = p^{(0)}(\mathbf{x}). \quad (4.5)$$

That is, the leading order pressure  $p^{(0)}$  is continuous in  $\Omega$  and independent of  $\mathbf{y}$ .

$\epsilon^1$  Fine-scale cell problems

We equate the same powers of  $\epsilon^1$  in (3.7–3.9) and account for (4.5) to obtain

$$\nabla_{\mathbf{y}} \cdot (\mathbf{K}_0 \nabla_{\mathbf{y}} p_0^{(1)}) = -\nabla_{\mathbf{y}} \cdot (\mathbf{K}_0 \nabla_{\mathbf{x}} p^{(0)}) + \nabla_{\mathbf{y}} \cdot \mathbf{f}_0^{(0)} \text{ in } \Omega_0, \quad (4.6)$$

$$\nabla_{\mathbf{y}} \cdot (\mathbf{K}_I \nabla_{\mathbf{y}} p_I^{(1)}) = -\nabla_{\mathbf{y}} \cdot (\mathbf{K}_I \nabla_{\mathbf{x}} p^{(0)}) + \nabla_{\mathbf{y}} \cdot \mathbf{f}_I^{(0)} \text{ in } \Omega_I, \quad (4.7)$$

$$(\mathbf{K}_I \nabla_{\mathbf{y}} p_I^{(1)} - \mathbf{K}_0 \nabla_{\mathbf{y}} p_0^{(1)}) \cdot \mathbf{n} = (\mathbf{K}_0 \nabla_{\mathbf{x}} p^{(0)} - \mathbf{K}_I \nabla_{\mathbf{x}} p^{(0)}) \cdot \mathbf{n} + (\mathbf{f}_I^{(0)} - \mathbf{f}_0^{(0)}) \cdot \mathbf{n} \text{ on } \Gamma, \quad (4.8)$$

$$p_I^{(1)} = p_0^{(1)} \text{ on } \Gamma. \quad (4.9)$$



The problem (4.6–4.9) reads as a diffusion-type periodic cell problem for the first order pressures supplemented by transmission (jump) interface conditions. The problem is driven by both fine scale variations of the hydraulic conductivity tensor and by the fine scale divergences of the leading order body forces. The latter contributions vanish when the forces are homogeneous on the fine scale  $\mathbf{y}$  in the whole cell (that is, the force is continuous across the interface between phases and independent of  $\mathbf{y}$ ). In this case, the cell problem (4.6–4.9) reads as the classical auxiliary problem which arises from asymptotic homogenization of the diffusion problem, and admits a unique solution up to a  $\mathbf{y}$ -constant function, see, e.g., [4, 7, 19]. The differential problem (4.6–4.9) exhibits additional contributions involving the forces  $\mathbf{f}_I$  and  $\mathbf{f}_0$ . However, the problem (4.6–4.9) is characterized by the proper compatibility between the driving forces in the cell portions (i.e.,  $\nabla_{\mathbf{y}} \cdot \mathbf{f}_0^{(0)}$  and  $\nabla_{\mathbf{y}} \cdot \mathbf{f}_I^{(0)}$ ) and the interface conditions (4.8), as we show in the following remark.

**Remark 3** (Integral identity of the cell problem (4.6–4.9)). *We first observe that exploiting Darcy's law (2.3), the power series representation (3.4), and relationship (3.10), the leading order velocities  $\mathbf{u}_0^{(0)}$  and  $\mathbf{u}_I^{(0)}$  are given by*

$$\mathbf{u}_0^{(0)} = \mathbf{K}_0 \nabla_{\mathbf{y}} p_0^{(1)} + \mathbf{K}_0 \nabla_{\mathbf{x}} p_0^{(0)} - \mathbf{f}_0^{(0)} \quad (4.10)$$

and

$$\mathbf{u}_I^{(0)} = \mathbf{K}_I \nabla_{\mathbf{y}} p_I^{(1)} + \mathbf{K}_I \nabla_{\mathbf{x}} p_I^{(0)} - \mathbf{f}_I^{(0)}. \quad (4.11)$$

We now rewrite equations (4.6–4.8) in terms of the leading order velocities (4.10)–(4.11), i.e.,

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}_0^{(0)} = 0 \text{ in } \Omega_0, \quad (4.12)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}_I^{(0)} = 0 \text{ in } \Omega_I, \quad (4.13)$$

$$\mathbf{u}_I^{(0)} \cdot \mathbf{n} = \mathbf{u}_0^{(0)} \cdot \mathbf{n} \text{ on } \Gamma. \quad (4.14)$$

Summing up the integral over their respective cell portions of relationships (4.12)–(4.13) we obtain

$$0 = \int_{\Omega_0} \nabla_{\mathbf{y}} \cdot \mathbf{u}_0^{(0)} d\mathbf{y} + \int_{\Omega_I} \nabla_{\mathbf{y}} \cdot \mathbf{u}_I^{(0)} d\mathbf{y} = \int_{\partial\Omega_0} \mathbf{u}_0^{(0)} \cdot \mathbf{n}_{\partial\Omega_0} dS + \int_{\partial\Omega_I} \mathbf{u}_I^{(0)} \cdot \mathbf{n}_{\partial\Omega_I} dS, \quad (4.15)$$

where we have applied the divergence theorem with respect to the fine scale variable  $\mathbf{y}$  and  $\mathbf{n}_{\partial\Omega_0}$  and  $\mathbf{n}_{\partial\Omega_I}$  represent the unit outward normal vectors to the boundary of  $\Omega_0$  and  $\Omega_I$ , respectively. The right hand side of (4.15) can be rewritten as a surface integral on the interface  $\Gamma$  only by considering that the velocities  $\mathbf{u}_0^{(0)}$  and  $\mathbf{u}_I^{(0)}$  are  $\mathbf{y}$ -periodic. Therefore, every contribution on the periodic boundaries  $\partial\Omega_0 \setminus \Gamma$  and  $\partial\Omega_I \setminus \Gamma$  reduces to zero, so that

$$\int_{\partial\Omega_0} \mathbf{u}_0^{(0)} \cdot \mathbf{n}_{\partial\Omega_0} dS + \int_{\partial\Omega_I} \mathbf{u}_I^{(0)} \cdot \mathbf{n}_{\partial\Omega_I} dS = \int_{\Gamma} (\mathbf{u}_I^{(0)} - \mathbf{u}_0^{(0)}) \cdot \mathbf{n} dS, \quad (4.16)$$

where we have observed that

$$\mathbf{n}_{\partial\Omega_0} = -\mathbf{n}_{\partial\Omega_I} = -\mathbf{n} \quad (4.17)$$

on the interface  $\Gamma$ . The continuity condition (4.14) indeed guarantees that the right-hand side of (4.16) reduces to zero as dictated by equation (4.15).  $\square$

Since the problem is linear and the vector function  $\nabla_x p^{(0)}$  is  $\mathbf{y}$ -constant, we state the following *ansatz* for the solution

$$p_0^{(1)}(\mathbf{x}, \mathbf{y}) = \mathbf{g}_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_x p^{(0)}(\mathbf{x}) + \tilde{g}_0(\mathbf{x}, \mathbf{y}), \quad (4.18)$$

$$p_I^{(1)}(\mathbf{x}, \mathbf{y}) = \mathbf{g}_I(\mathbf{x}, \mathbf{y}) \cdot \nabla_x p^{(0)}(\mathbf{x}) + \tilde{g}_I(\mathbf{x}, \mathbf{y}). \quad (4.19)$$

Relationship (4.18)–(4.19) are the solutions of the problem (4.6–4.9) (up to a  $\mathbf{y}$ -constant function), provided that the auxiliary vectors  $\mathbf{g}_0(\mathbf{x}, \mathbf{y})$ ,  $\mathbf{g}_I(\mathbf{x}, \mathbf{y})$  solve the following cell problems

$$\nabla_y \cdot (\nabla_y \mathbf{g}_0 \mathbf{K}_0^T) = -\nabla_y \cdot \mathbf{K}_0^T \text{ in } \Omega_0, \quad (4.20)$$

$$\nabla_y \cdot (\nabla_y \mathbf{g}_I \mathbf{K}_I^T) = -\nabla_y \cdot \mathbf{K}_I^T \text{ in } \Omega_I, \quad (4.21)$$

$$(\nabla_y \mathbf{g}_I \mathbf{K}_I^T - \nabla_y \mathbf{g}_0 \mathbf{K}_0^T) \cdot \mathbf{n} = (\mathbf{K}_0 - \mathbf{K}_I)^T \mathbf{n} \text{ on } \Gamma, \quad (4.22)$$

$$\mathbf{g}_I = \mathbf{g}_0 \text{ on } \Gamma. \quad (4.23)$$

The auxiliary scalar functions  $\tilde{g}_0(\mathbf{x}, \mathbf{y})$  and  $\tilde{g}_I(\mathbf{x}, \mathbf{y})$  solve

$$\nabla_y \cdot (\mathbf{K}_0 \nabla_y \tilde{g}_0) = \nabla_y \cdot \mathbf{f}_0^{(0)} \text{ in } \Omega_0, \quad (4.24)$$

$$\nabla_y \cdot (\mathbf{K}_I \nabla_y \tilde{g}_I) = \nabla_y \cdot \mathbf{f}_I^{(0)} \text{ in } \Omega_I, \quad (4.25)$$

$$(\mathbf{K}_I \nabla_y \tilde{g}_I - \mathbf{K}_0 \nabla_y \tilde{g}_0) \cdot \mathbf{n} = (\mathbf{f}_I^{(0)} - \mathbf{f}_0^{(0)}) \cdot \mathbf{n} \text{ on } \Gamma, \quad (4.26)$$

$$\tilde{g}_I = \tilde{g}_0 \text{ on } \Gamma, \quad (4.27)$$

where  $\mathbf{g}_I$ ,  $\mathbf{g}_0$ ,  $\tilde{g}_I$ , and  $\tilde{g}_0$  are  $\mathbf{y}$ -periodic and uniqueness can be achieved for example by fixing the integral average of the auxiliary variables over the periodic cell  $\Omega$ .

Next we obtain a closed coarse scale problem for the leading order pressure  $p^{(0)}$ .

$\epsilon^2$  The coarse-scale model

We are now ready to achieve our main result by equating the same powers of  $\epsilon^2$  in (3.7)–(3.8), which yields

$$\begin{aligned} & \nabla_x \cdot (\mathbf{K}_0 \nabla_x p_0^{(0)}) + \nabla_x \cdot (\mathbf{K}_0 \nabla_y p_0^{(1)}) + \\ & \nabla_y \cdot (\mathbf{K}_0 \nabla_x p_0^{(1)}) + \nabla_y \cdot (\mathbf{K}_0 \nabla_y p_0^{(2)}) = \nabla_x \cdot \mathbf{f}_0^{(0)} + \nabla_y \cdot \mathbf{f}_0^{(1)} \text{ in } \Omega_0, \end{aligned} \quad (4.28)$$

$$\begin{aligned} & \nabla_x \cdot (\mathbf{K}_I \nabla_x p_I^{(0)}) + \nabla_x \cdot (\mathbf{K}_I \nabla_y p_I^{(1)}) + \\ & \nabla_y \cdot (\mathbf{K}_I \nabla_x p_I^{(1)}) + \nabla_y \cdot (\mathbf{K}_I \nabla_y p_I^{(2)}) = \nabla_x \cdot \mathbf{f}_I^{(0)} + \nabla_y \cdot \mathbf{f}_I^{(1)} \text{ in } \Omega_I, \end{aligned} \quad (4.29)$$

$$(\mathbf{K}_I \nabla_y p_I^{(2)} - \mathbf{K}_0 \nabla_y p_0^{(2)}) \cdot \mathbf{n} = (\mathbf{K}_0 \nabla_x p_I^{(1)} - \mathbf{K}_I \nabla_x p_0^{(1)}) \cdot \mathbf{n} + (\mathbf{f}_I^{(1)} - \mathbf{f}_0^{(1)}) \cdot \mathbf{n} \text{ on } \Gamma. \quad (4.30)$$

We now perform an integral average of equations (4.28)–(4.29) over their respective cell portions  $\Omega_0$  and  $\Omega_I$ , respectively, i.e., we apply the following cell average operator:

$$\langle (\bullet) \rangle_{\Omega_\alpha} = \frac{1}{|\Omega|} \int_{\Omega_\alpha} (\bullet) \, d\mathbf{y}, \quad \alpha = 0, I, \quad (4.31)$$

where  $|\Omega|$  is the volume of the periodic cell. We therefore obtain

$$\begin{aligned} & \nabla_x \cdot \left( \langle \mathbf{K}_0 \rangle_{\Omega_0} \nabla_x p_0^{(0)} \right) + \nabla_x \cdot \left\langle \mathbf{K}_0 \nabla_y p_0^{(1)} \right\rangle_{\Omega_0} - \nabla_x \cdot \left\langle \mathbf{f}_0^{(0)} \right\rangle_{\Omega_0} = \\ & - \left\langle \nabla_y \cdot \left( \mathbf{K}_0 \nabla_x p_0^{(1)} + \mathbf{K}_0 \nabla_y p_0^{(2)} - \mathbf{f}_0^{(1)} \right) \right\rangle_{\Omega_0}, \end{aligned} \quad (4.32)$$

$$\begin{aligned} & \nabla_x \cdot \left( \langle \mathbf{K}_I \rangle_{\Omega_I} \nabla_x p_I^{(0)} \right) + \nabla_x \cdot \left\langle \mathbf{K}_I \nabla_y p_I^{(1)} \right\rangle_{\Omega_I} - \nabla_x \cdot \left\langle \mathbf{f}_I^{(0)} \right\rangle_{\Omega_I} = \\ & - \left\langle \nabla_y \cdot \left( \mathbf{K}_I \nabla_x p_I^{(1)} + \mathbf{K}_I \nabla_y p_I^{(2)} - \mathbf{f}_I^{(1)} \right) \right\rangle_{\Omega_I}. \end{aligned} \quad (4.33)$$

Summing up equations (4.32)–(4.33) and applying the divergence theorem with respect to the fine scale variable  $\mathbf{y}$  yields

$$\begin{aligned} & \nabla_x \cdot \left( (\langle \mathbf{K}_0 \rangle_{\Omega_0} + \langle \mathbf{K}_I \rangle_{\Omega_I}) \nabla_x p^{(0)} \right) + \nabla_x \cdot \left\langle \mathbf{K}_0 \nabla_y p_0^{(1)} \right\rangle_{\Omega_0} + \nabla_x \cdot \left\langle \mathbf{K}_I \nabla_y p_I^{(1)} \right\rangle_{\Omega_I} \\ & = \nabla_x \cdot \left\langle \mathbf{f}_0^{(0)} \right\rangle_{\Omega_0} + \nabla_x \cdot \left\langle \mathbf{f}_I^{(0)} \right\rangle_{\Omega_I}, \end{aligned} \quad (4.34)$$

as the contributions over the periodic boundaries  $\partial\Omega_0 \setminus \Gamma$  and  $\partial\Omega_I \setminus \Gamma$  cancel out due to  $\mathbf{y}$ -periodicity, and the remaining one on the interface  $\Gamma$  reduce to zero accounting for interface condition (4.30) and exploiting (4.17). Equation (4.34) can be further rearranged exploiting ansatz (4.18) and (4.19) into a coarse scale Darcy's-type problem for leading pressure  $p^{(0)}$ , that is

$$\nabla_x \cdot \left( \hat{\mathbf{K}}(\mathbf{x}) \nabla_x p^{(0)}(\mathbf{x}) \right) = \phi(\mathbf{x}), \quad (4.35)$$

where the coarse scale hydraulic conductivity tensor  $\hat{\mathbf{K}}(\mathbf{x})$  and the volume source  $\phi(\mathbf{x})$  are defined as

$$\hat{\mathbf{K}}(\mathbf{x}) = \langle \mathbf{K}_0 \rangle_{\Omega_0} + \langle \mathbf{K}_I \rangle_{\Omega_I} + \left\langle \mathbf{K}_0 (\nabla_y \mathbf{g}_0)^\top \right\rangle_{\Omega_0} + \left\langle \mathbf{K}_I (\nabla_y \mathbf{g}_I)^\top \right\rangle_{\Omega_I}, \quad (4.36)$$

and

$$\phi(\mathbf{x}) = \nabla_x \cdot \left\langle \mathbf{f}_0^{(0)} \right\rangle_{\Omega_0} + \nabla_x \cdot \left\langle \mathbf{f}_I^{(0)} \right\rangle_{\Omega_I} - \nabla_x \cdot \left\langle \mathbf{K}_0 \nabla_y \tilde{\mathbf{g}}_0 \right\rangle_{\Omega_0} - \nabla_x \cdot \left\langle \mathbf{K}_I \nabla_y \tilde{\mathbf{g}}_I \right\rangle_{\Omega_I}, \quad (4.37)$$

respectively.

The homogenized problem (4.35) is to be solved on the coarse scale only, and the fine scale information is encoded in the components of the effective hydraulic conductivity tensor (4.36), which can be computed solving the standard diffusion-type cell problems given by (4.20–4.23). The effective source driving the macroscale problem is given by (4.37) and comprises both the contribution given by the coarse scale divergences of the leading order body forces' cell average, as well as additional terms involving the solution of the cell problem (4.24–4.27). The source term  $\phi(\mathbf{x})$  actually reads as the coarse scale divergence of the effective volume load driving the leading order coarse scale fluid flow, as shown below.

#### 4.1. Coarse scale velocity profile

We now aim to obtain an effective governing equation which relates the leading order velocity field to the leading order pressure and body forces. We proceed by applying the average operator (4.31) to the zero-th order velocity fields given by equations (4.10)–(4.11) and considering the ansatz (4.18)–(4.19) to obtain

$$\left\langle \mathbf{u}_0^{(0)} \right\rangle_{\Omega_0} + \left\langle \mathbf{u}_I^{(0)} \right\rangle_{\Omega_I} = -\hat{\mathbf{K}} \nabla_x p^{(0)} - \left\langle \mathbf{K}_0 \nabla_y \tilde{\mathbf{g}}_0 \right\rangle_{\Omega_0} + \left\langle \mathbf{f}_0^{(0)} \right\rangle_{\Omega_0} - \left\langle \mathbf{K}_I \nabla_y \tilde{\mathbf{g}}_I \right\rangle_{\Omega_I} + \left\langle \mathbf{f}_I^{(0)} \right\rangle_{\Omega_I}, \quad (4.38)$$

where  $\hat{K}$  is the effective hydraulic conductivity tensor we have introduced in equation (4.36). Equation (4.38) represents the leading order coarse scale Darcy's law and can be rewritten as

$$\mathbf{u}_C = -\hat{K}\nabla_x p^{(0)} + \tilde{\mathbf{f}}, \quad (4.39)$$

where  $\mathbf{u}_C$  is the coarse scale leading order velocity defined by

$$\mathbf{u}_C := \langle \mathbf{u}_0^{(0)} \rangle_{\Omega_0} + \langle \mathbf{u}_I^{(0)} \rangle_{\Omega_I} \quad (4.40)$$

and the effective body force  $\tilde{\mathbf{f}}$  is given below

$$\tilde{\mathbf{f}} = \langle \mathbf{f}_0^{(0)} \rangle_{\Omega_0} + \langle \mathbf{f}_I^{(0)} \rangle_{\Omega_I} - \langle \mathbf{K}_0 \nabla_y \tilde{g}_0 \rangle_{\Omega_0} - \langle \mathbf{K}_I \nabla_y \tilde{g}_I \rangle_{\Omega_I} \quad (4.41)$$

Finally, we observe that the governing equations (4.35) can be rewritten exploiting Darcy's coarse scale law (4.39) and accounting for (4.41) as follows

$$\nabla_x \cdot \mathbf{u}_C = 0, \quad (4.42)$$

as

$$\nabla_x \cdot \tilde{\mathbf{f}}(\mathbf{x}) = \phi(\mathbf{x}). \quad (4.43)$$

The differential model (4.35) written in terms of the leading order pressure therefore represents a coarse scale, mass conserving Darcy's flow driven by the effective body force given by equation (4.41).

## 5. Discussion of the results and applicability of the model

The new mathematical model reads as a Darcy's-type system of PDEs, with effective hydraulic conductivity and body force given by (4.36) and (4.41), respectively. The effective hydraulic conductivity is to be computed by solving the cell problem (4.20–4.23), and coincides with the classical one arising from asymptotic homogenization of the diffusion problem, see, e.g., [4, 7, 19]. The major novelty resides in the effective volume load driving the coarse scale fluid flow (or equivalently, in its coarse scale divergence, which appears as a source in the differential problem (4.35)). The latter is not merely given by the cell average of the applied body force, but involves additional contributions which are directly related to the solution of the cell problem (4.24–4.27), which reads as a well-defined differential problem solely driven by the fine scale divergence of the force. Whenever the latter vanishes (e.g., when the body force is homogeneous on the fine scale), the solution of the problem (4.24–4.27) is a constant and therefore the effective volume load reduces to the cell average of the body force. In this case, the problem (4.35) in terms of the pressure reads formally as the classical homogenized diffusion problem driven by the coarse scale divergence of the average body force. The third and fourth terms on the right hand side of (4.41) explicitly account for the interplay between fine scale material heterogeneities (encoded in the hydraulic conductivity tensors  $\mathbf{K}_I$  and  $\mathbf{K}_0$ ) and fine scale variations of the body force. In fact, when the medium is homogeneous on the fine scale, that is, characterized by a continuous hydraulic conductivity tensor of the form  $\mathbf{K}(\mathbf{x})$  in the whole  $\Omega$ , we have (cf. (4.41))

$$\langle \mathbf{K}_0 \nabla_y \tilde{g}_0 \rangle_{\Omega_0} + \langle \mathbf{K}_I \nabla_y \tilde{g}_I \rangle_{\Omega_I} = \mathbf{K}(\mathbf{x}) \left( \langle \nabla_y \tilde{g}_0 \rangle_{\Omega_0} + \langle \nabla_y \tilde{g}_I \rangle_{\Omega_I} \right) = \frac{1}{|\Omega|} \mathbf{K}(\mathbf{x}) \int_{\Gamma} (\tilde{g}_0 - \tilde{g}_I) \mathbf{n} \, dS = 0, \quad (5.1)$$

where we have exploited  $y$ -periodicity on  $\partial\Omega$ , relationship (4.17), and continuity of the auxiliary pressure as per equation (4.27).

As observed in Remark 1, the inhomogeneous volume load that appears in the Darcy's problem (2.5) formally translates into a diffusion problem driven by a locally unbounded source. However, the homogenized problem (4.35) is actually well-defined, as the hydraulic conductivity reads as the standard homogenized diffusivity tensor, which is therefore symmetric and positive definite. The effective volume source comprises given regular quantities and the solution of the newly arising cell problem (4.24–4.27), which satisfies the appropriate compatibility condition (see Remark 3). Asymptotic homogenization of differential problems involving locally unbounded terms is not commonly treated, although a relevant example can be found in [35]. In the latter work, the authors derive the effective governing equations for linear elastic composite materials subject to a locally unbounded body force, which is given by a standard Helmholtz decomposition into the gradient and the curl of a scalar and vector potential, where the latter are assumed to be regular and periodic on the fine scale. They further show that the model sketched in [36] (Pages 94–96), where the authors apply the asymptotic homogenization technique to a homogeneous elastic material subject to an explicitly locally unbounded force, can be viewed as a particular case of [35] for constant stiffness tensor and under a consistent set of assumptions.

Our formulation is general with respect to the choice of the body force, which could be for example related to an external magnetic or electric field applied to ferrofluids and electrolytes, respectively. The most common ferrofluids are stable suspensions which are sensitive to the application of an external magnetic field due to the presence of small diameters magnetic particles, see, e.g., [34]. Ferrofluids have been successfully used for commercial purposes [28], and their flow in porous media is relevant to a large variety of scenarios. These include, for example, environmental engineering and applications related to the injection of chemical in contaminated underground zones [17], as well as enhancement of oil recovery [10]. Another relevant scenario of interest is transport of nanoparticles in biological tissues, which can be used for hyperthermia cancer treatment [18]. In the latter case, the host matrix represents the tissue interstitial matrix, the inclusions the tumor heterogeneities, and application of a magnetic field translates into an inhomogeneous body force which involves the magnetization and the applied field.

Although the main focus of our study resides in heterogeneous porous media, our results can also be relevant to general diffusion problems (and, with minor modifications, to problems governed by the same differential operators in space, such as the heat equation) when the source is prescribed in terms of the divergence of a given inhomogeneous flux.

In the next Section we conclude our manuscript highlighting possible future perspectives.

## 6. Conclusions and further perspectives

We have derived the new governing equations describing the homogenized behavior of heterogeneous porous media subject to an inhomogeneous body force. We have assumed that spatial variations of the hydraulic conductivity and body force occur on two well-separated scales and have derived our results by means of asymptotic homogenization. The results are presented in Section 4 and the homogenized system of PDEs (4.35), which comprises the source term (4.37), is well-defined, so that the problem (2.5) is proved to be homogenizable.

The major novelty of our formulation resides in the contributions that appear in the effective source and body force, which are explicitly related to the interplay between material heterogeneities and fine scale variations of the body force which appear in the cell problem (4.24–4.27). Our findings are discussed in Section 5 and the model is applicable to a large variety of physical scenarios involving the action of a body force driving porous media flow.

Since we have embraced a formal two-scale expansions approach, adopting rigorous convergence approaches such as those illustrated in [7] represents a natural challenging development from a theoretical viewpoint.

Furthermore, we believe that three (or multi)-scales homogenization approaches recently developed for elliptic problems in the context of elastic composites (see, e.g., [2, 14, 31–33]) can be adopted to describe the behavior of porous media characterized by several hierarchical levels of organization.

Our formulation is conceived for simple porous media governed by the heterogeneous Darcy’s law and the next natural step is the analysis of analytical and numerical solutions of the model in the context of practical applications such as those discussed in Section 5. This way, we could be able to quantify the impact of inhomogeneous body forces on heterogeneous porous materials, and validate the results against experimental data.

Example of numerical solutions of cell problems in three dimensions and related computational strategies have been recently provided in the context of practical applications, such as tumor modeling, composite materials, and mineralized (bone and tendons) tissues, see, e.g., [9, 15, 20], [23, 26, 35], [25], respectively.

Furthermore, our framework constitutes a preliminary starting point for the study of the effects of body forces on complex hierarchical porous media, which are encountered for example in geomechanics (see, e.g., [3]), as well as transport and growth in solid tumors [8, 21, 30, 37].

Generalizations of our model accounting for the coupling with different physical systems (such as blood vessels and fractures) can pave the way for investigation of real-world problems.

For example, application of electric fields to electrolytes also results in a body force driving the Darcy’s flow and proportional to the applied electric field, see, e.g., [5]. A body force driving the fluid flow in porous media can also be shown to arise from upscaling of growth processes in biological tissues, see, e.g., [27].

One of the most interesting scenarios in the context of biological applications resides in tumor modeling and cancer hyperthermia [16, 18]. In this case, magnetic nanoparticles are injected through the vessels and a sufficiently strong magnetic field causes the particles to heat. Therefore, the fluid flowing in the tissue becomes a suspension of nanoparticles which is sensitive to the application of the magnetic field. Thus, an appropriate extension of our model can in perspective provide more accurate predictions on the fluid (and, in turn, drug) transport profile in cancerous tissues subject to hyperthermia treatment.

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### Conflict of interest

The authors declare that they have no conflict of interest.

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