A Fully Probabilistic Design for Tracking Control for Stochastic Systems with Input Delay

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***Abstract*—This paper studies model reference adaptive control (MRAC) for a class of stochastic discrete time control systems with time delays in the control input. In particular, a unified fully probabilistic control framework is established to develop the solution to the MRAC, where the controller is the minimiser of the Kullback-Leibler Divergence (KLD) between the actual and desired joint probability density functions of the tracking error and the controller. The developed framework is quite general, where all the components within this framework, including the controller and system tracking error, are modelled using probabilistic models. The general solution for arbitrary probabilistic models of the framework components is first obtained and then demonstrated on a class of linear Gaussian systems with time delay in the main control input, thus obtaining the desired results. The contribution of this paper is twofold. First, we develop a fully probabilistic design framework for MRAC, referred to as MRFPD, for stochastic dynamical systems. Second, we establish a systematic pedagogic procedure that is based on deriving explicit forms for the required predictive distributions for obtaining the causal form of the randomised controller when input delays are present.**

***Index Terms*—Fully Probabilistic Design (FPD), Input delay, Model reference adaptive control (MRAC), Tracking error.**

I. Introduction

Many engineering and industrial systems are fraught with several sources of uncertainties including functional uncertainties, random noises and disturbances introduced by measurement devices and other surrounding environmental conditions. Under these situations where high levels of uncertainties exist, the design and derivation of an optimal control law becomes very complicated. The control solution becomes even more challenging when time delays in the input and/or the state are present. For linear stochastic and deterministic systems, the optimal linear quadratic regulation (LQR) and tracking control problems have received paramount attention since the 1960s [1]–[4]. Owing to their relevance to networked control and distributed and decentralised control, optimal stochastic and deterministic linear quadratic solutions have been extended to consider delay control systems [5], [6]. Depending on the type of delay, the nature of the system equations, and the performance criterion used for optimisation, some progress was made towards the development of the control solution to these delay systems. For example, the study in [7] has focused on the time optimal criterion, while the quadratic cost function has been used in [8]. On the other hand, some studies of optimal control in time delay systems considered delays in the state only [9], [10], others considered delays in the control input only [11], [12], and others considered delays in both control input and state [13]. The optimal tracking problem for discrete time systems with single input delay [14] and multiple state and input delays [13] was also discussed in the literature. However, despite the fact that large amount of research has been devoted to solve the control problem for systems with delays, a general solution that considers the stochastic nature of the system dynamics and the delays in its input and/or state is still lacking.

On the other hand, the fully probabilistic design (FPD) control method that uses the Kullback-Leibler divergence as a performance measure provides a systematic approach to obtaining the general control solution for stochastic systems subject to random inputs and deterministic systems characterised by functional uncertainty [15]. However, in its original form [16] the FPD method insists on zero delay between the input and the state of the system. On account of this, we recently extended the FPD method such that it considers a class of stochastic systems that involves a lagged and an unlagged control inputs [17]. This recent development considers the problem of designing randomised controllers that shape the joint probability density function of the system state. Nonetheless, the characterisation of the pdf of the system state can be difficult for many real world systems that work under high levels of uncertainty and stochasticity. Furthermore, in many real engineering systems the controller objective is to make the output of the system dynamics follow a predefined desired output value, thus emphasising the importance of the tracking error rather than the actual system state.

Concequently, the objective of the current paper is to develop a general probabilistic framework to obtain the tracking control solution to the stochastic control problems with input delay where the system state is required to track a predefined desired state as obtained from a reference model. The formulation in this paper aims at designing a randomised controller that shapes the pdf of the tracking error distribution as opposed to original formulation of the FPD method that considers designing randomised controllers that shape the pdf of the system state [15]–[17]. This method will be referred to as model reference fully probabilistic design control (MRFPD) method. An additional objective of the current paper is to extend the derivation of the considered tracking control problem, such that a randomised controller can be derived when the main input to the system is lagged by a number of time units. The solution to this problem is very challenging as the randomised controller has to be designed to adhere to the causality constraint. To reemphasise, the formulation of the

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control problem in this paper will be based on shaping the pdf of the tracking error, where the considered class of stochastic systems is assumed to be controlled by a lagged control input. This new formulation means that the solution methodology developed in [17], which considers shaping the pdf of the system state and assumes that stochastic systems are affected by lagged and unlagged control inputs, is not appropriate for the considered model reference tracking problem where the stochastic system is controlled by a lagged control input only. Thus, a different approach which is based on probabilistic inference for evaluating the predictive distributions of the tracking error and randomised controller will be followed in the current paper. The proposed framework is demonstrated on a class of linear stochastic control systems that also have input time delays where the obtained feedback control law is shown to be linear in the tracking error with an extra term that takes into consideration additional constraints imposed by the tracking objective. Similar to the conventional FPD controller, the feedback gain matrix of the derived randomised controller satisfies a generalised Riccati equation but now with tracking error (instead of state in the conventional method), and control input penalisation specified by the covariances of the actual, as well as, ideal distributions of the tracking error and control input respectively.

The proposed framework offers several improvements and advances the current results that have been obtained for time delay systems. Firstly, the obtained solution of the randomised controller is for the more general case of a tracking control problem rather than the special case of a regulation problem. Secondly, the derived randomised controller is more exploratory due to its probabilistic nature, and accounts for stochasticity and functional uncertainty in the system dynamics. It aims at achieving a narrow tracking error distribution centred around zero error state, thus guaranteeing that the system has tracked the desired trajectory and at the same time indicating that the uncertainty in the tracked trajectory is small. Thirdly, the causal optimal control law is given explicitly and the optimal distribution of the randomised controller is derived analytically.

To summarise, this paper generalises the solution to the optimal stochastic control problem with input delays by designing a randomised controller, rather than a deterministic one. The derivation of a randomised controller is facilitated at very early stages, by defining the performance measure to be optimised as the KLD between the actual joint pdf and an ideal joint pdf of the tracking performance of the system dynamics. The presented derivations consider the general tracking control problem where the system state is required to follow a predefined desired state specified by a reference model. The consideration of input delay and derivation of randomised controllers for MRAC problems under a FPD are innovative and considered for the first time in the current paper.

II. TRACKING CONTROL PROBLEM WHEN THERE IS NO DELAY

A. FPD aims of the tracking Control Problem

This section considers the development of a fully probabilistic design framework for MRAC for stochastic dynamical systems. The objective here is to design a randomised control strategy that will make the joint pdf of the system tracking error and control input follows a predefined desired pdf. In control, this tracking problem is usually specified in terms of a reference model and solved using MRAC [18]–[20]. MRAC is a powerful approach which can be implemented in a straight forward manner and at the same time can guarantee robustness to parameter variations, noise and unmodelled dynamics. This method can be naturally integrated to the FPD method and will be referred to as MRFPD. In conventional MRAC a cost function is defined as a function of tracking error between the state of the plant and a reference model, and the controller parameters are adjusted such that this cost function is minimised. The tracking error equation is given by,

$$e_{t+1} = x_{t+1} - x^r_{t+1}, \tag{1}$$

where $x_t \in R^n$ is the system state, $e_t \in R^n$ is the tracking error and $x^r_t \in R^n$ is the state of a reference model to be tracked by the system. On the other hand, for the class of stochastic systems considered in this paper, the system state, x_{t+1} in Equation (1) is not deterministic thus can only be characterised by its pdf. In particular, the pdf of the stochastic system state is given by,

$$s(x_{t+1}|x_t, u_{t+1}), \tag{2}$$

where $s(.|.)$ denotes the conditional pdf. Given this definition of the pdf of the system state, the density function of the tracking error can be obtained using probability theory as follows,

$$s_e(x_{t+1}, x^r_{t+1}) = s(e_{t+1} + x^r_{t+1}|e_t + x^r_t, u_{t+1}). \tag{3}$$

Thus, for these stochastic systems the MRAC can be reformulated following a probabilistic approach to design a randomised controller, $c(u_{t+1}|e_t)$ that reshapes the pdf of the tracking error. The basic block diagram of this proposed approach which is referred to in this paper as MRFPD is shown in Figure 1. As shown in the figure, the randomised controller can be derived by minimising the KLD between the joint pdf of the tracking error and the controller and a predefined ideal joint pdf,

$$\mathcal{D}\left(f|| \, ^I f\right) \equiv \int f(\mathbb{E}) \ln\left(\frac{f(\mathbb{E})}{^I f(\mathbb{E})}\right) d\mathbb{E}, \tag{4}$$

where $f(\mathbb{E}) = \prod_{t=0}^{T} s(e_{t+1}|u_{t+1}, e_t) c(u_{t+1}|e_t)$ is the joint distribution of the tracking error and control input, $^I f(\mathbb{E}) = \prod_{t=0}^{T} {^I s}(e_{t+1}|u_{t+1}, e_t) \, ^I c(u_{t+1}|e_t)$ is the ideal distribution of the tracking error and control input, $\mathbb{E} = \{e_{t+1}, \ldots, e_T, u_{t+1}, \ldots, u_T\}$ is the closed loop observed data sequence, and $T \leq \infty$ is a given control horizon.

Following the same procedure of the conventional FPD [15], using these newly defined ideal pdfs of the tracking error and randomised controller, the minimum cost function resulting from the minimisation of (4) with respect to admissible control sequence, u_{t+1}, $t \in \{0, \ldots, T\}$ can then be shown to be given by the following recurrence equation,

$$\begin{aligned} -\ln(\gamma(e_t)) = &\min_{c(u_{t+1}|e_t)} \int s(e_{t+1}|u_{t+1}, e_t) \\ c(u_{t+1}|e_t) \times &\Bigg[\underbrace{\ln\left(\frac{s(e_{t+1}|u_{t+1}, e_t)c(u_{t+1}|e_t)}{{}^I s(e_{t+1}|e_t, u_{t+1})\, {}^I c(u_{t+1}|e_t)}\right)}_{\equiv \text{partial cost} \implies U(e_{t+1}, u_{t+1})} \\ &- \underbrace{\ln(\gamma(e_{t+1}))}_{\text{optimal cost-to-go}} \Bigg] d(x_{t+1}, u_{t+1}), \end{aligned} \tag{5}$$

where $-\ln(\gamma(e_t))$ is the expected minimum cost–to–go function. The above equation constitutes the recurrence equation of the dynamic programming solution to the MRFPD control problem. Its derivation can be obtained following the same procedure discussed in [15].

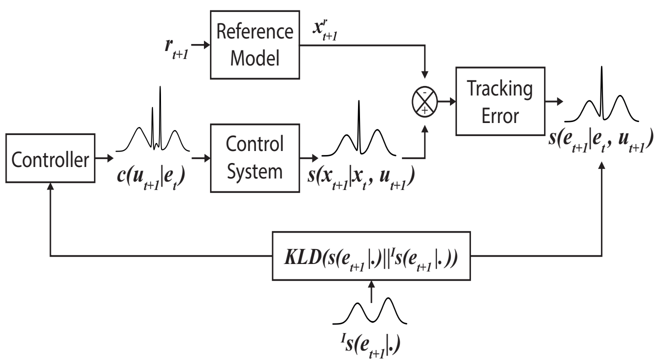

Fig. 1. Architecture of model reference fully probabilistic design.

B. MRFPD General Solution

The general solution of the optimisation used to minimise the cost-to-go function (5) with respect to control law, $c(u_{t+1} \mid e_t)$ can be shown to be given by the following proposition.

Proposition 1: The pdf of the optimal controller minimising the cost–to–go function (5) subject to the conditional distribution of the tracking error of the stochastic system, $s(e_{t+1}|u_{t+1}, e_t)$ is given by,

$$\begin{aligned} &c^*(u_{t+1}|e_t) = \\ &\frac{{}^I c(u_{t+1}|e_t)\exp[-\beta_1(u_{t+1}, e_t) - \beta_2(u_{t+1}, e_t)]}{\gamma(e_t)}, \\ &\gamma(e_t) = \int {}^I c(u_{t+1}|e_t)\exp[-\beta_1(u_{t+1}, e_t) \\ &- \beta_2(u_{t+1}, e_t)]du_{t+1}, \\ &\beta_1(u_{t+1}, e_t) = \\ &\int s(e_{t+1}|u_{t+1}, e_t)\left[\ln\frac{s(e_{t+1}|u_{t+1}, e_t)}{{}^I s(e_{t+1}|u_{t+1}, e_t)}\right]de_{t+1}, \\ &\beta_2(u_{t+1}, e_t) = -\int s(e_{t+1}|u_{t+1}, e_t)\ln(\gamma(e_{t+1}))de_{t+1}, \end{aligned} \tag{6}$$

for $t = 0, 1, \ldots, T$ and $\gamma(e_T) = 1$.

***Proof*:** This proposition can be proven by adapting the proof of Proposition 2 in [21].

As discussed earlier, the randomised control solution given in Equation (6) provides a general solution for stochastic systems subject to random inputs and deterministic systems characterised by functional uncertainty [22]. Other control approaches such as LQR and H2 [23] are deterministic approaches which are based on the certainty equivalence principle. Additionally, contrary to the linear quadratic Gaussian control [24] and the minimum variance control [25] approaches the solution given in Equation (6) is not restricted by the Gaussian assumption of the pdf of the tracking error dynamics or its ideal distribution. It provides the general solution for any arbitrary pdfs. Furthermore, the derived controller (6) is a randomised controller thus, it is more explorative than the conventional deterministic controllers. However, the evaluation of the analytic solution for this randomised controller is not possible except for the special case of linear and Gaussian pdfs. Therefore, to facilitate the understanding of the proposed control framework, the rest of the paper will focus on the development of the required solutions for the case where the pdfs of the tracking error, $s(e_{t+1}|u_{t+1}, e_t)$ and its ideal distribution, ${}^I s(e_{t+1}|u_{t+1}, e_t)$ are assumed to be Gaussian.

Besides, please note that although the general solution of the MRFPD controller given in Equation (6) has similar form to the conventional FPD [16], it is rather now dependent on the pdf of the tracking error and its characterisation as can be obtained from Equation (3). This will result in a different form of the derived randomised controller that considers the constraints imposed by the tracking control objective as will be seen in the next sections.

Finally, please note that the definition of the tracking error as given in Equation (1) does not impose any restrictions on the nature of the model of the reference state. This means that the reference model can be stochastic or deterministic. However, further development in this paper will be based on using a deterministic reference model as in traditional MRAC.

C. Solution to the MRFPD for Linear Stochastic Systems with no Delays

The aim of this section is to derive the analytic solution of the conditional pdf of the randomised controller for the tracking problem of linear stochastic systems with zero delay between the input and state. Here, the pdf of the system dynamics are given by,

$$s(x_{t+1}|x_t, u_{t+1}) = \mathcal{N}(Ax_t + Bu_{t+1}, \Sigma), \tag{7}$$

where $A \in R^{n\times n}$, and $B \in R^{n\times m}$ are constant state and control matrices respectively, and Σ is the covariance of the state distribution. The sought tracking control problem considered here, is to design a control strategy that will make the state of the system follow a predefined reference model specified by a reference input r_t as follows,

$$x^r_{t+1} = A^r x^r_t + B^r r_{t+1}, \tag{8}$$

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where $A^r \in R^{n\times n}$, and $B^r \in R^{n\times m}$ are constant matrices. As discussed earlier, because of the stochastic nature of the considered class of stochastic systems, the proposed framework derives randomised controllers that minimise the discrepancy between the joint pdf of the tracking error and control input and a predefine ideal distribution of tracking error as stated in proposition 1. Using equations (7), (8) and (3) the distribution of the tracking error can be shown to be given by,

$$s(e_{t+1}|e_t, u_{t+1}) = \mathcal{N}(Ae_t + Bu_{t+1} + \tilde{x}_t, \Sigma), \quad (9)$$

where $\tilde{x}_t = (A - A^r)x^r_t - B^r r_{t+1}$. The sought tracking control problem considered here, is to design a control strategy that will achieve a narrow tracking error distribution centred around zero, thus, guaranteeing an accurate tracking of the system state to the desired value. As such, the ideal distribution of the tracking error of the system is assumed to be given by,

$$^{I}s(e_{t+1}|e_t, u_{t+1}) = \mathcal{N}(0, \Sigma). \quad (10)$$

In addition, the ideal distribution of the controller is assumed to be given by,

$$^{I}c(u_{t+1}|x_t) = \mathcal{N}(\hat{u}_{t+1}, \Gamma), \quad (11)$$

where $\hat{u}_{t+1}$ and Γ are the mean and covariance of the ideal distribution of the control input respectively. In tracking control problems, the mean of the ideal distribution of the control input, $\hat{u}_{t+1}$ can be evaluated from the expected value of the tracking error as follows,

$$\lim_{t\to\infty} < e_{t+1} > = \lim_{t\to\infty} < Ae_t + Bu_{t+1} + \tilde{x}_t >,$$
$$\hat{u}_{t+1} = -B^{\dagger}\tilde{x}_t, \quad (12)$$

where here $< . >$ means the expected value, and $B^{\dagger}$ is the pseudo inverse of the matrix B. Please note that in the proposed MRFPD the system state is regulated to a value different than zero. As such, the mean value of the ideal distribution of the controller, $\hat{u}_{t+1}$ cannot be taken equal to zero and should be calculated as stated in equation (12). The mean value, $\hat{u}_{t+1}$ can be interpreted as a bias to drive the system state to the required desired state value x^r_{t+1}. Having obtained these distributions of the stochastic system, the solution to the tracking control problem can then be obtained by evaluating the optimal performance index $-\ln(\gamma(e_{t+1}))$. This in turn requires the calculation of $\beta_1(u_{t+1}, e_t), \beta_2(u_{t+1}, e_t)$, and $\gamma(e_t)$ as defined by Equation (6) and yields the probabilistic optimal feedback control law specified in the following theorem.

Theorem 1: The randomised optimal controller minimising the optimal cost-to-go function (5) subject to the pdf of the system tracking error (9) and ideal distributions given by (10) and (11) is,

$$c^*(u_{t+1}|e_t) = \mathcal{N}(-K_{t+1}e_t + L_{t+1}, \Gamma_{t+1}), \quad (13)$$

where

$$\begin{aligned}
\Gamma_{t+1} &= [\Gamma^{-1} + B^T(\Sigma^{-1} + M_{t+1})B]^{-1}, \\
K_{t+1} &= \Gamma_{t+1}B^T(\Sigma^{-1} + M_{t+1})A, \\
L_{t+1} &= -\Gamma_{t+1}[0.5B^T g^T_{t+1} + B^T(\Sigma^{-1} + M_{t+1})\tilde{x}_t \\
&\quad - \Gamma^{-1}\hat{u}_{t+1}]. \quad (14)
\end{aligned}$$

In addition,

$$\begin{aligned}
-\ln(\gamma(e_{t+1})) = 0.5(&e^T_{t+1}M_{t+1}e_{t+1} + g_{t+1}e_{t+1} \\
&+ \omega_{t+1}), \quad (15)
\end{aligned}$$

with

$$\begin{aligned}
M_t = A^T\Big\{&(\Sigma^{-1} + M_{t+1}) - (\Sigma^{-1} + M_{t+1})B \\
&\Gamma_{t+1}B^T(\Sigma^{-1} + M_{t+1})\Big\}A, \quad (16)
\end{aligned}$$

$$\begin{aligned}
g^T_t = A^T\Big\{&g^T_{t+1} + 2(\Sigma^{-1} + M_{t+1})\tilde{x}_t \\
&-2(\Sigma^{-1} + M_{t+1})B\Gamma_{t+1}[0.5B^T g^T_{t+1} \\
&+B^T(\Sigma^{-1} + M_{t+1})\tilde{x}_t - \Gamma^{-1}\hat{u}_{t+1}]\Big\}, \quad (17)
\end{aligned}$$

$$\begin{aligned}
\omega_t = &\hat{u}^T_{t+1}\Gamma^{-1}\hat{u}_{t+1} + \tilde{x}^T_t(\Sigma^{-1} + M_{t+1})\tilde{x}_t \\
&+\omega_{t+1} - [0.5B^T g^T_{t+1} + B^T(\Sigma^{-1} + M_{t+1})\tilde{x}_t \\
&-\Gamma^{-1}\hat{u}_{t+1}]^T\Gamma_{t+1}[0.5B^T g^T_{t+1} - \Gamma^{-1}\hat{u}_{t+1} \\
&+B^T(\Sigma^{-1} + M_{t+1})\tilde{x}_t] + \mathrm{tr}(M_{t+1}\Sigma) + g_{t+1}\tilde{x}_t \\
&+\ln|I + (B\Gamma^{0.5})^T(\Sigma^{-1} + M_t)(B\Gamma^{0.5})|, \quad (18)
\end{aligned}$$

is the quadratic cost function. Here ω_{t+1} is some positive constant, i.e ω_{t+1} is tracking error independent.

Proof: The theorem will be proven by backward induction. This will be achieved by verifying the quadratic form of $-\ln(\gamma(e_{t+1}))$ defined in Equation (15) for $t = T, T-1, \ldots, 0$. As can be seen from proposition 1, the starting value $-\ln(\gamma(e_T)) = 0$ adheres to this form for $M_T = 0, g_T = 0$ and $\omega_T = 0$. For $t \leq T$, the definitions of the functions $\beta_1(u_{t+1}, e_t), \beta_2(u_{t+1}, e_t)$ and $\gamma(e_t)$ given in Equation (6) imply the form of the exponent of the optimal control law. This can be shown through the evaluation of these functions which yields the recursive equations for evaluating M_t, g_t and ω_t as defined in Equations (16)-(18) respectively.

The evaluation of $\beta_1(u_{t+1}, e_t)$ can be easily obtained as follows,

$$\begin{aligned}
&\beta_1(u_{t+1}, e_t) = \int \mathcal{N}(Ae_t + Bu_{t+1} + \tilde{x}_t, \Sigma) \\
&\Big\{-0.5[e_{t+1} - (Ae_t + Bu_{t+1} + \tilde{x}_t)]^T\Sigma^{-1}[e_{t+1} \\
&-(Ae_t + Bu_{t+1} + \tilde{x}_t)] + 0.5e^T_{t+1}\Sigma^{-1}e_{t+1}\Big\}de_{t+1} \\
&= 0.5(Ae_t + Bu_{t+1} + \tilde{x}_t)^T\Sigma^{-1}(Ae_t + Bu_{t+1} + \tilde{x}_t) \quad (19)
\end{aligned}$$

The evaluation of $\beta_2(u_{t+1}, e_t)$ is based on the assumed quadratic form of $-\ln(\gamma(e_{t+1}))$ defined in Equation (15). With the assumed quadratic form of $-\ln(\gamma(e_{t+1}))$ we obtain,

$$\begin{aligned}
&\beta_2(u_{t+1}, e_t) = 0.5\int \mathcal{N}(Ae_t + Bu_{t+1} + \tilde{x}_t, \Sigma) \\
&\Big\{0.5(e^T_{t+1}M_{t+1}e_{t+1} + g_{t+1}e_{t+1} + \omega_{t+1})\Big\}de_{t+1} \\
&= 0.5(Ae_t + Bu_{t+1} + \tilde{x}_t)^T M_{t+1}(Ae_t + Bu_{t+1} + \tilde{x}_t) \\
&+ 0.5[\mathrm{tr}(M_{t+1}\Sigma) + \omega_{t+1} + g_{t+1}(Ae_t + Bu_{t+1})]. \quad (20)
\end{aligned}$$

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The function $\gamma(e_t)$ can also be calculated as given in proposition 1. It is given by,

$$\begin{aligned}\gamma(e_t) &= \int \mathcal{N}(\hat{u}_{t+1},\Gamma)\exp\Big\{-0.5\Big([Ae_t+Bu_{t+1}+\tilde{x}_t]^T \\ &\Sigma^{-1}[Ae_t+Bu_{t+1}+\tilde{x}_t]+[Ae_t+Bu_{t+1}+\tilde{x}_t]^T \\ &M_{t+1}[Ae_t+Bu_{t+1}+\tilde{x}_t]+g_{t+1}(Ae_t+Bu_{t+1}+\tilde{x}_t) \\ &+\omega_{t+1}+\mathrm{tr}(M_{t+1}\Sigma)\Big)\Big\}du_t \\ &=\exp\Big\{-0.5\Big(\hat{u}_{t+1}\Gamma^{-1}\hat{u}_{t+1}+e_t^TA^T(\Sigma^{-1}+M_{t+1})Ae_t \\ &+\tilde{x}_t^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t+e_t^TA^T(g_{t+1}^T+2(\Sigma^{-1}+M_{t+1})\tilde{x}_t) \\ &+\mathrm{tr}(M_{t+1}\Sigma)+\omega_{t+1}+g_{t+1}\tilde{x}_t-[B^T(\Sigma^{-1}+M_{t+1})Ae_t \\ &-\Gamma^{-1}\hat{u}_{t+1}+B^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t+0.5B^Tg_{t+1}^T]^T\Gamma_{t+1} \\ &[B^T(\Sigma^{-1}+M_{t+1})Ae_t-\Gamma^{-1}\hat{u}_{t+1}+B^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t \\ &+0.5B^Tg_{t+1}^T]+\ln|I+(B\Gamma^{0.5})^T(\Sigma^{-1}+M_t)(B\Gamma^{0.5})|\Big)\Big\}. \end{aligned} \tag{21}$$

Noting that $-\ln(\gamma(e_t)) = 0.5(e_t^TM_te_t+g_te_t+\omega_t)$, it can be seen that the identity is satisfied for M_t, g_t and ω_t as defined in Equations (16), (17), and (18) respectively. This proves the claimed quadratic nature of the performance function.

Similarly, the optimal control law can be obtained as specified in proposition 1. Using Equations (19), (20), (21) and (11) in the first equation of Proposition 1 and simplifying, yields,

$$\begin{aligned}c^*(u_{t+1}|e_t) &= \exp\Big\{-0.5\Big(\Big[u_{t+1}^T[\Gamma^{-1} \\ &+B^T(\Sigma^{-1}+M_{t+1})B]u_{t+1}+2u_{t+1}^T\{0.5B^Tg_{t+1}^T \\ &+B^T(\Sigma^{-1}+M_{t+1})Ae_t+B^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t \\ &-\Gamma^{-1}\hat{u}_{t+1}\}+[B^T(\Sigma^{-1}+M_{t+1})Ae_t-\Gamma^{-1}\hat{u}_{t+1} \\ &+B^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t+0.5B^Tg_{t+1}^T]^T\Gamma_{t+1} \\ &[B^T(\Sigma^{-1}+M_{t+1})Ae_t-\Gamma^{-1}\hat{u}_{t+1} \\ &+B^T(\Sigma^{-1}+M_{t+1})\tilde{x}_t+0.5B^Tg_{t+1}^T] \\ &+\ln|I+(B\Gamma^{0.5})^T(\Sigma^{-1}+M_{t+1})(B\Gamma^{0.5})|\Big)\Big\}. \end{aligned} \tag{22}$$

Completing the square in Equation (22) for u_{t+1} gives the randomised controller defined in Equation (13) with K_{t+1}, L_{t+1} and Γ_{t+1} as defined in Equation (14).

Compared to the conventional FPD, the optimal cost-to-go function of the MRFPD is now dependent on the tracking error, e_t rather than the state x_t. It also has an additional linear term, g_{t+1}, as can be seen from Equation (15). Additionally, the mean of the derived randomised controller is shifted by L_{t+1} as can be seen from Equation (13). The manifestation of these terms in the optimal cost-to-go function and randomised controller is the consequence of the defined control objective for the system state to track a desired state value. These two terms account for the constraints imposed by the tracking control objective. Finally, The derived control gain, K_{t+1} and Riccati equation, M_{t+1} are similar in form to the conventional FPD and LQR, where the tracking error penalisation is now equal to the covariance of the ideal distribution of the system tracking error, Σ^{-1} and input penalisation equal to the ideal randomised controller covariance matrix, Γ^{-1}. In fact, the derived randomised controller (13) can be interpreted as a standard LQR controller to which a white noise with zero mean and Γ_{t+1} covariance is added. Compared to traditional LQR, the additional terms in Equation (18) increase the value of the attained cost-to-go function, thus can be viewed as an extra cost for the use of the randomised controller. As has also been discussed earlier, similar results could be obtained if a stochastic reference model is used. Other generalisation are also possible, for instance an ideal tracking error pdf can be specified with a different covariance matrix than the covariance matrix of the actual tracking error.

All the parameters of the derived randomised controller (13) can be computed using standard methods following either the backward or forward approaches. Using the derived randomised controller, the conditional distribution of the system tracking error conditioned on previous error values can be obtained as follows,

$$s(e_{t+1}|e_t) = \int s(e_{t+1}|e_t,u_{t+1})c^*(u_{t+1}|e_t)du_{t+1}, \tag{23}$$

where $s(e_{t+1}|e_t,u_{t+1})$ defined in Equation (9) is the Gaussian distribution of the system tracking error, and $c^*(u_{t+1}|e_t)$ defined in Equation (13) is the derived randomised controller. Substituting Equations (9) and (13) in Equation (23) and integrating over u_{t+1} yields,

$$s(e_{t+1}|e_t) = \mathcal{N}(V_{t+1}e_t+H_{t+1}, B\Gamma_{t+1}B'+\Sigma), \tag{24}$$

with,

$$\begin{aligned} V_{t+1} &= A-BK_{t+1}, \\ H_{t+1} &= BL_{t+1}+\tilde{x}_t. \end{aligned} \tag{25}$$

Again here, since K_{t+1} and L_{t+1} are computable, V_{t+1} and H_{t+1} can also be computed. Following the methodology discussed in [26], the solution to the probabilistic model (24) can be easily verified to be given by,

$$s(e_t|e_0) = \mathcal{N}\Big(\mathfrak{E}_t, \Upsilon_t\Big), \tag{26}$$

with,

$$\begin{aligned} \mathfrak{E}_t &= \Phi(t,1)e_0+\sum_{i=2}^{t}\Phi(t,i)H_{i-1}+H_t, \\ \Upsilon_t &= \sum_{i=2}^{t}\Phi(t,i)Q_{i-1}\Phi^T(t,i)+Q_t, \\ Q_i &= B\Gamma_iB'+\Sigma, \\ \Phi(n,m) &= V_nV_{n-1}\ldots V_m. \end{aligned} \tag{27}$$

The above result can be also obtained by evaluating the joint pdf $s(e_t,e_{t-1},\ldots,e_1|e_0)=\prod_{i=0}^{t-1}s(e_{i+1}|e_i)$ and then integrating over $e_1,e_2,\ldots e_{t-1}$. Using this result, the open loop control is given by,

$$c^*(u_{t+1}|e_0) = \int c^*(u_{t+1}|e_t)s(e_t|e_0)de_t. \tag{28}$$

Using Equation (13) in the above equation we get,

$$c^*(u_{t+1}|e_0) = \mathcal{N}(-D_{t+1}e_0 + E_{t+1}, K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}), \quad (29)$$

with,

$$D_{t+1} = K_{t+1}\Phi(t,1),$$
$$E_{t+1} = -K_{t+1}\left(\sum_{i=2}^{t}\Phi(t,i)H_{i-1} + H_t\right) + L_{t+1}. \quad (30)$$

III. Solution to the MRFPD for Linear Stochastic Systems with Input Delays

The previous section presented the solution to the MRFPD for linear systems assuming that the delay between the control input and the system state is zero. However, for stochastic systems where the delay between the control input and the system state is greater than zero the design of a randomised controller becomes challenging. To elaborate, consider the following class of linear stochastic systems with input delay $\tau > 0$,

$$s(x_{t+1}|x_t, u_{t+1-\tau}) = \mathcal{N}(Ax_t + Bu_{t+1-\tau}, \Sigma), \quad (31)$$

where τ is an integer value. The initial values, x_0, and $\{u_{-\tau+1}, u_{-\tau+2}, \ldots, u_0\}$ that are needed in order to trace the system state forward through time are known. As can be seen from this equation, the main challenge here is the strict causality requirement of the designed randomised controller. For the considered MRFPD, strict causality means that the designed randomised controller must be in the form of,

$$c(u_{t+1} \mid e_t, \ldots, e_0, u_t, \ldots, u_{-\tau+1}) = c(u_{t+1} \mid e_t),$$

where here, the postulated equivalence means that the controller satisfies the Markov property. For deterministic control problems, the optimal causal controller can be obtained by using the smith predictor theory [27] to provide τ-step prediction of the future state $x_{t+\tau}$. Unfortunately, for the considered MRFPD method which is based on a fully probabilistic framework the deterministic τ-step Smith predictor cannot be used directly and need to be generalised such that the τ-step conditional distribution of the system tracking error can be obtained. This procedure is discussed in the following.

Similar to the MRFPD for stochastic systems with zero delay that is discussed in Section II-C, the MRFPD control objective here is to design a randomised controller $c(u_{t+1-\tau}|e_t)$. Note that the conditioning of $u_{t+1-\tau}$ on e_t indicates a non-causal randomised controller. As can be seen from the pdf of the system error, $s(e_{t+1}|e_t, u_{t+1-\tau})$ this randomised controller will not of course begin to affect the distribution of the system error until time $t+\tau$. However, as will be shown in further development, the causal randomised controller can be obtained by evaluating the predictive conditional distribution of the system tracking error. On the other hand, since the system is controlled by a lagged version of the control input, the obtained randomised controller for the stochastic systems with no input delays can be applied, but advanced in time by τ units. This can be done for $t > \tau$, but for the time interval $0 \le t \le \tau$, the system state would have been already affected by initial control values prior to $t = 0$. Therefore, a predictor of the tracking error distribution based on past controls needs to be identified. Consequently, the predictive pdf of the system tracking error at time τ can be obtained by considering the joint pdf of the system tracking error,

$$s(e_\tau, e_{\tau-1}, \ldots, e_1, u_0, \ldots, u_{1-\tau}|e_0) = \prod_{t=0}^{\tau-1} s(e_{t+1}|e_t, u_{t+1-\tau})c^*(u_{t+1-\tau}|e_t). \quad (32)$$

Integrating the above joint pdf over all control inputs and states from time 0 up to $\tau - 1$ yields the required predictive distribution,

$$s(e_\tau, |e_0) = \int\cdots\int \prod_{t=0}^{\tau-1} s(e_{t+1}|e_t, u_{t+1-\tau}) c^*(u_{t+1-\tau}|e_t)du_{t+1-\tau}de_t \quad (33)$$

Hence,

$$s(e_\tau|e_0) = \mathcal{N}(\mathscr{E}_\tau, \Omega_\tau), \quad (34)$$

with,

$$\mathscr{E}_\tau = A^\tau e_0 + \sum_{i=0}^{\tau-1} A^{\tau-j-1}[B < u_{i+1-\tau} > +\tilde{x}_j],$$
$$\Omega_\tau = \sum_{i=0}^{\tau-1} A^{\tau-j-1}\Sigma(A^{\tau-j-1})^T + \sum_{i=0}^{\tau-1} A^{\tau-j-1}B\Gamma_{i+1-\tau}(A^{\tau-j-1}B)^T. \quad (35)$$

Then the open loop control for $t \ge 0$, can be obtained by noticing that the lagged controller will be the same as that given in Equation (29) for the unlagged one, but now dependent on e_τ rather than e_0,

$$c^*(u_{t+1}|e_\tau) = \mathcal{N}(-D_{t+1}e_\tau + E_{t+1}, K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}). \quad (36)$$

Using the predictive distribution of the tracking error defined in Equation (34), the above equation can be rewritten as follows,

$$c^*(u_{t+1}|e_0) = \int c^*(u_{t+1}|e_\tau)s(e_\tau|e_0)de_\tau,$$
$$= \mathcal{N}(-D_{t+1}\mathscr{E}_\tau + E_{t+1}, D_{t+1}\Omega_\tau D_{t+1}^T + K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}). \quad (37)$$

The objective then is to obtain the causal randomised controller, $c^*(u_{t+1}|e_t)$. To achieve this objective, we need to write e_0 which affects the mean value of the distribution of the randomised controller $c^*(u_{t+1}|e_0)$ in terms of e_t. This can be done by evaluating the predictive distribution, $s(e_t|e_0)$ in the interval $0 \le t \le \tau$. Following the same procedure for obtaining the predictive distribution of e_τ, the predictive distribution of e_t can be shown to be given by,

$$s(e_t|e_0) = \mathcal{N}(\mathscr{E}_t, \Omega_t), \quad (38)$$

with,

$$
\begin{aligned}
\mathscr{E}_t &= A^t e_0 + \sum_{i=0}^{t-1} A^{t-j-1}[B < u_{i+1-\tau} > + \tilde{x}_j], \\
\Omega_t &= \sum_{i=0}^{t-1} A^{t-j-1}\Sigma(A^{t-j-1})^T \\
&\quad + \sum_{i=0}^{t-1} A^{t-j-1}B\Gamma_{i+1-\tau}(A^{t-j-1}B)^T,
\end{aligned} \tag{39}
$$

where $< u_{i+1-\tau} >$ is the expected value of the control input at time $i+1-\tau$. Using transformation of random variables, the pdf of the initial tracking error, e_0 can be obtained from that of e_t to give,

$$
\begin{aligned}
& s(e_0|e_t) = \mathcal{N}(\mathscr{E}_0, \Omega_0), \\
& \mathscr{E}_0 = A^{-t}e_t - A^{-t}\sum_{i=0}^{t-1} A^{t-j-1}[B < u_{i+1-\tau} > + \tilde{x}_j], \\
& \Omega_0 = A^{-t}\Omega_t(A^{-t})^T.
\end{aligned} \tag{40}
$$

Using Equations (40) and (37), we obtain,

$$
\begin{aligned}
c^*(u_{t+1}|e_t) &= \int c^*(u_{t+1}|e_0)s(e_0|e_t)de_0, \\
&= \mathcal{N}(< u_{t+1} >, \Gamma'_{t+1}),
\end{aligned} \tag{41}
$$

with,

$$
\begin{aligned}
& < u_{t+1} > = -D_{t+1}A^\tau A^{-t}e_t \\
& + D_{t+1}A^\tau A^{-t}\sum_{i=0}^{t-1} A^{t-j-1}[B < u_{i+1-\tau} > + \tilde{x}_j] \\
& - D_{t+1}\sum_{i=0}^{\tau-1} A^{\tau-j-1}[B < u_{i+1-\tau} > + \tilde{x}_j] + E_{t+1},
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
& \Gamma'_{t+1} = D_{t+1}A^\tau\Omega_0(D_{t+1}A^\tau)^T + D_{t+1}\Omega_\tau D_{t+1}^T \\
& + K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}.
\end{aligned} \tag{43}
$$

Note that when $t = \tau$, the distribution of randomised controller given in the above equation reduces to,

$$
\begin{aligned}
& c^*(u_{t+1}|e_t) = \mathcal{N}(-D_{t+1}e_t + E_{t+1}, \\
& D_{t+1}\Omega_t D_{t+1}^T + K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}).
\end{aligned} \tag{44}
$$

On the other hand, when $t \geq \tau$, we have,

$$
c(u_{t+1}|e_{t+\tau}) = \mathcal{N}(-K_{t+1}e_{t+\tau} + L_{t+1}, \Gamma_{t+1}). \tag{45}
$$

To obtain the randomised controller in that interval $t \geq \tau$, the τ-step predictive distribution of the future error needs to be evaluated. In particular, we need to evaluate $s(e_{t+\tau}|e_t)$. This can be obtained from Equation (24) to give,

$$
s(e_{t+\tau}|e_t) = \mathcal{N}\left(\tilde{\mathfrak{E}}_t, \tilde{\Upsilon}_t\right), \tag{46}
$$

with,

$$
\begin{aligned}
& \tilde{\mathfrak{E}}_t = \Phi(\tau,1)e_t + \sum_{i=2}^{\tau}\Phi(\tau,i)H_{i-1} + H_\tau, \\
& \tilde{\Upsilon}_t = \sum_{i=2}^{\tau}\Phi(\tau,i)Q_{i-1}\Phi^T(\tau,i) + Q_\tau \\
& + \Phi(\tau,1)Q_t\Phi^T(\tau,1),
\end{aligned} \tag{47}
$$

and where Q_i and $\Phi(n,m)$ have the same definitions as before. Consequently, the causal controller, $c^*(u_{t+1}|e_t)$ for $t \geq \tau$ can be obtained as follows,

$$
\begin{aligned}
& c^*(u_{t+1}|e_t) = c^*(u_{t+1}|e_{t+\tau})s(e_{t+\tau}|e_t), \\
& = \mathcal{N}(-K_{t+1}\tilde{\mathfrak{E}}_t + L_{t+1}, K_{t+1}\tilde{\Upsilon}_t K_{t+1}^T + \Gamma_{t+1}).
\end{aligned} \tag{48}
$$

Now it is straight forward to complete and summarise the obtained results.

Proposition 2: Equations (41) and (48) give the randomised optimal control for $0 \leq t \leq \tau$ and for $t \geq \tau$, respectively. Substituting for D_{t+1} and E_{t+1} from Equation (30), combining the second and third terms in (42) and the first and second terms in (43), and rewriting give the mean and covariance of the randomised optimal control law as follows.

For $0 \leq t \leq \tau$

$$
\begin{aligned}
& < u_{t+1} > = -K_{t+1}\Phi(t,1)A^\tau A^{-t}e_t \\
& + K_{t+1}\Phi(t,1)\sum_{i=0}^{\tau-1} A^{\tau-j-1}[B < u_{i+1-\tau} > \tilde{x}_j] \\
& - K_{t+1}\left(\sum_{i=2}^{t}\Phi(t,i)H_{i-1} + H_t\right) + L_{t+1},
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
& \Gamma'_{t+1} = K_{t+1}\Phi(t,1)\Omega_t(K_{t+1}\Phi(t,1))^T \\
& + K_{t+1}\Upsilon_t K_{t+1}^T + \Gamma_{t+1}.
\end{aligned} \tag{50}
$$

For $t \geq \tau$

$$
\begin{aligned}
& < u_{t+1} > = -K_{t+1}\Phi(\tau,1)e_t \\
& - K_{t+1}\left(\sum_{i=2}^{\tau}\Phi(\tau,i)H_{i-1} + H_\tau\right) + L_{t+1},
\end{aligned} \tag{51}
$$

$$
\Gamma'_{t+1} = K_{t+1}\tilde{\Upsilon}_t K_{t+1}^T + \Gamma_{t+1}. \tag{52}
$$

IV. Numerical Example

Consider the system (31) with

$$
\begin{aligned}
& A = \begin{bmatrix} 0.9935 & 0.0093 \\ -0.0156 & 0.9912 \end{bmatrix}, \; B = \begin{bmatrix} -0.00001865 \\ -0.0011 \end{bmatrix}, \\
& \Sigma = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \; \tau = 5, u_{-4} = -0.2, \\
& u_{-3} = 0, \; u_{-2} = -0.1, \; u_{-1} = -0.2, \; u_0 = 0.1.
\end{aligned}
$$

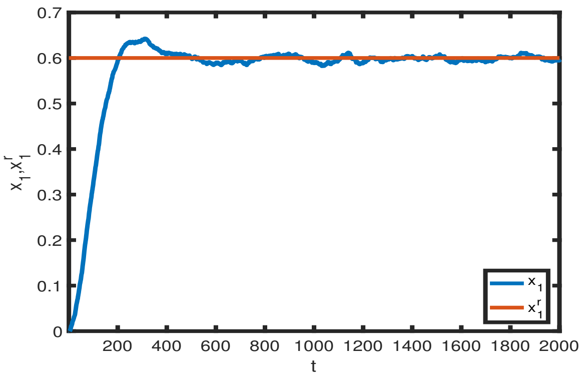
We also specify,

$$
\Gamma = 0.07, \; x_{t+1}^r = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}^T.
$$

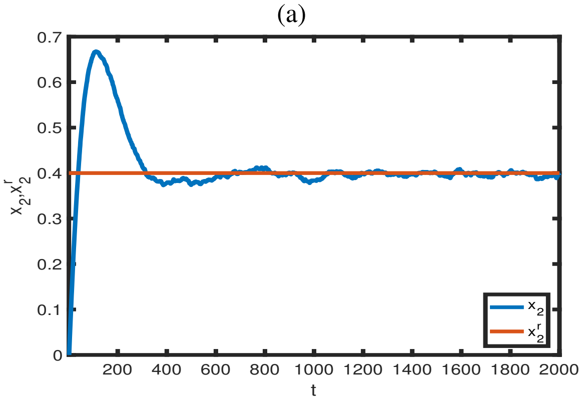
The gain and linear term of the proposed optimal randomised controller are calculated following Equation (14) and their steady state values were found to be,

$$
K = \begin{bmatrix} 0.2428 & -2.3914 \end{bmatrix}, \quad L = -11.6340.
$$

The simulation result of the designed controller is shown in Figure 2.



(a)



(b)

Fig. 2. Example 2: The state of the controlled system. (a) the first state of the controlled system, (b) the second state of the controlled system.

V. Conclusion

In this paper, the optimal control for stochastic discrete time systems with input delay has been considered. A fully probabilistic controller has been designed and presented for this class of stochastic systems with input delay. In addition, the conventional solution of the fully probabilistic randomised controller is extended to consider tracking problems with the objective that the system state follows a predefined desired state value. The derived solution is tested on a numerical example. Numerical simulation proved the efficacy of the derived randomised controller and showed that it can influence the system dynamics such that their state is maintained at the pre-specified desired value. Future work will discuss and demonstrate the extension of the fully probabilistic control method to systems that have multiple input and or state delays.

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