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# Fast-Converging Tatonnement Algorithms for the Market Problem

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**Abstract**

Why might markets tend toward and remain near equilibrium prices? In an effort to shed light on this question from an algorithmic perspective, this paper defines and analyzes two simple tatonnement algorithms that differ from previous algorithms that have been subject to asymptotic analysis in three significant respects: the price update for a good depends only on the price, demand, and supply for that good, and on no other information; the price update for each good occurs distributively and asynchronously; the algorithms work (and the analyses hold) from an arbitrary starting point.

Our algorithm introduces a new and natural update rule. We show that this update rule leads to fast convergence toward equilibrium prices in a broad class of markets that satisfy the weak gross substitutes property. These are the first analyses for computationally and informationally distributed algorithms that demonstrate polynomial convergence.

Our analysis identifies three parameters characterizing the markets, which govern the rate of convergence of our protocols. These parameters are, broadly speaking:

1. A bound on the fractional rate of change of demand for each good with respect to fractional changes in its price.
2. A bound on the fractional rate of change of demand for each good with respect to fractional changes in wealth.
3. The relative demand for money at equilibrium prices.

We give two protocols. The first assumes global knowledge of only the first parameter. For this protocol, we also provide a matching lower bound in terms of these parameters. Our second protocol assumes no global knowledge whatsoever.

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# 1 Introduction

The impetus for this work comes from the following question: why are well-functioning markets able to stay at or near equilibrium prices?[1] This raises two issues: what are plausible price adjustment mechanisms and in what types of markets are they effective?

This question was originally considered by Walras in 1874, when he suggested that prices adjust by tatonnement: upward if there is too much demand and downward if too little [26]. Since then, the study of market equilibria, their existence, stability, and their computation has received much attention in Economics, Operations Research, and most recently in Computer Science. Of late, this has led to a considerable number of polynomial time algorithms for finding approximate and exact equilibria in a variety of markets with divisible goods. However, these algorithms do not seek to, and do not appear to provide methods that might plausibly explain these markets' behavior.

We argue here for the relevance of this question from a computer science perspective. Much justification for looking at the market problem in computer science stems from the following argument: If economic models and statements about equilibrium and convergence are to make sense as being realizable in economies, then they should be concepts that are computationally tractable. Our viewpoint is that it is not enough to show that the problems are computationally tractable; it is also necessary to show that they are tractable in a model that might capture how a market works. Unless one has a controlled economy, markets surely do not perform overt global computations, using global information.

In formalizing the tatonnement model, economists have proposed models to capture aspects of how a market might work; and convergence of several of these formalizations has been demonstrated for some types of markets [1, 2, 18, 24]. However, there is no demonstration that these proposed models converge reasonably quickly. Indeed, without care in the specific details, they won't.[2]

We propose a model that captures important characteristics of models proposed in the economic literature, features that are lacking from previous algorithms subject to asymptotic analysis. Namely, our algorithms consist of price updates satisfying the following three criteria: the price update for a good depends only on the price, demand, and supply for that good, and on no other information about the market; the price update for each good occurs distributively and asynchronously; the algorithms can start with an arbitrary set of prices. We show that our update protocols converge quickly in markets that satisfy the weak gross substitutes property. In the process, we identify three natural parameters characterizing markets that govern the rate of convergence.

Another distributed price update protocol was given in a STOC 2005 paper of Codenotti, McCune, and Varadarajan [4]. However, it does not meet our criteria, which is not surprising as it was not seeking to address the question raised in our paper.[3]

At this point it will be helpful to define the market problem.

**The Market Problem.** A market consists of set of goods $G$, with $|G| = n$, and agents $A$, with $|A| = m$. The goods are assumed to be infinitely divisible. Each agent $l$ starts with an allocation $w_{il}$ of good $i$. Each agent $l$ has a utility function $u_l(x_{1l}, \cdots, x_{nl})$ expressing its preferences: if $l$ prefers a basket with $x_{il}$ units (possibly a real number) of good $i$, to the basket with $y_{il}$ units, for $1 \le i \le n$, then $u_l(x_{1l}, \cdots, x_{nl}) > u_l(y_{1l}, \cdots, y_{nl})$. Each agent $l$ intends to trade goods so as to achieve a personal optimal combination (basket) of goods given the constraints imposed by their initial allocation. The trade is driven by a collection of prices $p_i$ for good $i$, $1 \le i \le n$. Agent $l$ chooses $x_{il}$, $1 \le i \le n$, so as to maximize $u_l$, subject to the basket being affordable, that is: $\sum_{i=1}^{n} x_{il}p_i \le \sum_{i=1}^{n} w_{il}p_i$. Prices $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ are said to provide an *equilibrium* if, in addition, the demand for each good is bounded by the supply: $\sum_{j=1}^{m} x_{il} \le \sum_{l=1}^{m} w_{il}$. The market problem is to find equilibrium prices.[4]

**Standard notation:** $w_i = \sum_l w_{il}$ is the supply of good $i$. $x_i = \sum_l x_{il}$ is the demand for good $i$, and $z_i = x_i - w_i$ is the excess demand for good $i$ (which can be positive or negative). $v_l = \sum_i w_{il}p_i$ is the wealth of buyer $l$ given prices $\mathbf{p}$. Note that while $w$ is part of the specification of the market, $v$, $x$ and $z$ are functions of the vector of prices: $v$ directly so, and $x$ and $z$ as determined by individual agents maximizing their utility functions subject to $v$.

[1] We are not concerned with the question of whether this assertion is indeed correct.

[2] Of the referenced papers, only one formalization [24] is a discrete algorithm, and, as we make more specific later, it may not converge quickly.

[3] For this protocol first transforms the market using information about all the goods (*global* information); it uses a global approximation parameter in the price updates; it uses synchronous price updates; and it starts with an initial price point that is restricted to lie within a bounded region containing the equilibrium point.

[4] Equilibria exist under quite mild conditions (see [16] §17.C, for example).

We follow standard practice[5] and view the actions of individual buyers and sellers as being encapsulated in the price adjustments for each good. More specifically, we imagine that there is a separate, "virtual" price setter for each good in the market. Henceforth, for ease of exposition, we describe price setters as if they were actual entities, although in reality they are virtual entities induced by agents' trades.

**Market Properties.** In an effort to capture the distributed nature of markets and the likely limited computational power of individual interactions and consequently of each of the virtual price setters, we impose several constraints on procedures we wish to consider:

1. Limited information: the (virtual) price setter for good $i$ knows only the price, supply, and excess demand of good $i$, both current and past history. Thus the price updates can depend on this information only. Notably, this precludes the use not only of other prices or demands, but also of any information about the specific form of utility functions.
2. Simple actions: The price setters' procedures should be simple.
3. Asynchrony: Price updates for different goods are allowed to be asynchronous.
4. Fast Convergence: The price update procedure should converge quickly toward equilibrium prices from any initial price vector.

We call procedures that satisfy the first three constraints *local*, by contrast with centralized procedures that use complete (global) information about the market.

Next, we discuss the motivations for these constraints.

Constraint (1) stems from the plausible assertion that not everything about the market will be known to a single price setter. While no doubt some information about several goods is known to a price setter, it is a conservative assumption to assume less is known, for any convergence result carries over to the broader setting. Further, it is far from clear how to model the broader setting.

Constraint (2), simplicity, is in the eye of the beholder. Its presence reflects our view that without further information, this is both generally applicable and plausible.

Constraint (3), asynchrony, is an inherent property of independent price adjustments. Since the price setter of good $i$ reacts only to trade in good $i$, the price adjustment of good $i$ occurs independently of other price adjustments.

Constraint (4) arises in an effort to recognize the dynamic nature of real markets, which are subject to changing supplies and demands over time. However, surely much of the time, markets are changing gradually, for otherwise there would be no predictability. A natural approximation is to imagine fixed conditions and seek to come close to an equilibrium in the time they prevail — hence the desire for rapid convergence.

## 1.1 Our Contribution

We describe new and natural *local* price update protocols that converge quickly toward equilibrium prices starting from arbitrary initial prices: the longer they run, the closer they come. To specify this more precisely we need to define the computational model, our complexity measure, and our measure for approximation quality.

**Computational Model:** Since we are proposing a model for how a market might reach equilibrium, instead of how one might compute an equilibrium given all the information about the market, our computational model is a bit different from the standard computer science model. Our model is based on iterations, defined below.

*Iteration r:*

1. Simultaneously for each good $i$ in some subset $G^r$ of goods, the price setter for good $i$ updates the price of good $i$ using knowledge only of $p_i$, $z_i$, and the history of $p_i$ and $z_i$.
2. Given new prices $\mathbf{p}^r$, agents compute the wealth they could achieve by selling all their goods, and express their interest in a set of goods that maximizes their utility subject to their current wealth. Thus, utility functions are revealed only implicitly and partially through the aggregate demands for goods subject to a price vector.

It might seem more natural that the price setter for a good $i$ that has not yet updated the price would use an old value of the excess demand, or perhaps some convex combination of the excess demands seen since the last price update. As it turns out, our analysis works for any of these variants, yielding the same tight bounds without any change to the argument. This provides further evidence that the update procedure proposed here is robust.

[5] See Varian [25] §21.5.

**Complexity measure:** As is standard for asynchronous algorithms, we measure the complexity in rounds. The basic unit of time is a price update iteration as specified above. A *round* comprises a minimum length sequence of iterations in which every price updates at least once. The rounds are specified uniquely by defining them beginning from a fixed start time.

**b-Bounded asynchrony:** Sometimes it is useful to limit the extend of the asynchrony. We define $b$-bounded asynchrony to impose the requirement that in a single round any price updates at most $b$ times.

**Approximation quality:** The main approach in the Computer Science literature has been to define the quality of an equilibria as $1-\epsilon = \min_l\{u_l(A_l)/u_l(Opt_l)\}$, where the minimization is over all agents $l$ in the market, $u_l$ is $l$'s utility function, $A_l$ is the allocation of goods agent $l$ receives and $Opt_l$ is its preferred affordable allocation at the prevailing prices (i.e over all agents, what is the worst allocation percentage-wise measured in terms of the utility functions). This does not seem a feasible approach in our setting, where no allocation mechanism is specified, where there is no direct knowledge of the agents' utilities, and our algorithms are just responding to excess demands and not to the degree to which agents wish to change their allocations. Instead, we simply measure the distance from the equilibrium prices, $p_i^*$, directly: $\max_i |p_i^* - p_i|/p_i^*$.

**Our update algorithms:** We analyze protocols where price setters use the rule

$$p_i \leftarrow p_i(1+\lambda_i \min\{1, z_i/w_i\}). \tag{1}$$

The price of money remains at one.

This is a novel proposal for a price update. For example, it differs significantly from the update suggested by Uzawa [24] in that it scales a bounded excess demand by the current price. These differences are crucial for enabling a proof of rapid convergence. In particular, the min term prevents overreaction to large values of $z_i$; these can be unbounded in their effect in Uzawa's algorithm. The scaling by $p_i$ can also improve the rate of convergence significantly.

We begin by analyzing the protocol when $\lambda_i$ is fixed for all goods $i$, and given by a simple characteristic of the market. Our motivation for this is two-fold. First, in stable markets, it seems reasonable that the appropriate stepsizes for the price adjustments are known (within constant factors). Second, such an analysis has not been done before, and it is helpful to understand this case first.

We then analyze an *oblivious* protocol where the appropriate choice of $\lambda_i$ is not known at the start, and is therefore repeatedly adjusted to ensure that it is eventually small enough for convergence. This is intended to capture markets that are adjusting to new conditions.

The performance of both algorithms depends on several global parameters of the market. These relate to how effectively surpluses and scarcities signal the level of price inequities. The parameters are denoted by $\alpha, \beta, E$, where $\alpha, \beta \le 1$ and $E \ge 1$. The parameter $\alpha$ reflects the agents' collective utility for money relative to other goods, at equilibrium prices. [6] As we show (see Section 3), the smaller the relative utility, the slower the convergence rate. The parameter $E$ bounds the elasticity of demand with respect to prices. A large $E$ implies that small adjustments to the prices may have a large effect on demand, thus for convergence, it is necessary to have smaller price adjustments. As we show in this paper, it suffices that $\lambda_i = O(1/E)$. Finally, $\beta$ parametrizes a lower bound on the rate of change of demand with respect to changes in wealth. If $\beta$ is small, then even a large change in price might have only a small effect on demand. Thus, any procedure basing price updates on excess demand would require more updates when $\beta$ is smaller.

In suitable markets (specified in Section 3), we show that our first algorithm improves the accuracy of the least accurate price by at least one bit in $O(E/(\alpha\beta))$ rounds. (For small prices, this means doubling the price; for large ones, this means halving it; and for prices $p_i$ close to equilibrium price $p_i^*$, this means halving $|p_i^* - p_i|$.) In the Appendix we show that there are examples for which this complexity bound is tight for our update procedures.

Our second algorithm is *oblivious* in that it does not assume that a convergent value for $\lambda_i$ is known. Instead the protocol for good $i$ gradually reduces the value of $\lambda_i$. To obtain a complexity bound, we need to assume $b$-bounded asynchrony, and then we obtain roughly the square of the above complexity. Specifically, to obtain a $d$-bit improvement in accuracy requires $O(\frac{b}{\alpha^2\beta^2}(E^2+d^2))$ rounds.

## 1.2 Previous Work

To the best of our knowledge, asynchronous price update algorithms have not been considered previously. Further, there has been no complexity analysis of even synchronous tatonnement algorithms with this type of limited information. While

[6] In Section 2 we define the concept of a numeraire. We identify money as the numeraire, since it is a natural choice. The parameter $\alpha$ reflects the relative utility for the numeraire, which in this case is money.

Uzawa [24] gave a synchronous algorithm of this type, he only showed convergence, and did not address speed of convergence.

The existence of market equilibria has been a central topic of economics since the problem was formulated by Walras in 1874 [26]. Tatonnement was described more precisely as a differential equation by Samuelson [20]:

$$dp_i/dt = \mu_i z_i. \tag{2}$$

The $\mu_i$ are arbitrary positive constants that represent rates of adjustment for the different prices; they need not all be the same. Arrow, Block, and Hurwitz, and Nikaido and Uzawa [1, 2, 18] showed that for markets of gross substitutes the above differential equation will converge to an equilibrium price.

Unfortunately, for general utility functions (i.e. that do not lead to gross substitutability), the equilibrium need not be stable and the differential equation (and thus also discretized versions) need not converge [21]. Partly in response, Smale described a convergent procedure that uses the derivative matrix of excess demands with respect to prices [23]. Following this, Saari and Simon [19] showed that any price update algorithm which uses an update that is a fixed function of excesses and their derivatives with respect to prices needs to use essentially all the derivatives in order to converge in all markets. This is viewed as being an excessive amount of information, in general.

There are really two questions here. The first is how to find an equilibrium, and the second is how does the market find an equilibrium. The first question is partially addressed by the work of Arrow et al. and Smale, and addressed further in papers in operations research (notably Scarf [22] gives a (non-polynomial) algorithm for computing equilibrium prices), and theoretical computer science, where there are a series of very nice results demonstrating equilibria as the solutions to convex programs, or describing combinatorial algorithms to compute such equilibria exactly or approximately. (An early example of a polynomial algorithm for computing market equilibria for restricted settings is [7]. An extensive list of references is given in the survey [5].)

We are interested in the second question. The differential equations provide a start here, but they ignore the discrete nature of markets: prices typically change in discrete increments, not continuously. In 1960, Uzawa showed that there is a choice of $\lambda$ for which an obvious discrete analog of (2) does converge [24]. However, determining the right $\lambda$ depends on knowing properties of the matrix of derivatives of demand with respect to price, or in other words, this requires global information.

More recently, three separate groups have proposed three distinct discrete update algorithms for finding equilibrium prices and showed that their algorithms converge in markets of gross substitutes [15, 6, 4]. However, all of these algorithms use global information. With the exception of [4], none of this work gives (good) bounds on the rate of convergence. The algorithm in Codenotti et al. [4] appears to be a local algorithm (albeit not asynchronous); however, it begins by modifying the market by introducing a fictitious player with some convenient properties that capture global information about the market and have a profound effect on market behavior. Translating their algorithm back into the real market, it is neither simple nor local.

There are some auction-style algorithms for finding approximate equilibria which also have a distributed flavor but depend on buyer utilities being separable over the set of goods [10, 11]. However, these algorithms are not seeking to explain market behavior and not surprisingly do not obey natural properties of markets.[7]

The design and analysis of procedures and convergence to equilibria has been a recent topic of study for game theoretic problems as well. Examples include studying convergence in some network routing and network design games [3, 9, 12, 17]. In partial contrast, it is known that finding equilibria via local search (e.g., via best response dynamics) is PLS-complete in many contexts [14, 8]. Recently, Hart and Mansour [13] gave communication complexity lower bounds to show that in general games, players with limited information require an exponential (in the number of players) number of steps to reach an equilibrium.

## 1.3 Road map

In Section 2 we give some relevant definitions. In Section 3 we specify our protocols and results. In Section 4.1 we prove an upper bound on the rate of convergence of the fixed protocol for gross substitute markets with parameters $\alpha, \beta, E$. In Section 4.2, we prove an upper bound on the rate of convergence for the oblivious protocol. In the Appendix, we prove lower bounds on the rate of convergence of the fixed protocol.

[7] These algorithms start their computation at a non-arbitrary set of artificially low prices; global information is used for price initialization; and they work with a global approximation measure — each price update uses the goal approximation guarantee in its update.

## 2 Definitions and Notation

A market satisfies the *gross substitutes property* if for any good $i$, increasing $p_i$ leads to increased demand for all other goods. The market satisfies *weak gross substitutes* if the demand for every other good increases or stays the same. Examples of markets that may satisfy the gross substitutes property include markets of raw materials, energy, airline seats, toll roads. A broad enough market will not satisfy this property. Consider, for example, the market for bread and jam.

Next, we state some common concepts/assumptions regarding the market problem.

**Walras' Law:**[8] For any price vector $\mathbf{p}$,

$$\sum_{i \in G} z_i(\mathbf{p}) p_i = 0. \tag{3}$$

**Homogeneity of degree 0:**[9] For all price vectors $\mathbf{p}$ and scalars $\alpha > 0$, $x(\mathbf{p}) = x(\alpha\mathbf{p})$.

**Numeraire:** Under the assumption of homogeneity, if there is at least one equilibrium price vector, then there is an entire ray of equilibria. It is convenient to use normalization to remove this duplication. A common form of normalization used in the economics literature is the concept of the numeraire: choose one good as the *numeraire*; set its price to 1; scale all other prices accordingly.[10] We use money as the numeraire, and use the index \$ to denote this good. Usually the choice of a good to be the numeraire is viewed as arbitrary. However, as we will see, the rate of convergence of our algorithms also depends on how pervasively the numeraire is present throughout the market, and consequently we do not view it as an arbitrary choice.

**Uniqueness of Equilibria:** It is well-known that under normalization, markets of gross substitutes have a unique equilibrium.[11] Since we focus on markets satisfying gross substitutes, the markets we consider have a unique equilibrium. Throughout the paper we will use the superscript $^*$ to denote a characteristic of an equilibrium. For example, $\mathbf{p}^*$ is the equilibrium price vector; $\mathbf{x}^*$ is the equilibrium demand.

### The Parameters

Here, we define the three parameters $E, \alpha, \beta$ appearing in our analysis.

**Elasticity of Demand and the Parameter $E$:** The *price elasticity of demand* is the fractional rate of change of demand with respect to price: $\frac{\partial x_i/\partial p_i}{x_i/p_i}$. Under the assumption of gross substitutes, this is negative. The parameter $E$ is an upper bound on the absolute value of this quantity over all goods and all prices:

$$E = -\min_{i,\mathbf{p}} \frac{\partial x_i/\partial p_i}{x_i/p_i}$$

In general $E$ could be unbounded (e.g., when utility functions are linear). Intuitively, it is clear that the larger $E$ is, the smaller the price adjustments should be for a given level of excess demand; as a result $\lambda$ needs to be chosen correspondingly small enough so that adjustments ensure convergence. Were the value of the fractional derivative consistent for all prices this would not matter. However, when $E$ is large, $(\partial x_i/\partial p_i)/(x_i/p_i)$ cannot be large for all prices and goods[12]. The outcome is an $O(1/E)$ convergence speed.

**Normal Goods and the Parameter $\beta$:** Good $i$ is said to be *normal* for agent $l$ if $\partial x_{il}(\mathbf{p}, v_l)/\partial v_l \geq 0$, where $v_l$ is the wealth of agent $l$.[13] We impose the slightly stronger constraint that states for all $(\mathbf{p}, v_l)$ there is a $\beta > 0$ such that $\partial x_{il}(\mathbf{p}, v_l)/\partial v_l \geq \beta x_{il}/v_l$. In words, it says that the fractional rate of change of demand with respect to wealth is lower bounded by a strictly positive value. We call this the *wealth effect*.

[8]See Mas-Colell [16], page 23.

[9]Ibid.

[10]Ibid., page 24.

[11]Ibid., page 613.

[12]For if $E > 1$ for all prices and goods, imagine starting at equilibrium prices and then reducing the price for one good; eventually all interested buyers would be purchasing only that good; any further price reductions would induce a rate of change of demand for that good with $E \leq 1$.

[13]See Mas-Colell [16], p. 25. This is considered a reasonable constraint for broad categories of goods, such as "food": i.e. as wealth increases, spending on food generally increases, although spending on specific types of food may decrease.

**The Numeraire and the Parameter $\alpha$:** A separate parameter $\alpha_i$ is defined for each good $i$. In words, $\alpha_i$ is the fractional amount of the supply of good $i$ that is purchased using the portion of the wealth due to the allocation of money (the numeraire) alone, at equilibrium prices. Intuitively, $\alpha$ is a measure of the utility of money relative to other goods at equilibrium prices. Formally,

$$\alpha_i = \frac{1}{w_i} \sum_l \frac{x^*_{il} w_{\$l}}{\sum_k w_{kl} p^*_k}.$$

We then define $\alpha = \min_i \alpha_i$.

To see why the $\alpha$ could have a natural effect on the convergence rate in markets with a numeraire (such as money), consider the following example market: a market in which there is no allocation of money. Then doubling all prices (given homogeneity of degree zero) would have no effect on demand. It is now plausible, and turns out to be the case, that if only very little money is present in the market (i.e. at equilibrium, the value of the money is very small compared to that of the other goods), then the effect of price changes on demand is muted (or viewed inversely, even if the prices are quite far from equilibrium, the excess demands, and hence the signal they provide, are small).[14]

We note that in markets with CES utilities, $E = s$ and $\beta = 1$, while in markets with Cobb-Douglas utilities $E = 1$ and $\beta = 1$.[15]

# 3 Protocols and Results

The price update protocol for good $i \neq \$$ in the fixed market setting is

$$p_i \leftarrow p_i(1 + \frac{1}{\lambda_i} \min\{1, \frac{z_i}{w_i}\}). \tag{4}$$

where $\lambda_i$ is a small enough fixed value for good $i$. We show that $\lambda_i \leq \{\frac{1}{6E}, \sqrt{5} - 2\}$ for all $i$ is small enough to prove fast convergence.[16]

Our convergence result depends on a natural, but slightly technical, notion of distance of a price to the equilibrium price. We define the *distance* between prices $p_i$ and $p^*_i$ to be $\frac{p^*_i}{p_i}$ if $p^*_i \geq 3p_i$, $\frac{p^*_i - p_i}{p^*_i}$ if $p_i \leq p^*_i < 3p_i$, and $\frac{p_i - p^*_i}{p^*_i}$ if $p^*_i < p_i$.[17] The motivation for this definition is that if there is a big gap between $p_i$ and $p^*_i$, then our goal is to reduce the ratio, while if $p_i$ is close to $p^*_i$, then our goal is to reduce their difference. We let $\eta_i(p_i)$ denote this distance for good $i$, and $\eta(\mathbf{p}) = \max_i \eta_i(p_i)$. For simplicity of notation, we drop the argument of $\eta$ when it is clear from context.

**Theorem 3.1** *The price update protocol given by (4), in weak gross substitutes markets with parameters $\alpha, \beta, E$, and initial prices $\mathbf{p}^\circ > 0$, yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ in $O(\frac{1}{\alpha\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds, where $\lambda_i \leq \{\frac{1}{6E}, \sqrt{5} - 2\}$ for all i, and $\lambda = \min_i \lambda_i$.*

We prove this theorem in Section 4.1. Although prior work has shown that for suitably small choices of $\lambda_i$ there is a tatonnement-style price update protocol that converges to equilibrium prices, there has been no prior successful effort to devise and analyze a protocol for which convergence is rapid. Theorem 3.1 provides the first polynomial convergence guarantee for any tatonnement-style protocol with independent price updates, even with $\lambda_i$ at a fixed value.

As the following theorem asserts, this bound is tight for protocol (4). This is proved in the Appendix.

[14] One might be tempted to argue that one should measure the quality of an approximate equilibria in terms of the excess demands rather than the error in the prices, but this will have no effect on the rate of convergence, although it can change the percentage error.

[15] A *CES (Constant Elasticity of Substitution)* utility function has the following form: $(\sum_j a_j x_j^\rho)^{\frac{1}{\rho}}$, for $a_j \geq 0$. A market with agents that have CES utility functions with $0 \leq \rho \leq 1$ satisfies the gross substitutes property.

In the definition of the CES function, $\rho = \frac{s-1}{s}$, where $s$ is the elasticity of substitution. Intuitively, $s$ measures the degree to which demand for goods shift in response to a change in price. A high value of $s$ corresponds to a highly volatile market. Thus, when agents have near *linear utilities* with $\rho$ approaching 1, small changes in price can lead to large swings in demand.

An agent has a *Cobb-Douglas* utility function if it seeks to spend fixed fractions of its wealth on the various goods, i.e. agent $l$ seeks to spend $a_{il}v_l$ on good $i$, The utility function $\prod_l (x_{il})^{a_{il}}$, where $\sum_l a_{il} = 1$, $a_{il} \geq 0$ achieves this.

[16] It might not be necessary that $\lambda_i$ be this small for all $i$ to get convergence. This depends on the individual changes in rates of demand with respect to price.

[17] The combination of the assumptions of weak gross substitutes and the wealth effect prevent $p^*_i = 0$. To see this, start with equilibrium prices and reduce all prices (except money) by factor $f > 1$. The demand for all goods other than money increases by at least $f^\beta$, including the goods with price zero. But the price of these goods has not changed, and other prices have only decreased. This contradicts the weak gross substitute property. Accordingly, it seems reasonable to assume that $\mathbf{p}^\circ > 0$.

**Theorem 3.2** *For all $\frac{1}{3} \geq \alpha > 0, 1 \geq \beta > 0, E \geq 1$, in weak gross substitutes markets with parameters $\alpha, \beta, E$, there are initial prices $\mathbf{p}^\circ > 0$ such that for any final prices $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$, the price update procedure takes $\Omega(\frac{1}{\alpha\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds.*

If the market is liable to change, it is helpful to have a more flexible update protocol. We consider the following. To start, $\lambda_i = \frac{1}{2}$. Let $n_i$ be the number of updates to $p_i$.

$$p_i \leftarrow p_i + \frac{1}{2^{\lceil \log_4 n_i \rceil}} p_i \min\{1, \frac{z_i}{w_i}\} \tag{5}$$

**Theorem 3.3** *The price update protocol (5) with $b$-bounded asynchrony, in weak gross substitutes markets with parameters $\alpha, \beta, E$, and initial prices $\mathbf{p}^\circ > 0$, yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ after $O(\frac{b}{\alpha^2\beta^2}(E + \log \frac{\max\{1, \eta(p^\circ)\}}{\delta})^2)$ rounds.*

We prove this theorem in Section 4.2.

# 4 Proofs of Convergence

In the next two subsections we prove Theorems 3.1 and 3.3. The proof of Theorem 3.3 depends heavily on the lemmas developed for the proof of Theorem 3.1.

## 4.1 The Fixed Protocol

In this section we prove Theorem 3.1.

For simplicity of notation, we assume throughout this section and the remainder of the paper that $w_i = 1$ for all goods.[18] The implications of this is that now the updates have the form

$$p_i \leftarrow p_i(1 + \lambda_i \min\{z_i, 1\});$$

the excess demand $z_i = x_i - w_i \geq 0 - 1 = -1$ for all goods $i$, for any set of prices; and $\alpha_i = \sum_l \frac{x^*_{il} w_{\$l}}{\sum_k w_{kl} p^*_k}$.

Consider the update rule $p_i \leftarrow p_i(1 + \lambda_i \min\{z_i, 1\})$. We want to show that this rule "improves" the worst price by a constant factor in one round. In particular, this means that if $z_i$ is small, then it is roughly proportional to $\frac{p^*_i - p_i}{p^*_i}$. To demonstrate this we bound $x_i$ by a polynomial in $\frac{p_i}{p^*_i}$ which yields an $O(\frac{|p^*_i - p_i|}{p^*_i})$ bound for $|z_i|$ when $p_i$ is close to $p^*_i$.

Recall the definitions of $\alpha$, $\beta$, and $E$ given in Section 2.

**Lemma 4.1** *Let $\mathbf{p}$ be a price vector, $\mathbf{x}$ the corresponding demand vector. Let $\mathbf{x}'$ be the demand vector when $p_i$ is replaced with $p'_i < p_i$, and all other prices are unchanged. Then*

$$\frac{x'_i}{x_i} \leq \left(\frac{p_i}{p'_i}\right)^E$$

**Proof:** Using the definition of $E$, we have that $\frac{\partial}{\partial p_i}(p_i^E x_i) = E p_i^{E-1} x_i + p_i^E \frac{\partial x_i}{\partial p_i} \geq E p_i^{E-1} x_i - E p_i^{E-1} x_i = 0$. Thus $p_i^E x_i$ is an increasing function of $p_i$. Consequently, $(p'_i)^E x'_i \leq p_i^E x_i$ or $\frac{x'_i}{x_i} \leq (\frac{p_i}{p'_i})^E$. ∎

**Corollary 4.2** *Let $\mathbf{p}, \mathbf{q}$ with $\mathbf{q} \leq \mathbf{p}$ be two price vectors. Then*

$$\frac{x_i(\mathbf{q})}{x_i(\mathbf{p})} \leq \left(\frac{p_i}{q_i}\right)^E.$$

**Proof:** Let $\widetilde{\mathbf{q}}$ be the price vector $\mathbf{q}$ with $q_i$ replaced by $p_i$. By weak gross substitutes, $x_i(\widetilde{\mathbf{q}}) \leq x_i(\mathbf{p})$. Thus, using Lemma 4.1, we have that $\frac{x_i(\mathbf{q})}{x_i(\mathbf{p})} \leq \frac{x_i(\mathbf{q})}{x_i(\widetilde{\mathbf{q}})} \leq (\frac{p_i}{q_i})^E$. ∎

**Lemma 4.3** *If the wealth $v_l$ of buyer $l$ is multiplied by $a \geq 1$ with no change in prices, (e.g., by increasing $w_{il}$ uniformly for all $i$), then the relationship of the new demand $\mathbf{x}'$ to the old demand $\mathbf{x}$ satisfies $\frac{x'_{il}}{x_{il}} \geq a^\beta$.*

[18] This is without loss of generality and may be attained by changing the units of good $i$.

**Proof:** Using the definition of $\beta$, we have that $\frac{\partial}{\partial v_l}(v_l^{-\beta} x_{il}) = -\beta v_l^{-\beta-1} x_{il} + v_l^{-\beta} \frac{\partial x_{il}}{\partial v_l} \geq -\beta v_l^{-\beta-1} x_{il} + \beta v_l^{-\beta-1} x_{il} = 0$. Thus $v_l^{-\beta} x_{il}$ is an increasing function of $v_l$. Consequently, $(av_l)^{-\beta} x'_{il} \geq v_l^{-\beta} x_{il}$ or $\frac{x'_{il}}{x_{il}} \geq a^{\beta}$. ■

For each buyer $l$, define $y_l$ to be the fraction of buyer $l$'s wealth at equilibrium prices due to its allocation of money. That is,

$$y_l := \frac{w_{\$l}}{\sum_k w_{kl} p_k^*}.$$

Note that (since we have assumed $w_i = 1$)

$$\alpha_i = \sum_{\ell} y_l x_{i\ell}^*. \tag{6}$$

Without loss of generality, let $1 = \text{argmin}_i \frac{p_i}{p_i^*}$ and $n = \text{argmax}_i \frac{p_i}{p_i^*}$. We define upper and lower bounds on demand as a function of $p_1$ and $p_n$, as follows

$$\ell(p_1) := 1 + \alpha \left[ \left( \frac{p_1^*}{p_1} \right)^{\beta} - 1 \right]$$

$$u(p_n) := \left( 1 - \alpha \left[ 1 - \left( \frac{p_n^*}{p_n} \right) \right] \right)^{\beta}$$

Let $r_i := \frac{p_i}{p_i^*} \frac{p_1^*}{p_1} \geq 1$ and $s_i := \frac{p_i}{p_i^*} \frac{p_n^*}{p_n} \leq 1$.

**Lemma 4.4** *(i) If $p_1 \leq p_1^*$, then for all goods $i$, $x_i \geq \ell(p_1) r_i^{-E}$.*
*(ii) If $p_n \geq p_n^*$, then for all goods $i$, $x_i \leq u(p_n) s_i^{-E}$.*

**Proof:** We first prove (i). Let $f := \frac{p_1^*}{p_1} \geq 1$. Suppose that all prices, including the price of money, were reduced from the equilibrium values by a factor of $f$. By homogeneity, this has no effect on the demands. Now imagine restoring the price of money to $p_\$ = 1$ in two stages: first by changing the seller's price (and thus the wealth), and then by changing the buyer's price (and thus the demand for money relative to other goods). In the first stage $p_\$$ remains at $1/f$, but the wealth of each buyer $\ell$ is increased to equal what it would be were $p_\$ = 1$. Let $\widetilde{v}_\ell$ denote this wealth and let $\widetilde{x}_i$ denote the demands after the first stage. The second stage sets $p_\$ = 1$ but does not alter the wealth any further. Recall, $v_l = \sum_i w_{il} p_i^*$.

By Lemma 4.3, $\widetilde{x}_{i\ell} \geq x_{i\ell}^* \, (\frac{\widetilde{v}_\ell}{v_\ell/f})^{\beta}$ for all goods $i$. In the second stage, weak gross substitutes implies the demand for good $i \neq \$$ either increases or remains the same. As

$$\widetilde{x}_{i\ell} \geq x_{i\ell}^* \, (\frac{\widetilde{v}_\ell}{v_\ell/f})^{\beta},$$

and

$$\frac{\widetilde{v}_\ell}{v_\ell/f} \;=\; \frac{\frac{v_\ell}{f} + w_{\$\ell}(1-\frac{1}{f})}{\frac{v_\ell}{f}} \;=\; 1 + (1-\frac{1}{f})\frac{w_{\$\ell}}{v_\ell/f} \;=\; 1 + (f-1)y_\ell,$$

it follows that $\widetilde{x}_{i\ell} \geq [1 + (f-1)y_\ell]^{\beta} \, x_{i\ell}^*$ and $\widetilde{x}_i \geq \sum_\ell x_{i\ell}^* [1 + (f-1)y_\ell]^{\beta}$.

Given that $a^{\beta}$ is a concave function of $a$ for $0 \leq \beta \leq 1$, this expression is minimized by setting $y_\ell$ to extreme values (0 or 1). The expression for $\alpha_1$ in (6) limits the extent to which $y_l$ can be set to 1, yielding

$$\widetilde{x}_i \;\geq\; [(1-\alpha_1) + \alpha_1(1+f-1)^{\beta}] x_i^* \;=\; (1-\alpha_1) + \alpha_1 f^{\beta} \;\geq\; 1 + \alpha(f^{\beta} - 1) = \ell(p_1).$$

Finally, all prices are increased to their actual values $p_i$, by multiplying by $r_i$. Corollary 4.2 implies that

$$x_i \;\geq\; \widetilde{x}_i r_i^{-E} \;\geq\; \ell(p_1) r_i^{-E}.$$

The proof of (ii) is in the Appendix. ■

The proof of the next lemma uses the following fact, which is proved in the Appendix using Taylor series expansions.

**Fact 4.5** *(a) If $\delta \geq -1$ and either $a \leq 0$, or $a \geq 1$, then $(1+\delta)^a \geq 1 + a\delta$.*
*(b) If $0 \geq \delta \geq -\frac{1}{2}$ and $0 < a < 1$, then $(1+\delta)^a \geq 1 + 2a\delta$.*
*(c) If $\delta \geq -1$ and $0 \leq a \leq 1$, then $(1+\delta)^a \leq 1 + a\delta$.*
*(d) If $-\frac{1}{2} \leq \delta < 1$ and $0 < a\delta \leq \frac{1}{2}$, then $(1+\delta)^a \leq 1 + 2a\delta$.*

In what follows, let $p_i$ denote the current value of price $i$ and $p'_i$ denote its value following its next update.

**Lemma 4.6** *If, for all $i$, $\lambda_i \leq \min\{1/6E, \sqrt{5}-2\}$ then:*

*(i) If $p_1 \geq p_1^*$, then $p'_i \geq p_i^*$.*

*(ii) If $p_1 < p_1^*$ then $p'_i \geq p_i^* \frac{p_1}{p_1^*}[1+\lambda\ \min\{1, \ell(p_1)-1\}]$.*

*(iii) If $p_n \leq p_n^*$, then $p'_i \leq p_i^*$.*

*(iv) If $p_n > p_n^*$, then $p'_i \leq p_i^* \frac{p_n}{p_n^*}[1-\lambda(1-u(p_n))]$.*

**Proof:** We begin by showing (i) and (ii).

**Case 1** $z_i \geq 1$.

$\frac{p'_i}{p_i^*} = \frac{p_i}{p_i^*}(1+\lambda) \geq \frac{p_1}{p_1^*}(1+\lambda)$.

If $p_1 \geq p_1^*$, then $p'_i \geq p_i^*$ and so (i) is true for $p'_i$. Otherwise (ii) is true for $p'_i$.

**Case 2** $z_i < 1$ and $p_1 \geq p_1^*$.

Then $p_i \geq p_i^*$. In this case, Corollary 4.2 implies that $x_i \geq \left(\frac{p_i^*}{p_i}\right)^E$, and

$$\begin{aligned}\frac{p'_i}{p_i^*} &= \frac{p_i}{p_i^*}(1+\lambda z_i) = \frac{p_i}{p_i^*}(1+\lambda x_i - \lambda) \\ &\geq \frac{p_i}{p_i^*}\left[1+\lambda\left(\frac{p_i^*}{p_i}\right)^E - \lambda\right]\end{aligned}$$

If $\frac{p_i}{p_i^*} \geq 2$ then $\frac{p'_i}{p_i^*} \geq 2[1-\lambda] \geq 1$ as $\lambda \leq \frac{1}{2}$.

Otherwise, using Fact 4.5 (b),

$$\begin{aligned}\frac{p'_i}{p_i^*} &\geq \frac{p_i}{p_i^*}[1+\lambda(1+2E(\frac{p_i^*}{p_i}-1))-\lambda] \\ &= \frac{p_i}{p_i^*}[1-\lambda 2E(1-\frac{p_i^*}{p_i})] \geq 1 \quad \text{if } \lambda \leq \frac{1}{2E}\end{aligned}$$

so that (i) holds for $i$.

**Case 3** $p_1 < p_1^*$ and $\frac{p_i}{p_i^*}(1-\lambda) \geq \frac{p_1}{p_1^*}(1+\lambda)$.

Then $\frac{p'_i}{p_i^*} \geq \frac{p_i}{p_i^*}(1-\lambda) \geq \frac{p_1}{p_1^*}(1+\lambda)$, so that (ii) holds for $i$.

**Case 4** $z_i < 1$, $p_1 < p_1^*$, and $\frac{p_i}{p_i^*}(1-\lambda) < \frac{p_1}{p_1^*}(1+\lambda)$.

The analysis of this case uses techniques similar to Case 2. Details, as well as the proofs of (iii) and (iv), are in the appendix.

■

**Corollary 4.7** *Let $p_i$ be the price of good $i$ at the start of a round and $p'_i$ be the price at the end of the round. If $\lambda_i \leq \frac{1}{6E}$ for all $i$, and $\lambda = \min_i \lambda_i$ then*

*(i) If $p_1 \geq p_1^*$, then all $p'_i \geq p_i^*$.*

*(ii If $p_1 < p_1^*$ then all $p'_i \geq p_i^* \frac{p_1}{p_1^*}[1+\lambda\ \min\{1, \ell(p_1)-1\}]$.*

*(iii) If $p_n \leq p_n^*$, then all $p'_i \leq p_i^*$.*

*(iv) If $p_n > p_n^*$, then all $p'_i \leq p_i^* \frac{p_n}{p_n^*}[1-\lambda(1-u(p_n))]$.*

**Proof:** This follows from a simple induction over the sequence of updates in the round, using Lemma 4.7 to bound the effect of a single update. ■

**Remark:** Suppose that instead of using the current value of $z_i$ for calculating the update to $p_i$, the price setter uses a value of $z_i$ from some point since the last update to $p_i$, or any convex combination of such values. The analysis of Lemma 4.6 and Corollary 4.7 is readily modified to cover this case, but instead of assuring progress from round to round, it now assures progress every second round. We give details in the Appendix.

**Lemma 4.8** *If $\lambda_i \leq \frac{1}{6E}$ for all $i$ and $\lambda = \min_i \lambda_i$,*

*(i) If $\min_i \frac{p_i}{p_i^*} \leq \frac{1}{3}$, then $\min_i \frac{p_i}{p_i^*}$ is doubled in at most $\frac{3}{\alpha\beta\lambda}$ rounds.*

*(ii) If $\frac{1}{3} \leq \min_i \frac{p_i}{p_i^*} \leq 1$, then $1 - \min_i \frac{p_i}{p_i^*}$ is reduced by half in at most $\frac{3}{\alpha\beta\lambda}$ rounds.*

*(iii) If $\max_i \frac{p_i}{p_i^*} > 1$, then $\max_i \frac{p_i}{p_i^*} - 1$, is reduced by half in at most $\frac{1}{\alpha\beta\lambda}$ rounds.*

**Proof:** As before, we fix the indices so that $\frac{p_1}{p_1^*} = \min_i \frac{p_i}{p_i^*}$ and $\frac{p_n}{p_n^*} = \max_i \frac{p_i}{p_i^*}$.

We first prove (i) and (ii). From Corollary 4.7, we have that if $\min_i \frac{p_i}{p_i^*} < 1$, then in one round $\frac{p_i'}{p_i^*} \geq \frac{p_1}{p_1^*}[1 + \lambda \min\{1, \ell(p_1) - 1\}]$. Now,

$$\begin{aligned} l(p_1) - 1 &= \alpha[(\frac{p_1}{p_1^*})^{-\beta} - 1] \geq \alpha[1 + \beta(1 - \frac{p_1}{p_1^*}) - 1] \qquad \text{using Fact 4.5(a)} \\ &= \alpha\beta(1 - \frac{p_1}{p_1^*}) \end{aligned}$$

Thus $\frac{p_i'}{p_i^*} \geq \frac{p_1}{p_1^*}[1 + \lambda\alpha\beta(1 - \frac{p_1}{p_1^*})]$. Hence, if $\frac{p_1}{p_1^*}$ remains bounded by $\frac{1}{3}$, then in at most $\frac{3}{\lambda\alpha\beta}$ rounds, $\min_i \frac{p_i}{p_i^*}$ doubles, showing (i). Otherwise, we have that

$$1 - \frac{p_i'}{p_i^*} \leq 1 - \frac{p_1}{p_1^*}[1 + \lambda\alpha\beta(1 - \frac{p_1}{p_1^*})] = (1 - \frac{p_1}{p_1^*})[1 - \lambda\alpha\beta\frac{p_1}{p_1^*}].$$

Since $\frac{p_1}{p_1^*} \geq \frac{1}{3}$, we have that in at most $\frac{3}{\lambda\alpha\beta}$ rounds, $(1 - \min_i \frac{p_i}{p_i^*})$ is reduced by half, showing (ii).

The proof of (iii) is in the Appendix.

■

**Theorem 4.9** *If $\lambda_i \leq \min\{\frac{1}{6E}, \sqrt{5} - 2\}$ for all $i$, $\lambda = \min_i \lambda_i$, for initial price vector $\mathbf{p}^\circ$ the tatonnement process described by (4) yields price vector $\mathbf{p}$ satisfying $\eta(\mathbf{p}) \leq \delta$ in $O(\frac{1}{\alpha\beta\lambda}(\log \frac{\eta(\mathbf{p}^\circ)}{\delta}))$ rounds.*

**Proof:** This follows immediately from Lemma 4.8. ■

## 4.2 The Oblivious Protocol

Recall the oblivious protocol and the definition of $\eta$ given in Section 3. In this section, we restate the main result for this setting, and prove it.

**Theorem 4.10** *In markets obeying weak gross substitutes, with parameters $\alpha$, $\beta$, $E$, and initial prices $\mathbf{p}^\circ$, if prices updates follow (5) and are $b$-bounded asynchronous, then $\eta(\mathbf{p}) \leq \delta$ after $O(\frac{b}{\alpha^2\beta^2}(E + \log \frac{\max\{1,\eta(\mathbf{p}^\circ)\}}{\delta})^2)$ rounds.*

**Proof:** We analyze the number of rounds required in two phases. Phase I ends when $\lambda_i \leq \frac{1}{6E}$ for all goods $i$. Phase II ends when $\eta(\mathbf{p}) \leq \delta$. In Phase I, we bound four things: the number of rounds, the number of updates to $p_i$, the multiplicative increase to $\eta(p)$ over $\eta(p^\circ)$ by the end of the phase, and the value of the smallest $\lambda_i$.

As $\lambda_i = \frac{1}{2^r}$ after $\Theta(4^r)$ price updates, the number of price updates required to get $\lambda_i \leq \frac{1}{6E}$ is $\Theta(4^{\log 6E}) = \Theta(E^2)$. Since each round updates the price of each good at least once, this implies it takes $O(E^2)$ rounds to obtain $\lambda_i \leq \frac{1}{6E}$, for all $i$.

By $b$-bounded asynchrony, in any one round, the price of a good could be updated at most $b$ times. Thus, the total number of updates to $p_i$ by the end of Phase I is $O(bE^2)$. Using the above reasoning in reverse, this implies that the smallest $\lambda_i$ at the end of Phase I is $\Omega(\frac{1}{Eb^{1/2}})$.

While $\lambda_i = \frac{1}{2^r}$ there are $\Theta(4^r)$ multiplicative updates to the price $p_i$ that are at least $(1-\lambda)$ and at most $(1+\lambda)$. Thus, the ratio of $p_i$ to $p_i^*$ worsens by at most $4^{2^r}$. By the time $\lambda_i \leq \frac{1}{6E}$, this total increase to the ratio is at most $4^{24E}$. (Note $4^{24E} \geq 4^{2+4+\ldots+(2^k)}$ for $k = \lceil \log(6E) \rceil$.) After this point, the proof of Lemma 4.6 implies that the ratio for the price of good $i$ does not get worse than the worst ratio of the remaining goods. Thus, at the end of Phase I, the worst ratio of price to equilibrium price is bounded by $\max\{1, \eta(\mathbf{p}^\circ)\}4^{24E}$.

We now bound the number of rounds in Phase II. Phase II starts with price vector $\mathbf{p}'$, the price vector at the end of Phase I. Note that from the arguments for Phase I, that $\eta(\mathbf{p}') \leq \max\{1, \eta(\mathbf{p}^\circ)\}4^{24E}$. For simplicity, we assume Phase II starts with $\lambda = \min_i \lambda_i = \frac{1}{Eb^{1/2}}$ having just halved. Thus for $i_{\min} = \text{argmin}\lambda_i$, $\lambda_{i_{\min}}$ remains at this value for the next $E^2 b$ updates which occur over at least $\sigma_1$ rounds, for some $\sigma_1 \in [E^2, E^2 b]$. Lemma 4.8 implies that after every $3/(\alpha\beta\lambda)$ round $\eta_i(p_i')$ decreases by a factor of at least $\frac{1}{2}$. Thus, over the course of these $\sigma_1$ rounds, $\eta(\mathbf{p}')$ decreases by a factor of at least $2^{-\alpha\beta\sigma_1/(3Eb^{1/2})}$.

After $\sigma_1$ updates with $\lambda = \min_i \lambda_i = \frac{1}{3Eb^{1/2}}$, $\lambda$ may be halved, and the process is iterated. In general, let $\lambda = \min_i \lambda_i = \frac{1}{Eb^{1/2}} 2^{-r}$. Let $\kappa_r = \alpha\beta\sigma_r/(3Eb^{1/2})$ for $\sigma_r \in [E^2, E^2 b]$. There are at least $\sigma_r 4^r$ rounds with $\lambda = \frac{1}{Eb^{1/2}} 2^{-r}$, after which $\eta(\mathbf{p}')$ has decreased by at least a factor of $2^{-\kappa_r 2^r}$. Thus when $2^r = \frac{1}{\kappa_r}(48E + \log \frac{\max\{1, \eta(\mathbf{p}^\circ)\}}{\delta})$, $\eta(\mathbf{p}) \leq \delta$. This happens after $O(\sigma_r 4^r)$ rounds. Choosing $\sigma_r \in [E^2, E^2 b]$ to maximize this expression yields $\sigma_r = E^2$ and $\sigma_r 4^r = \frac{9b}{(\alpha\beta)^2}(48E + \log \frac{\max\{1, \eta(\mathbf{p}^\circ)\}}{\delta})^2$; thus the total number of rounds in Phase II is $O(\frac{b}{\alpha^2\beta^2}(E + \log \frac{\max\{1, \eta(\mathbf{p}^\circ)\}}{\delta})^2)$. This dominates the number of rounds in Phase I. ∎

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# 5 Appendix

This section contains missing (parts of) proofs in the paper.

## 5.1 Fact 4.5

We prove Fact 4.5(a)–(c), using a simplified version of Taylor's Theorem. First, we restate the fact:

(a) If $\delta \geq -1$ and either $a \leq 0$, or $a \geq 1$, then $(1+\delta)^a \geq 1 + a\delta$.
(b) If $0 \geq \delta \geq -\frac{1}{2}$ and $0 < a < 1$, then $(1+\delta)^a \geq 1 + 2a\delta$.
(c) If $\delta \geq -1$ and $0 \leq a \leq 1$, then $(1+\delta)^a \leq 1 + a\delta$.

**Theorem 5.1 (Taylor)** *If $f$ is a twice differentiable function in the interval $[0, x]$ (or $[x, 0]$, if $x < 0$) then there is a $\xi \in [0, x]$ ( or $\xi \in [x, 0]$, if $x < 0$) such that*

$$f(x) = f(0) + f'(0) * x + \frac{f''(\xi)}{2} x^2.$$

Let $f(x) = (1+x)^c$. Then $f'(x) = c(1+x)^{c-1}$, $f''(x) = c(c-1)(1+x)^{c-2}$, $f(0) = 1$, and $f'(0) = c$. Thus we have that

$$(1+x)^c = 1 + cx + \frac{c(c-1)}{2}(1+\xi)^{c-2}x^2.$$

If the last term is nonnegative, then we have that $(1+x)^c \geq 1 + cx$. The last term is nonnegative provided that $x \geq -1$ (implying $\xi \geq -1$) and either $c \geq 1$ or $c \leq 0$. This is (a).

If the last term is nonpositive, then we have that $(1+x)^c \leq 1 + cx$. The last term is nonpositive provided that $x \geq -1$ and $0 \leq c \leq 1$. This is (c).

(b) holds if the last term is at least $cx$. This is true since $cx < 0$ and $0 < \frac{(c-1)}{2}(1+\xi)^{c-2}x < 1$ when $0 \geq \xi \geq x \geq -\frac{1}{2}$ and $0 < c < 1$.

■

Fact 4.5 (d) is obtained by bounding the limit of the infinite Taylor series for $(1+\delta)^a$.

## 5.2 Lemma 4.4

Here, we give the second half of the proof of Lemma 4.4.

The proof for (ii) uses similar arguments. Here, let $f := \frac{p_n^*}{p_n} \leq 1$. Suppose all prices, including that for money, were increased from the equilibrium values by a factor of $\frac{1}{f}$. All demands are unchanged. Again, in the first two stages, the wealth is reduced to $\widetilde{w}$, and then $p_\$$ is reset to 1. After Stage 1, by Lemma 4.3,

$$\widetilde{x}_{i\ell} \leq x^*_{i\ell} \left( \frac{\widetilde{v}_\ell}{f\ v_\ell} \right)^\beta \text{ for all } i,$$

where

$$\frac{\widetilde{v}_\ell}{f\ v_\ell} \ = \ \frac{f\ v_\ell - (f-1)w_{\$\ell}}{f\ v_\ell} \ = \ 1 - (f-1)\frac{w_{\$\ell}}{f\ v_\ell} \ = \ 1 - (1 - \frac{1}{f})y_\ell.$$

Thus $\widetilde{x}_{i\ell} \leq [1 - y_\ell + y_\ell/f]^\beta\ x^*_{i\ell}$ and $\widetilde{x}_i \leq \sum_\ell x^*_{i\ell}\ [1 - y_\ell + y_\ell/f]^\beta$. This sum is maximized when all $y_\ell$ are equal (to $\alpha_i$). Thus

$$\widetilde{x}_i \ \leq \ x^*_i(1 - \alpha_i + \alpha_i/f)^\beta \ \leq \ (1 - \alpha(1 - 1/f))^\beta.$$

Finally, each price $p_i$, $i \neq n$, is reduced (multiplied) by $s_i$ to obtain the target price $\mathbf{p}$. The result, invoking Corollary 4.2, is

$$x_i \ \leq \ \widetilde{x}_i s^{-E} \ \leq \ u(p_n)s_i^{-E}.$$

■

## 5.3 Lemma 4.6

Here we give the analysis for Case 4 and the second half of the proof of Lemma 4.6.

**Case 4** $\quad z_i < 1$, $p_1 < p_1^*$, and $\frac{p_i}{p_i^*}(1-\lambda) < \frac{p_1}{p_1^*}(1+\lambda)$.

In this case, by Lemma 4.4, $x_i \geq \ell(p_1)r_i^{-E}$. If $\lambda \leq \frac{1}{5E}$, then $r_i \leq \frac{1+\lambda}{1-\lambda} \leq \frac{3}{2}$ and $E(r_i - 1) \leq \frac{1}{2}$. Thus, Fact 4.5(d) with $\delta = r_i - 1$ implies that $\ell(p_1) \leq x_i r_i^E \leq 2(1 + 2E(r_i - 1)) \leq 4$.

$$\begin{aligned}
\frac{p'_i}{p^*_i} &\geq \frac{p_i}{p^*_i}[1 + \lambda(\ell(p_1)r_i^{-E} - 1)] \\
&\geq \frac{p_1}{p^*_1}r_i[1 + \lambda\ell(p_1)(1 - E(r_i - 1)) - \lambda] \qquad \text{using Fact 4.5 (a)} \\
&\geq \frac{p_1}{p^*_1}[1 + (r_i - 1) + \lambda(\ell(p_1) - 1) - r_i(r_i - 1)\lambda E\ell(p_1)] \\
&\geq \frac{p_1}{p^*_1}[1 + \lambda(\ell(p_1) - 1) + (r_i - 1)(1 - \lambda r_i E\ell(p_1))]
\end{aligned}$$

so that (ii) holds for $i$ if $\lambda \leq \frac{1}{6E}$ (using $l(p_1) \leq 4$ and $r_i \leq \frac{3}{2}$).

Next, we show (iii) and (iv).

**Case 1** $\quad p_n(1+\lambda) \leq p^*_n$.

By the update, $p'_i \leq p_i(1+\lambda)$; so $\frac{p'_i}{p^*_i} \leq \frac{p_i}{p^*_i}(1+\lambda) \leq \frac{p_n}{p^*_n}(1+\lambda) \leq 1$, and (iii) holds.

**Case 2** $\quad \frac{p_i}{p^*_i}(1+\lambda) \leq \frac{p_n}{p^*_n}(1-\lambda)$.

Then $\frac{p'_i}{p^*_i} \leq \frac{p_i}{p^*_i}(1+\lambda) \leq \frac{p_n}{p^*_n}(1-\lambda)$.
If $p_n \leq p^*_n$, then (iii) holds for $i$; otherwise (iv) holds for $i$.

**Case 3** $\quad p^*_n/(1+\lambda) < p_n \leq p^*_n$, and $\frac{p_i}{p^*_i}(1+\lambda) \geq \frac{p_n}{p^*_n}(1-\lambda)$.

Note that $p_i \leq p^*_i$.

By Corollary 4.2, $x_i \le \left(\frac{p_i}{p_i^*}\right)^{-E}$. Note that $1-\frac{p_i}{p_i^*} \le 1-\frac{p_n}{p_n^*}\frac{1-\lambda}{1+\lambda} \le 1-\frac{1-\lambda}{(1+\lambda)^2} \le 3\lambda$ for $\lambda \le 1$. Similarly, $1-\frac{p_i}{p_i^*} \le 1/2$ if $\lambda \le \sqrt{5}-2 < .2362$. Thus

$$\begin{aligned}
\frac{p_i'}{p_i^*} &\le \frac{p_i}{p_i^*}(1+\lambda z_i) \\
&\le \frac{p_i}{p_i^*}[1+\lambda\left(\frac{p_i}{p_i^*}\right)^{-E} - \lambda] \\
&\le \frac{p_i}{p_i^*}[1+\lambda(1+2E(1-\frac{p_i}{p_i^*}))-\lambda] && \text{if } E(1-\tfrac{p_i}{p_i^*}) \le \tfrac{1}{2} \text{ and } \delta = -(1-\tfrac{p_i}{p_i^*}) \ge -\tfrac{1}{2} \text{ by Fact 4.5(d);} \\
& && \lambda \le \min\{\sqrt{5}-2,\ \frac{1}{6E}\} \text{ suffices.} \\
&\le 1-(1-\frac{p_i}{p_i^*})+2\lambda E(1-\frac{p_i}{p_i^*}) \\
&\le 1-(1-\frac{p_i}{p_i^*})(1-2\lambda E) \\
&\le 1 \text{ if } \lambda \le \frac{1}{2E}.
\end{aligned}$$

So (iii) holds for $i$.

**Case 4** $\quad p_n > p_n^*$ and $\frac{p_i}{p_i^*}(1+\lambda) \ge \frac{p_n}{p_n^*}(1-\lambda)$.

In this case, $1-s_i \le 1-\frac{1-\lambda}{1+\lambda} \le 2\lambda$; $1-s_i \le \frac{1}{2}$ if $\lambda \le \frac{1}{3}$; and $u(p_n) \le 1$. Lemma 4.4 implies that $x_i \ \le\ u(p_n)s_i^{-E}$.

$$\begin{aligned}
\frac{p_i'}{p_i^*} &\le \frac{p_i}{p_i^*}(1+\lambda z_i) \\
&\le \frac{p_n}{p_n^*}s_i(1+\lambda u(p_n)s_i^{-E}-\lambda) \\
&\le \frac{p_n}{p_n^*}s_i[1+\lambda u(p_n)(1+2E(1-s_i))-\lambda] && \text{if } E(1-s_i) \le \tfrac{1}{2} \text{ and } \delta = -(1-s_i) \ge -\tfrac{1}{2} \text{ by Fact 4.5(d);} \\
& && \lambda \le \min\{\frac{1}{3}, \frac{1}{4E}\} \text{ suffices.} \\
&= \frac{p_n}{p_n^*}[1-(1-s_i)][1+\lambda u(p_n)-\lambda+\lambda u(p_n)2E(1-s_i))] \\
&\le \frac{p_n}{p_n^*}[1+\lambda u(p_n)-\lambda-(1-s_i)(1+\lambda u(p_n)-\lambda)+(1-s_i)\lambda u(p_n)2E \\
&= \frac{p_n}{p_n^*}[1+\lambda u(p_n)-\lambda-(1-s_i)(1+\lambda u(p_n)-\lambda-\lambda u(p_n)2E)] \\
&\le \frac{p_n}{p_n^*}[1+\lambda u(p_n)-\lambda] \text{ for } \lambda \le \frac{1}{3E}.
\end{aligned}$$

Hence (iv) holds for $i$. ∎

## 5.4 Remark

To set this up, we need to consider the smallest and largest values $p_i/p_i^*$ over the course of a round: let $\widetilde{p}_i^{\,r} = \min_{\text{round } r} p_i$. Without loss of generality, suppose that $1 = \text{argmin}_i\{\widetilde{p}_i/p_i^*\}$. We show that:

(1) If ${\widetilde{p}_i}^{2r} \ge p_i^*$ then ${\widetilde{p}_i}^{2r+2} \ge p_i^*$.

(2) If ${\widetilde{p}_i}^{2r} < p_i^*$ then ${\widetilde{p}_i}^{2r+2} \ge p_i^*\frac{{\widetilde{p}_1}^{2r}}{p_1^*}[1+\min\{1, l({\widetilde{p}_1}^{2r})-1\}]$.

and analogous claims with respect to $\widetilde{p}_n$, where $n = \text{argmax}_i\widetilde{p}_i/p_i^*$.

The analysis of updates in round $2r+1$ and subsequently obeys Lemma 4.6 with ${\widetilde{p}_1}^{2r}$ replacing $p_1$ and $p_i'$ being the updated value (strictly, we need to argue inductively that $\widetilde{p}_1$ can only improve in the following sense: for all $i$, during round $2r+1$ or later, $\widetilde{p}_i \ge p_i^*\frac{{\widetilde{p}_1}^{2r}}{p_1^*}$). As a result, by the start of round $r+2$ all prices have been updated, obey (1) and (2) above, and continue to do so.

## 5.5 Lemma 4.8

In this section we give the rest of the proof of Lemma 4.8.

We next prove (iii). From Lemma 4.6, we have that if $\max_i \frac{p_i}{p_i^*} > 1$, then in one round $\frac{p_i'}{p_i^*} \le \frac{p_n}{p_n^*}[1 - \lambda\,(1 - u(p_n))]$. We bound $u(p_n)$ from above:

$$\begin{aligned} u(p_n) &= (1-\alpha[1-\frac{p_n^*}{p_n}])^\beta \\ &\le 1-\alpha\beta(1-\frac{p_n^*}{p_n}]) \qquad \text{using Fact 4.5(c).} \end{aligned}$$

Hence

$$\begin{aligned} \frac{p_i'}{p_i^*} - 1 &\le \frac{p_n}{p_n^*}[1-\lambda\alpha\beta(1-\frac{p_n^*}{p_n})] - 1 \\ &= \frac{p_n}{p_n^*}(1-\lambda\alpha\beta) + \lambda\alpha\beta - 1 \\ &= (\frac{p_n}{p_n^*} - 1)(1-\lambda\alpha\beta) \end{aligned}$$

and $\frac{p_n}{p_n^*} - 1$ is reduced by at least half in at most $\frac{1}{\lambda\alpha\beta}$ rounds. ■

## 5.6 Lower Bounds

We give two lower bounds; the first shows that the convergence rate is tight for $\alpha$ and $E$ together, the second shows it is tight for $\alpha$ and $\beta$ together, and then a simple observation combines the two bounds, showing the convergence rate is tight for all three parameters at once.

**Lemma 5.2** *There is a market $M_{\alpha 1}$ with $E = 1$, $\beta = 1$, such that for any $\alpha \le 1$ with $\lambda = 2$ there is a sequence of price updates using protocol (4) that does not converge.*

**Proof:** Consider the following market $M_{\alpha 1}$ with buyers having identical Cobb-Douglas utilities. Suppose there are two goods $G_1$ and $G_2$ plus money in the market. Let $p_1^* = p_2^* = a$ be the equilibrium price. Then $\alpha = 1/(2a+1)$. Suppose that the two prices are always updated simultaneously. Suppose that the initial prices are $p_1 = a(1+\delta)$ and $p_2 = a(1-\delta)$. Then, as we will show, each update swaps the prices, i.e. the first updates set $p_1' = a(1-\delta)$ and $p_2' = a(1+\delta)$, and so on.

By definition, $p_1 + p_2 + 1 = 2a + 1$. We note that $x_1 p_1 = x_2 p_2 = a x_\$$ (as spending on each good is in the ratio $a : a : 1$). Further, by Walrus' Law,

$$x_1 p_1 + x_2 p_2 + x_\$ = p_1 + p_2 + 1.$$

Substituting gives $(2a+1)x_\$ = 2a+1$; therefore $x_\$ = 1$.

$x_1 \le 2$ so long as $p_1 \ge a/2$, which holds for $|\delta| \le \frac{1}{2}$. Then $p_1' = p_1[1+\lambda(x_1 - 1)] = p_1 + \lambda p_1 x_1 - \lambda p_1 = a(1+\delta) + \lambda a - \lambda a(1+\delta) = a(1+\delta-\lambda\delta) = a(1-\delta)$.

Similarly, $p_2' = p_2[1+\lambda(x_2-1)] = p_1 + \lambda p_2 x_2 - \lambda p_2 = a(1-\delta) + \lambda a - \lambda a(1-\delta) = a(1-\delta+\lambda\delta) = a(1+\delta)$. ■

**Comment:** If $\lambda > 2$, say $\lambda = 1 + 1/\delta$, $\delta < 1$, then in the above scenario, $p_1' = 0$. This implies that the algorithm specified by protocol (4) can fail for any $\lambda > 2$.

**Corollary 5.3** *In Market $M_\alpha$ the protocol (4) requires $\Theta(1/\alpha)$ rounds to improve the accuracy of the price by one bit.*

**Proof:** By Lemma 5.2 and the subsequent comment, $\lambda < 2$.

Consider the following scenario. Initially, $p_1 = p_2 = ad$, $d > 1$. Suppose $p_1$ and $p_2$ are updated simultaneously. Then

$$x_\$ = \frac{p_1 + p_2 + 1}{2a+1} = \frac{2p_1 + 1}{2a+1}$$

and

$$2p_1 x_1 = p_1 + p_2 + 1 - x_\$ = 2p_1 + 1 - \frac{2p_1+1}{2a+1}.$$

Thus

$$\begin{aligned}
p_1' &= p_1(1+\lambda(x_1-1)) = p_1 + \lambda p_1 x_1 - \lambda p_1 \\
&= p_1 + \frac{\lambda}{2}\left[2p_1 + 1 - \frac{2p_1+1}{2a+1}\right] - \lambda p_1 \\
&= p_1 + \frac{\lambda}{2}\left[\frac{2a+1-2p-1}{2a+1}\right] \\
&= p_1 - \frac{\lambda a(d-1)}{2a+1} \\
&\geq p_1 - \frac{2ad}{2a+1} = p_1(1-2\alpha)
\end{aligned}$$

■

Next, we show that to first order, the same scenario arises in markets with CES utilities, for arbitrary $\rho$; recall that $E = 1/(1-\rho)$.

**Lemma 5.4** *There is a market $M_{\alpha E}$ with $\beta = 1$, such that for any $\alpha \leq 1$ and $E > 1$ with $\lambda = 2/E$ the price update protocol (4) does not converge in any bounded time.*

**Proof:** Consider the following market $M_{\alpha E}$ with buyers having identical CES utilities. Suppose there are two goods $G_1$ and $G_2$ plus money in the market. Let $p_1^* = p_2^* = a$ be the equilibrium price. Then $\alpha = 1/(2a+1)$. Suppose that the two prices are always updated simultaneously. Suppose that the initial prices are $p_1 = a(1+\delta)$ and $p_2 = a(1-\delta)$. Then, as we will show, to first order, each update swaps the prices, i.e. the first updates set $p_1' = a(1-\delta+O(\delta^2))$ and $p_2' = a(1+\delta+O(\delta^2))$, and so on.

By definition, $p_1 + p_2 + 1 = 2a+1$. We note that as the utilities are CES with parameter $\rho$, $x_1 p_1^E = x_2 p_2^E = a^E x_\$$. Thus $x_1(1+\delta)^E = x_\$$ or $x_1 = x_\$(1-E\delta+O(\delta^2))$, assuming $E\delta \leq \frac{1}{2}$. Likewise, $x_2 = x_\$(1+E\delta+O(\delta^2))$. Further, by Walras' Law,

$$x_1p_1 + x_2p_2 + x_\$ = p_1 + p_2 + 1 = 2a+1.$$

Substituting gives

$$\begin{aligned}
x_1p_1 + x_2p_2 + x_\$ &= x_\$(1-E\delta+O(\delta^2))a(1+\delta) + x_\$(1+E\delta+O(\delta^2))a(1-\delta) + x_\$ \\
&= x_\$(2a+1) + O(a\delta^2)
\end{aligned}$$

Thus $x_\$ = 1 + O(\delta^2)$, and so $x_1 = 1 - E\delta + O(\delta^2)$ and $x_2 = 1 + E\delta + O(\delta^2)$.

Then

$$\begin{aligned}
p_1' &= p_1(1+\lambda(x_1-1)) \\
&= a(1+\delta)[1+(2/E)(1-E\delta+O(\delta)^2-1)] \\
&= a(1-\delta+O(\delta)^2) = p_2 + O(a\delta^2)
\end{aligned}$$

Likewise, $p_2' = p_1 + O(a\delta^2)$.

Choosing $\delta$ arbitrarily small ensures arbitrarily poor progress. ■

**Comment:** If $\lambda > 2/E$, then following the above analysis yields $p_1' = a(1-\delta(E\lambda-1)+0(\delta^2))$ and $p_2' = a(1+\delta(E\lambda-1)+0(\delta^2))$. Thus, by starting with $\delta$ suitably small, we can ensure that the prices move away from equilibrium for an arbitrary amount of time (although conceivably this eventually reverses).

**Corollary 5.5** *In Market $M_{\alpha E}$, for $\alpha \leq \frac{1}{3}$ and $E > 1$, our protocol requires $\Theta(E/\alpha)$ rounds to improve the accuracy of the price by one bit.*

**Proof:** By Lemma 5.4 and the subsequent comment, $\lambda < 2/E$.

Consider the following scenario. Initially $p_i = \gamma p_i^* = \gamma a$, for $i = 1, 2$, with $\frac{1}{2} \leq \gamma < 1$. Now $x_1 = x_2$, $x_i(\gamma a)^E = a^E x_\$$. Walrus' Law then yields $x_\$(1+2\gamma^{-E+1}a) = 2\gamma a + 1$. So

$$x_i = \frac{2\gamma a + 1}{2\gamma a + \gamma^E}.$$

Hence

$$\begin{aligned}
p_1' = p_2' &= \gamma a\left(1+\lambda\frac{2\gamma a+1-2\gamma a-\gamma^E}{2\gamma a+\gamma^E}\right) \\
&= \gamma a\left(1+\frac{\lambda(1-\gamma^E)}{2\gamma a+\gamma^E}\right) \\
&\le \gamma a\left(1+\frac{2/E}{2\gamma a}\right) \\
&\le \gamma a(1+2/(Ea)) \quad \text{if } \gamma \ge \tfrac{1}{2} \\
&\le \gamma a(1+\Theta(\alpha/E))
\end{aligned}$$

■

We turn to the lower bound for $\alpha$ and $\beta$.

**Lemma 5.6** *For any $\alpha, \beta \le 1$, there is a market $\widetilde{M}_{\alpha\beta}$ such that our price update protocol requires $\Theta(1/(\alpha\beta\lambda))$ rounds to improve the worst price by one bit for $\mathbf{p}$ near to $\mathbf{p}^*$.*

**Proof:** This result is shown by giving a market in which the demand for $\mathbf{p}$ near to $\mathbf{p}^*$ matches what is allowed by the wealth effect. Consequently, for small $\max_i |p_i - p_i^*|$, the convergence rate matches the upper bounds.

Suppose that there are two goods $G_1$ and $G_2$ plus money in the market (in fact, one would suffice). Let the equilibrium price be $p_1^* = p_2^* = a$; so $\alpha = 1/a$. All buyers will have the same utility function.

We define the demand for good $i \neq \$$ to be

$$x_i(\mathbf{p}) = 1 + \frac{\beta}{2a+1}\left(\frac{p_i^*}{p_i} - 1\right).$$

To show that this is a legitimate demand, given the prices $\mathbf{p}$, we show that there is enough wealth to buy $x_i$: $p_i x_i = p_i + \frac{\beta}{2a+1}(p_i^* - p_i)$. If $p_i \ge p_i^*$ then there is enough money from the sale of good $i$ to purchase $x_i$. If $p_i < p_i^*$, then this quantity is less than $p_i + \frac{1}{2a+1}(p_i^* - p_i) < p_i + \frac{1}{2}$ and we can use in addition at most half the wealth from money to purchase $x_i$ for $i = 1, 2$. Next, we observe that this demand obeys the weak gross substitutes property. First note that this expression is independent of $p_j$, so that $\frac{\partial z_i p_i}{\partial p_j} \ge 0$. To show that $\frac{\partial z_i p_i}{\partial p_i} \le 0$, observe that

$$\frac{\partial z_i p_i}{\partial p_i} = z_i + p_i\frac{\partial z_i}{\partial p_i} = \frac{\beta}{2a+1}\left(\frac{p_i^*}{p_i} - 1\right) - p_i\frac{\beta}{2a+1}\frac{p_i^*}{p_i^2} = -\frac{\beta}{2a+1} < 0.$$

Finally, we show that $\frac{\partial z_\$ p_\$}{\partial p_i} \ge 0$. Demand for money is $x_\$ = 1 + p_1 + p_2 - x_1 p_1 - x_2 p_2$. Thus $\frac{\partial z_\$ p_\$}{\partial p_i} = 1 - \frac{\partial z_i p_i}{\partial p_i} > 0$.

Next, we define a utility function $u(\mathbf{x})$ yielding this demand.

$$u(\mathbf{x}) = \sum_{i=1}^{2} \beta a \log\left[\frac{2a+1}{\beta}(x_i - 1) + 1\right] + x_\$$$

Observe that

$$\frac{\partial u}{\partial x_i} = \frac{a(2a+1)}{\frac{2a+1}{\beta}(x_i-1)+1}$$

$$\frac{\partial u}{\partial x_\$} = 1$$

We know that at an optimal $\mathbf{x}$, $\frac{\partial u}{\partial x_i}/p_i = \eta_i \frac{\partial u}{\partial x_\$}$, for suitable fixed $\eta_i$. Evaluating this expression at $\mathbf{p}^*$ yields $\eta_i = a(2a+1)/p_i^* = 2a+1$. So

$$\frac{a(2a+1)}{\frac{2a+1}{\beta}(x_i-1)+1} = p_i(2a+1)$$

Or

$$\frac{a}{p_i} = \frac{2a+1}{\beta}(x_i - 1) + 1, \qquad \text{assuming } a = p_i^*$$

or

$$\beta\left(\frac{p_i^*}{p_i}-1\right)=(2a+1)(x_i-1)$$

or

$$x_i=1+\frac{\beta}{2a+1}\left(\frac{p_i^*}{p_i}-1\right)$$

as claimed.

As

$$z_i=\frac{\beta}{2a+1}\left(\frac{p_i^*}{p_i}-1\right)=\beta\alpha\left(\frac{p_i^*}{p_i}-1\right)$$

the update $p_i'=p_i(1+\lambda z_i)$ gives $p_i'=p_i(1+\alpha\beta\lambda(\frac{p_i^*}{p_i}-1))$ which is only a constant factor faster progress than what is claimed in our upper bound. In particular, setting $p_1=p_2$ and having simultaneous updates shows that in this setting the upper and lower bounds are tight up to constant factors. ■

Finally, we combine the bounds of the two lemmas.

**Theorem 5.7** *There is a market $M_{\text{all}}$ such that for any $\beta\leq 1,\alpha\leq\frac{1}{3}$ and $E\geq 1$ our price update protocol requires $\Theta(E/(\alpha\beta))$ rounds to improve the worst price by one bit for $\mathbf{p}$ near to $\mathbf{p}^*$.*

**Proof:** $M_{\text{all}}$ is the disjoint union of $M_{\alpha E}$ and $\widetilde{M}_{\alpha\beta}$. For $M_{\alpha E}$ forces $\lambda=O(1/E)$. Then the convergence rate follows from Lemma 5.6. ■