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Amit Chakrabarti
Dartmouth College

Subhash Khot
Georgia Institute of Technology

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Abstract

We consider the combinatorial problem of embedding a tree metric into the real line with low distortion. For two special families of trees — the family of complete binary trees and the family of subdivided stars — we provide embeddings whose distortion is provably optimal, up to a constant factor. We also prove that the optimal distortion of a linear embedding of a tree can be arbitrarily low or high even when it has bounded degree.

1 Introduction

For a variety of reasons, both mathematical and computational, the study of finite metric spaces has been the focus of much recent research. The most important questions in this study have centered around the issue of embedding a finite metric space into a given target space, preserving all distances up to a small factor, called the *distortion* of the embedding (for a precise definition, see Section 2). Embeddings with low distortion into low dimensional Minkowski spaces have proven particularly important, their study having led to the best known algorithms for multicommodity flow [LR88, LLR94], nearest neighbor searching [IM98], data clustering [LLR94] and several other computational problems.

It has been known for a while that a special class of metric spaces, called *tree metrics*, admit embeddings of much better quality than worst case metrics. For example, an n-point tree metric can be embedded isometrically (i.e., without distortion) into a $O(\lg n)$-dimensional Minkowski space [LLR94]. Moreover, for a given d, it can be embedded into d-dimensional Euclidean space with distortion $\widetilde{O}(n^{1/(d-1)})$ [G99]. In light of this, it is natural to consider embeddings of tree metrics into a *one* dimensional space.

The one dimensional space is the most basic Minkowski space; in this case, all Minkowski norms are equal and we can speak of this space simply as the *real line*. We shall refer to an embedding with the real line as target space as a *linear embedding*. In this paper, we shall be concerned with the distortion of linear embeddings of tree metrics. A seemingly related concept is the *dilation* (sometimes also called the *bandwidth*) of trees, which has been studied for special

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†Department of Computer Science, Dartmouth College, Hanover, NH 03755.

‡College of Computing, Georgia Institute of Technology, 801 Atlantic Drive, Atlanta, GA 30332.

classes of trees [D85, HKMU98]. However, the notion of dilation ignores the metric defined by a tree; in particular, a low dilation embedding may contract a distance in the tree metric quite badly and could therefore lead to a high distortion embedding.

We note at the outset that a uniform distortion bound for all tree metrics is not too interesting. Indeed, it is very easy to show that every n-vertex tree has a linear embedding with distortion $O(n)$ and that this is optimal in the worst case (Theorem 2.7). Therefore, in this paper, we take the point of view that non-trivial distortion bounds, both lower and upper, for special families of trees are of interest. With this in mind, we informally state our main results.

- The complete binary tree with n vertices has a linear embedding of distortion $O(n/\lg n)$ and this bound is easily shown to be optimal (Theorem 3.2).
- The tree consisting of $\sqrt{n}$ paths of length $\sqrt{n}$ each, all emanating from a common vertex (henceforth called the "subdivided star"), has a linear embedding of distortion $O(n^{3/4})$ (Theorem 4.1).
- The above distortion upper bound is in fact optimal; we know of no straightforward argument that establishes a lower bound stronger than $\Omega(n^{1/2})$ (Theorem 5.3).
- A bound on the maximum degree cannot tell us much about the optimal distortion of linear embeddings. The proofs of the two previous results can be generalized to show that even with maximum degree 3 (and thus, even when restricted to binary trees), any function in $\Omega(1) \cap O(n)$ can be made the optimal distortion for some tree (Theorem 6.1).

We would like to emphasize that these are *combinatorial* results, rather than algorithmic ones, such as a result on how to efficiently compute or estimate the optimal distortion of a given tree. To the best of our knowledge, all algorithmic work on embedding metrics into a line has concentrated on *average* distortion [R03, BDG+05], whereas we focus on the more classical notion on worst-case distortion. Combinatorial results are often at the heart of the analysis of corresponding algorithmic results. Therefore, we hope that our results, or the ideas behind their proofs, find eventual application in algorithms.

The remainder of this paper is organized as follows. In Section 2 we formally define the relevant concepts and prove a few easy general results on linear embeddings of trees. In Section 3 we describe a linear embedding of complete binary trees and analyze its distortion. In Section 4 we do the same for the "subdivided star." In Section 5 we prove that the upper bound established in Section 4 is in fact optimal. We end with a couple of "miscellaneous" results, in Section 6.

2 Definitions and Preliminaries

All logarithms are to base 2. Our target metric space, the real line, will be denoted by $\mathbf{R}$. The vertex set of a tree T will be denoted by $V(T)$. For $u, v \in V(T)$, we shall denote by $d_T(u, v)$ the number of edges in the unique path from u to v in T, and shall call this quantity the distance between u and v in T. It is clear that this distance is a metric on the vertices of T.

Definition 2.1 *An embedding φ of a finite metric space $\mathcal{M}$ into a (not necessarily finite) target metric space $\mathcal{N}$ is an injective function $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$. For any such φ we define*

$$C(\varphi) = \max_{\substack{u,v\in\mathcal{M}\\ u\neq v}} \frac{d_{\mathcal{M}}(u,v)}{d_{\mathcal{N}}(\varphi(u),\varphi(v))}, \quad E(\varphi) = \max_{\substack{u,v\in\mathcal{M}\\ u\neq v}} \frac{d_{\mathcal{N}}(\varphi(u),\varphi(v))}{d_{\mathcal{M}}(u,v)}, \quad D(\varphi) = C(\varphi)E(\varphi).$$

We call $C(\varphi)$ *the contraction,* $E(\varphi)$ *the expansion and* $D(\varphi)$ *the distortion of the embedding* φ*.*

Definition 2.2 *A linear embedding of a tree* T *is an injective function* $\varphi : V(T) \longrightarrow \mathbf{R}$ *where the target space* $\mathbf{R}$ *is understood to have the metric* $d(x, y) = |x - y|$ *for* $x, y \in \mathbf{R}$.

A linear embedding φ of a tree is said to be *contractionless* if $C(\varphi) = 1$. Since by rescaling we can always turn any embedding into a contractionless embedding, without affecting its distortion, we only consider contractionless embeddings henceforth and worry only about their expansions. *From now on, linear embeddings will be tacitly assumed to be contractionless.*

Lemma 2.3 *In any linear embedding* φ *of a tree* T*, the edges suffer the maximum expansion. In other words,*

$$E(\varphi) = \max_{\substack{u,v \in V(T) \\ u,v \text{ adjacent}}} |\varphi(u) - \varphi(v)|.$$

Proof: This is immediate from the triangle inequality. □

We shall now associate an embedding with every ordering of the vertices of T. This is done by mapping the first vertex in the ordering to 0 and then mapping each successive vertex to as small a positive real number as possible, keeping in mind that the embedding must be contractionless.

Definition 2.4 *Given an ordering of the vertices of a tree* T *by a sequence* $U = (u_0, u_1, u_2, \ldots)$, $u_i \in V(T)$*, we define an embedding* ψ_U *by*

$$\begin{aligned} \psi_U(u_0) &= 0, \\ \psi_U(u_i) &= \max_{0 \le j < i} \Big(\psi_U(u_j) + d_T(u_i, u_j) \Big), \quad \textit{for } i \ge 1. \end{aligned}$$

We call ψ_U *the linear embedding associated with* U.

Note: Let $s(u_i) := d_T(u_{i-1}, u_i)$ be called the *shift* of the vertex u_i. The triangle inequality on d_T allows us to replace the second equation with

$$\psi_U(u_i) = \psi_U(u_{i-1}) + s(u_i), \quad \text{for } i \ge 1.$$

It is clear that for any ordering of the vertices, the embedding obtained in this manner is the one with least expansion, amongst all embeddings that arrange the vertices on the line according to that ordering.

Definition 2.5 *An ordering* U *is called a* connected ordering *if the subtree induced by the vertices in every prefix of* U *is connected.*

For a connected ordering U, it is clear that for each $i > 0$ there exists a unique $j < i$ such that $u_j u_i$ is an edge of T. We shall call the vertex u_j the *predecessor* of u_i and denote it by $p(u_i)$. For vertices u and v, we write $u \prec v$ if u precedes v in the ordering U. From Lemma 2.3 we have

$$D(\psi_U) = E(\psi_U) = \max_{i > 0} \sum_{v:\ p(u_i) \prec v \preceq u_i} s(v). \tag{1}$$

Note that the functions p and s, and the relation $\prec$, all depend on the connected ordering U.

We now prove a few preliminary general results about linear embeddings of trees.

Definition 2.6 *Let φ be an embedding of a tree T and let $S \subseteq V(T)$. The span of S under φ is defined to be*

$$\max_{v \in S} \varphi(v) - \min_{v \in S} \varphi(v).$$

The span of φ is defined to be the span of $V(T)$.

Theorem 2.7 *Every n-vertex tree has a linear embedding with span at most $2n$, and hence with distortion at most $2n = O(n)$. This is optimal in the worst case.*

Proof: The proof is by induction on n. Let T be a tree with $n+1$ vertices. Let u be a leaf on T and let v be the vertex adjacent to it. Consider a linear embedding of $T \setminus \{u\}$ and move each vertex lying to the right of v in this embedding by a distance of 2. Then embed u at a distance 1 to the right of v. The resulting embedding of T is easily seen to be contractionless because the embedding of $T \setminus \{u\}$ was, and its span is the old span plus 2.

For the optimality, note that the star $K_{1,n-1}$ on n vertices cannot be linearly embedded with distortion less than n. $\square$

Theorem 2.8 *Any linear embedding of an n-vertex tree has distortion at least $(N_T(v,r) - 1)/2r$, for any vertex v and any positive integer r. Here $N_T(v,r)$ denotes the number of vertices of T at distance at most r from v.*

Proof: By contractionlessness, the set of vertices of T at distance at most r from v has span at least $N_T(v,r) - 1$. Therefore, some two vertices in this set are a distance of at least $N_T(v,r) - 1$ apart in the embedding. But this pair is at a distance of at most $2r$ in T. $\square$

3 The Complete Binary Tree

Let $B = B(k)$ denote the complete binary tree with $n = 2^k - 1$ vertices. In this section we describe an ordering of the vertices of B and bound from above the distortion of the corresponding embedding. We begin with a useful lemma.

Lemma 3.1 *The embedding corresponding to the symmetric ordering of the vertices of $B(k)$ has span at most $2^{k+1} \leq 3n$.*

Proof: Let s_k be the span in question. Clearly $s_2 = 2$ and $s_{k+1} = 2s_k + 2k$ for $k \geq 2$. A simple inductive argument shows that $s_k \leq 2^{k+1} - 3k$. $\square$

In the remainder of this section, we assume for the sake of simplicity that k is a power of 2; it will be clear that everything works for general k as well. For v a vertex of B, let B_v denote the subtree of B rooted at v and let r be the root of B. Let `visit_and_mark(v)` be a procedure that outputs the vertices of B_v in symmetric order, and "marks" these vertices too. Let B_{top} denote the complete binary tree obtained by taking only levels 0 through $\lg k$ of B (the root being of level 0), so that B_{top} has k leaves. When looping over a set of vertices, it is assumed that the vertices are examined from left to right.

```
output vertices of B_top in symmetric order;
for i := 1 to k/2 do begin
    for v := leftmost 2^(i+1) unmarked vertices at level (i + lg k) do
        visit_and_mark(v);
    for v := the remaining vertices at level (i + lg k) do
        output(v);
end;
```

Note that the ordering U that this procedure outputs is *not* a connected ordering. However, Lemma 3.1 makes the distortion of ψ_U quite simple to analyze.

By Lemma 2.3, we only need to upper bound the expansion under ψ_U of the edges of B. By Lemma 3.1, the expansion of an edge of B_{top} is at most $6k < O(n/\lg n)$. Since each call to the procedure `visit_and_mark` above marks a subtree with at most $(2^{k-\lg k-1} - 1)$ vertices, Lemma 3.1 also implies that an edge with both end points marked suffers an expansion of at most $2^{k-\lg k} = 2^k/k = O(n/\lg n)$.

For any other edge e, suppose it connects level $i - 1 + \lg k$ to level $i + \lg k$, where $i \geq 1$. Let S_0 be the set of leaves of B_{top} and for $1 \leq i \leq k/2$ let S_i denote the set of unmarked vertices at level $i + \lg k$. The set of vertices that, in the embedding ψ_U, lie between the end points of edge e is the union of a subset of $S_{i-1} \cup S_i$ and the vertices of at most 2^{i+1} marked subtrees, each rooted at level $i + \lg k$. The distance between two adjacent marked subtrees in the embedding is at most $2k$. From Lemma 3.1 we can infer that the span of the images of the vertices in S_i is at most $12|S_i|$. Thus, the expansion suffered by e is at most

$$2^{i+1}\left(2k + \frac{2^{k-i+1}}{k}\right) + 12|S_{i-1}| + 12|S_i|.$$

But from the definition of S_i we get the recurrence $|S_{j+1}| = 2|S_j| - 2^{j+2}$ and $|S_0| = k$, solving which gives $|S_i| = 2^i(k - 2i)$. Now, an easy application of differential calculus shows that

$$|S_i| \leq \frac{2\lg e}{e} \cdot 2^{k/2}.$$

Thus, the above expression for the expansion of e is at most

$$k \cdot 2^{i+2} + \frac{2^{k+2}}{k} + 26 \cdot 2^{k/2}.$$

Recalling that $i \leq k/2$ we see that this expression is in $O(n/\lg n)$. Therefore, $D(\psi_U) = O(n/\lg n)$.

Theorem 3.2 *The complete binary tree with n nodes can be linearly embedded with distortion $O(n/\lg n)$, and this bound is tight.*

Proof: The upper bound follows from the above discussion, since ψ_U is a suitable embedding. For the lower bound, we simply apply Theorem 2.8 with $r = \lg n$. □

4 The Subdivided Star: An upper bound

Consider the tree $T = T_n$ which consists of $\lceil\sqrt{n}\rceil$ paths $P_1, P_2, \ldots, P_{\lceil\sqrt{n}\rceil}$ emanating from a common vertex. Each path consists of $\lceil\sqrt{n}\rceil$ edges. The paths are vertex disjoint except for the one common end point. This tree will be called the "subdivided star" with parameter n. Note that it is indeed a subdivision of a star. Also, it has $\Theta(n)$ vertices, maximum degree $\lceil\sqrt{n}\rceil$ and diameter $\Theta(\sqrt{n})$. We shall call the paths $P_1, P_2, \dots$ the *branches* of T.

For ease of notation let's assume that $n = m^4$ for some positive integer m. It will be clear that our asymptotic results hold in the case of general n as well. Let us denote the common end point of the branches by v_0. For $1 \le i, j \le m^2$, let v_{ij} denote the vertex on branch P_i which is at a distance j from v_0. It will sometimes be convenient to let $v_{i0} = v_0$ for all i.

In this section we describe an embedding of the tree T, by presenting a certain ordering of the vertices of T and then using the ordering-embedding correspondence from Definition 2.4. We then upper bound the distortion of this embedding.

Suppose we have a procedure `visitpath(i,j)` that outputs the sequence of vertices $v_{ij}, v_{i,j+1}, \dots, v_{i,m^2}$ on the branch P_i. Let `output(i,j)` be a procedure that simply outputs one vertex v_{ij}. Let σ be the sequence of vertices that the following code fragment outputs:

```
output the vertex v0;
for k := 1 to m do begin
    for i := (k-1)*m + 1 to k*m do
        visitpath(i, k);
    for i := k*m + 1 to m*m do
        output(i, k);
end;
```

Clearly σ gives a connected ordering. Consider the embedding ψ_σ. We claim that $D(\psi_\sigma) = O(m^3) = O(n^{3/4})$.

Before we begin the proof of this claim, let us augment the code fragment that produces σ with some calls to a procedure `color(i,j,c)` which "colors" the vertex v_{ij} with the color `c`. The symbols **red** and **blue** are used in the augmented code fragment below to denote the two distinct colors we use.

```
output the vertex v0;
for k := 1 to m do begin
    for i := (k-1)*m + 1 to k*m do begin
        visitpath(i, k);
        if i + 1 <= m*m then
            color(i + 1, k, red);
    end;
    for i := k*m + 1 to m*m do begin
        output(i, k);
        if i + 1 <= m*m then
            color(i + 1, k, blue)
        else
            color(k*m + 1, k + 1, blue);
    end;
end;
```

We are now ready to prove our claim.

Theorem 4.1 *The embedding ψ_σ satisfies $D(\psi_\sigma) = O(n^{3/4})$.*

Proof: We make four observations, each verifiable simply by studying the code fragment.

1. For any vertex v, other than v_0, there are $O(m^3)$ vertices that lie between $p(v)$ and v in the sequence σ. Of these, at most $O(m^2)$ are blue vertices and at most $O(m)$ are red.

2. For any uncolored vertex v we have $s(v) = 1$.

3. For any blue vertex v, $s(v) \leq 1 + 2k(v)$, where $k(v)$ is the value that k had had when v got colored. Therefore $s(v) = O(m)$.

4. For any red vertex v we have $s(v) \leq \mathrm{diam}(T) = O(m^2)$. Indeed, this is true for *any* vertex, by definition of s.

The proof follows from these four observations and equation (1). □

5 The Subdivided Star: A tight lower bound

In this section we show that the upper bound in the previous section is tight up to a constant factor. Note that an application of Theorem 2.8 cannot give us a bound stronger than $\Omega(\sqrt{n})$. We reemphasize the fact that all our embeddings are required to be contractionless.

Let φ be an embedding of T into the real line. Scanning from left to right along the line, we encounter the images of the vertices of T in some order, say $U = (u_0, u_1, \ldots, u_n)$. If two vertices of T, other than v_0, are adjacent in the sequence U, not adjacent in T, but lie on the same branch of T, then we say that the embedding φ has a *gap*. Our first step is to remove all gaps from φ and the following lemma lets us do just that.

Lemma 5.1 *There is an embedding* $\hat{\varphi}$ *of* T *that has no gaps and that satisfies* $D(\hat{\varphi}) \leq D(\varphi)$.

Proof: Suppose v_{ij} and v_{ik} are adjacent in U but $j < k - 1$. We simply move the images of all v_{il} with $j < l < k$ so that they lie in between $\varphi(v_{ij})$ and $\varphi(v_{ik})$. This gives us a new candidate embedding which we call φ'. Assume WLOG that $\varphi(v_{ij}) < \varphi(v_{ik})$. We can write out φ' explicitly:

$$\begin{aligned}\varphi'(v_{il}) &= \varphi(v_{ij}) + (\varphi(v_{ik}) - \varphi(v_{ij})) \cdot \frac{l-j}{k-j}, \qquad j < l < k \\ \varphi'(v) &= \varphi(v), \qquad \text{for all other vertices } v\end{aligned}$$

We shall show that φ' is a valid contractionless embedding and that its distortion is no more than that of φ. We say that a vertex v has *moved* if $\varphi'(v) \neq \varphi(v)$; note that the only vertices which have moved are v_{il} with $j < l < k$. Recall that φ was a contractionless embedding and so the ratio

$$\alpha := \frac{|\varphi(v_{ik}) - \varphi(v_{ij})|}{k-j}$$

lies between 1 and $D(\varphi)$. For any pair of distinct vertices u and v, three cases arise:

- If u and v have both moved, then clearly $|\varphi'(u) - \varphi'(v)| = \alpha \cdot d_T(u, v)$.
- If neither u nor v has moved, then $|\varphi'(u) - \varphi'(v)| = |\varphi(u) - \varphi(v)|$.

- If $u = v_{il}$ has moved and v has not, assume WLOG that $\varphi(v) < \varphi(v_{ij})$. Then $|\varphi'(u)-\varphi'(v)| = |\varphi'(u)-\varphi(v_{ij})|+|\varphi(v_{ij})-\varphi(v)| \geq d_T(u,v_{ij})+d_T(v_{ij},v) \geq d_T(u,v)$. Next, the path in T from u to v must pass either through v_{ij} or through v_{ik}; WLOG assume that it passes through v_{ik}. Then $|\varphi'(u)-\varphi'(v)| \leq |\varphi'(u)-\varphi(v_{ik})|+|\varphi(v_{ik})-\varphi(v)| \leq \alpha d_T(u,v_{ik})+D(\varphi)d_T(v_{ik},v) \leq D(\varphi)d_T(u,v)$.

Putting these cases together proves that $D(\varphi') \leq D(\varphi)$. By repeatedly eliminating gaps one at a time in this way, we obtain the required embedding $\hat{\varphi}$. □

From now on we discard the original embedding and work with $\hat{\varphi}$. We redefine φ to be $\hat{\varphi}$ from now on; note that the sequence U would also have changed, we shall mean the new sequence whenever we refer to U hereafter.

Consider a maximal contiguous subsequence of U whose vertices form a path in $T-\{v_0\}$. Call the set of vertices in such a subsequence a *segment* of T. We also call the singleton set $\{v_0\}$ a segment. V is thus partitioned into segments. We define the *weight* $W(s)$ of a segment s to be the sum of the sizes of all segments, not including v_0 but including s, encountered on the path from v_0 to s. We denote the set of all segments of T by $S(T)$.

If s and s' are distinct segments we write $s \prec s'$ if $\varphi(u) < \varphi(u')$ whenever $u \in s$ and $u' \in s'$. Clearly $\prec$ is a total order on S. By abuse of notation, if $s \in S$ and $v \notin s$ is a vertex of T, we write $s \prec v$ if $\varphi(u) < \varphi(v)$ for all $u \in s$. The notation $v \prec s$ is defined similarly. An edge of T is called a *link* if its end points belong to distinct segments.

We say that a branch of T is of *type* t if the branch contains exactly t segments other than $\{v_0\}$. For each integer k, $1 \leq k \leq \lceil \lg m^2 \rceil$, let T_k denote the subtree of T consisting of exactly those branches which have type t for some t with $2^{k-1} \leq t < 2^k$. Let b_k be the number of branches in T_k. Let $S(T_k)$, the set of segments of T_k, inherit the total order $\prec$ from $S(T)$.

Lemma 5.2 *Let (u,v) be a link in T_k and suppose v is closer to v_0 than u. Then*

$$D(\varphi) \;\geq \sum_{s\in S(T_k):\; u\prec s\preceq v} 2W(s) \tag{2}$$

if $\varphi(u) < \varphi(v)$, and

$$D(\varphi) \;\geq \sum_{s\in S(T_k):\; v\preceq s\prec u} 2W(s) \tag{3}$$

otherwise.

Proof: Assume WLOG that $\varphi(u) < \varphi(v)$, the proof is essentially unchanged in the other case. Let $s_1 \prec s_2 \prec \cdots \prec s_t$ be the set of segments between u and v, with $u \prec s_1$ and $v \in s_t$. Let w_i be the vertex in s_i that is farthest from the vertex v_0. Note that $\varphi(w_1) < \varphi(w_2) < \cdots < \varphi(w_t)$. From our definition of weight, it follows that

$$d_T(v_0, w_i) = W(s_i).$$

Since φ has no gaps, for each $i \geq 1$, w_{i+1} is in a different branch of T from w_i. Thus

$$\varphi(w_{i+1}) - \varphi(w_i) \geq d_T(w_i, v_0) + d_T(v_0, w_{i+1}) = W(s_i) + W(s_{i+1}).$$

Also

$$\varphi(w_1) - \varphi(u) \geq d_T(u, w_1) > d_T(v, w_1) = W(s_1) + W(s_t),$$

since ν is closer to ν_0 than u. Adding up all the inequalities we have just obtained and noting that $w_t = \nu$ gives

$$\varphi(\nu) - \varphi(u) \geq \sum_{i=1}^{t} 2W(s_i).$$

But $d_T(u, \nu) = 1$. This gives us the desired lower bound on $D(\varphi)$. $\square$

Let $s_{\min}$ and $s_{\max}$ be the minimal and maximal elements of $S(T_k)$ according to the total order $\prec$. Let L be the set of links encountered on the path from a vertex in $s_{\min}$ to a vertex in $s_{\max}$. For each link in L we obtain an inequality, either from (2) or from (3). Summing all these inequalities gives

$$|L| \cdot D(\varphi) \;\geq\; \sum_{s \in S(T_k) - \{s_{\min}, s_{\max}\}} 2W(s). \tag{4}$$

Consider a branch of T_k which contains exactly t segments $s_0, s_1, \ldots, s_{t-1}$ ordered by distance from ν_0 in T_k, s_0 being closest to ν_0. Recalling that $\sum_{i=0}^{t-1} |s_i| = m^2$, we obtain

$$\begin{aligned}
W(s_0) + W(s_1) + \cdots + W(s_{t-1}) &= \sum_{i=0}^{t-1} (t-i)|s_i| \\
&\geq \sum_{i=0}^{t-1} \Big((t-i) + |s_i| - 1 \Big) \\
&\geq m^2 + t^2/4 - 1 \\
&\geq m^2 + 2^{2(k-1)}/4 - 1.
\end{aligned}$$

Using this fact in (4) gives

$$|L| \cdot D(\varphi) \;\geq\; 2b_k(m^2 + 2^{2(k-1)}/4 - 1) - 2W(s_{\min}) - 2W(s_{\max}).$$

Now consider only those k for which $b_k \geq 6$. Using $W(s_{\min}) \leq m^2$, $W(s_{\max}) \leq m^2$ and $|L| \leq 2 \cdot 2^k$ gives

$$2^{k+1} D(\varphi) \;\geq\; b_k(m^2 + 2^{2(k-1)}/4). \tag{5}$$

We are now ready to prove our lower bound.

Theorem 5.3 *Any embedding φ of the subdivided star satisfies* $D(\varphi) = \Omega(n^{3/4})$.

Proof: Suppose there is a k such that $b_k \geq 2^{k+1}m/16$. Then by (5) we would have $D(\varphi) \geq (m/16)(m^2 + 2^{2(k-1)}/4) \geq m^3/16$. Next, suppose there is a k such that $b_k \geq m^2/2^{k-\lceil \lg m \rceil + 1}$. Then (5) gives us $D(\varphi) \geq b_k(m^2/2^{k+1} + 2^{k-5}) \geq b_k \cdot 2^{k-5} \geq m^2 \cdot 2^{\lceil \lg m \rceil - 6} \geq m^3/64$. We shall now show that a k of at least one of these two types must exist, which will imply $D(\varphi) \geq m^3/64 = \Omega(m^3) = \Omega(n^{3/4})$.

If there is no k of either of these two types, then on the one hand

$$\sum_{k=1}^{\lceil \lg m \rceil} b_k < \frac{m}{16} \sum_{k=1}^{\lceil \lg m \rceil} 2^{k+1} < m^2/2,$$

and on the other hand

$$\sum_{k=\lceil \lg m\rceil+1}^{\lceil \lg m^2\rceil} b_k < \sum_{k=\lceil \lg m\rceil+1}^{\lceil \lg m^2\rceil} \frac{m^2}{2^{k-\lceil \lg m\rceil+1}} \leq m^2\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) \leq m^2/2.$$

Therefore, $b_1+b_2+\cdots+b_{\lceil \lg m^2\rceil} < m^2$ which is a contradiction, since by definition this sum equals m^2. This completes the proof. □

6 Other results

We conclude this paper with two more results on linear embeddings of trees, which we can prove with the tools developed in the previous section. We view these as negative results; for instance the first result shows that the maximum degree in a tree does not tell us much about the distortion of the optimal linear embedding; even if this maximum is restricted to be at most 3, this optimal distortion can be arbitrarily low or high. The second result shows that our sublinear distortion embedding of the complete binary tree cannot extend to arbitrary binary trees.

Theorem 6.1 *Let $f(n)$ be a nondecreasing function of n with $f(n) = \Omega(1)$ and $f(n) = O(n)$. Then there is a family of trees $\{T_i\}$, where T_n has n vertices, with maximum degree 3 and with the following property. There is a linear embedding of T_n with distortion $O(f(n))$, and any linear embedding of T_n has distortion $\Omega(f(n))$.*

Proof: Let the tree T_n be a comb with $\lceil n/f(n)\rceil$ teeth, each tooth being a path of length $\lceil f(n)\rceil$ except perhaps one which is of shorter length (in order to make the total T_n an n-vertex tree). Clearly the maximum degree of T_n is 3. It is very easy to embed T_n in the line with distortion $O(f(n))$, simply by embedding the teeth of the comb isometrically and separately. On the other hand, arguments similar to the ones used in Section 5 prove that any linear embedding of T_n must have distortion $\Omega(f(n))$.

Details will be provided in the final version of the paper. □

Corollary 6.2 *There is an n-vertex binary tree such that any linear embedding of it has distortion $\Omega(n)$.* □

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