11-20-2002

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Dartmouth Computer Science Technical Report TR2002-438

November 20, 2002

Abstract

The Lovász theta function has attracted a lot of attention for its connection with diverse issues, such as communicating without errors and computing large cliques in graphs. Indeed this function enjoys the remarkable property of being computable in polynomial time, despite being *sandwitched* between clique and chromatic number, two well known hard to compute quantities.

In this paper I provide a closed formula for the Lovász function of a specific class of sparse circulant graphs thus generalizing Lovász results on cycle graphs (circulant graphs of degree 2).

Keywords: *Lovász theta-function, Circulant graph, Linear programming*

1 Introduction

Consider a graph G whose vertices represent letters from a given alphabet, and where adjacency indicates that two letters can be "confused". The zero-error *capacity* of G is the number $\Theta(G)$ of messages that can be communicated without any error. This notion was introduced in 1956 by Shannon [13], and has generated a lot of interest over the years. It was understood quite early that the exact determination of the Shannon capacity is a very difficult problem, even for small and simple graphs. In 1979 Lovász [8] introduced a related function, to become soon

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thereafter known as Lovász theta function or Lovász number, with the aim of estimating the Shannon capacity.

The Lovász theta function (that will be denoted, here, by $\theta(G)$, and called theta function for short) is computable in polynomial time, although it is "sandwiched" between the clique number $\omega(G)$ and the chromatic number $k(G)$, whose computation is NP-hard. Because of this remarkable property and also of its relevance to communication issues, the Lovász number is widely studied (see the survey by Knuth [5] and the bibliography therein).

Despite a lot of work in the field, very little is known about classes of graphs for whose theta function either a formula or a very efficient algorithm is available. A rare example of such a result is Lovász's formula $\theta(C_n) = \frac{n\cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}}$ for n-cycles, with n odd [8]. Recently Brimkov et al. [2, 3] obtained formulae for the more general cases of circulant graphs with chord length two and three.

In this paper I use a geometric approach to establish and prove a closed formula for the theta function of circulant graphs of degree four, when the displacement j is even. The formula itself was already identified in [3] but not fully proven.

Here I close the issue establishing that, for j even and $n > 2(j+1)j$, the following holds:

$$\theta(C_{n,j}) = \frac{n}{1 + \frac{\cos\left(\frac{2\pi(i+1)j}{n}\right) - \cos\left(\frac{2\pi(i+1)}{n}\right) + \cos\left(\frac{2\pi i}{n}\right) - \cos\left(\frac{2\pi ij}{n}\right)}{\cos\left(\frac{2\pi(i+1)}{n}\right)\cdot\cos\left(\frac{2\pi ij}{n}\right) - \cos\left(\frac{2\pi i}{n}\right)\cdot\cos\left(\frac{2\pi(i+1)j}{n}\right)}} \tag{1}$$

for $i = \lfloor \frac{nj}{2(j+1)} \rfloor$.

In order to make this document self-contained I will review some of the results already established in [3].

2 Preliminaries

2.1 Some graph-theoretical notions and facts

Let us recall some well-known definitions from graph theory. Given a graph $G(V, E)$, its *complement graph* is the graph $\bar{G}(V, \bar{E})$, where $\bar{E}$ is the complement of E to the set of edges of the complete graph on V. An *automorphism* of the graph G is a permutation p of its vertices such that two vertices $u, v \in V$ are adjacent iff $p(u)$ and $p(v)$ are adjacent. G is *vertex symmetric* if its automorphism group is vertex transitive, i.e., for given $u, v \in V$ there is an automorphism p such that $p(u) = v$.

A graph $G'(V', E')$ is an *induced subgraph* of $G(V, E)$, if E' contains all edges from E that join vertices from $V' \subseteq V$. G is called *perfect* if $\omega(G_A) = k(G_A)$, $\forall A \subseteq V$, where G_A is the induced subgraph of G on the vertex set A.

An $n \times n$ matrix $A = (a_{i,j})_{i,j=0}^{n-1}$ is called *circulant* if its entries satisfy $a_{i,j} = a_{0,j-i}$, where the subscripts belong to the set $\{0, 1, \ldots, n-1\}$ and are calculated modulo n. In other words, any row of a circulant matrix can be obtained from the first one by a number of consecutive cyclic shifts, and thus the matrix is fully determined by its first row. A *circulant graph* is a graph with a circulant adjacency matrix. The expression $C_{n,j}$ will denote a circulant graph of degree four, with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{(i, i+1 \bmod n), (i, i+j \bmod n), i = 0, 1, \ldots, n-1\}$, where $1 < j \leq \frac{n-1}{2}$ is the *chord length.* See for illustration Fig. 1a presenting the circulant graph $C_{13,2}$.

Several equivalent definitions of the Lovász number are available [5]. Presented here is one which requires only little technical machinery.

Definition 1 *Given a graph G, let $\mathbf{A}$ be the family of matrices A such that $a_{ij} = 0$ if v_i and v_j are adjacent in G. Let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ be the eigenvalues of A. Then $\theta(A) = \max_{A \in \mathbf{A}} \{1 - \frac{\lambda_1(A)}{\lambda_n(A)}\}$.*

For various results related to the theta function I refer to [5]. In particular, the following proposition holds.

Proposition 1 (see [5]) *For every graph G with n vertices, $\theta(G) \cdot \theta(\bar{G}) \geq n$. If G is vertex symmetric, then $\theta(G) \cdot \theta(\bar{G}) = n$.*

2.2 LP formulation and related geometric constructions

Taking advantage of the particular properties of circulant matrices whose eigenvalues can be expressed in closed formulae and so generalizing the approach in [5], the validity of the following minmax formulation of the θ-function of circulant graphs of degree 4 can be easily derived.

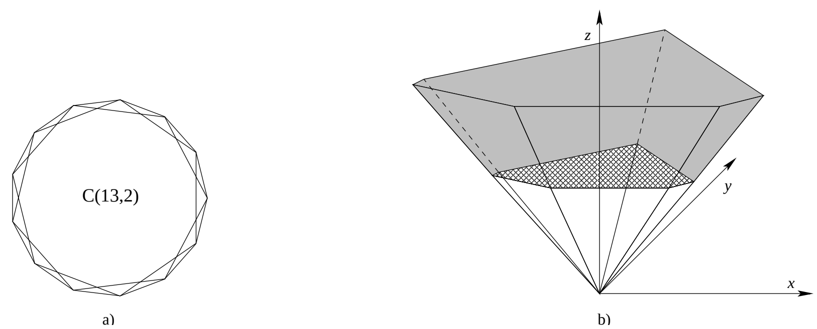

Figure 1: a) The circulant graph $C_{13,2}$. b) The truncated polyhedral cone related to $C_{13,2}$, cut at $z = 2$.

Lemma 1 (see [2]) *Let* $f_0(x, y) = n + 2x + 2y$ *and, for some fixed value of* j, $f_i(x, y) = 2x \cos \frac{2\pi i}{n} + 2y \cos \frac{2\pi ij}{n}$, $i = 1, 2, \ldots, n-1$. *Then*

$$\theta(C_{n,j}) = \min_{x,y} \max_{i} \left\{ f_i(x, y), i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor \right\}. \tag{2}$$

This in turn is equivalent to the following Linear Programming problem, that I will refer to, from now thereon:

$$\theta(C_{n,j}) = \quad \min\{z \ : \ f_i(x, y) - z \leq 0, \ i = 0, 1, \ldots, \lfloor \tfrac{n}{2} \rfloor, z \geq 0\}. \tag{3}$$

Observe that the equalities $f_1(x, y) - z = 0, \ldots, f_{n-1}(x, y) - z = 0$ define planes through the origin. Having in mind the specific coefficients of these planes in the different ortants, as well as the relations between the coefficients of two consecutive planes, one can see that the set $\max_i\{f_1(x, y), \ldots, f_{n-1}(x, y)\}$ is a polyhedral surface, namely a polyhedral cone C with apex at the origin. The cone belongs to the positive halfspace $z \geq 0$ and the Oz axis is contained inside the cone. The faces of the cone are portions of certain planes with equations $z = f_i(x, y)$, where i is in the range $1 < i \leq n - 1$. The rays of C are intersections of planes, obtained for no more than $n - 1$ pairs of indices i_1, i_2, where $1 \leq i_1, i_2 \leq n - 1$. The other intersections are not of interest, since they all fall "below" the conic surface $\max_i\{f_i\}$ and thus are not part of it.

Now, consider the plane $f_0(x, y) = n + 2x + 2y$. Its intersection with the cone C produces a new polyhedral surface, that is a truncated cone. This is the upper part of the cone C, i.e., the one above the plane f_0 (see Fig. 1b).

Clearly, the intersection points of the plane f_0 with C are the possible candidates for solution of the problem. The theta function is the intersection point with minimal z.

Consider the intersection of C and f_0. This intersection is the boundary of some 2D convex polyhedron P (possibly unbounded). As mentioned above, the solution is at some of the vertices of this intersection. Let this be the point $A = (x_0, y_0, z_0)$ (and thus $\vartheta = z_0$) and assume that we have intersected C by the plane $z = z_0$ (parallel to the xy-plane). The intersection is a (bounded) convex polygon Q_{z_0}. By construction, it follows that the polyhedron P and the polygon Q_{z_0} intersect at a single point, i.e., the point $A = (x_0, y_0, z_0)$. Point A will be determined using the sides of Q_{z_0}, rather than the sides of P. Since the coefficients of x and y of the plane $z = n + 2x + 2y$ are equal (indeed they are both equal to 2), then it is not difficult to see that A will be the vertex of Q_{z_0}, obtained as the intersection of the two sides of Q_{z_0} which "sandwich" the straight line in $z = z_0$ passing through A, and with a slope of 45 degrees. These lines have equations $2x\cos\alpha + 2y\cos(\alpha j) = z_0$ and $2x\cos\beta + 2y\cos(\beta j) = z_0$, where $\alpha = \frac{2\pi i_1}{n}$ and $\beta = \frac{2\pi i_2}{n}$, for some indices i_1 and i_2. Once i_1 and i_2 are known, z_0 can be computed by solving the linear system

$$\begin{cases} z = 2x\cos\alpha + 2y\cos(j\alpha) \\ z = 2x\cos\beta + 2y\cos(j\beta) \\ z = n + 2x + 2y. \end{cases}$$

Note that one can use any horizontal intersection of the cone, since all such intersections are homothetic to each other.

Through a nontrivial analysis of the structure of the admissible region defined by the linear constraints, it was possible to obtain closed formulae for some special cases of circulant graphs of degree four [2]. For example, I report the least complex formula for the simplest case $j = 2$:

$$\theta(C_{n,2}) = n\left(1 - \frac{\frac{1}{2} - \cos(\frac{2\pi}{n}\lfloor\frac{n}{3}\rfloor) - \cos(\frac{2\pi}{n}(\lfloor\frac{n}{3}\rfloor + 1))}{(\cos(\frac{2\pi}{n}\lfloor\frac{n}{3}\rfloor) - 1)(\cos(\frac{2\pi}{n}(\lfloor\frac{n}{3}\rfloor + 1)) - 1)}\right). \tag{4}$$

3 Proof of the Formula

As anticipated before, the proof is based on a geometric approach. It will be useful to introduce the following definitions. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{j+1}\}$ be a set of adjacent intervals covering $[0, \pi]$

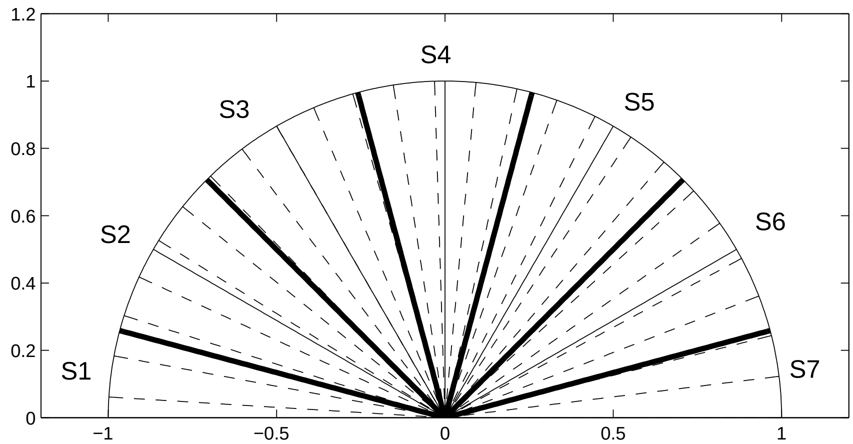

Figure 2: Pictorial description of the S-intervals for $j = 6$. The dashed lines correspond to angles $\frac{2k\pi}{n}$, for $n = 51$ and $1 \leq k \leq \frac{n-1}{2}$.

defined as

$$S_1 = [\pi - \frac{\pi}{2j}, \pi], S_{j+1} = [0, \frac{\pi}{2j}], \text{ and } S_{k+1} = [\pi - (2k+1)\frac{\pi}{2j}, \pi - (2k-1)\frac{\pi}{2j}],$$

for $k = 1, 2, \ldots, j-1$. So, S_1, S_{j+1} are intervals of width $\frac{\pi}{2j}$, whereas $S_2, S_3, \ldots, S_j$ are intervals of width $\frac{\pi}{j}$ (see Fig. 2). The $j/2$ even numbered ones, S_{2k}, for $k = 1, 2, \ldots, j/2$, are those in which $\cos(j\alpha)$ is negative. Let $\beta^{(k)}$ denote the angle corresponding to the center of S_{2k}. Thus $\cos(j\beta^{(k)}) = -1$ for all k. Notice that each interval contains no more than $\lceil \frac{n}{j} \rceil$ lines.

Let us focus on $S_1 \cup S_2$ and on the following sequence of lines $l_{i_1}, l_{i_1+1}, \ldots, l_s$, where l_{i_1} is the line whose angle, α_{i_1} is the closest to the center of S_2, and l_s is the line whose angle is the largest within S_1, i.e., $s = \lfloor n/2 \rfloor$. It is not hard to see that those lines define a set C_1 of segments that, together with the x and y negative axes, bind a convex polygon Q.

The same idea can be applied to the other even numbered intervals, S_{2k}, $k = 2, 3, \ldots, j/2$, to define the sequence of lines $l_{i_k}, l_{i_k+1}, \ldots, l_{s_k}$, where l_{i_k} is the line whose angle is the closest to the center of S_{2k}, whereas l_{s_k} is the line whose angle is the largest in S_{2k-1}. It turns out that for all k only l_{i_k} might intersect Q. Furthermore, this would occur only when the angle of l_{i_k}, α_{i_k}, satisfies

$$\mid \alpha_{i_k} - \beta^{(k)} \mid < \mid \alpha_{i_1} - \beta^{(1)} \mid \leq \pi/n \ . \tag{5}$$

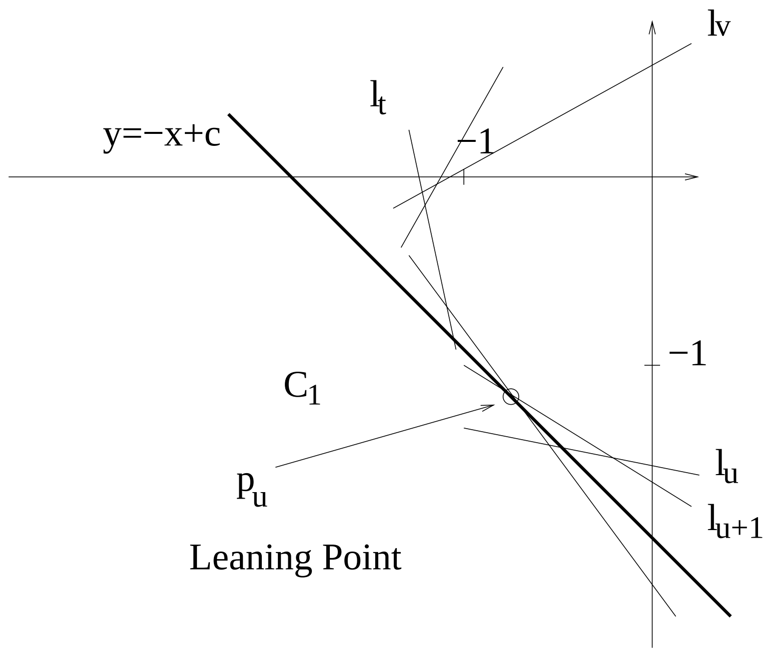

Figure 3: Chain Property. Note that in this case also the Leaning Property holds for $k = u + 1$.

As a consequence, the search for the solution can be restricted to the vertices of the polygon formed by the two axes, the lines in $S_1 \cup S_2$ plus, possibly, the lines whose angles verify property 5.

Relying on formulation 3 I will focus on the geometric unintuitive regularities of the polygon defined by the lines l_k of equation

$$x \cos(\alpha_k) + y \cos(j\alpha_k) \quad = \quad 1, \tag{6}$$

with $\alpha_k = \frac{2\pi}{n}k$. Let $a(k) = 1/\cos(\alpha_k)$ and $b(k) = 1/\cos(2\pi kj/n)$ be their x and y coordinates (axes cuts) respectively. I will refer to angle α_k as to the angle of line l_k.

Let us recall the following definitions that will be be instrumental to our analysis.

Definition 2 (Chain Property) *Let $u < v$ be two positive integers. We say that the sequence of lines $\{l_i\}, i = u, u+1, \ldots, v$, possesses the* Chain Property *if the intersection points p_i between consecutive lines l_i and l_{i+1} are all vertices of the convex polygon Q defined by $x, y \leq 0$, and $\forall i \in \{u, u+1, \ldots, v\},\ x \cdot \cos \alpha_i + y \cdot \cos j\alpha_i \leq 1$.*

Definition 3 (Leaning Property) *Let $u < v$ be two positive integers. We say that the sequence $\{l_i\}, i = u, u+1, \ldots, v$, possesses the* Leaning Property *if it possesses the Chain Property and, in addition, there exists an index k, $u \leq k \leq v$, such that line l_k forms, with the Oy axis, an angle larger than 45^o and l_{k+1} forms, with the Oy axis, an angle smaller than 45^o.*

The sense of the Chain Property is that all the intersection points p_i lie on a convex curve while the Leaning Property implies, in addition, that Q *leans* on a line of equation $y = -x + c$ for a proper $c < 0$ (see Fig. 3). As we will see, this makes point p_u a candidate solution for the problem in Q_{z_0}. Observe that the Leaning Property holds in the case $j = 2$ and that allowed us to establish the closed formula (4) for $\theta(C_{n,2})$ [2].

Interestingly it is possible to prove that the Leaning Property holds for appropriate subsequences of lines and in addition the leaning vertex p_u belongs to Q_{z_0}. This means that p_u is never cut off by any other line not in $\{l_{i_1}, l_{i_1+1}, \ldots, l_s\}$ and that leaves it as the only solution to the problem.

Theorem 1 *Let n and j be integer numbers. Assume that j is even and $n > 2(1+j)j$ and let L_b^a denote the line of equation $x \cos \frac{2\pi b}{a} + y \cos \frac{2\pi b j}{a} = 1$. Then for all $m \geq n$ the sequences of lines $\{L_i^m\}$, for $\lceil \frac{m(j-1)}{2j} \rceil \leq i \leq \frac{m-1}{2}$, all possess the Chain Property.*

Proof. The idea behind the proof is the following. We can consider a continuous family of lines that includes all the sequences $\{L_i^m\}$ and determine its envelope curve as the locus of the intersections of neighboring lines. This curve of the *loci* has the property of being tangent to all the lines in all the sequences $\{L_i^m\}$. Then I show that this curve of the *loci* is convex in the interval of interest.

Let $g(t)$ be the line of equation $x \cos t + y \cos jt = 1$ and let $(x_t(s), y_t(s))$ be the intersection point between $g(t)$ and $g(t+s)$. Note that $L_i^m = g(2\pi i/m)$. Let us prove the following facts:

1. The following two limits

$$x_t = \lim_{s \to 0} x_t(s) \text{ and } y_t = \lim_{s \to 0} y_t(s)$$

   exist and determine a parameterized curve $X(t) = (x_t, y_t)$.

2. for $t \in \left(\pi - \frac{\pi}{j}, \pi\right)$, $y_t(s)$ is monotonically increasing for $s \in [t, \pi]$.

3. The curve of the loci $X(t)$ is convex for $t \in \left(\pi - \frac{\pi}{j}, \pi\right)$.

This will be enough as we can observe that, for all m, lines $\{L_i^m\}$ are all tangent to $X(t)$, have angles within $\left(\pi - \frac{\pi}{j}, \pi\right)$, and in the limit, as m goes to infinity, the "chain" of segments, connecting intersection points between consecutive lines, converges to $X(t)$ itself.

1. By Kramer rule

$$\begin{aligned} y_t(s) &= \frac{\cos t - \cos(t+s)}{\cos j(t+s)\cos t - \cos jt \cos(t+s)} \\ &= \frac{\cos t - \cos t \cos s + \sin t \sin s}{\cos t(\cos jt \cos js - \sin jt \sin js) - \cos jt(\cos t \cos s - \sin t \sin s)} \end{aligned}$$

Since $\cos s \sim 1 - s^2/2$ and $\sin s \sim s$, we can rewrite the expression as

$$\begin{aligned} y_t(s) &\sim \frac{\frac{s^2}{2} \cdot \cos t + s \cdot \sin t}{s \cdot [-j \cos t \sin jt + \cos jt \sin t] + s^2 \cdot \frac{1-j^2}{2} \cos t \cos jt} \\ &\sim \frac{\frac{s}{2} \cdot \cos t + \sin t}{[-j \cos t \sin jt + \cos jt \sin t] + s \cdot \frac{1-j^2}{2} \cos t \cos jt} \ . \end{aligned}$$

Analogously we can determine the x solution:

$$\begin{aligned} x_t(s) &= \frac{-\cos jt + \cos(j(t+s))}{\cos j(t+s)\cos t - \cos jt \cos(t+s)} \\ &= \frac{\cos jt \cos js - \sin jt \sin js - \cos jt}{\cos t(\cos jt \cos js - \sin jt \sin js) - \cos jt(\cos t \cos s - \sin t \sin s)} \end{aligned}$$

Again $\cos js \sim 1 - (js)^2/2$ and $\sin js \sim js$, and so we can rewrite the expression as

$$\begin{aligned} x_t(s) &\sim \frac{-\frac{(js)^2}{2} \cdot \cos jt - js \cdot \sin jt}{s \cdot [-j \cos t \sin jt + \cos jt \sin t] + s^2 \cdot \frac{1-j^2}{2} \cos t \cos jt} \\ &\sim \frac{-\frac{j^2}{2} s \cdot \cos jt - j \sin jt}{[-j \cos t \sin jt + \cos jt \sin t] + s \cdot \frac{1-j^2}{2} \cos t \cos jt} \ . \end{aligned}$$

Thus, since $-j \cos t \sin jt + \cos jt \sin t$ does not vanish in $(\pi - \frac{\pi}{j}, \pi)$, the following limits exist

$$\begin{aligned} x_t = \lim_{s \to 0} x_t(s) &= \frac{-j \cdot \sin jt}{-j \cos t \sin jt + \cos jt \sin t} \\ y_t = \lim_{s \to 0} y_t(s) &= \frac{\sin t}{-j \cos t \sin jt + \cos jt \sin t} \end{aligned}$$

Examples of $X(t)$ are given in Fig. 1. See also Fig. 3 for the behavior of the lines with respect to $X(t)$.

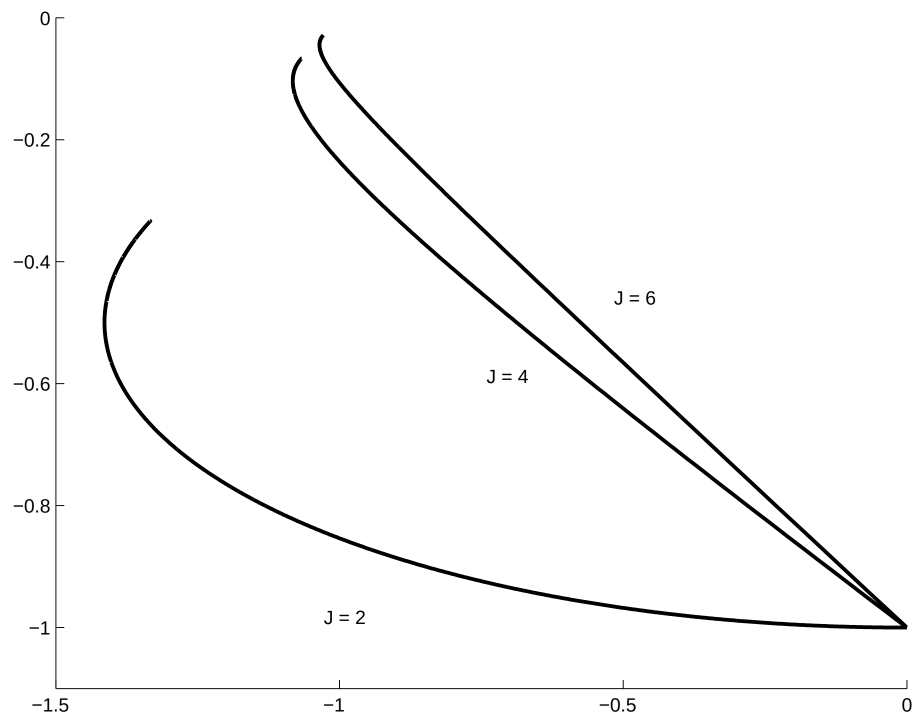

Figure 4: Curves of the *loci* $X(t)$, for $j = 2, 4, 6$.

2. This claim can be proven by showing that $dy_t(s)/ds > 0$ in the interval of interest. Let

$$\begin{aligned} A &= -j\cos t \sin jt + \cos jt \sin t \\ B &= \frac{1-j^2}{2} \cos t \cos jt \ . \end{aligned}$$

By applying the rules of differentiation we obtain

$$\begin{aligned} \frac{dy_t(s)}{ds} &= \frac{\frac{\cos t}{2}(A+sB) - (\frac{\cos t}{2}s + \sin t)B}{(A+sB)^2} \\ &= \frac{\frac{\cos t}{2}A + \frac{\cos t}{2}sB - \frac{\cos t}{2}sB - B\sin t}{(A+sB)^2} \\ &= \frac{\frac{\cos t}{2}A - B\sin t}{(A+sB)^2} \ . \end{aligned}$$

Now, let us study the sign of the numerator in the points in which $A + sB$ does not vanish. In particular it is necessary to show that it is positive. By substituting A and B for their expressions we obtain

$$\frac{\cos t}{2}(-j\cos t \sin jt + \cos jt \sin t) \quad \geq \quad \left(\frac{1-j^2}{2}\right) \sin t \cos t \cos jt$$

$$\begin{aligned} -j(\cos t)^2 \sin jt &\geq -j^2 \sin t \cos t \cos jt \\ (\cos t)^2 \sin jt &\leq j \sin t \cos t \cos jt \end{aligned}$$

Remembering that $\cos t < 0$ for $t \in (\pi - \frac{\pi}{j}, \pi)$, we can divide by $\cos t$ and change sign:

$$\cos t \sin jt \geq j \sin t \cos jt \ .$$

The left hand side is always positive in $(\pi - \frac{\pi}{j}, \pi)$, whereas the right hand side is negative in $(\pi - \frac{\pi}{j}, \pi - \frac{\pi}{2j})$, but positive in $(\pi - \frac{\pi}{2j}, \pi)$. So, the critical case is the second part of the interval. For that let us resort to asymptotic analysis, by Taylor expansions around π. If we substitute $x = \pi - t$ we can study the equivalent inequality

$$\cos x \sin jx \geq j \sin x \cos jx$$

for $x \in [0, \frac{\pi}{2j}]$. A positive lower bound to the left hand side is given by

$$\cos x \sin jx > \left(1 - \frac{x^2}{2}\right)\left(jx - \frac{(jx)^3}{6}\right)$$

whereas an upper bound to the right hand side is given by

$$j \sin x \cos jx < jx \left(1 - \frac{(jx)^2}{2} + \frac{(jx)^4}{4!}\right)$$

And now the following must be shown

$$\left(1 - \frac{x^2}{2}\right)\left(jx - \frac{(jx)^3}{6}\right) \geq jx \left(1 - \frac{(jx)^2}{2} + \frac{(jx)^4}{4!}\right)$$

or equivalently

$$\begin{aligned} 1 - \frac{(jx)^2}{6} - \frac{x^2}{2} + \frac{(jx)^2 x^2}{12} &> 1 - \frac{(jx)^2}{2} + \frac{(jx)^4}{4!} \\ -\frac{j^2}{3} - 1 + \frac{(jx)^2}{6} &> -j^2 + \frac{j^4 x^2}{12} \\ x^2 \left(\frac{j^2}{6} - \frac{j^4}{12}\right) &> \frac{j^2}{3} + 1 - j^2 \\ x^2 \frac{j^2(2 - j^2)}{12} &> \frac{j^2 + 3 - 3j^2}{3} \\ x^2 \frac{j^2(j^2 - 2)}{12} &< \frac{2}{3} j^2 - 1 \end{aligned}$$

Since the left hand side is a positive and increasing quantity in x, the worst case is attained at $x = \frac{\pi}{2j}$, which corresponds to $t = \pi - \frac{\pi}{2j}$. Performing the substitution we have

$$\begin{aligned} \frac{\pi^2}{4j^2}\frac{j^2(j^2-2)}{12} &< \frac{2}{3}j^2 - 1 \\ \frac{\pi^2}{48}(j^2-2) &< \frac{2}{3}j^2 - 1 \\ \left(\frac{2}{3} - \frac{\pi^2}{48}\right)j^2 &> 1 - \frac{\pi^2}{24} \end{aligned}$$

But this is true for $j \geq 2$ since the left hand side is greater than or equal to 1.8442 whereas the right hand side is 0.5888.

3. Now, let us see the convexity of $X(t) = (x_t, y_t)$, for $t \in (\pi - \frac{\pi}{j}, \pi)$. But this follows immediately from point (2) and the fact that lines $g(t)$ move "clockwise", as t runs from $\pi - \frac{\pi}{j}$ to π. The reason is that the slope of line $g(t)$, $-\frac{\cos t}{\cos jt}$, is monotonically decreasing for $\pi - \frac{\pi}{j} < t < \pi - \frac{\pi}{2j}$, and for $\pi - \frac{\pi}{2j} < t < \pi$.

□

The following lemma establishes a condition for the Leaning Property to hold:

Lemma 2 *Let j be an even number and let $n > 2(1+j)j$. Then the sequence of lines $\{l_i\}$, for $\lceil \frac{n(j-1)}{2j} \rceil \leq i \leq \frac{n-1}{2}$ possesses the Leaning Property.*

Proof Theorem 1 assures that $\{l_i\}$ possesses the Chain Property. So, now let us prove that, if $n > 2(1+j)j$, two conditions hold:

1. Line $l_{\lceil \frac{n(j-1)}{2j} \rceil}$, that has the smallest angle greater than or equal to $\pi - \pi/j$, the center of S_2, makes with the Oy axis an angle $\gamma \geq 45$ degrees;

2. Line l_t whose angle is the largest in S_2 forms instead with the Oy axis an angle $\gamma_t \leq 45$ degrees.

The claim will then follow from the fact that the Chain Property implies that the angle formed by line l_i with Oy strictly decreases with the index i and so at some point it will necessarily cross the border of 45 degrees (see Fig. 3, for $u = \lceil \frac{n(j-1)}{2j} \rceil$ and $v = \lfloor \frac{n}{2} \rfloor$).

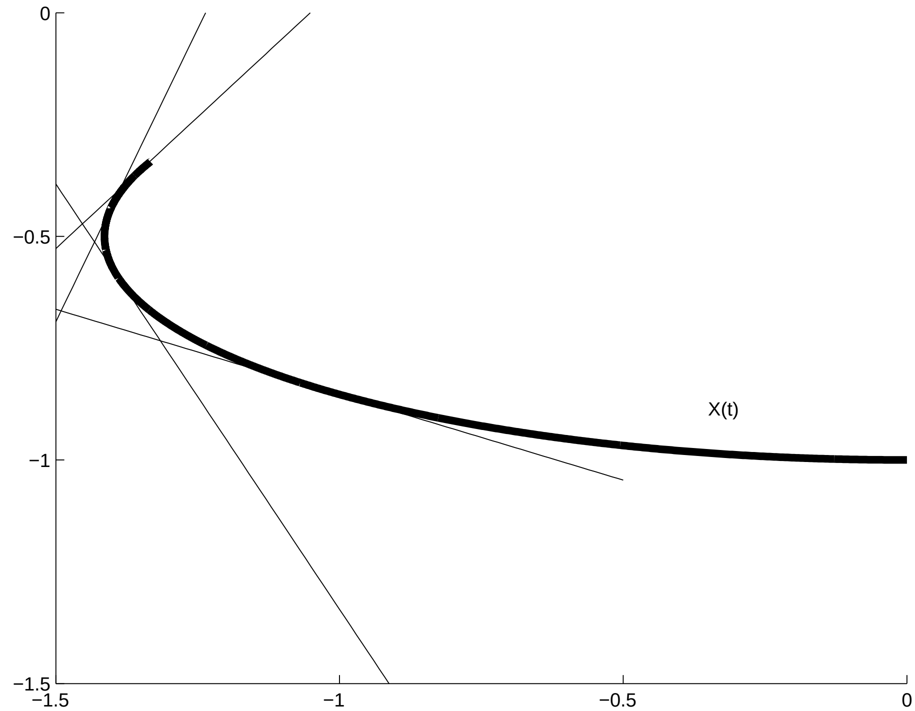

Figure 5: Simple case $j = 2$, $n = 20$. Notice how the lines are tangent to $X(t)$.

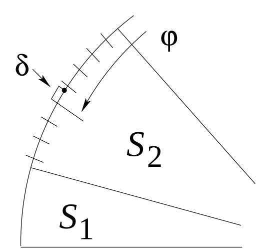

Figure 6: Smallest angle greater than the center of S_2 (black dot).

1. Let $A = (1/a, 0)$ and $B = (0, 1/b)$ be the intersection points of $l_{\lceil \frac{n(j-1)}{2j} \rceil}$ with the axes Ox and Oy, respectively. The angle in question will be larger than 45 degrees if $|a| < |b|$.

   Let $\phi = \alpha_{\lceil \frac{n(j-1)}{2j} \rceil} = \frac{2\pi}{n} \lceil \frac{n(j-1)}{2j} \rceil$ and $\delta = \phi - (\pi - \pi/j)$. We can see that $|a| = |\cos(\phi)|$ and $|b| = |\cos(j\delta)|$. Thus to prove the lemma, it is necessary to show that $|\cos(\phi)| < |\cos(j\delta)|$ (see Fig. 6).

   Note that if $\delta = 0$ then the lemma is clearly true. Let $\delta \neq 0$. We can safely assume that

$j > 2$, so that $\pi - (\pi - \pi/j) < \pi/2$. Then our problem reduces to solving the inequality $\cos(\pi - \phi) < \cos(j\delta)$ or equivalently:

$$\cos\left(\pi - \frac{2\pi}{n}\left\lceil\frac{n(j-1)}{2j}\right\rceil\right) < \cos\left(j\left(\frac{2\pi}{n}\left\lceil\frac{n(j-1)}{2j}\right\rceil - (\pi - \pi/j)\right)\right).$$

Since function cos is monotone decreasing in $[0, \pi/2]$ the above expression is verified for

$$2\left\lceil\frac{n(j-1)}{2j}\right\rceil - n < n(j-1) - 2j\left\lceil\frac{n(j-1)}{2j}\right\rceil,$$

and a fortiori for

$$2\left(\frac{n(j-1)}{2j} + 1\right) - n < n(j-1) - 2j\left(\frac{n(j-1)}{2j} + 1\right)$$

from which the claim follows.

2. By definition $t = \arg\max\{2\frac{\pi}{n}i \mid 2\frac{\pi}{n}i < \pi - \frac{\pi}{2j}\} = \lfloor\frac{n}{2} - \frac{n}{4j}\rfloor$. As before, let $A = (1/a, 0)$ and $B = (0, 1/b)$ be the intersection points of l_t with the axes Ox and Oy, respectively. The angle in question will be smaller than 45 degrees if $|a| > |b|$, i.e., if

$$\left|\cos\left(\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right| > \left|\cos\left(j\cdot\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right|.$$

But this can be easily verified by observing that, for $j > 2$, it is certainly true that

$$\left|\cos\left(\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right| > \cos\frac{\pi}{4} > \left|\cos\left(j\cdot\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right|.$$

(Intuitively, as n increases, one gets $\left|\cos\left(\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right| \approx |\cos(\pi - \frac{\pi}{2j})| = \cos\frac{\pi}{2j}$, and $\left|\cos\left(j\cdot\frac{2\pi}{n}\cdot\left\lfloor\frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right| \approx |\cos(j(\pi - \frac{\pi}{2j}))| = 0$.) □

The main theorem can now be fully proven.

Theorem 2 *Let n and j be integer numbers. Assume that j is even and $n > 2(1+j)j$. Then $\theta(C_{n,j}) = z_0$, where (x_0, y_0, z_0) is the only solution to the following 3×3 linear system:*

$$\begin{cases} n + 2x + 2y & = & z, \\ 2x\cos(\frac{2\pi k}{n}) + 2y\cos(\frac{2\pi kj}{n}) & = & z, \\ 2x\cos(\frac{2\pi(k+1)}{n}) + 2y\cos(\frac{2\pi(k+1)j}{n}) & = & z, \end{cases}$$

for $k = \lfloor\frac{nj}{2(j+1)}\rfloor$. And by Kramer's rule this gives formula 4.

Proof The hypothesis, Theorem 1 and Lemma 2 imply that the sequence of lines $\{L_i^n = l_i\}$, for $\left\lceil \frac{n(j-1)}{2j} \right\rceil \leq i \leq \frac{n-1}{2}$, possesses the Leaning Property. So to prove the claim I need to show that the leaning vertex p_u of Q is the intersection point $P(k)$ between lines l_k and l_{k+1} and it is not cut off by any other lines defining Q_{z_0}.

Let us start with the first claim. The Leaning Property reduces the problem to identifying the two angles across the zero of the trigonometric equation $\cos(j\alpha) - \cos(\alpha) = 0$. Basic trigonometric calculations show that the only solution to the given equation within the second half of interval S_2 is given by

$$\alpha = \frac{\pi j}{j+1}.$$

To see this, let us first exploit the following classical identity

$$\cos(j\alpha) - \cos(\alpha) = -2\sin\frac{\alpha(j+1)}{2} \cdot \sin\frac{\alpha(j-1)}{2} = 0 \ .$$

Then let us solve it for each term in the interval $[\pi - \pi/j, \pi - \pi/2j]$, obtaining the following candidates

$$\alpha_{1,s} = \frac{2\pi s}{j+1} \quad \text{and} \quad \alpha_{2,h} = \frac{2\pi h}{j+1},$$

for h, s nonnegative integers. Thus the first term nullifies for all integers s such that

$$\frac{j^2-1}{2j} < s < \frac{j^2 + \frac{j}{2} - \frac{1}{2}}{2j} \ .$$

We can see that there must exist only one solution given by

$$s = \left\lceil \frac{j^2-1}{2j} \right\rceil = \left\lfloor \frac{j^2 + \frac{j}{2} - \frac{1}{2}}{2j} \right\rfloor = \frac{j}{2} \ .$$

The second term, instead, nullifies for all integers h such that

$$\frac{(j-1)^2}{2j} < h < \frac{j^2 - \frac{3}{2}j + \frac{1}{2}}{2j} \ ,$$

and we can easily see that no such integer values can exist since, for $j \geq 4$,

$$\left\lceil \frac{(j-1)^2}{2j} \right\rceil = \left\lfloor \frac{(j-1)^2}{2j} + 1 \right\rfloor = \left\lfloor \frac{j^2+1}{2j} \right\rfloor = \frac{j}{2} > \left\lfloor \frac{j^2 - \frac{3}{2}j + \frac{1}{2}}{2j} \right\rfloor \ .$$

So the first of the two lines we are looking for will be found determining the largest integer k such that $\frac{2\pi k}{n} < \alpha$. This determines line l_k. The second one will be just the next one: l_{k+1}.

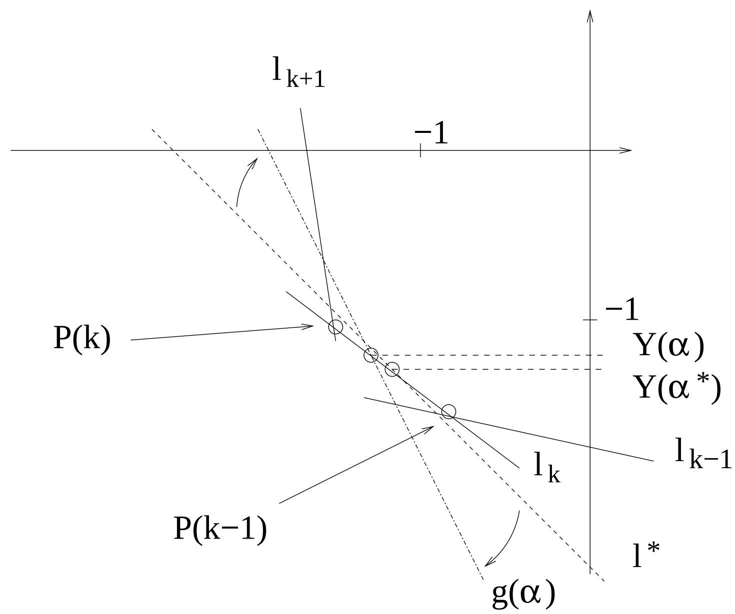

Figure 7: Motion of $g(\alpha)$: $Y(\alpha)$ increases with α.

The second claim is a little more complex. First observe that the only lines that might interfere are those whose angles verify property 5. Now the worst possible case is represented by a line l' that cuts the Oy axis in the point $(0,-1)$ and the Ox axis in the point $(1/\cos(\beta^{(2)}+2\pi/n),0)$. That would have equation $x\cos(\beta^{(2)}+2\pi/n)-y=1$, where $\beta^{(2)}=\pi-3\pi/j$, $\cos(j\cdot\beta^{(2)})=-1$. This comes from the fact that the two values $1/\cos(\beta^{(2)}+2\pi/n)$ and -1 are upper bounds to the x and y cuts of any possible "harmful" line, where, by harmful, I mean a line that verifies property 5.

Let $P(i)=(A_i,B_i)$ be the intersection point between line l_i and line l_{i+1}. Let l^* be the line of equation $x\cos\alpha^*+y\cos j\alpha^*=1$, where α^* verifies $\cos\alpha^*=\cos j\alpha^*$. This line belongs to the family $y=-x+c$ and must pass between points $P(k-1)$ and $P(k)$.

To see this, consider the family of lines

$$L=\{g(\alpha):\ x\cdot\cos\alpha+y\cdot\cos j\alpha=1 \mid \alpha\in[\alpha_k,\alpha_{k+1}]\}.$$

Clearly, $L_i^n=l_i=g\left(\frac{2\pi i}{n}\right)$. Now, imagine line $g(\alpha)$ that moves with continuity from l_k to l_{k+1}, as α ranges from α_k to α_{k+1}, and focus on the ordinate $Y(\alpha)$ of the intersection point between

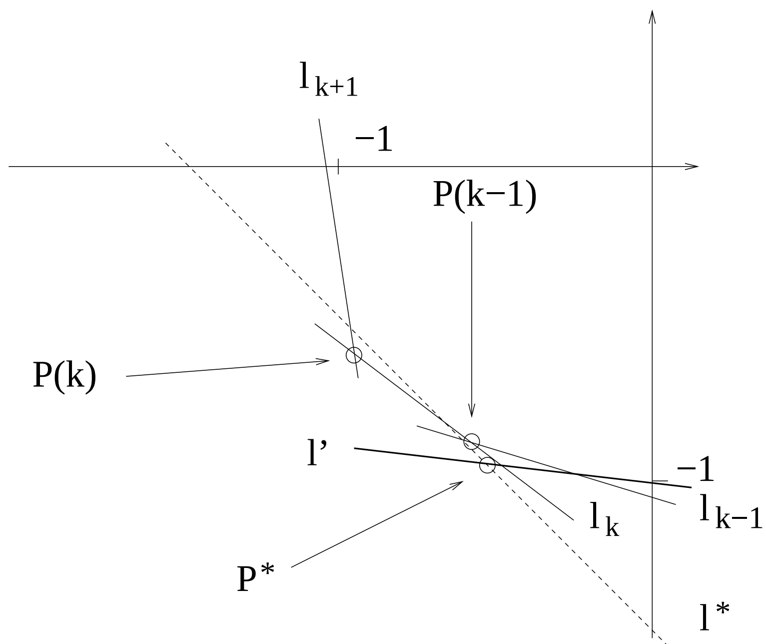

Figure 8: $P(k)$ is not cut off by l'.

l_k and $g(\alpha)$ (see Fig. 7). Since the Chain Property holds for all m, then $Y(\alpha)$ must increase monotonically from B^+_{k-1} to B^-_k. Furthermore, $\alpha_k \leq \alpha^* \leq \alpha_{k+1}$ and so l^* must cut somewhere the segment joining $P(k-1)$ and $P(k)$.

Let $P^* = (A^*, B^*)$ be the intersection point between l' and l^*. It will be proven that $B^* < B_{k-1}$. Thus, since $B_{k-1} < B_k$, a fortiori, $P(k)$ cannot be compromised by line l' (see Fig. 8).

Solving the intersection problems for lines l', l^* and lines l_{k-1}, l_k, we obtain

$$\begin{aligned} B_{k-1} &= \frac{\cos\alpha_{k-1} - \cos\alpha_k}{\cos\alpha_{k-1}\cos j\alpha_k - \cos j\alpha_{k-1}\cos\alpha_k}, \\ B^* &= \frac{\cos\alpha^* - \cos(\pi - 3\pi/j + 2\pi/n)}{-\cos\alpha^* - \cos j\alpha^* \cos(\pi - 3\pi/j + 2\pi/n)}. \end{aligned}$$

Let us now compare the numerators. Clearly, it holds that

$$\cos\alpha_{k+1} < \cos\alpha^* < \cos\alpha_k < \cos\alpha_{k-1} < \cos(\pi - 3\pi/j + 2\pi/n) < 0,$$

$$|\cos\alpha_{k+1}| > |\cos\alpha^*| > |\cos\alpha_k| > |\cos\alpha_{k-1}| > |\cos(\pi - 3\pi/j + 2\pi/n)| > 0,$$

so it must be $\cos\alpha^* - \cos(\pi - 3\pi/j + 2\pi/n) < \cos\alpha_{k-1} - \cos\alpha_k$. Let us now compare the denominators. Clearly, it holds that

$$\cos j\alpha_{k-1} < \cos j\alpha_k < \cos j\alpha^* < \cos j\alpha_{k+1} < 0 \ ,$$

$$|\cos j\alpha_{k-1}| > |\cos j\alpha_k| > |\cos j\alpha^*| > |\cos j\alpha_{k+1}| > 0,$$

and so $\frac{\cos j\alpha_{k-1}}{\cos j\alpha^*} > 1$ and $\frac{\cos\alpha_{k-1}}{\cos\alpha^*} < 1$. It is necessary to show that

$$-\cos\alpha^* - \cos j\alpha^* \cos(\pi - 3\pi/j + 2\pi/n) > \cos\alpha_{k-1}\cos j\alpha_k - \cos j\alpha_{k-1}\cos\alpha_k \ ,$$

or equivalently

$$-\cos\alpha^* + \cos j\alpha_{k-1}\cos\alpha_k > \cos\alpha_{k-1}\cos j\alpha_k + \cos j\alpha^* \cos(\pi - 3\pi/j + 2\pi/n) \ .$$

Dividing by $-\cos\alpha^* > 0$ and remembering that $\cos\alpha^* = \cos j\alpha^*$, the above inequality translates into

$$1 - \frac{\cos j\alpha_{k-1}}{\cos\alpha^*}\cos\alpha_k > \frac{\cos\alpha_{k-1}}{\cos\alpha^*}(-\cos j\alpha_k) - \cos(\pi - 3\pi/j + 2\pi/n) \ .$$

But this follows from the fact that

$$1 > \frac{\cos\alpha_{k-1}}{\cos\alpha^*}(-\cos j\alpha_k) > 0$$

and

$$\frac{\cos j\alpha_{k-1}}{\cos\alpha^*}(-\cos\alpha_k) > -\cos\alpha_k > -\cos(\pi - 3\pi/j + 2\pi/n) > 0 \ .$$

□

Acknowledgements

I wish to thank Bruno Codenotti and Valentin Brimkov for inspiring this work and Luis Caffarelli for suggesting a way to prove theorem 1.

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