The Cybernetics of nonzero sum games — the prisoner's dilemma reinterpreted as a pure conflict game with nature, with empirical applications

by

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Abstract

In this thesis a new solution concept is developed for n-player, nonzero sum games. The solution concept is based in reinterpreting the n-player nonzero sum game into 2-player zero sum games. The n-player nonzero sum game is first rewritten as an n + 1 player coalition game. The definition of zero sum payment is that one player pays the other what he gets in a given outcome (coalition of the n + 1 player game). Who pays whom depends on the coalition. More than one 2-player zero sum interpretation game always results from the procedure, and criteria are established to select one of the zero sum interpretation games. The solution concept defines results identical to the minimax concept when applied directly to zero sum 2-player games.

When applied to 2-player prisoner's dilemma games, the solution procedure assigns mixed strategies to the prisoners, thereby "resolving" the dilemma. The mixed strategies vary with the payoffs (up to a linear transformation). For prisoner's dilemma matrices which have been used in large numbers of gaming experiments, the solution concept predicts dynamically, i.e., by play number, the "fraction of cooperative choices" for (approximately) the first 30 plays. In addition, the mixed strategy appears in a game between each subject (prisoner) and the n + 1st player (district attorney), suggesting that the subjects have been playing against the experimenter. Empirical evidence for this conclusion is given. A theorem is proved for n-player prisoner's dilemma games.

Game theory is reviewed to show the roots of this solution concept in the heuristic use of zero sum n-player games in the von Neumann and Morgenstern theory, and in rational decision making models, e.g., "games against Nature." The empirical and formal difficulties of the equilibrium point solution concept for nonzero sum games are discussed. Detailed connections between game theory and cybernetics are described.

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CONTENTS

0.0 Introduction and summary

Cybernetics, the study of control and communication in animals and machines,[70] employs mathematical decision making models for normative prescriptions of control processes[71] and descriptive models of the environment to be controlled.[17, 97] One important mathematical decision making model is the theory of games of strategy.[71] However, for formal reasons, the effectiveness of this type of model is limited to its most elegant portion — the theory of zero sum two player games, where the interests of the players are totally opposed. This part of game theory links with linear programming,[72] the theory of neural[73, 78] nets, and statistical decision making processes.[74] Solutions to games of this type can be fully characterized and computed (although the latter is sometimes difficult). However, there have been few applications of this portion of game theory to real world problems because few problems satisfy the zero sum restriction.[75] Worse still, the theory of many player games and nonzero sum games is unsatisfactory for various reasons.

In the case of zero sum games of more than two players, the theory seems to say too much — it does not give precise enough a statement (although there does seem to be some empirical relevance to its conclusion). In the case of nonzero sum games of two or more players, we may distinguish two cases, according to whether the players can negotiate or not. If they can negotiate, we can rethink the game as a zero sum game, and the previous comment applies. If they cannot negotiate, the theory is unsatisfactory for both formal and empirical reasons. Formally, the theory leads to a paradox — the prisoner's dilemma paradox[4] — and empirically, the paradox (as well as other predictions of the theory) appears to be false. The theory of nonzero sum games is based on the idea of strategy, where as the theory of zero sum games (two player, many player, and negotiated nonzero sum games which are rethought as zero sum games) is based on the idea of conflict. The argument of this thesis is that the idea — heuristic — of pure conflict is far more powerful in terms of effectiveness than is the idea of strategy. The contribution of this thesis to game theory is the development of a solution concept based on the heuristic of pure conflict for nonzero sum games.

This paper will develop a new solution concept for nonzero sum games which will "resolve" the prisoner's dilemma paradox by making the result probabilistic. The basic idea is to convert the nonzero sum game into a zero sum two player game between each prisoner and Nature.[84] The new solution concept has empirical applications, and by restating the nonzero sum game into zero sum games, can draw upon the applications and theorems of zero sum two players games. Thus, the new solution concept may open-up nonzero sum games for use in successful descriptive models of the world. One such model — of the dynamics and statics of psychological experiments on prisoner's dilemma games — will be discussed in detail.

A very brief description of the new solution concept follows.

0.1 The prisoner's dilemma

The prisoner's dilemma paradox derives from the parable of a district attorney having in custody two prisoners whom he knows committed a certain crime, but he lacks proof. The D.A. (district attorney) separates the prisoners so that they cannot talk to each other and gives each prisoner a chance to turn state's evidence, i.e., confess on the other. If one confesses and the other does not, the one who confesses gets off (gets the maximum payoff) and the other gets the maximum sentence (minimum payoff). If both confess, both get an intermediate sentence for co-operating with the police (third best payoff). If neither confesses, each gets a light sentence on some trumped up charge (second best payoff). A numerical example satisfying the constraints is the following:

0.1.1

Prisoner I	Prisoner II: Confess	Prisoner II: Not confess
Confess	0,0	2,-2
Not confess	-2,2	1,1

The first number in each cell refers to the payoff to player 1 and the second number is that to player 2. Prisoner I chooses a row and Prisoner II chooses a column, and the outcome is the cell where the row and column intersect.

The outcome representing mutual confession is considered the solution since, if the players arrive at it, they would not want to change their strategies; either prisoner who did so would reduce his payoff — providing the other did not change his strategy. This combination of choices is called an equilibrium point.[5] There is a great deal of empirical evidence to the effect that the prisoner's dilemma outcome fails to occur when it might be expected to occur[43,45,50] and thus we think a new solution concept is called for. The next section will briefly describe the alternative solution concept called zero sum interpretation. The development of this solution concept, some of its results, and the elaboration of it from its sources in game theory and cybernetics will be the concern of this thesis.

0.2 The resolution by zero sum interpretation

If players I and II reinterpreted the prisoner's dilemma game directly into a zero sum game between themselves, nothing different from the equilibrium point solution would occur. This has been shown by Scodel et al,[44] and recently elaborated by Shubik.[83] We shall try a slightly different approach by introducing the idea of winning and losing coalitions. Suppose that we include the D.A. as a dummy player, and say that when a prisoner confesses he joins a winning coalition with the D.A. against the other prisoner. If both prisoners confess, we can say that each forms a separate coalition with the D.A., and that both of the separate coalitions occur. Examining game (1), we see that if only one player confesses, he wins 2 and the other loses 2. If both confess, each obtains 0 in game (1), and we see that 0 = 2-2, i.e. each obtains the <u>sum</u> of what he wins when he is in a winning coalition together with what he loses when he is outside of a winning coalition. Finally, we can treat the mutual non-confession outcome as the formation of a coalition against the D.A. We can say that the D.A. loses 2 in this case. Now we can interpret this zero sum <u>three</u> player game into zero sum 2-player games.

First, we notice that a zero sum 2-player game between the two prisoners again gives us nothing new. This is because the upper right hand cell must contain 2 since the defeated prisoner must certainly pay out 2 and the victorious prisoner wins 2. Also, the lower left hand corner must contain -2 for a similar reason. The upper left hand corner must be 0 since it is the sum of the other two zero sum 2-player interpretations. These facts make the upper left hand corner a saddlepoint independently of the value of the lower right hand corner. This is because it is simultaneously the maximum of its column and the minimum of its row, and no value for the lower right hand corner could be this, since it would have to be simultaneously less than -2 and greater than 2.

In any case, what value(s) could be in this cell? Both prisoners are in the same coalition (against the D.A.). We could say that the payments would be interpreted as zero sum between the two players if one payed the other his amount in the coalition. In other words, we could give the lower right hand box entries of <u>either</u> 1 or -1. As we have seem, of course, neither entry would alter the solution to <u>this</u> zero sum interpretation game. Thus, this approach to zero sum interpretation still gives us nothing new — until we consider zero sum 2- player games between each player and the D.A.

Considering a zero sum 2-player game between prisoner I and the D.A., and writing prisoner I as the row player and the D.A. as the column player, where the matrix cells represent the same outcomes as in matrix (1), we can see that the lower left hand cell must contain the entry 0 and the lower right hand cell must contain the entry 1. This is because the D.A. clearly obtains nothing (in the game of matrix (1)) when prisoner I is defeated, and he clearly pays 1 to prisoner I when prisoner I is victorious <u>and</u> the D.A. is defeated. Now the upper right hand cell must contain <u>either</u> 0 or 2, since these are the amounts that one player in the coalition of the D.A. and prisoner I pays to the other. Finally, the upper left hand cell must contain the sum of the zero sum interpretations of the lower left cell and the upper right cell, i.e., either 0 + 0 = 0 or 0 + 2 = 2. So, we have the following zero sum 2-player matrices to consider:

0.2

$$\text{(a)}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{(b)}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(c)}\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{(d)}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrices (a) and (c) have saddlepoints on the upper left hand corner and thus give us nothing new. Matrix (b) gives us a saddlepoint on the lower left hand corner. This means that if we define the probabilities on the zero sum interpretation game as identical to those in the original nonzero sum game (which we <u>must</u> do for the idea of zero sum interpretation to make sense) we get an inconsistent result. This is because we get a probability weight of 1 on the cell representing the defeat of prisoner I, but if this had been a game between the D.A. and prisoner II, the probability weight of 1 would be on the cell representing the defeat of prisoner II (with prisoner

II as the row player and the D.A. as the column player as in (b). Moreover, if both prisoners, in their respective zero sum interpretation games choose not to confess, the probability weight of 1 should be on the lower right corner. But, if we were to allow the D.A. to do what prisoner II might do, then the D.A's strategy is not optimal, and thus the whole point of zero sum interpretation is lost. Game (b) clearly represents an impossible inconsistency, and must be ignored. These considerations leave only game (d). Notice that its solution is a mixed strategy, which can be computed by assigning probability weights of x to row 1, 1-x to row 2, y to column 1 and 1-y to column 2. This gives the value of the games to prisoner I

$$= \quad 2xy + (1(1-x)(1-y) = 3xy - y - x + 1.$$

Factoring out y, we obtain $y(3x - 1) - x + 1$.

By setting $x = 1/3$, the coefficient of y becomes 0 and the value of the game to prisoner I is equal to 2/3. Notice also that a similar calculation would yield $y = 1/3$, and thus the D.A. uses a strategy which would correspond perfectly with an optimal strategy for prisoner 2 in his corresponding game (d).

By using the trick of coalitions and zero sum interpretation within and between coalitions, we have obtained a new result! It would seem natural to define the solution to the original prisoner's dilemma game as the probability weightings obtained in game (d), since if both players (prisoners) use these strategies they are mutually better off: They give for each player, $1/3(1/3)0+1/3(2/3)2+2/3(1/3)(-2)+2/3(2/3)1 = 4/9$, which is greater than 0, the maximum both obtain from any of the other consistent zero sum interpretations.

The reason that players reinterpret the original game along the lines we have indicated is extra-theoretic, but this heuristic can be taken in the same spirit as that of "perfect bargaining" in von Neumann and Morgenstern's theory of negotiated nonzero sum games.[3] The idea of "perfect bargaining" is extra theoretic in their theory. If we employ our heuristic, we obtain a uniform theory of zero sum interpretation games, since, if the solution is applied to a game which is already a zero sum 2-player game, our result will be identical to that of the von Neumann and Morgenstern minimax concept. This is shown below.

0.3 Comments on this solution concept

Incidentally, the "strategy" of the D.A. which we have used is not the same as that used below, where the D.A. uses a strategy against player I which corresponds with that of the n-1 other real players in their corresponding games with the same index number, providing this strategy gives at least the zero sum 2- player game value v against player I's maximin strategy. The two ways of computing the D.A.'s strategy work out to the same thing for 2-player symmetrical prisoner's dilemma games. But for n-player nonzero sum games (where $n > 2$), only the strategy derived below will work.

An interesting point proved below is that all n-player symmetrical prisoner's dilemma games of the type of matrix (1), i.e., where mutual confession has a value of 0 for all players, have zero sum interpretation mixed strategies which are independent of the number of players.

This is interesting because it may be that players will not know how many other players there are in a game of this type. Also, the probability of the occurrence of the coalition of all real players against the D.A. turns out to approach the limit e^{-b} where e = 2.7 .... b is equal to the negative of the amount lost by the D.A. when no one confesses. What is striking about this result is that the exponential function is often intuitively associated with prisoner's dilemma type situations, such as the spread of a fashion,[17] but had not previously been deduced from a formal solution to the game. Finally, the solution concept employs a normalization, which, essentially, does no more than put games in the form suggested by matrix (1), i.e., where mutual confession is represented by summing the payoffs associated with the separate 2-player coalitions of a real player and the D.A. Normalization formulae are given for all 2-player prisoner's dilemma games below.

This solution concept has empirical applications to prisoner's dilemma gaming experiments, i.e., the fraction of co-operative choices for the first 30 (roughly) plays can often be described. The strategy occurs in a zero sum 2-player game between each player and the D.A., who can be thought of, in the context of the experiments, as the experimenter. Thus we conclude that in the "early play" (under 30), the subjects have been playing against the experimenter rather than interacting with each other. The topic will be discussed in detail below.

0.4 Historical note and organization of the paper

The mathematical theory of games of strategy was developed by John von Neumann in 1928,[1] although some earlier work had been done by Emil Borel[2] and published in 1927. Borel had conjectured some special cases of the minimax theorem, which, in full generality, von Neumann proved. The classic work, The Theory of Games and Economic Behavior[3] by John von Neumann and Oscar Morgenstern, appeared in 1943. The work through 1956 is surveyed by Luce and Raiffa, Games and Decisions,[4] which is, to our knowledge, the last published, comprehensive survey of the field. This thesis developes a particular point of view; the heuristic underlying the von Neumann and Morgenstern theory can be extended to cover the set of games they did not discuss (non-negotiated, nonzero sum games - which we shall define later). We shall therefore review their theory, and such other work as seems pertinent (e.g. Nash's[5] theory) in order to fully understand their underlying heuristic and present the further development of it. As we proceed, we shall link each section of the theory of games with its equivalence or analogue in cybernetics. Finally, in section 7, we shall discuss in detail the solution concept described in sections (0.2) and (0.3).

1. Zero sum 2-player games.

1.1 Definition of a game.

The introduction of a few technical terms will help clarify exactly what we mean by a "game". By "game" we mean "the totality of rules which describes it" (von Neumann and Morgenstern p. 49), but we may conveniently distinguish some important features of these rules. First, there is some finite number of players, which is designnated by an integer. The term n-player game, therefore, always refers to the number of players, (as opposed to, say, the number of options open to eacher player). In this section, we shall be concerned exclusively with 2-player games. Second, at each moment in the game, players are faced with moves. Now, a move is not the actual choice a player makes, but the entire complex of possible choice allowed for by the rules of the game at the specific point in the game. The moves are numbered 1, ..., r, and at each number, one and only one player has an opportunity to make a choice. The moves constitute all possible positions which could be open to the player at this point and which could, in principle, be enumerated before the game even began. There are two types of moves, personal and chance. A personal move designates a choice for a player; for a chance move (or referee's move) the rules must specify the probability of each alternative occuring. Third, at each instance of a personal move, a player makes a choice from among the possible alternatives open to him at that move. Now, the series of actual choices made by all the players in the course of the game from beginning to end (i.e. from the first move to the last) constitutes a play of the game. Since the actual choices made are limited, but not determined, by the rules, a game may have a number (possibly a very large number) of distinct plays. When a play is completed, the rules of the game specify conclusively what payments must be made to each player. These payments, therefore, are a function of the actual choices made by each and every player in the course of the specific play of the game. If each choice (taken at each move) of the game is designated successively as $d_1, \ldots, d_r$, then the payoff to player k is a function of the d_j, and is designated as

1.1.1 $f_k\ (d_1, \ldots, d_r),\ k = 1, \ldots, n.$

Finally, we need to say something about the amount of information each player has when he actually makes his choice from the alternatives at each move. This is a subject of great complexity, but for our purposes, we need only distinguish two possibilities. First, the player is fully informed of all previous choices of the other players. Chess is an example of a game of this type. Second, some or all of the choices that have previously occured were made in secret, i.e. the players are not fully informed of all that has preceded their own choices. Poker is an example. Games of the first type, games of perfect information, are often thought of as being of a particularly "rational" character, while the others are generally thought of as relying much more heavily on luck or skill. Although von Neumann and Morgenstern have shown that some features of say, poker, (for example bluffing) are matters of choosing wisely (what is normally termed "strategy") an extremely important theorem - also proved by von Neumann and Morgenstern - shows that games of perfect information do indeed possess a property which makes them particularly "rational" (a term we shall define in the next section).

1.2 Utilities, strategies and game trees.

Although the theory of games can be described without ever raising the issue of utility (by pretending that the payoff matrix represents money, and that the players are interested in maximizing their expected amount of money), the theory of games is, in principle, based on the notion of utility. The idea is that players are interested in maximizing the expected value of their utility, where the word "utility" can refer to anything, including money. The easiest way to describe the theory of utility is in anecdotal form.

Imagine that you are on the late train to Lands End and the Buffet Car offers a limited selection of beverages: tea, whisky, castor oil. After looking over the selection, you quickly rank your preferences in decending order: 1) whisky, 2) tea, 3) castor oil. So, you order a whisky, to which the attendent replies, "I'm sorry Sir, but on these late night special buffet services, its not as easy as that. I'll give you a sure thing of tea against a brown paper bag which has a 50-50 chance of containing either a labelled small bottle of whisky or a (labelled) small bottle of castor oil. Which do you want for 10p?" Assuming that you don't return to your seat at this point, you may reply, "Tea, certainly!" To this the attendent replies, "75-25 whiskey against castor oil." And you may say, "I'll still take the tea." "95-05 whisky against castor oil." "I'm not sure," you reply.

Writing this algebraically, we can see what happened at the final offer of the Buffet Car attendant, where we write

$$U_t = \text{utility of tea}$$

$$U_w = \text{utility of whisky}$$

$$U_{co} = \text{utility of castor oil.}$$

Now the ranking was, U_w, U_t, U_{co}, and if we set $U_w = 1$ and $U_{co} = 0$, we get the final offer:

$$U_t = .95(U_w) + .05(U_{co})$$

$$U_t = .95(1) + .05(0)$$

$$U_t = .95$$

If there had been a fourth item on the menu, say orange squash, with a U_{os} associated with the choice and ranked, say, just before castor oil, then we could use the same end points of the gamble $pU_w + (1-p)U_{co}$ and find the point where this equates with U_{os}. We know that it will be closer to castor oil than was tea. In this way, a numerical ordering of preferences can be achieved for any number of alternatives. The anecdote, incidently, is not as far fetched as it might seem, for the technique of determining someone's intensity of preferences according to the theory of utility is done in essentially this way (although it need not be done on the late night train). Notice that the ordering is specific to a particular context (in this case the train). Under other circumstances, a different ordering might result. Notice also that the zero and the unit value (maximum value) were chosen arbitrarily for this customers's values. In other words, there is no interpersonal comparison of utility implied by this procedure. Since the zero and unit point are arbitrary, they can be altered by multiplication by a positive scalar, and addition of a constant to all the values. This means that the utility is preserved up to a linear transformation of sA + B, where A is a matrix of utility values, s a scalar matrix, and B a constant matrix.

This definition of utility is known as an interval scale. The ranking alone is known as an ordinal scale. And, if we made a comparison between two (or more) persons -- which means fixing the zero and units at the same values for everyone, we would have a cardinal scale. Axiomatic systems to arrive at the utility system we have described are presented in Luce and Raiffa and von Neumann and Morgenstern. The two basic assumptions in either of these axiomatic systems are, first, the transitivity of the preferences, i.e., if A is preferred to B and B to C, then A is preferred to C, and, second, the independence of irrelevant alternatives, i.e., we were able to determine a utility for tea without having to worry about orange squash, and vice versa.

If fact, because of the obvious practical difficulties and certain conceptual ones, the theory of utility has had very little to do with the development of game theory. It does provide a logical basis for the concept of mixed strategies in zero sum two player games, but even here, the fact that the game is zero sum means that the assumption of cardinality has, essentially, been made. (We say "essentially" because the two player's utility functions could be related by a linear transformation). The theoretical difficulties centre on the original ordering of preferences, i.e., why should real persons be able to do this, and why should the preferences be transitive? In other words, if one is with Liz, perhaps he would rather be with Alice, and vice versa. The practical difficulties in the use of utility have, we hope, been suggested by our anecdote.

Having described the concept of utility, we are now in a position to define exactly what we mean by rationality. This is simply the independent maximization on one's utility index. If utility is linear with money, (i.e., if a graph of money against utility produces a straight line) then this definition of rationality amounts to maximizing, independently, the expected amount of money. By using the word "expected" we allow for the possibility of the money being discounted by its liklihood of being obtained. In any case, there would seen to be at least one actual social situation where utility <u>is</u> linear with money — the fiduciary relationship. That is, a trustee is under a legal obligation to prudently maximize the funds of the person whose money is being managed. (See Riker (6) for a fuller treatment of this idea). We shall assume, for the remainder of this thesis, that we are always speaking in terms of money rather than utility, and that each player is out to maximize his expected amount of money.

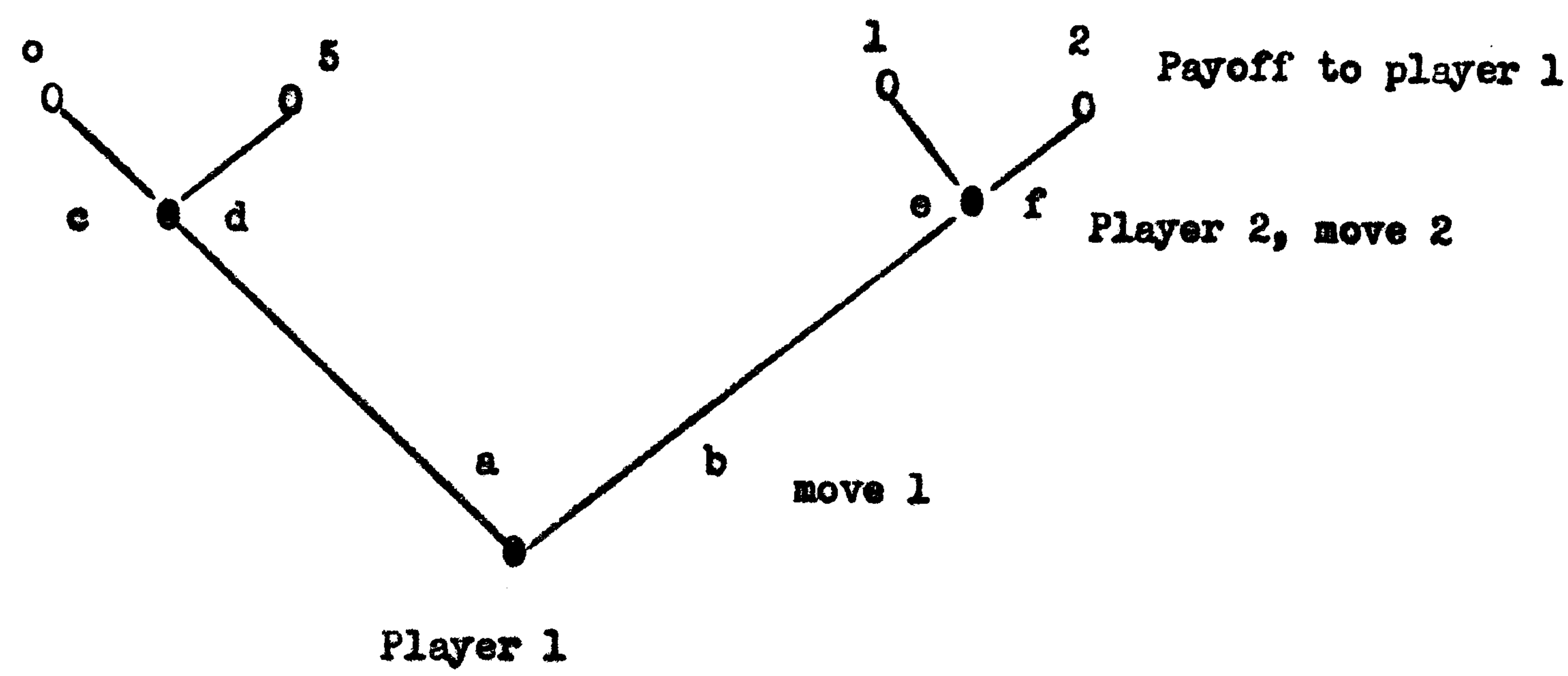
We can see that the function 1.1.1 always defines a number which is an exact amount of money, i.e.

1.2.1 $$f_k(d_1, \ldots, d_r) \gtreqless 0 \quad \text{for } k = 1, \ldots, n.$$

We have now given an exact meaning to the end point of a game, (i.e. the d_r, and the payments f_k), and we have described the motivation of the players, viz, each player k, will choose among the alternatives open to him at each move such that $f_k(d_1, \ldots, d_r)$ is maximized. Consider an extremely simple two player game, where each player moves once only and then the game is over. Each player k will choose to achieve the maximum of $f_k(d_1, d_2)$.

Suppose player 1, makes his choice first and then player 2 makes his choice. We could draw a diagram to show this situation. It is shown in figure 1.2.1.

Figure 1.2.1



A black dot indicates a set of alternatives, a clear circle, an end point to the game to which specify monetary payoffs are attached and expressed in terms of the payoffs to player 1. We shall assume that player 1 receives from (if the number is positive) or pays out (if the number is negative) these amounts to player 2. Games which have this property constitute an exceptionally important set of games and are known as zero sum games, since the sum of all payments in any play of the game is zero. In terms of economics, they constitute problems of distribution as opposed to problems of production. They are defined formally for two player games as follows:

1.2.2 $$f_1(d_1, \ldots, d_r) = -f_2(d_1, \ldots, d_r)$$

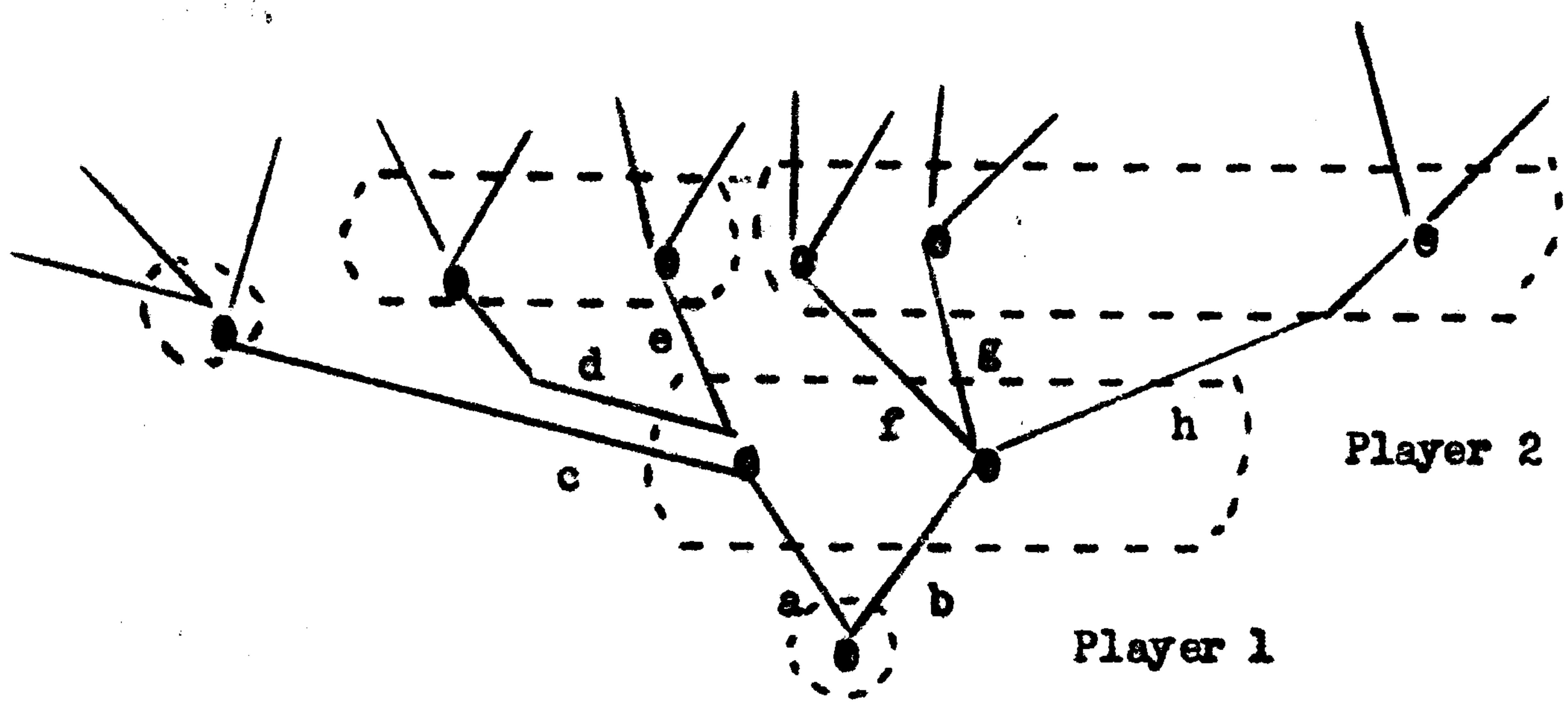
Now in figure 1.2.1, the diagram, which is called a <u>game tree</u>, for obvious reasons, shows that player 1 chooses first, that he has a choice among two alternatives, and that player 2 then makes his choice. Player 2 also only has two alternatives from which to choose, although the diagram shows four branches. This is because, of course, player 2 <u>will find himself</u> at one or the other of the two points designated by the two black dots at move 2.

On what basis will the players make their choices? Considering, first, player 1, he will look at the game tree - assuming of course that he has it available for consultation - and see that if he chooses alternative a, then depending on what player 2 does, he will either draw or win 5. If he chooses alternative b, however, he will obtain either 1 or 2 depending on the choice of player 2. Obviously he would want to obtain 5, but he cannot count on the stupidity of player 2, who would prefer to hold player 1 to zero. Therefore, player 1 must look at the possible choices of player 2. If player 2 found himself at the black dot on the left he would certainly choose alternative c and draw with player 1. If on the other hand, player 2 found himself at the black dot on the right, he would certainly choose alternative e and hold player 1 to 1, i.e. lose 1 to player 1. It is now clear how player 1 should choose. He must count on player 2's rationality and therefore pick alternative b. Player 2, realizing this, must choose alternative e and pay one unit to player 1, i.e.

1.2.3 $$f_1(b,e) = -f_2(b,e) = 1 = v$$

Now the thinking we have just described can be written more succinctly: For player <u>2</u>: Choose e against player 1's choice of b, and choose c against player 1's choice of a. For player 1: Choose alternative b for an expected value of 1, and choose alternative a for an expected value of zero. These prescriptions are strategies. The number expressed by 1.2.3 is the value of the game to player 1, and is designated as the lower case English v. Observe, that although this game is zero sum, in this case the number $v = 1 \neq 0$. The game we have just described had a deterministic solution, i.e. it was obvious exactly what two intelligent players would do and each had a definite course of action, i.e. a single clear path through the game tree which seemed "best". Such games are called strictly determined. We might also note that we have solved this game, i.e. found a set of good strategies and a value for each player, entirely on the basis of a game tree, with its branches describing the game in every detail. A description of a game using a game tree which describes every possible event is called the extensive form of the game. Throughout the discussion of the game pictured in figure 1.2.1, we have implicitly assumed that player 2 knew before making his choice exactly what player 1 had done. We noted earlier that this situation does not always obtain, e.g. in poker. Game trees can be constructed to show exactly the information each player possesses when he makes his choice. The usual procedure is merely to place dotted lines around each set of nodes of the game tree. These dotted enclosures tell us exactly what the player knows about his move. A dotted circle around a single node means that the player knows unambiguously that he is at that node. But if the dotted enclosure contains two or more nodes, then the player does not know at which of these nodes he happens to be. This situation is shown in figure 1.2.2.

Figure 1.2.2



At move 1, player 1 knows that the game is beginning. At move 2, player 2 does <u>not</u> know whether player 1 chose branch a or b. Observe that for this to be the case, the two nodes inside player 2's information set must have the same number of branches emerging from them. At move 3, player 1 knows if he is at the node at the end of choice c (because there are three branches from it), and he knows if he is choosing at the end of "d" or "e", or if he is choosing at the end of "f", "g" or "h". The game continues, but we need not be concerned with the remainder. Figure 1.2.2 was already considerably more complex than was figure 1.2.1, and this gives us a hint at how complicated game trees can become. In fact, for games like chess and checkers they are <u>indescribably</u>[7] complicated. For example, the number of possible alternative routes through the game tree in checkers is estimated[8] at 10^{40}. Clearly this is beyond our capability even to enumerate, let alone work through the strategic possibilities, which would take 10^{21} centuries if the alternatives at each move were considered three to a millimicrosecond. In fact, of course, the "curse of dimensionality" as Bellman has called it[9], takes effect long before we attempt to construct a game tree for a game as complicated as checkers. We must not get the impression however, that the extensive form of a game is merely an elaborate, if useless, curiosity. An extremely important result can be proved on the basis of the extensive form of games. In fact, we have already demonstrated a specific case of the result in our discussion of the game of figure 1.2.1. We recall that in that game, the players had perfect information as to each others' actual choices, i.e. player 2 could see exactly the choice which player 1 made. Since player 1 chose first, he could not of course see the choice made by player 2 before he himself chose, but he did know that player 2 would see his choice, and this fact was decisive in the thinking of player 1. The general result, of which the game of figure 1.2.1 was an instance, cannot be stated without one more definition. A <u>pure strategy</u> is any single set of instructions, covering an entire play of the game, for a player 1 which defines his exact choice at each of his personal moves in the play of the game. The general result is as follows:

1.2.4 Every zero sum two player game of perfect information and expressed in extensive form has a good pure strategy for each player and a value v.

Theorem 1.2.4 can be rephrased:

1.2.5 A sufficient condition for a zero sum two player game in extensive form to have a value v and a good (optimal) pure strategy for each player is that the game have perfect information.

Observe that perfect information is sufficient, but not necessary, for the game to have optimal strategies and a value v. For a necessary condition, see Dalkey.[10]

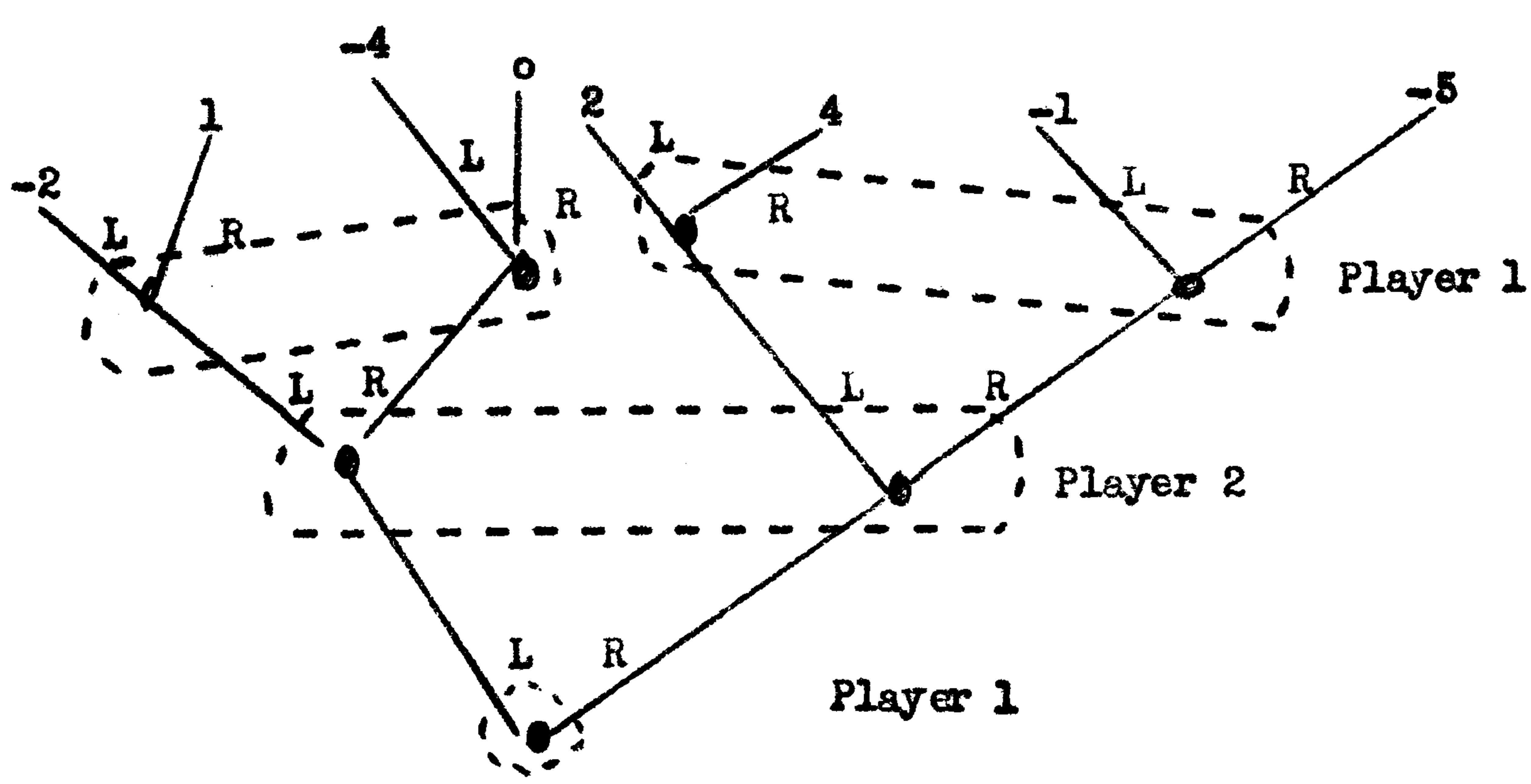
There are several proofs of this theorem (see, for example, von Neumann and Morgenstern, Section 15) and we shall sketch only the most intuitive of them. However, this proof is satisfactory for our purposes, and in fact there is only one objection to it — it requires the assumption of rationality on the part of both players. We have already seen that this assumption played a role in our discussion of the game of figure 1.2.1. Now the theory of zero sum two player games is specifically designed not to require this assumption. However, as we shall see, the way in which the theory avoids the assumption is hardly advantageous, i.e., the power of the theory is only apparent if the assumption is made. At any rate, the proof can be stated simply and verbally. At the next to the last move (i.e. move r-1) of the play of the game, the player making a choice knows that his opponent will minimize if he is player 2 (or maximize if he is player 1) the $f_k(d_1, \ldots, d_r)$. Therefore, the final move can be deleted because the value of the game is known at move r-1. Therefore, move r-1 becomes the final move of the game. However, the player choosing at r-2 similarly knows what will happen at r-1 and thus the value is known at r-2. So, move r-1 can be deleted and r-2 can be considered the terminal move. Obviously, this process can be carried back to the first personal move in the game, and a value and optimal pure strategy determined for each player, which is what we said we would prove. As we said earlier, the only objection to the proof is that it requires that both player be rational. The role of perfect information is also apparent - each player knows exactly where he is in the game tree and therefore knows what to expect next and how to optimize at the moment of choice.

Theorem 1.2.5 tells us that all games of perfect information are strictly determined; but it gives no practical guide for finding the true path through the game tree. Certainly one device which would seem to be necessary in this regard would be a way to list all possible pure strategies for each player, together with their expected values. In this way the various pure strategies could be compared. The technique of doing this will be discussed in the next section.

1.3 Normalization, matrix games, mixed strategies

Consider the following game tree, where, it should be noticed, both players are utterly lacking in information about the choices of the other player.

Figure 1.3.1



Player 1 chooses first from among the set (L,R) for left and right. Next, player 2, not knowing which choice player 1 made, chooses either L or R, after which player 1, not knowing the choice player 2 made, chooses L or R. The payoffs for the possible paths are given at the ends of the tree.

Player 2 clearly has only two alternatives from which to choose, either left or right. Player 1, however, has the possibility of choosing among four possible ways. He can choose L at move 1, and L at move 3; we shall write this as (L,L). Similarly his other three possible strategies are as follows: (L,R), (R,L), (R,R). The game tree gives us the value for the possible combinations of strategies, and these can be enumerated in the form of a rectangular table, or *matrix* of payoffs. We shall list player 2's strategies as columns and player 1's strategies as rows:

Figure 1.3.2

Player 2

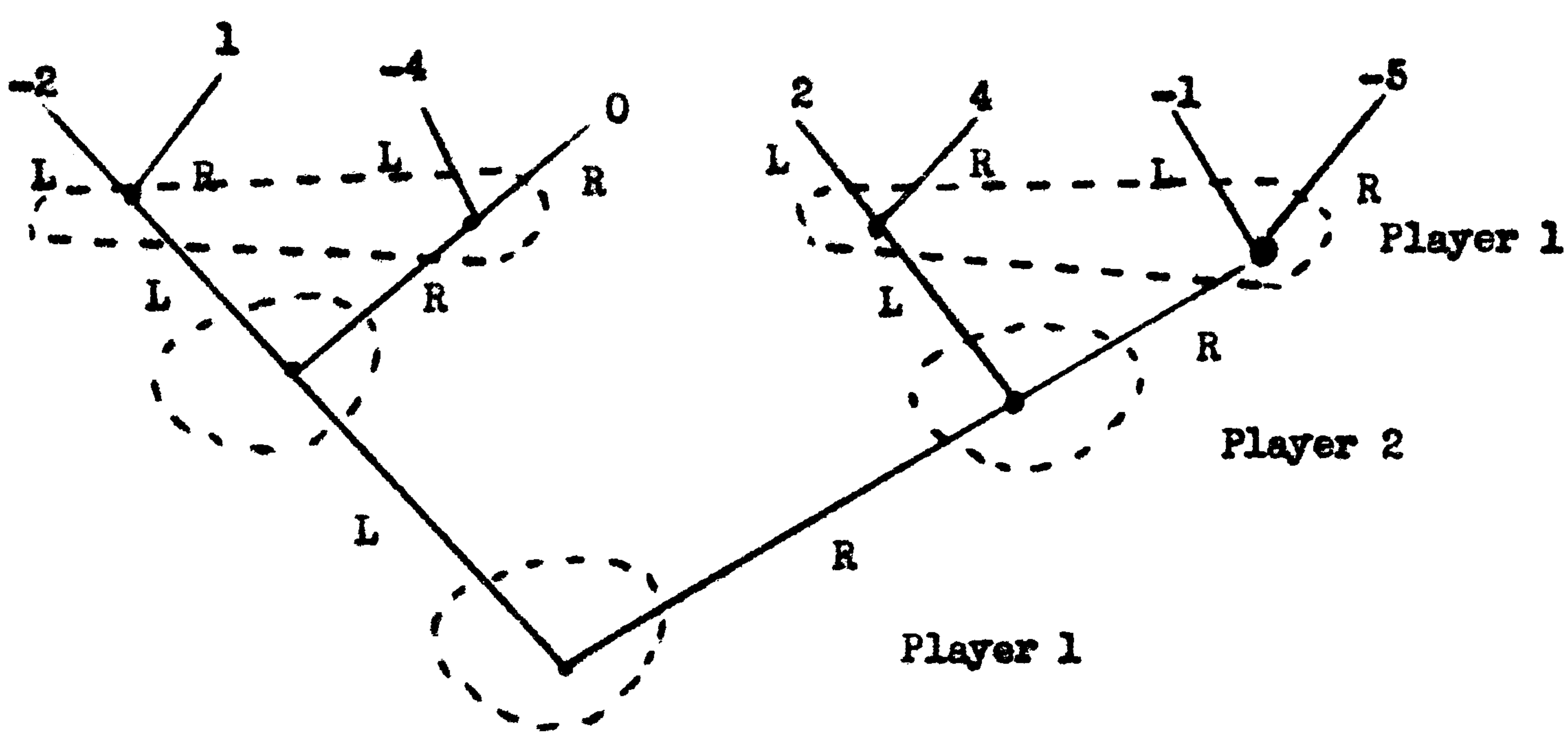
Player 1	L	R	Row minima
(L,L)	-2	-4	-4
(L,R)	1	0	0
(R,L)	2	-1	-1
(R,R)	4	-5	-5
Column maxima	4	0	

minimum of column maxima = maximum of row minima = 0 = v

Once the game is in normalized (matrix) form, the players are assumed to choose simultaneously and in secret. This game can be evaluated, now, very simply. Player 2 can see that if he chooses R, he need never do any worse than draw with player 1. That is, from the standpoint of player 2, each element in column 2 is less than or equal to the corresponding element in column 1. For situations of this kind, we say that column 1 dominates column 2. Player 2, of course, wants the smallest number possible to result in this game, therefore, domination tells him which column to avoid, i.e. he should choose column 2. Player 1 can see that player 2 will avoid column 1, therefore he must pick that row the second element of which is as large as possible. This turns out to be row 2 (i.e. the strategy (L.R)). The value of this game therefore is $v = 0$. When a matrix game has the property of an optimal pure strategy for each player, i.e. when the game is strictly determined, it is said to have a saddle point in pure strategies. (A matrix element is a saddle point if it is simultaneously the minimum of its row and the maximum of its column). Although a best way can be found for each player to play the game of figure 1.3.1 from an inspection of the game tree, the normalized form of the game (figure 1.3.2) provides a much easier way to analyse the game. Of course, making easier the decision making task was precisely what we hoped to achieve with the normalization. Incidentally, the game of figures 1.3.1 and 1.3.2 is strictly determined but lacks perfect information. We see that perfect information cannot, by itself, be necessary for strict determinateness. Consider the game tree of figure 1.3.1 again, and now imagine that player 2 can see what player 1 does in his first choice, but player 1 is still ignorant of what player 2 does at move 2.

The game is now represented as follows:

Figure 1.3.3



Player 2 now has some additional strategic choices which correspond with his additional information. That is, if player 1 has chosen L at move 1, then player 2 can choose either L or R at move 2 and if player 1 has chosen R at move 1, then player 2 again can choose L or R at move 2. Therefore, player 2's set or pure strategies is as follows: Strategy 1: (L against L, L against R); Strategy 2: (L against L, R against R); Strategy 3: (R against L, L against R), Strategy 4: (R against L, R against R). In short, each strategy specifies a choice against <u>any</u> choice of player 1.

Player 1, on the other hand, is still ignorant of the choice of player 2 at move 2. Therefore, at move 3, he still does not know precisely where he is in the game tree (except that he knows that he is at move 3, and he knows whether he is on the left or right half of the tree). His strategic choices, therefore, are the same as before: (L,L), (L,R), (R,L), (R,R). The normalized form of the game of figure 1.3.3 is now a four by four payoff matrix.

Figure 1.3.4

Player 2

Player 1	(L ag* L) (L ag R)	(L ag L) (R ag R)	(R ag L) (L ag R)	(R ag L) (R ag R)	row minima
(L,L)	-2	-2	-4	-4	-4
(L,R)	1	1	0	0	0
(R,L)	2	-1	2	-1	-1
(R,R)	4	-5	4	-5	-5
Column maxima	4	1	4	0	

* ag = against.

maximum of row minima = 0

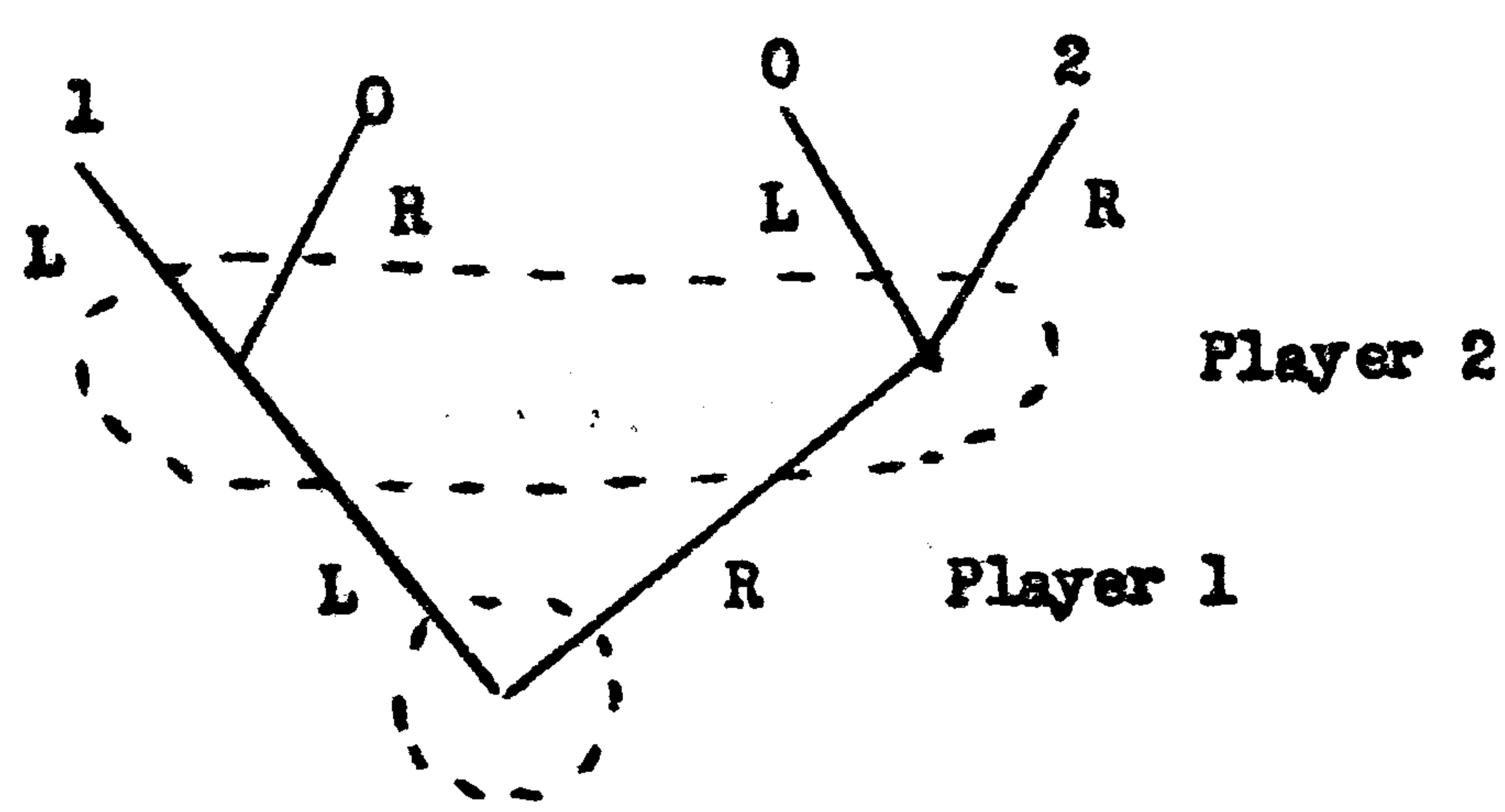
minimum of column maxima = 0

This matrix turns out to be strictly determined in exactly the same way as was that of figure 1.3.2, i.e. player 2 finds column 4 (which is identical to column 2 in figure 1.3.2) to be best, and player 1 is <u>again</u> forced to choose row 2 merely to draw even. Now this is interesting, because it shows that the addition of new information somewhat complicates player 2's decision making task (as well as player 1's) without at the same time improving his reward, i.e. the value v. This situation is not true in general, and as one should expect, "finding out" the other player's choice can increase the payoff to the player finding out. To show this however, we shall have to introduce a new concept - that of *mixed strategies*. These are used when no pure strategy seems a good reply to the other player's best pure strategy. A mixed strategy is a probability weighting (summing to unity) on a player's pure strategies. Some of the weights may be zero, and a pure strategy is clearly just a trivial case of a mixed strategy. The value v is then the expected value, and is defined as follows where a_{ij} is the i, jth element of a payoff matrix, x_i is the *ith* component of player 1's mixed strategy vector and y_j is the jth component of player 2's mixed strategy vector:

1.3.1 $$v = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_i y_j$$

Now consider the following game tree.

Figure 1.3.5



Clearly, both players make their choices uninformed of the choice of the other player. The payoff matrix is as follows:

Figure 1.3.6

Player 2

Player 1		L	R
	L	1	0
	R	0	2

If player 1 chooses row 2 in an attempt to obtain 2, he may assume that player 2's best reply to this would be to choose column 1 for a value of 0. On the other hand, if player 1 chooses row 1, in an attempt to obtain 1, he may rightly assume that player 2's best reply to this choice would be to choose column 2, for a value of 0. Now, since this game is normalized the choices are made in secret, so of course, player 2 cannot <u>know</u> which choice player 1 will make, but he cannot even intelligently guess the choice either, and the same is true for player 1. Obviously, there is no single best way for either of the players to choose. Moreover, neither <u>wants</u> the other to be able to guess what he will choose. This is the reason that we introduce mixed strategies. Player 1 chooses a probability weighting of x on row 1 and 1-x on row 2. Similarly, player 2 chooses a weighting of y on column 1 and 1-y on column 2. The value v is therefore given by

1.3.2 $v = xy(1) + 0x\ (1-y) + 0y\ (1-x) + 2\ (1-x)\ (1-y)$

$v = 3\ xy - 2x - 2y + 2$

1.3.3 $v = x(3y - 2) - 2y + 2$

If player 2 chooses $y = \frac{2}{3}$ then the coefficient of x is zero, and no matter what player 1 does, player 2 need not pay him more than

1.3.4 $v = -2y + 2 = -2\left(\frac{2}{3}\right) + 2 = \frac{2}{3}$

Similarly, player 1 can guarantee himself at least this amount by choosing $x = \frac{2}{3}$. The optimal mixed strategies therefore become:

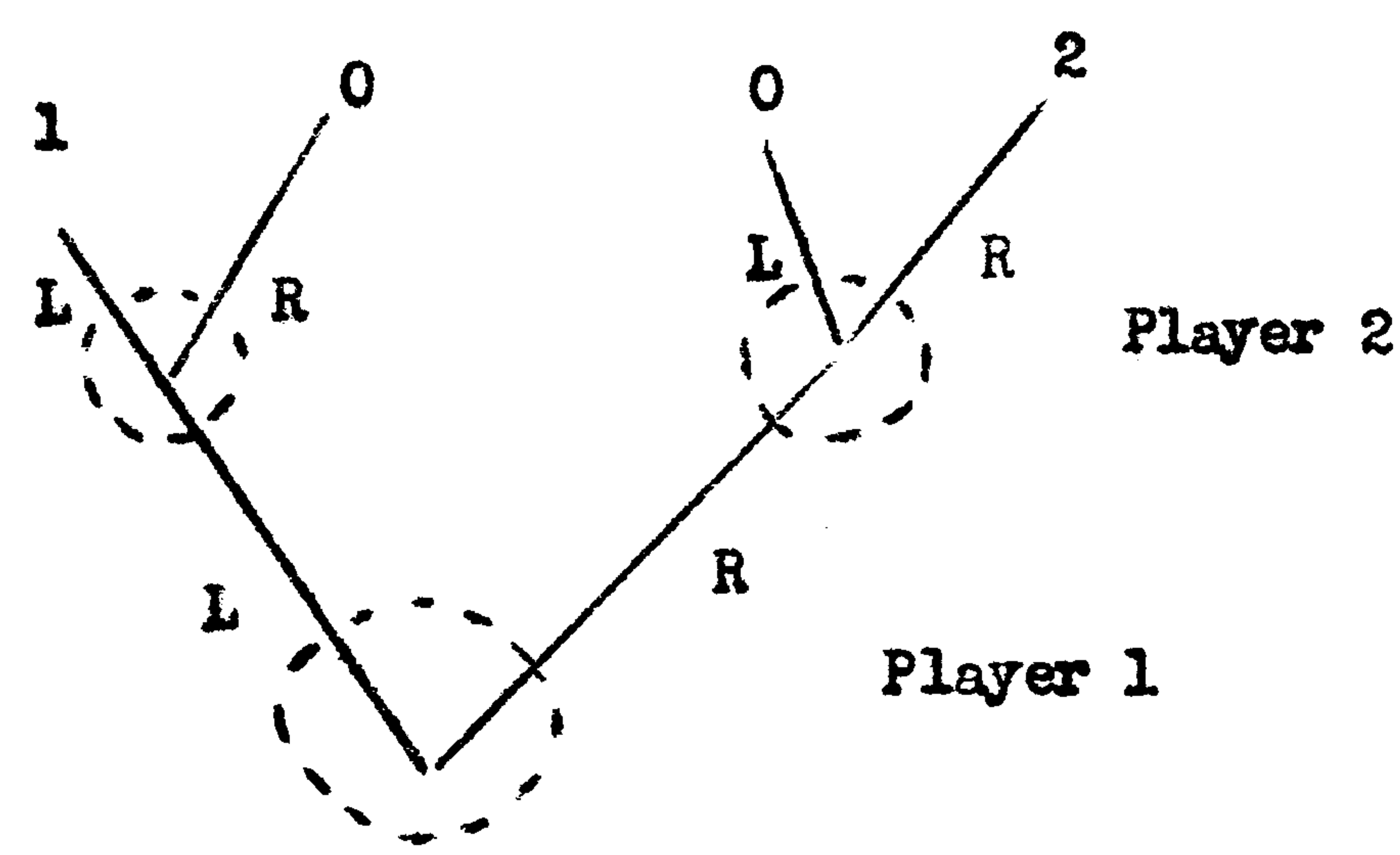
For player 1: $\left(\frac{2}{3}, \frac{1}{3}\right)$

For player 2: $\left(\frac{2}{3}, \frac{1}{3}\right)$

And the value $v = \frac{2}{3}$

Let us now consider the game tree of figure 1.3.5 such that player 2 chooses after player 1 <u>and</u> is somehow informed of the choice of player 1 (he cheats):

Figure 1.3.7



Player 1 still has only two pure strategies, L or R. Player 2, however, now has four pure strategies, as shown on the following payoff matrix:

Figure 1.3.8

Player 2

Player 1	L ag* L L ag R	L ag L L ag R	R ag L L ag R	R ag L R ag R	row minima
L	1	1	0	0	0
R	0	2	0	2	0
column maxima	1	2	0	2	

*ag = against

maximum of row minima = 0

minimum of column maxima = 0

v = 0

This game is strictly determined with v = 0, player 2 using a pure strategy of column 3 and player 1 having two equivalent strategies of either row 1 or row 2 or <u>any</u> mixture of the two. (The value v will be the same). Therefore, by knowing the choice of player 1, player 2 can hold player 1 to a draw - instead of losing $\frac{2}{3}$ to him. Intuitively, we suspected that increasing a players' information in a game (relative to the other players information) would improve his situation. This has now been formally demonstrated. Increasing a player's information actually increases the size of the payoff matrix because it makes more numerous the available strategies. An incidental remark of McKinsey's[11] is well worth quoting in this regard:

> "... and it is true almost universally, besides, that the less knowledge we have, the easier we find it to make up our minds (a deaf man has less trouble deciding on a wife than has a man with normal hearing)".

A final point with regard to normalization of games should now be briefly mentioned. We are referring to chance (referee's) moves in the game. Since these are assumed to be made by the referee, we may just as well assume that they are all made in advance of the actual play and then disclosed to the players at the actual moment they occur in the course of the play. In this way, they can be put in the game tree (i.e. the probabilities of the various branches occurring can be put in the game tree) in advance of the play, and all possible pure strategies can still be mapped out before the actual play. The way this works is discussed in some detail in McKinsey, Chapter 5. An interesting alternative approach to including chance (referee's) moves in a game <u>without</u> writing out a game tree is contained in a version of poker discussed by Kemeny, Snell and Thompson[12] (pp. 378 - 381).

In this game, the cards are dealt by a referee at the beginning of the game, one card to each player. The cards are either high (H) or low (L) with no further gradations assumed. (For example, red cards could be low and black cards high). The equiprobable deals are, of course, the following four sets, where the first letter is player 1's card and the second is player 2's card: (H,H) (H,L) (L,H) (L,L). Each player can see his own but not his opponent's card. Player 1 is allowed to choose first and he can either "see" or "raise". If he chooses to see, the higher hand wins or equal hands split the pot - player 2 has no choices in this case. If player 1 chooses to "raise", he adds an amount b to the pot (which already contains an ante, a, from each player). Only if player 1 chooses "raise", does player 2 exercise choices; he can either "fold", in which case player 1 wins the pot, without revealing his hand, or player 2 can "call", by adding the same amount b to the pot. Then the cards are compared and the higher hand wins or, for equal hands, the pot is divided evenly between the players. Player 1's strategies are the four possible combinations of seeing and raising against his own high or low card (e.g. raise against a high card, see against a low card - which we can write raise-see). Player 2's strategies are the four combinations of fold and call (e.g. fold against receipt of a high card, call against receipt of a low card). Since the four possible deals are assumed to be equiprobable, we can easily compute the expected value of each combination of strategies, for example, (see-raise) versus (fold-fold).

For the deal (H,H) Player 1 sees and the pot is split.
" " " (H,L) Player 1 sees and wins the pot, i.e., a.
" " " (L,H) Player 1 raises and player 2 folds, player 1 wins a.
" " " (L,L) Player 1 raises and player 2 folds, and so player 1 wins a.

The expected value of these two strategy combinations (to player 1) can thus be computed: $\frac{1}{4}x0 + \frac{1}{4}xa + \frac{1}{4}xa + \frac{1}{4}xa = \frac{3a}{4}$

The number $\frac{3a}{4}$ would be entered in the appropriate place on the 4x4 payoff matrix. In this way, the probabilities involved in the random deal can be taken into account in the normalized form of the game even without the use of a game tree. The reader is referred to Kemeny, Snell and Thompson[12] for the remainder of the analysis of this poker game.

1.4 Mixed strategies and the minimax theorem.

We introduced the concept of mixed strategies in an attempt to elucidate a key issue of game theory - the role played by information. We shall now discuss mixed strategies for their own sake. One point, however, has already been mentioned in the discussion of figure 1.3.5 and 1.3.6 - secrecy. The game described by figure 1.3.5 did not allow player 2 to know the choice of player.1. If he had been able to know, he could take advantage of this fact, as was seen in figure 1.3.8. Therefore, if the rules of the game allow secrecy the players are well advised to exploit it. In this case, player 1, by keeping his choice secret, i.e. by randomizing it with appropriate probability weightings, could win something from player 2. Moreover player 2 would not want to announce, in advance, his <u>choice</u> to player 1, since player 1 could then exploit this knowledge to his own advantage. Now, for strict determinateness, announcing an optimal strategy in advance will make no difference - since the other player has already decided that the announced strategy was going to be used anyway. This is why, when we gave player 2 more information in the game tree of figure 1.3.3 than he had had in that of figure 1.3.1, we still obtained the same saddle point and optimal strategies. Secrecy plays no essential role in strictly determined games. But it is the key feature of mixed strategies. The specific choice in this case is so secret that even the player making it does not know in advance what it will be. He can, if he chooses, announce the probability weightings, since he can lose nothing if he does this.

We shall examine the algebra of mixed strategies in more detail largely using the notation of Vajda.[13] We have defined the mixed strategy expected value as

1.4.1 $$v = \sum_i \sum_j x_i y_j a_{ij} \text{ where } x = (x_1, \ldots, x_m), \sum x_i = 1, \text{ and } y = (y_1, \ldots, y_n) \quad \sum y_j = 1.$$

Now suppose that player 2 chooses a pure strategy which minimize the payoff of player 1's mixed strategy. That is, suppose player 2 chooses

1.4.2 $$\min_j \sum_i a_{ij} x_i .$$

Player 1 can assume that player 2 will certainly want to achieve this minimum. Therefore player 1 may as well assume that it will occure and choose his mixed strategy x to maximize this value:

1.4.3 $$\max_x \min_j \sum_i a_{ij} x_i = v_1 .$$

Similarly, we can obtain the analogous expression for player 2:

1.4.4 $$\min_y \max_i \sum a_{ij} y_j = v_2 .$$

Now the values v_1 and v_2 involve a pure strategy against the best mixture of the opponent. We can also establish them against <u>any</u> strategy: $v_1 = \min_j \sum_i \bar{x}_i a_{ij}$ and $v_2 = \max_i \sum_j \bar{y}_j a_{ij}$, where $\bar{x}$ and $\bar{y}$ are any strategies at all. So, we always have

1.4.5 $$v_1 \leq \sum_i \sum_j \bar{x}_i \bar{y}_j a_{ij} \quad \text{and}$$

1.4.6 $$v_2 \geq \sum_i \sum_j \bar{x}_i \bar{y}_j a_{ij}$$

Since the right hand sides of 1.4.5 and 1.4.6 are equal, we always have

1.4.7 $$v_1 \leq v_2.$$

If we can prove that $v_1 = v_2$, we will prove two things. First, we will prove that mixed strategies can <u>always</u> give a value v (for non strictly determined games) which is identical in spirit to the value v for strictly determined games, i.e. a value v below which player 1 cannot be reduced no matter what player 2 does and more than which player 2 need not pay player 1 no matter what player 1 does. Second, we will prove the minimax theorem, the most important theorem in game theory, which establishes precisely what we have said in the preceding remark. Von Neumann and Morgenstern proved the theorem by showing that, with appropriate mixed strategies it is never possible for $v_1 < v_2$, and we shall now present this proof.

The proof of the minimax theorem in von Neumann and Morgenstern is not based on the original proof by von Neumann[1], rather it is a simplification of a proof by J. Ville[98]. Further simplification of the treatment in von Neumann and Morgenstern can be found in the work of Vajda[13] and of Owen.[22] The presentation here is based on von Neumann and Morgenstern, Vajda and Owen. To prove the theorem, we shall first have to prove two lemmas.

The Theorem of the Supporting Hyperplanes

1.4.8 Let p vectors $\vec{X}^1, \ldots, \vec{X}^p$ be given. Then a vector $\vec{Y}$ either belongs to the convex C spanned by $\vec{X}^1, \ldots, \vec{X}^p$ or there exists a hyperplane which contains $\vec{Y}$ such that all of C is contained in one half space produced by that hyperplane.

Proof:

The case where $\vec{Y}$ belongs to C is trivial, so we shall assume that $\vec{Y}$ does not belong to C. Consider a point $\vec{Z}$ of C which lies as near as possible to $\vec{Y}$, i.e., where $|\vec{Z} - \vec{Y}| = \sum_{i=1}^{n} (z_i - y_i)^2$ is a minumum. Now consider any other point $\vec{U}$ to C. For every t $(0 \leqq t \leqq 1)$, $t\vec{U} + (1 - t)\vec{Z}$ belongs to the convex C. Since $\vec{Z}$ has the minimum property mentioned above,

$$|t\vec{U} + (1 - t)\vec{Z} - \vec{Y}|^2 \geqq |\vec{Z} - \vec{Y}|^2$$

$$|(\vec{Z} - \vec{Y}) + t(\vec{U} - \vec{Z})|^2 \geqq |\vec{Z} - \vec{Y}|^2$$

$$\text{i.e., } \sum_{i=1}^{n} ((z_i - y_i) + t(u_i - z_i))^2 \geqq \sum_{i=1}^{n} (z_i - y_i)^2$$

This gives, by elementary algebra

$$2\sum_{i=1}^{n} (z_i - y_i)(u_i - z_i)t + \sum_{i=1}^{n} (u_i - z_i)^2 t^2 \geqq 0 .$$ If we first

divide by t, we next see that as t converges to 0, this expression converges to

$$2\sum_{i=1}^{n} (z_i - y_i)(u_i - z_i) \geqq 0,$$ and dividing by 2

$$\sum_{i=1}^{n} (z_i - y_i)(u_i - z_i) \geqq 0.$$ Having shown that the

portion on the left is greater than zero, we see that

$$u_i - y_i = (u_i - z_i) + (z_i - y_i),$$ which means

$$\sum_{i=1}^{n} (z_i - y_i)(u_i - y_i) = \sum_{i=1}^{n} (z_i - y_i)^2 = |\vec{Z} - \vec{Y}|.$$

Now, $|\vec{Z} - \vec{Y}| > 0$ since $\vec{Z} \neq \vec{Y}$ (since $\vec{Z}$ is in C but $\vec{Y}$ is not).

Thus $$\sum_{i=1}^{n} (z_i - y_i)(u_i - y_i) > 0,$$

i.e., $$\sum_{i=1}^{n} (u_i(z_i - y_i) - y_i(z_i - y_i)) > 0.$$

So, $$\sum_{i=1}^{n} (z_i - y_i) u_i > \sum_{i=1}^{n} (z_i - y_i)y_i.$$

Set $a_i = z_i - y_i$, and $a_1 = \dots = a_n = 0$ is not possible because $\vec{Z} \neq \vec{Y}$. Put $\sum_{i=1}^{n} a_i y_i = b$, which is a hyperplane to which $\vec{Y}$ belongs. Set $x_i = u_i$ and we have

1.4.9 $\sum_{i=1}^{n} a_i x_i$ b, which is the half space produced by the hyperplane, and $\vec{U}$ belongs to this half space. Since $\vec{U}$ was an arbitrary element of C, the proof is complete.

The second lemma is the following:

<u>The theorem of the alternative for matrices</u>

1.4.10 Consider an mxn matrix $A = (a_{ij})$. Then the following alternatives are mutually exclusive:

1) The point $\vec{0}$ (the zero vector) is contained in the convex hull C of the m + n points:

$$a_1 = (a_{11}, \dots, a_{m1})$$

...

$$a_n = (a_{1n}, \dots, a_{mn})$$

and

$$e_1 = (1, 0, \dots, 0)$$

$$e_2 = (0, 1, \dots, 0)$$

$$e_m = (0, \dots, 0, 1)$$

and thus there exist numbers y_j such that $y_j \geq 0$,

$$\sum y_j = 1, \text{ and } \sum_{j=1}^{m} a_{ij}y_j \leq 0, \text{ for } i = 1, \ldots, n.$$

2) The point 0 is not contained in the convex hull of the m + n points, and there exist numbers $x_1, \ldots x_m$ such that

$$x_i > 0$$

$$\sum_{i=1}^{m} x_i = 1$$

$$\sum_{i=1}^{m} a_{ij}x_i > 0 \quad \text{for } j = 1, \ldots, n.$$

Proof:

Cases 1 and 2 are mutually exclusive:

Multiply $\sum_{i=1}^{m} a_{ij}x_i > 0$ by y_j and sum over $j = 1, \ldots, n$.

This gives $\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}x_iy_j > 0$. Now multiply

$\sum_{j=1}^{n} a_{ij}y_j \leq 0$ by x_i and sum over $i = 1, \ldots, m$.

This gives $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_iy_j \leq 0$. This is a contradiction.

Proof of case 1:

Since 0 is contained in C, it is a convex combination of the m + n points. Thus there exist numbers $t_j \geq 0$, with j = 1, ..., m + n, and adding to unity, such that

$$\sum_{j=1}^{n} t_j a_{ij} + t_{n+i} = 0, \text{ for } i = 1, \ldots, m .$$

(Thus a column vector of 0 is generated with the appropriate element from e contributing its weight). So, we see that

$$\sum_{j=1}^{n} t_j a_{ij} = -t_{n+i} \leq 0. \quad \text{If } t_1 = \ldots = t_n = 0, \ t_n + 1 = 0,$$

which is not possible since, as we have just remarked, it comes from the unit vectors, which are linearly independent. Thus we see that at least one of the t_j is positive. So, $\sum^{n} t_j > 0$, and we can put

1.4.10 $$y_j = \frac{t_j}{\sum_{j=1}^{n} t_j},$$ and these values will satisfy

the conditions of case 1.

Proof of case 2:

Since 0 does not belong to C, we can use the theorem of the supporting hyperplane to say that there exist numbers $s_1, \ldots, s_m$ such that

$$\sum_{i=1}^{m} s_i a_{ij} > 0 \qquad \text{for all } j.$$

Now this is true for every one of the m + n vectors, so $s_i > 0$ for all i. So, we can set

$$x_i = \frac{s_i}{\sum_{i=1}^{m} s_i}, \quad \text{and these are the numbers}$$

specified in case 2.

Proof of the minimax theorem:

We can use the previous result to prove this theorem. Recall that we know already that $v_1 \leqq v_2$.

Case 1 gives us y such that, for all i

1.4.12 $\sum_j a_{ij} y_j \leqq 0.$ There fore, also for

$$\max_i \sum_j a_{ij} y_j \leqq 0$$

So,

1.4.13 $$v_2 = \min_y \max_i \sum_j a_{ij} y_j \leqq 0.$$

Case 2 gives us x such that for all j,

1.4.14 $$\sum_i a_{ij} x_i > 0,$$ and thus, also for

$$\min_j \sum_i a_{ij} x_i > 0.$$

So,

1.4.15 $$v_1 = \max_x \min_j \sum_i a_{ij} x_i > 0.$$

So,

1.4.16 $v_1 < 0 < v_2$ is not possible.

Now take any k and alter the matrix a_{ij} to become $a_{ij} - k$. In the same way as before, it is not possible to have

1.4.17 $v_1 < k < v_2$.

Thus, we have only the possibility that $v_1 = v_2$, which completes the proof of the theorem.

It is clear that the only thing that the minimax theorem does is prove the existence of a value v for mixed strategy zero sum two player games. Now this value v is optimal in the sense of being at least v against the best that the other player can do. What about the other player's worst? In general, optimal strategies do not exploit the other player's mistakes. Optimal strategies are conservative strategies. This is the reason we said much earlier in this thesis that the power of the concept of optimal strategies is apparent <u>only</u> if both players are assumed to be rational (i.e. good) players. This topic is discussed at some length in Luce and Raiffa, as well as in von Neumann and Morgenstern (see, in particular, section 17.11). Another problem is that the minimax theorem gives us no hint as to a way to find the optimal strategies[87]. This can be done quite simply for 2x2 games using the device of 1.3.2 to 1.3.4. In fact, for 2x2 mixed strategy games, general solutions are readily available and can be found, for example, in von Neumann and Morgenstern (section 18.2.5); Kemeny Snell and Thompson; and Rapaport[14]. The reader can derive them easily enough himself if he remembers the fact that the optimal <u>mixed</u> strategy is good (i.e. gives at least the value v) against any pure strategy of the opponent. Thus, if the following is a mixed strategy matrix (for which a necessary and sufficient condition is that the diagonals be separated, i.e. $a > b$ and $d > c$, or $c > a$ and $b > d$,) we can easily compute the optimal strategies:

Figure 1.4.1

Player 2

Player 1		
	a	b
	c	d

If player 1 chooses (x, 1-x) and player 2 chooses column 1, we have,

1.4.18 $$x\,(a) + c\,(1-x) \geq v$$

Similarly, if player 2 chooses column 2, we have

1.4.19 $$x\,(b) + d\,(1-x) \geq v$$

It is sufficient that these expressions are in fact equal to v_1 so we can set them equal to each other and obtain

1.4.20 $$x\,(a) + c\,(1-x) = x\,(b) + d\,(1-x)$$

From this, we can solve for x to obtain

1.4.21 $$x = \frac{d-c}{a-c-b+d}$$

The formula for y can be similarly obtained, and, of course, 1-x and 1-y can be even more easily obtained! These formulae - which we shall not compute here - constitute a general solution for 2x2 mixed strategy games. (See (1.5.14) for general solutions to all matrix games).

A few properties of optimal strategies should at least be mentioned. We have already noted the concept of one column (or row) majorizing another. This is also known as domination. There are a couple of varieties or domination which should be pointed out. In addition to the one we have already mentioned (every element in one row strictly greater than every element in another row), there is also the case of at least one element greater than its corresponding element and the other elements equal. Finally, there is the domination of a convex combination of two rows which has a greater value than a pure strategy of another row. Another property of optimal strategies is that any row or column which appears in an optimal strategy is, by itself as a pure strategy, an equalizer against the other player's optimal mixed strategy. This means that when used as a pure strategy against the other player's optimal mixed strategy, the row (or column) will produce the value v. The various properties of optimal strategies are discussed at length in Karlin.[15]

One interesting property of mixed strategies might be mentioned, although the reader may have spotted it already. Reconsider the game of figure 1.3.6

Figure 1.3.6

1	0
0	2

Player 1's optimal strategy was, we may recall ($\frac{2}{3}$, $\frac{1}{3}$). This means he chooses row 1 with probability $\frac{2}{3}$ and row 2 with probability $\frac{1}{3}$. Yet, his highest payoff, 2, is in row 2! The intuitive reason for this reverse weighting is to avoid being taken advantage of over something which seem obviously desirable. This is merely avoiding the "fools rush in" mentality. It is described as "inhibition", by Young.[16]

We should note that the solution to a zero sum two player game matrix holds up to a linear transformation. This means that if A is a matrix, s is a positive scaler matrix and B a constant matrix, the good strategies are unaffected by a transformation of the form $sA + B$.

The theory of zero sum two player games is a normative theory, i.e. it tells one that there always exists a method of play which will guarantee a value v no matter how good a player the opponent may be. If the opponent is a bad player the theory says that the amount v can still be obtained, but the theory does not indicate that the opponent's weaknesses can be taken advantage of. Optimal strategies, therefore are normatively good strategies (this word was used by von Neumann and Morgenstern rather than "optimal"). Whether or not the theory is also descriptive of real human beings - good players, average ones or even bad ones - is a subject about which the theory says nothing.

1.5 Information theory, cybernetics and zero sum two player games.

Let us look again at two of our examples. The first example is that of figure 1.3.2.

Figure 1.3.2

Player 2

Player 1	L	R	Row minima	
(L,L)	-2	-4	-4	
(L,R)	1	0	0	Maximum minimum
(R,L)	2	-1	-1	
(R,R)	4	-5	-5	
column maxima	4	0		

minimum maximum = maximum minimum
v = 0

In this game, we may recall, the players were utterly uninformed about each other's choices. When we allowed player 2 to see the choice player 1 had made at move 1, we obtained the following payoff matrix:

Figure 1.3.4

Player 2

Player 1	L ag* L L ag R	L ag. L R ag. R	R ag. L L ag. R	R ag. L R ag. R	row minima
	-2	-2	-4	-4	-4
	1	1	0	0	0
	2	-1	2	-1	-1
	4	-5	4	-5	-5
Column maxima	4	1	4	0	

*ag = against

Maximum of row minima = 0

Minimum of column maxima = 0

Thus, we saw that increasing player 2's information changed <u>nothing</u> in this strictly determined game. Now, consider the other example, that of figure 1.3.6.

Figure 1.3.6

Player 2

Player 1	L	R
L	1	0
R	0	2

In the extensive form of this game, we may recall, each player made his choice uninformed of the choice of the other player. They employed mixed strategies, which turned out to be the same for each player, namely $x = (\frac{2}{3}, \frac{1}{3})$, $y = (\frac{2}{3}, \frac{1}{3})$, and $v = \frac{2}{3}$. When we gave player 2 information about the choice of player 1, however, the situation altered radically.

Player 1.3.8

Player 2

	L ag* L L ag R	L ag L R ag R	R ag L L ag R	R ag L R ag R	row minima
L	1	1	0	0	0
R	0	2	0	2	0
column maxima	1	2	0	2	

*ag = Against.

Maximum of row minima = 0

Minimum of column maxima = 0

$v = 0$

The game, in short, became strictly determined and player 2, who obtained the new information, was able to increase his payoff from $-\frac{2}{3}$ to 0.

It is clear how the two cases differ. In the first case, although player 2 obtained new information, he was just as certain as he had been before obtaining the information exactly what player 1 would do. But in the second game, the one which had a mixed strategy solution (i.e. the game of figure 1.3.6), the addition of new information removed player 2's uncertainty. The removal of uncertainty, therefore, is the key ingredient which distinguishes the two games. It is reflected in the fact that the value v increased for the player whose uncertainty was removed. (This fact has been used by Farquharson[88] to show the importance of the secret ballot in voting).

Now information theory treats information as "that which removes uncertainty."[17] The average amount of information obtained after the receipt of a message has been given a precise definition by Shannon[18] and we shall look at this definition in a moment. At this point, however, two things are worth noting. First, something may be called "information" in game theory even if it removes no uncertainty. Second, to put the matter most plainly, information theory talks about the amount of uncertainty in a system while game theory tells us if we can make any money out of the uncertainty. Notice that increasing a player's information also increases his number of strategies. However, the increase in his strategies only sometimes leads to an increase in payoff — when the information which produced these new strategies also removed uncertainty. If no uncertainty is removed, the new information merely complicates the problem for the person obtaining it. If the problem got so complicated he could no longer understand it, the new information could reduce his payoff — by making him play less efficiently. In deterministic games, ignorance may be bliss!

Ashby[17] has described what he has called the Law of Requisite Variety, which we shall now relate to the theory of games. Consider a table of outcomes:

Figure 1.5.1

		R		
		α	β	γ
	1	f	f	k
	2	k	e	f
	3	m	k	a
	4	b	b	b
D	5	c	q	c
	6	h	h	m
	7	j	d	d
	8	a	p	j
	9	l	n	h

(from Ashby, Table 11/5/1)

R specifies a choice of strategy (α, β, γ) for each choice of D. Since no letter is repeated in each column, the variety (number of possible outcomes) cannot be less than

1.5.1 $\dfrac{\text{D's variety}}{\text{R's variety}}$ = m/n where m is the number of

rows and n the number of columns. Thus the variety in the outcome V_o cannot be less than $\frac{V_d}{V_r}$ (variety of the disturbance divided by the variety of the regulator). Measured logarithmically, this gives us

1.5.2 $V_o = V_d - V_r$.

This formula shows that a decrease in the variety of the outcome can only be achieved by an increase in the variety of the regulator. As Ashby says, "Only variety can destroy variety."

Ashby then derives essentially the same result from the work of Shannon, using some results from his Mathematical Theory of Communication. Shannon's measure of the average amount of information communicated by a received message is given by

1.5.3 $$H = - \sum p_i \log p_i.$$

Using this formula, Shannon establishes a few identies for information such as

1.5.4 $$H(D) + H_D(R) = H(R, D)$$
$$H(R) + H_R(D) = H(R, D)$$

where $H_D(R)$ is the entropy of R when the disturbance is known, and $H(R, D)$ is the entropy of two information sources, R and D. Shannon also shows (p. 51) that

1.5.5 $$H(R, D) \leq H(R) + H(D)$$

This is intuitively obvious since the uncertainty of a joint event should be less than the sum of the uncertainties of the individual events, unless the two events are independent (i.e., where $p(i,j) = p(i)p(j)$), in which case the entropies should be equal.

Now, Ashby's condition that no letter be repeated in the same column means that

1.5.6 $$H_R(E) \geq H_R(D)$$

And the two identies above can be set equal to each other:

1.5.7 $$H(D) + H_D(R) = H(R) + H_R(D)$$

Substituting $H_R(E)$ for $H_R(D)$, we get

1.5.8 $$H(D) + H_D(R) \leq H(R) + H_R(E)$$

Now from the identies we have

1.5.9 $$H(R) + H_R(E) = H(R,E)$$

Thus

1.5.10 $$H(D) + H_D(R) \leq H(R,E)$$

But we know that

1.5.11 $$H(R,E) \leq H(R) + H(E).$$

So,

1.5.12 $$H(D) + H_D(R) \leq H(R) + H(E)$$

Which gives

1.5.13 $$H(E) \geq H(D) + H_D(R) - H(R)$$

This result is essentially the same as the earlier expression of the Law of Requisite Variety.

The interesting thing is that an analogous result exists in zero sum two player game theory.[66] Although the game theoretic result is more useful, it answers the same question: How many strategies do I need to do as well as possible against a malevalent and efficient opponent? The answer is: No fewer strategies than he uses.

1.5.14 Consider a rectangular game with payoff matrix A, and consider an X which is strategy for the row player and a Y which is strategy for the column player. A necessary and sufficient condition that the X and Y are (extreme point — meaning that all optimal strategies can be derived from them as convex combinations) optimal strategies is that there exist a submatrix B of A, of order r, such that

$$J_r(\text{adj } B)J_r^t \neq 0, \quad \text{and}$$

$$v = \frac{\det B}{J_r(\text{adj } B)J_r^t}$$

$$\dot{X} = \frac{J_r(\text{adj } B)}{J_r(\text{adj } B)J_r^t}$$

$$\dot{Y} = \frac{J_r(\text{adj } B)^t}{J_r(\text{adj } B)J_r^t}$$

where $\dot{X}$ is the vector obtained from X by deleting the elements corresponding to the rows deleted to obtain B from A, $\acute{Y}$ is the vector obtained from Y by deleting the elements corresponding to the columns deleted to obtain B from A, J_r^t is a column vector of length r containing only 1's, and adj B is the adjoint of B. (The adjoint of a matrix B is the transpose of the matrix obtained from B by replacing each element by its cofactor, where the cofactor is $(-1)^{i+j}\det(M_{ij}(B))$, and $M_{ij}(B)$ is the minor of B obtained by striking out the ith row and jth column).

We shall not prove this theorem here (It is proved in Mckinsey, Section 3.3) since our purpose is merely to point to its existence, and to some of its features. Notice that each strategy has the same number of components in it.

A numberical example, from McKinsey, (p. 79) illustrates the theorem.

$$A = \begin{matrix} 2 & 4 & 0 \\ 1 & 0 & 4 \end{matrix}$$

The three 2x2 submatrices (these are the largest square submatrices possible in this case) are as follows:

Figure 1.5.3

$$B = \begin{matrix} 2 & 4 \\ 1 & 0 \end{matrix} \qquad C = \begin{matrix} 2 & 0 \\ 1 & 4 \end{matrix} \qquad D = \begin{matrix} 4 & 0 \\ 0 & 4 \end{matrix}$$

It is easy to show that submatrix B produces a Y strategy with a negative component, and thus is not acceptable. Submatrix C is more satisfactory:

The det C = 8, adj C = $\begin{matrix} 4 & 0 \\ -1 & 2 \end{matrix}$, J_r(adj B) = (3 2)

and $J_r(\text{adj B})^t$ = (4 1). Thus, applying the formulas, we obtain

1.5.15 $$X = (3/5, \; 2/5)$$
$$Y = (4/5, \; 0, \; 1/5)$$
$$v = 8/5$$

This result can be proved to be a solution to matrix A by trying the X against each column of A to see if it yields $\geq$ v, and similarly by trying the Y against each row of A to see if it yields $\leq$ v. If both X and Y satisfy their respective inequalities, they are optimal strategies. It can be shown that the submatrix D does not yield a solution to A. The optimal strategies characterized by this theorem can be expressed in terms of the strategies for the Disturbance and the Regulator of Ashby's Law of Requisite Variety. The regulator must be able to use at least as many pure strategies as the Disturbance uses, <u>when the Disturbance is at its most efficient</u>. If the Disturbance is optimal (malevolent) but not efficient (not using a minimum number os strategies), the Regulator need not use more than his minimum optimal number. The requisite number of strategies corresponds to the number of components in the smallest optimal maximin strategy. The value of the game is the essential variable which the Regulator is trying to keep under control. The number of elements in the smallest square submatrix which provides optimal strategies corresponds to the outcomes E in Ashby's sense. Ashby's term $H_D(R)$ equals zero since this corresponds to the entropy of the Regulator's strategy in the minorant game, which is strictly determined.

There is only one objection to this analogy — why should the Disturbance be malevalent? We shall not attempt to answer this question now. It will come up again when we discuss the theory of zero sum interpretation, and we shall deal with it then. The idea that one player exercises control strategies and the other disturbance strategies has recently been further developed by Banerji.[19] The correspondence between a zero sum two player game with perfect information and in extensive form with the activities of a prey and predator is described by Ashby.

Reconsider the function of 1.1.1, i.e.

$$f_k(d_1, \ldots, d_r), \text{ for } k = 1,2$$

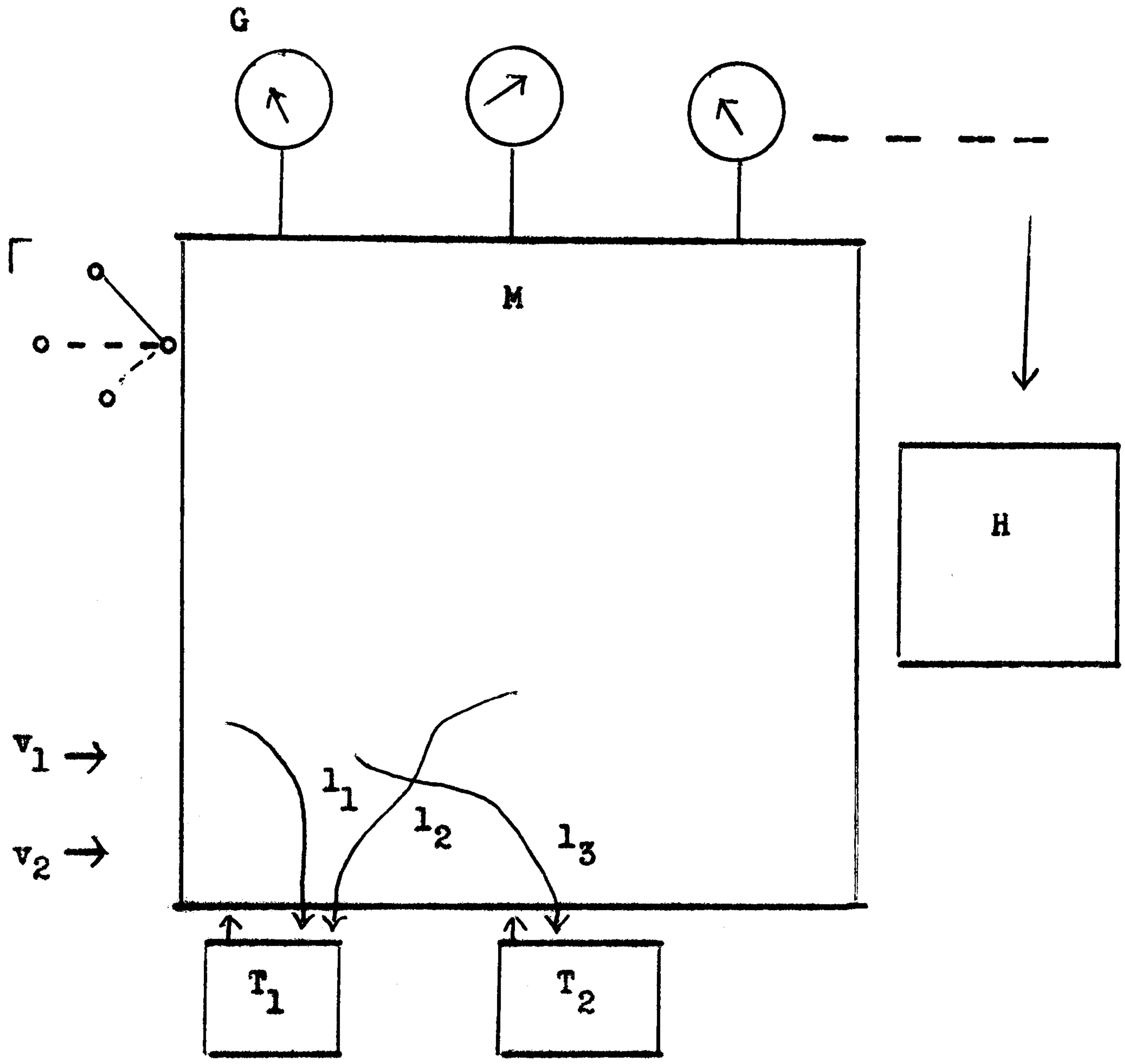
Consider the case where the predator comes within view of the prey and call that d_1^1. The prey's first response is d_2^2. The predator then stops to take note of the reaction of the prey, d_3^1. The process continues until the prey escapes or is eaten. This of course is exactly what the function of 1.1.1 describes:

$$f_k(d_1^1, d_2^2, d_3^1, \ldots, d_n^k) \; k = 1,2$$

Note that we have left open the question of who has the last word by designating the last move as d_n^k.

Ashby has also pointed out that a game in extensive form can be shown to be isomorphic with certain machines with input. The following diagram is from Ashby's figure 12/22/1:

Figure 1.5.2



The above diagram is the machine with input. The players T_i know its internal structure (i.e. its game tree). The switch Γ determines which game is played. The v_i are chance moves (referee's moves). The players T_i are determinate (because the game is in extensive form) dynamic systems coupled to M both by reciving information through definite channels l_i and by making determinate acts on M. Now, the activities of the T's and the v_i acting through the machinery of M bring about controls on the dials G. The referee, H, reads the dials and makes the appropriate payments to the T's.

If the players use optimal minmax strategies, they will presumably be satisfied with the payments received from H, and the system will be ultrastable.

A link has been indicated by George between the theory of neural nets and the theory of two player, zero sum games.[78] This has also been discussed by Jan Gecsei,[73] who has succeeded in constructing a neuron network model to generate optimal strategies. The model is too complicated to be discussed in this paper, however, the technique for approximating solutions to games which he employs will be described.in a later section (1.7).

Notice that because minmax strategies are not permanently optimal, and for complicated games, not always computable, they belong to the general field of <u>heuristics</u>. The term has been defined by George:[96]

> The word 'heuristic' is used to represent a general rule of thumb ('ad hoc rule' or 'shortcut' are synonymous) which is meant as a good rough guide, but which is not an algorithm, and is therefore liable to error. An example would be that of saying 'Never allow your mother-in-law to live with you or 'In summer it is sunny', etc. These are 'better than average' guiding principles to which there are obvious exceptions, and they are used when an algorithm is either impossible or uneconomic. For example, to work out exactly how to play 'perfect' chess (by an algorithm) might take a thousand years, so we play by heuristics; the better the heuristics the better the player. Heuristics are like generalisations of hypotheses, and intelligent behaviour leads to the adapting and modifying of heuristics in the light of experience ....

In this context "heuristic" refers to the applications — both normative and descriptive — of the zero sum model. Formally, the theory specifies an algorithm, since a minmax solution always exists to a zero sum two player game, although, of course, as the quotation shows, the solution is not always computable. An application of zero sum 2-player games has been made to optimal word length in a natural language (English) by Mandelbrot.[99]

Before leaving the topic of cybernetics and two player zero sum games, mention should be made of the single player decision making models known as "games against Nature." These are part of the field known as decision making under conditions of uncertainty. These are zero sum two player games where one player is a real player and the other player is Nature or the state of the world. (In part 7 we shall also talk about games against Nature, i.e., zero sum two player games where one player is a real player and the other player is an imaginary player. The only differences between the games against Nature of section 7 and those of the present section are the way they are obtained and their exact use in the decision making process; the spirit of the two is quite similar). Games against Nature are discussed by Milnor,[102] and we shall make only a few comments.

First, since these are essentially one player games (the second player is just a fiction) the theory of zero sum two player games is used as a heuristic device for decision making. Thus they properly belong to the field of heuristic decision making, as has been pointed out by George.[71] Second, although a maxmin strategy for the real player (with the payoff matrix representing the possible monetary payoff for each course of action against each state of the world) is one type of decision model, some of these decision models involve transformation of the payoff matrix. We shall discuss an example known as "minmax regret."[103] Consider the following payoff matrix, where the rows represent courses of action for the decision maker, the columns states of the world, and the numbers the monetary payoff to the (row) player:

6	4	1
7	8	3
2	5	9

Suppose the decision maker decides to obtain 9 (the most he can get in this case). He must choose row three and assume that the state of the world will be as represented by column three. But what if the state of the world turns out to be that represented by column one! The row player would obtain 2 instead of 9. Had the row player correctly guessed the state of the world, he would have chosen row two and obtained 7. Thus, the difference between what he actually obtained 2 and what he might have obtained 7 is 7 - 2 = 5, which is defined to be the regret suffered by the decision maker over his choice. We can compute the regret for each state of the world by replacing each matrix entry by the difference between it and the largest entry in the column. For the above matrix, we would obtain the following:

1	4	8
0	0	6
5	3	0

A cautious decision maker may wish to guarantee that his regret will be the minimum. Therefore, he need only examine each row for its maximum regret, and choose whichever row has the minimum of the maximum regrets:

	maximum regret
row one	8
row two	6
row three	5 minimizes the maximum regret.

This decision making concept suffers from the weakness of not being independent of irrelevant alternatives. The important points from our perspective are, first, the game is essentially a one player game; the second player, Nature, is assumed to be acting in a way to impose the maximum damage on the real player. Thus, the game is effectively interpreted as a zero sum two player game with all winnings obtained from Nature. Second, given the zero sum interpretation, an ad hoc rule is used to transform the payoff matrix, before the decision is made, and the decision is made on the basis of the transformed matrix. In section 7, we shall develop a solution concept which is in somewhat the same spirit — reinterpreting a nonzero sum game as a zero sum two player game where one player is Nature, and, in doing this, making use of rules to change the payoff matrix entries.

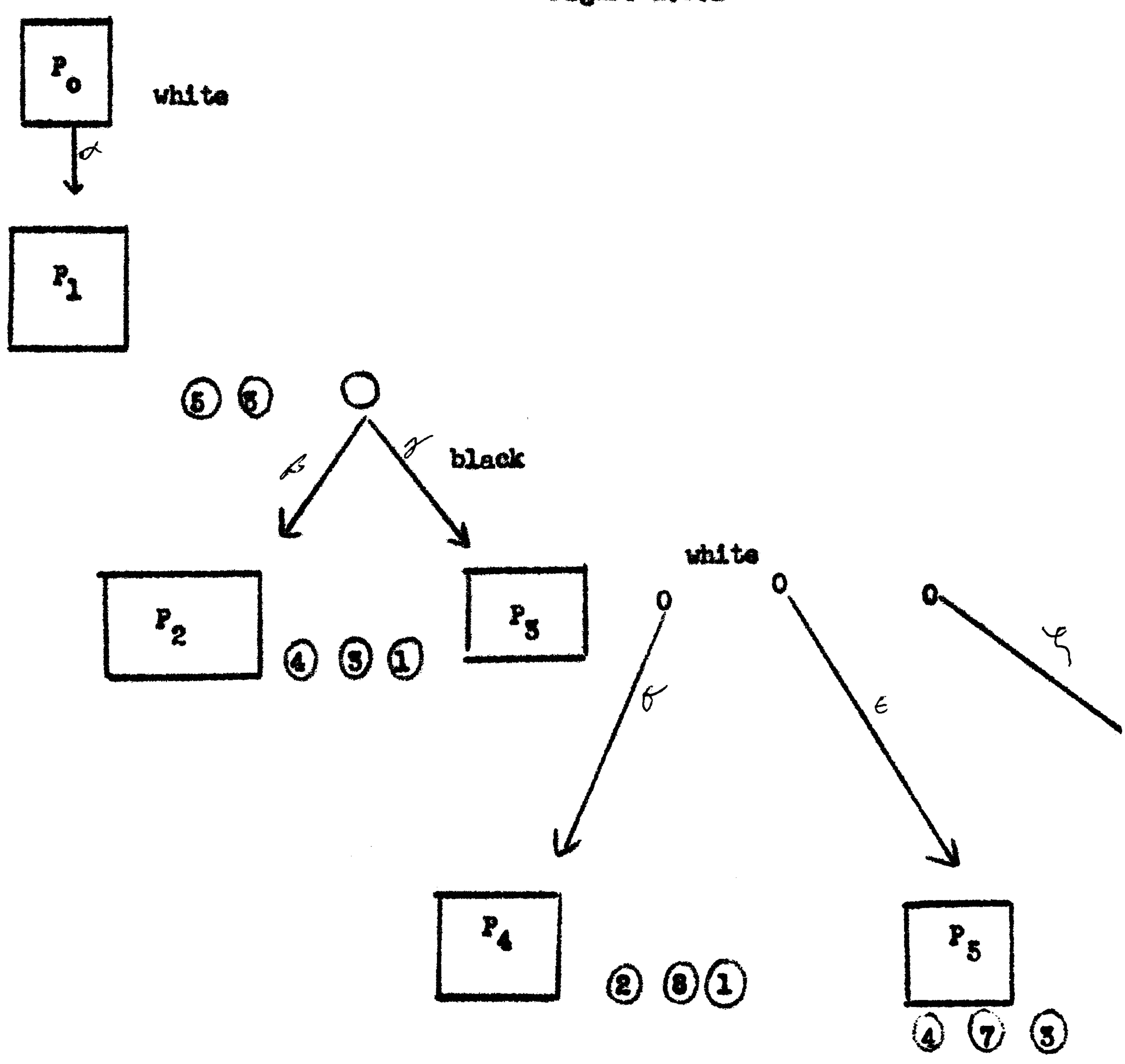
1.6 Game playing programs

We have discussed zero sum two player games with perfect information and have looked at the theorem which says that they always are strictly determined. We have also noted that game trees can be indescribably complicated. For example the game tree for the most famous and prestigious example of a game of this type, i.e. chess, involves somewhere around 10^{20} different paths through the tree - a figure estimated by Claude Shannon.[20] And the theory of games does not give us a clue as to how we should play it. For a way to play infallibly winning chess - and there may be more than one - we would need to explicate enough of the game tree to get from first to last move and allow for all detours taken by the opponent. Obviously, this is again an indescribably complicated task, even though it is clearly a smaller job than explicating the entire game tree. Thus, there seems to be no hope of turning chess into a trivial game within the foreseeable future. We can, however, program a computer to look ahead in the game tree a few moves and evaluate, in some way, the possible alternatives which are available. Now this is not, strictly game theory. Rather it is the manufacturing of chess playing behaviour. Since the behaviour is manufactured, i.e., a human artifact, it is artificial, and since chess playing is an activity normally associated with intelligent behaviour, these chess playing programs are a branch of artificial intelligence. Nevertheless, these programs utilize some game theoretic concepts (as well as some heuristic tricks) and therefore fall into a sort of borderline zone between game theory and heuristic programing. The fact that these programs are essentially heuristics, rather than "pure" mathematics, does not mean that they should be despised by those who call themselves game theoreticians.

We have already seen that the optimal strategies in a zero sum two player game are not permanently optimal, i.e. they employ a heuristic-admittedly a high powered one - limitation in the search for strategies. Later, we shall see that the theory of n-player games uses, in a fundamental way, a zero sum two player interpretation heuristic. Thus, the entire theory of games will be seen to be shot through with heuristic devices.

Thus, we are well advised to look at a chess playing program. There have been several, beginning with Shannon's[20, 54] in 1949. We shall briefly describe that of Newell, Shaw and Simon.[57] The basic idea of the program is as follows. First there are various well known goals associated with chess e.g. king safety, center control. The goals, in short, are already tricks associated with "good" (i.e. even passing) chess. They are however, not specified by the rules of the game. Now, the goals are called upon by an analysis at the beginning of each move which determines what situation obtains in the game at that move (i.e. what branches lead from that node). Having established the situation, the program specifies a set of appropriate goals. More than one goal is considered at a time - a list is made with the most critical ones at the top. From this list, the rest of the program is controlled. A "move generator" proposes alternatives. (The word "move" is used here in its popular, not game theoretic sense). These again have been largely written into the program from the viewpoint of chess playing experience. For example, if the goal is "center control", the move generator will propose P-Q4 for a good opening choice. Having proposed that choice, another routine will evaluate it to determine its acceptability. This will be based on a value for the proposed choice which is a vector having one element for each goal. This element is sometimes a number - the standard piece values, e.g. 9 for a queen - and sometimes the element simply expresses the presence or absence of the property specified by the goal. Now the value vector is determined by looking ahead in the branches leading from the given node, and the question is, of course, how far ahead? To determine this, a trick used by Turing[21] is adopted in this program. This involves exploring ahead as far as it takes to reach a "dead position". A dead position is one which is "stable", which is taken to mean that it is at the end of a sequence of choices at which all exchanges have taken place such as an exchange of queens. In short, no more choices would be made which would significantly effect the particular goal under consideration, such as material balance. The search continues until the position is found to be dead for all the goals considered in the value vector. Thus, the program does not search ahead for a fixed number of moves, but merely continues to search until the value vector is well defined. This could happen with a relatively short search, or it could be quite an extensive search, depending entirely on the position from which the search is made and the goals in the value vector. Also, this process may result in several competing series of choices being explored. For example, for a proposed choice, the value vector may be dead, i.e. have a value, for n-1 components at the first continuation (i.e. at the opponent's reply). However, the nth component may not be dead, i.e. may not be assigned a value, and thus, the continuations must be further explored until all components are assigned values. This process is illustrated in the following diagram (which is figure 4 of Newell, Shaw & Simon):

Figure 1.6.1



The proposed choice for white P_o is α . From this we obtain P_1, which, however, leaves the third component in the value vector unspecified. From this point, we see that black will choose between γ and β . β will specify a dead position, and γ will allow white three further alternatives, δ and ϵ , both of which specify dead positions, and a third continuation ζ which is not shown. Thus, we see how the program specifies continuations. It remains to discuss how these static value vectors are compared. In other words, which one is chosen. Vectors are compared component wise, with the first element dominating and then, if found equal, the next element dominating, and if found equal, the next .... The decision procedure which is used is that of minimax. Going back to figure 1.6.1, we see that at P_1 black would certainly not choose γ , for this would allow white to choose ϵ for a value of (4, 7, 3). Black would therefore choose β and hold white to (4, 3, 1). This value is therefore the minimax value of the choice α . The <u>final</u> choice is made by establishing an acceptance level for a given choice, computing minimax values for various goal specifications in order of importance and taking the first choice which reaches the specified acceptance level. If none are found, the best of the evaluated choices is taken. The estimated time per move for this program is between one and ten hours.

Having read a brief description of the Newell, Shaw and Simon program, the reader might expect a description of its modification in the light of experience and, perhaps, a reference to its most recent successor, which has just won a chess tournament.

> "Unfortunately this is not possible, since there seem to be no survivors of the various chess programs known to have been written ... which are capable of really competent chess. On this criterion it appears that no significant advance has been made since the early days, in spite of a widespread and strongly-held belief to the contrary." 105

The statement we have just quoted was written in 1966. By 1967, a program was written, that of Greenblatt,[106] which was able to play passable chess against good amateur players. This program employs approximately 50 heuristics, has some book moves written into it, and explores to a fixed depth (but has some flexibility on width of search, i.e., evaluation of possible choices at a given position). However, one authority on computer-chess, Levy,[107] doubts that significant progress will be made beyond the achievement of the Greenblatt program, except perhaps in the direction of hardware which speeds up the evaluation. Levy's assessment appeared in 1971. For an example of a recent program, see that of Scott,[85] which we shall not discuss here. Incidentally, the game of checkers has also been made the object of artificial intelligence programs,[8, 67] but we shall not discuss this topic here.

1.7 Linear programming and Brown's approximation algorithm

Any zero sum two player game in matrix form can be solved using the technique of linear programming, i.e., every two player zero sum game problem is also a linear programming problem. The reverse is also true, but is of no particular interest. The conversion is perfectly straightforward. Consider an mxn matrix with all positive elements:

1.7.1

$$\begin{matrix} a(1,1) & a(1,2) & \dots & a(1,j) & \dots & a(1,n) \\ \vdots & & & & & \\ a(i,1) & a(i,2) & \dots & a(i,j) & \dots & a(i,n) \\ \vdots & & & & & \\ a(m,1) & a(m,2) & \dots & a(m,j) & \dots & a(m,n) \end{matrix}$$

If player 2 (column) uses a pure strategy of column j, and player 1 uses a mixture $(x_1, x_2, \dots, x_m)$, where $\sum x_i = 1$, then we have $\sum_{i=1}^{m} x_i a_{ij} = g$, say. Now, for every pure column against $\vec{x}$, at least one g will be the smallest. Call this $\bar{v}$. Thus, for every column j, $\sum x_i a_{ij} \geqq \bar{v}$. We can divide by $\bar{v}$ to obtain

$\sum \frac{x_i a_{ij}}{\bar{v}} \geqq 1$, and for the x_i we get $\sum \frac{x_i}{\bar{v}} = 1/\bar{v}$.

Clearly, the row player wants to maximize $\bar{v}$, which is the same as minimizing its reciprocal, $1/\bar{v}$. Thus, our last two summations can be taken to be a standard problem of linear programming, with the latter summation the objective function to be optimized, and the former summation the constraints to which the objective function is subject. This can be solved by the simplex method.

Thus set $x_i' = \frac{x_i}{\bar{v}}$ and $m = 1/\bar{v}$, and we obtain a familiar problem of linear programming:

Subject to

1.7.2 $$\sum x_i' a_{ij} \geq 1 \text{ for all } j,$$

minimize

$$\sum x_i' = m.$$

With the use of negative slack variables to bring about equalities and artificial variables for a starting basis, we can use the simplex method to solve this game. The dual problem for the column player can be analogously defined. There are several other methods for converting a game into a linear programming problem, but we shall not discuss the matter further. We could also show that any linear programming problem can be converted into a (squew symmetric) game matrix, but we will not show this here. It is discussed very clearly in Dorfman, et al.[75] The classic papers, one by D. Gale, H. Kuhn and A.W. Tucker,[108] another by G. Dantzig,[72] and a third by R. Dorfman[109] were presented at a conference in 1951.

The solution to a zero sum two player game in matrix form can be approximated by the algorithm of Brown,[68, 69] Moreover, this algorithm can be used by a player who knows nothing about game theory. He only needs a maxim: "The future will be like the past." At the first play, each of the two players arbitrarily selects any strategy at all, and then, on the second play each chooses by maximizing on the other's choice at play number 1. At play number 3, each player chooses according to the total of his expectations from the previous plays. This is done by summing the expectations associated with each pure strategy choice. As an example, imagine a 2x2 matrix. The row player says to himself, "If I had chosen row 1 when he chose (say) column 1, I would have obtained x plus my previous balance of X associated with my choice of row 1. If I had chosen row 2 when he chose column 1, I would have obtained y plus my previous balance of Y associated with my choice of row 2. I see that (say) X + x is greater than Y + y, so I shall choose row 1 on the next play." The column player reasons similarly. If this procedure is followed, as the number of plays increases, the ratio of choices of row 1 to row 2 will approach the optimal maxmin mixture, and similarly for the ratio of column 1 to column 2 (which will approach the optimal minmax mixture). For games which are 2x2, and larger any convergent series will converge to an optimal strategy.[86]

As a numerical example, consider the following matrix:

-2	-3
-3	-1

Starting the procedure arbitrarily with a choice of row 1 and column 1 for players 1 and 2 respectively, we obtain the following table for the first five plays:

1.7.3

Play	Player 1 chooses	Total expectation of Player 2 by choosing column 1	2	Player 2 chooses	Total expectation of Player 1 by choosing row 1	2
1	1	2	3	1	-2	-3
2	1	4	6	2	-5	-4
3	2	7	7	2	-8	-5
4	2	10	8	2	-11	-6
5	2	13	9	1	-13	-9

This table could be made as long as we wish. There are, of course, three other tables which could be constructed by starting in one of the other three matrix elements. Any rule could be used to break ties as in play 4 for player 2, where his previous expectation did not give him an unambiguous choice. The rule we have used in this example is, "If in doubt, do what I did the last time." However, the rule could have alternatively required doing the opposite of the previous choice, or could have required flipping a coin when in doubt. Von Neumann[90] has developed a probabilistic approximation technique which has been largely ignored in the literature.

2 Zero sum n-player games

The theory of zero sum two player games is extremely elegant mathematically. It suffers, however, from an almost fatal defect — irrelevance to most social situations.[92, 93, 100] Even many parlor games involve $n > 2$ players, e.g., most versions of poker. Von Neumann and Morgenstern were perfectly aware of this fact, and devoted nearly two thirds of their book to games which were not of pure conflict between two players. These are of two types - zero sum games with $n \geq 3$, and nonzero sum games with $n \geq 2$. We shall briefly discuss the former in this section. As we shall see, however, neither of these topics is unrelated to our previous discussion. Indeed, they depend heavily on the theory of zero sum 2-player games.

2.1 Coalitions

Suppose three players numbered 1,2,3 decide to play the following zero sum game:

In secret, each writes the number of another player on a piece of paper. The three pieces of paper are then handed to a referee. If two players choose each other's number, the referee announces that they form a coalition; and the third player must pay this coalition the sum of £1, which they may divide any way they like. Either exactly one coalition will form, or none will form (e.g. 1 chooses "2", 2 chooses "3", 3 chooses "1"). The obvious three questions are, how should a player play this game; which coalition can be expected to form; what distribution of payments can be expected within the winning coalition?). The von Neumann and Morgenstern n-player theory of games fails to answer the first two questions, but does give an interesting answer to the third. We shall discuss the above three player game in some detail, and as we procede we shall make use of some convenient notation. Having discussed this game, we shall extend the formalization and definitions to all $n \geq 3$. In following this approach, we are exactly following the line of discussion used by von Neumann and Morgenstern. For an alternative and less intuitive order of presentation - see Luce and Raiffa, section 7.1 and chapters 8 and 9.

The game we have just described can be given some convenient notation. We know that the following sets can be formed:

2.1.1

$$(1,2) \quad \text{and } (3)$$
$$(1,3) \quad \text{and } (2)$$
$$(2,3) \quad \text{and } (1)$$
$$(1,2,3)$$

and of course the empty set $\emptyset$ (for the sake of completeness). In addition, we know that each of these sets have a value, which we shall call v:

2.1.2

$$v\ ((1,2)) = 1, \quad v\ ((3)) = -1$$
$$v\ ((1,3)) = 1, \quad v\ ((2)) = -1$$
$$v\ ((2,3)) = 1, \quad v\ ((1)) = -1$$
$$v\ ((1,2,3)) = 0$$
$$v\ (\emptyset) = 0$$

This value v can be characterized as follows for each sets of players, i.e. each v(S):

2.1.3

$$v(S) = \begin{cases} 1 & \text{if S has 2 elements} \\ -1 & \text{if S has 1 element} \\ 0 & \text{if S has 3 elements} \\ 0 & \text{if S has no elements} \end{cases}$$

Now if the set of all players is designated by I, then we can properly speak of sets S and their compliments -S within I and from 2.1.3, we can observe that we always have

$$v(S) = -v(-S) \tag{2.1.4}$$

The above equation expresses the fact that the game with coalitions is a zero sum two player game between the players S and -S.

Therefore, if the players of S can "cooperate fully" with each other against the players in -S, the minimax theorem of section 1 can be seen to apply, and thus we see the importance of the 2-player zero sum theory to the n-player zero sum theory. Now, suppose that (S) = (1) and (T) = 2, we observe that 2.1.5 is true.

[The symbol $S \cup T$ refers to a new set composed of all S and all of T. The symbol $S \cap T$ refers to a new set composed only of what is common to both S and T, and to no other parts of S and T].

$$v(S \cup T) \geq v(S) + v(T) \text{ for } S \cap T = \emptyset \tag{2.1.5}$$

The expression 2.1.5 says that players have an incentive to form coalitions.

In the rules of the three player game we first described, the players were not allowed to communicate with each other when they chose the numbers. We may assume, however, that before the game began, they as a group, or perhaps a subset of them, say two close friends, discussed what they were going to do. That is, they decided which numbers to pick. This is the same as saying that they decided which coalition would form. The question therefore becomes, what will the considerations be which cause them to make up their minds as to which coalition will form. So far we have mentioned two possible considerations - the £1 to be divided, i.e. how it will be divided, and the possibility of two good friends in the negotiations. We shall dispose of the later of these first. In fact, it disposes of itself - if there are two and only two close associates in the game, then we may expect that they will naturally choose each other's numbers and discriminate against the third player by excluding him, from whatever further discussion is involved in the division of the take, about which we might expect a "friendly rivalry", i.e. each will try to get as much as he can. An interesting possibility suggests itself - what if there are three good friends? Perhaps two of them are "better friends" with each other than with the third and thus would form a coalition but nevertheless would not want to alienate the third, i.e. they might "kick back" some or all of the £1 plus perhaps, even pay him a premium, so as not to hurt his feelings. These are interesting speculations and, what is even more interesting, they turn up quite unexpectedly in the rigorous mathematical analysis of this game - but more of that later.

Let us consider the first case - where none of the players are friends and the only possible influence on the formation of a coalition is division of the £1 between the members of the winning coalition. The reader may object that the division is self evident, and although we could demonstrate the strategic bargaining property which makes it "self evident" (as opposed to some principle of "fair division") we shall instead generalize considerably the game and discuss the strategic aspect for the following game, of which the game of 2.1.3 is merely an example. The new game is as follows: If players 1 and 2 form a coalition they obtain the amount c, and no more, from player 3. If players 1 and 3 form a coalition, they obtain b, and no more, from player 2. If players 2 and 3 form a coalition, they obtain the amount a, and no more from player 1. The reader can see that the game of 2.1.3 is merely the situation we have just described with a=b=c=1. However, a,b, and c need not be equal. Suppose player 1 demands the amount x if he forms a coalition with player 2, and he demands the same amount if he forms the coalition with player 3. Thus, in the first case, player 2 can expect to obtain c-x, and in the second case, player 3 can expect to obtain b-x. Put another way, player 1 will keep x, and give a <u>side payment</u> of b-x (or c-x) to his partner. Players 2 and 3 can evaluate the claim of player 1 in the following light. If 2 and 3 form a coalition with each other, perhaps they can each obtain more than c-x and b-x respectively. Put another way, if the sum of their offers, i.e. c-x and b-x are <u>less</u> than the total which they can obtain if they form a coalition with each other against player 1, then they must be expected to reject player 1's offer. That is, player 1's offer will certainly be rejected if:

2.1.6 $$(c-x) + (b-x) < a.$$

Thus, the claim of x on the part of player 1 will certainly be rejected <u>unless it satisfies</u>

2.1.7 $$(c-x) + (b-x) \geq a.$$

We may rewrite the inequality of 2.1.7 as follows

2.1.8 $$x \leq \frac{-a+b+c}{2}$$

Clearly, the <u>maximum</u> claim α which player 1 can reasonably make is the one where equality holds in 2.1.8

2.1.9 $$\alpha = \frac{-a+b+c}{2}$$

Repeating the above arguments for players 2 and 3 respectively, we obtain

2.1.10 The maximum amount player 2 can reasonably claim is

$$\beta = \frac{a-b+c}{2}$$

2.1.11 The maximum amount player 3 can reasonably claim is

$$\gamma = \frac{a+b-c}{2}$$

Observe that the word "reasonable" in the above expressions does <u>not</u> refer to "fairness" or to any egalitarian principles whatever. Rather, it refers exclusively to the expectation that if more than α, β or γ is claimed, the claim will certainly be rejected. The reader can further verify that the following holds:

2.1.12 $$\alpha + \beta = c, \quad \alpha + \gamma = b, \quad \beta + \gamma = a$$

That is, the claims exhaust, but do not exceed, the values of the respective coalitions.

We see, therefore, that although the expected division of the spoils in game 2.1.3 was somehow "obvious", other 3 player zero sum games can be constructed for which this is hardly the case. For example, set

$$v((1,2)) = 1,\ v((1,3)) = \tfrac{1}{4},\ v((2,3)) = \tfrac{1}{8}\ .$$

We shall not perform the computations here. Notice that the formulae of 2.1.9, 2.1.10, 2.1.11 establish one point conclusively: Whenever a player is a member of a winning coalition, he can always reasonably expect the same amount. But, what incentive does a player have to enter a coalition, that is, why not refuse to cooperate with the other players. Clearly, if player 1 refuses to cooperate, he obtains $-a$, if player 2 does so, he obtains $-b$, if player 3, the amount is $-c$. Thus players 1 and 2 can always get $-(a+b)$ if they don't cooperate with each other. If they do cooperate, they could get $c \geq -a-b$. Now consider the two cases of $c \geq -a-b$, which we rewrite as

$$\Delta = a+b+c \geq 0. \tag{2.1.13}$$

If $\Delta = 0$, then $c = -a-b$, and clearly the two players have no incentive to form a coalition. Such games are said to be <u>inessential</u>. If $\Delta > 0$, then $c > -a-b$, and the two players <u>have</u> an incentive to form coalitions. In this case, the game is said to be <u>essential</u>. We can say more about this case. If a player, say player 1, forms a coalition, he can obtain α. If the game is essential, then $\alpha - (a) > 0$ (since he gets $-a$ if he does not join a coalition). Now, $\alpha + a$ turns out to be equal to $\frac{\Delta}{2}$. Repeating the argument for each player, we find that $\frac{\Delta}{2} \geq 0$ is always the inducement for joining a coalition in a three player zero sum game.

We have thus been able to fully describe (non-discriminatory) payoff distributions for __any__ essential zero sum three player game. Which distribution obtains? The answer is not given by the theory. Indeed, no one distribution is stable by itself, since it was found only as an answer to other possible distributions. Therefore, the __set__ of reasonable distributions itself, rather than any one of its elements, constitutes the "solution" to this game in the sense we have been describing. We can list each of these possible distributions as a payoff vector relative to a certain coalition. Thus, we have found that for the coalition (1,2), the payoff vector is $(\alpha, \beta, -c)$. The entire non-discriminatory solution is as follows:

2.1.14	coalition	payoff vector
	(1,2)	$(\alpha, \beta, -c)$
	(1,3)	$(\alpha, -b, \gamma)$
	(2,3)	$(-a, \beta, \gamma)$

where the α, β, γ, a,b,c, are as in 2.1.9, 2.1.10, 2.1.11

Which of these payoff distributions will obtain, that is, which coalition will form, is __outside the scope of the theory__. This solution is somewhere between a prescription of what one can reasonably ask for, and a description of what one can reasonably expect to find in a zero sum three player game. That is, it is not clear whether the theory is __normative__ or __descriptive__, although one suspects that it is most probably the latter type of theory.

We should note that we were able to derive the possible reasonable distributions entirely from a consideration of the v(S). Clearly, the v(S) is of central importance to the theory, indeed, the entire n-player theory hinges on it. A v(S) of the type we have just described is clearly a mathematical set function, and is known as a __characteristic__ function. Its properties for zero sum n-player games are as follows:

2.1.15 $$v(\emptyset) = 0$$
$$v(S) = -v(-S)$$
$$v(S \cup T) \geq v(S) + v(T) \text{ for } S \cap T = 0$$

We have seen one instance (for n=3) of the way in which the characteristic function determines everything. Von Neumann and Morgenstern have succeeded in developing a theory, for all n, based on the characteristic function described in 2.1.15. The reader is referred to chapter Vadd VI of their book for a detailed discussion (or to Luce and Raiffa, chapt. 8, McKinsey, chapt. 15, Owen[22], chapt VIII). We shall now take up the formal properties of the payoff vectors.

2.2 Imputations, solutions, normalization.

The payoff vectors, called imputations, exhibit certain properties which we can conveniently list. First, we noted that for essential games, we never had the payment to a member of a winning coalition as less than what he could obtain alone. Thus, if α_i is the payoff to player i, member of the winning coalition S, we have

2.2.1 $$\alpha_i \geq v((i)), \text{ for } i = 1, \ldots, n$$

Second since the game was zero sum, each imputation satisfied the following property:

2.2.2 $$\sum_{i=1}^{n} \alpha_i = 0$$

Third, we can notice that in no case were the sum of the α_i to the members of S in the imputation greater than the amount that this set of players could steal from -S:

2.2.3 $$\sum_{i \text{ in } S} \alpha_i \leq v(S)$$

If the imputations are thought of as possible promises of rewards which will be paid when the game is over, expression 2.2.3 says that the promises must not be more than the coalition can deliver. Thus, if a set S exists for a given imputation such that 2.2.3 is satisfied, the set S is said to be effective.

A fourth property was exhibited by the imputations, namely, they <u>dominated all</u> imputations not among them and were dominated by none among them. Domination is a relationship which is always relative to a specific set S. It has the following three properties:

2.2.4
a) S is not emply
b) S is effective for the imputation, say $\vec{\alpha}$
c) $\alpha_i > \beta_i$ for all i <u>in</u> S

If an imputation $\vec{\alpha}$ dominates another, say $\vec{\beta}$, relative to a particular effective set S, this is conventionally written as

$$\vec{\alpha} \succ \vec{\beta}$$

The entire set of imputation vectors in 2.1.14 constituted the <u>solution</u>, as we have seen. We can therefore generalize to an exact definition for a solution, where the set of imputation vectors in the <u>solution</u> is written as V:

2.2.5 No $\vec{\beta}$ in V is dominated by an $\vec{\alpha}$ in V,

2.2.6 Every $\vec{\beta}$ not in V is dominated by some $\vec{\alpha}$ in V.

Now, V represents a standard of behaviour, within which various particular permutations and combinations are possible. This is, on reflection, a reasonable definition of a solution, because it isolates what is stable about a given social situation - namely, the underlying values and types of distribution. Indeed, recent political thinking on the part of the "New Left" seems to view social situations in precisely this way. That is, the entire standard of behaviour is what is called into question and who happens to be in the winning coalition at any given moment is of little importance.

Thus, whether George Wallace, Richard Nixon or Hubert Humphrey would be elected president of the U.S. in 1968 seemed unimportant to those with this political persuasion, (some of whom voted for Pigasus, a pig, as an (ironical) alternative, i.e. as a choice outside the standard of behaviour which included Wallace, Nixon and Humphrey[23]).

Another point about this solution concept should certainly be mentioned. The permutations and various imputations in V all exist, in a virtual sense, even though only one may obtain in reality. The other imputations must be included as part of the solution because they have helped to shape the reality. From this viewpoint, the bias of some historians to trace back actual events and seek "causes" in empirical evidence may be highly misplaced.[24] For, the "cause" of an event may never have occured at all.

We should take note of the existence and number of these n-player game solutions. Von Neumann did not know, that is could not prove, whether a solution even existed for games of any n. He considered this the main unsolved problem in game theory (See Tucker and Luce[55]). We now know that this question has been answered in the negative, since a ten player game has been found which has no solution.[101] Various catagories of games exist for which solutions can be described relatively easily. The reader is referred to the discussion in Luce and Raiffa, Chapter 9, Owen, chapter VIII, Tucker and Luce, the Introduction, and von Neumann and Morgenstern, where a very large portion of the book is concerned with the investigation of solutions. Second, there may be more than one solution V, each with its set of specific imputations, for any given game. This turns out to be the case for zero sum three player games, where in addition to the main solution of 2.1.14, there exists another solution, which can be found by graphical means, which is a discriminatory solution of the type we originally intuited. This V turns out to be an infinite set composed of the three permutations of an infinite set of discriminatory distributions. Moreover, if an imputation is $(c, a, -c-a)$ where $-1 \leq a \leq 1-c$, and where c is the discriminatory assignment, restricted to $\frac{1}{2} > c \geq -1$, the payments to players 2 and 3 are no longer subject to the bargaining influence of the first player, and thus, themselves become essentially unrestricted - although they must satisfy 2.2.1 and 2.2.2 (shown above with a normalization such that $v((i)) = -1$). Thus, we see that for those games for which we have solutions, we seem to have more than we know what to do with!

One important topic of the von Neumann and Morgenstern n-player zero sum theory should be mentioned: strategic equivalence and normalization. Two games are said to be strategically equivalent for a given characteristic function if the same strategic considerations apply to either one. Thus, if we multiplied the characteristic function v by a positive constant c, we do nothing but change the scale of the unit of measurement (as in changing £ to \$). This is expressed as follows:

2.2.7 $$v(S) \overset{S}{=} cv(S) \text{ for all subsets } S \text{ of } I.$$

Observe that the constant must be positive or else it changes winning coalitions into losing ones (i.e. turns a game of strategy into a potlatch). Second, we can add a fixed payment a_i to the payoff to each player in S:

2.2.8 $$v(S) + \sum_{i \text{ in } S} a_i \overset{S}{=} v(S).$$

We can combine these two operations

2.2.9 $$v(S) \overset{S}{=} cv(S) + \sum_{i \text{ in } S} a_i$$

Now, with the formula of 2.2.9 we find that equivalence classes can be established for games, and we need only look at a convenient representative from each class. The normalization we referred to earlier in a parenthetical remark on discriminatory solutions was,

2.2.10 $$v((i)) = -1,\ v(I) = 0 \text{ for every } i \text{ in } I.$$

This normalization is discussed in some detail in von Neumann and Morgenstern, (where it is called the reduced form of the game), and is also described in Luce and Raiffa, section 8.3. Another normalization is also common. This is, 2.2.11 $v((i)) = 0,\ v(I) = 1$

Games in this normalization are said to be constant sum games. They can always be obtained from their strategically equivalent zero sum version.

2.3 Criticism of the theory, alternatives.

We have described the von Neumann and Morgenstern theory of n-player zero sum games in sufficient detail to consider a few of the criticisms which have been levelled against it. Some of these have already been mentioned, e.g. the large number of solutions which have been found for games of various n. (See Luce and Raiffa for a brief account, von Neumann and Morgenstern for a detailed, but incomplete, account of some specific cases, e.g. n=4). However, as we have noted, this abundance of solutions for a given n is not necessarily undesireable. The solution concept, after all, isolates the various standards of behaviour for a given number of players, and this result is of considerable value in itself. Therefore, the criticism about the abundance, i.e. non-uniqueness of solutions, is really a call for a theory with a greater resolving power than this one has. We feel, nevertheless, that although there is certainly a need for a theory which will for example, tell one which coalition can be expected to occur, there is also a need for a theory which isolates the possible underlying standards of behaviour, and this theory appears to do the job.[89] There is incidentally no shortage of partial results of alternative theories. The reader is referred to Luce and Raiffa, chapters

7, 8, 9 and 10. An excellent discussion (and collection of papers) can also be found in Tucker and Luce.

Another criticism, closely related to the first, is that the theory "seems to prejudge the problem". (Tucker and Luce[55] p. 2). In short, why should the players divide into two opposing coalitions? Von Neumann and Morgenstern go into the argument in some detail, and we shall not rehash it here. The reader is referred to their section 2.4.2 for a detailed mathematical, and heuristic, case for the coalition theory. We emphasize the word heuristic for a very good reason. It has appeared before in this thesis when we noted that the zero sum 2-player theory involves a heuristic limitation on the search for solutions. That is clearly the case in the n-player zero sum theory as well. (Although there is considerably less success in the later case). Indeed, even with their imposition of the zero sum two player assumption, they are still compelled to adduce *ad hoc* argumentation over the determinants of both the specific standard of behaviour, i.e. the specific solution V, and the occurrence of the specific coalition S within the V. In any case, the imposition of the reinterpretation of the n-player zero sum game as a two player zero sum coalition game certainly is a heuristic trick. This *is* a criticism only if the heuristic is a bad heuristic. Now, a good case can be made for the formation of the winning coalition S, but what of the formation of its compliment -S? Why should it form if it has no incentive to do so? The answer, although it may be a weak one, is that -S forms so that it can effectively exploit any mixed strategy in the game of S versus -S.

A third criticism of the theory is that it is static, when what is really needed is a dynamic theory. This objection is closely related to the criticism that the theory does not tell us which coalition can be expected to occur. Fourth, the theory has been criticized over the assumptions of an infinitely divisible, conserved and transferable utility quantity (e.g. money). This criticism could just as easily be (and has been) levelled against the 2-player zero sum theory. It is far too complicated to discuss here. The reader is referred to Luce and Raiffa. Finally, the theory has been criticized because it fails to make explicit the actual bargaining procedure upon which the whole justification for coalitions rests.

Even within the assumptions of the theory, numerical examples can be shown which shake one's faith in the "reasonableness" of the solution - no matter how one defines "reasonableness". Consider the following example of Kemeney's[25]:

$$v((1,2)) = 100, \ v((1,3)) = 10, \ v((2,3)) = 1.$$

The non-discriminatory solution has the following imputations (which the reader may verify directly with the formulae 2.1.9, 2.1.10, 2.1.11): (54.5, 45.5, -100), (54.5, -10, -44.5), (-1, 45.5, -44.5). In fairness to the solution concept, however, we might expect a discriminatory solution in a game with such wild assymetrices in the values of the various coalitions. Kemeny's example is by no means unrealistic, for actual social situations exist where the players have exactly one equal vote each, but the wealth that each represents is widely unequal. The U.S. Senate is an example. Each state popularly elects two senators, who are supposed to represent the interests of the state. (Since there are 50 states, this is a 100-player game). The senator from Nevada represents less than a half million persons, while the senator from neighboring California represents over 20 million persons, and one of the wealthiest political entities in the world. Of course, the discriminatory nature of the standard of behaviour in the Senate is notorous, and is institutionalized in a senority system which gives among other things, key committee assignments to those senators who have been in the Senate the longest. It is interesting to note who gets discriminated against - the "wealthy players", i.e. the senators from the wealthy (Northern and Western) states. These tend to change more frequently than do the senators from Southern rural states[26] where the electoral procedures are sometimes scandalously corrupt and depraved, systematically disenfranchising whole segments of the potential electorate, e.g. blacks in Mississippi.[27] It is interesting to speculate that if the standard of behaviour were changed to a non-discriminatory one, the senators from large states would still not do well, as Kemeny's example suggests (for a 3-player game). Nevertheless, they might be able to do considerably better than at present, where the large industrial states seem to be robbed to the maximum possible extent - witness the well known "decay" of the big cities in the U.S., virtually all of which are in the large industrial states.

Mention should be made at this point of the Shapley value[28] and of the suggestion by Shapley and Shubik[29] that this value can be applied to legislatures to determine the power distribution involved in legislative voting schemes. A key notion involved in this application of the Shapley value is that the formation of coalitions in legislatures is treated as random. Of course, we know that this is not the case, e.g. political parties interfere with pure randomness. In any case, it is possible to compute voting power ratios of, say, individual members of the U.S. House of Representatives, the U.S.

Senate, and the President, and the power ratio turns out to be 2:9:350. The formula for the Shapley value is as follows:

$$\phi_i(v) = \sum_{\substack{s \text{ a subset} \\ \text{of } I}} \gamma_n(s) \left[v(S) - v(S-(i)) \right]$$

where s is the number of elements in S and $\gamma_n(s) = \frac{(s-1)!\,(n-s)!}{n!}$ [Note that $v(S)-v(S-(i)) = 0$ for all S which do not include i].

This formula is based on three axioms. First, the individual values will be the same in any permutation of players of the original game. Second, the sum of the values ϕ_i for all i is precisely the value of the coalition of all players (in o,1 normalization). Third, if two independent games are combined, the two values for each player must be summed, i.e. $(\phi\ (v+w) = \phi v + \phi w$.

Another alternative theory, which is really merely an alteration of the von Neumann and Morgenstern theory, should be mentioned. This is the concept of ψ- stability advanced by Luce[30], and discussed in detail in Luce and Raiffa, Chapter 10. What ψ- stability involves is a "rule of admissable coalition changes", which defines for each coalition structure the set of permitted changes in the structure. The example given in Luce and Raiffa (p. 167) is a rule which allows a given coalition structure T to change by the addition of one more player. Thus for the structure T, i.e., all possible structures of T in a three player game, we have the following chart:

2.3.7

T	$\psi(T)$
((1), (2), (3))	(1), (2), (3), (1,2) (1,3), (2,3)
((1,2), (3))	(1,2), (3), (1,2,3), (1,3), (2,3)
((1,3), (2))	(1,3), (2), (1,2,3), (1,2), (2,3)
((2,3), (1,))	(2,3), (1), (1,2,3), (1,2), (1,3)
((1,2,3))	(1,2,3)

The main point about the function is that it must be imposed <u>ad hoc</u> onto the coalition structure. Determining the relevant ψ is an empirical problem.

So far we have mentioned the function ψ. Now we shall mention the part of the concept relating to stability. For a given structure T, an imputation $\vec{\alpha}$ and T are said to be ψ - stable (for a given game with characteristic function v and admissable coalition rule ψ) if two conditions hold. First, the imputation is effective for every S in T, i.e., $v(S) \leqq \sum_{i \text{ in } S} \alpha_i$ for every S in ψ (T). Second, $\psi_i > v((i))$ if player i is in a nontrivial coalition and receives the amount α_i.

2.4 Cybernetics, n-player, zero sum games, and experimental evidence.

We have already discussed the major connection between the two theories, i.e. the heuristic nature of the n-player solution concept with its zero sum two player interpretation. This use of the interpretation of the game into two opposing coalitions served the function of limiting the search for possible solutions. Second, a connection has been noted by Pask and von Foerster[31] between a self organizing system and an n-player zero sum game. The reader is referred to their experimental and theoretical discussions. Third, Pask[32] has noted the connection between the game theoretic concept of coalitions, in particular, expression 2.1.5 and the "social" activity of certain simple organisms, in this case, slime mold. The key idea is that slime molds can be thought of as making decisions to form coalitions which satisfy the property 2.1.5. Thus, they can be thought to constitute a self organizing system. Finally, a very readable and interesting experimental study on three player constant sum games has been presented by Riker.[33]

3. Non-zero sum n-player negotiated games.

3.1 Significance of non-zero sum games.

The previous two sections have dealt with games which exhibit a particularly restrictive but important property - the fact that the payments to all players sum to zero. An example of the two player case was suggested, namely a "game" of survival between a prey and its predator. And an example of a many player zero sum game (actually of its strategically equivalent constant sum version) was the U.S. Senate. The theory thus has some bearing on reality. However, its applications are limited; the overwhelming number of social situations are such that it is possible for all members of a society (i.e. all players of a game) to be mutually better or worse off if they pursue certain actions. Or it is possible for one or several members of society to be worse off in a manner quite incommensurate with the well being of the other members of society. Thus, not all decisions are exclusively concerned with problems of <u>distribution</u>. Many involve, at least in part, problems of production (amplification) and destruction. Now, if a game has even one outcome which exhibits the latter quality, which we shall call nonzero sum, than the entire game must be regarded as nonzero sum, even if all the other outcomes are strictly zero sum. On reflection, one finds that most social situations one nonzero sum in character. Thus, we shall now notice another feature of the theory which was not obvious before - its potential will become more apparent, and its shortcomings more glaring. Put another way, since the zero sum situations were basically artificial (e.g. actual parlor games) we had no objection to their solutions being in some sense artificial, e.g. good strategies in two player games and the isolation of non-descriminatory standards of behaviour in n-player - even though the "real world" games, such as legislatures, seem clearly to operate with discriminatory standards of behaviour. Since the "games" will begin to correspond with reality, we shall begin to want the solutions to do so as well. In this connection an important question must be raised.

3.1.1 Does knowledge of the theory falsify the conclusions of the theory? Von Neumann and Morgenstern went to great lengths to face and (successfully) answer this question. For it is clear that with two "rational" players, (rational in the almost super human sense required), knowledge of the theory certainly does not falsify its conclusions. Although, for the n-player theory, the "solution" is so inclusive, that one suspects any reality (among "rational" players) must be included. Now, a closely related question, has not been properly raised:

3.1.2 Does knowledge of the theory guarantee its confirmation? That is, is the theory a normatively self-fulfilling prophecy?

One suspects that this is the real question which von Neumann and Morgenstern have answered. This answer was satisfactory for zero sum two player games largely because they are inherently artificial and thus an artificial (in some sense) solution was satisfactory, i.e. even if the theory compelled players to use strategies they might not otherwise have used. But for nonzero sum games, which may closely correspond to real world events, a theory which is a self-fulfilling prophecy could turn out to be disastrous. If this turns out to be the case, and we shall see that, indeed, it does, we may have to attempt to disentagle the question of 3.1.1 from that of 3.1.2. We shall discuss solution concepts which attempts to do this, (those of Howard[34] and Shubik.[79]) However, we shall find that those attempts will merely alter the "reality" to fit the solution, and therefore will have limited applications. Thus we shall be compelled to consider a solution concept which cannot successfully answer 3.1.1, (but can successfully give a "no" answer to 3.1.2) for nonzero sum games, but nevertheless is successful at answering 3.1.1 for zero sum games. In other words, we shall show that for nonzero sum games, these questions are not relevant.

The author may be accused of presenting the immediately preceding discussion (on 3.1.1 and 3.1.2) somewhat prematurely, since we shall not discuss its main features until section 4. However, we feel that the reader should be alert to the problems now and consider all of the following discussion in their light (or shadow). For, as we shall see, one of the solution concepts discussed in section 4 is very closely connected (at least intuitively) with the solution concept which we shall now discuss.

3.2. The von Neumann and Morgenstern treatment of nonzero sum games - the extra player.

Consider the following two player game: Each player, chooses in secret a number from the set (1,2). If both choose the number "1", each player receives $\frac{1}{2}$. Otherwise, each player receives the amount -1, i.e. each loses 1. Notice that if each chooses "1", and receives α_i, then $\sum \alpha_i = 1$. However, if each chooses 2, or if one player chooses "2" and the other chooses "1", $\sum \alpha_i = -2$. Since $1 \neq -2 \neq c$, this game is neither zero sum, nor constant sum. It is clearly nonzero sum. In this game, a "good way of playing" seems perfectly obvious and trivial, i.e. each player should choose "1". However, we have introduced no game theoretic decision making procedures by which this "obvious" solution can be obtained. Von Neumann and Morgenstern's entire theory, as we have seen, is based either directly on zero sum two player games or on the reinterpretation of an n (≥ 3) player zero sum game as a zero sum two player coalition game by means of the characteristic function. What von Neumann and Morgenstern have proposed for this game is, not surprisingly, another zero sum reinterpretation. They suggest that an "extra" player be added to the set of players. Thus an n-player game becomes an n+1 player game. The extra player, i.e. player n+1, cannot enter into the bargaining or pre-game discussions in any way whatever; nor does he have any strategic choices, i.e. control over any of the variables in the sequence $d_1, \ldots, d_r$ which constitute the extensive play of the game. However, he is assigned an amount by the function $f_{n+1}(d_1, \ldots, d_r)$, and in fact, it is

3.2.1 $$f_{n+1}(d_1, \ldots, d_r) = -\sum_{k=1}^{n} f_k(d_1, \ldots, d_r)$$

Thus, we see that the addition of the extra player turns this nonzero sum n-player game into an n+1 player zero sum game. Having obtained this zero sum game, we can apply the theory of the previous sections to obtain its solution.

The complication is this - player n+1 is a total fiction who cannot participate in any way in the bargaining preceding the formation of coalitions, etc. However, we saw that this situation occured in the case of zero sum three player games as well. This happended when two players chose to discriminate against a third player and assign him some amount $\frac{1}{2} > c \geq -1$. If we restrict ourselves to discriminatory solutions, with player n+1 the victim, the zero sum n-player theory is adequate for defining some kind of a solution. Two questions remain. First, what will be the exact value of c? Second, what will be the divisions of payments between players 1 and 2? The first question can be answered quite simply, we always have $c = -1$. That is, the totality of all real players always acts to maximize the $v((1, \ldots, n))$. This is always assumed to be possible because of the assumption of absolutely perfect bargaining, negotiations and communications among the real players preceding the actual play. The second question, as we recall, was left open in the discussion of the discriminatory solutions. For 2 player games, each obtains $\alpha_i \geq -1$, but otherwise they are in pure opposition to each other for the further division of the money. They play a zero sum two player game between themselves to determine the final distribution. This, in summary, is the von Neumann and Morgenstern theory of negotiated nonzero sum games. The exact, final distribution is always left to considerations which are extra theoretical - for example, who is the better negotiator. Since real players can pay each other compensations - side payments - they can arrive at outcomes, i.e. coordinate choices, which achieve a maximum value for the coalition S. Threats, bluffs, promises, punishments - all enter into the negotiation procedure. Clearly, no real player need accept less than the $v((i))$, i.e., what he can get without joining a coalition of real players. The set of negotiated outcomes, the <u>negotiation set,</u> is composed of all outcomes with $\alpha_i \geq v((i))$, and for which the $\sum \alpha_i = -v((n+1))$, where it will be recalled, $v((n+1))$ was assumed to be at its minimum. The condition we have just described is called Pareto optimality:

3.2.2. $$\sum_{i=1}^{n} \alpha_i = -v((n+1)), \text{ for } v((n+1)) \text{ at its minimum.}$$

Thus, the negotiation set is composed of all outcomes satisfying 3.2.2. In the example of a nonzero sum game which we discussed earlier, the only Pareto-optimal outcome was designated by the payoff vector $(\frac{1}{2},\frac{1}{2},-1)$. Observe that the final number of this vector may be omitted since it assigns a value to a completely fictional player. We could have written the payoff equally well as $(\frac{1}{2},\frac{1}{2})$. This is an ordered pair of numbers where the first number is the payoff to player 1 and the second is the payoff to player 2. The game discussed earlier could be written in the form of a matrix:

Figure 3.2.1

Player 2

		1	2
Player 1	1	$\frac{1}{2}, \frac{1}{2}$	-1, -1
	2	-1, -1	-1, -1

Writing 2-player nonzero sum games in this way is often convenient, as we shall see in the next section.

3.3. An example, and general soltuions for the 2-player case.

If the 2-player nonzero sum negotiated game is <u>not</u> in normalized form, as was the preceding example, its solution in the sense we have just considered can nevertheless be fully described, and from the preceding discussion is seen to be the set of all ordered pairs (α_1, α_2) which satisfy the following conditions:

3.3.1
$$\alpha_1 \geq v((1))$$
$$\alpha_2 \geq v((2))$$
$$\alpha_1 + \alpha_2 = v((1,2))$$

(These formulae are from von Neumann and Morgenstern section 60.2.2) The formulae of 3.3.1 can be used to compute the solution to any game of the type we have been describing.

For example,

Figure 3.3.1

Player 2

Player 1	1	2
1	0,0	-3, 2
2	-1, 3	5, -1

This is really two matrices:

Figure 3.3.2

For player 1

1	0	-3
2	-1	5

Figure 3.3.3

For player 2

1	0	3
2	2	-1

Observe that the matrix has been transposed, i.e. the ij<u>th</u> element is replaced by the ji<u>th</u> element. This has been done so that the maximin, by convention an operation for the <u>row</u> player, can be computed.

The v((i)) can be easily computed. It is the maximin value of each matrix, i.e. that value below which a player can never be reduced no matter what the other does.

For player 1, it is $v((1)) = -\frac{1}{3}$ with the mixed strategy $(\frac{2}{3}, \frac{1}{3})$.

For player 2, it is $v((2)) = 1$ with mixed strategy $(\frac{1}{2}, \frac{1}{2})$. The $v((1,2))$ is the maximum amount obtained by summing each set of ordered pairs of the matrix of figure 3.3.1 (because only this matrix is defined in terms of the ordered pairs (α_1, α_2)). Thus, the $v((1,2)) = 4 > 2 > 0 > -1$. And the solution is the system of all ordered apirs (α_1, α_2) satisfying the following:

3.3.2

$$\alpha_1 + \alpha_2 = 4$$

$$\alpha_1 \geq -\tfrac{1}{3}$$

$$\alpha_2 \geq 1$$

The difference between the maximum which both can obtain if they cooperate fully, 4, and the minimum both obtain if they fail to cooperate is

$$4 - (1 - \tfrac{1}{3}) = \frac{10}{3}.$$

If we represent (following McKinsey) the outcome of the negotiations as a number $1 \geq \theta \geq 0$, we can write the α_1 as follows:

3.3.3

$$\alpha_1 = -\tfrac{1}{3} + \frac{10}{3}\theta$$

$$\alpha_2 = 1 + \frac{10}{3}(1 - \theta).$$

3.4 Discussion and criticism of the solution concept, alternatives

There are four main criticisms which can be levelled against this part of the theory. One criticism deals with the heuristic nature of the solution. Another concerns the ad hoc nature of the role of negotiations; the third concerns the crude way in which the stop at the characteristic function can gloss over glaring asymmetries in a game; and the fourth centers on the vital role of interpersonal comparisons of utility.

First, we shall discuss the fact that the solution is clearly a heuristic trick. This may or may not be a criticism, depending on one's viewpoint. The zero sum interpretation technique is a method used by von Neumann and Morgenstern to obtain a more or less unified theory of games (see their discussion on this point, section 56.5). Indeed, this solution concept is really the solution to the n+1 player zero sum extension game to the original n-player game. Von Neumann and Morgenstern's ultimate justification for their procedure, in addition to that of preserving a unified theory, is based on its economic applications. That is, they argue that the value of their procedure must finally rest on its success or failure in "real world" economic and socialogical applications. This is, of course, the last defense for an intuitively "unnatural" procedure. But it is a reasonable defense if the theory does have successful applications since one's "obvious" inituitive assessment of a situation must certainly be wrong in general, or else we would have had highly successful mathematical social theories long ago.

The second comment on the theory is that the pre-game negotiations, which play such a decisive role in the von Neumann and Morgenstern theory are not part of the theory, and since the final distribution depends on the outcome of the negotiations, the solution seems hopelessly imprecise. Moreover, the credibility of threats is assumed in the theory, for "since there exists perfect information for all players, there can never be any doubt". (Von Neumann and Morgenstern, p. 541). Not everyone agrees with von Neumann and Morgenstern on this point, and an interesting book by Schelling[35] goes into the point in some detail. Luce and Raiffa also summarize the problem very well (section 6.4). The problem boils down to this - if a threat has to be carried out, the threatener may be worse off than he would have been if he had not threatened in the first place. Thus, a threat may not be credible. But, if threats and promises are not fully believable, we no longer have a situation of perfect preplay negotiations, and therefore the relevance of the whole theory is called into question. What is needed is clearly a theory of negotiations, and what is also obviously suggested is the need for a theory of non-negotiated nonzero sum games as well. Now, von Neumann and Morgenstern have offered no extension of their zero sum heuristic, to the case of non-negotiated nonzero sum games. One such extension has recently been developed by the present author[36] and will be described in detail in section 7 of this thesis.

A third criticism of the solution concept centers on the way the characteristic function ignores particularly striking asymmetries of a game. The following example from McKinsey illustrates the point:

Figure 3.4.1

Player 2

		1	2
Player 1	1	0,-1000	10,0

The first player has only one "choice", the second player has 2 choices. Obviously, the game favors player 1, who can never do worse than zero, and may obtain 10. Player 2, who must make the choice, can try to extract some of player 1's 10 in exchange for choosing choice 2, but any threat to choose choice 1 would presumably not be believed since it involves such a staggering loss to player 2. Thus, there seems no way that player 2 can threaten player 1 and obtain some share of the money. The characteristic function of the game does not show this fundamental asymmetry:

3.4.1
$$v(\emptyset) = 0$$
$$v((1)) = v((2)) = 0$$
$$v((1,2)) = 10$$

A solution is any (α_1, α_2) with $\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 10$.

Another criticism, which should certainly be mentioned even though we have deliberately suppressed the problem throughout this paper, is the question of interpersonal comparison of utility. Much of the work that has been done on negotiated games has been done to avoid interpersonal comparisons. The reader is referred to Luce and Raiffa for the relevant discussion.

Alternative approaches to the theory of negotiated nonzero sum games exist and are discussed in Luce and Raiffa. Most of these are called arbitration schemes and involve axiomatic formulations which will give a unique payoff vector which satisfies various criteria of fairness. The reader is referred to the discussion in Luce and Raiffa, and to Braithwaite.[94]

4. Non-negotiated nonzero sum games

Although much more detail could be added to the discussion we have just given, no von Neumann and Morgenstern solution to any other type of game could be presented, for their book ends, more or less, with the section we have just described. As we have said already, however, a most important type of game (perhaps <u>the</u> most important type) has not been discussed - nonzero sum games where the players cannot negotiate before the play or, as has been suggested, cannot believe each other even if they <u>can</u> negotiate. A widely discussed theory for this type of game does exist, however, and we shall take it up now. Our discussion will be primarily concerned with 2-player games, although some of the considerations will be enlarged to include n-player games as well.

The 2-player zero sum, n-player, and nonzero sum game solution concept was, as we have said repeatedly a zero sum interpretation solution concept. The procedure we will now consider is not of that type at all. It is based on a quite different definition - that of the <u>equilibrium point</u>.

4.1 Equilibrium points.

An equilibrium point is a payoff function to player i on a set of pure strategies S and defined as follows:

4.1.1 An n-tuple $(S_1, \ldots, S_i, \ldots, S_n)$ is an equilibrium point if

$$M_i(S_1, \ldots, S_i, \ldots, S_n) \geqq M_i(S_1, \ldots, \bar{S}_i, \ldots, S_n)$$

for $i = 1, \ldots, n$.

Definition 4.1.1 says that if all players use their respective strategies $(S_1, \ldots, S_i, \ldots, S_n)$, and thus player i uses strategy S_i, player i cannot increase his payoff by changing to some other strategy $\bar{S}_i$. If this holds for every player i, then the strategy n-tuple $(S_1, \ldots, S_n)$ is a set of equilibrium strategies, and is also called an equilibrium point. This definition of an equilibrium point is due to Nash,[5] and so these equilibrium points are often called Nash equilibrium points. The corresponding definition for mixed strategies can be obtained merely by substituting σ_i for S_i where σ_i is the mixed strategy of player i. Nash has proved that for every finite game at least one equilibrium point exists in mixed strategies.

Observe that in the definition of 4.1.1 we said nothing about whether or not the game was zero sum or nonzero sum. The reason is that the definition covers both categories and therefore may be viewed as a more general concept than that of optimal maximin or minimax strategies. Nash's theorem produces identical results to the minimax theorem for 2-player zero sum games and seems to extend these results to other types of games. Two difficulties may be mentioned now, and both apply to nonzero sum games and n-player zero sum games. First, a game may have more than one equilibrium point, but these points may not be worth the same to each player. Thus, if M_i^1 is the payoff to player i from equilibrium point 1, and M_i^2 is the payoff to him from equilibrium point 2, each of the following is possible: $M_i^1 \gtreqless M_i^2$. Since the equilibrium points may be worth different amounts to different players, one player may prefer one equilibrium point and another player may prefer some other equilibrium point. Secondly, the strategies in the equilibrium points may not be interchangeable, thus, if one equilibrium point is $(S_1, \ldots, S_i, \ldots, S_n)$ and another is $(r_1, \ldots, r_i, \ldots, r_n)$, the n-tuple $(S_1, \ldots, r_i, \ldots, S_n)$ may not be an equilibrium point. This problem could be, presumably, obviated if the players were able to agree beforehand on which equilibrium point n-tuple be used. However, we have assumed a total absence of negotiation, bargaining etc., and there thus seems to be no way by which the n-tuple could be chosen.

One advantage of speaking in terms of equilibrium points is that there is no need to assume interpersonal comparisons of utility. Each player chooses his strategy on the basis of his own payoff, and he is concerned not with the payments to the other players, but with the strategies, σ_i, of the other players. Now, of course, these σ_i are arrived at on the basis of each player's utility, and thus, knowledge of the σ_i usually does in fact require knowledge of the other player's utilities. [We shall see how this works in section 4.2].

Luce and Raiffa discuss the significance of this definition of equilibrium, and their discussion (section 5.7) is worth taking note of. They seem to feel that the definition an equilibrium point 4.1.1 corresponds to the notion of equilibrium in society in the following sense: One can imagine the members of society "floundering about" with various courses of action (strategies) until the society finally settles into a set of strategies where no one sees any margin in opposing the general tide of society's opinion. In this way, a social equilibrium is created which seems to correspond strikingly with the definition 4.1.1. It is interesting to note the difference between this notion of social equilibria and the von Neumann and Morgenstern idea of sets of imputations as expressing a standard of behaviour where the imputations within the set are in equilibrium. The Nash concept of equilibrium is a much more rigid notion. In the von Neumann concept, opposing a particular configuration of social forces is not going outside the equilibrium, unless it also challenges the underlying standard of behaviour. One is strongly tempted to see in these two definitions of stability - von Neumann's for negotiated games and Nash's for non-negotiated ones - two different historical periods - von Neumann's reflecting the musical chairs governmental changes of the Weimar Republic, Nash's reflecting the totalitarian rigidity of the Joe McCarthy era in the U.S.

Luce and Raiffa give various definitions of "solutions" to two player non-zero sum games, and they should be noted.

A 2-player non-negotiated game is solvable, in Nash's sense, if every pair of equilibrium pairs are interchangeable. If a game is solvable in this sense, its solution is its set of equilibrium pairs. Thus we see that whether or not a game is defined to have a solution is quite a different matter from whether or not the game has an equilibrium point (which it always does).

It is clear thatequilibrium points may dominate one another. In the two player case, we say that (S_1, S_2) jointly dominates (r_1, r_2) if $M_1(S_1, S_2) > M_1(r_1, r_2)$ and $M_2(S_1, S_2) > M_2(r_1, r_2)$. If an equilibrium pair is not jointly dominated by another one, it is said to be jointly admissable. Thus another possible definition of a solution (for 2-player games) is the following: A non-negotiated game has a solution in the strict sense if an equilibrium pair exists among the jointly admissable strategy pairs, and, all jointly admissable equilibrium pairs are interchangeable and equivalent.

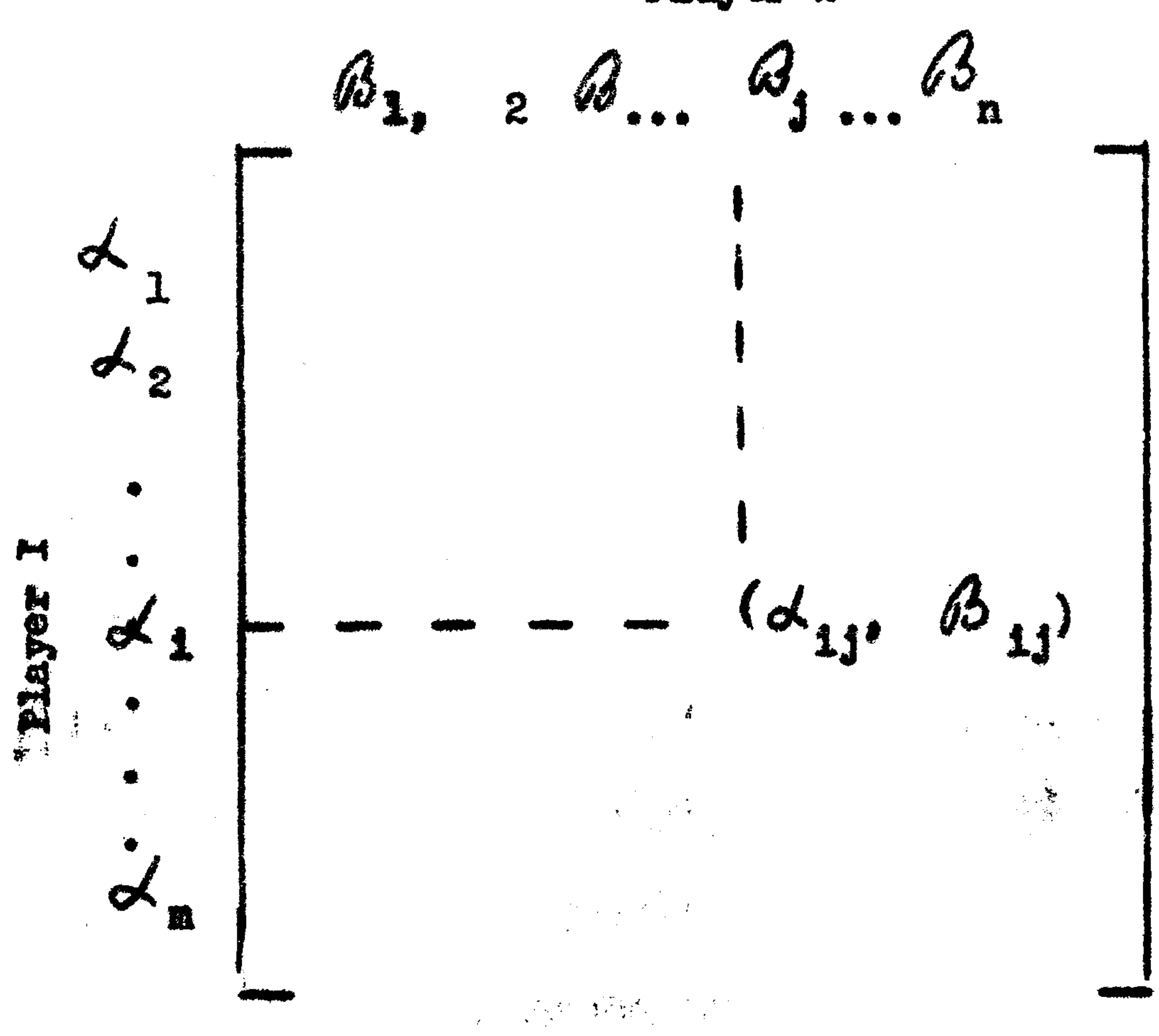
Finally, we might note that we have said nothing about how the equilibrium points can be found, we have only said that at least one exists. A constructive proof for the existence of an equilibrium point for any 2-player nonzero sum game is given by Lemke and Howson.[104] Since the proof is constructive, it provides a technique for finding one equilibrium point for any two player nonzero sum matrix.

4.2 Bimatrix form, examples.

We have already looked at an example of a negotiated nonzero sum game matrix (figure 3.3.1) and we can observe that the same matrix could have been used to describe the game if it had been non-negotiated. We can easily generalize this matrix form:

Figure 4.2.1

Player 2



Game matrices in this form are sometimes called game in bimatrix form (Owen). Observe that the element $(\alpha_{ij}, \beta_{ij})$ is an ordered pair.

We shall consider a few examples of non-negotiated nonzero sum games. There are three classic examples, each of which has a parable attached. The first, usually known as "battle of the sexes" clearly was given its parable in the days before the women's liberation movement got underway: A husband wants to go to a boxing match with his wife, who wants to go to the ballet with her husband instead. Each prefers a decision to do one or the other together over a no-agreement outcome.

The payoff matrix, from Luce and Raiffa (section 5.3) is as follows:

Figure 4.2.2

Husband \ Wife	boxing match β_1	ballet β_2
boxing match α_1	2,1	-1, -1
ballet α_2	-1, -1	1,2

Observe that each of outcomes (α_1, β_1) and (α_2, β_2) is an equilibrium point, but the utility values for the two points are not equivalent for the husband (or the wife). This may be interpreted to mean that, if the husband somehow found himself at the ballet with his wife, he would not want to leave her there so that he could go to another part of town to watch the boxing match. A similar interpretation could be given for the wife if she found herself at the prize fight. But there is no indication how they should find themselves at either the fight or the ballet in the absence of pre-play negotiation. Observe that say, the husband might want his strategy to be "found out" by the wife providing she could not see to it that he knew her strategy as well. Thus the husband might be tempted to ask a good friend to phone her, and deliver the message that the husband was closeted in an important business conference at an undisclosed restaurant and would meet his wife at the boxing match at eight. This subject is discussed in some detail by Schelling under the label <u>coordination</u> of strategies.

Luce and Raiffa observe that heavy handed methods (such as the strategem of the preceding paragraph) may actually change the utility entries in the payoff matrix. For our purposes, however, we shall assume the payoffs to be fixed throughout any pre-game message delivering. There is also a mixed strategy equilibrium point to this game. The husband uses the strategy $(\frac{3}{5}, \frac{2}{5})$, the wife uses the strategy $(\frac{2}{5}, \frac{3}{5})$. Observe that these are different from the maximin mixed strategies, which are $(\frac{2}{5}, \frac{3}{5})$ and $(\frac{3}{5}, \frac{2}{5})$ respectively. A mixed equilibrium strategy holds the <u>other</u> player to a certain amount, where as a maximin strategy guarantees oneself a certain amount. What is interesting in this distinction is that the equilibrium strategy does not, supposedly, involve interpersonal comparisons of utility. But, to compute the equilibrium mixed strategies, one must <u>know</u> the other player's utilities. On the other hand, to compute the maximin, one need only know his own utilities.

Another well known example of a non-zero sum game is "chicken", made famous in a Hollywood movie entitled "Rebel Without a Cause", starring James Dean. This movie is strongly recommended (it sometimes appears on late night television) for a visual demonstration of the following game: Two cars of equal speed are driven at full throttle toward a cliff. The first driver to jump out of his rapidly accelerating car is given the epithet "chicken". Both drivers, however, are expected to jump by the last possible moment, and if this happens, they are both considered equally courageous. However, if one jumps out first, the other is not only courageous but dominant (assuming he jumps out before the car goes over the cliff) and thus is totally triumphant. If, however, they both wait too long, they both go over the cliff and are not only dead but foolish. In the Hollywood version, a random element was thrown in just to make things interesting, and one of the drivers - not James Dean - inadvertently went over the cliff with his car because he caught his sleeve on the door handle. A typical payoff matrix for chicken, from Ells and Sermat[37] is the following:

Figure 4.2.3

jump	5,5	3,7
wait	7,3	0,0

Observe that the strategy combinations (α_1, β_1) and (α_2, β_2) are not equilibrium points. However, (α_1, β_2) - player 1 is chicken and player 2 is dominant - and (α_2, β_1) - player 2 is chicken and player 1 is dominant - are equilibrium points. There is also a mixed strategy equilibrium point composed of the two equivalent strategies: $(\frac{3}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{2}{5})$. This is obtained, for example for player 2, as follows: We write down player 1's utility values, and compute the mixed strategy:

Figure 4.2.4

Player 2

Player 1	y	1-y
x	5	3
1-x	7	0

$$v_1 = 5xy + 3\,(x(1-y)) + 7(y(1-x))$$

$$v_1 = -5xy + 3x + 7y =$$

$$v_1 = x(-5y+3) + 7y$$

Observe that if player 2 chooses $y = \frac{3}{5}$, the coefficient of x is zero, and hence with $y = \frac{3}{5}$, $v_1 = \frac{21}{5}$. Thus, if player 2 uses $y = \frac{3}{5}$, player 1 cannot increase his own payoff with any change of strategy. (Player 1 similarly can hold player 2 to $\frac{21}{5}$ by setting $x = \frac{3}{5}$). A mixed strategy solution has a certain plausibility with regard to the game of chicken since one strongly suspects that the outcome is by no means deterministic before the game begins. Whether the empirically true mixed strategy is the one we have just found is another matter altogether. In fact, it clearly is not; see reference 37.

We should observe that we have been able to describe - in a very crude way - two actual social situations with nonzero sum non-negotiated games, and there seems to be some intuitive accuracy in the description of the situation using non-negotiated game matrices. Whether there is any intuitive or empirical accuracy in the outcome of the situation as defined by the equilibrium point concept is, however, an open question. For example, the mixed strategy equilibrium pair does not appear to correspond with any empirically observed frequencies in games of chicken performed under laboratory conditions.[37]

Before discussing this point further, we shall look at the most widely discussed nonzero sum game.

5. Prisoner's dilemma games: a paradox.

The third nonzero sum game parable (from Luce and Raiffa) is as follows: A prosecuting attorney (District Attorney or Queen's Council) has under interrogation two prisoners whom he strongly suspects are guilty of a crime, but he lacks proof. The only way he can convict is with a confession. The prosecuting attorney puts the suspects in seperate rooms so they cannot communicate with each other and then suggests to each separately that the courts will "go easier" on either one who confesses, if his partner keeps quiet. The one who keeps quiet if the other confesses gets the maximum sentence, i.e. "takes the rap". If neither confesses the D.A. can still prosecute on some trumped-up charge, e.g. concealing evidence, and this payoff is equal to or worse than the payoff to the long confessors. If both confess, the courts will still show some leniency, but each will get an intermediate sentence, e.g. perhaps life imprisonment instead of the death penalty.

Each suspect could reason as follows - "If I confess then I get the lightest sentence, for turning states' evidence if the other guy keeps quiet. On the other hand if he confesses as well, I get a long prison term. But if I don't confess, and he does, I get the electric chair. Of course, if he also doesn't confess, I get the punishment associated with the trumped-up charge. I guess I had better confess".

Since they both can reason that way, they both confess, and get life in prison. Now, had they both been irrational, i.e. not gone through the reasoning process we have just described, they both could have kept quiet, and received the sentence associated with the trumped-up charge. That is, two irrational players could do better than two rational ones! This is the paradox of "rational" decision making. It is sometimes said that it contrasts individual rationality with collective rationality (Rapaport[38]). This interpretation is relevant only if one accepts the equilibrium point concept as a definition of rationality. Since, as we shall see, we feel that the entire argument is misplaced and that the equilibrium point concept should be ignored altogether, we shall not consider the merits of this interpretation of the prisoners dilemma (i.e. of group versus individual rationality).

Notice, incidentally, that the conviction of a suspect is quite independent of his guilt or innocence, but depends entirely on the action of his "partner".

5.1 Payoff matrix.

A numerical example of the problem is the following:

Figure 5.1.1

Prisoner 2

Prisoner 1	β_1 confess	β_2 not confess
α_1 confess	-1,-1	2,-2
α_2 not confess	-2,2	1,1

We see that (α_1, β_1) is an equilibrium point - and the only one - because, having arrived at it, neither player has any incentive to change his strategy. Observe also, that, for player 1, row 1 strictly dominates row 2, and for player 2, column 1 strictly dominates column 2. Thus, the choice of row 1 is player 1's maximin strategy, as is the choice of column 1 by player 2. Thus, we see that the dilemma holds whether we use maximin or equilibrium point concepts. Clearly, the problem allows for more than one numerical representation, and in fact, it occurs whenever certain inequalities hold. The following matrix form, where the numbers refer to the combination of row and column, will be helpful:

Figure 5.1.2

Player 2

Player 1		
	$u_i\ (1,1)$	$u_i\ (1,2)$
	$u_i\ (2,1)$	$u_i\ (2,2)$

$i = 1,2$

The inequalities of the prisoner's dilemma are as follows:

5.1.1 For player 1,

$$u_1\ (1,2) \geq u_1(2,2) > u_1(1,1) > u_1(2,1)$$

For player 2,

$$u_2(2,1) \geq u_2(2,2) > u_2(1,1) > u_2(1,2).$$

N-player versions of the game are possible, but each player's payoff would have to be represented on a separate matrix, where row 1 (confess) strictly dominates row 2 (not confess): An example of the matrix for player 1 in a 3-player game is the following:

Figure 5.1.3

Player 1		2&3 confess	2 confesses	3 confesses	no one else confesses
	confess	0	3	3	6
	not confess	-6	-3	-3	2

Clearly, we could construct matrices for the payoffs, to a single player in an n-player game for arbitrarily large n merely by adding additional columns.

5.2 The paradox

We have observed that two irrational players could do better than two rational ones in this game. Recalling our questions 3.1.1 and 3.1.2, we may observe that the theory is designed to give a "no" answer to the first question; Does knowledge of the theory indicate to players that they should not follow the theory, i.e. falsify the conclusions of the theory? It succeeds in doing this so well, that it gives a "yes" answer to the second question: Does knowledge of the theory guarantee the conclusions of the theory, i.e. is the theory a self-fulfilling prophecy?

We have also observed that for zero sum two player games there is no reasonable objection to a "yes" answer to the second question. This is so for two reasons. First the value v although perhaps contrived in the sense that it may not be in the minds of real game players is nevertheless "natural" in the sense of describing totally efficient play when the player's interests are totally opposed. Second, since zero sum two player games are essentially unnatural situations, we do not object to the solution being in some sense unnatural.

But a self-fulfilling prophecy for nonzero sum games becomes unacceptable for precisely the reasons that it is acceptable in zero sum games. That is, there may be joint maxima to be realized or joint minima to be avoided, and we might expect superhumanly efficient players to be able, at least to a small extent, to take advantage of them. Secondly, since the situations the games describe may be in some sense "real", we do not want "knowledge" to leave us in a worse position than ignorance would have done. Rather than allowing us to further control our environment, the "knowledge" appears to damn us hopelessly.

We should take stock of the equilibrium point notion for nonzero sum games. We have seen that for a given game there may be two or more equilibrium points which may not be equivalent or interchangeable. Now we have seen that they may not even be desireable! Still they are presumably self fulfilling prophecies and desireable or not, if the prophecies are true of the world, we must deal with them. Thus, the real question seems to be, is this self fulfilling prophecy really fulfilled? If not, then considering the formal difficulties, we may as well ignore them and find some other solution concept for nonzero sum games.

5.3 Empirical evidence

The data is still incomplete, but it is beginning to look like prisoner's dilemma outcomes simply do not occur, in general, in prisoner's dilemma situations. To show this, we shall very briefly examine one historical event, one sociological phenomenon and some psychological experiments.

Perhaps no situation appears to typify a prisoner's dilemma better than an arms race between two nations. If we substitute "arm" for "confess" and "not arm" for "not confess" in the matrix of figure 5.1.1, we seem to get a crude approximation of an arms race situation. The point has been discussed in a number of books by experts in international relations, e.g. Morton Kaplan.[39]

Without going into historical detail on arms races, a topic far beyond the scope of this thesis, we need only observe that even within the crude policy dichotomy of arm-not arm, the "arm" choice does not receive a strategy probability weighting of 1.

"To a considerable extent, the governments are carrying out arms control measures on a unilateral basis. The limitation of military budgets, the choice of weapons systems designed primarily for defense or retaliation rather than for attack (e.g. mobile Minuteman missiles and the Polaris submarines), the development of communications and command systems designed to slow down the response to ambiguous danger signals - all of these represent some aspect of a self-imposed control".[40]

At least the first of these items, budgetary restraint, amounts to a "not arm" choice some of the time. The present U.S. - Soviet arms race, however catastrophic it is, is still not as bad as it would be if both sides unfailingly adopted every weapons proposal which came along. Thus it is not a true prisoner's dilemma; the motivation is slightly different. If this is the case, the international system might be expected to be somewhat less conflict oriented than might be suggested by the prisoner's dilemma intuition. A recent study seems to confirm precisely this point.[41]

Our conclusion would seem to be that the motivation suggested by the prisoner's dilemma problem is inaccurate and therefore misleading.

We do have a most interesting case history in recent times which seems to suggest that, at least for one side, the prophecy of the prisoner's dilemma was false. This is the development of the atomic bomb. We know that the allies (that is primarily the U.S.) developed an atomic bomb out of fear that the Nazi's might do so first.[42] Thus the U.S. seems to have viewed itself as in a prisoner's dilemma. What about the Nazis? The evidence is inconclusive, but we do know that, for _whatever_ reason, they did _not_ attach overriding importance to the fear of the U.S. first developing an atomic bomb, because they did not seriously begin work on their project until quite late in the war. We do _not_ need to know their reasoning; we only need to observe that whatever the reason - they did not act as if they were in a prisoner's dilemma even though they were aware of the U.S. Manhatten project!

Please observe that we are _not_ presenting any historical thesis, rather we are merely trying to offer alternative intuitions to what has become, after more than twenty years of Cold War, established dogma.

The second example of a prisoner's dilemma is that of a panic of escape from a crowded burning theatre. This subject is discussed in some detail in an account by Roger Brown.[43] It is clearly an n-player version of the prisoner's dilemma with "rush exits" substituted for "confess" and "take turns" substituted for "not confess". The only point we need make is that in fact panics do not always occur in fire situations in crowded buildings. Brown discounts the influence of the members of the crowd being able to watch each other for signs of bolting since he argues, quite persuasively, that with "the smoke and confusion it would be hard to tell what anyone else is up to". Brown suggests two reasons for the fact that panics do not always occur in situations where they might do so. First, the members of the theatre audience may not immediately perceive the situation as a prisoner's dilemma, i.e. "it takes time for the values in the matrix to become clear to everyone in the house". Second, the fire may be perceived as of such minor proportions that it does not really constitute a prisoner's dilemma. Both of these explanations seem plausible, although the first considerably more so than the second, which, in fact, would seem to contradict the first. Indeed, we shall extend Brown's intuition: Perhaps prisoner's dilemma situations are not always perceived as prisoner's dilemmas by the participants. To this extension we might add that since the dilemma is a result of a "solution" to a game, i.e. a decision rule applied to a payoff matrix which satisfies certain inequalities, perhaps the decision rule which in fact is applied does not invariably produce the dilemma. In short, whatever decision rule(s) is (are) applied by real human beings, only sometimes do these decision rules produce prisoner's dilemma outcomes, i.e. joint minima. All of this suggests for a descriptive theory, the development of a decision rule which produces prisoner's dilemma outcomes in some situations but not in others. One strongly intuits that the occurance of the "panic" may depend heavily on the exact numerical entries in the payoff matrix. The basic inequalities of 5.1. may not tell the whole story. For a non-game theoretic discussion of panics, see Schultz.[91]

At the level of sociological or historical analysis, there seems to be no easy way to investigate this conjecture. There is, however, a great deal of evidence accumulating in the field of experimental social psychology which confirms the conjecture. This evidence is the accumulation of data from actual experiments where subjects, usually but not always undergraduates, actually play prisoner's dilemma for money or points (but not jail sentences, at least not in the experiments). Admittedly, as is often the case, there has been a wide variety of phenomena under investigation. Experiments have been conducted, to mention only a few variables, using different sets of instructions, lengths of play, communication situations, populations of subjects (age, sex, race, nationality, large university, small college), scoring units, degrees of friendships or disliking among the subjects, types of strategies from the (simulated) "other player", and payoff matrices. Great care must therefore be taken when speaking about "the" empirical results. Nevertheless, throughout most of the pure gaming experiments (as opposed to "simulation" experiments where the subjects are told to pretend to be someone else, e.g. Khrushchev) a certain standard set of instructions and gaming situation has emerged, often in the control group. The subjects generally do not know who one another are, and do not communicate with each other before or during the experiment. They are seated individually in booths, or at partitioned tables, which display a prisoner's dilemma matrix or payoff description, and two switches - one to indicate the cooperative choice (not confess) the other to indicate the non-cooperative choice (confess). These choices are almost never labelled in a way to suggest cooperative or non-cooperative connotations, but are usually designated "red" or "black" or "a" or "b". The instructions given to the subjects are usually designed to be utterly devoid of game-like connotations. The game is usually repeated, and, after each play, the subjects are informed of the choice of the other player, and, in some way, the winnings and losses involved in the play are called to the subject's attention. Each player is then allowed to choose again (if the game is repeated). After the experiment, the subjects are given some amount of money (rarely more than a few dollars; and usually less than two dollars) which generally depends on their overall winnings, and losses in the repeated game. In most experiments, men play against men and women against women.

The results of these experiments should, at a minimum, raise doubts on the empirical relevance of the prisoner's dilemma intuitions. The fact is, in actual play, even if the game is played only once, subjects almost never choose the non-cooperative choice (confess) or the cooperative one, every time. In fact, if one averages the choices for the subjects in a single or repeated play experiment the fraction of cooperative choices (as opposed to cooperative outcomes, which require simultaneous cooperative choices) is amazingly consistent for a given payoff matrix and number of plays (say up to 40 plays). For example, the following matrix has been used in at least nine[44] separate gaming experiments and the typical fraction of cooperative choices[45] is roughly 20 to 40% by, say, 40 plays. Typically, the fraction begins close to 50% and <u>declines</u> to the overall averages we shall be giving.[77]

Figure 5.3.1

Player 2

Player 1	Defect	Cooperate
Defect	1,1	5,0
Cooperate	0,5	3,3

Interestingly, the prisoners dilemma matrix of figure 5.1.1 has also produced roughly the same fraction of cooperative choices.[46] Two other matrices which have produced the same fraction are the following:

Figure 5.3.2

Player 2

Player 1	Defect	Cooperate
Defect	1,1	4,0
Cooperate	0,4	3,3

Reference 47

Figure 5.3.3

Player 2

Player 1	Defect	Cooperate
Defect	-5,-5	10,-10
Cooperate	-10,10	5,5

Reference 48

Interestingly, in one of the very rare one trial runs of matrix of figure 5.3.3, the fraction of cooperative choices was exactly $\frac{1}{3}$, i.e. very close to the middle of the 20 to 40% range. Two matrices which typically exhibit a larger fraction of cooperative choices are the following, which typically give a 50 to 60% fraction of cooperative choices:

Figure 5.3.4

	Defect	Cooperate
Defect	3,3	5,2
Cooperate	2,5	4,4

Reference 49

Figure 5.3.5

Player 2

Player 1	Defect	Cooperate
Defect	3,3	7,1
Cooperate	1,7	5,5

Reference 37

Now, gaming experiments have been conducted with runs as long as 300 plays,[50] but the results for very long runs do not seem as consistent as for relatively short runs (say under 40). Nevertheless, in long runs, various dynamic interaction effects become apparent. For example, one such phenomena, "locking-in" on the mutually cooperative or mutually competitive outcomes, is "most clearly observable if the game is played several hundred times".[51] We suggest that for long runs, it is quite possible that boredom, aggression, etc. have significantly altered the player's perception of the payoff matrix. Whether the game remains a prisoner's dilemma, in any sense whatever, is open to doubt. Our comments, therefore, are strictly confined to the "early play", i.e. under 40 plays.

The "early play" empirical judgement would seem to be the following. First, players rarely if ever choose either the "confess" choice 100% of the time or the "not confess" choice 100% of the time. Second, the consistency of fraction of choices for given matrices and the differences in fraction of choices between different matrices strongly suggests that the actual numerical entries in the payoff matrix are critical. Third, there is typically a decline in the fraction of cooperative choices which more or less stops by 30 - 40 plays.

The preceding discussion of the prisoner's dilemma and the other nonzero sum games has shown both formalistic and empirical grounds for rejecting the equilibrium point (or maximin) concept as the definition, in any sense, of the solution to non-negotiated non-zero sum games. For prisoner's dilemma games, the solution is an undesirable self fulfilling prophecy which in general fails to occur empirically. Moreover, in gaming experiments with "chicken", the equilibrium point concept appears to fail to predict actual choices as well. Thus, the equilibrium point concept is normatively undesireable and apparently empirically false. To advise people to act on the basis of the dilemma because other people will do so appears to be patently bad advice.

All of this suggests the need for a new solution concept for non-negotiated, nonzero sum games, and in particular, a solution concept which somehow deals with the prisoner's dilemma problem in a new way. Would this be a solution concept which would tell us when the dilemma occurs and when it does not? The answer to the question is that the question is based on the wrong intuitions. We already have established the fact that the dilemma often does not occur when it might do so. Thus, in those situations where the joint minimal outcome does occur, our understanding is obscured by calling the situations "prisoner's dilemmas". The point is, a new solution concept should not even raise the question of a "prisoner's dilemma".

The dilemma would not be "resolved" in this theory, it simply would not appear. The intuitions which motivate joint minimal outcomes, in those situations where they can be expected to occur, would not be the same intuitions as motivate those outcomes in equilibrium point solutions to prisoner's dilemmas.

To realize a solution concept of this type, we apparently would have to go back to nonzero sum games, and begin again with a non-equilibrium point solution concept. The candidate for such a solution concept has already been discussed repeatedly in this paper - zero sum interpretation. Before we discuss that theory, however, we would be well advised to examine some alternatives to it, which operate within the framework of equilibrium point theory.

6. The metagame resolution of the prisoner's dilemma,[34, 38] and Shubik's solutions[79] - equilibrium point resolutions

6.1 Conditional strategies

Reconsider the matrix of figure 5.1.1:

Figure 6.1.1

Player 2

Player 1	confess	not confess
confess	-1,-1	2,-2
not confess	-2,2	1,1

Pretend that player 2 could choose after player 1 had chosen, and could see player 1's choice, i.e. player 2 knows the choice of player 1. We have seen in figure 1.3.8 the effect of such information - it increases the available strategies to player 2. Instead of the alternatives "confess" and "not confess", he can now choose among the following four strategies:

1. Confess if 1 confesses, confess if 1 does not confess.
2. Confess if 1 confesses, do not confess if 1 does not confess.
3. Do not confess if 1 confesses, confess if 1 does not confess.
4. Do not confess if 1 confesses, do not confess if 1 does not confess.

By using the numerical entries for the various strategy combinations of figure 6.1.1 we can obtain the new matrix:

Figure 6.1.2

		Player 2			
	If 1 chooses "c", then 2 chooses	confess	confess	not confess	not confess
	If 1 chooses "NC", then 2 chooses	confess	not confess	confess	not confess
Player 1	confess	-1,-1	-1,-1	2,-2	2,-2
	not confess	-2,2	1,1	-2,2	1,1

We observe that player 2 still has a dominating pure strategy of column 1, and player 1 still has his old dominating pure strategy of row 1. Thus, no new equilibria are introduced. Now suppose that we are faced with the game described by figure 6.1.2 (the 2 x 4 game), and pretend that player 1 can see the choice made by player 2. We know that this will increase the available strategies to player 1, - it will give him all possible responses to player 1's four possible strategies. This amounts to sixteen pure strategies. For example, one of these would be the following: Confess against his first choice, not confess against his second choice, confess against his third choice, confess against his fourth choice. We shall write this as (C,NC,C,C).

The new matrix thus becomes that of figure 6.1.3.

Figure 6.1.3

Player 2

Player 1				
If 1 chooses "C", then 2 chooses	confess	confess	not confess	not confess
If 1 chooses "NC", then 2 chooses	confess	not confess	confess	not confess
C,C,C,C	-1,-1	-1,-1	+2,-2	+2,-2
NC,C,C,C	-2,+2	-1,-1	2,-2	2,-2
C,NC,C,C	-1,-1	1,1	2,-2	2,-2
C,C,NC,C	-1,-1	-1,-1	-2,2	2,-2
C,C,C,NC	-1,-1	-1,-1	2,-2	1,1
NC,NC,C,C	-2,2	+1,+1	2,-2	2,-2
NC,C,NC,C	-2,2	-1,-1	-2,2	2,-2
NC,C,C,NC	-2,2	-1,-1	2,-2	1,1
C,NC,NC,C	-1,-1	1,1	-2,2	2,-2
C,C,NC,NC	-1,-1	-1,-1	-2,2	1,1
C,NC,C,NC	-1,-1	1,1	2,-2	1,1
NC,NC,NC,C	-2,2	1,1	-2,2	2,-2
NC,NC,C,NC	-2,2	1,1	2,-2	1,1
NC,C,NC,NC	-2,2	-1,-1	-2,2	1,1 X
C,NC,NC,NC	-1,-1	1,1	-2,2	1,1
NC,NC,NC,NC	-2,2	1,1	-2,2	1,1

Observe that we have interpreted player 2's strategy choices as specific instructions to either confess or not according to whether player 1 confesses or not. That is, they are conditional on the choice of "C" or "NC" of player 1. On the other hand, player 1's choices are conditional on the choice of column number of player 2. In this way, meaning can be given to the matrix and we can fill in all of the boxes on the basis of the number associated with the strategy combinations of the players in the matrix of figure 6.1.1. The two new equilibrium points, as well as the old one, are circled. They are found as follows: Hold the strategy of player 1 fixed and look at the payoffs to player 2 for 2's own alternative strategies. If no other strategy offers a higher payoff to player 2, repeat the procedure for player 1, i.e. hold player 2's strategy fixed and search player 1's alternative strategies. If no other strategy offers a higher payoff then this combination of strategies is an equilibrium point (really, a metaequilibrium, since it occurs in the metagame). The three circled equilibrium points are the only ones on this matrix. For example, the cell we have labelled with an "X" is not an equilibrium point since both players have an incentive to change their strategies.

The game of figure 6.1.2 is called the 2-metagame, and that of figure 6.1.3 is the 1, 2-metagame. Howard has proven that further expansions will yield no new equilibrium points. Thus, once each of the players has been named once no new equilibria are introduced. Therefore, we never need investigate the 2,1,2-metagame.

Observe that the dilemma is resolved, that is, the joint maximal outcome can always be realized by the metagame procedure. Notice also that we could have given player 1 the four choices and player 2 the sixteen choices, but in each complete expansion (expanding once for each player) one player must have four and the other sixteen choices. The solution is defined as those meta-equilibria which are meta-rational in both expansions. Thus, the joint maximal outcome in prisoner's dilemma games is meta-rational no matter how it is viewed, i.e., no matter which player expands first. For practical purposes — and recall that this is what nonzero sum games are about — it may, nevertheless, matter a great deal who expands first. This is because meta-strategies are ways to enforce certain outcomes, rather than models of thought. We shall discuss the asymmetries in the expansion shortly.

If the metagame concept is applied to zero sum 2-player games, the meta-rational strategies do not change the value v of the game. They can be, however, slightly different from optimal minimax strategies. This point is discussed by Howard.[52] For a textbook treatment of metagame theory, see Saaty.[56]

6.2 Discussion of the metagame concept

The difficulties with the solution concept for nonzero sum games such as the prisoner's dilemma should be mentioned. First, and most obvious, new strategies, i.e., the conditional strategies, are introduced which simply were not in the original game of figure 5.1.1 (or 6.1.1). Harris[58] has argued - and we agree - that these conditional strategies change the nature of the game and use the referee (the District Attorney) in a way contrary to the rules of the game. (Harris' original comments[59] have sparked a three way controversy involving Harris,[62] Rapoport,[60, 61] and Howard[63, 64] (and we might note that the comments of these gentlemen, in this controversy, are not always gentlemenly). Rapoport has argued that a referee always decodes normalized strategies and thus plays no new role in the metagame of the prisoner's dilemma. This, however, seems hardly to be the case since the meta-strategies can be contrary to the original rules of the game. Thus, in the prisoner's dilemma parable, we can imagine the cigar smoking D.A. grimly entering the room containing Prisoner 1 and asking, "Do you confess?" To this, Prisoner 1 replies, "I do, unless he (Prisoner 2) says that he confesses if I do and does not confess if I don't confess. If he says that, then I don't confess." The D.A. (perhaps somewhat surprised by this reply) walks into the adjoining room and asks Prisoner 2, "Well, what about it, do you confess?" Prisoner 2 replies, "I confess if he (Prisoner 1) confesses and don't confess if he does not confess."

The reader can make his own assessment as to whether the D.A. would accept these replies. In short, the referee must be willing to accept statements of strategies which were not specified in the original rules of the game.

A second criticism is that the solution concept turns a fundamentally symmetrical game into an asymmetrical one, i.e., in any actual play of the metagame, one player has four strategy choices and the other sixteen. Moreover, each must know which player he is, i.e. whether he is player 1 or player 2.

The question is whether the asymmetry is of no importance. That is, does the asymmetry merely reflect certain features of the table but no significant <u>strategic</u> considerations? Put another way, would we rather be player 1 or player 2? The answer is decisively that we would rather be player 1. We would expect that the player with more variety (choices) at his disposal is favored, although we have seen in the matrix of figure 1.3.4 that this is not necessarily the case. The advantage only occurs in the case of the metagame resolution of the prisoner's dilemma, if one of the player makes a mistake. Thus, consider the situation if player 1 uses his third strategy, and player 2 makes a mistake and chooses his third or fourth strategy. Player 2 loses 2, and player 1 wins 2. Now consider the situation if player 2 uses his second strategy, and player 1 makes a mistake, i.e. uses some other strategy. Inspection of the table shows that either there is no change in payoffs, or both players lose 1. Thus, player 2's optimal strategy is only optimal in the sense of guaranteeing at least some number (-1), but player 1's optimal strategy is permanently optimal, i.e. it always maximizes on any choice of player 2. But the asymmetry is even more drastic. If player 2 uses strategy 2, then player 1 is protected against his own mistakes, i.e. no mistake will give him less than -1. But if player 1 uses strategy 3, player 2 is not protected against his own mistakes, i.e. a mistake on his (player 2's) part could give him -2. Thus, player 1 becomes the odds-on favorite, but he was given no such advantages in the original game of figure 6.1.1. Again we ask, which player in the game of figure 6.1.3 would the reader prefer to be, player 1 or player 2? Now, look at the game of figure 6.1.1. Which player would the reader prefer to be, player 1 or player 2? Thus, the real question: is the metagame of figure 6.1.3 the same game as the original prisoner's dilemma game of figure 6.1.1? If the answer is "no", then the symmetrical prisoner's dilemma of figure 6.1.1 has still not been "resolved".

Within the context of equilibrium point theory, however, the metagame concept achieves something - it allows two players, who know their roles, to reason independently to a joint maximal outcome. This could be the basis for a new normative theory of equilibrium point nonzero sum games. (Howard argues that the theory is descriptive, but there seems to be no clear evidence that it describes anything. The reader is referred to the report of his experiments in reference (52)).

The solution is decidely artificial, but since it is not also obviously undesireable, there seems to be no obvious objection to its artificial character. The judgement as to whether or not the metagame theory may be useful for practical situations, e.g. 2-player negotiations[95] where the players do not trust each other, (Harris, in "Paradox Regained" indicates that Rapoport has privately said that this is where the greatest value of metagame theory lies) would seem to depend very heavily on two things. First, the players would have to agree on the order of expansion of the game. As we have seen, this is not a trivial matter.

Howard's example of the Vietnam war[52] is illuminating in this respect. He treats each side (the U.S. and the Vietcong) as having pure strategies of "escalation" and "not escalation" in a prisoner's dilemma game. For the metagame resolution, we may as well quote Howard: "One side must have a policy of Tit-for-Tat, and the other must react with 'Not Escalate' against 'Tit-for-Tat', 'Escalate' against 'Always Escalate' or 'Tat-for-Tit' moreover, of course, these policies must be credible." (52, p. 112). (Emphasis added). Howard leaves open the question of which side adopts the respective policies: It does not tax one's thinking, however, to see that in this case, the question of which side will have a permanently optimal strategy - if it is raised in the negotiations - may be viewed by the participants as going to the heart of the problem.

Second, the judgement as to the value of an application of metagame theory to a particular negotiation would depend decisively on whether or not the use of conditional strategies significantly alters the objective situation about which the sides are negotiating. Observe, in this regard that the problem of credibility of the conditional strategies assumes overwhelming importance. This is because each player must be able to make his conditional strategies believable to the other players. How this could be done is an empirical matter, but as we have seen in section 3.4, this could drastically alter (perhaps in an unpleasant way) the original objective situation. There is no issue of credibility in the non-negotiated prisoner's dilemma situation, since each player either does one thing or another thing.

6.3 Shubik's solutions

Shubik[79] has suggested that there may be several valid solutions to prisoner's dilemma type games largely because the prisoner's dilemma does not exist as a single problem: "... the only paradox (if there is any) is that so much concern has been lavished on mistaking a class of games for a single game and on trying to use too simple a construct to explain too much." (p. 91) Shubik is concerned with the "real world" (and experimental environment) relevance of solutions to prisoner's dilemma games, which, as we have said repeatedly, is precisely the importance of nonzero sum games. He describes two solution concepts, one involving an infinite repetition of the prisoner's dilemma where

Nature (i.e., a random device) decides if the game will terminate at any given play or be repeated. The players get the sum of their payoffs P from each play. Thus, if the game __is__ played five times, the players get

6.3.1 $$\left(\sum_{t=1}^{5} P_1 t, \sum_{t=1}^{5} P_2 t\right)$$

If the game is infinitely repeated (i.e., until the randomizing device assigns p = 0 of continuing at time t) we have

6.3.2 $$\left(\sum_{t=1}^{\infty} p^t P_1, t, \sum_{t=1}^{\infty} p^t P_2, t\right) \text{ were } 0 \leq p \leq 1 \text{ is}$$

the probability of continuing to the next day, and 1 - p is the probability of terminating at time t. Note that each payoff is discounted by its liklihood of occuring. Neither player now knows when the game will end. Therefore, the argument that a known finite number of repétitions of the prisoner's dilemma always produces a prisoner's dilemma outcome no longer applies. (The argument, essentially identical to the proof of theorem 1.2.5 says that on the final play, each player will choose to confess, since he does not fear retaliation. So the final play's outcome is known and the next to the last play can be treated as the final play, but again the previous remark applies, and so on to the first day). Thus, with no known end to the game, perfect cooperation becomes a possible equilibrium state. This can be combined with various threat strategies, as we shall see in a moment.

Shubik's second resolution model involves the previous infinite sum expression, but this time always interpreted as an infinite play of the game. The p^t is now considered to represent a discount rate on the payoff matrix. Why the discount rate? This is merely a way of saying that money made in the future is worth less than money made at the present time. In the previous solution, this was because the game might end, and thus the probability of its continuing got pregressively smaller. In the present model, the discount rate means that, as Shubik says, "'Pie in the plate' is preferred to 'pie in the sky'".[80]

As an example of the second solution concept, consider the following matrix:

6.3.3

0,0	-5, 10
10,-5	5, 5

Suppose that the player choose to cooperate on the first play, i.e., choose row 2 and column 2 respectively. The present worth of 5, for each player, is

6.3.4 $$5(p + p^2 + p^3 + \ldots.) = \frac{5p}{1-p}$$

Now assume a discount rate $p = .9$ and suppose that player 2 defects on play one. He obtains 10. Suppose player 1 has a strategy, "If he defects, then I defect for k periods and then resume choice 2. If he continues to defect, I choose row 1 (defect) for k+1 periods, and then resume choice 2." So, if $k = 1$, then player 2's winning of $10 + 0 > 5 + 4.5$. But, if $k = 2$, then $10 + 0 < 5 + 4.5 + 4.05$. Thus defection does not pay when $k = 2$, and mutual cooperation is stable. The infinite payoff to player 2, if he insists on optimaly defecting is given by

$$6.3.5 \qquad 10(p_1 + p_2^3 + p_3^6 + p_4^{10} + p_5^{15} + p_6^{21} + p_i^{k_i} + \ldots .)$$

where $k_i = i - 1 + k_{i-1}$. If $p = .9$, this expression reaches the limit 29.42. The present worth of $5 = \frac{5p}{1-p} = 45$.

Thus, in an infinite play, no one has an incentive ever to play choice 1, providing at least one player uses the threat strategy with $k = 2$.

The first solution concept is due to R.J. Aumann, but is reported only in Shubik (reference 79). The second solution concept is an application of Shubik's work on games of social survival.[80,81] Notice that the threat strategies need not be verbally communicated. They can be learned in the course of announcing the result of each play. Since the game is played on infinity of times, there is no shortage of time for learning.

Rapoport[82] has pointed out some of the difficulties of these solution concepts, all of which center around the fact that they radically alter the nature of the original problem. That is, threat strategies, infinite repetitions (with known probabilities) and discount rates are simply not part of the original problem. Moreover, we have seen that single play experiments (figure 5.3.3) seem to produce no significant difference in fraction of cooperative choices than do experiments involving 30 or 40 plays. Thus, even from the point of view of empirical psychological explanation, these models are not helpful. We make this claim only for the early play. Perhaps for long runs of experimental games, these models may have some relevance.

The objections we have discussed to the metagame concept and to the two solutions discussed by Shubik are in addition to the far more fundamental empirical objection to the equilibrium point concept itself, and in particular its use in prisoner's dilemma situations. We shall now look at an entirely different approach to non-negotiated nonzero sum games — a non-equilibrium point solution concept in which (for two player game) the prisoner's dilemma outcome and intuition simply do not arise. The need for a new type of solution seems clear, for we have seen that the prisoner's dilemma does not tell us the truth about ourselves, and in fact, its use as a guide to intuition may be doing us great harm, even though it is put forward as a painful, but helpful, insight which will aid our survival. The prisoner's dilemma appears, in fact, to be a hoax.

7. Zero sum interpretation theory

We have seen that for negotiated nonzero sum games, the von Neumann and Morgenstern solution added an extra player, to make all differences zero sum, and then had the n real players "cooperate fully" against him. "Cooperating fully" involved a perfect interplay of threats, haggling, negotiations, etc. The question is, could a zero sum interpretation theory be constructed which specifically excluded any possibility of negotiations? A "yes" answer to this question has been developed by the present author,[36] and we shall now discuss this solution concept in some detail.

7.1 Two player zero sum games

Since no pregame negotiations are possible, the notion of cooperating fully against the n + 1st player must be revised. For it, we shall substitute the concept of playing independently against the n + 1st player. Instead of a coalition of n players playing an imaginary zero sum 2-player game against an imaginary extra player, we shall develop a theory which has each of the n real players playing separate imaginary zero sum two player games against the (still imaginary) extra player. These imaginary games will correspond element for element with the real game. The motivation for the players' thinking in these terms is extra theoretic, i.e. we must assume that the real players view their decision making task in terms of the zero sum 2-player games between each player and the extra player. Of course, this assumption prejudges the argument. It should be thought of in the same spirit as the technique used by von Neumann and Morgenstern for treating n-player nonzero sum and zero sum games, where the negotiations are extra theoretic. We can, however, bolster our assumption to a certain extent; we can also consider a wide range of zero sum 2-player games between each and every pair of real players, to see if the real players might not prefer pure conflict with each other over pure conflict with the extra player.

Since we have assumed that the motivation for the zero sum interpretation is extra-theoretic, and that all decisions are made in terms of the imaginary zero sum 2-player games, one decision rule would seem to be compelling - if a combination of n separate zero sum 2-player games against the extra player produces a joint maximum for the n real players, this combination of imaginary games should constitute "the solution". That is each real player should act in the real game <u>as if</u> he were playing his "best" (joint maximum) zero sum 2-player game against the extra player. This joint maximum-realizing combination of imaginary zero sum games (actually of the <u>strategies</u> for the real player in these games) must be subject to the single requirement imposed by the one to one correspondence between the elements of the imaginary games and those of the real game: the probabilities which "occur" in the imaginary games must correspond one to one with the probabilities which occur in the real game. Thus, the imaginary games are true zero sum 2-player interpretations of the real game. We shall therefore, consider our task completed if we can show the "existence" of a joint maximum-realizing combination of imaginary zero sum 2-player games. It will turn out that for 2-player prisoner's dilemma games such a combination always exists, and that the joint maximum solution for each real player is always a mixed strategy (which incidentally, corresponds strikingly with the fractions of cooperative choices obtained in the gaming experiments). Whether or not a similar combination exists for n-player ($n > 2$) prisoner's dilemma games depends critically on the exact numerical payoffs. We shall show an important case where the combination does exist, and for which the mixed strategy solutions are independent of the number of players. We shall also show that in the special case where the <u>real</u> nonzero sum game happens to be in fact a zero sum 2-player game, our solution concept produces results identical - both in terms of strategies and the value v - with those of the von Neumann and Morgenstern minimax concept.

7.2 n + 1 player game

Notice that the addition of the extra player serves only to turn the n-player nonzero sum game into an n + 1 player (and <u>not</u> a 2-player) zero sum game, and that the <u>imaginary</u> zero sum 2-player games we have been discussing are subsequently derived from the n + 1 player zero sum game. This is the critical difference between our solution concept and von Neumann and Morgenstern's treatment of negotiated nonzero sum games. We shall find it convenient to treat the n + 1 player game as a coalition game, where each real player has the choice of joining or not joining a coalition with the extra player (corresponding with "confessing" and "not confessing"). That is, if i_o is the extra-player, and i_k is any real player, then each real player can choose to form his own separate coalition with the extra player. We shall write this coalition as (i_o, i_k). If two or more real players choose to form their coalitions with the extra player, we shall treat this as the union of the separate two player coalitions: $(i_o, i_k) \cup (i_o, i_j)$. In this way, the isolation of the separated real players, with the obvious possibility of "overcompensating" the imaginary extra player, is brought out. Inefficiency, in short, is represented as cumulative. Thus, in a two player prisoner's dilemma game, if (i_o, i_1) corresponds to the situation where player 1 confesses, we could have a payoff n-tuple as follows: $(\frac{1}{2}, \frac{1}{2}, -1)$, where the first element is the payment to the extra player i_o, the second element is the payment to player 1, and the third element is that to player 2. The situation when player 2 chooses "confess" is as follows: (i_o, i_2), $(\frac{1}{2}, -1, \frac{1}{2})$. Since mutual confessions is $(i_o, i_1) \cup (i_o, i_2)$ and represents compounded inefficiency, we must sum the respective payoff n-tuples to obtain $(1,-\frac{1}{2},-\frac{1}{2})$. Any symmetrical prisoner's dilemma game can be normalized into either this form, where the extra player wins some amount if a player confesses, or into the analagous form, where the extra player pays out some amount if a real player confesses. The first of these cases represents a loss to nature due to inefficiency. The second represents a subsidy from nature, which may be thought of as increased productivity resulting from competition. Normalization is essential if we are to determine the exact role played by the extra player (for example, the D.A.) in a prisoner's dilemma game. The normalized matrices for the two possible cases in 2-player prisoner's dilemma games are as follows, together with their respective normalization formulae:

Figure 7.2.1

Case 1

Player 1	confess	$u_1(1,2)-1$	$u_1(1,2)$
	not confess	-1	$u_1(2,2)$

Each element of the original prisoner's dilemma matrix M (for player 1) becomes

$$\frac{u_1(m,n)}{u_1(1,2) - u_1(1,1)} + \frac{u_1(2,1)}{u_1(1,1) - u_1(1,2)} - 1$$

Figure 7.2.2

Case 2

Player 1			
	confess	$1-u_1(2,1)$	1
	not confess	$u_1(2,1)$	$u_1(2,2)$

Each element of the original prisoner's dilemma matrix M (for (player 1) becomes

$$\frac{u_1(m,n) - u_1(1,2)}{u_1(1,1) - u_1(2,1)} + 1.$$

Analogous expressions can be obtained for player 2. The matrix of figure 6.1.1 normalies as follows:

Figure 7.2.3

Player 2

Player 1	confess	not confess
confess	$-\frac{2}{3}, -\frac{2}{3}$	$\frac{1}{3}, -1$
not confess	$-1, \frac{1}{3}$	0,0

Written as a coalition game, this would be

7.2.1

	coalition	n-tuple
1)	(i_0, i_1)	$(\frac{2}{3}, \frac{1}{3}, -1)$
2)	(i_0, i_2)	$(\frac{2}{3}, -1, \frac{1}{3})$
3)	$(i_0, i_1) \cup (i_0, i_2)$	$(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3})$
4)	(i_1, i_2)	$(0,0,0)$

Notice that the elements of each n-tuple sum to zero. We see also that the description of the game in 7.2.1 is merely a list of coalitions and associated n-tuples. We shall designate the list by $\mathcal{L}$

and each component of it by $\mathcal{L}$. Observe also that for each coalition (i_o, i_k), we have $v((i_o, i_k)) = -v((i_j))$.

The 3 player symmetrical game of figure 5.1.3 could be normalized (to $v((i_k)) = -1$) and written as a zero sum coalition game as follows:

7.2.2

coalition	n-tuple
(i_o, i_1)	$(0, 1, -\frac{1}{2}, -\frac{1}{2})$
(i_o, i_2)	$(0, -\frac{1}{2}, 1, -\frac{1}{2})$
(i_o, i_3)	$(0, -\frac{1}{2}, -\frac{1}{2}, 1)$
$(i_o, i_1) \cup (i_o, i_2)$	$(0, \frac{1}{2}, \frac{1}{2}, -1)$
$(i_o, i_1) \cup (i_o, i_3)$	$(0, \frac{1}{2}, -1, \frac{1}{2})$
$(i_o, i_2) \cup (i_o, i_3)$	$(0, -1, \frac{1}{2}, \frac{1}{2})$
$(i_o, i_1) \cup (i_o, i_2) \cup (i_o, i_3)$	$(0, 0, 0, 0)$
(i_1, i_2, i_3)	$(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

An n + 1 player (i.e. the n real players, $i_1, \ldots, i_n$ and player i_o) coalition game is characterized by the following:

7.2.3

i) Players = $I = (i_o, i_1, \ldots, i_n)$.

ii) Each player of $(i_1, \ldots, i_n)$, i.e. $I - (i_o)$ has two strategic choices "join i_o", "not join i_o". Player i_o has no strategic choices. Therefore, the game has 2^n outcomes.

iii) Each coalition (i_o, i_k), $(k \neq 0)$, has a unique payoff n-tuple ^{k} where $^k = (a_o^k, a_1^k, \ldots, a_n^k)$, and the superscript k indicates that player i_k chose "join i_o".

iv) The coalition $(i_1, \ldots, i_n)$ has a unique n-tuple associated with it and is designated without a superscript as $= (a_o, a_1, \ldots, a_n)$.

v) For $(i_o, i_k)\ U(i_o, i_j)$, $k \neq j \neq o$, we have $^k + \ ^j$

vi) For $(i_o, i_k)\ U(i_o, i_j)\ U \ldots U(i_o, i_n)$ we have $^k + \ ^j + \ldots + \ ^n$, with $k \neq j \neq 0 \neq \ldots \neq n$

vii) Each n-tuple ^{k} satisfies

$$a_k^k = - \sum_{\substack{i=o \\ i \neq k}}^{n} a_i^k \quad \Big| \quad 1 \geq a_k^k \geq 0.$$

viii) Each coalition's value is such that $v((i_o, i_k)) = -v((i_j, \ldots, i_n))$ where $k \neq j \neq o \neq \ldots \neq n$

and

$v((i_1, \ldots, i_n)) = -v((i_o))$

ix) $o > v((i_k)) \geq -1$, for $k = 1, \ldots, n$. $v((i_o))$ is unrestricted.

x) The entire list of coalitions, union of coalitions and associated payoff n-tuples is designated by $\mathcal{L}$ and each coalition and associated n-tuple by ℓ .

7.3 Zero sum interpretation of each outcome

We shall now consider the zero sum interpretations of each outcome. These are the numbers which are used to fill in the 2-player zero sum interpretation matrices. Since the nonzero sum game has been converted into a coalition game (by including the District Attorney, player i_o), any zero sum interpretation payments between two players will always be between members of different coalitions or the same coalition(s). We shall call any coalition containing any player and i_o a winning coalition, and the coalition of all real players we shall also call a winning coalition (even if it wins a negative amount). All other coalitions — including one player "coalitions" — we shall call losing coalitions. Every zero sum interpretation matrix will represent every outcome of the game in terms of payments between two specific players. Some matrix boxes will represent the cases where one of these two players is a winner and the other a loser, and vice versa for other matrix boxes. For those cases — for these boxes — the loser pays the winner what the winner gets. In other boxes, both players are winners or both losers, i.e., both are members of the same coalition. The zero sum interpretation of payments within a coalition (winning or losing) are that one player pays the other what he gets. This gives two possible interpretations depending on who pays whom. A third interpretation within coalitions is to take the difference between the other two interpretations. So, for payments between members of the same coalition (winning or losing) we have a total of three possible zero sum interpretations. Since a zero sum game matrix has only one number in it, we would need three matrices with the appropriate box containing the respective entry. Some matrix boxes represent the occurance of several separate coalitions. We sum their respective zero sum interpretations and enter them in the appropriate matrix box. Again, this may call for several matrices since a coalition may not have a unique zero sum interpretation. Remember, incidentally, that in a zero sum two player game, the column player's gains are represented as a negative number, and his losses as a positive number. We shall adopt the convention of writing i_o as the column player.

Consider as an example, a zero sum 2-player interpretation of a 2-player prisoner's dilemma game. How many zero sum interpretation matrices are there between each real player say i_o and the District Attorney i_o? Between i_1 and i_o, there are nine separate zero sum interpretation matrices, (and there exists the same number between i_2 and i_o). This is because the coalition (i_o, i_1) has three zero sum interpretations $(a_1^1, -a_o^1, a_1^1 - a_o^1)$. Now the mutual confession outcome is represented by $(i_o, i_1) \cup (i_o, i_2)$. The first of these coalitions has the three zero sum interpretations we have just listed. The second of these coalitions (i_o, i_2) has a single zero sum interpretation between i_o and i_1 — i_o is a winner and i_1 is a loser, so the zero sum interpretation is a_o^2. This would be written as a negative number if i_o is the column player. This gives us for $(i_o, i_1) \cup (i_o, i_2)$ the following <u>three</u> zero sum interpretations between i_1 and i_o: $(a_1^1 - a_o^2)$, $(a_1^1 - a_o^2 - a_o^1)$, $(-a_o^1 - a_o^2)$. Next we have the case of (i_o, i_2). We have already considered this as part of the union of coalitions. The matrix entry is simply $-a_o^2$. Finally, we have the coalition (i_1, i_2). In this case i_o is a loser and i_1 is a winner. The matrix entry is simply a_1. Thus we have 3x3x1x1 = 9 separate matrices required to represent all of the variety. We shall see what these matrices look like in the next section.

7.4 The imaginary zero sum 2-player games and the solution axioms

We are now in a position to derive zero sum interpretation solutions to nonzero sum games. This is because, in section 7.1, we have already described the nature of the solution and the imaginary games. Instead of immediately deriving some results, however, we shall present a few more formal structures which will be convenient.

First, we have noted that the imaginary zero sum 2-player games relate to the original nonzero sum game on a one to one basis, i.e. matrix element for element. We may as well define some generalized matrix forms. The matrix form of figure 7.4.1 (see next page) will suffice for games between each real player and player i_o. We shall call these games A_h^k where k refers to player i_k ($k \neq 0$) and h refers to the game number. The games must be numbered because, as we have seen, some matrix elements will have more than one zero sum interpretation and thus there will in general be more than one game A_h^k between each real player i_k ($k = 1, \ldots, n$) and player i_o.

<u>THE FOLLOWING THREE PAGES SHOULD BE READ AS ONE TABLE</u>

Figure 7.4.7

Game A_h^k (For any n):

Player i_1

n $U(i_o, i_k)$ $k = 1$	n - 1 $U(i_o, i_k)$ $k = 1$ $k \neq n$	n $U(i_o, i_k)$ $k = 1$ $k \neq n - 1$	n $U(i_o, i_k)$ $k = 1$ $k \neq n - 2$	...
n $U(i_o, i_k)$ $k = 2$ $k \neq 1$	n - 1 $U(i_o, i_k)$ $k = 2$ $k \neq 1, n$	n $U(i_o, i_k)$ $k = 2$ $k \neq 1, n-1$	n $U(i_o, i_k)$ $k = 2$ $k \neq n - 2, 1$	...

...	n $U(i_o, i_k)$ $k = 1$ $k \neq 2$	$n - 2$ $U(i_o, i_k)$ $k = 1$ $k \neq n, n-1$	$n - 1$ $U(i_o, i_k)$ $k = 1$ $k \neq n, n-2$	...	$n - 1$ $U(i_o, i_k)$ $k = 1$ $k \neq n, 2$
...	n $U(i_o, i_k)$ $k = 3$ $k \neq 2, 1$	$n - 2$ $U(i_o, i_k)$ $k = 2$ $k \neq 1, n,$ $n - 1$	$n - 1$ $U(i_o, i_k)$ $k = 2$ $k \neq 1, n,$ $n - 2$	...	$n - 1$ $U(i_o, i_k)$ $k = 3$ $k \neq 1, n, 2$

$\bigcup_{k=1}^{n} (i_o, i_k)$, $k \neq n-1, n-2$	$\bigcup_{k=1}^{n} (i_o, i_k)$, $k \neq n-1, n-3$	...	$\bigcup_{k=1}^{n-3} (i_o, i_k)$, $k \neq n, n-1, n-2$	$\bigcup_{k=1}^{n-2} (i_o, i_k)$, $k \neq n, n-1, n-3$	...	...	(i_o, i_k), $k = 1$
$\bigcup_{k=2}^{n} (i_o, i_k)$, $k \neq 1, n-1, n-2$	$\bigcup_{k=2}^{n} (i_o, i_k)$, $k \neq 1, n-1, n-3$	...	$\bigcup_{k=2}^{n-3} (i_o, i_k)$, $k \neq 1, n, n-1, n-2$	$\bigcup_{k=2}^{n-2} (i_o, i_k)$, $k \neq 1, n, n-1, n-3$	...	...	$(i_1, \ldots, i_n)$

Where $\bigcup_{k=1}^{n} (i_o, i_k) = (i_o, i_1) \cup (i_o, i_2) \cup (i_o, i_3) \cup \ldots \cup (i_o, i_n)$

For imaginary zero sum 2-player games between two real players, we must first recall that each real player has two and only two choices. (This restriction has been removed in recent unpublished work, but we shall not discuss this generalization here). The imaginary zero sum games between any two real players are therefore limited to 2x2 games. If n = 2, this presents no problem, for larger n, however, we must have a system of games where the specific elements must depend on the actions of players not in the game. The total number of distinct matrix blanks could become quite large even for relatively small n. The number is obtained by first noting there are $\frac{n!}{(n-2)!2}$ distinct sets of two players each. Moreover, between each pair of players there are 2^{n-2} possible matrix blanks, corresponding to the actions of the other (n-2) players. Altogether, this gives us for any n

7.4.1 $$\frac{n!}{(n-2)!\,2}\,2^{n-2} \quad \text{distinct 2x2}$$

zero sum games between each and every distinct pair of players in the set $(i_1, \ldots, i_n)$.

In fact, it turns out that all of these games never have to be enumerated since most of them can never satisfy the solution criteria discussed in section 7.1. For the sake of formal completeness, however, we must take note of the entire system of these games. We shall index these games as $B_h^{j,k}$ where h again is the game number, and j and k are real player numbers.

Each of these games will have the following form:

Figure 7.4.2

If players i_1, ..., i_n choose "join i_o" ("not joint i_o")
then

Player i_k \ Player i_j	join i_o	not join i_o
join i_o	$(i_o,i_k) \cup (i_o,i_j) \cup \ldots \cup (i_o,i_n)$	$(i_o,i_k) \cup \ldots \cup (i_o,i_n)$ with i_j member of losing coalition.
not join i_o	$(i_o,i_j) \cup \ldots \cup (i_o,i_n)$ with i_k member of losing coalition.	$(i_j,i_k, \ldots)$, which is winning if and only if it is the coalition $(i_1, \ldots, i_n)$ and otherwise it is losing

One comment which we should make at this point concerns the possibility that the matrices of Figures 7.4.1 or 7.4.2 could be trivial matrices, i.e. matrices where every element is the same number. In this case, obviously, any pure or mixed strategy for either and both players is as good as any other. We shall exclude trivial matrices of this type from our considerations since they convey no useful information to the real player.

The solution axioms which define the solution concept described in section 7.1 can be listed. It is less cumbersome to state the solution concept verbally as in 7.1 then axiomatically, as we shall now present it. However, the axioms are helpful for computational purposes. The first axiom relates the imaginary zero sum games A_h^k and $B_h^{j,k}$ to the real game $\mathcal{L}$. If we write $A_h^k\ (m,n)\,\ell$ to mean the probability assigned by the (m,n)th element of game A_h^k to outcome ℓ, we have

7.4.2 Either $A_h^k\ (m,n)\,\ell = p\ (\ell)$ or

$B_h^{j,k}\ (m,n)\ell = p(\ell)$, for each and every player of

$(i_1, \ldots, i_n)$

and for all ℓ.

Player i_k ($k \neq o$) uses his optimal von Neumann and Morgenstern minimax (or maximin) strategy. Player i_o uses any strategy against i_k's optimal strategy which yields the value v of the zero sum game in the sense of von Neumann and Morgenstern.

Notice that each player k's hth game must define the same $p(\ell)$ for all ℓ. Therefore, we need only work with any one player's hth game.

Now, the probabilities assigned by the games A_h^k or $B_h^{j,k}$ also necessarily assign each player a certain utility value in game $\mathcal{L}$. We can write this as

$$U_{i_k}\left[\ A_h^k\ (m,n)\ell\ \right] \text{ for all } \ell \text{, and}$$

$$U_{i_k}\left[\ B_h^{j,k}\ (m,n)\ell\ \right] \text{ for all } \ell.$$

If the game A_h^k or $B_h^{j,k}$ which assigns the joint maximum utility is designated as A_1^k or $B_1^{j,k}$, our definition of the joint maximum realizing combination of imaginary games becomes

7.4.3 Either a or b is the case:

a) $U_{i_k}\left[A_1^k(m,n)\ell\right] \geq U_{i_k}\left[A_q^k(m,n)\ell\right]$ or $U_{i_k}\left[B_q^{j,k}(m,n)\ell\right]$

for all strategically non-trivial A_q, B_q $(q = 1, \ldots, r)$,
for all ℓ, and
for each and every i_k (where $k = 1, \ldots, n$)

b) $U_{i_k}\left[B_1^{j,k}(m,n)\ell\right] \geq U_{i_k}\left[B_q^{j,k}(m,n)\ell\right]$ or

$U_{i_k}\left[A_q^k(m,n)\ell\right]$

for all strategically non-trivial A_q, B_q
$(q = 1, \ldots, r)$
for all ℓ, and
for each and every i_k $(k = 1, \ldots, n)$

Thus, we have in 7.4.2 specified the exact relation between the real game $\mathcal{L}$ and the imaginary two player zero sum games A_h^k and $B_h^{j,k}$. In 7.4.3 we specified the joint maximum realizing combination of games A_h^k and $B_h^{j,k}$. Now, if each real player uses a pure or mixed strategy S_{i_k}, then the probabilities on $\mathcal{L}$ are given by $s_{i_1} \dots s_{i_n} = p(\ell)$ for all ℓ and for all i_k $(k = 1, \dots, n)$.

($s_{i_1} \dots s_{i_n}$ should be interpreted as follows, e.g. for n = 3:

$(s_1^1 s_1^2 s_1^3, s_2^1 s_1^2 s_1^3, s_1^1 s_2^2 s_1^3, s_1^1 s_1^2 s_2^3, s_2^1 s_2^2 s_1^3, s_2^1 s_1^2 s_2^3,$

$s_1^1 s_2^2 s_2^3, s_2^1 s_2^2 s_2^3)$ where the superscript is the player number, and the subscript is the strategy vector element number).

Thus our final axiom merely combines axiom 7.4.2 with the above expression:

7.4.4 Either $p(\ell) = s_{i_1} \dots s_{i_n} = A_h^k(m,n)\,\ell$ or

$p(\ell) = s_{i_1} \dots s_{i_n} = B_h^{j,k}(m,n)\,\ell$

for all ℓ, and for all uncorrelated strategy vectors

s_{i_k} $(k = 1, \dots, n)$

With these three axioms we can define a solution to any game in the normalization of 7.2.3:

7.4.5 We define a solution to any game in the normalization of 7.2.3 if and only if axioms 7.4.2, 7.4.3, and 7.4.4 are satisfied. One immediate result which we present without proof concerns the s_{i_e}, i.e. the "strategy" of the extra player in the games A_h^k. It turns out that only one type of strategy can satisfy the solution axioms. This result, which is trivial to prove, is as follows, where S_{i_o}, i_k designates a strategy vector for player i_o in a game A_h^k with player i_k:

7.4.6 $$S_{i_o}, i_k = S_{i_1} \ldots S_{i_j} \ldots S_{i_n}, \text{ for } j \neq k$$

Intuitively, 7.4.6 says that player i_o is a perfect transmitter of the actions of all real members of society, except i_k. Thus player i_o exercises no independent choices and in fact is an exact expression of the "general will". Nature, player i_o, uses passively optimal strategies, i.e. inefficient choices are not used. But beyond this, Nature does not use permanently optimal or even good strategies - strategies which will exploit a real player's mistakes or even hold a real player to a certain value in the zero sum interpretation game. This amounts to a partial answer to Weiner's objection to game theory based on Einstein's maxim "The Lord is subtle, but he isn't simply mean."[76]

7.5 A theorem for almost perfectly competitive n-player prisoner's dilemma games.

Consider the case where $a_o^k = o$. This means that nature neither pays out nor receives any amount when a member of a winning coalition. Thus, all "competitive" payments are strictly among the real players. There is, however, a payoff vector associated with the coalition $(i_1, \ldots, i_n)$ with $a_k > o$ for $k \neq o$. The three player prisoner's dilemma game of 7.2.2 is an example. We shall prove a theorem for this class of games of any n (≥ 2). We shall show that a good zero sum interpretation strategy is always mixed and is independent of the number of players.

The theorem can now be stated:

7.5.1 For any symmetrical prisoner's dilemma game in the normalization of 7.2.3, with $n \geq 2$, $a_o^k = o$ and $a_k > o$, we have

$$S_{i_k} = \left(\frac{a_k}{1 + a_k} , \frac{1}{1 + a_k} \right)$$

$$U_{i_k} = (a_k) \left(\frac{1}{1 + a_k} \right)^n$$

Proof

By reference to Figure 7.4.1 we find that the games A_h^k are 2xr matrices, where $r = 2^{n-1}$. By application of the zero sum interpretation definitions of section 7.3., row 2 of these matrices always has zero for every element (2,1), ..., (2, r-1). Element (2,r) always contains a_k which is $> O$ by assumption. Row 1 of these matrices always is composed of either o or 1 for every element (1,1), ..., (1,r). If row 1 dominates row 2, we have by application of 7.4.5, $U_{i_k} = O$ for <u>every</u> player $(i_o, i_1, \ldots, i_n)$. If row 2 dominates row 1, then we have $v = O \neq S_{i_k} A_1^k S_{i_o}$ since the only way row 2 could dominate is if every element of row 1 is zero. Therefore, we do not have row 2 dominating row 1 in our solution 7.4.5. If neither row dominates, then the S_{i_k} are mixed unless for some column q we have each element $= O$. But then we have $v = O \neq S_{i_k} A_h^k S_{i_o}$. Therefore, games A_h^k with some column having only zero components will not satisfy

7.4.5. These considerations leave only games A_h^k of the following form to be considered:

Figure 7.5.1

$$\begin{pmatrix} 1, \ldots, 1, o \\ o, \ldots, o, a_k \end{pmatrix}$$

Since a_k is fixed for any given game, these games are unique, i.e., for any a_k, there is only one such game. If it is solved, the maximum solution turns out to be $S_{i_k} = \left(\frac{a_k}{1 + a_k}, \frac{1}{1 + a_k} \right)$. Since the coalition games of 7.5.1 are perfectly symmetrical, and all competitive payments are strictly among $(i_1, \ldots, i_n)$, i.e. $a_o^k = o$, all competitive outcomes sum to 0, and we have $U_{i_k} = (a_k)\left(\frac{1}{1 + a_k} \right)$. Since $a_k > 0$, the $U_{i_k} > 0$, and this game A_h^k becomes A_1^k. We need only prove that no game $B_h^{j,k}$ is better than A_1^k, i.e. that

$$U_{i_k}\left[A_1^k \,(m,n)\, \ell \right] > U_{i_k}\left[B_h^{j,k} \,(m,n)\ell \right] \quad \text{for all games } B_h^{j,k}$$

and for all ℓ.

Only two possible outcomes to games $B_h^{j,k}$ could be considered in our solution: a saddlepoint on the purely cooperative outcome, or a saddle point on the purely competitive outcome, i.e. only these two outcomes can satisfy axioms 7.4.3 and 7.4.4 simultaneously. (Otherwise, the solution will be biased against at least one player. If he chooses a game $B_h^{j,k}$ biased in <u>his</u> favor, there will be competing sets of p (ℓ), with no basis for choosing between them). Now, the purely competitive outcome saddle point would assign

$$U_{i_k} = 0 < \left(\frac{1}{1 + a_k} \right)^n (a_k)$$ and therefore would not be considered.

This leaves only the saddle point at the purely cooperative outcome. This occurs in games of the following form:

Figure 7.5.2.

If all other players choose "not join i_o", then

Player i_2

Player i_1	join i_2	not join i_2
join i_o	$(i_o, i_1)U(i_o, i_2)$	(i_o, i_1)
not join i_o	(i_o, i_2)	$(i_1 \ldots, i_n)$

Now, element (1,1) is always fixed at 0 on this matrix; element (1,2) is always $\frac{1}{n-1}$; element (2,1) is always $\frac{-1}{n-1}$. This makes element (1,1) a saddle point independently of the value of (2,2). For (2,2) to be a saddlepoint, it would have to be simultaneously the minimum of its row and the maximum of its column. This means it would have to be both less than $\frac{-1}{n-1}$, and greater than $\frac{1}{n-1}$, which is not possible. Hence, (2,2) cannot be a saddlepoint. This completes the proof of the theorem.

An example of the application of this theorem is to game 7.2.2:

	coalition	payoff n-tuple
(i)	(i_o, i_1)	$(0, 1, -\frac{1}{2}, -\frac{1}{2})$
(ii)	(i_o, i_2)	$(0, -\frac{1}{2}, 1, -\frac{1}{2})$
(iii)	(i_o, i_3)	$(0, -\frac{1}{2}, -\frac{1}{2}, 1)$
(iv)	$(i_o, i_1) \cup (i_o, i_2)$	$(0, \frac{1}{2}, \frac{1}{2}, -1)$
(v)	$(i_o, i_1) \cup (i_o, i_3)$	$(0, \frac{1}{2}, -1, \frac{1}{2})$
(vi)	$(i_o, i_2) \cup (i_o, i_3)$	$(0, -1, \frac{1}{2}, \frac{1}{2})$
(vii)	$(i_o, i_1) \cup (i_o, i_2) \cup (i_o, i_3)$	$(0, 0, 0, 0,)$
(viii)	(i_1, i_2, i_3)	$(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Our solution to this game is as follows:

7.5.2

$P(i) = 9/64$

$P(ii) = 9/64$

$P(iii) = 9/64$

$P(iv) = 3/64$

$P(v) = 3/64$

$P(vi) = 3/64$

$P(vii) = 1/64$

$P(viii) = 27/64$

$S_{i_k} = (\frac{1}{4}, \frac{3}{4})$ for $K = 1,2,3$

$U_{i_k} = 9/64$ for $K = 1,2,3$

$U_{i_o} = -27/64$

The main point about the solution described by the theorem is that we have proved that a good zero sum interpretation strategy is independent of the number of players for $n \geq 2$, and $a_o^k = o$. This means that in a symmetrical game where $a_o^k = o$ and $a_k > o$, the decision making task of the real player is extremely simple, he need only know his own utilities and nothing else. The addition of new players does not complicate the decision making task. Indeed, a player need not even know how many other players there are in a game of this type, which seems reasonable for a non-negotiated game.

7.6 The limit e^{-b}

From theorem 7.5.1, we know that the probability of the occurrence of the coalition $(i_1, \ldots, i_n)$ is given by $\left(\frac{1}{1 + a_k}\right)^n$ when the game is of the prisoner's dilemma type and $a_o^k = o$. Since $a_k = \frac{b}{n}$ where $b = -a_o$, we can write this as $\left(\frac{1}{1 + \frac{b}{n}}\right)^n$. This expression approaches a well known limit[53] as n becomes very large. If b = 1, we have $\lim_{n \to \infty} \left(\frac{1}{1 + 1/n}\right)^n = e^{-1}$. In general, for any b (constrained by our assumption however to $b > o$) we have

$$\lim_{n \to \infty} \left(\frac{1}{1 + \frac{b}{n}}\right)^n = e^{-b} \quad (\text{where } e = 2.718 \ldots).$$

We may take this to be the probability of a cataclysmic human event, e.g. a successful revolution. (Compare Ashby, Section 4/23, for an analogous point about dynamic systems).

7.7 Two player symmetrical prisoner's dilemma games

General solution formulae can be derived for this case, but we shall not derive them here, although we shall present, without proof, an S_{i_k} for this case:

7.7.1 For 2-player, symmetrical prisoners dilemma games in the normalization of 7.2.3 with a^k_o o, we have,

$$S_{i_k} = \left(\frac{a_k + 1 - a^k_k}{a_k + 1}, \quad \frac{a^k_k}{a_k + 1} \right).$$

This formula is obtained by solving a general game with a^k_o o, and in the normalization of 7.2.3. Notice that if $a^k_o = o$, then $a^k_k = 1$, and the formula is the same as that in theorem 7.5.1. The reader will find that the second element of this strategy vector defines a result strikingly close to the fraction of cooperative choices obtained in the early plays of the consistent gaming experiments, such as those discussed in section 5.3. For example, the matrix of figure 5.3.1, which is

Figure 7.7.1

1,1	5,0
0,5	3,3

normalizes to the following:

Figure 7.7.2

$-\frac{3}{4}, -\frac{3}{4}$	$\frac{1}{4}, -1$
$-1, \frac{1}{4}$	$-\frac{1}{4}, -\frac{1}{4}$

The formula 7.7.1 gives us

$$\frac{a_k^k}{a_k + 1} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} \text{ as the}$$

probability weight on row 2 (or column 2). Correspondingly accurate probability weightings can be obtained with formula 7.7.1 for the other matrices of section 5. These are discussed in detail in reference 36 but the reader may compute the solutions himself using the formula 7.7.1, applied to the normalized games. The most interesting aspect of the formula 7.7.1 is that it is obtained in a game A_1^k, i.e., in an imaginary zero sum 2-player game between a real player and the extra player, who in the context of the experimental situation must be interpreted as the experimenter. Thus our conclusion is that in the early plays of the experiments, the real players have been playing against the experimenter, and not interacting with each other.

There is some evidence for this conclusion in the literature on experimental gaming. One experimenter, D.W. Conrath,[65] allowed the subjects to pass messages to each other between plays. The messages were censored to prevent offers of sidepayments or identification of the persons for later settlements. Conrath's comments should be quoted:

> "One additional point is worth noting. As we (experimenters) have defined it, the conflict to be resolved exists between two (or more) subjects. Some of the communication, however, indicated otherwise. A number of the subjects attempted to elicit cooperation by essentially asking the other party or parties to form a coalition with them against the experimental environment (or the experimenter). They would describe the game situation not as one of conflict between themselves, but as one of conflict between them and their environment. Experimental results, then, may not reflect how two subjects will behave toward each other in an isolated conflict situation. Rather, the results may indicate how well the subjects, once they have sized up the situation, can transform the conflict presumed to be between them to one in which they are on the same side. In a nonzero sum game environment this may be a common mechanism for conflict resolution, the transformation of an interpersonal conflict to one involving a game against 'nature'."

We have accounted for the overall fraction of cooperative choices, and given some evidence for the reinterpretation of the prisoner's dilemma game into a game against the experimenter. We shall now account for the decline in the fraction of cooperative choices which generally occurs in the first 30 plays. Our explanation is that the subjects in the experiments have been approximating a solution to the zero sum interpretation game (A_1^k) using an approximation technique such as Brown's algorithm discussed in section 1.7.

By averaging the four possible tables which one can generate using this algorithm, we can obtain various decline patterns, depending on the tie breaking rule:

If in doubt, choose the same as on the previous choice:

Play	5	10	15	20	25	30	35
% row 2	60	55	45	40	37	35	37

If in doubt, choose the opposite from the previous choice:

Play	5	10	15	20	25	30	35
% row 2	55	43	33	39	41	34	31

Thus, we appear to have a good idea of what happens in the early play of the prisoner's dilemma game. Notice that the dynamics of the experiments are merely approximating a static result. Moreover, as a comment on Professor Rapoport's remark about players not "getting the hang" of the game until after about 100 plays, we see that in fact humans approximate to a zero sum interpretation solution roughly as fast as a computer would, i.e., in as many iterations as a computer would require.

The solution axiom 7.4.5 in fact specifies a rote procedure by which a solution can be found. It is this rote procedure used on generalized games which yields the general formula, as we have seen in the proof of theorem 7.5.1. The rote procedure, of course, can be applied directly to a numerical example. One such example will now be worked out in detail. The game is as follows:

Figure 7.7.3

Player i_2

Player i_1

	$-\frac{1}{2}, -\frac{1}{2}$	$\frac{1}{2}, -1$
	$-1, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$

The coalition game is the following:

7.7.2

payoff vector	coalition
$(\frac{1}{2}, \frac{1}{2}, -1)$	(i_o, i_1)
$(\frac{1}{2}, -1, \frac{1}{2})$	(i_o, i_2)
$(1, -\frac{1}{2}, -\frac{1}{2})$	$(i_o, i_1) \cup (i_o, i_2)$
$(-1, \frac{1}{2}, \frac{1}{2})$	(i_1, i_2)

The generalized matrix form for games A_h^1 is as follows:

Figure 7.7.4

Player i_o

Player i_1		
	$(i_o, i_1) \cup (i_o, i_2)$	(i_o, i_1)
	(i_o, i_2)	(i_1, i_2)

By application of the zero sum interpretation definitions of section 7.3, we find that there are nine possible games A_h^k for each $K = 1,2$. These can conveniently be listed.

7.7.3

Game	Element: (1,1)	(1,2)	(2,1)	(2,2)
A_1^1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_2^1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_3^1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
A_4^1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_5^1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_6^1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
A_7^1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_8^1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
A_9^1	-1	0	$-\frac{1}{2}$	$\frac{1}{2}$

Game A_1^1, the solution game, is as follows:

Figure 7.7.5

Player i_o

Player i_1		
	0	$-\frac{1}{2}$
	$-\frac{1}{2}$	$\frac{1}{2}$

The strategy is $S_1 = (\frac{2}{3}, \frac{1}{3})$. (Game A_1^2 is the transpose at A_1^1 and A_h^2 is the transpose of the matrix of Figure 7.7.4).

The games $B_h^{1,2}$ can similarly be listed:

7.7.4	Game	Element: (1,1)	(1,2)	(2,1)	(2,2)
	$B_1^{1,2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	$B_2^{1,2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	$B_3^{1,2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0

The solution is as follows:

7.7.5

$$U_{i_k}\left[A_1^k(m,n)\,\ell\right] = -\frac{5}{18} > -\tfrac{1}{2} =$$

$$U_{i_k}\left[A_2^k(m,n)\,\ell\right] = U_{i_k}\left[A_3^k(m,n)\,\ell\right] =$$

$$U_{i_k}\left[B_h^{1,2}(m,n)\,\ell\right]$$

for all ℓ, for $h = 1,2,3$, and for $k = 1,2$.

Games A^k_4, A^k_5, A^k_6, A^k_7, A^k_8, A^k_9, fail to satisfy axiom 7.4.2 since $S_{i_1} A^k_h S_{i_o} \neq v$ for $h = 4, 5, 6, 7, 8, 9$, and for

$k = 1,2.$

An example of the failure to satisfy axiom 7.4.2 is game A^1_4:

Figure 7.7.6

Player i_o

Player i_1

	Minimax:	1	0
	S_{i_o} :	0	1
0		$-\frac{1}{2}$	$-\frac{1}{2}$
1		$-\frac{1}{2}$	$\frac{1}{2}$

$v = -\frac{1}{2}$

$S_{i_1} A^k_4 S_{i_o} = \frac{1}{2} \neq v$

Thus the p (ℓ) and the S_{i_k} are as follows:

7.7.6

$p(i) = 2/9$

$p(ii) = 2/9$

$p(iii) = 4/9$

$p(iv) = 1/9$

$S_{i_1} = S_{i_2} = (2/3, 1/3)$

$U_{i_1} = U_{i_2} = -\frac{5}{18}$

Incidentally, this solution is unique, because game A_1^k is unique, and its solution is unique by the proof in von Neumann and Morgenstern Section 18.2.5.

That the solution concept 7.4.5 reduces totally to the 2-player zero sum minimax concept when applied to 2-player zero sum games should not be surprising but we shall now demonstrate that fact. Consider the following game:

7.7.7

	coalition	payoff vector
i)	(i_o, i_1)	(0, 1, - 1)
ii)	(i_o, i_2)	(0, -1, 1)
iii)	$(i_o, i_1) \cup (i_o, i_2)$	(0, 0, 0)
iv)	(i_1, i_2)	(0, 0, 0)

The solution to this game, the computation of which we will not show here, is as follows:

7.7.8

$$p(i) = 0$$

$$p(ii) = 0$$

$$p(iii) = 1$$

$$p(iv) = 0$$

$$S_{i_1} = S_{i_2} = (1,0)$$

$$U_{i_1} = U_{i_2} = 0$$

The 2-player nonzero sum game embedded in the coalition game of 7.7.7 turns out to be a zero sum 2-player game, which we present with the solution in the sense of 7.4.5:

Figure 7.7.7

Player i_2

Player i_1

	1	0
1	0	1
0	-1	0

However, as the reader can easily verify, the solution to this game in the sense of 7.4.5 is identical to the von Neumann and Morgenstern minimax solution to this game.

8. Summary — zero sum interpretation and cybernetics

We have seen how the von Neumann and Morgenstern theory of n-player and nonzero sum games is based on a zero sum 2-player heuristic and we have also noted that the zero sum 2-player theory itself involves a heuristic limitation in the search for strategies. We have also seen that when the zero sum interpretation is abandoned, as in the Nash theory of non-negotiated games, an empirically false and intuitively objectionable paradox results — the prisoner's dilemma (for finite play). Much of game theory, therefore, i.e., all those portions which deal with zero sum games, can properly be thought of as belonging to the general field of heuristics, i.e., models which approximate results because no algorithm exists which is generally recognized to be satisfactory. The theory of zero sum interpretation of nonzero sum games, therefore, is part of the general field of heuristics in cybernetics.

We have seen that the zero sum heuristic is a very powerful one linking with linear programming, statistical decision making, decision making under uncertainty, Ashby's theory of regulation and control, the concept of requisite variety, the theory of neurals nets, artificial intelligence game playing programs, and negotiated n-player games. The extension of this heuristic to non-negotiated, nonzero sum games was clearly called for by the empirical importance of nonzero sum games. The development of this extension is the contribution of this thesis to game theory. We have seen that the extension developed in this paper has both intuitive and regorous empirical applications to the statics and dynamics of prisoner's dilemma gaming experiments. Moreover, formally, the dilemma simply does not appear for two player, and some n-player, games. We have seen that this is important since the prisoner's dilemma appears to be an unreal problem, i.e., motivated by faulty intuition. The solution concept may open cybernetics to the range of activity which can be described by nonzero sum games.

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