FORMAL LANGUAGES, PART THEORY, AND CHANGE

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David Bernard Benson

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ABSTRACT

A general definition of interpreted formal language is presented. The notion "is a part of" is formally developed and models of the resulting part theory are used as universes of discourse of the formal languages. It is shown that certain Boolean algebras are models of part theory.

With this development, the structure imposed upon the universe of discourse by a formal language is characterized by a group of automorphisms of the model of part theory. If the model of part theory is thought of as a static world, the automorphisms become the changes which take place in the world. Using this formalism, we discuss a notion of abstraction and the concept of definability. A Galois connection between the groups characterizing formal languages and a language-like closure over the groups is determined.

It is shown that a set theory can be developed within models of part theory such that certain strong formal languages can be said to determine their own set theory. This development is such that for a given formal language whose universe of discourse is a model of part theory, a set theory can be imbedded as a submodel of part theory so that the formal language has parts which are sets as its discursive entities.

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I. INTRODUCTION

The study of formal languages in the abstract has applications in mathematical logic, linguistics, and information science. In linguistics formal languages are used to approximate natural languages [2, 3, 4, 5].

In information science, formal language theory has been used to describe computer programming languages, for example [33], design question answering systems [8], and has potential applications in furthering the understanding of information system behavior. Possibly the greatest benefit to be gained from abstract formal language theory is the understanding of the limitations of formal languages. Recursive function theory [18, 9, 22] has made a major contribution to our understanding of these limitations, but we feel that much remains to be gained by considering both the syntax and semantics of formal languages abstractly.

Formal syntax has been quite thoroughly explored, for example [11, 15, 19, 25]. Ginsberg [11] has an extensive bibliography on formal syntax. Formal language semantics have been defined [17, 32, 33], but previously little was known about the properties of formal languages with both a syntactic and semantic component. This dissertation considers certain properties of such formal languages. Following Thompson [32], we define a formal language as a collection of semantic transformations on some universe of discourse. We then show that each collection of semantic transformations gives rise to an associated syntax which is the syntax of the formal language.

To provide a uniform and highly homogeneous universe of discourse for all formal languages, we consider part theory in Chapter III. Part theory is based on an axiomitization of the notion "is a part of", as in "the leg is a part of the table." We can consider a formal language to give a particular structure to the universe of parts, and we characterize this structure by a group of automorphisms of the universe of parts.

Our intuition indicates that for each pair of formal languages there is a language powerful enough to describe anything which is describable in either of the original languages, and that there is a language in which the only describable entities are those describable in both original languages. For example, the computer programming languages Algol and Fortran are similar, containing some similar syntactic constructions to describe the same process. The algebraic expressions of the two languages are in this category. A language consisting of just algebraic expressions is a lower bound language to Algol and Fortran. As a more powerful language in which we can express anything expressible in either Algol or Fortran, we take the assembly language of the computer. This is an upper bound language for Algol and Fortran. The problem of finding upper and lower bound languages for a pair of formal languages is partially solved by considering the lattice of groups of automorphisms and the Galois connection between the groups associated with formal languages and the semantic transformation closure on the groups.

The intuition also indicates that the universe of discourse of a formal language can always be considered to be a model of set theory. In Chapter V we show that it is possible to imbed a set theory within a sufficiently large model of part theory so that a given formal language simultaneously has both parts and sets as discursive entities. The central result of this dissertation is that certain strong formal languages determine their own set theory.

Finally, in Chapter VI we consider various research problems which the material in this dissertation has suggested. One of the more interesting of these is the possibility of defining formally our intuitive feeling about the distance between languages. For example, we feel that Algol and Fortran are much closer to each other than either is to Cobol. If we can realize this intention, our understanding of languages will be greatly enhanced.

II. FORMAL LANGUAGES

We present here material sufficient to define the general notion of a formal language as it will be treated in the sequel. A fuller treatment, including motivational material, is in Thompson [32], to which this chapter owes its genesis.

A language is often considered to have two components, syntactic and semantic. The syntax determines which sequences of words are well-formed or grammatical. The semantics establishes the meanings of the grammatical sequences. For us, a formal language will refer to a formalized semantics, from which a syntax can be derived.

Universe of Discourse

A formal language must talk about something, its universe of discourse. Since we are attempting to formalize notions of language, we will assume we have at our disposal a set theory, which we will use in the usual informal mathematical manner. In order to distinguish between this "outside" set theory and any particular model of set theory which may be the universe of discourse of a given formal language, we will refer, throughout the dissertation, to "outside" sets as classes and sets in any given model of set theory as sets. Since we will follow the Zermelo - Fraenkel axiomitization of set theory, the class-set distinction of Bernays - Gödel will not be needed.

The universe of discourse of a formal language is a model of a relational system, $U = < U_o, R_1, R_2, ..., R_n, ... >$ where U_o is an abstract class and the R_i are finitary relations defined on U_o. Hence, the objects interrelated by a formal language are elements of U_o.

In mathematical model theory there is usually some relationship between the formal language and the relations, R_i, of the model U, but no such restriction is implied here.

Suppose the universe of discourse is given as the relational system $U = < U_o, R_1, R_2, ..., R_n >$. Furthermore, suppose we have a model of set theory $M = < M_o, e >$, where e is the relation "is a member of". Then we can always model the given relational system within M, and we may, if we so desire, consider any formal language to have a model of set theory as its universe of discourse.

Semantic Categories

A semantic category is to be thought of as a collection of objects having some property in common. For example, if the model U is a ring, the zero-divisors of U can be thought of as a semantic category.

Since a formal language is to have a finite "computable" character (as opposed to a natural language, which may not be definable or computable), we will insist that a given formal language possess a finite number of semantic categories. Each semantic category may contain an infinite number of objects, and the semantic categories need not be disjoint. Furthermore, in the "metatheory" there may be an infinite number of categories, each corresponding to a property of $U = \langle U_o, R_1, \ldots, R_n, \ldots \rangle$, but only a finite collection of these are in a given formal language over U .

Let n be an integer, i. e., $n \in \omega$. Then $K = \{C_i | i \in n\}$ is a collection of semantic categories if each C_i is a subclass of U_o .

Semantic Transformations

The semantic transformations of a given formal language are the means for moving from object to object in the universe of discourse.

For example, if the universe of discourse is a model of set theory, then the function s such that $s(x) = x + \{x\}$ is such a semantic transformation, where + denotes set union.

Again, we will insist that a formal language be based on only a finite number of semantic transformations.

A semantic transformation is in general quite complex. It may take a sequence of objects into another sequence of objects and thus both its domain and its range may be subdirect products of semantic categories. Here range means the image of the domain under the given semantic transformation.

Let m be an integer.

Definition: $T = \{\tau_j | j \in m\}$ is a collection of semantic transformations over $K = \{C_i | i \in n\}$, if for each $j \in m$ the following hold:

i) There exists a domain function, d_j , from an integer δ_j to n , $d_j : \delta_j \rightarrow n$,

ii) There exists a range function, r_j , from an integer ρ_j to n , $r_j : \rho_j \rightarrow n$, and

iii) τ_j is a function from a subdirect product of

$$C_{d_j(1)} \times C_{d_j(2)} \times \ldots \times C_{d_j(\delta_j - 1)}$$

to a subdirect product of

$$C_{r_j(1)} \times C_{r_j(2)} \times \ldots \times C_{r_j(\rho_j - 1)} .$$

The domain functions, d_j , and the range functions, r_j , select the particular semantic categories used to form the direct products containing the domain and range of each semantic transformation.

The "property" that a semantic category is to correspond to, is the property of being a projection of the domain or image of a semantic transformation. We can say that a semantic category is a domain or image class of a semantic transformation.

Structural Semantic Transformations

A semantic transformation may be closely related to its universe of discourse in that it depends only upon the given relational structure. To make this notion precise, consider any permution, γ , of U_o . We write $x\gamma$ for the value of γ when applied to x . We say that γ commutes with a semantic transformation, τ , if

$$\tau(x_1\gamma, \ldots, x_\delta\gamma) = (y_1\gamma, \ldots, y_\rho\gamma)$$

whenever

$$\tau(x_1, \ldots, x_\delta) = (y_1, \ldots, y_\rho) .$$

Definition: A semantic transformation is structural if it commutes with every permutation that preserves the relations R_1, R_2, ... , of U .

As an example, let $M = \langle M_o, e, 0\rangle$ be a model of set theory, where e is the binary relation "is a member of" and 0 is the empty set. The proper automorphisms of M are determined by permutations of the individuals of the set theory which are not the empty set, and any such permutation determines an automorphism. Then the function List (x) = $\{\{y\} | y \, e \, x\}$ is structural. That is, the value of List (x) depends only on the set-theoretical structure of x and not on whether x is a particular set or individual.

If the model of set theory is pure, that is, the only individual is the empty set, then there are no proper automorphisms of the model; every semantic transformation is structural. This is a consequence of the extensionality of pure set theory, and suggests that one of the reasons for the general acceptance of set theory as a foundation for mathematics is the completely structural character of a pure set theory.

Constructive Semantic Transformations

A semantic transformation may be constructive. By this we mean that the semantic transformation can be defined in terms of given primitive semantic transformations by given methods of construction. To formalize this notion in any particular case, one must decide upon the primitive semantic transformations and what methods will be considered constructive.

The following example is a straight-forward generalization of the methods of recursive function theory, as found in Kleene [18] or Davis [9], and illustrates one notion of constructivity.

The universe of discourse is a model of set theory. A semantic transformation, τ, is constructive if there exists a finite class of defining equations from which the values of τ can be effectively computed in terms of the arguments of τ. The defining equations are written in terms of the primitive functions which follow: set union, written $+$; set intersection, written $\cdot$; set difference, $-$; and the singleton function, s. $s(x)$ is that set whose sole member is x, that is, $s(x) = \{x\}$. Clearly these primitive functions are structural. To give power to the recursive definitions, we include a choice function, c, among the primitive functions. The choice function is undefined on the individuals of the model. If the model of set theory includes two or more individuals other than the empty set, no choice function is structural.

We can now give the defining equations for the semantic transformation List(x).

$$\text{List}(x) = s(s\,c\,x) + \text{List}(x - s\,c\,x)$$

$$\text{List}(0) = 0$$

where s c x means $s(c(x))$ and 0 is the name of the empty set. We include equations of the form

$$\text{List}(a) = a$$

for every individual a in U.

To show that List is structural and constructive on finite sets, we first require that U have but a finite number of individuals, in which case the set of defining equations is finite. Now, by induction, List $(\{x_1, \ldots, x_n\}) = \{\{x_1\}, \ldots, \{x_n\}\}$, independent of the choice function, c . Hence, List is well defined and structural. Since List is only defined on finite sets, its domain is the semantic category of finite sets and its range is the class of those finite sets whose elements are singleton sets.

If a formal language is defined over a model of set theory, it seems reasonable to insist that the semantic transformations of the language depend only on the set-theoretical structure of the model, that is, that they be structural, and further, that they be constructive as illustrated above. Thompson [32] has taken this approach. In Chapter III we indicate that the structural semantic transformations are essentially trivial when the universe of discourse is a model of part theory.

Referents

Continuing to define the general notion of a formal language, we require a set of referents. Each referent is an entry point into the semantic structure of the formal language, the object that a word of the formal language names.

Definition: Let $K = \{C_i \mid i \in n\}$ be a collection of semantic categories. If X is a subclass of the union of the C_i , then X is a collection of referents.

Definition of a Formal Language

A formal language is a triple $<T, K, X>$ over a given universe of discourse U such that:

i) K is a finite collection of semantic categories over U ,

ii) T is a finite collection of semantic transformations over K , and

iii) X is a finite collection of referents over K .

Syntax

We discuss the relationship between a formal language and the usual notions of syntax as given in [32], [3], or [33].

A syntax is a quadruple <V, W, G, D> where: V and W are finite classes of abstract symbols such that $W \subseteq V$. V is the vocabulary of the syntax and W is the class of terminal symbols, or words. G is a disjoint union of G_v and G_w , each of which is a finite class of grammar rules. A grammar rule is an ordered pair of strings over V , written $\alpha \rightarrow \beta$, where β may be substituted for any occurrence of α as a substring in a string to produce a new string. The rules of G_v are over V-W , and the rules of G_w are of the form $v \rightarrow w$, where $v \in V\text{-}W$ and $w \in W$. D is a finite class of distinguished strings.

The language of the syntax <V, W, G, D> is that class of strings over W which can be produced from D by repeated application of the grammar rules.

The connection between a formal language and a syntax is given by the following correspondence.

If there exists a one-to-one correspondence, ϕ , between:

i) X and W ,

ii) K and V-W such that if $x \in X$ and $x \in C_i$ then

$\phi(C_i) \rightarrow \phi(x)$ is a grammar rule of G_W, and

iii) T and G_V such that if $\tau \in T$ has range in $C_{j_1} \times \ldots \times C_{j_\rho}$ and domain in $C_{i_1} \times \ldots \times C_{i_\delta}$ then $\phi(\tau)$ is the grammar rule

$$\phi(C_{j_1})\,\phi(C_{j_2}) \ldots \phi(C_{j_\rho}) \rightarrow \phi(C_{i_1}) \ldots \phi(C_{i_\delta}) ,$$

then the syntax $<V, W, G, D>$ is a proposed syntax for the formal language $<T, K, X>$. If, in addition, for each string $d \in D$ there is a semantic transformation with range in $C_{j_1} \times \ldots \times C_{j_\rho}$ such that $\phi(C_{j_1}) \ldots \phi(C_{j_\rho}) = d$ then $<V, W, G, D>$ is a syntax for the formal language $<T, K, X>$.

One may wish to impose additional restrictions on D, the class of preferred strings, so that for each $d \in D$, starting from referents, and by repeated applications of semantic transformations in T, it is possible to reach some $<y_1, \ldots, y_\rho> \in C_{j_1} \times \ldots \times C_{j_\rho}$ which corresponds to $d \in D$.

A significant aspect of the correspondence between the formal language and its syntax is the relationship between the repeated applications of grammar rules and composition of the semantic transformations.

We can describe the correspondence, ϕ, between the syntax and semantics as a contravariant functor [24], as indicated by the following diagrams:

$$\begin{array}{ccc} x & \in & C \\ \phi \downarrow & & \downarrow \phi \\ w & \leftarrow & v \end{array}$$

and

$$\tau: C_{i_1} \times \ldots \times C_{i_\delta} \to C_{j_1} \times \ldots \times C_{j_\rho}$$

$$\phi\downarrow \qquad\qquad \phi\downarrow$$

$$\phi(\tau): v_{i_1} \quad \ldots \quad v_{i_\delta} \leftarrow v_{j_1} \quad \ldots \quad v_{j_\rho} \; .$$

Note that the language of the syntax is in general larger than the class of meaningful strings, where "meaningful" means: mapping the strings into sequences of referents via ϕ and applying semantic transformations to obtain sequences of objects in the correct semantic categories to correspond to an element of D . The syntactic language is larger since the range of a semantic transformation may be a subdirect product of its image semantic categories, thus disallowing certain compositions of semantic transformations which appear syntactically correct. Similarly, if the domain of a semantic transformation is the subdirect product of semantic categories, certain strings of words may be syntactically correct, while their semantic counterpart will not be in the domain of any semantic transformation.

This is reasonable in view of Chomsky's example [2, p. 15]

"Green ideas sleep furiously. ",

which, while syntactically correct, is usually considered to be meaningless.

A grammar rule is called context-free if it is of the form $v \to \beta$ where v is a single symbol of V-W . If a semantic transformation is in correspondence with a context-free grammar rule, the semantic transformation is also called context-free. A context-free semantic transformation resembles the usual mathematical function, as is illustrated by the following commuting diagram.

$$\begin{array}{lccc} \tau: & C_{i_1} \times \ldots \times C_{i_\delta} & \rightarrow & C_\rho \\ & \downarrow \phi & & \downarrow \phi \\ \phi(\tau): & v_{i_1} \quad \ldots \quad v_{i_\delta} & \leftarrow & v_\rho \end{array}$$

The above diagram also illustrates that we can consider a grammar rule as the abstraction of the corresponding semantic transformation to the range and domain of the semantic transformation. We can say that the syntax is the surface effect of a semantic system.

What we have called here a formal language can also be considered the abstract semantics for the usual notion of a syntactic language, as in Ginsberg [11]. Since we have insisted upon a one-to-one correspondence between the formal language (semantics) and a syntax for it, we can construct a syntax for a given formal language if necessary, and so we will consider only formal languages in the sequel.

Derived Semantic Transformations

Semantic transformations can be composed in a manner similar to functional composition, although in a more complex fashion. The compositions of semantic transformations into derived semantic transformations is entirely analogous to the use of several grammar rules in the production of one syntactic string from another.

As a syntactic example, suppose we have the grammar rules

$$\alpha \rightarrow \beta\gamma\delta$$

$$\gamma \rightarrow \eta$$

where α , β , γ , δ are strings over the non-terminal vocabulary, V-W . Then we can derive $\beta\eta\delta$ from α by applying the rules in the above order. We will mirror just this kind of process in composing semantic transformations.

Consider the following direct product of semantic categories

$$A_1 \times \ldots \times A_a \times B_1 \times \ldots \times B_b \times C_1 \times \ldots \times C_c$$

where either a or b may be 0.

If we have a semantic transformation, τ_1 , with domain a subdirect product of

$$B_1 \times \ldots \times B_b$$

and range a subdirect product of

$$D_1 \times \ldots \times D_d \quad ,$$

then we may "derive" a subdirect product, K , of

$$A_1 \times \ldots \times A_a \times D_1 \times \ldots \times D_d \times C_1 \times \ldots \times C_c$$

corresponding to the image of τ_1 . If this subdirect product overlaps the domain of a semantic transformation, τ_2 , that is, the domain of τ_2 is a subdirect product of

$$A_1 \times \ldots \times A_a \times D_1 \times \ldots \times D_d \times C_1 \times \ldots \times C_c$$

which has a non-null intersection with K , then we may compose τ_1 and τ_2 to obtain a derived semantic transformation whose domain is the subdirect product of

$$A_1 \times \ldots \times A_a \times B_1 \times \ldots \times B_b \times C_1 \times \ldots \times C_c$$

determined by τ_1 and the extent of the intersection with the domain of τ_2, and whose range is a subdirect product of

$$E_1 \times \ldots \times E_e$$

where the range of τ_2 is also a subdirect product of $E_1 \times \ldots \times E_e$.

Because a derived semantic transformation depends on the subdirect product structure of the participating semantic transformation, derivations which are syntactically correct (that is, the appropriate direct products of semantic categories overlap), may not be semantically allowed. A study of syntax alone may not enable one to determine what phrases of a language are meaningful. If the domain and range of every semantic transformation are <u>direct</u> products of semantic categories, then every syntactically correct phrase is meaningful, at least in the sense that a derived semantic transformation can be applied to the objects corresponding to the words of the phrase.

The analysis of syntax in terms of semantic transformations suggests why linguists are currently using transformational grammars to study the regularities of natural language. Transformational grammars [4, 5] allow for more complex rules for transforming one string into another than simply replacing the occurrence of one string for another, as we have described above.

Definable Objects

What objects in a universe of discourse are definable by a formal language over that universe? There does not appear to be a unique answer to this question, so we will define and discuss two possible answers.

First we will agree that the referents of a given formal language are definable objects. Each referent corresponds to a word in the vocabulary of the syntax, i.e., objects with a name. Now suppose we have sequence of definable objects, for example a sequence of referents, $(r_1, \ldots, r_n)$, together with a semantic transformation, τ, defined on $(r_1, \ldots, r_n)$ such that

$$\tau(r_1, \ldots, r_n) = x .$$

Then we can agree that x is also a definable object. For example, the object defined by "All red ships are red." is the object corresponding to the word "true."

However, if $(r_1, \ldots, r_n)$ is a sequence of definable objects and

$$\tau(r_1, \ldots, r_n) = (x_1, \ldots, x_k)$$

it is not clear that each of $x_1 \ldots, x_k$ is a definable object, if we assume they are not otherwise definable. This obscurity leads to the two definitions of definable objects. The first declares that the x_i are not definable; the second admits each of the x_i as a definable object, but only in the context of the remaining $x_1, \ldots, x_k$.

Definition: An object, x, of a given universe of discourse, U, is context-free definable relative to the formal language < T, K, X> if it is a referent in X or if there is a sequence of referents $(r_1, \ldots, r_n)$ and a semantic transformation, τ , derived from the transformations of T , such that

$$\tau(r_1, \ldots, r_n) = x .$$

Definition: An object, x , of a given universe of discourse, U , is contextually definable relative to the formal language < T, K, X> if it is a referent in X or if there is a sequence of referents $(r_1, \ldots, r_n)$ and a semantic transformation, τ , derived from the transformations of T , such that

$$\tau(r_1, \ldots, r_n) = (x_1, \ldots, x_n)$$

and for some $i \leq n$, $x_i = x$.

In the first definition, an object may be context-free definable although some of the semantic transformations in T used to define the object are not context-free semantic transformations. In the second definition, an object, x_i , is contextually definable only if the objects entering into the derivation of $(x_1, \ldots, x_n)$ from $(r_1, \ldots, r_n)$ are all contextually definable. We have the obvious corollary of the above definitions that every context-free definable object is contextually definable.

The following example illustrates the notion of definable object. Suppose "2" is the name of a definable object in the universe of discourse, and suppose there is a semantic transformation whose value is the square root of its argument. Then, 1.414... is a definable object of the given universe.

Sentences

So far we have not discussed the question of what constitutes a sentence of a formal language. Syntactically, a sentence, as a phrase, is a distinguished string of the syntax, usually consisting of one symbol. As for words, a sentence is any string of words, i.e., any string over W , which can be produced by the repeated application of grammar rules from the distinguished string. This definition of a sentence, while mathematically productive (for example, see Ginsberg [11]), seems to be imposed on the syntax instead of arising naturally from the given language structure. In Chapter VI, we suggest possibilities for defining a sentence which may be more "natural".

We mirror the above definition of a sentence in the semantic structure by selecting a distinguished semantic category, S . The syntactic counterpart of S is a distinguished string in D . If there is a sequence of referents $(r_1, \ldots, r_n)$ and a derived semantic transformation, τ , such that

$$\tau(r_1, \ldots, r_n) \in S$$

we say that the string of words corresponding to $(r_1, \ldots, r_n)$ is a sentence, and that τ is a sentential semantic transformation. If we feel that a sentence must be either "true" or "false", then the semantic category S is a class consisting of two objects, say 0 and 1 . In a multi-valued logic system, S is a class consisting of as many distinct objects as there are values to the logic, and if a sentence possesses a "probability" of being true, then S is the unit interval of the real line. Because of the apparent artificiality of the above definition of a sentence, we will consider definable objects and the semantic counterpart of phrases, rather than sentences, to be the basic units of meaning in the sequel.

Analytic vs. Productive Language Models

We have treated the semantic transformations of a formal language as analytic transformations. That is, as the recipient of a sequence of words would analyze the phrase to discover its meaning. At the same time, the syntax has been treated productively. Starting with a distinguished phrase, grammar rules are repeatedly applied until a string of words over the terminal vocabulary results. However, by letting the semantics determine the syntax, we are in effect using the syntactic structure in an analytic manner as well.

A complete definition of a formal language should include a method for synthesizing sequences of utterable words. It should provide for transforming definable objects into sequences of referents, which in turn correspond to strings of words. One possibility is to insist that every semantic transformation be invertible, so that it could be used either analytically or productively; another is that different collections of semantic transformations are used to produce phrases and to recognize phrases.

We will concentrate on analytic formal languages, and leave their exact relationship to productive formal languages an open problem.

III. PART THEORY

We mentioned that the universe of discourse can always be considered to be a model of set theory. This implies that we are willing to make certain strong ontological assumptions about the universe of discourse, namely that various infinite sets exist and that the world is atomic and almost well-founded. By atomic, we mean that there are entities which have no proper subsets other than the empty set and that every entity is a union of atomic sets. By almost well-founded, we mean that the axiom of regularity holds for all sets except possibly certain individuals which have no members but are not the empty set.

Presumably there is no difficulty about accepting finite sets. However, to have a set theory, we must admit some very large sets, such as the continuum, and impredicatively defined sets. Even the set of integers may be suspect due to the following reasoning. We define the integers by stating certain properties of the successor function. This statement is a linguistic process and we can argue that the set of integers is actually just the linguistic statement defining the integers. For mathematics, this distinction makes no difference, for we still have entities satisfying the axioms of set theory. However, in the view of set theory in which only linguistic elements exist, the only sets are those which can be defined by a formula of set theory. This is very close to the constructivist point of view [6, 13] in which the continuum, for example, does not seem to match our intuition regarding the real line. This linguistic view of set theory can be said to regard large sets as fictions. As Cohen observes, "The great defect with this view is that it leaves unexplained why this fiction is successful and how a presumably incorrect intuition has led us to such a remarkable system." [6, p. 150]. The material in Chapter V suggests an explanation of why this fiction is successful.

The second difficulty with set theory is its atomic character. While obviously useful for mathematics, it is not clear that we can adequately model the real world in set theory partly because of this property. If we model an electron as a set we must decide what its elements are to be. For different purposes we model it differently, for example as a collection of quantum states or as a point charge. It is not clear that a single model of an electron can subsume all the models of an electron that we may wish to make, especially considering that new properties of electrons may be discovered, for example, superconductivity. What appears to be required is a model of the world which allows for new properties of entities to be discovered and new interrelationships to be explored.

The fact that almost everything must be constructed on the empty set is another problem with set theory. This is closely related to atomicity, and leads to the same conclusion that every interrelationship among entities modeled by sets and individuals is inherent in the set theory, and there is no room for discoveries. Suppose the set theory is pure, that is, there are no individuals. Then every set is well-founded and let us consider modeling the electron again. An electron in the model is some complicated set, but all of whose elements are set-theoretical constructions built on top of nothing, that is, on the empty set. If we find this notion philosophically unpalatable, then we can consider including some individuals in our set theory. Now the electron set can have individuals as members of members of ... . However, there is nothing within set theory which allows us to distinguish between individuals and thus between sets with the same structure over "different" individuals. So the electron set's elements are set-theoretical constructions built on top of nothing or on top of indistinguishable structureless somethings. This does not appear any more acceptable than a pure set theory.

Since accepting set theory as the universe of discourse requires accepting atomicity, well-foundedness, and the existence of large and strangely defined sets, and these lead to philosophical difficulties, we ask what can be accomplished by making weaker ontological assumptions than those required by set theory. We will assume only that an entity can be part of another entity. The sole predicate of the theory is the notion "x is a part of y", formalized as $x \pi y$. The theory of the part-whole relationship as developed in the next section requires only three axioms and one axiom schema, indicative of the weaker ontology. Earlier discussions of the theory of parts appear in Goodman [14] and Tarski [29]. Tarski's axiom system depends on the availability of a set theory. The approach here is to develop part theory independently of set theory. Goodman considers his part theory to be a calculus of individuals. Our axiom system is similar in intent to Goodman's, although more completely formalized. We assume that the only entities are the "individuals".

The theory of parts includes a means of constructing entities from parts described by a formula, in analogy to the axiom of replacement of set theory. However, the entities formed in this way do not appear to have a different character than other parts. In set theory, our intuition becomes shakier as we move from the integers to the countable ordinals to the continuum and impredicatively defined sets. In part theory, the world is far more homogeneous and every part has about the same credibility as a "real" entity. Since a model of part theory need not have atoms, we can label certain parts of an electron as distinguished and then find proper parts of the distinguished parts to explore and so on.

Well-foundedness does not apply to the theory of parts, and we can label a particular part as an electron without having specified anything about its structure. Because of homogeneity, the electron looks approximately like any other part, but we accept this situation as follows. The "structure" of any part is imposed on it by an observer. As the formal development shows, the homegeneity of parts means that any two parts can have the same "structure". Another way of saying this is that every part can have any structure and the particular structure of a part, say an electron, is that one selected by an observer. This ontological position is developed formally in Chapter IV, with a formal language replacing an "observer". As we will see, we can recover a set theory from within a model of part theory, demonstrating the existence of any particular set-theory-based structure that we please.

As a method of exploring linguistic and mathematical notions, the theory of parts seems to be a powerful tool. It also seems to be in line with some current thought in the philosophy and history of science [26, 20]. As to whether it is a completely adequate model of reality, the answer is of course no. It does seem to offer possibilities for furthering our understanding of formal linguistics and the relationship between a language and its universe of discourse.

Axioms and Basic Theorems

This axiomitization of part theory was developed by F. B. Thompson. Some of the theorems in this section are due to F. B. Thompson, the remainder to R. Lambert [21]. The theorems are stated here without proof, but we attempt to give the intuition behind the axioms and basic theorems of the system.

Axiom 1: $\forall a \forall b[a = b \leftrightarrow \forall c(c \pi a \leftrightarrow c \pi b)]$.

a is the same entity as b if and only if they share all their parts in common. This axiom of extensionality for parts could alternately be taken as a definition of equality in part theory. Note that a and b are not necessarily parts of any entity.

Axiom 2: $\forall a \forall b[a \pi b \leftrightarrow \exists c(a \pi c) \,\&\, \forall d(d \pi a \rightarrow d \pi b)]$.

a is a part of b if and only if a is part of something and every part of a is a part of b . This axiom is a strong form of transitivity for parts.

The following three theorems establish that π is a partial ordering.

Theorem 3: $\forall a[\exists c(a \pi c) \rightarrow a \pi a]$.

Theorem 4: $\forall a \forall b[a \pi b \,\&\, b \pi a \rightarrow a = b]$.

Theorem 5: $\forall a \forall b \forall c[a \pi b \,\&\, b \pi c \rightarrow a \pi c]$.

Definition 6: Let $F(a; x_o, \ldots, x_{n-1})$ be a formula of the lower predicate calculus in which b , c , d and e are not free. The only predicate in F is π and $x_o, \ldots, x_{n-1}$ are the names of n entities. Then, $c \pi P_a[F(a; x_o, \ldots, x_{n-1})]$ if and only if

$$\forall d[d \pi c \rightarrow \exists a \exists e[F(a; x_o, \ldots, x_{n-1}) \,\&\, e \pi a \,\&\, e \pi d]] \,\&\, \exists f(c \pi f) .$$

$P_a[F(a)]$ is that entity formed by "conglomerating" all the a such that $F(a)$. In forming the conglomerate, parts other than the a satisfying $F(a)$ may be parts of the conglomerate and the definition specifies which parts are to be included. In words, c is a part of the conglomerate if every part of c meets some a satisfying $F(a)$. This definition and the following axiom schema hold a position in the theory of parts analogous to the axiom of replacement in Zermelo-Fraenkel set theory. While the axiom of replacement guarantees that the range of a function is a set, here the following axiom schema guarantees that each conglomerate exists and is the least upper bound to the collection of a satisfying $F(a)$. That is, we can find an entity whose parts are just those defined by the formula F together with all of their parts and the various combinations of these.

Axiom Schema 7: $\exists b \forall c[c \pi b \leftrightarrow c \pi P_a[F(a; x_o, \ldots, x_{n-1})]]$.

Theorem 8: $\exists! b \forall c [c \pi b \leftrightarrow c \pi P_a [F(a; x_o, \ldots, x_{n-1})]]$.

The conglomerate is unique and thus is the least upper bound, with respect to the partial ordering π , to the collection of a satisfying F(a) .

Axiom 9: $\forall a \forall b [\neg a \pi b \; \& \; \exists c(a \pi c) \rightarrow \exists d(d \pi a \; \& \; \forall e \neg [e \pi d \; \& \; e \pi b])]$.

If a is not a part of b and a is part of something, then there is a part of a which is disjoint from b , that is, has no part in common with b . This axiom guarantees that a and b are distinguishable by some part which they do not share. The following theorem illustrates this.

Theorem 10: $\forall a \forall b [a \pi b \leftrightarrow \forall c(c \pi a \rightarrow \exists d [d \pi c \; \& \; d \pi b]) \; \& \; \exists e(a \pi e)]$

a is a part of b if and only if every part of a has a part in common with b and a is part of something.

Theorem 11: $\forall b [b = P_a [a \pi b]]$

This theorem shows that the theory of parts is well-formed; b is the conglomerate or least upper bound of the collection of its parts and since every part can be defined by a formula, every part has roughly the same degree of credibility.

Axioms 1, 2, and 9 together with axiom schema 7 constitute the main collection of axioms of the theory of parts. The remaining definitions and theorems develop the theory far enough to make it clear that a Boolean algebra can be a model of part theory.

Definition 12: $1 = P_a [a = a]$.

1 is the universe of parts, obtained by taking the conglomerate over a tautology.

Theorem 13: $\forall a[a \pi 1 \longleftrightarrow \exists b(a \pi b)]$.

Every entity which is a part is a part of the universe and vice versa.

Definition 14: $b + c = P_a[a \pi b \vee a \pi c]$.

This defines the union of the parts b and c , but does not guarantee that b + c is a part of the universe.

Theorem 15: $b + c = P_a[a = b \vee a = c]$.

This theorem illustrates the nature of conglomerating. $P_a[a = b \vee a = c]$ has as parts every part of b and every part of c , as the following theorem shows.

Theorem 16: $\forall a \forall b \forall c[a \pi b \vee a \pi c \rightarrow a \pi b + c]$.

Theorem 17: $\forall b \forall c \forall d[(\forall a[a \pi b \vee a \pi c \rightarrow a \pi d] \& [b+c \pi 1]) \rightarrow b+c \pi d]$.

If b + c is a part of the universe, then b + c is the least upper bound of b and c .

Definition 18: $b \cdot c = P_a[a \pi b \& a \pi c]$.

This defines the intersection of two parts, which may not always exist as a part of the universe.

Theorem 19: $\forall a \forall b \forall c[a \pi b \& a \pi c \rightarrow a \pi b \cdot c]$.

$b \cdot c$ is the greatest lower bound of b and c .

Theorem 20: $\forall b \forall c[b \cdot c \pi 1 \rightarrow b \cdot c \pi c \;\&\; b \cdot c \pi b]$.

If $b \cdot c$ is part of the universe, then it is a part of b and a part of c .

Definition 21: $0 = P_a[a \neq a]$.

Zero is the conglomerate over a contradiction. Zero is not part of any entity and has no parts as the following two theorems show. This is one of the more pleasing aspects of the theory of parts. Zero is the only entity without parts and can be said to be "nothing at all". This is contrasted with the empty set which has many relations with the remaining sets, such as being a subset of every set and being a member of certain sets.

Theorem 22: $\neg[0 \pi 1]$.

Zero is not a part of the universe, so by Theorem 13, it is not a part of any entity.

Theorem 23: $\forall a[(\forall b \neg (b \pi a)) \leftrightarrow a = 0]$.

Zero has no parts, including itself, and is the only such entity.

Theorem 24: $\forall b \forall c[\neg \exists a(a \pi b \;\&\; a \pi c) \leftrightarrow b \cdot c = 0]$.

b and c have no part in common if and only if their intersection is zero. This theorem is another indication that the theory is well-formed.

Theorem 25: $\exists b \exists c[b \pi 1 \;\&\; c \pi 1 \;\&\; b \neq c] \rightarrow$

$\neg \forall b \forall c[b \pi 1 \;\&\; c \pi 1 \rightarrow b \cdot c \pi 1]$.

Theorem 26: $\forall b[b+0 = b]$.

Theorem 27: $\forall b[b\cdot 0 = 0]$.

The above three theorems illustrate the relations between zero and other entities.

Theorem 28: $\forall a[a\pi b\cdot c \rightarrow a\pi b \; \& \; a\pi c]$.

Definition 29: $\tilde{b} = P_a[\forall c(c\pi a \rightarrow \neg\, c\pi b)]$.

This defines the complement of b , which is unique by theorem 8 .

Theorem 30: $\tilde{0} = 1 \; \& \; \tilde{1} = 0$.

Theorem 31: $\forall a \forall b \neg [a\pi b \; \& \; a\pi\tilde{b}]$.

b and its complement are disjoint.

Theorem 32: $\forall a[a+\tilde{a} = 1]$.

Part Theory and Boolean Algebra

The theorems of the previous section indicate that a model of part theory is also a model of a Boolean algebra. What Boolean algebras are models of part theory? The answer is only very uniform Boolean algebras. We will show that a Boolean algebra which is a direct product of an atomic Boolean algebra and an atomless Boolean algebra is a model of part theory. We let $a\pi b$ correspond to $(a \leq b \; \& \; a \neq 0)$ in the Boolean algebra and proceed to prove the axioms of part theory as theorems of Boolean algebra. As we will demonstrate, all of the axioms are straight-forward to prove, except axiom schema 7. The proof of axiom schema 7 requires that the least upper bound of

$$\{a \mid F(a; x_o, \ldots, x_{n-1})\}$$

exist in the Boolean algebra. This existence proof is obtained by applying the method of elimination of quantifiers [31] to $F(a; x_o, \ldots, x_{n-1})$, reducing this formula to a standard form for which it is possible to show the existence of the least upper bound. Since the proof outlined here is long, it has been relegated to the appendix. The proof uses, critically and in two different places, that the Boolean algebra is a <u>direct</u> product of atomic and atomless factors. This makes it most likely that the direct product condition is necessary in order that a Boolean algebra be a model of part theory. Now, assuming that $\sum_{F(a)} a$ exists, we prove the axioms of part theory as Boolean algebraic theorems.

Let $a \pi b$ be interpreted as $(a \leq b \;\&\; a \neq 0)$ in the Boolean algebra. The operations in the proofs to follow are Boolean algebraic and the 0 and 1 of part theory will be interpreted as the 0 and 1 of the Boolean algebra.

Axiom 1: $\forall a \forall b[a = b \longleftrightarrow \forall c(c \pi a \longleftrightarrow c \pi b)]$.

Proof: Assume $a = b$. Then $c \leq a$ if and only if $c \leq b$. Now assume $\forall c(c \pi a \longleftrightarrow c \pi b)$. In particular, $(a \leq a \;\&\; a \neq 0) \rightarrow (a \leq b \;\&\; a \neq 0)$ and $b \leq b \rightarrow b \leq a$, hence $a = b$.

Axiom 2: $\forall a \forall b[a \pi b \longleftrightarrow \exists c(a \pi c) \;\&\; \forall d(d \pi a \rightarrow d \pi b)])$

Proof: Assume $a \pi b$. Then $a \leq a$ and every d less than or equal to a is also less than or equal to b . Now assume $\exists c(a \pi c) \& \forall d(d \pi a \rightarrow d \pi b)$. Then if $\neg(a \leq b)$, there is an $e \leq a$ such that $\neg(e \leq b)$, contradiction.

Axiom schema 7: $\exists b \forall c[c \pi b \longleftrightarrow c \pi P_a[F(a)]]$ where we have dropped the parameters $x_o, \ldots, x_{n-1}$ in F for clarity.

We require two lemmas before proceeding with the proof of axiom schema 7.

Lemma 1: $\sum_{e \leq a} a$ exists and is equal to a.

Proof: We know that a is an upper bound to the set $\{e | e \leq a\}$. Since $a \in \{e | e \leq a\}$, it is the least upper bound.

Lemma 2: If $\sum_{F(a)} a$ exists, then

$$\sum_{F(a)} a = \sum_{F(a)} [\sum_{e \leq a} e] = \sum_{e \leq a \& F(a)} e$$

Proof: This is a simple application of infinite associativity, as in Sikorski [28, p. 59].

We turn to the proof of axiom schema 7. Recall that $c \pi P_a [F(a)]$ if and only if

$$\forall d[d \pi c \rightarrow \exists a \exists e[F(a) \& e \pi a \& e \pi d]] \& \exists f(c \pi f) \ .$$

Consider $b = \sum_{F(a)} a$, which exists by the proof in the appendix. By lemma 2, $b = \sum_{e \leq a \& F(a)} e$. Consider any $c \leq b$ which is not 0 . First of all, $\exists f(c \pi f)$ since for all c , $c \leq 1$. Now consider any $d \leq c$ which is not 0 . Suppose for all e in $\{e | e \leq a \& e \neq 0 \& F(a)\}$ it is the case that $e \not\leq d$. Then d is disjoint from each e so that

$d - b \neq 0$, contradicting $d \leq b$. Hence, $\exists a \exists e[F(a) \& e \pi a \& e \pi d]$.

Now suppose there exists a c satisfying $\forall d[d \pi c \rightarrow \exists a \exists e(F(a) \& e \pi a \& e \pi d)]$. We will show that $c \leq b = \sum_{F(a)} a$. Suppose the contrary. Then $x = c - b$ is not zero and $x \leq c$. But then $\exists a \exists e(F(a) \& e \leq a \& e \leq x \& e \neq 0)$ which implies that $e \leq b$, and $e \leq b \& e \leq x$ contradicts $x \cdot b = 0$.

The Model

If we accept "is a part of" as the fundamental notion for discussing informational entities, then we must face the question of how many parts the universe possesses. The answer must be an infinite number. Consider some part, say a sheet of paper. It possesses conceptual parts like the top two-thirds and the margin. These parts are potentially infinite. Are there any parts which possess no proper subparts, that is, are there any parts which are atoms? Assuming that there are no atoms, we can always divide any part of the universe into smaller subparts. For example, an electron can be divided into its mass, momentum, position, charge, and so on, while the electron's mass can be divided into rest mass and energy mass, and so on as long as we please. There is no claim here that the parts into which we divide the electron are unique or necessarily useful for physical theory. The only claim is that we always find a proper part of any part of an electron.

The assumption of atomlessness can be formalized as an additional axiom of part theory as follows:

$$\forall a[a \neq 0 \rightarrow \exists b(b \pi a \& b \neq a)] .$$

With this axiom, only the atomless, therefore infinite, Boolean algebras are models of part theory. We know from the Löwenheim-Skolem theorem that any set of axioms has a countable model. In particular, the theory of atomless parts has a countable model. Without becoming involved in model-theoretic considerations we present a countable atomless Boolean algebra, P , as a model of part theory. Consider half-open intervals (x, y] such that $0 \leq x$, $y \leq 1$ and such that x and y are rational. Each such interval is a part where $(x_1, y_1] \ \pi \ (x_2, y_2]$ if and only if $x_1 \geq x_2$ and $y_1 \leq y_2$. (0, 1] is the universal part of part theory and the unit of the Boolean algebra. Finite unions of parts are parts. Furthermore, this model is isomorphic to every countable atomless Boolean algebra and to the free Boolean algebra on a countable number of generators [10, p. 54]. P is the smallest of the class of models we consider. The remaining sections of this chapter discuss properties of P . Larger models, under suitable conditions, also possess these properties.

Size

The size of each part of P can be defined by defining a measure on P. Such a measure exists. For example, a normed finitely additive effective measure can be obtained as follows [10, p. 56]: since each part of P is uniquely expressible as a finite union of disjoint parts, say, $p = \sum_{i \in n} (x_i, y_i]$, we define the measure, μ , on each part as $\mu(p) = \sum_{i \in n} (y_i - x_i)$, where here Σ means addition and $y_i - x_i$ is the length of the interval $(x_i, y_i]$. Define the measure of the zero of P to be 0 .

The size of a part can be thought of as the importance of the part, its probability, or as its "physical" size. Also, the measure can be used to define a metric on P by setting the distance between two parts, p_1 and p_2 , in P to the measure of their symmetric difference, $d(p_1, p_2) = \mu(p_1 \dot{-} p_2)$, where $p_1 \dot{-} p_2 = (p_1 - p_2) + (p_2 - p_1)$.

We prove a lemma useful in the following section.

Lemma: Given $\epsilon > 0$ and a sequence of disjoint parts $p_1, p_2, \ldots$, if $\mu(p_i) \geqslant \epsilon$ for each i , then the sequence is finite, $p_1, \ldots, p_n$.

Proof: $\mu(p_i + p_j) = \mu(p_i) + \mu(p_j)$ for all $i \neq j$ since p_i and p_j are disjoint. The measure of the union of at most $1/\epsilon$ of the p_i is equal to 1 . Since the measure of all parts is less than or equal to 1 , there are only a finite number of disjoint parts with measure $\geqslant \epsilon$.

Limit Points and Measure

Let Y be a subclass of a model of part theory. A point is a limit point of Y if there is a sequence of points in Y which, eventually, do not exclude any part of the limit point and eventually do not include any point not part of the limit point. Formally, we have

Definition: y is a limit point of Y if there is a sequence $y_1, y_2, \ldots$ of elements of Y such that for any $z \neq 0$, there exists an N for which $j > N$ implies $z \cdot (\overline{y \dot{-} y_j}) \neq 0$.

To show that this is equivalent to the sentence above the definition, let $z \leqslant y$, $z \neq 0$. Then for some N and all $j > N$,

$$z \cdot (\overline{y \dot{-} y_j}) = z \cdot [(y \cdot y_j) + (\tilde{y} \cdot \tilde{y}_j)] = z \cdot y_j \neq 0 \ .$$

Hence z is not excluded. Similarly, let $z \leq \tilde{y}$, $z \neq 0$. Then for some N and all $j > N$,

$$z \cdot (\overline{y \dot{-} y_j}) = z \cdot \tilde{y} \cdot \tilde{y}_j = z \cdot \tilde{y}_j \neq 0$$

so that z is not included.

Let P be the model of part theory previously defined and let μ be the normed finitely additive effective measure we have defined on it. We defined the distance between x and y in P by $d(x, y) = \mu(x \dot{-} y)$. We wish to show that a point is a limit point of a sequence in P if and only if it is a limit point in the metric space determined by the measure. Hence the topology determined by the definition of limit point given above and the metric topology determined by the measure can be made to coincide.

Theorem: y is a limit point of Y if and only if $\mu(y \dot{-} y_i) \to 0$ as $i \to \infty$ for some sequence $y_1, y_2, \ldots$ in Y .

Proof: Suppose $\mu(y \dot{-} y_i) \to 0$ as $i \to \infty$ for some sequence $y_1, y_2, \ldots$. Consider any $z \neq 0$. z has non-zero measure since μ is effective. There exists an N such that for all $j > N$, $\mu(y \dot{-} y_j) < \mu(z)$, and so for all $j > N$, $z \cdot (\overline{y \dot{-} y_j}) \neq 0$. For suppose for some $k > N$, $z \cdot (\overline{y \dot{-} y_k}) = 0$. Then $z \leq (y \dot{-} y_k)$ thus $\mu(z) \leq (y \dot{-} y_k)$, contradiction. This shows that y is a limit point of $\{y_1, y_2, \ldots\}$.

Suppose y is a limit point defined by the sequence $y_1, y_2, \ldots$ and that $\mu(y \dot{-} y_i)$ does not converge to 0 . We will show that for some $\epsilon > 0$ there is an infinite subsequence of y_i such that $\mu(y \dot{-} y_i) \geq \epsilon$. For if $\mu(y \dot{-} y_i)$ does not converge to 0 there is an

$\epsilon > 0$ such that for all M there exists an $i > M$ such that $\mu(y \dot{-} y_i) \geq \epsilon$. Since y is a limit point, there exists an N such that for all $j > N$ $(y \dot{-} y_i) \nleq (y \dot{-} y_j)$. We form the infinite subsequence as follows: Let M = 1 . Then there is an i_1 such that $\mu(y \dot{-} y_{i_1}) \geq \epsilon$. i_1 determines an N such that for all $j > N$ $(y \dot{-} y_{i_1}) \nleq (y \dot{-} y_j)$. Now let $M = \max(i_1, N)$ to determine i_2 . Continuing in this way, we have $i_k < i_{k+1}$ and for each k, $\mu(y \dot{-} y_{i_k}) \geq \epsilon$. By the lemma of the previous section, the $(y \dot{-} y_{i_k})$ of this subsequence are not disjoint. In fact, the lemma shows there must be a $z \neq 0$ such that $z \leq (y \dot{-} y_{i_k})$ for an infinite number of k . This contradicts the assumption that y is a limit point, proving that $\mu(y \dot{-} y_i) \to 0$ as $i \to \infty$ if y is a limit point.

Automorphisms

An automorphism on P is a one-to-one function, g , from P onto P such that $g(x) \pi g(y)$ iff $x \pi y$. If P is countable and atomless, P has 2^ω automorphisms [10, p. 50]. If P is a model of a static world, then the automorphisms model the changes which take place in that world. If an automorphism g interchanges two parts, then the change which has taken place is that interchange. This notion of change is very general and does not lead directly to a notion of time. However, by metrizing the group of automorphisms of P , we can consider time to be a continuous map from the real line to the group of automorphisms. If g and h are automorphisms, define $d^*[g, h] = \sup_P d[g(x), h(x)]$.

There are some philosophical difficulties associated with this notion of change, which the following example illustrates. Suppose we have a red pencil, which burns and changes to black soot. If the redness of the pencil is a part of the pencil, then under the automorphism changing the red pencil into black soot, the "red" part is transformed into some part of the soot, and the soot is not red. Hence the automorphism does not preserve sensual redness. Furthermore, some part of the red pencil must be transformed into the "black" part of the soot even though it appears that the red pencil has no black part. For the time being, we just accept this difficulty as indicating that our notion of change is a rather crude one. We will return to this problem in Chapter VI, after having considered the relationship between part theory and formal languages.

IV. GROUPS OF AUTOMORPHISMS

In this chapter we develop some of the consequences of considering a formal language to have a model of part theory as its universe of discourse. One of these consequences is that the structural semantic transformations are the Boolean functions on the model of part theory, including the infinite Boolean functions defined in terms of the abstraction operation considered in this chapter. The interesting semantic transformations are not *a priori* structural but impose a richer structure on the model of part theory than it originally possessed. In this sense we can say that a formal language determines the structure of its universe of discourse.

This structure can be characterized by a group of automorphisms. The results in this chapter stem from considering the groups of automorphisms associated with formal languages. With each formal language we associate the largest group of automorphisms under which the formal language is invariant. The characterization of a formal language by its associated group is imperfect in the sense that several formal languages may be associated with a given group. However one may then say that these several languages all give the same structure to the universe of discourse.

In considering this structure, we see that the parts definable by the formal language are those to which it gives additional structure, and undefinable parts are left unrestricted, except in so far as they are parts of definable entities. The following example illustrates several matters which are treated formally in the remainder of the chapter.

Suppose we have a formal language which discusses the interrelationships of objects in a room. The referent words of the language may be "desk", "table", and "chair" together with function words which select particular semantic transformations, such as "to the left of" and "under". In this formal language, the left two-thirds of a desk is an undefinable part. Without additional referents and possibly additional semantic transformations, this entity is indescribable in the given formal language. In this case a desk is an atom in the algebra of definable parts.

Continuing this example, suppose in the room which is our universe of discourse every desk is to the left of a table. If we permute the desks then the structure "every desk is to the left of a table" remains invariant. These permutations are in the group of the formal language provided they preserve all of the structure determined by the formal language. Assuming the permutations of desks do preserve the language, we see that two desks cannot be distinguished if they are permuted one into the other. As far as this formal language is concerned they are indistinguishable and the syntactic entities which name or describe them are synonomous.

<u>The Group of a Formal Language</u>

Let $F = \langle T, K, X \rangle$ be a formal language over a model of part theory, P .

Definition: Let G be a subgroup of the group of all automorphisms of P . G is the group of the formal language F if for every basic or derived semantic transformation, τ , of F and every sequence $(x_1, \ldots, x_n)$ of referents in X such that

$$\tau(x_1, \ldots, x_n) = (y_1, \ldots, y_k)$$

and for every g in G ,

$$\tau(g(x_1), \ldots, g(x_n)) = (g(y_1), \ldots, g(y_k)) .$$

Under this definition, G is the group of the formal language F if every automorphism in G commutes with every semantic transformation of F whenever the semantic transformation is defined. We may symbolize the fact that the automorphism g commutes with the semantic transformation τ , in the sense of the definition, by $g\tau = \tau g$. The group G is said to leave the formal language F invariant since for each change in G and each semantic transformation of F we obtain the same result no matter whether the change or the semantic transformation is done first.

We now define indistinguishability of parts.

Definition: Δ is an orbit of P under G if Δ is a subclass of P and for every x and y in Δ there is a g in G such that $g(x) = y$.

Definition: Let GF be the group of the formal language F . If two parts are in the same orbit under GF , then they are indistinguishable by F .

As another example of indistinguishability suppose we have a formal language which includes the word "Scott" and the phrase "author of Waverly". "Scott" is interpreted as some part x and "author of Waverly" as some other part, y. Now if there is an automorphism g in the group of the formal language such that $g(x) = y$ then "Scott" and "author of Waverly" are indistinguishable by the the formal language under consideration. To say "Scott is the author of Waverly" is to assert that x and y are equivalent under the group of the formal language and x may or may not be equal to y. Now suppose there is a phrase involving the word "Scott" corresponding to the semantic transformation τ on the sequence of referents $(x, z_1, \ldots, z_k)$ such that $\tau(x, z_1, \ldots, z_k) = (w_1, \ldots, w_n)$. Then $\tau(y, g(z_1), \ldots, g(z_k)) = (g(w_1), \ldots, g(w_n))$. This shows that we can replace "Scott" by "author of Waverly" in the given phrase and the new phrase's meaning is indistinguishable from the original phrase.

Transformation Closure of a Group

The structure determined by a formal language may also be characterized by the collection of all semantic transformations invariant under the group of the formal language. The semantic transformation closure of a group of automorphisms is the collection of all semantic transformations which commute with every automorphism in the group. The formal definition follows.

Definition: LG is the semantic transformation closure of G if LG is the collection of all semantic transformations, τ, satisfying the following property: for each pair of sequences of parts $(x_1, \ldots, x_n)$

and $(y_1, \ldots, y_k)$ such that

$$\tau(x_1, \ldots, x_n) = (y_1, \ldots, y_k)$$

then for each g in G we have

$$\tau(g(x_1), \ldots, g(x_n)) = (g(y_1), \ldots, g(y_k)) .$$

If GF is the group of a formal language F then LGF includes every basic and derived semantic transformation of F when these transformations are considered to be restricted to definable parts. LGF may be thought of as the collection of all semantic transformations admissable under GF . While we can think of LGF as constructible by infinite methods from F , LGF is not in general finitely constructible from F as the following argument shows.

Given a group of automorphisms, G , consider the collection of all parts in P which are fixed under the action of G . These form a subalgebra of P , say B . Now consider all the functions from B to B . Each such function is admissable under G and so is a semantic transformation in LG . If B is infinite, then the collection of all functions from B to B is uncountable and hence so is LG .

Roughly speaking, in LGF there is a semantic transformation from almost any sequence of definable parts into almost any other sequence. Somewhat more precisely, LGF includes every potential, applicable semantic transformation, given the particular synonymities of the group G . In LGF we can get to any definable part from the referents of F . Can we get to any other parts by the application of semantic transformations in LGF which are not derivable from the basic semantic transformations of F? In general the answer is yes, however, in the next section, we describe how F can be strong enough so that everything definable by LGF is also definable by F.

Definability

Definition: Let F be a formal language $<T, K, X>$. Then let DF be the collection of contextually definable parts relative to the formal language F. Let DLGF be the collection of contextually definable parts relative to the class of referents X and the collection of semantic transformations LGF.

Clearly DF is a subclass of DLGF. We are interested in determining when DF = DLGF. This requires considering the topology of P.

Theorem: Every automorphism of P is a homeomorphism when P is endowed with the metric topology induced by the measure μ.

Proof: Let g be an automorphism of P. Suppose $\lim x_i = p$ and $\lim g(x_i) \neq g(p)$. Then $\lim (g(x_i) - g(p)) = c \neq 0$. Hence for some N and for all $j > N$, $c \, \pi \, (g(x_j) - g(p))$, so that $g^{-1}(c) \, \pi \, (x_j - p)$ and $g^{-1}(c) \neq 0$, contradicting $\lim x_i = p$. Thus g preserves all the limit points in P.

If there is a sequence of parts x_i in DF such that $\lim x_i = y$, then we say that DF has y as a limit point. If all the limit points of DF are in DF, then DF is closed. If y is a limit point of DF and g is any automorphism, then the image of DF under g possesses g(y) as a limit point. So if DF has a limit point, there is a semantic transformation in LGF from at least one definable part of DF to the limit point. For if $\lim x_i = y$, and every x_i is in DF , then a semantic transformation, τ , defined only on the x_i such that $\tau(x_i) = y$ commutes with every g in G . This shows that limit points of DF are in DLGF and suggests the following theorem.

Theorem: If DF is closed in the measure topology, then DLGF is equal to the subalgebra generated by DF .

Proof: Since DLGF properly contains DF , there is a semantic transformation in LGF from a sequence of parts in DF to a sequence of parts in DLGF - DF . This in turn implies the existence of a semantic transformation in LGF from a sequence of parts in DF to a single part in DLGF - DF . So suppose we have $\tau(x_1, \ldots, x_n) = y$ where $x_1, \ldots, x_n$ are in DF and y is not in the subalgebra generated by DF . Then we will show that there is an automorphism, g , in GF such that $x_1, \ldots, x_n$ are fixed under g while $g(y) \neq y$. This means that τ is not a semantic transformation in LGF and y is not in DLGF .

The proof is completed by considering the various possible Boolean algebraic relationships between y and the subalgebra generated by DF .

(i) y is disjoint from every x in DF . Then there are two subcases. First, y is the largest part disjoint from every x in DF . In this case $\tilde{y} = \Sigma DF$ and since DF is closed, ΣDF is in DF and so y is in the subalgebra generated by DF . Second, y is not the complement of ΣDF . Then there is some z such that $y \pi z$ and z is disjoint from every x in DF. Consider the group of all automorphisms of the principal ideal generated by z. These automorphisms can be extended to automorphisms of P by considering them to act trivially on the parts of $\tilde{z}$. Since every x in DF is a part of $\tilde{z}$, these automorphisms are the identity on every definable part, including $x_1, \ldots, x_n$. Furthermore, some of these automorphisms move y. Since these automorphisms fix the definable parts, this group is a subgroup of G. We have obtained a contradiction, which shows that if $\tau(x_1, \ldots, x_n) = y$, then either $\tilde{y} = \Sigma\, DF$ or y has a part in common with some x in DF.

(ii) Now if y is not covered by parts in DF, then there is some part of y to which (i) applies. So we will assume that y is a part of some x in the subalgebra generated by DF. Since y is, by assumption, not in the subalgebra generated by DF, it must be the case that some part of y is a proper part of an atom of the subalgebra, say z. Now consider the group of automorphisms of (z) as extended to automorphisms of P. The analysis in (i) applies to show that for some g in GF, y is moved while $x_1, \ldots, x_n$ are not.

The fact that DF is closed has been used implicitly in the proof. For if DF were not closed, then limit points of DF would not be in the algebra generated by DF although they are preserved by GF.

Closures on Formal Languages

With the above theorem in mind, we study two closures on formal languages.

The question of when DF = DLGF is of some importance if we feel that the essentially infinite processes, the undefinable semantic transformations, described as LGF , should be admitted as the completion of a formal language. That is, if we speak a completely formal language as described here, then our "intuition" roughly corresponds to the undefinable transformations in LGF . If this seems reasonable, then for DF to equal DLGF means we have a language powerful enough to formally define everything we can "intuitively" define beginning with the synonymities and structure described by G .

As we show, a language is "intuitively" complete when every limit point of the definable parts is determined by a single semantic transformation. An example is the set of all integers considered as a limit determined by the successor function.

Given a formal language $<T, K, X>$ over a model of part theory, P , we can always extend the formal language to include those semantic transformations which are invariant under every automorphism of P . These semantic transformations include part theoretic union, intersection, and complement, a selection function ζ , and all of the projection functions δ_i^n . The selection function $\zeta(x, y, z)$ is equal to y if $x = 0$ and is equal to z otherwise. The projection function δ_i^n projects onto the $i^{\underline{th}}$ component of a sequence of length n . $\delta_i^n(x_1, \ldots, x_n) = x_i$.

We define BF , where $F = <T, K, X>$, to be the collection of semantic transformations derivable from T together with the Boolean operations on P, ζ , and the δ_i^n . Because of the projection functions, any part definable by BF from X is context-free definable.

If we considered only a finite number of projection functions, we could extend F to another formal language, F' , whose basic semantic transformations are those in T together with ζ , the finite collection of projection functions, and the Boolean operations on P . In either case the definable parts, DBF or DF' , are their own subalgebras.

The second closure on a formal language involves an abstraction operator. The abstraction operator takes the conglomerate over semantic transformations rather than formulas. We then show that the semantic transformations derived from BF by abstraction are in LGF . In the following development we write xg for $g(x)$ and $\bar{y}$ for $y_1, \ldots, y_n$.

Theorem: If g is an automorphism of a model of part theory, then

$$P_z[F(z)]g = P_{zg}[F(z)]$$

where F(z) is a formula with z free.

Proof: $\forall d[d \pi y \leftrightarrow dg \pi yg]$ since g is an automorphism. Thus we have

$$yg \pi P_{zg}[F(z)]$$

$$\leftrightarrow \forall d[dg \pi yg \rightarrow \exists z \exists e(F(z) \;\&\; eg \pi zg \;\&\; eg \pi dg)] \;\&\; \exists f(yg \pi f)$$

$$\leftrightarrow \forall d[d \pi y \rightarrow \exists z \exists e(F(z) \;\&\; e \pi z \;\&\; e \pi d)] \;\&\; \exists f(y \pi f)$$

$$\leftrightarrow y \pi P_z[F(z)] \;.$$

$$\leftrightarrow yg \pi P_z[F(z)]g \;.$$

Theorem: If $F(z, \bar{y})$ is a formula of the lower predicate calculus whose atomic formulas are of the form $t_1(\bar{x}) = t_2(\bar{x})$ and t_1, t_2 are semantic transformations, then

$$P_z[F(z, \bar{y})] = P_z[\tau(z, \bar{y}) = 0]$$

where $\tau(z, \bar{y})$ is a semantic transformation.

Proof: (i) Any atomic formula $t_1(\bar{x}) = t_2(\bar{x})$ can be reduced to

$t_1(\bar{x}) \dot{-} t_2(\bar{x}) = 0$, where $\dot{-}$ denotes symmetric difference.

(ii) $t_1(\bar{x}) = 0 \;\&\; t_2(\bar{x}) = 0$ iff $t_1(\bar{x}) + t_2(\bar{x}) = 0$.

$t_1(\bar{x}) = 0 \vee t_2(\bar{x}) = 0$ iff $t_1(\bar{x}) \cdot t_2(\bar{x}) = 0$.

(iii) $t(\bar{x}) \neq 0$ iff $P_z[t(\bar{x}) = 0] = 0$, where z is not free in $t(\bar{x})$.

(iv) $\exists y[t(y, \bar{x}) = 0]$ iff $P_y[t(y, \bar{x}) = 0] \neq 0$

iff $P_z[P_y[t(y, \bar{x}) = 0] = 0] = 0$, where z is not free in $t(\bar{x})$. Note that here $\exists$ means there is a part.

Thus any formula is reducible to $\tau(\bar{x}) = 0$ for the appropriate τ involving the original semantic transformations of the formula, and the additional operations $\dot{-}$, $+$, $\cdot$, and $P_z[\ ]$.

We can now define the abstraction of a semantic transformation.

Definition: If $\tau(y, \bar{x})$ is a semantic transformation, define $\tau_q(\bar{x}) = P_y[\tau(y, \bar{x}) = 0]$.

The following lemma shows that if τ is in LG for some G , then so τ_q, if it exists.

Lemma: Let g be an automorphism of a model of part theory. If $\tau(y, \overline{x})g = \tau(yg, \overline{x}g)$ then $\tau_q(\overline{x})g = \tau_q(\overline{x}g)$.

Proof: Suppose $\tau(y, \overline{x})g = \tau(yg, \overline{x}g)$. By the theorem above we have

$$\tau_q(\overline{x})g = P_y[\tau(y, \overline{x}) = 0]g = P_{yg}[\tau(y, \overline{x}) = 0] = P_y[\tau(yg^{-1}, \overline{x}) = 0],$$

and by definition,

$$\tau_q(\overline{x}g) = P_y[\tau(y, \overline{x}g) = 0] .$$

If we can show that $\tau(g^{-1}y, \overline{x}) = 0 \leftrightarrow \tau(y, g\overline{x}) = 0$ then $\tau_q(x)g = \tau_q(\overline{x}g)$. Recalling that $xg = 0 \leftrightarrow x = 0$, we have $\tau(yg^{-1}, \overline{x}) = 0$ iff $\tau(yg^{-1}, \overline{x})g = 0$ and $\tau(yg^{-1}, \overline{x})g = \tau(y, \overline{x}g)$, completing the proof.

The following is an intuitive justification for considering the abstraction operation as a linguistic process. If $y_1, \ldots, y_n$ are definable parts such that $\tau(y_i, \overline{x}) = 0$, we know the structure of $y_1, \ldots, y_n$ in the context $\overline{x}$ as determined by the "formula" or semantic transformation $\tau(y, \overline{x}) = 0$. The intuition is that we know enough to abstract to that part which is the conglomerate of all parts, y , with the structure determined by $\tau(y, \overline{x}) = 0$, although all of these parts may not be definable.

For example, if we know a few men, say John, Jack, and Joe, we can abstract to "man", without having met all men. While the extension of "man" is presumably the class of all living men, the abstraction we obtain by conglomerating is somewhat closer to the intension of man, that is, the conglomerate of those entities with the structure of any man. Furthermore we can reach the abstraction "man" without having enumerated all men.

It is reasonable to assume that formal languages of any strength include quantifiers and the sentential connectives. The sentential connectives correspond to the Boolean semantic transformations and the quantifiers correspond to the abstraction operation, as illustrated in the following paragraph.

Suppose, in a given formal language, we have a grammatical string of words, $w_1 \ldots w_n$, corresponding to the sequence of referents $(x_1, \ldots, x_n)$. Further suppose that $w_1 \ldots w_n$ is a logical sentence, either true or false, whose truth value is determined by a semantic transformation τ with value 0 just in case $(x_1, \ldots, x_n)$ corresponds to a true sentence. τ can be thought of as a characteristic function on sequences of length n. We assume that there is another such string $w'_1 \ldots w'_m$ with associated characteristic function τ'. Then "$w_1 \ldots w_n$ and $w'_1 \ldots w'_m$" is true just in case $\tau(x_1, \ldots, x_n) \cdot \tau'(x'_1, \ldots, x'_m) = 0$. Consider the phrase "There is a w_1 such that $w_1 \ldots w_n$." This is a true sentence just in case there is a part y such that $\tau(y, x_2, \ldots, x_n) = 0$. Now $\exists y[\tau(y, x_2, \ldots, x_n) = 0]$ if and only if $P_y[\tau(y, x_2, \ldots, x_n) = 0] \neq 0$. We may rewrite the latter as $\tau_q(x_2, \ldots, x_n) \neq 0$. This shows that quantification on the syntactic level corresponds to abstraction on the semantic level.

The closure BF of a formal language F includes the Boolean semantic transformations. To include the abstraction operation, and thus the quantifiers, we define the abstraction closure of a formal language.

Definition: If F is a formal language, then PF is the smallest class including BF and, if τ is in PF , then τ_q is in PF .

Suppose for every limit point, y , of DPF there is a sequence x_i converging to y and the collection x_i is exactly the image of some τ in PF . If each x_i is part of y we have $y = P_z[\exists x(\tau(x) = z)]$ and if y is part of each x_i , $y = P_z[\forall x \forall w(\tau(x) = w \rightarrow z \cdot w = z)]$. Thus DPF is closed. By the above lemma, the formulas inside $P_z[\ ]$ can be replaced by semantic transformations. Under these conditions we have DPF = DLGF . If some other part theoretic relationship holds between the sequence x_i and y , then this method of obtaining the limit point by abstraction does not appear to work. In any case, if there is a limit point of DPF for which no sequence converging to it is definable by a finite number of semantic transformations, then DPF is not closed.

Galois Connection

In developing the Galois connection [7] between the groups of formal languages and the semantic transformation closures of the groups, we assume that the formal languages all have the same collection of referents, X . If we have two formal languages $F_1 = \langle T_1, K_1, X \rangle$ and $F_2 = \langle T_2, K_2, X \rangle$, by the union of the two languages, $F_1 + F_2$, we mean the formal language $\langle T_1 + T_2, K_1 + K_2, X \rangle$. CF denotes the collection of all derived semantic transformations of F . By the union of two groups, $G_1 + G_2$, we mean the smallest group including both G_1 and G_2 .

Since the proofs of the lemmas which establish the Galois connection are straight-forward, we use the following notation. By $\tau \in F$ we mean that τ is a basic semantic transformation of F . By $g\tau = \tau g$, we mean that the semantic transformation τ commutes with the automorphism g in the manner used to define the group of a formal language.

The first seven lemmas develop the Galois connection for arbitrary groups of automorphisms and their semantic transformation closures.

Lemma 1: $G_o \subseteq GLG_o$.

Proof: $g \in G_o \rightarrow \forall \tau \in LG_o(g\tau = \tau g) \rightarrow g \in GLG_o$.

Lemma 2: $LGLG_o = LG_o$.

Proof: $\tau \in LG_o \rightarrow \forall g \in GLG_o(g\tau = \tau g) \rightarrow \tau \in LGLG_o$.

$$\tau \in LGLG_o \rightarrow \forall g \in GLG_o(g\tau = \tau g)$$

$$\rightarrow \forall g \in G_o(g\tau = \tau g) \rightarrow \tau \in LG_o \ .$$

Lemma 3: $G_o \subseteq G_1 \rightarrow LG_o \supseteq LG_1$.

Proof: Assume $G_o \subseteq G_1$. Then $\tau \in LG_1 \rightarrow \forall g \in G_1(g\tau = \tau g)$

$$\rightarrow \forall g \in G_o(g\tau = \tau g) \rightarrow \tau \in LG_o \ .$$

Lemma 4: $G_1 + G_2 \subseteq G(LG_1 \cdot LG_2)$.

Proof: $g \in G_1 + G_2 \rightarrow \forall \tau \in LG_1 \cdot LG_2 (g\tau = \tau g) \rightarrow g \in G(LG_1 \cdot LG_2)$.

Lemma 5: $L(G_1 + G_2) = LG_1 \cdot LG_2$.

Proof: $\tau \in L(G_1 + G_2) \leftrightarrow \forall g \in G_1 + G_2 (g\tau = \tau g)$

$\leftrightarrow \forall g \in G_1 (g\tau = \tau g) \; \& \; \forall g \in G_2 (g\tau = \tau g)$

$\leftrightarrow \tau \in LG_1 \; \& \; \tau \in LG_2 \leftrightarrow \tau \in LG_1 \cdot LG_2$.

Lemma 6: $LG(LG_1 \cdot LG_2) = LG_1 \cdot LG_2$.

Proof: Apply lemma 5, lemma 2, and then lemma 5 again.

Lemma 7: $L(G_1 \cdot G_2) \supseteq LG_1 + LG_2$.

Proof: $\tau \in LG_1 + LG_2 \rightarrow \forall g \in G_1 \cdot G_2 (g\tau = \tau g) \rightarrow \tau \in L(G_1 \cdot G_2)$.

The remaining results complete the Galois connection for groups over formal languages and their semantic transformation closures.

Lemma 8: $GLGF = GF$.

Proof: Since we have lemma 1, it only remains to show that $GLGF \subseteq GF$. Suppose there is a $g \in GLGF$ not in GF . Then g does not commute with some semantic transformation in CF , and since $CF \subseteq LGF$, g does not commute with every semantic transformation in $GLGF$, contradiction.

Lemma 9: $G(LGF_1 + LGF_2) = GF_1 \cdot GF_2$.

Proof: $g \in G(LGF_1 + LGF_2)$

$\leftrightarrow \forall \tau \in LGF_1 (g\tau = \tau g) \; \& \; \forall \tau \in LGF_2 (g\tau = \tau g)$

$\leftrightarrow g \in GF_1 \;\&\; g \in GF_2 \leftrightarrow g \in GF_1 \cdot GF_2$.

Lemma 10: $GF_1 \cdot GF_2 = G(F_1 + F_2)$.

Proof: $g \in GF_1 \cdot GF_2 \leftrightarrow \forall \tau \in F_1 (g\tau = \tau g) \;\&\; \forall \tau \in F_2 (g\tau = \tau g)$

$\leftrightarrow \forall \tau \in F_1 + F_2 (g\tau = \tau g) \leftrightarrow g \in G(F_1 + F_2)$.

Lemma 11: $G(LGF_1 \cdot LGF_2) \subseteq G(CF_1 \cdot CF_2)$.

Proof: $g \in G(LGF_1 \cdot LGF_2) \rightarrow \forall \tau \in LGF_1 \cdot LGF_2 (g\tau = \tau g)$.

Since $CF \subseteq LGF$ we have $CF_1 \cdot CF_2 \subseteq LGF_1 \cdot LGF_2$.

Thus $g \in G(LGF_1 \cdot LGF_2) \rightarrow \forall \tau \in CF_1 \cdot CF_2 (g\tau = \tau g) \rightarrow g \in G(CF_1 \cdot CF_2)$.

Lemma 12: $LGF_1 \cdot LGF_2 \subseteq LG(CF_1 \cdot CF_2)$.

Proof: Apply lemmas 3 and 2 to the result of lemma 11.

Theorem 13: $G(CF_1 \cdot CF_2) = G(LGF_1 \cdot LGF_2)$.

Proof: By lemma 12 we have

$LGF_1 \cdot LGF_2 \subseteq LG(CF_1 \cdot CF_2)$. So

$g \in G(CF_1 \cdot CF_2) \rightarrow \forall \tau \in LG(CF_1 \cdot CF_2)(g\tau = \tau g)$

$\rightarrow \forall \tau \in LGF_1 \cdot LGF_2 (g\tau = \tau g)$

$\rightarrow g \in G(LGF_1 \cdot LGF_2)$.

Together with lemma 11, this proves the theorem.

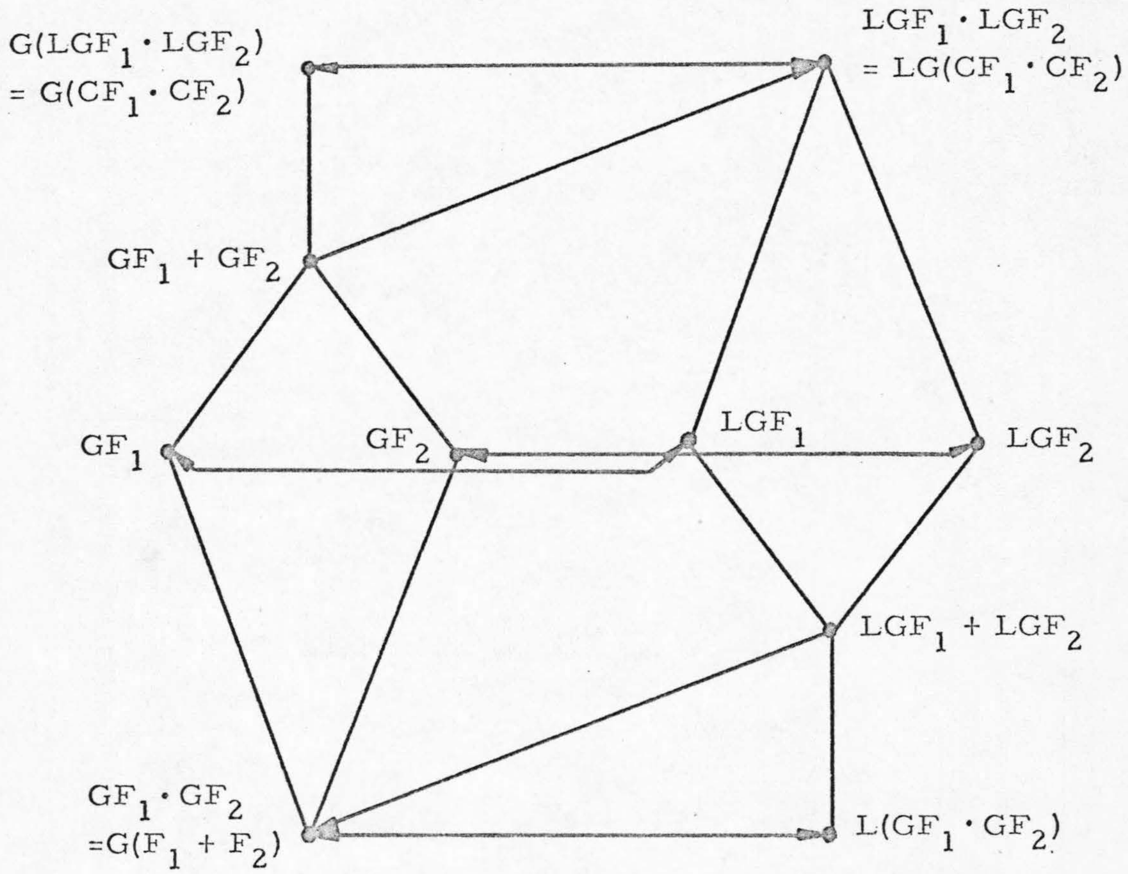

Diagram of the

Galois Connection

Between

Formal Languages

and Their

Semantic Transformation Closure.

V. SET THEORY

In Chapter II we mentioned that the universe of discourse of a formal language can always be considered a model of set theory and in Chapter IV we considered the universe of discourse a model of part theory. Of course we can always model part theory in set theory to obtain a set-theoretical universe of discourse for a formal language. Can we, for a given formal language, find a submodel of an appropriate model of part theory which is both a set theory and the universe of discourse of the language? This question is not trivial since part theory is weaker than set theory and since we are looking for a set theory which can serve as the universe of discourse of a given language. In this chapter we show that a set theory can be imbedded in appropriate models of part theory. This imbedding is not arbitrary, but the consequence of certain strong formal languages. Indeed one may say that the central result of this thesis is the fact that strong formal languages determine their own set theory, which is structurally determined by the formal language and intimately connected with the part-whole relation. This is very different from the rather <u>ad hoc</u> relationship between an "outside" set theory and a model of part theory.

We assume we have a semantic transformation s which satisfies the following two properties, where we write sx for s(x):

(i) $\forall x \exists z \forall y (x \neq y \rightarrow z \,\pi\, sx - sy)$,

(ii) $\forall x(sx \; \pi \; 1)$

The first property guarantees that s is one-to-one since there is a part which distinguishes sx from sy for all y . This semantic transformation is to be thought of as the singleton function of set theory, $sx = \{x\}$. s need only be defined on those parts which are "sets", but we can also assume that s is defined on all of the model of part theory. If s is not everywhere defined, then property (i) can be rewritten to hold only for those x such that sx is defined. We can think of s as a predicate true of (x, y) iff $sx = y$.

In order to develop the set theory, we require that the model of part theory, P , have the following property:

> If F(z) is any formula with z free over the predicates π and s , then $P_z[F(z)]$ exists in P .

This is considerably stronger than the original axiom schema of part theory, although maintaining its flavor. In particular, the countable atomless Boolean algebra does not have the above property for any s .

Now we can say that a part x is a set if it satisfies the following equation:

$$x = P_z[z \; \pi \; x \; \& \; \exists y(sy = z)] \; .$$

This equation will be denoted by Set(x) in the sequel. The equation guarantees that x is a set just in case it is the least upper bound of all the singletons which are part of x . We define set membership as follows:

$$x \in y \longleftrightarrow \mathrm{Set}(y) \; \& \; sx \; \pi \; y \; .$$

We can now prove that the singleton parts are sets.

Lemma: $\forall x[\mathrm{Set}(sx)]$

Proof: By property (ii) of s together with theorems 13 and 3 of part theory, $sx \; \pi \; P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)]$. It remains to show that $P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)] \; \pi \; sx$. Suppose there is some $w \; \pi \; P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)]$ such that w is disjoint from sx . But since every part of w must have all of its parts in common with the various sy in the conglomerate, there is at least one y, $y \neq x$, such that $sy \; \pi \; P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)]$. However, every part of sy must have a part in common with sx in order that sy be part of $P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)]$, so sy cannot satisfy property (i), and we have obtained a contradiction. Hence

$$sx = P_z[z \; \pi \; sx \; \& \; \exists y(sy = z)] \; .$$

The following lemma shows that singleton sets have no sets as proper parts.

Lemma: $\forall x[\exists y(x \; \pi \; sy \; \& \; x \neq sy) \rightarrow \neg \, \mathrm{Set}(x)]$

Proof: Assume $\exists y(x \; \pi \; sy \; \& \; x \neq sy)$. We must show that $x \neq P_z[z \; \pi \; x \; \& \; \exists y(sy = z)]$. First we note that since $x \; \pi \; sy$, x is not the zero of the part theory. Now suppose x is a set. Then every singleton set which is a part of x is a proper part of sy , contradicting property (i) of s .

We use the Zermelo-Fraenkel axiom system for set theory. We will show that those parts of P for which Set(x) holds satisfy the Zermelo-Fraenkel axioms, except for regularity and choice. At that point two alternatives are open to us. We can assume that the s function satisfies the axiom of regularity or we can develop a constructive theory of sets, in which the axiom of regularity can be proved. Not every part satisfying Set(x) is a constructive set, however. This latter approach has the advantage that Gödel [12, 13] has proved the axiom of choice for the constructive sets. Since Cohen [6] has given a clear exposition of the constructive method for Zermelo-Fraenkel set theory, our presentation is formal, completing the details of Cohen's presentation.

We can write a predicate of set theory which is true just in case its argument is constructive. With the aid of the imbedding to be presented, this predicate can also be written for part theory, and those parts which satisfy this predicate are the sets satisfying all the axioms of set theory, including regularity and choice.

We turn to the prooof of the axioms of set theory within part theory. We recall that

$$x \in y \leftrightarrow \mathrm{Set}(y) \,\&\, sx \,\pi\, y \ .$$

With the aid of this definition, we translate the axioms of set theory into statements of part theory for the proofs to follow. The order and presentation of the axioms of set theory follows Cohen [6].

1. Extensionality

$$\forall x \forall y \,[\mathrm{Set}(x) \,\&\, \mathrm{Set}(y) \rightarrow (\forall z[sz \,\pi\, x \leftrightarrow sz \,\pi\, y] \rightarrow x = y)]$$

Proof: Suppose x and y are sets and that every singleton part of x is a part of y . We will show that $\forall z(z \,\pi\, x \rightarrow z \,\pi\, y)$. Consider any

$z \pi x$. Then since x is a set, we have $z \pi P_z[z \pi x \& \exists w(sw = z)]$, that is, $\forall d[d \pi z \rightarrow \exists a \exists e(a \pi x \& \exists w(sw = a) \& e \pi a \& e \pi d)]$. Since every singleton part of x is a part of y we have $z \pi y$. The same proof in the other direction shows that $x = y$.

2. Null Set

$$\exists x[Set(x) \& \forall y(\neg\, sy \pi x)]$$

Proof: 0 is a set which has no parts. Thus the zero of part theory can be used as the null set.

3. Unordered Pairs

$$\forall x \forall y[Set(x) \& Set(y) \rightarrow \exists z(Set(z) \& \forall w[sw \pi z \leftrightarrow w = x \vee w = y])]$$

Proof: Given sets x and y, consider $z = sx + sy$, where $+$ is part theoretic union. By property (i) of s, the only singleton sets which are part of z are sx and sy. Also, z is a set since it is the least upper bound of sx and sy.

4. Union

$$\forall x[Set(x) \rightarrow \exists y(Set(y) \& \forall z[sz \pi y \leftrightarrow \exists t(Set(t) \& sz \pi t \& st \pi x)])]$$

Proof: Consider $y = P_t[st \pi x \& Set(t)]$. Suppose y is not a set. Then there is a $c \pi y$ such that c is disjoint from each singleton set in y. Call this part c_o. But every part, d, of c_o must have a part in common with some t_d such that $st_d \pi x \& Set(t_d)$. So c_o shares a part with some singleton part of t, hence of y. This contradiction shows that y is a set. Now, if we have some t and z such that $Set(t) \& sz \pi t \& st \pi x$, we have $sz \pi y$. For the reverse implication let sz be some singleton part of y for which it is not the case that $\exists t(Set(t) \& sz \pi t \& st \pi x)$. We will use property (i) of s to derive a contradiction. By property (i) there is a part of sz, say w, which has no part in common with any other singleton set. But every part d of w must have a part in common with some t_d such that $st_d \pi x \& Set(t_d)$. Hence w shares a part with some singleton part of some t, contradiction. Thus if $sz \pi y$ then $\exists t(set(t) \& sz \pi t \& st \pi x)$, completing the proof of the union axiom.

Note that Set(t) in $P_t[st \pi x \& Set(t)]$ can be rewritten as

$$[\forall d[d \pi t \rightarrow \exists a \exists e(F(a) \& e \pi a \& e \pi d)] \& t \pi 1]$$

$$\&[(\forall d[d \pi c \rightarrow \exists a \exists e(F(a) \& e \pi a \& e \pi d)] \& c \pi 1) \rightarrow c \pi t]$$

where

$$F(a) \leftrightarrow a \pi t \& \exists w(sw = a)$$

which shows that Set(t) is a formula over π and s alone, and so we have guaranteed the existence of $P_t[st \pi x \& Set(t)]$.

5. Infinity

$$\exists x[Set(x) \& s0 \pi x \& \forall y(sy \pi x \rightarrow s(y + sy) \pi x)]$$

Proof: Let $\Omega(x)$ be the following formula of part theory:

$$s0 \pi x \& \forall y(sy \pi x \rightarrow s(y + sy) \pi x).$$

Let $\omega = P_z[\Omega(z) \& \forall x(\Omega(x) \rightarrow z \pi x)]$. Clearly ω satisfies Ω and so it satisfies the axiom of infinity if it is a set. If ω is not a set then there is some $c \pi \omega$ such that c is disjoint from every singleton set in ω. In this case $\omega' = \omega - c$ satisfies Ω, and since ω satisfies Ω we have $\omega \pi \omega - c$, showing that $c = 0$, contradicting $c \pi \omega$. So we have

$\mathrm{Set}(\omega)$.

6_n. Replacement

To state the axiom of replacement, we enumerate the formulas of part theory over π and s with at least two free variables, $A_n(x, y; t_1, \ldots, t_k)$ where k depends on n . We may think of the $t_1, \ldots, t_k$ as parameters of the formula A_n . The axiom of replacement then reads

$$\forall t_1 \ldots t_k [\forall x(\mathrm{Set}(x) \rightarrow \exists! y[\mathrm{Set}(y) \ \& \ A_n(x, y; t_1, \ldots, t_k)]) \rightarrow \forall z \exists w \, B(z, w)]$$

where

$$B(z, w) \longleftrightarrow [\mathrm{Set}(z) \rightarrow \mathrm{Set}(w) \ \& \ \forall r(sr \ \pi \ w \longleftrightarrow$$

$$\exists q[\mathrm{Set}(q) \ \& \ sq \ \pi \ z \ \& \ A_n(q, r; t_1, \ldots, t_k)])] .$$

Fixing $t_1, \ldots, t_k$, if $A_n(x, y)$ determines y uniquely for each set x , $y = f(x)$, then the range of f , when f is restricted to the set z , is a set.

Proof: Suppose it is the case that $\forall x(\mathrm{Set}(x) \rightarrow \exists! y[\mathrm{Set}(y) \ \& \ A_n(x, y)])$ where we assume $t_1, \ldots, t_k$ are fixed and so have dropped them from our formal presentation. To show that $\forall z \exists w \, B(z, w)$, let z be any part satisfying $\mathrm{Set}(z)$. Define

$$w = P_a[\exists b(sb = a \ \& \ \exists q[\mathrm{Set}(q) \ \& \ sq \ \pi \ z \ \& \ A_n(q, b)])] .$$

w is a set since it is a conglomerate of singletons. Furthermore we have

$$\forall r(sr \ \pi \ w \longleftrightarrow \exists q[\mathrm{Set}(q) \ \& \ sq \ \pi \ z \ \& \ A_n(q, r)])$$

since if $sr \; \pi \; w$ then by the definition of w we have

$\exists q[Set(q) \; \& \; sq \; \pi \; z \; \& \; A_n(q, r)]$ while if we are given an r such that $\exists q[Set(q) \; \& \; sq \; \pi \; z \; \& \; A_n(q, r)]$ then $sr \; \pi \; w$, again by the definition of w . Hence the axiom of replacement holds.

7. Power Set

We define the subset relation by

$$z \subseteq x \longleftrightarrow \forall w(sw \; \pi \; z \rightarrow sw \; \pi \; x) \; .$$

Then the statement of the power set axiom we prove here is

$$\forall x \exists y \forall z[sz \; \pi \; y \longleftrightarrow z \subseteq x] \; .$$

This version of the power set axiom is much stronger than is required to show that the power set of every set exists. For we have here that the power parts of every part, set or not, are parts. As the proof indicates, the smallest power part is a set so that the power set axiom holds when it is restricted to those parts satisfying Set(x) .

Proof: Given a part x , consider $y = P_a[\exists b(sb = a \; \& \; b \subseteq x)]$. Since y is a conglomerate of singletons, we have Set(y) , whether x is a set or not. Clearly if z is a subset of x then $sz \; \pi \; y$. Now consider any singleton which is a part of y . Since the singletons are unique functions of their arguments by property (i) , any singleton part of y must satisfy the formula defining y . Hence y is exactly the power set of x .

8. Axiom of Regularity

$$\forall x[Set(x) \; \& \; x \neq 0 \rightarrow \exists y(Set(y) \; \& \; sy \; \pi \; x \; \& \; \forall z[sz \; \pi \; x \rightarrow \neg \, sz \; \pi \; y]))]$$

Parts satisfying this axiom are said to be well-founded. As was remarked earlier, the singleton function s has not been restricted enough to prove the axiom of regularity. However, with the seven axioms we have proved, it is possible to consider just the constructive sets, for which the axiom of regularity can be proved.

Constructive Set Theory

In order to describe the constructive set theory, we define the ordinals of the model of set theory.

Definition: A part x is transitive if

$$sz \,\pi\, y \,\&\, sy \,\pi\, x \rightarrow sz \,\pi\, x \ .$$

Definition: A part x is well-ordered by ϵ if ϵ orders x and if $y \,\pi\, x$ and y is a set then $\exists z[z \epsilon\, y \,\&\, \forall w(w \,\epsilon\, y \rightarrow \neg\, w \,\epsilon\, z)]$, where $a \,\epsilon\, b \leftrightarrow \mathrm{Set}(b) \,\&\, sa \,\pi\, b$.

Definition: A part α is an ordinal if α is a set well-ordered by ϵ and α is transitive.

Next we will define an ordinal sequence of sets which contain all the constructible sets. We will require a method which given a set, X , obtains the set, X' , of all sets constructible from X .

Definition: Let X be a set. We define the set X' by forming the union of X and the set of all sets y defined by a formula restricted to X . That is, consider any formula over π and s with at least one free variable,

$$A(z;\ t_1,\ \ldots,\ t_k)\ .$$

Then let $A_X(z;\ t_1,\ \dots,\ t_k)$ be the formula A with all bound variables restricted to sets in X. Let $x_1,\ \dots,\ x_k$ be fixed sets in X, and define

$$y = P_{sz}[sz\ \pi\ X\ \&\ A_X(z;\ x_1,\ \dots,\ x_k)]\ .$$

Now let $Y = P_{sy}[$ if $A(z;\ t_1,\ \dots,\ t_k)$ is a formula over π and s then $y = P_{sz}[sz\ \pi\ X\ \&\ A_X(z;\ x_1,\ \dots,\ x_k)]]$, and let $X' = X + Y$.

This definition still requires a complete formalization. This is done by defining a single formula B(X, z) which is satisfied just in case $z = X'$. We return to this after completing the presentation of constructive set theory.

Definition: If α is an ordinal larger than 0, define $M_\alpha = (\sum_{\beta<\alpha} M_\beta)'$, where the union is set theoretic. $\sum_{\beta<\alpha} M_\beta$ is guaranteed to exist by the replacement and union axioms. Define $M_0 = 0$.

Definition: A set x is constructible if there exists an ordinal, α, such that $x \in M_\alpha$.

It only remains to show that the Zermelo-Fraenkel axioms hold for the constructible sets. Since this is done in Cohen [6, p. 89], we do not repeat the proofs here. The only axiom we must independently check is the axiom of infinity since Cohen's proof appears to require the axiom of regularity. Since in our proof of the axiom of infinity we demonstrated the set of all integers, ω, we only need to note that ω is an ordinal and that $\omega \in M_{\omega+1}$, to show that ω is constructible.

We turn to formalizing the relation $Y = X'$. For each $r \geq 0$ let

X_r denote the set of all sets S of sequences of length n , $(x_1, \ldots, x_n)$, for which there is a formula $A(x_1, \ldots, x_n; t_1, \ldots, t_m)$ with exactly r quantifiers and $y_i \in X$ such that

$$S = \{(x_1, \ldots, x_n) | A_x(x_1, \ldots, x_n; y_1, \ldots, y_m)\} .$$

Then X' is the set of all sequences of length 1 which are in any X_r .

We show that the relation $Y = X_r$ is expressible in part theory with s . The relation $Y = X_o$ is expressed by defining the sets S which arise from the quantifier-free formulas by induction on the length of the formula. The following formulas are used in defining the relation $Y = X_o$. The notation here is that $<x, y>$ is the ordered pair composed of x and y and $<x, y, z> = <x, <y, z>>$.

$$H_1(X, Y) \leftrightarrow \forall x(x \in Y \rightarrow$$
$$\exists y[y \in Y \& \forall z(z \in y \rightarrow \exists u \exists v(z = <u, v> \& v \in x \& u \in X))$$
$$\& \forall u \forall v \exists z(u \in X \& v \in x \rightarrow z = <u, v> \& z \in y)])$$

$$H_2(X, Y) \leftrightarrow \forall x(x \in Y \rightarrow \forall t(t \in X \rightarrow$$
$$\exists y[y \in Y \& \forall z \exists u \exists v(z \in y \rightarrow u \in t \& v \in x \& z = <u, v>)$$
$$\& \forall u \forall v \exists z(u \in t \& v \in x \rightarrow z = <u, v> \& z \in y)]))$$

$$H_3(X, Y) \leftrightarrow \forall x(x \in Y \rightarrow \forall t(t \in X \rightarrow$$
$$\exists y[y \in Y \& \forall z \exists u \exists v(z \in y \rightarrow t \in u \& u \in X \& v \in x \& z = <u, v>)$$
$$\& \forall u \forall v \exists z(t \in u \& u \in X \& v \in x \rightarrow z = <u, v> \& z \in y)]))$$

$H_4(X, Y) \longleftrightarrow \forall x(x \in Y \rightarrow$

$\exists y[y \in Y \; \& \; \forall z \exists u \exists v \exists w(z \in y \rightarrow u \in v \; \& \; u \in X \; \& \; v \in X \; \& \; w \in x \; \& \; z = \langle u, v, w \rangle)$

$\& \; \forall u \forall v \forall w \exists z(u \in v \; \& \; u \in X \; \& \; v \in X \; \& \; w \in x \rightarrow z = \langle u, v, w \rangle \; \& \; z \in y)])$

conjunction:

$J_1(X, Y) \longleftrightarrow \forall x \forall y(x \in Y \; \& \; y \in Y \rightarrow x \circ y \in Y)$

negation:

$J_2(X, Y) \longleftrightarrow \forall x \forall y(x \in Y \; \& \; y \in Y \rightarrow x - y \in Y)$

rearrangement:

$I_1(X, Y) \longleftrightarrow \forall x(x \in Y \; \& \; \forall z \exists u \exists v(z \in x \rightarrow z = \langle u, v \rangle \; \& \; u \in X \; \& \; v \in X) \rightarrow$

$\exists y[y \in Y \; \& \; \forall z \exists u \exists v(z \in y \rightarrow z = \langle v, u \rangle \; \& \; \langle u, v \rangle \in x)$

$\& \; \forall u \forall v \exists z(\langle u, v \rangle \in x \rightarrow z = \langle v, u \rangle \; \& \; z \in y)])$

$I_2(X, Y) \longleftrightarrow \forall x(x \in Y \; \& \; \forall z \exists u \exists v \exists w(z \in x \rightarrow z = \langle u, v, w \rangle \; \& \; u \in X \; \& \; v \in X) \rightarrow$

$\exists y[y \in Y \; \& \; \forall z \exists u \exists v \exists w(z \in y \rightarrow z = \langle v, u, w \rangle \; \& \; \langle u, v, w \rangle \in x)$

$\& \; \forall u \forall v \forall w \exists z(\langle u, v, w \rangle \in x \rightarrow z = \langle v, u, w \rangle \; \& \; z \in y)])$

terminal elements:

$G_1(X, Y) \longleftrightarrow X \in Y$

$G_2(X, Y) \longleftrightarrow X \subseteq Y$

$G_3(X, Y) \longleftrightarrow \forall t(t \in X \rightarrow \exists y[y \in Y \; \& \; \forall z(z \in y \longleftrightarrow t \in z \; \& \; z \in X)])$

$G_4(X, Y) \longleftrightarrow \forall x(x \in X \; \& \; \forall z \exists u \exists v(z \in x \rightarrow z = \langle u, v \rangle \; \& \; u \in X \; \& \; v \in X) \rightarrow$

$\exists y[y \in Y \; \& \; \forall z \exists u \exists v(z \in y \longleftrightarrow u \in v \; \& \; z \in x \; \& \; z = \langle u, v \rangle)])$

We now define the formula $Q_o(X, Y)$ which is true if and only if $Y \supseteq X_o$ where X_o denotes the set of all sets S of n-tuples $\langle x_1, \ldots, x_n \rangle$ for which there is a quantifier-free formula $A(x_1, \ldots, x_n; t_1, \ldots, t_m)$ and there are $y_i \in X$ such that

$$S = \{\langle x_1, \ldots, x_n \rangle \mid A(x_1, \ldots, x_n; y_1, \ldots, y_m)\} .$$

$$\begin{aligned} Q_o(X, Y) \leftrightarrow\ & G_1(X, Y) \mathbin{\&} G_2(X, Y) \mathbin{\&} G_3(X, Y) \mathbin{\&} G_4(X, Y) \\ & \mathbin{\&} H_1(X, Y) \mathbin{\&} H_2(X, Y) \mathbin{\&} H_3(X, Y) \mathbin{\&} H_4(X, Y) \\ & \mathbin{\&} I_1(X, Y) \mathbin{\&} I_2(X, Y) \\ & \mathbin{\&} J_1(X, Y) \mathbin{\&} J_2(X, Y) . \end{aligned}$$

The following theorem shows that there is a formula of part theory over the predicates π and s which expresses the relation $Y = X_o$.

Theorem: $X_o = P_z[Q_o(X, z) \mathbin{\&} \forall x(Q_o(X, x) \rightarrow z \mathbin{\pi} x)]$.

Proof: By induction on the length of the formulas.

1. Atomic formulas:

i) $x_i \in y_j$. This case is handled by G_2 and H_2 , together H_1 which is required in "build up" to the $i+1^{\underline{st}}$ place from the $n^{\underline{th}}$ place and then from the $i-1^{\underline{st}}$ place to the first place. G_1 is required to start constructing the direct product.

ii) $y_j \in x_i$. This case is handled by G_3 and H_3 .

iii) $x_i \in x_j$. This case is handled by G_4 and H_4 together with I_1 and I_2 to allow rearrangement of the n-tuples so that we can have i different from $j+1$.

iv) $y_i \in y_j$. Trivial since this formula is either true or false, so adds no new n-tuples.

2. Induction:

i) If we have

$$S_A = \{<x_1, \ldots, x_n> \mid A(x_1, \ldots, x_n; y_1, \ldots, y_m)\}$$

and

$$S_B = \{<x_1, \ldots, x_n> \mid B(x_1, \ldots, x_n; y_1, \ldots, y_m)\}$$

then

$$S_A \cdot S_B = \{<x_1, \ldots, x_n> \mid A(\ldots) \,\&\, B(\ldots)\}$$

This is the purpose of J_1.

ii) If we have

$$S_A = \{<x_1, \ldots, x_n> \mid A(\ldots)\}$$

then

$$S_{\neg A} = \{<x_1, \ldots, x_n> \mid \neg A(\ldots)\}$$

This is the purpose of J_2. Since every element of an n-tuple is restricted to X, if we remove from the direct product, $\underbrace{X \times \cdots \times X}_{n\text{-times}}$, the n-tuples corresponding to A, we have the set of n-tuples corresponding to $\neg A$.

It remains to show that

$$P_z[Q_o(X, z) \,\&\, \forall x(Q_o(X, x) \rightarrow z \,\pi\, x)] \subseteq X_o \,.$$

Clearly every element of $P_z[\ldots]$ is a set of n-tuples, each n-tuple of which is over X. Since $P_z[\ \dots\ ]$ is the smallest set satisfying $Q_o(X, Y)$, every element of $P_z[\ \dots\ ]$ has been generated by one of the G, H, I, J "processes" defined above, and each process corresponds to a "construction" of the lower predicate calculus over ϵ. Hence each element of $P_z[\ \dots\ ]$ corresponds to some quantifier-free formula restricted to X.

We now define X_{r+1} in terms of X_r. This definition corresponds to adding one quantifier, $\exists$ or $\forall$, to the formulas defining the n-tuples in X_r, where the range of the quantifier is restricted to X.

$$x \epsilon X_{r+1} \leftrightarrow \exists t[Set(t) \,\&\, (t \epsilon X_r \rightarrow \forall z[z \epsilon x \rightarrow \exists w(w \epsilon X \,\&\, \langle w, z\rangle \epsilon t)])]$$
$$\vee \exists t[Set(t) \,\&\, (t \epsilon X_r \rightarrow \forall z[z \epsilon x \leftrightarrow \forall w(w \epsilon X \rightarrow \langle w, z\rangle \epsilon t)])].$$

Finally, we can define X' in terms of X formally by:

$$X' = X + P_{sz}[Set(z) \,\&\, (x \epsilon z \rightarrow x \epsilon X) \,\&\, \exists r(r \epsilon \omega \,\&\, z \epsilon X_r)] .$$

This completes our imbedding of set theory in a model of part theory.

Set Theories Under a Formal Language

We have developed some of the consequences of assuming that a formal language has a model of part theory as its universe of discourse. In particular, a formal language is characterized, albeit imperfectly, by the group of automorphisms which leave the language invariant. We also have shown that if the model of part theory has a strong property then we can imbed a set theory in the collection of parts. In this section we discuss the relationship between the group of a formal language and models of set theory. Assuming that the universe of discourse is strong enough to support a set theory, we would like to show that we can always find a set theory within the parts such that the formal language can be construed to have the collection of sets as its universe of discourse. That is, for all parts x definable by the formal language, x satisfies Set(x), given some fixed singleton function.

The simplest case is when the formal language F has such a singleton function as one of its basic semantic transformations. Then GF fixes every part x which satisfies Set(x) and which is well-founded, since a model of set theory has no proper automorphisms. It is worth noting that GF itself is larger than the identity automorphism since there are proper parts of each singleton which can be permuted among themselves without affecting the sets.

We digress to reconsider what property a model of part theory must possess to obtain a set theory. The existence of the conglomerate of every formula over π and s is certainly stronger than is required, since we need only use π in order to define ϵ. With the help of property (ii) of s we can show that $sx \; \pi \; y$ if and only if $sx \cdot y = sx$. Recalling the results of the previous chapter on the relationship between formulas and abstraction, we see that it is sufficient, for the imbedding of a set theory, to insist upon the existence of the abstraction of every semantic transformation in a formal language with a singleton function and the Boolean functions.

Lemma: $sx \; \pi \; y \longleftrightarrow sx \cdot y = sx$

Proof: By axiom 2 of part theory, $sx \pi y \rightarrow \forall z(z \pi sx \rightarrow z \pi y)$. So using theorem 19, $\forall z(z \pi sx \rightarrow z \pi sx \cdot y)$. By theorem 28, $\forall z(z \pi sx \cdot y \rightarrow z \pi sx)$. Combining the last two formulas and using axiom 1 we have $sx \pi y \rightarrow sx \cdot y = sx$. Now assume $sx \cdot y = sx$. By axiom 1, $\forall z(z \pi sx \rightarrow z \pi sx \cdot y)$. Combining these with property (ii) of s and applying axiom 2 we have $sx \pi y$.

To return to the discussion of set theories under formal languages, suppose F is such that GF has an infinite number of fixed points which form an atomic subalgebra of the algebra of parts. Then since the Skolem-Löwenheim theorem guarantees that there is a countable model of set theory, LGF contains a singleton function defined on the fixed points of GF and the singleton function defines a set theory.

Even if the fixed points of GF do not form an infinite atomic subalgebra, we can find a set theory under F. There is a subgroup, N, of GF which fixes every part definable by F. If F defines an infinite number of parts which generate an atomic subalgebra then LN contains a singleton function. Further, every part definable by F satisfies Set(x). N is a normal subgroup of GF, as we show below. The factor group GF/N can be thought of as the automorphisms of the sets which leave F invariant. If GF/N is larger than the identity then the singleton function is not invariant under GF/N and F is not strong enough to define a set theory. For example, suppose the only basic semantic transformation in F is $\text{List}(x) = \{\{y\} \mid y \in x\}$ where $\in$ is defined by a particular singleton function in LN. Then GF/N contains, among others, the automorphism which permutes

$\{^k 2\}^k$ and $\{^k 3\}^k$ for all $k \geq 1$ where $\{^1 x\}^1 = \{x\}$ and $\{^{k+1} x\}^{k+1} = \{^k \{x\}\}^k$.

The subalgebra generated by the definable parts may contain non-atomic parts. The only way to include the non-atomic parts in the set theory is to widen the definition of the predicate "Set" to include individuals and consider the non-atomic parts to be individuals of the set theory.

If the formla language F defines only a finite number of parts we can still find a set theory under F. For each atom x of the algebra of definable parts we can choose an infinite number of proper parts of x which form an atomic Boolean algebra when relativized to x. Then GF has a normal subgroup, N, which fixes the chosen algebras and LN has a set theory defining singleton function. In this set theory each part definable by F is an infinite set. This is not pleasing to the intuition, for one feels that if there are only a finite number of definable parts then the formal language ought to be discussing the interrelations among a few finite sets, rather than infinite sets. If the part theoretic union, y, of the definable parts does not reach the universal part, 1, then the intuition can be satisfied by developing the set theory within the complement of y, using the algebra of the definable parts as some of the finite sets.

We turn now to the proof that the subgroup N of GF is normal in GF, for any formal language F whose definable parts generate an atomic subalgebra.

Given a group G, which is a group over a formal language, and the collection, X, of parts contextually definable by the formal language, we obtain the collection of G-definable parts by taking the set-theoretic union of the orbits under G of every x in X. That is, if x is definable, then g(x) is G-definable for every g in G. Let B be the sub-Boolean algebra (submodel of part theory) generated by the G-definable parts.

Now consider the largest subgroup, N, of G which leaves every part in B fixed. To show that N is normal in G, we prove some lemmas.

Lemma: If x is in B, then for all g in G, g(x) is in B.

Proof: Let x be in the subalgebra B. Then x is the Boolean union, intersection or complement of G-definable parts. Hence g(x) is also a Boolean combination of G-definable parts. Thus g(x) is in B.

Lemma: If x is an atom of B, then for all g in G, g(x) is an atom of B.

Proof: Suppose x is an atom of B and for some g in G, g(x) is not an atom of B. Since g(x) is in B, g(x) possesses a proper part which is in B, say z. But then $g^{-1}(z)$ is a proper part of x which is in B, contradiction.

Notation: G restricted to x, $G|x$, is the collection of all automorphisms in G with the domain of each automorphism restricted to the parts of x.

Lemma: If x is an atom of B, then

$$N|x = G|x = A(x)$$

where A(x) is the full group of automorphisms of the principal ideal generated by x .

Proof: Since G is a group over a formal language, it is the largest group satisfying the commuting property on the definable parts. Since no proper part of x is definable, $G \mid x$ is $A(x)$. $N \mid x = G \mid x$ since N is, by definition, the largest subgroup of G leaving B fixed.

In the following lemma and theorem the automorphisms act on the right, so that xg is the value of g when applied to x .

Lemma: If x is an atom of B , then

$$\forall g \text{ in } G \quad \forall n \text{ in } N \quad \exists m \text{ in } N \quad \forall z \pi x[zgng^{-1} = xm] .$$

Proof: Consider any g in G , n in N , and suppose xg = y . We can consider any g in G taking x to y , when restricted to x , as the product of two maps. The first is a "translation" depending only on x and y , i_{xy} , and the second some p_g in A(y) . That is, $g \mid x = i_{xy} p_g$. Thus

$$gng^{-1} = i_{xy} p_g n p_g^{-1} i_{xy}^{-1}$$

where n is considered to be restricted to y , and g is considered to be restricted to x . Now $p_g n p_g^{-1}$ is an element of A(y) , say m' , and $i_{xy} m' i_{xy}^{-1}$ is an element of A(x) , say m . So we have $g n g^{-1} \mid x = m \mid x$ which proves the lemma.

Theorem: If B is atomic then N is normal in G .

Proof: Since all automorphisms preserve least upper bounds we have $x = \Sigma x_\alpha$ if and only if $xg = \Sigma x_\alpha g$ for every x in the model and every g in G . For every x in the model, we can write $x = \Sigma x_\alpha$ where each

x_α is a part of an atom of B. If we consider N to be restricted to the parts of the atoms, a_α, of B, then N is the direct product

$$A(a_1) \times \ldots \times A(a_\alpha) \times \ldots$$

since N fixes a_α and is unrestricted on the proper parts of a_α. Now consider n in N where N is thought of as acting on the full model. Then $xn = \Sigma x_\alpha n$ where, if a_α is the atom of B such that $x_\alpha \pi a_\alpha$, then $x_\alpha n \pi a_\alpha$. For any g in G we have $xg n g^{-1} = \Sigma x_\alpha g n g^{-1}$ and by the previous lemma, for each α there is an m_α in $A(a_\alpha)$ such that $xgng^{-1} = \Sigma x_\alpha m_\alpha$. Since N is the direct product of the full automorphism groups $A(a_\alpha)$, there is an m in N such that $m | a_\alpha = m_\alpha$, hence $gng^{-1} = m$ in N.

VI. CONCLUSIONS AND FURTHER RESEARCH

Formal Language Definitions

We defined, following Thompson [32], a general notion of formal language which includes the current notions of syntax, where we considered the syntactic elements of the language to arise from a functionally oriented semantic system. The syntax of a formal language is in general a Post production system and we mentioned that, in this formulation, certain strings may be syntactically correct but not meaningful because of semantic considerations. The semantic considerations have been formalized by allowing a semantic transformation to be defined on a subdirect product of semantic categories, that is, only on those elements of a direct product of semantic category satisfying certain constraints. If these constraints can be described using syntactic notions alone, then they may be considered structural constraints and the notions of transformational grammar [4, 5] may apply. However, some further study will be required before the relationship between formal languages and transformational grammars is completely understood.

Groups

By considering a formal language to have a model of part theory as its universe of discourse we can consider the formal language to be characterized by a group of automorphisms of the model. Since the groups are partially ordered by inclusion, we can preorder the formal languages by defining $F_1 \leq F_2$ whenever $GF_1 \supseteq GF_2$. The relation on formal languages is a preordering since two formal languages may be associated with the same group as the following example illustrates.

We define two formal languages over the same model of set theory which we may consider as imbedded in a given model of part theory. F_1 has as its sole semantic transformation a characteristic function defined on sets. $Char(x) = \{0\}$ if $x \neq 0$ and $Char(0) = 0$. The sole semantic category of F_1 is the class of all sets and the class of referents of F_1 is any finite collection of sets which includes 0 and $\{0\}$. F_2 has as its sole semantic transformation a slightly different characteristic function on sets. $Empty(x) = \{0\}$ if $0 \notin x$ and $Empty(x) = 0$ if $0 \in x$, where $\in$ is the membership relation for the model of set theory. GF_1 is generated by all permutations of the singleton sets which leave $\{0\}$ fixed, since all automorphisms of a Boolean algebra fix 0 and if $x \neq 0$, then $xg \neq 0$ and $Char(xg) = \{0\}$ which implies $g(\{0\}) = \{0\}$. Since every set can be expressed as a union of singletons, we see that GF_1 is as stated. The semantic transformation Empty has the special property that $\{0\}$ is in its range and $0 \in x$ if and only if $\{0\}$ is a subset of x . This implies that any automorphism in GF_2 must leave $\{0\}$ fixed, and if g is any auromorphism with this property, then $0 \in x$ iff $0 \in xg$ so we have $Empty(xg) = Empty(x)$. Thus $GF_1 = GF_2$.

A possible method of avoiding this difficulty is to consider the semigroup of endomorphisms which leave a given language invariant.

The lemmas establishing the nature of the Galois connection between GF and LGF also hold if endomorphisms are considered rather than automorphisms, so that these results are available, but further study is required to determine if each formal language uniquely determines a semigroup of endomorphisms.

The Galois connection between GF and LGF suggests that each pair of languages possesses an upper bound and a lower bound language. By an upper bound language for F_1 and F_2 we mean a language which can express anything expressible in F_1 and in F_2. $F_1 + F_2$, as defined in the section on the Galois connection, is such an upper bound language, although not necessarily a least upper bound. By a lower bound language for F_1 and F_2 we mean a language in which the only entities definable are also definable in F_1 and in F_2. If it is possible to find a finite collection of basic semantic transformations, T, such that $CT \subseteq CF_1 \cdot CF_2$, then from T we can form a lower bound language. If $CT = CF_1 \cdot CF_2$ then from T we can form a "greatest" lower bound language. T may not be unique, however, which implies that there is not, in general, a unique greatest lower bound. As an example, if $GF_1 \cdot CF_2$ is the collection of all Boolean functions, then as basic semantic transformations for T we can take intersection and complement, or the Boolean ring sum and product.

These considerations suggest studying the conditions under which two formal languages have reasonably unique least upper bound and greatest lower bound languages. Further, the Galois connection becomes considerably more difficult to analyze if F_1 and F_2 do not possess the same collection of referents. This analysis will be carried out and it may indicate the conditions under which least upper bound and greatest lower bound languages exist. One result that we do have that will be useful in studying least upper bound languages is that every pair of formal languages has a common set theory underlying them. The sets are the elements of the subalgebra generated by the parts definable in either language.

Another problem suggested by the many-one correspondence between formal languages and their groups, and also by the nature of the Galois connection, is determining the class of formal languages which are associated with a given group of automorphisms. At present, no work has been done on this problem.

Part Theory, Set Theory

The relatively weak ontology of assuming the universe to be part theoretic has enabled us to consider automorphisms of the model of part theory as the changes taking place in the world, and characterize the structure which the formal language discusses as a group of automorphisms. If the model of part theory is strong enough, for example a complete Boolean algebra, then a model of set theory is imbeddable in the model of part theory. At the same time, we have avoided assuming that the world is atomistic and in fact have implicitly assumed throughout Chapters IV and V that it is atomless. The fact that we can imbed a set theory implies that any particular atomic view of the universe can be accommodated. There remains the question, however, of the relationship between a non-atomic subalgebra of parts defined by a formal language and a set theory imbedded in the model of part theory. If there is room in the model for both the non-atomic subalgebra of parts and a set theory, then, letting N be the group of the set theory, LN will contain a function, f , from the non-atomic definable parts to the sets such that if $x \pi y$ then $f(x) \subseteq f(y)$. Then in LN we should be able to discuss these parts as if they were large sets. The exact situation here remains unclear.

The view of a set theory as a particular collection of parts may have value in the theory of sets itself. Cohen, in discussing models of set theory in which the axiom of choice fails, points out that a standard model of set theory has no proper automorphisms, but that "the basic idea of having some kind of symmetry remains." [6, p. 136]. The collection of parts fixed by a group N can, if it is large enough, be thought of as a set theory determined by a particular singleton function in LN . There are many such singleton functions in LN and it may be possible to consider Cohen's "forcing" as a method of selecting among these functions.

Another aspect of the relationship between a formal language and a set theory beneath it we wish to determine is when it is possible to find a set theory such that every basic semantic transformation of the formal language is recursively definable in terms of the operations of the set theory, as discussed in Chapter II. This result should be suggestive of the adequacy of a model of part theory as the sole universe of discourse for formal languages. When it is possible to find a set theory for which the formal language is recursively definable, any objections from the constructivists are met since the semantic transformations are effectively computable in terms of intuitively satisfying primitive operations.

Sentences

In Chapter II we indicated that the position of sentences as a subclass of the phrases of a language is unclear. We can a priori distinguish sentences by syntactic means, but this leaves open the question of why those particular phrases are distinguished from the remaining ones. It may be possible to consider sentences as distinguished by an algebraic property based on the universe of discourse. For example, it may be that certain systems of endomorphisms leave invariant the structure which the intuition says is associated with sentences but not with phrases of the language. Here the first step is to state the problem more clearly than we have been able to do.

Transformations of Qualities

We return to the problem posed by the red pencil burning into black soot. Suppose we have a formal language in which "The red pencil burns into black soot." is a sentence, corresponding to some derived semantic transformation. In this language "the red pencil" corresponds to some part x and "black soot" to some part y. The change described in the formal language by "the red pencil burns into black soot." is an automorphism, g, such that $g(x) = y$. This automorphism is not in the group of the formal language since the red pencil and the black soot, x and y, are distinguishable. If x and y are atoms of the subalgebra of definable parts, then there is no "red" part of x or "black" part of y as far as this formal language is concerned since x and y are structureless. Any automorphism taking x to y is an equally valid change irrespective of how the automorphism takes proper parts of x into proper parts of y . In this formal language, there is no red part to be transformed.

Suppose x and y are not atoms, x having a red part and y having a black one. The automorphism taking x to y need not preserve the structure of x since it is not in the group of the formal language, and so there is no requirement for the red part of x to map into any particular part of y . The distinguishability of x and y imply that the transformation of x into y is a change observable by a speaker of the given formal language. A change is observable because of the change in structure, where the structure is determined by the language. The sentence "The red pencil burns into black soot. " denotes that the entity with the structure of a red pencil has been transformed into an entity with the structure of black soot without specifying the transformation of substructures like the red part of the red pencil. We can, if we like, consider the red part of x to be transformed into the black part of y since this preserves some structure associated with color.

If we have a much more detailed formal language in which we can discuss the interaction of molecules and light, there is a sentence which explains the transformation of sensual red into sensual black in a way that preserves far more structure than our original sentence. Even here, however, the entire structure is not preserved since the change is observable.

This discussion has served to illustrate that in relation to a fixed formal language there is no difficulty associated with the transformation of the qualities of a part when the part is transformed. Either the entire structure of the part is preserved, in which case the part and its transform are indistinguishable by the formal language, or the entire structure is not preserved and the transform of a part is distinguishable from the part itself.

Groups as Function Spaces

We would like to formalize the intuitive notion of the distance between formal languages. There appears to be some possibility of doing so along the following lines.

Given a model of part theory, P , together with a measure μ on P , we can consider any group of automorphisms as a function space by defining, for any two automorphisms g and h ,

$$d^*[g, h] = \sup d[xg, xh]$$

where d is the metric defined by μ .

If each automorphism in a group G is uniformly continuous as a homeomorphism of P , then G is a topological group, as we prove below. G. Birkhoff [1, p. 169] describes a complete measure algebra, $\overline{M}$, unique up to isometric isomorphism, which is obviously a model of part theory. Since $\overline{M}$ is metrically complete, it is complete as a metric space. Furthermore, $\overline{M}$ is totally bounded so $\overline{M}$ is compact [16, p. 84] and thus every automorphism of $\overline{M}$ is uniformly continuous [16, p. 30]. If F is a formal language with $\overline{M}$

as its universe of discourse, GF is a toplogical group. The consequences of this fact may have interesting implications for the study of formal languages. In particular, it enhances the prospects for finding an intuitively reasonable metric topology on the lattice of groups of formal languages. If this can be done, then we can define the psuedo-distance between two formal languages as the distance between their associated groups. There remains, of course, the difficulty that two distinct formal languages may share the same group, so that the collection of formal languages is, under these assumptions, a psuedo-metric space. While it may be possible to circumvent this difficulty by considering semigroups of endomorphisms associated with formal languages rather than groups of automorphisms, further study is clearly required to analyze the situation.

We now show that if every automorphism in a group G is uniformly continuous as a map on P to P, then G is a topological group. We first prove that the map from G to G that takes each automorphism into its inverse is continuous in the d* topology.

Lemma: For all g in G and for all $\epsilon > 0$ there is a δ such that if $d^*[g, h] < \delta$ then $d^*[g^{-1}, h^{-1}] < \epsilon$.

Proof: First we note that if for all x, $d[xg, xh] < \epsilon$, then $d^*[g, h] \leq \epsilon$. So if we can show that for all ϵ there exists a δ such that $\forall x(d[xg, xh] < \delta) \rightarrow \forall x(d[xg^{-1}, xh^{-1}] < \epsilon)$ we can complete the proof since $d^*[g, h] < \delta$ implies $\forall x(d[xg, xh] < \delta)$ which in turn implies $d^*[g^{-1}, h^{-1}] \leq \epsilon$. Then, for each $\epsilon > 0$, pick δ such that

$d^*[g, h] < \delta$ implies $d^*[g^{-1}, h^{-1}] \leq \epsilon/2 < \epsilon$.

The uniform continuity of g^{-1} implies that for all ϵ there exists a δ such that $\forall x(d[xg, xh] < \delta) \rightarrow \forall x(d[xg^{-1}, xh^{-1}] < \epsilon)$. For if not, then there exists an ϵ_o such that for all δ we can find an h such that $\forall x(d[xg, xh] < \delta)$ & $\exists x(d[xg^{-1}, xh^{-1}] \geq \epsilon_o)$. The uniform continuity of g^{-1} guarantees that for ϵ_o there exists δ_o such that for all xg and all xh , $d[xg, xh] < \delta_o \rightarrow d[x, xhg^{-1}] < \epsilon_o$. Combining this with the supposition above we have for ϵ_o and δ_o an h_o such that $\forall x(d[xg, xh_o] < \delta_o)$ which implies $\forall x(d[x, xhg^{-1}] < \epsilon_o)$, via continuity. The latter statement can be rewritten, by setting $y = xh$, as $\forall y(d[yh^{-1}, yg^{-1}] < \epsilon_o)$, which is a contradiction, and completes the proof.

The following lemma shows that the map from G x G to G which takes (g_1, g_2) to g_1g_2 is continuous. Taken together, the two lemmas imply that G is a topological group.

Lemma: Given g_1 and g_2 in G , for all ϵ there exists δ_1 and δ_2 such that if $d^*[g_1, h_1] < \delta_1$ and $d^*[g_2, h_2] < \delta_2$ then $d^*[g_1g_2, h_1h_2] < \epsilon$.

Proof: Since $d^*[g_1g_2, h_1h_2] \leq d^*[g_1g_2, h_1g_2] + d^*[h_1g_2, h_1h_2]$, if we can find δ_1 and δ_2 such that each of $d^*[g_1g_2, h_1g_2]$ and $d^*[h_1g_2, h_1h_2]$ is less than or equal to $\epsilon/3$, then $d^*[g_1g_2, h_1h_2] < \epsilon$. Since $\sup d[xh_1g_2, xh_1h_2] = \sup d[yg_2, yh_2] = d^*[g_2, h_2]$, let $\delta_2 = \epsilon/3$. By the uniform continuity of g_2 we have that for all ϵ there exists δ such that for all x , $d[xg_1, xh_1] < \delta \rightarrow d[xg_1g_2, xh_1g_2] < \epsilon/3$. Given ϵ , we pick δ_1 such that $d^*[g_1, h_1] < \delta_1$ and thus by the sentence above

$\forall x(d[xg_1g_2, xh_1g_2] < \epsilon/3)$, implying $d^*[g_1g_2, h_1g_2] \leq \epsilon/3$.

APPENDIX

The appendix is in two sections. In the first, we prove in the affirmative the decision problem for the elementary theory of Boolean algebras which are direct products of atomic and atomless Boolean algebras. The main interest in this result is the decision method, which shows that every formula in the elementary theory is equivalent to a quantifier-free formula. In the second section, the results obtained from the decision method are used to prove the existence of the union of all elements of a direct product Boolean algebra which satisfy a given elementary formula. This result shows that every direct product Boolean algebra possesses a very restricted completeness property, namely that the union of a class of elements exists when the class is definable by elementary formulas.

Tarski [30] has proved in the affirmative the decision problem for general Boolean algebras, but the results, as they appear in the literature, do not enable one to prove that $\sum_{F(a)} a$ exists. Since this union must exist in order that a Boolean algebra be a model of part theory, we require the decision method presented below.

The Decision Method

This section follows the following outline. We define a standard form for formulas and proceed to show that Boolean connections of these formulas can be reduced to standard form. Next we show that a standard form formula involving a quantifier is equivalent to a standard form formula without the quantifier. Using these results, we argue that any formula of the lower predicate calculus whose only predicate is "$\leq$" is equivalent to a formula in standard form. Finally we prove that this decision method is only valid for Boolean algebras which are direct products of atomic and atomless Boolean algebras. Implicitly we are using the fact that every Boolean algebra is the subdirect product of atomic and atomless Boolean algebras.

Let 2^n be the collection of all functions from n to $\{0, 1\}$. Let $R \subseteq 2^n$, $\overline{R} = 2^n - R$. For $r \in 2^n$ let $r^1 = \{i \mid r(i) = 1\}$, $r^0 = \{i \mid r(i) = 0\}$. Given $x_0, \ldots, x_{n-1}$, let $y_r = \prod_{i \in r^1} x_i \cdot \prod_{i \in r^0} \tilde{x}_i$.

Definition: $F_R(x_0, \ldots, x_{n-1})$ is in S-form iff

$$F_R(x_0, \ldots, x_{n-1}) = \bigwedge_{r \in \overline{R}} (y_r = 0) \;\&\; \bigwedge_{r \in R} [y_r \neq 0 \;\&\; At(k_r, \ell_r, m_r; y_r)]$$

where $At(k, \ell, m; y)$ is a predicate satisfied by k, ℓ, m, and y just in case

y is atomic iff $k = 1$,

y is not atomic iff $k = 0$,

y has $\geq n$ atoms iff $\ell = 1$ and $n = m$,

and y has exactly n atoms iff $\ell = 0$ and $n = m$.

We may write F_R for $F_R(x_0, \ldots, x_{n-1})$. We will consider k and ℓ to be elements of the two element Boolean algebra, $\{0, 1\}$.

Professor F. B. Thompson pointed out this standard form (S-form) for use in proving the decision problem. Essentially, the y_r are the atoms of the subalgebra finitely generated by

$x_o, \ldots, x_{n-1}$. As we will show here, the elementary formulas can only specify unions, intersections, complements, and the atomic structure of the y_r and hence of the x_i .

Lemma: If $r, s \in 2^n$ and $r \neq s$, then $y_r \cdot y_s = 0$.

Proof: For some $i \in n$, x_i occurs in y_r uncomplemented and occurs in y_s complemented, or vice versa. Since y_r and y_s are intersections of the x_i, $y_r \cdot y_s = 0$.

We define formulas to be used in the next theorem.

$$G_S(a, b, c; x_o, \ldots, x_{n-1}) = \bigwedge_{s \in \overline{S}} (y_s = 0) \ \& \ \bigwedge_{s \in S} [y_s \neq 0 \ \& \ At(a, b, c; y_s)] \ .$$

$$H_S(x_o, \ldots, x_{n-1}) = G_S(1, 1, 1; x_o, \ldots, x_{n-1}) \vee G_S(0, 1, 0; x_o, \ldots, x_{n-1}) \ .$$

$$I^r_S(x_o, \ldots, x_{n-1}) = \bigvee_{f \in A_r} [\bigwedge_{s \in \overline{S}} (y_s = 0) \ \& \ \bigwedge_{s \in S} [y_s \neq 0 \ \& \ At(f_1(s), f_2(s), f_3(s); y_s)]]$$

where

$$\begin{aligned} A_r = \{f \mid f = \langle f_1, f_2, f_3 \rangle \ & \& \ f_1, f_2 \in 2^S \ \& \ f_3 \in \omega^S \\ & \& (f_1(s) = 1 - k_r) \rightarrow (f_2(s) = 1 \ \& \ f_3(s) = 1 - k_r) \\ & \& (f_1(s) = k_r \ \& \ f_2(s) = 0) \rightarrow (f_3(s) < m_r) \\ & \& (f_1(s) = k_r \ \& \ f_2(s) = 1) \rightarrow (f_3(s) = m_r + 1 \ \& \ \ell_r = 0)\} \ . \end{aligned}$$

I^r_S is a disjunction of formulas, one of which will hold if $\neg At(k_r, \ell_r, m_r; y_r)$. The class A_r is the means of selecting the

the atomic conditions, one of which must hold if $At(k_r, \ell_r, m_r; y_r)$ does not.

We may drop $(x_o, \ldots, x_{n-1})$ when writing G, H, and I for compactness of expression.

Theorem: (Not) Let $F_R(x_o, \ldots, x_{n-1})$ be a formula in S-form. Let

$$N_R(x_o, \ldots, x_{n-1}) = \bigvee_{r\in\overline{R}} [\bigvee_{r\in S\subseteq 2^n} H_S] \vee \bigvee_{r\in R} [\bigvee_{r\notin S\subseteq 2^n} H_S \vee \bigvee_{r\in S\subseteq 2^n} I_S^r].$$

Then $\neg F_R(x_o, \ldots, x_{n-1}) \leftrightarrow N_R(x_o, \ldots, x_{n-1})$.

Proof: $\neg F_R(x_o, \ldots, x_{n-1}) \leftrightarrow$

$$\bigvee_{r\in\overline{R}} (y_r \neq 0) \vee \bigvee_{r\in R} [y_r = 0 \vee \neg At(k_r, \ell_r, m_r; y_r)] .$$

(i) $y_r \neq 0$ iff $\bigvee_{r\in S\subseteq 2^n} H_S$ since $\bigvee_{r\in S\subseteq 2^n} H_S$ implies $y_r \neq 0$ by the construction of H_S and $y_r \neq 0$ gives no condition on the remaining members of R , or on their atomic structure. H_S has been defined so that any condition on the remaining members of R is satisfied by one of the H_S in $\bigvee_{r\in S\subseteq 2^n} H_S$.

(ii) $y_r = 0$ iff $\bigvee_{r\notin S\subseteq 2^n} H_S$. This case is like (i). All possible conditions on the remaining members of R are covered by the disjunction.

(iii) $[y_r \neq 0 \ \& \neg At(k_r, \ell_r, m_r; y_r)] \leftrightarrow \bigvee_{r\in S\subseteq 2^n} I_S^r$.

If y_r is not zero and does not have the atomic structure (k_r, ℓ_r, m_r) , then it has some other atomic structure and every possibility is in the disjunction of the I_S^r. If some disjunction in one of the I_S^r is true, then $y_r \neq 0$ and y_r does not have the atomic structure (k_r, ℓ_r, m_r).

Theorem: (Or) If D_1 and D_2 are disjunctions of S-form formulas, then $D_1 \vee D_2$ is a disjunction of S-form formulas.

To prove a similar result for conjunctions, we first require a lemma which enables us to increase the number of variables on which a formula depends. With this result, we can then assume that the two formulas in a conjunction are both over the same variables.

We define certain classes in order to state the lemma. We first require the notation that if $s \in 2^m$ and $m > n$, then $r = s \mid n$ means that r is the restriction of the function s to the first n integers.

Definitions: For $R \subseteq 2^n$ and $n < m$,

$$R^* = \{S \subseteq 2^m \mid \forall r \in R \, \exists s \in S (r = s \mid n) \ \& \ \forall r \in \overline{R} \, \forall s [\, r = s \mid n \rightarrow s \in \overline{S}\,]\} .$$

The class R^* will be used to expand the number of variables from n to m. Roughly, R^* will allow us to expand in every possible manner.

The following definitions lead to a class of triples of functions, each triple determining an atomic structure for an "expanded" y_r.

$$E_r = \{s \in 2^m \mid r = s \mid n\} .$$

$$\underline{K}_r = \{K_r \in 2^{E_r} \mid \prod_{s \in E_r} K_r(s) = k_r\} ,$$

$$\underline{L}_r = \{L_r \in 2^{E_r} \mid \sum_{s \in E_r} L_r(s) = \ell_r\} ,$$

and $$\underline{M}_r = \{M_r \in \omega^{E_r} \mid \sum_{s \in E_r} M_r(s) = m_r\} .$$

$$Q_S = \{<f_1, f_2, f_3> \mid f_1(s) = K_{s|n}(s) \in \underline{K}_{s|n} \,\&\, f_2(s) = L_{s|n}(s) \in \underline{L}_{s|n}$$

$$\& \, f_3(s) = M_{s|n}(s) \in \underline{M}_{s|n} \,\&\, s \in S\} .$$

Lemma: (Adding Variables) Let $F_R(x_o, \ldots, x_{n-1})$ be a formula in S-form. Let $A_R(x_o, \ldots, x_{m-1}) =$

$$\bigvee_{S \in R^*} \bigvee_{<f_1, f_2, f_3> \in Q_S} [\bigwedge_{s \in \overline{S}} (y_s = 0)$$

$$\& \bigwedge_{s \in S} [y_s \neq 0 \,\&\, At(f_1(s), f_2(s), f_3(s); y_s)]] .$$

If $n < m$ then $F_R(x_o, \ldots, x_{n-1}) \leftrightarrow A_R(x_o, \ldots, x_{m-1})$.

Proof: Assume $F_R(x_o, \ldots, x_{n-1})$. Let $x_n, \ldots, x_{m-1}$ take on any particular values. Then at least one of the conjunctions in A_R holds since.

(i) $y_r = 0 \rightarrow y_s = 0$ for $r = s|n$

and (ii) if $y_r \neq 0 \,\&\, At(k_r, \ell_r, m_r; y_r)$ then each piece of y_r determined by an s such that $r = s|n$ must partake of the give atomic structure of y_r . In particular, if y_r is atomic, i.e., $k_r = 1$, then each $y_s \neq 0$ such that $r = s|n$ must be atomic. The construction of Q_S guarantees this by forcing $f_1(s) = 1$ for $s \in S$

such that $r = s|n$. If y_r is not atomic, i.e., $k_r = 0$, then there is some s such that $r = s|n$ and $y_s \neq 0$ and y_s is not atomic, again by the construction of Q_S. A similar argument holds for the ℓ and m parameters determining the number of atoms in any piece.

Now assume $A_R(x_o, \ldots, x_{m-1})$, hence at least one of the conjunctions in A_R holds. We will show that the restriction of this conjunction to n variables is satisfied.

(i) If $y_s = 0$ for all s such that $r = s|n$ then clearly $y_r = 0$.

(ii) If (i) is not the case for a given r then there is an s such that $r = s|n$ and $y_s \neq 0$. Hence $y_r \neq 0$ since y_r is the intersection of the first n variables (complemented or not) whose intersection with the remaining m - n variables is y_s. If $y_r \neq 0$, consider the terms

$$y_s \neq 0 \;\&\; At(k_s, \ell_s, m_s; y_s)$$

for which $r = s|n$ in the satisfied conjunction of A_R. Since $y_r = \sum_{r=s|n} y_s$, we have

$$\bigwedge_{r=s|n} [y_s \neq 0 \;\&\; At(k_s, \ell_s, m_s; y_s)] \rightarrow$$

$$[y_r \neq 0 \;\&\; At(\prod_s k_s, \sum_s \ell_s, \sum_s m_s; y_r)] .$$

Now by the construction of these terms in A_R, we have

$\prod_s k_s = k_r$, $\sum_s \ell_s = \ell_s$, and $\sum_s m_s = m_r$. Therefore $A_R \rightarrow F_R$.

Lemma 1: If $F_{R_1}(x_o, \ldots, x_{n-1})$ and $F_{R_2}(x_o, \ldots, x_{n-1})$ are formulas in S-form, then $F_{R_1}(x_o, \ldots, x_{n-1})$ & $F_{R_2}(x_o, \ldots, x_{n-1})$ is satisfied only if $R_1 = R_2$.

Proof: Suppose $R_1 \neq R_2$. Then there is an r in the symmetric difference of R_1 and R_2, say in R_1 but not in R_2. Hence $y_r \neq 0$ in F_{R_1} but $y_r = 0$ in F_{R_2}, hence F_{R_1} & F_{R_2} is false.

Notation: Let k_r^i, ℓ_r^i, m_r^i be the atomic structure parameters in F_R if $i = 1$ and in G_R if $i = 2$.

Lemma 2: If F_R and G_R are S-form formulas then F_R & G_R only if

$$\forall r \in R[k_r^1 = k_r^2]$$

$$\& \ \forall r \in R[(\ell_r^1 = \ell_r^2 = 0) \rightarrow m_r^1 = m_r^2]$$

$$\& \ \forall r \in R[(\ell_r^1 = 0 \ \& \ \ell_r^2 = 1) \rightarrow m_r^1 \geq m_r^2]$$

$$\& \ \forall r \in R[(\ell_r^1 = 1 \ \& \ \ell_r^2 = 0) \rightarrow m_r^1 \leq m_r^2] \ .$$

Proof: $k_r^1 = k_r^2$ for otherwise y_r is both atomic and not atomic. Similarly, if $\ell_r^1 = \ell_r^2 = 0$, then y_r has exactly $m_r^1 = m_r^2$ atoms. If $\ell_r^1 = 0$ and $\ell_r^2 = 1$, then y_r has exactly m_r^1 and greater than or equal to m_r^2 atoms, hence $m_r^1 \geq m_r^2$. Similarly if $\ell_r^1 = 1$ and $\ell_r^2 = 0$.

To state the theorem for conjunctions we use the formula

$$C_R(x_o, \ldots, x_{n-1}) = \bigwedge_{r \in \overline{R}} [y_r = 0]$$

$$\& \bigwedge_{r \in R} [y_r \neq 0 \ \& \ At(k_r, \ell_r^1 \cdot \ell_r^2, \max(m_r^1, m_r^2); y_r)]$$

Theorem: (And) If $F_{R_1}(x_o, \ldots, x_{n-1})$ and $F_{R_2}(x_o, \ldots, x_{n-1})$ are formulas in S-form, then $F_{R_1}(x_o, \ldots, x_{n-1})$ & $F_{R_2}(x_o, \ldots, x_{n-1})$ iff $C_{R_1}(x_o, \ldots, x_{n-1})$ and lemmas 1 and 2 hold for F_{R_1} and F_{R_2}.

Proof: Assume F_{R_1} & F_{R_2}. Then $R_1 = R_2$ by lemma 1 and the atomic structures of F_1 and F_2 are interrelated as described in lemma 2. Let $R = R_1$. It remains to show that $C_R(x_o, \ldots, x_{n-1})$ is satisfied. Since $R_1 = R_2$, if $y_r = 0$ in F_1 and F_2 by the construction of C_R the same term is present in C_R. Similarly if $y_r \neq 0$ in F_1 and F_2. The only remaining question is the atomic structure of $y_r \neq 0$ in C_R. Since lemma 2 holds, the parameters $(k_r, \ell_r^1 \cdot \ell_r^2, \max(m_r^1, m_r^2))$ do describe the atomic structure of y_r. For, $k_r^1 = k_r^2$, and if $\ell_r^1 = \ell_r^2 = 0$, then $\max(m_r^1, m_r^2) = m_r^1 = m_r^2$, while if $\ell_r^1 = 0$ and $\ell_r^2 = 1$, $\max(m_r^1, m_r^2) = m_r^1 \geq m_r^2$.

Now assume that $R_1 = R_2 = R$ and the conditions on the atomic structure parameters are as described in lemma 2, and that $C_R(x_o, \ldots, x_{n-1})$ is satisfied. Then it is clear that F_{R_1} is satisfied and so is F_{R_2} hence F_{R_1} & F_{R_2} is satisfied.

We use the following classes and formula to state the elimination of quantifiers theorem. If $S \subseteq 2^{n+1}$, then

$R = \{ r \in 2^n \mid r = s|n \text{ for } s \in S \}$, $E_r = \{ s \mid r = s|n \;\&\; s \in S \}$,

$R^+ = \{ r \in R \mid \exists!\, s(s \in S \rightarrow r = s|n) \}$, and $R^* = \{ r \in R \mid r = s|n \rightarrow s(n) = 1 \}$.

For $r \in R$, let $k_r = \prod_{s \in E_r} k_s$, $\ell_r = \sum_{s \in E_r} \ell_s$, and $m_r = \sum_{s \in E_r} m_s$.

Note that E_r contains at most two elements for each r.

Define $W_R(x_o, \ldots, x_{n-1}) = \bigwedge_{r \in \overline{R}} [y_r = 0]$ & $\bigwedge_{r \in R} [y_r \neq 0 \;\&\; At(k_r, \ell_r, m_r; y_r)]$.

Theorem: (Elimination of Quantifiers) If $F_S(x_o, \ldots, x_{n-1}, x_n)$ is a formula in S-form then $\exists x_n F_S(x_o, \ldots, x_n) \leftrightarrow W_R(x_o, \ldots, x_{n-1})$.

Proof: Assume $W_R(x_o, \ldots, x_{n-1})$. We will show that $\exists x_n F_S(x_o, \ldots, x_n)$. As a notational convenience, we write x^o for $\tilde{x}$ and x^1 for x .

If $r \in \overline{R}$, for any x , $x^i \cdot y_r = 0$ for $i = 0, 1$, since $y_r = 0$. By the definition of R , $r \in \overline{R}$ implies any extension of r to n+1 variables is a member of $\overline{S}$. Hence for each s such that $s|n \in \overline{R}$, $[y_s = 0]$ is satisfied.

Consider $r \in R^+$. We must show that

$$[y_r \cdot x_n^{1-i} = 0] \,\&\, [y_r \cdot x_n^i \neq 0 \,\&\, At(k, \ell, m; y_r \cdot x_n^i)]$$

in F_S is satisfied for each $r \in R^+$, where i depends on r . Since $r \in R^+$, we know that

$$[y_r \neq 0 \,\&\, At(k_r, \ell_r, m_r; y_r)]$$

since W_r is satisfied, and by construction, $k = k_r$, $\ell = \ell_r$, and $m = m_r$. Let $z_o = \sum_{r \in R^*} y_r$. For each $r \in R^+$, $z_o^i \geq y_r$. For if $i_r = 0$ then

$$y_r \cdot \tilde{z}_o = \prod_{t \in R^*} (y_r \cdot \tilde{y}_t) \,.$$

Now $r \notin R^*$ since $i_r = 0$, and since all the y's are disjoint, $\tilde{y}_t \geq y_r$, thus $y_r \cdot \tilde{z}_o = y_r$. Now suppose $i_r = 1$. Then

$$y_r \cdot z_o = \sum_{t \in R^*} (y_r \cdot y_t)$$

and since this time r is a member of R^* and $y_r \cdot y_t = 0$ if $t \neq r$, we have $y_r \cdot z_o = y_r$.

Since $z_o^i \geq y_r$, we have $y_r \cdot z_o^{1-i} = 0$. So if we set $x_n = z_o$, all of the terms in F_S under consideration are satisfied. Combining the results for $r \in \overline{R}$ and $r \in R^+$, we see that $\bigwedge_{s \in \overline{S}} [y_s = 0]$ is satisfied.

The remaining terms of F_S to be satisfied are of the form

$$[y_r \cdot \tilde{x}_n \neq 0 \;\&\; At(k_1, \ell_1, m_1; y_r \cdot \tilde{x}_n)]$$

$$\&\; [y_r \cdot x_n \neq 0 \;\&\; At(k_2, \ell_2, m_2; y_r \cdot x_n)]$$

for $r \in R - R^+$. In W_R the corresponding terms are

$$[y_r \neq 0 \;\&\; At(k_1 \cdot k_2, \ell_1 + \ell_2, m_1 + m_2; y_r)] \;.$$

Here it is necessary to divide y_r into two pieces such that each has the atomic structure specified in F_S. To show that this is possible, we will consider cases.

(i) $k_1 = k_2 = 1$. That is, both $y_r \cdot \tilde{x}_n$ and $y_r \cdot x_n$ are atomic. If $\ell_1 = \ell_2 = 0$, then y_r has exactly $m_1 + m_2$ atoms, so there is a $z_r \leq y_r$ with exactly m_2 atoms, hence $At(k_1, \ell_1, m_1; y_r \cdot \tilde{z}_r)$ and $At(k_2, \ell_2, m_2; y_r \cdot z_r)$. If $\ell_1 = 1$ and $\ell_2 = 0$, then y_r has $\geq m_1 + m_2$ atoms, and again there is a $z_r \leq y_r$ with exactly m_2 atoms such that $At(k_1, \ell_1, m_1; y_r \cdot \tilde{z}_r)$ and $At(k_2, \ell_2, m_2; y_r \cdot z_r)$. Similarly if $\ell_1 = \ell_2 = 1$ or if $\ell_1 = 0$, $\ell_2 = 1$.

(ii) $k_1 = 0$ and $k_2 = 1$. Then y_r is not atomic. Now if $\ell_1 = \ell_2 = 0$, then y_r has exactly $m_1 + m_2$ atoms, and if we let

z_r be the union of exactly m_2 atoms of y_r, we satisfy the appropriate terms in F_S. If $\ell_1 = 0$ and $\ell_2 = 1$, then y_r has $\geq m_1 + m_2$ atoms, and if we let z_r be the union of all but m_1 atoms of y_r, again we satisfy the terms in F_S. Similarly for the remaining combinations of ℓ_1 and ℓ_2.

(iii) $k_1 = 1$ and $k_2 = 0$. This case is like (ii) except that we find some $\tilde{z}_r \leq y_r$.

(iv) $k_1 = k_2 = 0$. Then y_r is not atomic and we must find a $z_r \leq y_r$ such that both $y_r \cdot z_r$ and $y_r \cdot \tilde{z}_r$ are not atomic with the appropriate numbers of atoms. Since the Boolean algebra is the direct product of an atomic and an atomless Boolean algebra, this is always possible.

Let $z = z_o + \sum_{r \in R - R^+} z_r$. Then z satisfies $F_S(x_o, \ldots, x_{n-1}, z)$. For the proof in the other direction, assume $\exists x_n F_S(x_o, \ldots, x_{n-1}, x_n)$. Then z as constructed above also satisfies $F_S(x_o, \ldots, x_{n-1}, z)$, and so $W_R(x_o, \ldots, x_{n-1})$ is satisfied.

It remains to show that every formula of the lower predicate calculus with $\leq$ as the only predicate is equivalent to a disjunction of S-form formulas. This is done by induction on the length of a formula.

Terms: $x_o \leq x_1$ is equivalent to a disjunction of certain $F_S(x_o, x_1)$. This disjunction includes the various possible atomic conditions on $x_o^i \cdot x_1^j$ in the form $At(1, 1, 1; x_o^i \cdot x_1^j)$ or $At(0, 1, 0; x_o^i \cdot x_1^j)$. The F_S are of the form $x_o \cdot \tilde{x}_1 = 0$ in conjunction with $x_o \cdot x_1$, $\tilde{x}_o \cdot x_1$, $\tilde{x}_o \cdot \tilde{x}_1$ either equal to zero or not equal to zero in all possible ways, and whenever $x_o^i \cdot x_1^j \neq 0$, one of the two atomic conditions. The disjunction of these twenty-five S-form formulas is satisfied if and only if $x_o \leq x_1$.

Negation: If D is a disjunction of formulas in S-form, say

$$D = F_1 \vee \dots \vee F_p$$

then $\neg D \leftrightarrow \neg F_1 \,\&\, \dots \,\&\, \neg F_p \leftrightarrow N_1 \,\&\, \dots \,\&\, N_p$, where N_i is related to F_i by the "not" theorem. By redistributing among the $N_1, \dots, N_p$, we obtain a disjunction of conjunctions of formulas in S-form, provided that all the F_i are over the same variables. The adding variables lemma guarantees that we can find the appropriate formulas equivalent to the F_i.

Conjunction: If C is a conjunction of formulas in S-form, by the adding variables lemma, C is equivalent to a conjunction of S-form formulas all over the same variables. By then applying the "and" theorem sufficiently often we obtain an equivalent S-form formula or a contradiction.

Quantifiers: If D is a disjunction of S-form formulas, $D = F_1 \vee \dots \vee F_p$, then $\exists xD \leftrightarrow \exists xF_1 \vee \dots \vee \exists xF_p$ and $\forall xD \leftrightarrow \neg \exists x \neg D$. Since $\neg D$ is equivalent to a disjunction of S-form formulas, say $F_1' \vee \dots \vee F_q'$, then $\forall xD \leftrightarrow \neg(\exists xF_1' \vee \dots \vee \exists xF_q')$. By elimination of quantifiers, disjunctions of S-form formulas equivalent to $\exists xD$ and $\forall xD$ are obtained.

This completes the decision method for the elementary theory of direct product Boolean algebras. Next we show that the elimination of quantifiers theorem requires that the Boolean algebra be a direct product of an atomic and an atomless Boolean algebra. Assuming that the Boolean algebra is a subdirect product of an atomic and an atomless Boolean algebra possessing just the properties necessary for the proof of the elimination of quantifiers theorem, we prove that the Boolean algebra must be the direct product of its factors.

Let B be a subdirect product of A x N where A is an atomic Boolean algebra and N is an atomless Boolean algebra. If $b \in B$, we write $b = <a, n>$ where a and n are the projections of b onto A and N , respectively. Let B possess the following property:

(i) If $<a, n> \in B$ and a is the union of m atoms then for all $m' \leq m$ there is an $a' \leq a$ such that a' is the union of m' atoms and $<a', 0> \in B$.

B must possess this property for the proof of the elimination of quantifiers theorem to be valid.

Lemma: For all $a \in A$, $<a, 0>$ is in B .

Proof: For some $n \in N$, $<a, n>$ is in B . Then by (i) , $<a, 0> \in B$.

Lemma: If $<a, n> \in B$ then $<0, n> \in B$.

Proof: Using the previous lemma, $<a, 0> \in B$. Hence $<a, n> - <a, 0> = <0, n>$ is in B .

Theorem: B is the direct product A x N .

Proof: First, for all $n \in N$, $<0, n>$ is in B since for all $n \in N$ there is some $a \in A$ such that $<a, n> \in B$ and applying the lemma above gives the result. Now let a be any member of A , n any member of

N. Then $<a, 0> + <0, n> = <a, n> \in B$.

Existence of Unions

Let $F(x_o, \ldots, x_{n-1}, a)$ be a formula of the lower predicate calculus over the predicate "$\leq$". We will show that

$$\underset{F(x_o, \ldots, x_{n-1}, a)}{\Sigma a}$$

exists in any Boolean algebra which is the direct product of an atomic and an atomless Boolean algebra. By the decision method in the first section, it suffices to show that the union exists for formulas which are disjunctions of S-form formulas. Suppose $F_1, \ldots, F_n$ are S-form formulas such that the union exists for each formula. Then the union exists for the formula $F_1 \vee \ldots \vee F_n$ by the following argument. For each i, let $T_i = \{a \mid F_i(a)\}$ and let $T = \{a \mid F_1(a) \vee \ldots \vee F_n(a)\}$. Then T is the union of the T_i and from Sikorski [28, p. 59] we have

$$\sum_{t \in T_1} a_t + \ldots + \sum_{t \in T_n} a_t = \sum_{t \in T} a_t$$

where the existence of the left side implies the existence of the right.

It remains to show that the union exists for any S-form formula. Let $F_S(x_o, \ldots, x_{n-1}, a)$ be an S-form formula. The first n variables can be considered to be parameters since we are taking the union of all a satisfying $F_S(x_o, \ldots, x_{n-1}, a)$. So consider $x_o, \ldots, x_{n-1}$ to be fixed elements of the Boolean algebra such that F_S is true for at least one a. Under these conditions we will find an equivalent formula from which the union can be determined directly. We use an intermediate formula which is F_S rewritten to display the variable a .

For $s \in S$ and $r = s|n$, let $y_r = \prod_{i \in r^1} x_i \cdot \prod_{i \in r^o} \tilde{x}_i$. Then $F_S(x_o, \ldots, x_{n-1}, a)$ is equivalent to $B_S(y_o, \ldots, y_p; a)$ where

$$B_S(y_o, \ldots, y_p; a) = \bigwedge_{s \in \bar{S}} [y_{s|n} \cdot a^{s(n)} = 0]$$

$$\& \bigwedge_{s \in S} [y_{s|n} \cdot a^{s(n)} \neq 0 \; \& \; At(k_s, \ell_s, m_s; y_{s|n} \cdot a^{s(n)})] .$$

For each $r \in 2^n$ exactly one of the following four cases is a conjunction in B_S and the cases, repeated enough times, exhaust B_S . If y_r satisfies case j , we may index y_r as y_{jk} .

1. $(y_r \cdot a = 0) \; \& \; (y_r \cdot \tilde{a} = 0)$. In this case $y_r = 0$ and terms of this type imply no restriction on a . So B_S is equivalent to a formula with terms satisfying this case removed.

2. $(y_r \cdot \tilde{a} = 0) \; \& \; (y_r \cdot a \neq 0 \; \& \; At(k, \ell, m; y_r \cdot a))$. In this case we have $(a \geq y_r)$ and B_S is equivalent to a formula with terms of this type replaced by $(a \geq y_{2k})$ for an appropriate indexing set, $k \in \{1, \ldots, q\}$.

3. $(y_r \cdot a = 0) \; \& \; (y_r \cdot \tilde{a} \neq 0 \; \& \; At(k, \ell, m; y_r \cdot \tilde{a}))$. Here we have $(a \leq \tilde{y}_r)$ and B_S is equivalent to a formula with terms of this type replaced by $(a \leq \tilde{y}_{3k})$ for $k \in \{1, \ldots, t\}$.

In case these three cases exhaust B_S , we have the equivalent formula

$$(a \geq y_{21}) \; \& \ldots \& \; (a \geq y_{2q}) \; \& \; (a \leq \tilde{y}_{31}) \; \& \ldots \& \; (a \leq \tilde{y}_{3t}) .$$

Since $\tilde{y}_r \geq y_s$ for all $s \neq r$ and $y_r \neq y_s$ for all r and s, we have that the least upper bound of all a satisfying the above conjunction is the intersection of the $\tilde{y}_{3k}$, $\Sigma a = \tilde{y}_{31} \ldots \tilde{y}_{3t}$. If there are no terms of type 3 in B_S, $\Sigma a = 1$.

If B_S has terms of type 4 below, we use the fact that the Boolean algebra is a direct product to determine Σa. Let α be the projection onto the atomic factor, β the projection onto the atomless factor, and let b be the least upper bound of the a satisfying the above conjunction, that is, the least upper bound of the a satisfying all terms in B_S of the first three types. If terms of type 3 exist, $b = \tilde{y}_{31} \ldots \tilde{y}_{3t}$, otherwise $b = 1$. Let the y_r satisfying terms of type 4 below be indexed $y_{41}, \ldots, y_{4u}$. We will determine Σa by determining $\Sigma\alpha(a)$ and $\Sigma\beta(a)$.

4. $(y_r \cdot \tilde{a} \neq 0 \;\&\; At(k', \ell', m'; y_r \cdot \tilde{a}))$

$\&\; (y_r \cdot a \neq 0 \;\&\; At(k, \ell, m; y_r \cdot a))$

Since B_S holds, so does $At(k' + k, \ell' + \ell, m' + m; y_r)$. $\alpha(y_r \cdot a)$ is the union of m atoms of y_r if $\ell = 0$, the union of at least m atoms if $\ell = 1$. Consider the y_{4j} such that $At(k, 0, 0; y_{4j} \cdot a)$. Let them be indexed as $y_1, \ldots, y_v$. For each y_i, $\alpha(y_i \cdot a) = 0$ so $\alpha(a) \leq \alpha(\tilde{y}_i)$. If $v \neq 0$, let $c = \tilde{y}_1 \ldots \tilde{y}_v$, otherwise $c = 1$. Then for every a satisfying all terms of type 4 in B_S it is the case that $\alpha(a) \leq \alpha(c)$. Now let the y_{4j} which are not in $\{y_1, \ldots, y_v\}$ be indexed as $y_1, \ldots, y_w$. For each of these, $\alpha(y_j \cdot a)$ contains at least one atom. Thus, the least upper bound for $\alpha(a)$ such that a satisfies all terms of type 4 in B_S contains every atom in each of $y_1, \ldots, y_w$

and thus all of $d = [\alpha(y_1) + \ldots + \alpha(y_w)]$. By our previous results, $\alpha(b) \cdot \alpha(c)$ is an upper bound to $\alpha(a)$. Since all the y are disjoint, $d \leq \alpha(b) \cdot \alpha(c)$. We will show that $\alpha(b) \cdot \alpha(c)$ is the least upper bound. Suppose there is some $z < \alpha(b) \cdot \alpha(c)$ which is an upper bound to all $\alpha(a)$ such that a satisfies B_S . Consider $z' = \alpha(b) \cdot \alpha(c) - z$. z' meets some $y_r \neq 0$, or else is in the complement of the union of all the y_r . If z' meets some $y_r \neq 0$ then since $z' < \alpha(b) \cdot \alpha(c)$, the y_r must either satisfy case 2 or else be in $\{y_1, \ldots, y_w\}$. In either case there is an a such that $a \cdot z' \neq 0$, contradiction. Hence z' must be in the complement of the union of all the y_r . In this case, $y_r \cdot z' = 0$ and $y_r \cdot \tilde{z}' = y_r$ for all r . Therefore if a satisfies B_S then $a + z'$ satisfies B_S , hence z is not an upper bound. We have proved that

$$\Sigma \alpha(a) = \alpha(b) \cdot \alpha(c) .$$

To determine $\Sigma \beta(a)$ we consider four subcases.

(i) $k = 1$ and $k' = 1$. In this case y_r is atomic and $\beta(y_r \cdot a) = \beta(0)$ for any a . This subcase does not restrict $\beta(a)$ in any way and we need not consider it further.

(ii) $k = 1$ and $k' = 0$. That is, $y_r \cdot a$ is atomic and $y_r \cdot \tilde{a}$ is not atomic. We have the condition $\beta(y_r \cdot a) = \beta(0)$, or $\beta(a) \leq \beta(\tilde{y}_r)$. Let the y_{4j} in B_S which satisfy subcase (ii) be indexed $y_1, \ldots, y_x$ and let $e = \tilde{y}_1 \ldots \tilde{y}_x$.

(iii) $k = 0$ and $k' = 1$. That is, $y_r \cdot a$ is not atomic and $y_r \cdot \tilde{a}$ is atomic. We have the condition $\beta(y_r \cdot \tilde{a}) = \beta(0)$, or $\beta(a) \geq \beta(y_r)$. Let f be the union of the y_r satisfying this subcase. Any upper bound of $\beta(a)$ must include $\beta(f)$.

(iv) $k = 0$ and $k' = 0$. In this case $y_r \cdot a$ is not atomic and $y_r \cdot \tilde{a}$ is not atomic. Since any a such that $y_r \cdot a$ and $y_r \cdot \tilde{a}$ are not atomic satisfies this subcase, any upper bound of $\beta(a)$ includes all of $\beta(y_r)$. Since $\beta(y_r) \leq \beta(\tilde{y}_s)$ for all $s \neq r$, and $\beta(f) \leq \beta(b) \cdot \beta(e)$, we have

$$\Sigma \beta(a) = \beta(b) \cdot \beta(e)$$

by essentially the same proof as for the atomic factor.

Combining the atomic and atomless least upper bounds, we have

$$\Sigma a = < \alpha(b) \cdot \alpha(c) \text{ , } \beta(b) \cdot \beta(e) >$$

as the least upper bound for the a satisfying B_S .

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