YANG-MILLS THEORY IN SIX-DIMENSIONAL SUPERSPACE

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Abstract

The superspace approach provides a manifestly supersymmetric formulation of supersymmetric theories. For N=1 supersymmetry one can use either constrained or unconstrained superfields for such a formulation. Only the unconstrained formulation is suitable for quantum calculations. Until now, all interacting N>1 theories have been written using constrained superfields. No solutions of the nonlinear constraint equations were known.

In this work, we first review the superspace approach and its relation to conventional component methods. The difference between constrained and unconstrained formulations is explained, and the origin of the nonlinear constraints in supersymmetric gauge theories is discussed. It is then shown that these nonlinear constraint equations can be solved by transforming them into linear equations. The method is shown to work for N=1 Yang-Mills theory in four dimensions.

N=2 Yang-Mills theory is formulated in constrained form in six-dimensional superspace, which can be dimensionally reduced to four-dimensional N=2 extended superspace. We construct a superfield calculus for six-dimensional superspace, and show that known matter multiplets can be described very simply. Our method for solving constraints is then applied to the constrained N=2 Yang-Mills theory, and we obtain an explicit solution in terms of an unconstrained superfield. The solution of the constraints can easily be expanded in powers of the unconstrained superfield, and a similar expansion of the action is also given. A background-field expansion is provided for any gauge theory in which the constraints can be solved by our methods. Some implications of this for superspace gauge theories are briefly discussed.

Table of Contents

INTRODUCTION

Since the discovery of supersymmetry about a decade ago, there have been three main areas of investigation in the field. First, supersymmetry promises a truly unified theory of elementary particles, in which Bosons and Fermions can belong to the same multiplet, and the ad hoc nature of present grand unified theories is reduced. The investigation of such multiplets and their interactions from a group theoretic point of view is an active branch of theoretical physics, and many of the tools of grand unified theory and general relativity can be usefully applied. Thus one is led into topics such as superGUTS, Kaluza-Klein theory, superstring theory, supersymmetry breaking and so forth, which aim ultimately at showing how the currently perceived phenomenology of low energy particle physics could arise from a unified supersymmetric model.

A second subject of research is the mathematical nature of supersymmetry, which has stimulated new work in fields such as Lie group theory, differential geometry and topology.

The third aspect of supersymmetry which has attracted a great deal of attention is the useful effect it has on the quantum properties of a system. It was noticed fairly early on that the ultraviolet divergences in a supersymmetric theory were less severe than one would naively expect, and in some instances even seemed to be absent. This property is particularly useful in quantum gravity, where any divergences are non-renormalizable. It was not immediately obvious why supersymmetry caused such "miraculous" divergence cancellations, or to what order of perturbation theory they persisted, but later work has given a better grasp of the situation. The use of superspace techniques has led to a complete understanding of the simplest ("N=1") supersymmetry, and pointed the way to an understanding of the more complicated ("extended") types. However, a number of problems still need to be solved.

The following work deals mainly with aspects of supersymmetry of the second and third type. It is based on the superfield approach, and includes the solution of a problem that has been around for some years. The achievement has been to construct the first unconstrained, manifestly supersymmetric, interacting field theory in extended superspace, and this has set a portion of supersymmetry theory on firmer ground.

In chapter 1 we give a short introduction to the component-field approach to supersymmetry. It is reasonably self-contained. In chapter 2 we introduce the techniques of global superspace, and explain the difference between constrained and unconstrained formulations of a theory. Chapter 3 is a discussion of the unconstrained formulation of N=1 Yang-Mills theory from a novel point of view, and a development of the basic ideas which lead in chapter 4 to the construction of an unconstrained formulation of N=2 Yang-Mills theory. Chapter 4 also discusses the features of superspace in six-dimensional space-time, which is useful for a more compact and illuminating treatment of N=2 supersymmetry. The appendices describe our conventions and discuss various mathematical points.

Chapter 1

AN OVERVIEW OF SUPERSYMMETRY

A classical or quantum system is called "supersymmetric" if a symmetry relates the bosonic and fermionic sectors. Although there is no observational evidence for such systems, except perhaps in nuclear physics [1], it is not unreasonable to hope that a unified theory of particle physics has this property. Before the discovery of supersymmetry, it was believed impossible to implement symmetries relating different spins in Poincaré invariant particle theories, and a number of "no-go" theorems were proved, the most general of which was due to Coleman and Mandula [2]. These theorems are evaded in supersymmetry by having a symmetry algebra containing both commutation and anticommutation relations. Such structures were not considered in ref. [2]. To show how they arise, we shall construct the simplest supersymmetric model, using counting arguments and dimensional analysis, and later generalize it. For simplicity consider a field theory in four-dimensional spacetime.

1. The Wess-Zumino Multiplet [3]

The simplest four-dimensional system with bosonic and fermionic degrees of freedom is the scalar-spinor system, with spins 0 and $\frac{1}{2}$. A bose-fermi symmetry, however, requires equal numbers of such states, so we do a preliminary count to arrange this. If a one-particle state with helicity $\frac{1}{2}$ exists, then by the CPT theorem any local field theory describing this state must contain a state of helicity $-\frac{1}{2}$ too. Thus the simplest spin $\frac{1}{2}$ field theory has these two independent one-particle states, and the minimal supersymmetric theory needs two spin $\frac{1}{2}$ and two spin 0 states. Such a theory cannot be constructed using four-component Dirac spinors, which describe four spin $\frac{1}{2}$ states (e.g. helicity $\pm\frac{1}{2}$ electrons and positrons). There are two ways to remedy this, the usual one being to introduce Majorana spinors, which are four-component spinors with a reality condition ($\psi^* = \psi$). However, in four dimensions ("d=4"), a mathematically equivalent method is to use Weyl spinors ($\psi = \pm\gamma^5\psi$), and we do so in this work, since it simplifies many results. Expressions may be transcribed from one form to the other using a standard set of rules [4]. This point is examined more closely in chapter 4.

Consider a system of one complex scalar and one left-handed Weyl spinor $\psi = \gamma^5\psi$, described by the free action

$$S = \int d^4x \ \left[-\overline{\varphi}\Box\varphi + i\overline{\psi}\partial\!\!\!/\,\psi \right] \ . \tag{1}$$

The γ matrix notation is that of Bjorken and Drell, ref. [5]. The Feynman path integral prescription for fermion fields requires ψ to be an anticommuting Grassmann number. To demonstrate supersymmetry, we seek infinitesimal linear transformation $\varphi' = \varphi + \delta\varphi$, $\quad \psi' = \psi + \delta\psi$ with $\delta\varphi \propto \psi$ and $\delta\psi \propto \varphi$, which leaves S invariant. Clearly, the only way to turn ψ into a scalar $\delta\varphi$ is to introduce a spinor parameter ε, and the simplest choice is $\delta\varphi = \overline{\varepsilon}\psi$ ($\gamma^5\varepsilon = -\varepsilon$ since ψ is left handed). The no-go theorem of ref.[2] did not consider transformations with fermionic parameters.

Dimensional analysis now suggests a unique $\delta\psi$: denoting the mass dimension of a quantity by the symbol [ ], we have $[\varphi] = 1$, $[\psi] = \frac{3}{2}$ and $[\partial] = 1$. Thus $[\varepsilon] = -\frac{1}{2}$, and $\delta\psi$ must contain φ , ε and one derivative to be of the correct dimension. The only Lorentz-covariant possibility is $\delta\psi = \alpha(\gamma^\mu \varepsilon)\partial_\mu \varphi$ where α is some number. Trying the transformation

$$\delta\varphi = \bar{\varepsilon}\psi \qquad \delta\psi = \alpha \partial\!\!\!/ \,\varepsilon\, \varphi \qquad \delta\bar{\psi} = \bar{\alpha}\bar{\varepsilon}\partial\!\!\!/ \,\bar{\varphi}$$

on (1) gives

$$\delta S = \bar{\varepsilon}\Big[\int d^4x \,(-\bar{\varphi}\Box\psi + i\bar{\alpha}(\partial\!\!\!/ \,\varphi)\partial\!\!\!/ \,\psi)\ \Big] \ + \quad \text{complex conjugate} \quad .$$

Choosing $\alpha = -i$ therefore gives $\delta S = 0$ up to surface terms, which vanish if the fields decrease rapidly enough at infinity. (We will always make this assumption here, and not consider topological aspects of the theory.) Thus (1) is invariant under

$$\delta\varphi = \bar{\varepsilon}\psi \qquad \delta\psi = -i\,\partial\!\!\!/ \,\varepsilon\, \varphi \qquad \delta\bar{\psi} = i\,\bar{\varepsilon}\partial\!\!\!/ \,\bar{\varphi} \quad . \tag{2}$$

There are a number of subtleties that should be pointed out. First, in the usual interpretation of a lagrangian, the bose field φ is an ordinary C-number, but we have added to it the peculiar quantity $\bar{\varepsilon}\psi$, which is not a C-number since $(\bar{\varepsilon}\psi)^3 = 0$ from the Grassmann nature of ψ. So to be able even to implement these transformations, one must assume that φ takes values more general than C-numbers. We thus introduce a "Grassmann algebra," containing both fermionic and bosonic numbers. The C-numbers form a subalgebra of the bosonic sector. Fermions anticommute with each other, and all other products are commutative. A Grassmann algebra is a special case of a structure we shall encounter often here, a "(Z_2) graded algebra." The latter can contain objects more general than "numbers" (e.g., operators), which nevertheless split into a fermionic sector and a bosonic sector, and which obey the product rules (boson)(boson) = (fermion)(fermion) = (boson)

(boson)(fermion) = (fermion)(boson) = (fermion)

φ and ψ must therefore take values in the even (bosonic) and odd (fermionic) parts of a Grassmann algebra, respectively. This does not seriously affect any of the intuitive notions of bosonic fields, provided one recognizes that operations such as "integration over all field values," used in the path integral, are now definitions with the same formal status as those for fermionic integration. (The perturbative Green functions are still ordinary C-numbers, because they depend only on the vertices and free propagators, which are unaffected.)

The second subtlety is that we work with "infinitesimal transformations," which we know to be sufficient for studying the part of a Lie group connected to the identity, by the relationship of Lie groups to their Lie algebras. As a taste of things to come, it's worth showing how we "exponentiate" the infinitesimal transformations when they contain Grassmann parameters.

To produce a finite transformation from the infinitesimal form (2) we introduce a parameter t and solve the differential equations

$$\frac{d}{dt}\,\varphi(x,t\varepsilon) = \bar{\varepsilon}\psi(x,t\varepsilon) \qquad \varphi(x,0) \equiv \Phi(x) \quad ,$$

$$\frac{d}{dt}\,\psi(x,t\varepsilon) = -i\partial\!\!\!/\,\varepsilon\varphi(x,t\varepsilon) \qquad \psi(x,0) \equiv \Psi(x) \quad . \tag{3}$$

ε is now a finite fermionic parameter, and $\varphi(x,t\varepsilon)$ and $\psi(x,t\varepsilon)$ are interpreted as the result of repeatedly applying the infinitesimal transformation $\delta\varphi = \bar{\varepsilon}\psi\, dt$ $\delta\psi = -i\partial\!\!\!/\,\varepsilon\varphi\, dt$ to φ and ψ. These two ordinary differential equations have a unique solution, which is easily obtained by expanding φ and ψ as Taylor series in t. It is unnecessary to assume analyticity in t, since explicitly differentiating (3) with respect to t gives:

$$
\begin{array}{ll}
\varphi(0) = \Phi & \psi(0) = \Psi \\[6pt]
\dfrac{d\varphi}{dt}(0) = \bar{\varepsilon}\Psi & \dfrac{d\psi}{dt} = -i\not\partial\,\varepsilon\Phi \\[6pt]
\dfrac{d^2\varphi}{dt^2}(0) = -i\bar{\varepsilon}\not\partial\,\varepsilon\Phi & \dfrac{d^2\psi}{dt^2}(0) = -i\not\partial\,\varepsilon(\bar{\varepsilon}\Psi) \\[6pt]
\dfrac{d^3\varphi}{dt^3}(0) = -i(\bar{\varepsilon}\not\partial\,\varepsilon)(\bar{\varepsilon}\Psi) & \dfrac{d^3\psi}{dt^3}(0) = -(\not\partial\,\varepsilon)(\bar{\varepsilon}\not\partial\,\varepsilon)\Phi \\[6pt]
\dfrac{d^4\varphi}{dt^4}(0) = -(\bar{\varepsilon}\not\partial\,\varepsilon)^2\Phi & \dfrac{d^4\psi}{dt^4}(0) = -(\not\partial\,\varepsilon)(\bar{\varepsilon}\not\partial\,\varepsilon)(\bar{\varepsilon}\Psi) \\[6pt]
\dfrac{d^5\varphi}{dt^5}(0) = -(\bar{\varepsilon}\not\partial\,\varepsilon)^2(\bar{\varepsilon}\Psi) = 0 & \dfrac{d^5\psi}{dt^5}(0) = i(\not\partial\,\varepsilon)(\bar{\varepsilon}\not\partial\,\varepsilon)^2\Psi = 0.
\end{array}
\tag{4}
$$

The $\dfrac{d^5}{dt^5}$ and higher terms are zero because there are only four independent real Grassmann parameters in ε and $\bar{\varepsilon}$, so any product higher than a fourth power vanishes. One can therefore write down the full solution for φ and ψ, and set $t = 1$ since it only occurs as $t\,\varepsilon$.

$$
\varphi(\varepsilon) = \left[1 + \frac{1}{2!}(-i\bar{\varepsilon}\not\partial\,\varepsilon) + \frac{1}{4!}(-i\bar{\varepsilon}\not\partial\,\varepsilon)^2\right]\Phi \; + \; \left[\bar{\varepsilon} + \frac{1}{3!}(-i\bar{\varepsilon}\not\partial\,\varepsilon)\bar{\varepsilon}\right]\Psi \tag{5}
$$

$$
\psi(\varepsilon) = \left[1 + \frac{1}{2!}(-i\not\partial\,\varepsilon\bar{\varepsilon}) + \frac{1}{4!}(-i\not\partial\,\varepsilon\bar{\varepsilon})^2\right]\Psi + \left[(-i\not\partial\,\varepsilon) + \frac{1}{3!}(-i\not\partial\,\varepsilon\bar{\varepsilon})(-i\not\partial\,\varepsilon)\right]\Phi
$$

Thus for any finite ε we can find a finite one parameter group of supersymmetry transformations

$$
\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}' \;=\; G(t\,\varepsilon)\begin{bmatrix}\Phi\\ \Psi\end{bmatrix} \quad .
$$

One can prove, from the differential equations or by explicit multiplication, that $G(t\,\varepsilon)\,G(t'\varepsilon) = G((t+t')\varepsilon)$. The proof of invariance of the action proceeds exactly as before. Whether we use infinitesimal variations, or differentiation by t, the cancellation of terms takes place in the same way. Thus $\dfrac{dS}{dt} = 0$, and S is invariant under the group.

The set of all such transformations, for all possible ε and in all possible orders, forms a group in the standard sense. It is, however, most easily described by extending the Lie group concept to that of a "graded Lie group." We introduce supersymmetry generators $Q, \overline{Q} = Q^*$ by defining their action on any field Φ to be $\delta\Phi = (\overline{\varepsilon} Q + \overline{Q}\varepsilon)\Phi$. (Note that Q is a left-handed spinor.) In the example (2) one finds

$$Q\varphi = \psi \qquad \overline{Q}\varphi = 0 \qquad Q\psi = 0 \qquad \overline{Q}\psi = -i\partial\!\!\!/ \varphi \ . \tag{6}$$

Q is a fermionic operator in the sense that it anticommutes with odd Grassmann numbers. Notation for operators is described in the appendix.

Consider the commutator of two group transformations:

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)\varphi &= \delta_1 \overline{\varepsilon}_2\psi - \delta_2\overline{\varepsilon}_1\psi \\ &= -i(\overline{\varepsilon}_2\partial\!\!\!/ \varepsilon_1 - \overline{\varepsilon}_1\partial\!\!\!/ \varepsilon_2)\varphi \\ &= (i\,\overline{\varepsilon}_1\gamma^\mu\varepsilon_2 + h.c.\,)\partial_\mu\varphi = i\,\xi^\mu P_\mu\varphi \ .\end{aligned} \tag{7}$$

Acting on the bose field, the commutator of two supersymmetry transformations is a translation, but a translation by the nilpotent quantity $(i\,\overline{\varepsilon}_2\gamma^\mu\overline{\varepsilon}_1 + h.c.\,)$. To interpret this in terms of generators, observe that for fermionic generators Q and R and fermionic parameters ω and τ the commutation relations become anticommutation relations when we pull the parameters in front

$$[\,\omega Q, \tau R\,] = \omega\tau\{\,Q, R\,\} \ .$$

Thus, writing the spinor indices explicitly,

$$\{\,Q_\alpha, Q_\beta\}\varphi = \{\,\overline{Q}_\alpha, \overline{Q}_\beta\}\varphi = 0 \ , \tag{8}$$

$$\{ Q_\alpha , \overline{Q}_\beta \} = \tfrac{1}{2} (\gamma^\mu (1 - \gamma^5))_{\alpha\beta} \, i \, \partial_\mu \varphi \quad . \tag{9}$$

The chiral projection arises because Q is left-handed.

The simple relation $\{Q, \overline{Q}\} \sim \not\partial$ does not hold for the fermion field where we get

$$[\delta_1 , \delta_2] \psi = i \, ((\not\partial \, \varepsilon_{[1}) \overline{\varepsilon}_{2]} \psi \quad .$$

This can be rewritten, using a Fierz identity,

$$(\gamma^\mu \varepsilon_2) \overline{\varepsilon}_1 \psi \equiv - \overline{\varepsilon}_1 \gamma^\mu \varepsilon_2 \psi - \tfrac{1}{2} \overline{\varepsilon}_1 \gamma^\nu \varepsilon_2 \gamma_\nu \gamma^\mu \psi \quad ,$$

$$[\delta_1 , \delta_2] \psi = (i \, \overline{\varepsilon}_1 \gamma^\mu \varepsilon_2 + h.c. \,) \partial_\mu \psi + \tfrac{1}{2} (i \, \overline{\varepsilon}_1 \gamma^\nu \varepsilon_2 + h.c. \,) \gamma_\nu \partial_\mu \psi \quad . \tag{10}$$

Thus on the fermion field, the commutator of two supersymmetries is a translation by the same parameter as before, plus more. The additional piece is proportional to $\not\xi \not\partial \psi$, and thus to the equation of motion of ψ. It is easy to verify that $\delta\psi = \not\xi \not\partial \psi$ is an invariance of the free fermion action, independent of any supersymmetry arguments. However, if one tries to construct a conserved current corresponding to this invariance, using $J_\mu = \dfrac{\delta L}{\delta(\partial_\mu \psi)} \delta\psi$, we find that it vanishes on shell because $\delta\psi$ does. Thus these transformations (and quite generally, any transformations proportional to field equations) cannot be interpreted as physical invariances. In quantum language, they are invariances of the Green functions that are trivial on the S matrix, and therefore an artifact of the techniques used for calculating the S matrix.

2. Supersymmetries of the S Matrix.

Since the S matrix is an ordinary complex number, it is meaningless to apply transformations containing Grassmann parameters to it. However, if one keeps track of the supersymmetry transformations in the path-integral quantization procedure, the ultimate result is a Hilbert space representation of the superalgebra, not the whole supergroup. Supersymmetry then corresponds to having a set of symmetry operators acting on the Hilbert space of states, containing both bosonic operators B_i and fermionic operators F_a, with the graded commutation relations

$$[B_i, B_j] = C_{ij}{}^k\, B_k$$

$$[B_i, F_a] = C_{ia}{}^b\, F_b$$

$$\{F_a, F_b\} = C_{ab}{}^i\, B_i \qquad , \tag{11}$$

and which commute with the S matrix, $[B_i, S] = [F_a, S] = 0$. The B_i's by themselves form an ordinary Lie algebra, and could be used to generate a conventional group of symmetries, but the F's cannot.

The abstract structure (i) is called a (Z_2) graded Lie algebra, and satisfies a graded Jacobi identity, which takes various forms depending on how many fermionic operators are involved, e.g.,

$$[B_i, \{F_a, F_b\}] + \{F_a, [F_b, B_i]\} - \{F_b, [B_i, F_a]\} = 0 \quad ,$$

which follows immediately upon expanding the brackets.

We conclude that the no-go theorem of Coleman and Mandula is inadequate, because it assumes that the symmetry generators form a Lie algebra, whereas one can actually interpret more general graded Lie algebras. Haag, Lopuszanski and Sohnius (HLS) have generalized the Coleman-Mandula theorem to allow for this possibility [6]: the Lie algebra formed by the bosonic generators must satisfy the theorem, and the fermionic generators are then restricted by the new graded Jacobi identities. They showed (with certain assumptions) that all possible supersymmetries of the S matrix contain only the following operators:

[1] Poincaré generators $M_{\mu\nu}$ and P_μ.

[2] N fermionic generators Q_i , $i = 1,...,N$, and their complex conjugates $\overline{Q}^i$, which are Lorentz spinors.

[3] Some bosonic "central charges" Z_a, which are Lorentz scalars.

The graded Lie algebra then takes the generic form:

$$[M,M] \approx M \quad [M,Q] \approx Q \quad [M,P] \approx P \quad [M,Z] = 0$$

$$[P,Q] = [Z,Q] = 0 \qquad \{Q,\overline{Q}\} \approx P \quad \{Q,Q\} \approx Z$$

$$[P,P] = [Z,Z] = 0 \ . \tag{12}$$

The only freedom is how many Q's or Z's are included. One has to arrange that vector or spinor indices match on both sides of any such bracket relation, which is equivalent to insisting that Jacobi identities containing M's be satisfied. Also, reality properties must be maintained. A convenient way to remember this algebra is to assign mass dimensions [M]=0 , [Q]=$\frac{1}{2}$, [P]=[Z]=1.

It is also possible to augment the algebra with some "external charges" A whose only effect is to rotate the fermionic generators $[A,Q] \approx Q$. However, one must then insure that the additional Jacobi identities are satisfied, which restricts the A's to generating at most $U(N)$, or a subalgebra if Z's are present.

In many cases the Z's can be interpreted as corresponding to momentum component of a supersymmetry algebra in more than four dimensions.

One can summarize by saying that the effect of supersymmetry is to introduce a number ("N") of "supercharges." If N=1 we speak of simple supersymmetry, while N > 1 is called extended supersymmetry.

3. Physical Spectra

To examine the physical content of a supermultiplet, it is convenient to use SL(2,C) notation, where a left-handed spinor is represented by an object with a Greek index, ξ_α, and a right-handed spinor has a dotted index $\overline{\eta}_{\dot\alpha}$. This notation is described in detail in the appendix. Here we emphasize that a vector can be represented by a hermitian matrix $V^\mu \to V^\mu(\sigma_\mu)_{\alpha\dot\beta} \equiv V_{\alpha\dot\beta}$. The σ_μ are the Pauli matrices, so that $V_{\alpha\dot\beta} = \begin{bmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{bmatrix}$.

The most general supersymmetric extension of the Poincaré algebra according to the previous section is then obtained by introducing N spinorial generators $Q_{i\alpha}$ and their hermitian conjugates $\overline{Q}^i{}_{\dot\alpha}$, with the algebra

$$\{ Q_{i\alpha}, Q_{j\beta} \} = C_{\alpha\beta} Z_{ij} \qquad \{ Q_{i\alpha}, \overline{Q}^j{}_{\dot\beta} \} = \delta_i^j P_{\alpha\dot\beta} \qquad \overline{Q}^i{}_{\dot\alpha} = (Q_{i\alpha})^\dagger \ . \tag{13}$$

We will take the central charges Z^{ij} to be zero for simplicity. Note that this algebra follows uniquely up to rescalings from the rules in the previous section. We can if we wish also assume that the spinors Q are an $\underline{N}$ of $SU(N)$, and the $\overline{Q}$ are an $\underline{\overline{N}}$, which we have done implicitly by putting subscript $Q_{i\alpha}$ and superscript $\overline{Q}^i{}_{\dot\alpha}$. This algebra is known as the Superpoincaré algebra, and to find its irreducible representations ("irreps") one can use the method of "little groups" introduced by Wigner [7].

First note that P^2 is a Casimir operator in this algebra, since it commutes with all generators. Thus any irrep has P^2 = const = m^2. The basic result of Wigner, which follows ultimately from the theory of induced representations, is that for a particular m^2, irreps of the Poincaré group are classified by representations of the little group. If m=0, the relevant little group is the transverse $SO(2)$, so massless Poincaré irreps are classified by a single number, the helicity. Extending the Poincaré algebra as in (13) forces one to include a number of different helicity states in the same multiplet. While the following method [8] can be adapted to any m^2 and algebras including central charges, we content ourselves with obtaining just the massless representations for algebras with no central charges.

If we choose a helicity subspace so that the momentum is in the z-direction, then $P_{\alpha\dot{\beta}} = \begin{bmatrix} 2p^0 & 0 \\ 0 & 0 \end{bmatrix}$ and the algebra becomes

$$\{Q_{i1}, \overline{Q}^j_1\} = 2p^0\delta_i^j \qquad \{Q_{i2}, \overline{Q}^j_2\} = 0 \qquad \{Q_{i1}, \overline{Q}^j_2\} = 0 \quad . \tag{14}$$

Since $\overline{Q}^j{}_{\dot{\beta}} = (Q_{j\beta})^\dagger$, the second relation gives $Q_{i2} = 0$, because

$$0 = <\psi|\,\{Q_{i2}, \overline{Q}^i_2\}\,|\psi> = |\ \overline{Q}^i_2{}^\dagger|\psi>|^2 + |\ Q_{i2}|\psi>|^2$$

$$=> \ \overline{Q}^i_2\,|\psi> = Q_{i2}|\psi> = 0$$

for any state $|\psi>$. Thus one may ignore Q_{i2} and work with $Q_i \equiv \dfrac{Q_{i1}}{\sqrt{p^0}}$ and $\overline{Q}^i = (Q_i)^\dagger$, so that in the helicity subspace one is left with the Clifford algebra

$$\{Q_i, \overline{Q}^j\} = 2\delta_i^j \qquad \{Q_i, Q_j\} = 0 \quad . \tag{15}$$

This is isomorphic to an algebra of fermionic creation and annihilation operators, and its irreducible representations can be constructed from a particular helicity state $|s>$, assumed to satisfy $\overline{Q}^i\,|s> = 0$. By successively operating

Q_i, Q_j, ... ,Q_k on $|s>$ one produces a basis for an irrep. The set is finite and closed under operation of both Q_i and $\overline{Q}^i$, since $\overline{Q}^i$'s may be pushed to the right and eliminated using (15) and $\overline{Q}^i\,|s> = 0$. Using formulae in the appendix, it is straightforward to show that operating Q_i on any state reduces its helicity by $\frac{1}{2}$. Thus the following set of states carries an irreducible representation of the algebra (15):

$$
\begin{array}{llll}
|s> & & 1 \text{ state} & \\
|i, s-\tfrac{1}{2}> = Q_i\,|s> & & N \text{ states} & \\
|ij, s-1> = Q_i Q_j\,|s> & i \neq j & \frac{N(N-1)}{2} \text{ states} & \\
|i_1 i_2 \cdots i_n, s-\tfrac{n}{2}> = Q_{i_1} Q_{i_2} \cdots Q_{i_n}\,|s> & i_j \neq i_k & \binom{N}{n} \text{ states} & \\
|i_1 i_2 \cdots i_N, s-\tfrac{N}{2}> = Q_{i_1} Q_{i_2} \cdots Q_{i_N}\,|s> & i_j \neq i_k & \binom{N}{N} = 1 \text{ state.} & (16)
\end{array}
$$

It is worth noting that the states appear as the totally antisymmetric representations of U(N). Now, to describe multiplets that have no helicities greater than 2 one must arrange that $|s|$ and $|s-\frac{N}{2}| \leq 2$, which immediately demands $N \leq 8$. Also, one must take into account that, in a local, Lorentz invariant field theory, the CPT conjugate of any state will also occur.

Table (1) shows some of the more interesting multiplets. We remark on a few of them.

(a) The Wess-Zumino multiplet [3] of section 1.

(b) The Yang-Mills multiplet [9] is the main topic of chapter 3.

(c) The N=2 analogue [10,11] of (b), and the main topic of chapter 4.

(d) The "hypermultiplet" of Fayet [11] is the N=2 multiplet obtained from $|\, s=\frac{1}{2}>$. Examining the states required, we find that without CPT one has

$$|\,\tfrac{1}{2}> \quad |\, i,0,> \quad |\,-\tfrac{1}{2}> \quad .$$

Naively one might think there is no need to add a CPT conjugate set of states, since a helicity $-\frac{1}{2}$ state is already present. However, this is not true, because the SU(2) doublet of scalars cannot be self-conjugate, since a $\underline{2}$ of SU(2) is not a real representation (it is "pseudoreal"). Thus one requires a second, identical multiplet which is the CPT conjugate of the first. Yet another SU(2) symmetry relates the two. Thus the hypermultiplet has SU(2)×SU(2) symmetry, one SU(2) coming from the supersymmetry algebra.

(e) This N=3 multiplet can actually carry N=4 supersymmetry as a count of states suggests. Thus one does not use an "N=3 Yang-Mills" multiplet. In the same way, N=7 supergravity has the same particle content as N=8 supergravity, and is not an independent theory.

(f) N=4 Yang-Mills [12] and N=8 supergravity [14] are self-conjugate under CPT, which is possible because the $\underline{6}$ of SU(4) and the $\underline{70}$ of SU(8) are real, unlike the $\underline{2}$ of SU(2).

4. Reduction to Smaller N

It is often useful to observe that for n>m one may decompose any N=n multiplet into smaller N=m multiplets, since the N=m super Poincaré algebra is a subalgebra of the N=n algebra. For instance, the N=2 Yang-Mills multiplet can be considered an N=1 Yang-Mills multiplet plus an N=1 Wess-Zumino multiplet. The example we use later is that N=4 Yang-Mills is an N=2 Yang-Mills multiplet plus an N=2 hypermultiplet. One use of the observation is that, when general results for N=m supersymmetry can be proved (e.g., absence in four dimensions of multiloop counterterms for an N=2 system if all spins ≤ 1), there can be immediate implications for N > m supersymmetry, e.g., finiteness of N=4 Yang-Mills in this case.

5. Supersymmetric Actions

Since our main interest is dynamics, we return now to discussing how supersymmetry can be implemented in a field theory. A first step is to write a free supersymmetric action with the right particle content. Thus, since we are working with massless multiplets, the free action contains a Klein-Gordon lagrangian

$$-\tfrac{1}{2}\varphi\Box\varphi$$

for each spin 0, a Dirac or Weyl lagrangian

$$-i\bar{\psi}^{\dot{\beta}}\partial_{\alpha\dot{\beta}}\psi^{\alpha}$$

for each spin $\frac{1}{2}$, and a Maxwell lagrangian

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv \tfrac{1}{16}F^{\alpha\beta}F_{\alpha\beta} \text{ with } F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$$

for each spin 1. The Maxwell lagrangian is manifestly gauge invariant under $\delta A_{\mu} = \partial_{\mu}\lambda$ for scalar λ, which is necessary for the action to describe spin one particles. Gauge invariance is a property of the higher spin actions as well. For each spin $\frac{3}{2}$ particle we use the Rarita-Schwinger action, with a left-handed spinor-vector $\psi_{\mu} = \gamma^5\psi_{\mu}$:

$$L_{RS} = -i\,\varepsilon^{\mu\nu\sigma\tau}\,\overline{\psi}_\mu\gamma_\nu\partial_\sigma\psi_\tau\ .$$

It has the gauge invariance $\delta\psi_\mu = \partial_\mu\eta$ where η is a left-handed spinor $\eta = \gamma^5\eta$. This is the local supersymmetry of supergravity in the free theory limit.

For spin 2 gravitons, one can take the linearized form of the Einstein-Hilbert action, in the form where the fundamental field is the vierbein. This enables one to couple gravity to fermionic matter. The gauge invariance is local coordinate invariance × local SO(3,1).

In this survey of free actions it is worth including the action for an antisymmetric tensor field $A_{\mu\nu} = -A_{\nu\mu}$, which describes a spinless particle. One defines a field strength $V^\mu = \varepsilon^{\mu\nu\sigma\tau}\partial_\nu A_{\sigma\tau}$ and writes $S = \int \frac{1}{2} V^\mu V_\mu$, where V satisfies $\partial^\mu V_\mu = 0$. S has the gauge invariance $\delta A_{\mu\nu} = \partial_{[\mu}\lambda_{\nu]}$, and a little-group analysis shows that $A_{\mu\nu}$ is a zero-helicity field.

The physical bosonic and fermionic fields in these free lagrangians have dimensions 1 and $\frac{3}{2}$ respectively.

To construct a linearized supersymmetric action, one may adapt the reasoning of section 1: given the particle content of an irreducible multiplet, one writes a kinetic term for each particle, while maintaining SU(N) covariance. As an example, the N=2 Yang-Mills action is

$$S = \int d^4x \Big[-\tfrac{1}{4}F^2 - i\overline{\psi}_i^{\dot\beta}\partial_{\alpha\dot\beta}\psi^{\alpha i} - \tfrac{1}{2}\overline{\varphi}\Box\varphi \Big] \quad . \tag{17}$$

Dimensional analysis and covariance then determine the supersymmetry transformations up to unknown factors, which may be found by checking $\delta S = 0$. In practice this is easier than first obtaining the transformation rule by group theoretic means, which would involve checking commutation relations.

6. Adding Interactions

Linearized lagrangians describe free theories and are of limited interest. The simplest couplings are mass terms, which can be treated at the linear level. For the Wess-Zumino multiplet and the hypermultiplet they are easily introduced, since the number of degrees of freedom for massless and massive particles is the same. One could for instance give equal masses to the scalar and spinor in the Wess-Zumino multiplet by taking

$$S = \int d^4x \left[-\tfrac{1}{2}\overline{\varphi}\Box\varphi - \tfrac{1}{2}m^2\overline{\varphi}\varphi - i\overline{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^{\alpha} + \frac{m}{2}\left(\overline{\psi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}} + \psi^{\alpha}\psi_{\alpha}\right)\right] . \quad (18)$$

The mass terms are not invariant under the supersymmetry transformation (1), which does not involve m, but by modifying the transformation of the fermion field one can restore supersymmetry in the action:

$$\delta\varphi = \varepsilon^{\alpha}\psi_{\alpha} \qquad \delta\psi^{\alpha} = i\,\overline{\varepsilon}^{\dot{\beta}}\partial^{\alpha}{}_{\dot{\beta}}\varphi + m\,\varepsilon^{\alpha}\overline{\varphi} \ . \quad (19)$$

Moreover, if one now looks at the commutator of two supersymmetries, we find as before

$$\delta_{[1}\delta_{2]}\varphi = -i\,\overline{\varepsilon}^{\dot{\alpha}}_{[1}\varepsilon^{\alpha}_{2]}\,\partial_{\alpha\dot{\alpha}}\varphi \quad (20)$$

on the Bose field, but

$$\delta_{[1}\delta_{2]}\psi^{\alpha} = -i\,\overline{\varepsilon}^{\dot{\alpha}}_{[1}\varepsilon^{\beta}_{2]}\partial_{\beta\dot{\alpha}}\psi^{\alpha} + i\,\overline{\varepsilon}^{\dot{\alpha}}_{[1}\varepsilon^{\alpha}_{2]}\left(i\partial_{\delta\dot{\alpha}}\psi^{\delta} - m\,\overline{\psi}_{\dot{\alpha}}\right) \quad (21)$$

on the Fermi field. Thus once again the commutator is an ordinary translation, plus a term proportional to the fermion equation of motion. This situation persists when more interactions are present, since the HLS theorem tells us that any extra terms must vanish on shell. It means that the supersymmetry transformations are in general nonlinear. For instance, one can add an interaction term

$$S_{int} = \int -\lambda^2(\overline{\varphi}\varphi)^2 - \lambda m(\varphi\overline{\varphi}^2 + \overline{\varphi}\varphi^2) + 2\lambda(\varphi\psi^\alpha\psi_\alpha + \overline{\varphi}\overline{\psi}^{\dot\alpha}\overline{\psi}_{\dot\alpha}) \qquad (22)$$

and obtain an action invariant under the supersymmetry

$$\delta\varphi = \varepsilon^\alpha\psi_\alpha \qquad \delta\psi^\alpha = i\,\overline{\varepsilon}^{\dot\beta}\partial^\alpha{}_{\dot\beta}\varphi + m\,\varepsilon^\alpha\overline{\varphi} + \lambda\,\varepsilon^\alpha\overline{\varphi}^2 \ . \qquad (23)$$

This simple example shows the problems one has to face in constructing supersymmetric theories. In this approach, the action and the supersymmetries must be derived at the same time. Thus the coupling of two irreducible multiplets is accompanied by a non-trivial change in the supersymmetry transformation. Indeed, finding lagrangians and their supersymmetries has become quite an enterprise. The problem is particularly acute when one discusses supergravity, where even dimensional analysis is less useful because of the dimensionful coupling constant.

The interactions of the Yang-Mills multiplet are also worth discussing. One may for instance take a number of such multiplets and assume that the vectors are the gauge vectors for some (arbitrary) compact gauge group, which means they belong to its adjoint representation. A supersymmetric action is obtained by putting all the other fields in adjoint representations and using minimal coupling. For example, the N=1 Yang-Mills multiplet, comprising $A_{\alpha\dot\beta}$ and ψ^α, has action

$$S_{YM} = \frac{1}{g^2}\int d^4x \left[\frac{1}{16}F_{\alpha\beta}{}^*F^{\alpha\beta} - i\overline{\psi}^{\dot\alpha}\,{}^*\nabla_{\alpha\dot\alpha}\psi^\alpha \right] , \qquad (24)$$

where

$$\nabla_{\alpha\dot\beta} = \partial_{\alpha\dot\beta} + A_{\alpha\dot\beta} \qquad [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] = \frac{i}{2}C_{\alpha\beta}\overline{F}_{\dot\alpha\dot\beta} + \frac{i}{2}C_{\dot\alpha\dot\beta}F_{\alpha\beta} \ . \qquad (25)$$

A, F and ψ are Lie algebra valued, i.e., of the form $A^a X_a$ where the X_a are generators of the gauge group, and $X{*}Y$ is the invariant metric. $X_a {*} X_b = -trace X_a X_b$ in a matrix representation. This action is invariant under

$$\delta A_{\alpha\dot{\alpha}} = \varepsilon_\alpha \overline{\psi}_{\dot{\alpha}} - \overline{\varepsilon}_{\dot{\alpha}} \psi_\alpha \qquad \delta\psi^\alpha = -\tfrac{1}{8}\varepsilon_\beta F^{\alpha\beta} \ . \tag{26}$$

A new feature, not found in the matter multiplet, occurs when we consider the commutator of two supersymmetry transformations

$$[\delta_1, \delta_2] A_{\alpha\dot{\alpha}} = (\xi^{\beta\dot{\beta}} \partial_{\beta\dot{\beta}}) A_{\alpha\dot{\alpha}} - \nabla_{\alpha\dot{\alpha}} (\xi^{\beta\dot{\beta}} A_{\beta\dot{\beta}}) \qquad \xi^{\beta\dot{\beta}} = -\frac{i}{4} \varepsilon^\beta_{[1} \overline{\varepsilon}^{\dot{\beta}}_{2]} \ . \tag{27}$$

One obtains the expected translation, as well as a gauge transformation. Thus the supersymmetry is intimately bound to both the Lorentz symmetry and the gauge symmetry, and the generator of gauge transformations must be included in the supersymmetry algebra. This does not violate the HLS theorem since gauge transformations are not symmetries of the S matrix.

One can couple arbitrary numbers of scalar multiplets to a super Yang-Mills theory using minimal coupling, and the result is still supersymmetric, although the transformation rules change. However, it becomes cumbersome to keep writing out individual fields as we have done, and in the next chapter we describe a better method.

No introduction to supersymmetry is complete without mentioning supergravity. Since it is not considered in the following chapters, we merely comment that it is clear from the anticommutation relation $\{Q, \overline{Q}\} \sim \partial$ that a supersymmetric theory of gravity will have local supersymmetry, since Einstein gravity can be regarded as a theory with local Poincaré symmetry. Such a theory was constructed in 1976 [13], and since then supergravity theories for all $N \leq 8$ have been written down [14]. These come in various forms and in some cases may be coupled to matter and Yang-Mills multiplets. In addition, versions where the spin 1 particles are gauge vectors can be constructed, so this is a rich research area. Of these, the N=8 models are in many ways the most promising candidates for realistic physical theories. Nevertheless, the others serve as useful models for testing techniques, since the N=8 model is extremely complicated, has an intricate symmetry structure, and is not fully understood. Thus, for instance, the N=4 supergravity theory also contains particles of all spins up to two, and illustrates in simplified form many of the features of the N=8 model.

7. Quantum Properties of Supersymmetric Theories

It was noticed early in the development of supersymmetry that the particle contents and relations between coupling constants required to make a theory supersymmetric also result in improved ultraviolet behaviour in the quantum perturbation series. In particular, the β-function for four-dimensional N=4 Yang-Mills was calculated to three loops and found to be zero [15]. This is of significance for supergravity, since the non-renormalizability of Einstein gravity is a problem, and cancellations due to supersymmetry could circumvent this. One-loop calculations for gravity coupled to lower spin fields have been made [14], and it is found that S-matrix finiteness occurs only when the fields coincide with one of the supergravity multiplets. An intuitive argument for this is available: ordinary Einstein gravity is known to be finite at one loop because no on-shell counterterms of the correct dimension exist [16]. Thus, in a theory where gravity is a sector of an <u>irreducible</u> symmetry representation, there can be no one-loop counterterm, since by truncation it would provide one for pure gravity.

It is very difficult to investigate possible higher order divergence cancellations using ordinary component methods. The nonlinearity of the supersymmetry makes the resulting Ward identities extremely complicated. Fortunately, in some cases a way around this problem has been found. For certain theories one can find modified lagrangians where

(1) The algebra $\{Q, \overline{Q}\} = P$ is satisfied on all fields, bosonic and fermonic, without using the equations of motion.

(2) All supersymmetry transformations are linear in fields.

(3) The transformations are independent of coupling constants and masses. Kinetic, mass and interaction terms are separately invariant. Thus couplings are easy to implement.

These desirable properties are obtained by adding "auxiliary fields" to the theory.

8. Auxiliary Fields

An auxiliary field is one whose equation of motion is an algebraic equation rather than a differential equation in time. As a result, the field has no dynamical degrees of freedom, and its field equation is solved by some combination of other fields. (An example is the spin connection in gravity.)

The simplest four-dimensional example is the Wess-Zumino model. Consider the effect of adding a dimension 2 complex scalar field, B to the theory. At the linearized level we have

$$S = \int d^4x \left[-\overline{\varphi} \Box \varphi - i \overline{\psi}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi^\alpha + \overline{B} B \right] , \tag{28}$$

which has the same degrees of freedom as eq. (1), since the field equation for B is just $B = 0$. However, S is invariant under the supersymmetry transformation

$$\delta\varphi = \varepsilon^\alpha \psi_\alpha$$

$$\delta\psi^\alpha = i\bar{\varepsilon}^{\dot\alpha}\partial^\alpha{}_{\dot\alpha}\varphi - \varepsilon^\alpha B$$

$$\delta B = -i\bar{\varepsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^\alpha \quad . \tag{29}$$

The commutation relations are now

$$\delta_{[1}\delta_{2]}\varphi = \xi^{\beta\dot\beta}\partial_{\beta\dot\beta}\varphi \qquad \delta_{[1}\delta_{2]}\psi^\alpha = \xi^{\beta\dot\beta}\partial_{\beta\dot\beta}\ \ \psi^\alpha \qquad \delta_{[1}\delta_{2]}B = \xi^{\beta\dot\beta}\partial_{\beta\dot\beta}B \tag{30}$$

with $\xi^{\beta\dot\beta} = i\varepsilon^{\beta}_{[1}\bar{\varepsilon}^{\dot\beta}_{2]}$. This is an improvement, because the algebra $\{Q^\alpha, \overline{Q}^{\dot\beta}\} = P^{\alpha\dot\beta}$ is now satisfied both on and off shell. Other expressions invariant under these supersymmetry transformations include

$$S_m = \frac{m}{2}\int d^4x\ (\psi^\alpha\psi_\alpha + 2\varphi B) \tag{31}$$

and

$$S_{int} = \lambda\int d^4x\ (\varphi^2 B + \varphi\psi^\alpha\psi_\alpha) \tag{32}$$

and their complex conjugates.

One may now form an invariant interacting lagrangian

$$S = S_{kin} + S_m + \overline{S}_m + S_{int} + \overline{S}_{int} \quad . \tag{33}$$

The field equations are then

$$-\Box\varphi + m\overline{B} + \lambda(2\overline{\varphi}\overline{B} + \overline{\psi}^{\dot\alpha}\overline{\psi}_{\dot\alpha}) \doteq 0$$

$$i\partial_{\alpha\dot\alpha}\psi^\alpha - m\overline{\psi}_{\dot\alpha} + 2\lambda\overline{\psi}_{\dot\alpha}\overline{\varphi} = 0$$

$$\overline{B} = -m\varphi - \lambda\varphi^2 \quad . \tag{34}$$

B may be eliminated from the action (33) before we calculate the field equations for φ and ψ, because doing the path integral for an auxiliary field has the same effect as replacing it by its field equation:

$$\int \mathrm{dB}\, e^{aB + \frac{1}{2}B^2} = e^{-\frac{a^2}{2}} = e^{aB + \frac{1}{2}B^2}\Big|_{B=-a} \quad . \tag{35}$$

The symmetry of the action is not disturbed,

$$\delta S = \delta\varphi \frac{\delta S}{\delta\varphi} + \delta\psi \frac{\delta S}{\delta\psi} + \delta B \frac{\delta S}{\delta B} \quad ,$$

and since $\frac{\delta S}{\delta B} = 0$, δB can be arbitrary. The only consequence of eliminating B is in the symmetry algebra itself, which now becomes the more complicated relation (21) (for $\lambda = 0$).

There is no guarantee that auxiliary fields exist for an arbitrary supersymmetric theory, and there are actually good arguments against their existence in some cases. However, they have been found for all N=1 and N=2 theories [17], for N=4 conformal supergravity [18] and for a ten-dimensional version of N=4 Poincaré supergravity [19].

There is a powerful approach to the study of supersymmetry available for theories that have auxiliary fields, namely the covariant superfield method, which is the subject of the next chapter.

Table 1. Common Supersymmetric Multiplets

N	s	Name	Number of states and SU(N) Representation									Notes
			helicity									
			-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$	$+\frac{3}{2}$	2	
1	$\frac{1}{2}$	Wess-Zumino				1	1+1	1				(a)
1	1	N=1 Yang-Mills			1	1		1	1			(b)
1	$\frac{3}{2}$			1	1				1	1		
1	2	N=1 Supergravity	1	1						1	1	
2	$\frac{1}{2}$	Hypermultiplet				1+1	$2+\bar{2}$	1+1				(d)
2	1	N=2 Yang-Mills			1	$\bar{2}$	1+1	2	1			(c)
2	2	N=2 Supergravity	1	2	1				1	2	1	
3	1				1	$\bar{3}+1$	$3+\bar{3}$	1+3	1			(e)
4	1	N=4 Yang-Mills			1	$\bar{4}$	6	4	1			(f)
4	2	N=4 Supergravity	1	$\bar{4}$	6	4	1+1	$\bar{4}$	6	4	1	
8	2	N=8 Supergravity	1	$\bar{8}$	28	$\overline{56}$	70	56	28	8	1	(f)

Chapter 2

INTRODUCTION TO SUPERFIELDS

The superfield approach to supersymmetry was introduced by Salam and Strathdee [20]. Their suggestion was that, in analogy with flat spacetime being isomorphic to the quotient space of the Poincaré group and the Lorentz group, it should be possible to express global supersymmetry in terms of transformations of a "superspace" isomorphic to the quotient of the Superpoincaré and Lorentz groups. They showed how this could be done, and the ideas were developed further by many other authors [21]. At present, it is the only approach in which supersymmetry is always manifest, and thus appears to be particularly convenient for discussing supersymmetric quantum field theory, where the supersymmetry causes divergence cancellations that appear "miraculous" in component calculations.

While the superfield method works for simple (N=1) supersymmetry, it has encountered difficulties in the transition to extended supersymmetry. We must emphasize that this is not necessarily a flaw in the supersymmetry concept itself. Superfields are a convenient way of describing supersymmetric theories, and it is worthwhile to apply the method as widely as possible, but some theories may well be beyond their scope. It is the purpose of the present work to show that, for the N=2 case at least, the problems can be overcome. These results, in conjunction with later work by Howe, Stelle and Townsend [22] on the ghost sector of N=2 Yang-Mills, provide an explicit method for quantizing N=2 Yang-Mills in superspace.

In this chapter we introduce the superfield technique, using as examples the free N=1 Wess-Zumino multiplet and the free N=1 Yang-Mills multiplet. We show that there are two complementary approaches available, the "constrained" and the "unconstrained." The chapter ends with a table comparing the two.

1. Superspace

A superspace manifold has as coordinates

(a) 4 real bosonic variables x^μ

(b) 2 complex fermionic variables θ^α and their 2 complex conjugates $\overline{\theta}^{\dot{\alpha}}$.

These take values in the even and odd parts of a Grassmann algebra, respectively, so the θ's anticommute with each other and commute with the x's.

Superspace carries a natural representation of the (N=1) Superpoincaré group, eqs. (1a,b,c,d), where the generators are represented as follows:

$$L_{\mu\nu} = x_{[\mu}\partial_{\nu]} + (i\,\sigma_{\mu\nu\alpha}{}^{\beta}\theta^\alpha\partial_\beta + c.c.\ )$$

$$P_\mu = -i\,\partial_\mu$$

$$Q_\alpha = (\partial_\alpha - \frac{i}{2}\,\overline{\theta}^{\dot{\beta}}\partial_{\alpha\dot{\beta}})$$

$$\overline{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger = (\partial_{\dot{\alpha}} - \frac{i}{2}\,\theta^\beta\partial_{\beta\dot{\alpha}})\ . \qquad \text{(1a,b,c,d)}$$

The various partial derivatives are defined by

$$\partial_\mu x^\nu = \delta^\nu_\mu \qquad \partial_{\alpha\dot{\beta}} = \sigma^\mu_{\alpha\dot{\beta}}\partial_\mu$$

$$\partial_\alpha\theta^\beta = \delta^\beta_\alpha = C_\alpha{}^\beta \qquad \partial_{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} = C_{\dot{\alpha}}{}^{\dot{\beta}}\ . \qquad (2)$$

Using these expressions it is easy to see that

$$[L_{\mu\nu}, x^{\sigma}] = x_{[\mu}\eta_{\nu]}{}^{\sigma}$$

$$[L_{\mu\nu}, \theta^{\alpha}] = i\theta^{\beta}(\sigma_{\mu\nu})_{\beta}{}^{\alpha} \qquad [L_{\mu\nu}, \bar{\theta}^{\dot{\alpha}}] = i\bar{\theta}^{\dot{\beta}}(\bar{\sigma}_{\mu\nu})_{\dot{\beta}}{}^{\dot{\alpha}} \quad . \tag{3}$$

Thus x^{μ} transforms as a vector under the Lorentz group, and θ^{α} and $\bar{\theta}^{\dot{\alpha}}$ transform as spinors, as suggested by the notation. The important part of the algebra is the anticommutator of two fermionic generators, and one can check that we indeed get

$$\{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$$

$$\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\} = P_{\alpha\dot{\beta}} \quad . \tag{4}$$

This relation is valid implicitly in any superspace field theory. Thus, a component theory must have a complete set of auxiliary fields to have an off-shell superspace formulation. Theories for which auxiliary fields are not available can only have "on-shell" formulations in superspace; i.e., the fields must satisfy their equations of motion, and no action can be found. Some authors [23] have suggested ways to modify the algebra (4), in an effort to accommodate the more general cases that component methods can treat, but none has led to simplification.

At this point it is useful to assign dimensions to the coordinates: given $[\partial_{\mu}] = 1$, we need $[x_{\mu}] = -1$, and since $[\partial_{\alpha}] = [Q_{\alpha}] = \frac{1}{2}$, one obtains $[\theta^{\alpha}] = -\frac{1}{2}$. A nice geometric interpretation for the Q generators can be found if one looks at the effect on the coordinates

$$\delta\theta^{\beta} = [\varepsilon^{\alpha}Q_{\alpha} + \bar{\varepsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \theta^{\beta}] = \varepsilon^{\beta}$$

$$\delta\bar{\theta}^{\dot{\beta}} = [\, \varepsilon^{\alpha} Q_{\alpha} + \bar{\varepsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \,, \bar{\theta}^{\dot{\beta}} \,] = \bar{\varepsilon}^{\dot{\beta}} \quad . \tag{5}$$

Thus supersymmetry transformations can be regarded as translations in the fermionic directions in superspace. This point of view is especially useful for supergravity theory [24].

The Superpoincaré algebra as written above only plays an implicit role in the superspace approach, for the following reason: one can construct another set of fermionic operators called "covariant D operators"

$$D_{\alpha} = \partial_{\alpha} + \frac{i}{2}\bar{\theta}^{\dot{\beta}}\partial_{\alpha\dot{\beta}} \qquad \bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + \frac{i}{2}\theta^{\beta}\partial_{\beta\dot{\alpha}} \tag{6}$$

differing from the Q's only by the sign of the second term. These are spinors under the SL(2,C) subgroup like Q and $\bar{Q}$, and anticommute with them:

$$\{ D_{\alpha}, Q_{\beta} \} = \{ D_{\alpha}, \bar{Q}_{\dot{\beta}} \} = \{ \bar{D}_{\dot{\alpha}}, Q_{\beta} \} = \{ \bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0. \tag{7}$$

Thus "covariant" here means supersymmetrically covariant. The D's satisfy a relation similar to that of the Q's:

$$\{ D_{\alpha}, D_{\beta} \} = \{ \bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}} \} = 0$$

$$\{ D_{\alpha}, \bar{D}_{\dot{\beta}} \} = i\partial_{\alpha\dot{\beta}} \qquad\qquad (cf. \;\; \{ Q_{\alpha}, \bar{Q}_{\dot{\beta}} \} = -i\partial_{\alpha\dot{\beta}}) \,. \tag{8}$$

The existence of these operators can be understood by general arguments from the theory of quotient spaces.

2. Superfields

A "superfield" Φ is now defined as a function on superspace that can be expanded as a power series in the fermionic coordinates. Since the θ's anticommute and α can take only two values so that

$$\theta^\alpha \theta^\beta = -\tfrac{1}{2} C^{\alpha\beta}\theta^2 \qquad \theta^\alpha\theta^\beta\theta^\gamma = 0 \qquad \theta^2 = \theta^\alpha\theta_\alpha \ , \tag{9}$$

the above expansion terminates after only a few terms, and

$$\Phi(x,\theta,\overline{\theta}) = \varphi(x) + \psi_\alpha(x)\theta^\alpha + \overline{\chi}_{\dot\alpha}(x)\overline{\theta}^{\dot\alpha} + T(x)\theta^2 + U_{\alpha\dot\beta}(x)\theta^\alpha\overline{\theta}^{\dot\beta} + V(x)\overline{\theta}^2$$

$$+ \rho_\alpha(x)\theta^\alpha\overline{\theta}^2 + \overline{\sigma}_{\dot\alpha}(x)\overline{\theta}^{\dot\alpha}\theta^2 + A(x)\theta^2\overline{\theta}^2 \tag{10}$$

The expansion coefficients are called the components of the superfield. Each is a function of the bosonic coordinates x only, and hence defines an ordinary space-time field. We call the $\theta,\overline{\theta}$ independent term the "lowest" component and the $\theta^2\overline{\theta}^2$ the "highest." These coefficients are not invariant under the superalgebra (1), because the Q's contain explicit θ's and $\frac{\partial}{\partial\theta}$'s, which mix them with other components.

One can consider more complicated superfields with external Lorentz indices, e.g., $\Psi_\alpha(x,\theta,\overline{\theta})$, but we then also need to augment the Lorentz generators in (1) with a spin part $S_{\mu\nu}$. ($L_{\mu\nu} \to M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$) which rotates the external indices appropriately. In the superfield expansion, each component field also carries the extra indices. This leads typically to a representation of supersymmetry containing a large number of irreducible multiplets. Even the single complex scalar superfield (10), without any external indices, contains two Yang-Mills multiplets and two Wess-Zumino multiplets. This may be reduced to one of each by taking Φ real, but in general irreducibility can be achieved only by setting some components to zero. This must not violate supersymmetry, and the easiest way to ensure this is to use the D operators of (8) to "constrain" the superfield.

3. Constraints and Covariant Components

Consider a complex scalar superfield Φ with the constraint

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \quad . \tag{11}$$

(In supersymmetry theory, any superfield satisfying this constraint is called "chiral," and its complex conjugate, which satisfies $D_\alpha\overline{\Phi} = 0$, is called "antichiral.") Since the Q's and D's anticommute, eq. (11) is an invariant statement:

$$\overline{D}_{\dot{\alpha}}(\Phi + \delta\Phi) \;=\; \overline{D}_{\dot{\alpha}}(\varepsilon^\beta Q_\beta + \bar{\varepsilon}^{\dot{\beta}} Q_{\dot{\beta}})\,\Phi \;=\; (\varepsilon^\beta Q_\beta + \bar{\varepsilon}^{\dot{\beta}} Q_{\dot{\beta}})\,\overline{D}_{\dot{\alpha}}\Phi = 0.$$

Writing eq. (11) as $\partial_{\dot{\alpha}}\Phi = -\frac{i}{2}\,\theta^\beta\partial_{\beta\dot{\alpha}}\Phi$ shows that some components are now defined as spacetime derivatives of lower ones, and others are set to zero. Clearly it would be laborious to write out the solution in all detail, but there is a more elegant procedure. Note that any component in a superfield can be obtained by operating θ derivatives on it and evaluating the result at $\theta = 0$, e.g., in the above example $V_{\alpha\dot{\beta}}(x) = \partial_{\dot{\beta}}\partial_\alpha\Phi(x,\theta,\overline{\theta})\,|_{\theta=\overline{\theta}=0}$. An alternative is to define new components by using instead the *covariant* D operators. For example,

$$\widetilde{V}_{\alpha\dot{\beta}}(x) \equiv \overline{D}_{\dot{\beta}}D_\alpha\;\Phi(x,\theta,\overline{\theta})\,|_{\theta=\overline{\theta}=0} = V_{\alpha\dot{\beta}}(x) + \frac{i}{2}\,\partial_{\alpha\dot{\beta}}\varphi(x) \quad .$$

This definition differs from the old one only by terms proportional to spacetime derivatives of lower components. These extra terms will depend on the precise ordering of the D's and $\overline{D}$'s in our definition, but they are basically just an unimportant field redefinition. The same reasoning applies to other components as well, as can be seen by dimensional analysis, and it turns out that for most purposes the new definition is more convenient than the old one. Of course, with this definition it's no longer easy to write out the θ expansion of a superfield because the θ coefficients are now more complicated functions of the components. However, the whole point of the superfield method is to avoid writing out expressions component by component.

To summarize,

(a) We call a superfield "covariant" if it transforms under supersymmetry by

$$\delta\Phi = (\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}^{\dot\alpha}\bar{Q}_{\dot\alpha})\,\Phi \quad . \tag{12}$$

Φ may have external indices.

(b) If Φ is a covariant superfield, then so are $D_\alpha\Phi$ and $\bar{D}_{\dot\alpha}\Phi$ (but not $\partial_\alpha\Phi$ or $\partial_{\dot\alpha}\Phi$).

(c) Every component of a covariant superfield is the lowest component of some other covariant superfield; e.g. $\tilde{V}_{\alpha\dot\beta}(x)$ is the lowest component of

$$V_{\alpha\dot\beta}(x,\theta,\bar\theta) = \bar{D}_{\dot\beta}D_\alpha\Phi(x,\theta,\bar\theta) \ .$$

To simplify notation we usually denote a superfield and its lowest component by similar symbols, e.g., $\Phi(x,\theta,\bar\theta)$ and $\varphi(x)$ above, and sometimes even by the same symbol. It is usually obvious which we mean.

4. Component Supersymmetry Transformations

It is often useful, particularly when one wants to make contact with results in component supersymmetry, to be able to calculate the supersymmetry transformations of components. This is straightforward, since at $\theta=0$, Q and D are the same thing,

$$(\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}})\Phi|_{\theta=0} = (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}})\Phi|_{\theta=0} \quad . \tag{13}$$

The constrained complex superfield (11) will be used to demonstrate the reasoning. We first define

$$\varphi(x) = \Phi|_{\theta=0} \quad ,$$

$$\psi_\alpha(x) = D_\alpha \Phi|_{\theta=0} \quad ,$$

$$B(x) = \tfrac{1}{2} D^\alpha D_\alpha \Phi|_{\theta=0} \quad . \tag{14}$$

Any other component is a spacetime derivative of one of these, since any $\bar{D}$'s may be pushed to the right until they hit Φ and give 0, e.g.,

$$\bar{D}_{\dot{\alpha}} D_\beta \Phi|_{\theta=0} = (-D_\beta \bar{D}_{\dot{\alpha}} + i\partial_{\beta\dot{\alpha}})\Phi|_{\theta=0} = i\partial_{\beta\dot{\alpha}}\Phi|_{\theta=0} = i\partial_{\beta\dot{\alpha}}\varphi \quad .$$

The supersymmetry transformations are then deduced as follows:

$$\delta\varphi(x) = (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}})\,\Phi|_{\theta=0} = \varepsilon^\alpha D_\alpha \Phi|_{\theta=0} = \varepsilon^\alpha \psi_\alpha(x) \quad ,$$

$$\delta\psi_\beta(x) = (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}})(D_\beta \Phi)|_{\theta=0} = \varepsilon^\alpha(-\tfrac{1}{2}\, C_{\alpha\beta}) D^\gamma D_\gamma \Phi|_{\theta=0} + i\,\bar{\varepsilon}^{\dot{\alpha}} \partial_{\beta\dot{\alpha}} \Phi|_{\theta=0}$$

$$= -\varepsilon_\beta B(x) + i\bar{\varepsilon}^{\dot{\alpha}} \partial_{\beta\dot{\alpha}}\, \varphi(x) \quad ,$$

$$\delta B(x) = (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}})\ \frac{1}{2}\ D^\beta D_\beta \Phi|_{\theta=0} = i\bar{\varepsilon}^{\dot{\alpha}}\, \partial^\beta{}_{\dot{\alpha}}\, D_\beta\, \Phi|_{\theta=0} = i\bar{\varepsilon}^{\dot{\alpha}}\, \partial^\beta{}_{\dot{\alpha}}\, \psi_\beta(x) \quad .$$

(15a,b,c)

These are the fields and transformation laws found for the Wess-Zumino multiplet in the previous chapter. We thus have a compact description of the multiplet in terms of superfields. However, we still require a method for finding supersymmetric actions.

5. Superspace actions

Under a supersymmetry transformation the highest component of a superfield transforms into spacetime derivatives of lower components, so that its integral over spacetime is an invariant. If we now define an "integration" operation identical to fermionic differentiation, i.e.,

$$\int (d\theta)_\alpha \equiv \partial_\alpha \qquad \int (d\bar{\theta})_{\dot\alpha} \equiv \partial_{\dot\alpha} \quad , \tag{16}$$

where the parentheses are a reminder that the external indices of $d\theta$ are those of ∂ and not θ, then for any superfield S

$$\int d^4x \int (d\theta)^\alpha (d\theta)_\alpha (d\bar{\theta})^{\dot\beta} (d\bar{\theta})_{\dot\beta} \ \mathrm{S} \ \equiv \ \int\int d^4x \, d^2\theta \, d^2\bar{\theta} \ \mathrm{S}$$

is a supersymmetric invariant. This definition of fermionic integration is not as ad hoc as it may seem, because up to a scale it is the unique definition allowing fermionic integration by parts

$$\int d\theta f \ \frac{\partial}{\partial\theta} \ g = -\int d\theta \ \frac{\partial f}{\partial\theta} \ g \ (-1)^{|f|} \tag{17}$$

where the $(-1)^{|f|}$ accounts for the possibility of f being fermionic. $|f|$ is 0 for bosonic f and 1 for fermionic f. (16) does, however, have the strange consequence that a fermionic integration has dimension $+\frac{1}{2}$, whereas a bosonic integration has dimension -1.

For any constrained superfield S, the highest component is a total derivative, so the above expression is trivial. In this case we can instead use the highest dimension component that is not a total derivative. For example, any chiral superfield $\Phi(x,\theta,\bar{\theta})$ with $\bar{D}_{\dot\alpha}\Phi = 0$ has highest non-derivative component $D^2\Phi$, so we take as an invariant

$$S_c \equiv \int d^4x \int (d\theta)^\alpha (d\theta)_\alpha \ \Phi = \int d^4x \ d^2\theta \ \Phi \quad . \tag{18}$$

To prove this is invariant note that

$$S_c = \int d^4x \; d^2\theta \; \Phi = \int d^4x \; \partial^\alpha \partial_\alpha \, \Phi = \int d^4x \; D^\alpha D_\alpha \, \Phi$$

up to surface terms which we ignore. Therefore

$$\delta S_c = \int d^4x \; d^2\theta \; \delta\Phi = \int d^4x D^\beta D_\beta (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot\alpha} \bar{D}_{\dot\alpha}) \Phi$$

$$= 0 \qquad\qquad \text{since} \quad D^\beta D_\beta D_\alpha = 0 \quad \text{and} \quad \bar{D}_{\dot\alpha} \Phi = 0.$$

The product of chiral fields is chiral, so we can write the following action terms for the Wess-Zumino multiplet, corresponding to kinetic, mass and interaction terms

$$S_{kin} = \int d^4x \, d^2\theta \, d^2\bar\theta \; \bar\Phi \Phi$$

$$S_m = m \int d^4x \, d^2\theta \; \Phi^2 + m \int d^4x \, d^2\bar\theta \; \bar\Phi^2$$

$$S_{int} = \lambda \int d^4x \, d^2\theta \; \Phi^3 + \lambda \int d^4x \, d^2\bar\theta \; \bar\Phi^3 \; . \tag{19}$$

Other terms are also possible, but don't give renormalizable interactions. The techniques just demonstrated make it easy to expand such expressions in components, through the following sequence of steps:

$$\int d^4x \, d^2\theta \, d^2\bar\theta \; \bar\Phi\Phi = \left(\int d^4x \, d^2\theta \, d^2\bar\theta \; \bar\Phi\Phi\right)|_{\theta=0} \qquad \text{since no } \theta\text{'s are left anyhow}$$

$$= \left(\int d^4x \; \partial^\alpha \partial_\alpha \partial^{\dot\alpha} \partial_{\dot\alpha} \; \bar\Phi\Phi\right)|_{\theta=0} \qquad \text{by definition}$$

$$= \left(\int d^4x \; D^\alpha D_\alpha \bar{D}^{\dot\alpha} \bar{D}_{\dot\alpha} \bar\Phi\Phi\right)|_{\theta=0} \qquad \text{up to surface terms}$$

$$= \left(\int d^4x \; D^\alpha D_\alpha \; (\bar{D}^{\dot\alpha} \bar{D}_{\dot\alpha} \bar\Phi)\Phi \;\right)|_{\theta=0} \qquad \text{by chirality of } \Phi$$

$$= \left(\int d^4x D^\alpha [(2i\partial_\alpha{}^{\dot\alpha} \bar{D}_{\dot\alpha} \bar\Phi)\, \Phi + (\bar{D}^{\dot\alpha} \bar{D}_{\dot\alpha} \bar\Phi)\, D_\alpha \Phi]\right)|_{\theta=0} \qquad \text{using } D_\alpha \bar\Phi = 0$$

$$= \int d^4x \, ((-2\partial_\alpha{}^{\dot\alpha}\partial^\alpha{}_{\dot\alpha}\bar\Phi)\,\Phi - 4i\,(\partial_\alpha{}^{\dot\alpha}\bar D_{\dot\alpha}\bar\Phi)D^\alpha\Phi + (\bar D^{\dot\alpha}\bar D_{\dot\alpha}\bar\Phi)\,(D^\alpha D_\alpha\Phi)\,)\,|_{\theta=0}$$

$$= 4 \int d^4x \left[-\bar\varphi\Box\varphi - i\bar\psi^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^\alpha + \bar B B \right] \qquad \text{evaluating at } \theta = 0 \quad , \quad (20)$$

which is just the kinetic term of the previous chapter.

Similarly, the other two terms of eq. (19) are the familiar ones. In practice, one omits the $\theta = 0$ condition in these manipulations, since a careful examination of (20) reveals that θ-dependent terms would be surface terms.

$$\int d^4x \, d^2\theta \, \Phi^2 = \int d^4x \; D^\alpha D_\alpha \Phi^2 = \int d^4x \; (\, 2\Phi \, D^\alpha D_\alpha \Phi + 2 \, D^\alpha \Phi \, D_\alpha \Phi \,)$$

$$= 2 \int d^4x \; (\, 2\Phi B + \psi^\alpha \psi_\alpha \,) \quad . \qquad (21)$$

This shows the equivalence between the superfield and component approaches to supersymmetry, at least in this simple case. An important lesson is that the covariant derivatives D_α , $\bar D_{\dot\alpha}$ are the only operators used in all these manipulations.

6. Unconstrained Superfields

In many instances, particularly when quantizing a superspace theory, it is inconvenient or even impossible to work with constrained fields. As a simple example of the problem, consider ordinary electrodynamics, described by a field strength $F_{\mu\nu}$. Since this satisfies a constraint $\partial_{[\mu}F_{\nu\sigma]} = 0$, the components of F are not all independent, and it is difficult to quantize this form of electrodynamics. A simpler way is to "solve" the constraint in terms of a potential A_μ by $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. The constraint is now satisfied, but the theory has become a gauge theory since $\delta A_\mu = \partial_\mu \lambda$ is an invariance. However, we know how to quantize gauge theories, using Faddeev-Popov ghosts, say, and calculations can proceed.

In the case of the Wess-Zumino multiplet, one can also solve the constraint, eq. (11), by putting

$$\Phi = \overline{D}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}}\, Y \ , \tag{22}$$

where Y is an unconstrained scalar superfield. A gauge invariance arises, since putting

$$\delta Y = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \tag{23}$$

for an arbitrary spinor $\overline{\chi}^{\dot{\alpha}}$, gives

$$\delta\Phi = \overline{D}^{\dot{\beta}}\overline{D}_{\dot{\beta}}\ (\overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}) = 0 \ . \tag{24}$$

It is important to note that the solution to the constraint and the form of the gauge invariance both result from the identity $\overline{D}_{\dot{\alpha}}\overline{D}_{\dot{\beta}}\overline{D}_{\dot{\gamma}} = 0$, which can ultimately be traced to the relation $\{\overline{D}_{\dot{\alpha}},\overline{D}_{\dot{\beta}}\} = 0$. We argue in the next chapter that this relation is the essential element in all N=1 supersymmetric theories, and then in chapter 4 find the N=2 analogue.

The superspace terminology is similar to that for ordinary space. The gauge invariant superfield Φ is called the "field strength" of the Wess-Zumino multiplet, and (11) is called the constraint. The unconstrained superfield Y is called a "prepotential" (because something else was already called a potential), and $\overline{\chi}^{\dot{\alpha}}$ is called the "gauge parameter." Once it has been solved by (22), the constraint (11) can be called a Bianchi identity.

Let us write down the superspace kinetic term for the multiplet (14) using the unconstrained scalar field Y:

$$S_{kin} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\ \overline{Y}\, D^2\overline{D}^2\, Y \ \ , \tag{25}$$

where D^2 has been integrated by parts. This form is useful for finding field equations: since Y is unconstrained, we have immediately

$$\delta S_{kin} = 0 \quad => \quad D^2\overline{D}^2 Y = 0$$

$$=> \quad D^2\Phi = 0 \quad . \tag{26}$$

This could not have been found directly from the form (19) because Φ is constrained.

To verify that we have the correct field equations, project out components.

$$D^2\Phi = 0 \;=>\; D^2\Phi|_{\theta=0} = 2B(x) = 0 \quad ,$$

$$=> \overline{D}_{\dot\alpha} D^2\Phi|_{\theta=0} = 2i\,\partial^\alpha{}_{\dot\alpha} D_\alpha \Phi|_{\theta=0} = 2i\,\partial^\alpha{}_{\dot\alpha}\psi_\alpha = 0 \quad ,$$

$$=> \overline{D}^2 D^2\Phi|_{\theta=0} = -4\Box\Phi = 0 \quad , \tag{27}$$

where as usual we have commuted the $\overline{D}$'s to the right and used $\overline{D}_{\dot\alpha}\Phi = 0$. These are the component field equations. The easiest one to identify is that for the auxiliary field B, since it is algebraic, and occurs at the "bottom" of the field equation. The others necessarily follow by supersymmetry.

An interesting effect occurs in the term S_m (or S_{int}).

$$S_m = m\int d^4x\, d^2\theta \;\; (\overline{D}^2 Y)\,(\overline{D}^2 Y)$$

This can be manipulated using the identity

$$(\overline{D}^2 X)\,(\overline{D}^2 Y) = \overline{D}^2(X\,\overline{D}^2 Y) = \int d^2\overline{\theta}(X\overline{D}^2 Y) \;+\; \text{spacetime derivatives.} \tag{28}$$

Thus

$$S_m = m\int d^4x \;d^2\theta\; d^2\overline{\theta} \;\; Y\,\overline{D}^2\, Y \quad . \tag{29}$$

We start with a $\int d^2\theta$ in the constrained approach (i.e., a "subspace integral") but end up with a "full superspace integral" $\int d^2\theta d^2\overline{\theta}$ in the unconstrained approach.

7. Yang-Mills Multiplet [25]

One can repeat the above procedure to describe the Yang-Mills multiplet using superfields in an analogous way.

7.1. Constrained Approach [26]

The constrained superfield that describes the Yang-Mills multiplet is a chiral spinor W^α, with constraints

$$\overline{D}_{\dot\alpha} W_\beta = 0 \qquad\qquad D_\alpha W^\alpha = \overline{D}_{\dot\alpha} \overline{W}^{\dot\alpha} \quad , \tag{30a,b}$$

where $\overline{W}^{\dot\alpha} = (W^\alpha)^*$. From these constraints one can work out the components as before:

$$W^\alpha|_{\theta=0} = \psi^\alpha \qquad \text{(spin 1/2 field)},$$

$$D_{(\alpha} W_{\beta)}|_{\theta=0} = F_{\alpha\beta} \qquad \text{(field strength of spin 1 field)}, \tag{31a,b,c}$$

$$D_\alpha W^\alpha|_{\theta=0} = B \qquad \text{(scalar auxiliary field, real by (30b))}.$$

All other components are spacetime derivatives of these. We identify $F_{\alpha\beta}$ as the field strength of a spin 1 field, because operating with $D_\beta \overline{D}_{\dot\gamma}$ on both sides of (30b) gives the constraint

$$\partial_{\alpha\dot\gamma} F_\beta{}^\alpha - \partial_{\beta\dot\alpha} \overline{F}_{\dot\gamma}{}^{\dot\alpha} = 0 \quad , \tag{32}$$

which is the SL(2,C) version of $\partial_{[\mu} F_{\nu\sigma]} = 0$. Other relations can also be deduced. For instance, operating D_β on (30b) gives

$$D^2 W^\alpha = 2\,D_\alpha B = 2i\,\partial_{\alpha\dot\beta}\,\overline{W}^{\dot\beta} \quad . \tag{33}$$

$D_\beta W_\alpha = \frac{1}{2} F_{\beta\alpha} + \frac{1}{2} C_{\beta\alpha} B$ follows immediately from (31b,c).

To find an action formula, we note that W^α is chiral and has dimension 3/2, since its lowest component is a physical spinor. Thus the quantity $\int d^4x\, d^2\theta\; W_\alpha W^\alpha$ has the correct dimension for an action and is a supersymmetrical invariant. We can expand it as follows,

$$\begin{aligned}\int d^4x\, d^2\theta\;\, W_\alpha W^\alpha &= \int d^4x\; D^\beta D_\beta\;\, W_\alpha W^\alpha \,|_{\theta=0} \\ &= 2\int d^4x\; \big(W_\alpha (D^2 W^\alpha) - (D_\beta W_\alpha)(D^\beta W^\alpha)\big)\,|_{\theta=0} \\ &= \int d^4x\; \big(\, -4i\,\overline{\psi}^{\dot\beta}\partial_{\alpha\dot\beta}\psi^\alpha + \tfrac{1}{2}\, F_{\alpha\beta}F^{\alpha\beta} + \tfrac{1}{2}\, B^2 \,\big) \quad ,\end{aligned} \tag{34}$$

which is eq. (1.24) with B as an auxiliary field. Thus we take

$$S = \frac{1}{8}\int d^4x\, d^2\theta\;\, W_\alpha W^\alpha \tag{35}$$

as the action for the linearized Yang-Mills multiplet with a real scalar auxiliary field.

7.2. Unconstrained Approach [27]

Until now we have used the covariant objects ψ_α, $F_{\alpha\beta}$, B ; the gauge field $A_{\alpha\dot\beta}$ has not yet appeared. It appears when we change to an unconstrained formalism.

(30a) is solved by putting

$$W_\alpha = \overline{D}^2 L_\alpha \quad . \tag{36}$$

After some trial and error, we find that (30b) is solved by requiring further

$$L_\alpha = D_\alpha V \quad , \tag{37}$$

where V is a real scalar superfield. (30b) then becomes

$$D_\alpha \bar{D}^2 D^\alpha V = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V$$

which is true, since pushing D's towards the middle gives $D_\alpha \bar{D}^2 D^\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}$. Thus we can write the action in unconstrained form as

$$S = \frac{1}{8}\int d^4x\, d^2\theta\, (\bar{D}^2 D_\alpha V)(\bar{D}^2 D^\alpha V)$$

$$= -\frac{1}{8}\int d^4x\, d^2\theta\, d^2\bar\theta \quad V D_\alpha \bar{D}^2 D^\alpha V \quad . \tag{38}$$

The expression $W_\alpha = \bar{D}^2 D_\alpha V$ now has a gauge invariance

$$\delta V = D^2\rho + \bar{D}^2\bar\rho \tag{39}$$

for an arbitrary scalar superfield parameter ρ. It is interesting that this unconstrained superfield and its associated gauge invariance include as one component the spacetime gauge invariance of an abelian gauge theory. In fact

$$F_{\alpha\beta} = D_{(\alpha} W_{\beta)}|_{\theta=0} = D_{(\alpha} \bar{D}^2 D_{\beta)} V|_{\theta=0}$$

$$= i\partial_\alpha{}^{\dot\gamma}(D_\beta \bar{D}_{\dot\gamma} - \bar{D}_{\dot\gamma} D_\beta)\, V|_{\theta=0} + (\alpha <-> \beta) \quad , \tag{40}$$

by manipulating D's. If we identify

$$A_{\beta\dot\gamma} = (D_\beta \bar{D}_{\dot\gamma} - \bar{D}_{\dot\gamma} D_\beta)\, V|_{\theta=0} \quad , \tag{41}$$

a real vector, then we have

$$F_{\alpha\beta} = i\partial_{(\alpha}{}^{\dot\gamma} A_{\beta)\dot\gamma} \quad , \tag{42}$$

which is the standard expression in SL(2,C) form. Moreover, under the transformation (39) we get

$$\delta A_{\alpha\dot\beta} = (D_\alpha \overline{D}_{\dot\beta} - \overline{D}_{\dot\beta} D_\alpha)\ (D^2\rho + \overline{D}^2\overline{\rho}) = \partial_{\alpha\dot\beta}\ (iD^2\rho - i\overline{D}^2\overline{\rho})\ , \qquad (43)$$

again the standard result. We see that in a supersymmetric gauge theory, supersymmetry and gauge invariance are intimately connected.

In the next chapter interactions are introduced.

8. Extended Superfields

Extended superfields (N>1) work in much the same way. One adds an SU(N) index to each spinorial coordinate $\theta^\alpha \to \theta^{i\alpha}$, $\overline{\theta}^{\dot\alpha} \to \overline{\theta}_i{}^{\dot\alpha}$. The basic commutation relations of the D operators are then

$$\{D_{i\alpha}, D_{j\beta}\} = 0 \qquad \{D_{i\alpha}, \overline{D}^j{}_{\dot\beta}\} = i\,\delta_i^j\,\partial_{\alpha\dot\beta} \quad . \qquad (44)$$

Extended superfields have many more components than simple superfields do, because the θ expansions terminate only at $\theta^{2N}\overline{\theta}^{2N}$. Thus a general complex scalar extended superfield has 2^{4N} components, and this exponential increase with N causes trouble. To obtain irreducible multiplets, extended superfields must be severely constrained, which makes it difficult to convert to an unconstrained formalism. Indeed, the results presented in chapter 4 provide the first example of an unconstrained interacting theory in extended superspace.

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Summary of the important results of this chapter:

[1] Superfields provide a manifestly supersymmetric formulation of supersymmetric theories.

[2] All component results can be derived using the Q-covariant operators D and $\overline{D}$.

[3] Two approaches are available: "constrained" and "unconstrained." Table (2) compares them. They are the two extreme cases of a range of possible superspace formulations of any theory. By using different superfields, one may arrange for the multiplet to occur at different powers of θ. For instance, one may formulate the N=1 Yang-Mills multiplet in terms of the superfield L_α of eq. (36). L_α then satisfies a constraint, but is also subject to gauge transformations. The general rule is that unwanted components above the multiplet must be constrained to vanish, and unwanted components below the multiplet must be gauged away. The results of the next chapter can be interpreted as the formulation of the Yang-Mills multiplet in terms of the "potential" L_α, which has a geometric interpretation.

Table 2. Comparison of Constrained and Unconstrained Approaches

Constrained	Unconstrained
Basic superfields are "constrained field strengths"	Basic superfields are "unconstrained prepotentials"
Unwanted superfield components are set to zero by the constraints	Unwanted superfield components are gauge degrees of freedom
Useful for finding component transformations, component actions, etc.	Relation to component supersymmetry not manifest
For any irreducible multiplet, the field strength has the same external indices as the lowest dimension component field, because that component occurs at the bottom of the field strength	For any irreducible multiplet, the prepotential has the same external indices as the highest dimension component field, which occurs at the top of the prepotential
The constraint has the same external indices as the lowest dimension unwanted field strength component	The gauge parameter has the same external indices as the highest dimension unwanted prepotential component.
The superspace action often has a subspace integral	The superspace action is expressed as a full superspace integral
Can be applied to equations of motion	Theory is defined off shell
Very difficult or impossible to quantize	Can ba quantized using Faddeev-Popov procedure
Has been formulated for nearly all known supersymmetric theories	Has been applied to N=1 superfields and the free N=2 vector multiplet.

Chapter 3

YANG-MILLS THEORIES IN SUPERSPACE

In the previous chapters it was shown that supersymmetric gauge theories mix the supersymmetry and the gauge symmetry so that they are not separable in a covariant way. This feature led various authors [28] to consider constructing gauge theories as geometric theories in superspace, in the same way that pure Yang-Mills and Einstein gravity are treated in ordinary spacetime. This has proved quite successful, and leads to a good understanding of N=1 Yang-Mills and N=1 supergravity. Here the Yang-Mills case is discussed, beginning with the constrained approach.

1. Constrained Approach [29]

As in ordinary space, one begins by introducing superspace covariant derivatives ∇_α, $\nabla_{\dot\alpha}$ and $\nabla_{\alpha\dot\beta}$, postulated to be covariant under gauge transformations with some Yang-Mills group G. Thus let G be a compact Lie group, and $\{X_i\}$ a basis for its Lie algebra. Then for $K(x,\theta,\overline{\theta}) = K^i(x,\theta,\overline{\theta})\,X_i$ where the K^i are real superfields, the covariant derivatives transform under a gauge transformation as

$$\nabla'_\alpha = e^K \nabla_\alpha e^{-K} \qquad \nabla'_{\dot\alpha} = e^K \nabla_{\dot\alpha} e^{-K} \qquad \nabla'_{\alpha\dot\beta} = e^K \nabla_{\alpha\dot\beta} e^{-K} \quad . \tag{1}$$

We assume further that the ∇'s can be written in terms of potentials Γ

$$\nabla_\alpha = D_\alpha + \Gamma_\alpha \qquad \nabla_{\dot\alpha} = \overline{D}_{\dot\alpha} + \Gamma_{\dot\alpha} \qquad \nabla_{\alpha\dot\beta} = \partial_{\alpha\dot\beta} + \Gamma_{\alpha\dot\beta} \quad . \tag{2}$$

Any theory invariant under the transformations (1) clearly contains a spacetime Yang-Mills theory, since $\nabla_{\alpha\dot{\beta}}$ contains a spacetime covariant derivative at lowest order in θ. However, two other independent spacetime covariant derivatives exist as well, since one can construct more covariant objects by anticommuting ∇_α and $\nabla_{\dot{\alpha}}$

$$\mathrm{Re}(-i\{\nabla_\alpha,\nabla_{\dot{\beta}}\}) = \partial_{\alpha\dot{\beta}} + \cdots \quad ,$$

and

$$\mathrm{Re}(-i\{(\nabla_\alpha)^*,(\nabla_{\dot{\beta}})^*\}) = \partial_{\beta\dot{\alpha}} + \cdots \quad .$$

Thus the theory is not irreducible, and again the solution is to enforce some constraints. The simplest way is to equate the three independent quantities above, i.e., put

$$\nabla_{\dot{\alpha}} = -(\nabla_\alpha)^* \equiv \overline{\nabla}_{\dot{\alpha}} \qquad (\nabla_{\alpha\dot{\beta}})^* = \nabla_{\beta\dot{\alpha}} \quad , \tag{3}$$

and require

$$\{\nabla_\alpha,\nabla_{\dot{\beta}}\} = i\,\nabla_{\alpha\dot{\beta}} \quad . \tag{4}$$

(Complex conjugation for covariant derivatives is discussed in the appendix.)

Apart from covariant derivatives, one can also construct covariant field strengths by commuting them in various ways. In particular one has

$$\{\nabla_\alpha,\nabla_\beta\} = D_{(\alpha}\Gamma_{\beta)} + \{\Gamma_\alpha,\Gamma_\beta\} \equiv G_{\alpha\beta} \quad , \tag{5}$$

$$[\nabla_\alpha,\nabla_{\beta\dot{\gamma}}] = D_\alpha\Gamma_{\beta\dot{\gamma}} - \partial_{\beta\dot{\gamma}}\Gamma_\alpha + [\Gamma_\alpha,\Gamma_{\beta\dot{\gamma}}] \equiv R_{\alpha,\beta\dot{\gamma}} \quad . \tag{6}$$

The dimensions of G and R are 1 and $\frac{3}{2}$ respectively. Since $G_{\alpha\beta}$ has a covariant spacetime field $G_{\alpha\beta}(x)$ of dimension 1 as its lowest component, and there is no such field in the irreducible N=1 Yang-Mills multiplet, the theory is still reducible. Thus, to reproduce pure N=1 Yang-Mills, this component must be zero. Supersymmetry then implies that all other components of $G_{\alpha\beta}$ are zero as well because one can repeatedly use

$$\delta G(x,\theta,\overline{\theta})|_{\theta=0} = 0 \quad => \quad (\varepsilon^{\gamma}\nabla_{\gamma} + \overline{\varepsilon}^{\dot{\gamma}}\nabla_{\dot{\gamma}})\, G(x,\theta,\overline{\theta})|_{\theta=0} = 0$$

$$=> \quad \nabla_{\gamma} G|_{\theta=0} = \nabla_{\dot{\gamma}} G|_{\theta=0} = 0 \quad . \tag{7}$$

This is a general result, often used implicitly in what follows: if the lowest component of a covariant superfield vanishes, the entire superfield vanishes. (This is not true for higher components, which may be set to zero without eliminating the entire superfield. A chiral constraint is an example.)

Similarly, decomposing $R_{\alpha,\beta\dot{\gamma}}$ into SL(2,C) irreducible pieces gives

$$R_{\alpha,\beta\dot{\gamma}} = \widehat{R}_{\alpha\beta\dot{\gamma}} + C_{\alpha\beta}\overline{W}_{\dot{\gamma}} \quad , \tag{8}$$

and only $\overline{W}_{\dot{\gamma}}$ can be accommodated in the multiplet, as the single dimension $\frac{3}{2}$ physical spinor, so one should constrain $\widehat{R}_{\alpha\beta\dot{\gamma}} = 0$. One could proceed in this way and examine $\nabla_{\alpha} W_{\beta}$, $\nabla_{\dot{\alpha}} W_{\beta}$ etc., setting unwanted components to zero, but this is unnecessary. It is sufficient to postulate only the lowest dimension constraints on the covariant derivatives,

$$\{\nabla_{\alpha},\nabla_{\beta}\} = \{\nabla_{\dot{\alpha}},\nabla_{\dot{\beta}}\} = 0 \quad ,$$

$$\{\nabla_{\alpha},\nabla_{\dot{\beta}}\} = i\nabla_{\alpha\dot{\beta}} \quad . \tag{9}$$

Constraints on higher-order objects then follow automatically from the graded Jacobi identity (henceforth called the Bianchi identity)[30]. For example, since one has the following identity by expanding out the brackets

$$[\nabla_\alpha,\{\nabla_\beta,\nabla_{\dot\gamma}\}] + [\nabla_{\dot\gamma},\{\nabla_\alpha,\nabla_\beta\}] + [\nabla_\beta,\{\nabla_{\dot\gamma},\nabla_\alpha\}] = 0 \quad , \tag{10}$$

one can substitute in the above constraints (9) and find the implication:

$$0 = [\,\nabla_\alpha\,, i\nabla_{\beta\dot\gamma}\,] + [\,\nabla_{\dot\gamma}\,, 0\,] + [\,\nabla_\beta\,, i\nabla_{\alpha\dot\gamma}\,]$$

$$= iR_{\alpha,\beta\dot\gamma} + iR_{\beta,\alpha\dot\gamma} = 2i\widehat{R}_{\alpha\beta\dot\gamma} \quad . \tag{11}$$

Thus the constraint $\widehat{R}_{\alpha\beta\dot\gamma} = 0$ is not independent, and follows automatically from (9) via the Bianchi identities. Another way of saying this is that the lowest component of $\widehat{R}_{\alpha\beta\dot\gamma}$ is the second component of $G_{\alpha\beta}$ and vanishes if $G_{\alpha\beta}$ does.

Continuing in this way, the full implications of (9) are worked out, and the most efficient method is to proceed in order of increasing dimension. The only such Bianchi identity worth demonstrating is the dimension two

$$\{\nabla_\alpha,[\nabla_\beta,\nabla_{\gamma\dot\delta}]\} + [\nabla_{\gamma\dot\delta},\{\nabla_\alpha,\nabla_\beta\}] + \{\nabla_\alpha,[\nabla_\beta,\nabla_{\gamma\dot\delta}]\} = 0 \quad , \tag{12}$$

which implies that W_α is "covariantly chiral":

$$0 = \nabla_\alpha \overline{W}_{\dot\delta} \equiv \{\nabla_\alpha, \overline{W}_{\dot\delta}\} \qquad 0 = \nabla_{\dot\alpha} W_\beta \quad . \tag{13}$$

The following set of relations is obtained:

dim 1 $\qquad \{\nabla_\alpha,\nabla_\beta\} = 0 \qquad \{\nabla_\alpha,\nabla_{\dot\beta}\} = i\nabla_{\alpha\dot\beta} \qquad$ (14a,b)

dim $\frac{3}{2}$ $\qquad [\nabla_\alpha,\nabla_{\beta\dot\gamma}] = C_{\alpha\beta}\overline{W}_{\dot\gamma} \qquad [\nabla_{\dot\alpha},\nabla_{\beta\dot\gamma}] = -C_{\dot\alpha\dot\gamma}W_\beta \qquad$ (15a,b)

dim 2 $\qquad \nabla_\alpha \overline{W}_{\dot\beta} = \nabla_{\dot\alpha} W_\beta = 0 \qquad$ (16)

$$i[\nabla_{\alpha\dot\beta},\nabla_{\gamma\dot\delta}] = C_{\alpha\gamma}\nabla_{\dot\beta}\overline{W}_{\dot\delta} - C_{\dot\beta\dot\delta}\nabla_\alpha W_\gamma \quad . \tag{17}$$

Using our conventional SL(2,C) decomposition of an antisymmetric tensor,

$$[\nabla_{\alpha\dot\beta},\nabla_{\gamma\dot\delta}] = \frac{i}{2}\, C_{\alpha\gamma}\bar F_{\dot\beta\dot\delta} + \frac{i}{2}\, C_{\dot\beta\dot\delta}F_{\alpha\gamma} \quad , \tag{18}$$

allows one to express (17) as

$$\nabla_{(\dot\beta}\, \bar W_{\dot\delta)} = -\bar F_{\dot\beta\dot\delta} \qquad \nabla_{(\alpha}\, W_{\gamma)} = F_{\alpha\gamma} \quad , \tag{19a,b}$$

$$\nabla_\alpha W^\alpha = \nabla_{\dot\alpha}\bar W^{\dot\alpha} = B\,, \qquad \text{with B real.} \tag{20}$$

dim $\frac{5}{2}$

$$\nabla_{\dot\alpha} B = i\, \nabla_{\beta\dot\alpha} W^\beta \tag{21}$$

$$\nabla_{\dot\alpha} F_{\beta\gamma} = i \nabla_{(\beta\dot\alpha} W_{\gamma)} \tag{22}$$

$$\nabla_\alpha F_{\beta\gamma} = i C_{\alpha(\beta}\nabla_{\gamma)\dot\delta}\bar W^{\dot\delta} \quad . \tag{23}$$

Thus all component fields with dimension greater than 2 can be expressed as vector covariant derivatives of lower-dimensional components, and are not independent degrees of freedom. These relations may be compared with those for the constrained formulation of the vector multiplet in the previous chapter, sect.(2.7.1), eqs. (30-33). The two sets are identical except that here full Yang-Mills covariant derivatives have replaced D_α, $\bar D_{\dot\alpha}$ and $\partial_{\alpha\dot\beta}$. The correspondence has two possible interpretations:

(A) The previous results are linearizations of the new ones, obtained by discarding any terms quadratic or higher in the Γ's. Then, for instance,

$$\nabla_{\dot\alpha} W_\beta = 0 \;=>\; (\bar D_{\dot\alpha} + \Gamma_{\dot\alpha}) W_\beta = 0 \;=>\; \bar D_{\dot\alpha} W_\beta = 0$$

to lowest order.

(B) The results of sect.(2.7.1) describe an abelian gauge group. Since everything is in the adjoint representation, which is trivial,

$$\nabla_{\dot\alpha} W_\beta = \{\bar D_{\dot\alpha} + \Gamma_{\dot\alpha}, W_\beta\} = \bar D_{\dot\alpha} W_\beta \quad .$$

It is useful to check that the following results, especially those in the unconstrained theory, reduce to the free expressions in both cases.

Following the linearized case, consider as an action for the interacting multiplet

$$S = \frac{1}{8}\int d^4x d^2\theta \, W_\alpha * W^\alpha \quad . \tag{24}$$

This is manifestly gauge invariant (because $L \equiv W_\alpha * W^\alpha$ is a group invariant), but it is also supersymmetric because L is chiral

$$\bar{D}_{\dot\alpha} L = \nabla_{\dot\alpha} L \qquad \text{since } L \text{ is a scalar}$$

$$= \nabla_{\dot\alpha} \; W_\beta * W^\beta \;=\; 2(\nabla_{\dot\alpha} W_\beta) * W^\beta \;= 0 \qquad \text{since } \nabla \text{ is a derivation.}$$

Using the same reasoning as in sect. (2.5), one can evaluate S in components

$$S = \frac{1}{8}\int d^4x \, d^2\theta \; L = \frac{1}{8}\int d^4x \; D^\alpha D_\alpha \, L \;|_{\theta=0} \;=\; \frac{1}{8}\int d^4x \; \nabla^\alpha \nabla_\alpha L \;|_{\theta=0}$$

$$= \int d^4x \left[\frac{1}{16} F_{\alpha\beta} * F^{\alpha\beta} - \frac{i}{2} \, \overline{W}^{\dot\beta} * \nabla_{\alpha\dot\beta} W^\alpha + \frac{1}{16} B * B \right] \quad . \tag{25}$$

It is worth noting that the equation of motion $B=0$ remains true in the interacting case. A convenient way to find the component supersymmetry transformations, is to define a "Yang-Mills covariant supersymmetry transformation," which is a supersymmetry transformation plus a field-dependent gauge transformation, such that

$$\delta(\Phi \,|_{\theta=0}) = (\,(\varepsilon^\alpha \nabla_\alpha + \bar{\varepsilon}^{\dot\alpha} \nabla_{\dot\alpha}) \; \Phi\,)\,|_{\theta=0}$$

$$= (\varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot\alpha} \bar{D}_{\dot\alpha}) \Phi \,|_{\theta=0} + (\varepsilon^\alpha \Gamma_\alpha + \bar{\varepsilon}^{\dot\alpha} \Gamma_{\dot\alpha}) \Phi \,|_{\theta=0} \quad , \tag{26}$$

i.e., a supersymmetry transformation followed by a superspace gauge transformation with $K = (\varepsilon^\alpha \Gamma_\alpha + \bar{\varepsilon}^{\dot\alpha}\Gamma_{\dot\alpha})$. Eqs. (14-23) can then be interpreted as before as the supersymmetry transformations of the component fields.

Thus, just as all of matter supersymmetry theory could be reduced to relations between operators D_α and $\bar{D}_{\dot\alpha}$, the supersymmetric Yang-Mills theory follows from the commutation relations of the covariant derivatives ∇_α, $\nabla_{\dot\alpha}$ and $\nabla_{\alpha\dot\beta}$.

2. Minimal Coupling

One can couple the Yang-Mills system to additional matter multiplets, using the analogue of minimal coupling. Suppose one has a set of Wess Zumino multiplets φ^a, which in flat space satisfy $D_{\dot\alpha}\varphi^a = 0$ and transform under some unitary global representation of the group G. Then $\bar\varphi_a\varphi^a$ is a group scalar, and the action

$$S_{WZ} = \int d^4x\, d^2\theta\, d^2\bar\theta\ \bar\varphi\varphi \tag{27}$$

(where the index is implicit) is supersymmetric and globally invariant under group transformations $\delta\varphi = K\varphi$ for K a Lie algebra valued superfield independent of $x,\theta,\bar\theta$. It is not invariant under local K transformations because the chirality condition $\bar{D}_{\dot\alpha}\varphi = 0$ is not. This can be remedied by generalizing the chirality constraint to $\nabla_{\dot\alpha}\varphi = 0$; i.e., φ is now "covariantly chiral." We then define "covariant components" by

$$\begin{aligned} \varphi(x) &= \varphi\,|_{\theta=0} \\ \psi_\alpha(x) &= (\nabla_\alpha\varphi)\,|_{\theta=0} \\ B(x) &= (\tfrac{1}{2}\nabla^\alpha\nabla_\alpha\varphi)\,|_{\theta=0} \quad , \end{aligned} \tag{28}$$

in analogy with (II.14). The action (27) then becomes

$$S_{WZ} = \frac{1}{4}\int d^4x\, \nabla^\alpha\nabla_\alpha\nabla^{\dot\alpha}\nabla_{\dot\alpha}\, \bar\varphi\varphi$$

$$= \int d^4x \left[-\varphi\Box\varphi - i\bar\psi^{\dot\alpha}\nabla_{\alpha\dot\alpha}\psi^\alpha + \bar B B \right] \quad \text{with } \Box = -\tfrac{1}{2}\nabla^{\alpha\dot\beta}\nabla_{\alpha\dot\beta} = \nabla_\mu\nabla^\mu \,. \tag{29}$$

Various authors [31] have pointed out a relation between the constraint $\nabla_{\dot\alpha}\varphi = 0$ and the super Yang-Mills constraint (9). If we were to allow the more general rule $\{\nabla_\alpha, \nabla_\beta\} = G_{\alpha\beta}$, with $G_{\alpha\beta}$ arbitrary, then $\nabla_{\dot\alpha}\varphi = 0$ would imply φ vanishes:

$$\{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\}\varphi = F_{\dot\alpha\dot\beta}\varphi = 0 \;\Rightarrow\; \varphi = 0 \,. \tag{30}$$

Thus the need for a minimal coupling prescription for scalar multiplets can be regarded as an a priori reason for the constraints (9). This idea is usually referred to as "the preservation of representations": constraints that are consistent at the free level, and which are necessary for physical matter multiplet formulations should remain consistent when gauge covariant derivatives are introduced. It can be used to derive some of the constraints in N=1 supergravity. In the next section we show that the constraints on the covariant derivatives also allow one to find an unconstrained formulation of the Wess-Zumino multiplet that is just a covariantization $D \to \nabla$ of the free result. We then introduce a new idea: "The Yang-Mills constraints are such that the unconstrained formalism for superspace Yang-Mills can be regarded as the covariantization $D \to \nabla$ of the free result." The rest of chapter 3 is devoted to making this concept precise, and testing it on N=1 Yang-Mills. In chapter 4 it is applied to N=2 Yang-Mills.

The idea turns out to be at the heart of the superspace "non-renormalization theorem" of Grisaru and Siegel [32], because it is closely related to the background field method.

3. Unconstrained Formalism

The above formulation of N=1 Yang-Mills uses both constraints and a gauge invariance to remove unwanted component fields. However, to quantize interacting superspace gauge theories, one must find a completely unconstrained formalism where the constrained superfields are expressed in terms of prepotentials.

For the scalar multiplet one can note that

$$\nabla_{\dot\alpha}\varphi = 0 \;\Rightarrow\; \varphi = \nabla^{\dot\beta}\nabla_{\dot\beta}\chi \qquad . \tag{31}$$

Once again, the result $\{\nabla_\alpha, \nabla_\beta\} = 0$ allows one to replace the D operators with covariant ∇'s, since $\nabla^3 = 0$.

However, for the Yang-Mills sector the situation is more complicated. We require a parametrization, using unconstrained superfields, of all covariant derivatives satisfying $\{\nabla_\alpha, \nabla_\beta\} = 0$. One can at this stage regard (14b) as the definition of $\nabla_{\alpha\dot\beta}$, rather than a constraint. (The usual terminology is that (14b) is a "conventional" constraint.)

The way (14a) has customarily been solved [33] is by adapting the spacetime result that the commutator of two covariant derivatives vanishes if and only if they are gauge-equivalent to spacetime partial derivatives,

$$[\nabla_\mu, \nabla_\nu] = 0 \;\Leftrightarrow\; \nabla_\mu = e^K \partial_\mu e^{-K} \quad , \tag{32}$$

where K is some Lie algebra valued real scalar field. In this case, since ∇_α is not real, one can relax the reality restriction and write

$$\{\nabla_\alpha, \nabla_\beta\} = 0 \;\Leftrightarrow\; \nabla_\alpha = e^W D_\alpha e^{-W} \quad \text{and} \quad \nabla_{\dot\alpha} = e^{\overline{W}} \overline{D}_{\dot\alpha} e^{-\overline{W}} \qquad \overline{W} = (W)^* \,. \tag{33}$$

We use D_α instead of ∂_α here because (2) demands $\nabla_\alpha = D_\alpha + \Gamma_\alpha$:

$$e^{W} D_{\alpha} e^{-W} = D_{\alpha} + [W, D_{\alpha}] + \frac{1}{2!}[W,[W,D_{\alpha}]] + \cdots \qquad (34)$$

as required. Thus the prepotential for N=1 Yang-Mills is a complex Lie algebra valued scalar superfield $W = W^i X_i$. Note that only the real part of W can be gauged away by a superspace gauge transformation $\nabla_{\alpha} \to e^{K} \nabla_{\alpha} e^{-K}$ (K real), and the imaginary part has physical significance. One could work in the gauge $W = iV$, V real, and linearizing would then give the free results of sect. (2.7.2).

This approach describes N=1 Yang-Mills theory in superspace. However, at least two issues are not very satisfactory:

(A) Why must one introduce an exponential function, when the coupling to the Wess-Zumino multiplet is so simple?

(B) What is the generalization to extended supersymmetry, and in particular to N=2 Yang-Mills? Simple dimensional arguments rule out an exponential, and there is no analogy to pursue.

The approach we have developed [34] is more deductive and answers these questions. It was discovered in six-dimensional superspace, but turns out to be applicable to many problems. We begin by looking at ordinary space, and the problem of finding a "background field expansion."

4. Background Field Yang-Mills in Spacetime

One of the most useful approaches to quantizing Yang-Mills theories is the background field method [35], in which one divides the vector potential into a "classical piece" and a "quantum piece"

$$\partial_{\mu} + \Gamma_{\mu} = \partial_{\mu} + \mathbf{A}_{\mu} + A_{\mu} = \boldsymbol{\nabla}_{\mu} + A_{\mu} \quad . \qquad (34)$$

Background objects are in larger bold type. This decomposition is such that under gauge transformations

$$\delta\Gamma_\mu = \delta\mathbf{A}_\mu + \delta A_\mu = \nabla_\mu\lambda = \boldsymbol{\nabla}_\mu\lambda + [A_\mu,\lambda] \ . \tag{35}$$

This can be interpreted in two ways:

(a) As a "classical" gauge invariance, under which A_μ transforms covariantly

$$\delta\boldsymbol{\nabla}_\mu = \boldsymbol{\nabla}_\mu\lambda \qquad \delta A_\mu = [A_\mu,\lambda] \quad . \tag{36}$$

(b) As a "quantum" gauge transformation

$$\delta\boldsymbol{\nabla}_\mu = 0 \qquad \delta A_\mu = (\boldsymbol{\nabla}_\mu + A_\mu)\lambda \quad . \tag{37}$$

In a path integral quantization, only A_μ is integrated, and only the transformation (b) causes a problem that requires gauge fixing. However, the gauge may be fixed in such a way that the result is still covariant under type (a) transformations: for instance, one may use the gauge-fixing function $\boldsymbol{\nabla}_\mu A^\mu$ which is covariant under (a). It can thus be shown that a gauge invariant effective action results, and powerful conclusions about renormalization properties can be drawn. For this reason, background field expansions are very useful.

One can carry out such an expansion for the standard Yang-Mills action:

$$S = \int d^4x \Big(-\tfrac{1}{4}\mathbf{F}_{\mu\nu} * \mathbf{F}^{\mu\nu} + A_\mu * (\boldsymbol{\nabla}_\nu \mathbf{F}^{\nu\mu})$$

$$+ \tfrac{1}{2}A^\mu * (g_{\mu\nu}\boldsymbol{\nabla}^2 - \boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu + 2\mathbf{F}_{\mu\nu})A^\nu - \tfrac{1}{2}(\boldsymbol{\nabla}^\mu A^\nu) * [A_\mu,A_\nu] - \tfrac{1}{4}[A_\mu,A_\nu]^2 \Big) \tag{38}$$

obtained by substituting (34) into $S = -\frac{1}{4}\int d^4x\, F_{\mu\nu} * F^{\mu\nu}$. In the special case $\boldsymbol{\nabla}_\mu = \partial_\mu$, $\mathbf{F}_{\mu\nu} = 0$ one gets the familiar expression

$$S = \int d^4x\ \tfrac{1}{2}\, A^\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu - \tfrac{1}{2}\,\partial^\mu A^\nu[A_\mu,A_\nu] - \tfrac{1}{2}\,[A_\mu,A_\nu]^2 \quad . \tag{39}$$

Although the terms in (38) look like a "minimal coupling substitution" $\partial_\mu \rightarrow \boldsymbol{\nabla}_\mu$

in the flat background action (39), this prescription would not be precise because of the ordering ambiguity in non-commuting $\boldsymbol{\nabla}$'s.

There is a nicer way to generate the expansion (38) than just substituting (34) in the action: if we let $A_\mu \to tA_\mu$ where t is a real parameter, (38) is just the Taylor series in t. Moreover, since (34) can be written

$$\nabla_\mu = \boldsymbol{\nabla}_\mu + tA_\mu \quad , \tag{34'}$$

we have

$$\frac{d}{dt}\nabla_\mu = A_\mu \qquad \nabla_\mu(t=0) = \boldsymbol{\nabla}_\mu \tag{40}$$

$$\frac{d}{dt}F_{\mu\nu} = \frac{d}{dt}[\nabla_\mu,\nabla_\nu] = [\frac{d}{dt}\nabla_\mu,\nabla_\nu] + [\nabla_\mu,\frac{d}{dt}\nabla_\nu]$$

$$= [A_\mu,\nabla_\nu] + [\nabla_\mu,A_\nu] = \nabla_{[\mu}A_{\nu]} \quad . \tag{41}$$

Thus

$$\frac{d}{dt} - \frac{1}{4}\int d^4x \; F_{\mu\nu} {*F^{\mu\nu}} = -\int d^4x \; (\nabla_\mu A_\nu) {*(F^{\mu\nu})}$$

$$= \int d^4x \; A_\nu {*(\nabla_\mu F^{\mu\nu})} \quad . \tag{42}$$

Constructing higher derivatives in this way, we expand S in a Taylor series

$$S = S(t) = S(0) + tS'(0) + \frac{t^2}{2}S''(0) + \cdots \tag{43}$$

which reproduces (38).

We now focus attention on eq. (42), giving $\frac{dS}{dt}$. One recognizes the coefficient of A_μ as the field equation tensor, as it must be since by the chain rule

$$\frac{dS}{dt} = \int d^4x \;\; \frac{d\Gamma_\mu}{dt}\,\frac{\delta S}{\delta\Gamma_\mu} \quad , \tag{44}$$

which is a special case of

$$\delta S = \int d^4x \;\; \delta\Gamma_\mu \frac{\delta S}{\delta\Gamma_\mu} \quad .$$

One can use (42) to verify the gauge invariance of S, by substituting $A_\mu \to \nabla_\mu\lambda$.

$$\delta S_{gauge} = \int d^4x \; \nabla_\nu\lambda * \nabla_\mu F^{\mu\nu} = -\int d^4x \; \lambda * \nabla_\nu\nabla_\mu F^{\mu\nu} \quad . \tag{45}$$

This vanishes because the field equation tensor satisfies a Bianchi identity

$$\nabla_\nu(\nabla_\mu F^{\mu\nu}) = \tfrac{1}{2}\nabla_{[\nu}\nabla_{\mu]} F^{\mu\nu} = \; \equiv -\tfrac{1}{2}[F_{\mu\nu}, F^{\mu\nu}] = 0 \quad . \tag{46}$$

Thus the equation for $\frac{dS}{dt}$ exhibits neatly the relation between the constrained and unconstrained formulations of the theory. The constrained formulation uses the covariant objects ∇ and F, which can have a direct physical interpretation; e.g., in electrodynamics the components of $F_{\mu\nu}$ are the electric and magnetic fields. However, these objects are constrained by Bianchi identities. On the other hand, the unconstrained formulation uses gauge fields with no direct physical interpretation. The formulations are "dual" to each other in that

(a) The external indices carried by the gauge field are the "transpose" of those carried by the field equation; i.e., the two can be contracted to give a scalar.

(b) The gauge transformation is the transpose of the Bianchi identity satisfied by the field equation, in the sense of (45).

While these observations are elementary in this case, there are situations, particularly in superspace, where no manifestly covariant action is available. One can then use this duality to construct a covariant action.

5. Example: Three-dimensional Mass Term

As a simple example in ordinary space, consider Yang-Mills theory in three-dimensional spacetime. In the abelian case, a mass type term is available for the action:

$$S_m = \frac{m}{g^2}\int d^3x \ \varepsilon^{\mu\nu\sigma}\,\Gamma_\mu\partial_\nu\Gamma_\sigma = \frac{m}{2g^2}\int d^3x \ \varepsilon^{\mu\nu\sigma}\,\Gamma_\mu F_{\nu\sigma} \tag{47}$$

This is invariant under abelian gauge transformations because $F_{\mu\nu}$ is invariant, so $\delta\Gamma_\mu = \partial_\mu\lambda$ gives

$$\delta S_m = -\frac{m}{g^2}\int d^3x \ \lambda\,\varepsilon^{\mu\nu\sigma}\partial_\mu F_{\nu\sigma} = 0$$

because $\partial_{[\mu}F_{\nu\sigma]} = 0$ by the Bianchi identity. This reasoning breaks down for nonabelian transformations because F is no longer invariant, only covariant,

$$\delta F_{\mu\nu} = [F_{\mu\nu},\lambda] \quad .$$

However, we recognize the cancellation mechanism as that found in $\frac{dS}{dt}$. Moreover, one can regard (47) as the definition of $\frac{1}{2}\,\frac{dS_m}{dt}$ rather than S_m since it is quadratic.

Thus suppose a gauge invariant S_m exists in the nonabelian case, and suppose that making the definitions

$$\frac{d}{dt}\nabla_\mu = A_\mu \qquad\qquad \nabla_\mu|_{t=0} \equiv \boldsymbol{\nabla}_\mu \tag{48}$$

and differentiating S_m with respect to t gives

$$\frac{dS_m}{dt} = \int d^3x A_\mu * \varepsilon^{\mu\nu\sigma}F_{\nu\sigma} \quad . \tag{49}$$

This is consistent with a gauge invariant S_m, since gauge invariance is manifested in $\frac{dS}{dt}$ as invariance under the substitution $\delta A_\mu = \nabla_\mu\lambda$ *while holding all* *covariant objects fixed* . Thus even in the nonabelian case,

$$\delta \frac{dS_m}{dt} = \int d^3x \ \nabla_\mu \lambda * \varepsilon^{\mu\nu\sigma} F_{\nu\sigma} \ = \ -\int d^4x \ \lambda \ * \ \varepsilon^{\mu\nu\sigma} \nabla_\mu F_{\nu\sigma} \ = \ 0$$

by the Bianchi identity.

Eqs. (48-49) define S_m up to a piece depending only on the background, since by repeatedly differentiating (48) with respect to t to find the higher derivatives of S_m one can obtain

$$S_m = \mathbf{S}_m + tr \int d^3x \ \varepsilon^{\mu\nu\sigma} (A_\mu * \mathbf{F}_{\nu\sigma} + \tfrac{1}{2} A_\mu * \boldsymbol{\nabla}_{[\nu} A_{\sigma]} + \tfrac{1}{3} A_\mu * [A_\nu, A_\sigma]). \quad (50)$$

$\boldsymbol{\nabla}_\nu$ is the arbitrary background covariant derivative, and the expression is invariant under both

$$(1) \qquad \delta A_\mu = \boldsymbol{\nabla}_\mu \lambda + [A_\mu, \lambda] \qquad \delta \boldsymbol{\nabla}_\mu = 0 \qquad \text{(quantum transformation)} \quad (51)$$

and

$$(2) \qquad \delta \boldsymbol{\nabla}_\mu = \boldsymbol{\nabla}_\mu \lambda \qquad \delta A_\mu = [A_\mu, \lambda] \qquad \text{(classical transformation).} \quad (52)$$

Specializing to an empty space background gives

$$S_m = \int d^3x \ \varepsilon^{\mu\nu\sigma} (A_\mu * \partial_\nu A_\sigma + \tfrac{1}{3} A_\mu * [A_\nu, A_\sigma]) \quad , \qquad (53)$$

invariant under

$$\delta A_\mu = (\partial_\mu + A_\mu) \lambda \ , \qquad (54)$$

as can be checked explicitly. In the abelian case all commutators vanish and we get our original expression

$$\int d^3x \, \varepsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma = \tfrac{1}{2} \int d^3x \, \varepsilon^{\mu\nu\sigma} A_\mu F_{\nu\sigma} \quad . \qquad (55)$$

Thus we have a prescription for generalizing expressions invariant at the linearized level to fully gauge invariant quantities.

6. N=1 Yang-Mills Theory in Four Dimensions

One can adapt this reasoning to N=1 Yang-Mills theory, as described by the constrained covariant derivatives (14). If we look for a background field formulation by putting $\nabla_\alpha = \boldsymbol{\nabla}_\alpha + \Gamma_\alpha$ where $\boldsymbol{\nabla}_\alpha$ is a background covariant derivative which also satisfies (14), then, unlike the case in the previous section, Γ_α must satisfy the nonlinear constraint

$$\{ \boldsymbol{\nabla}_\alpha + \Gamma_\alpha , \boldsymbol{\nabla}_\beta + \Gamma_\beta \} = 0 \quad => \quad \boldsymbol{\nabla}_{(\alpha}\Gamma_{\beta)} + \{ \Gamma_\alpha , \Gamma_\beta \} = 0 \quad , \tag{56}$$

and is thus unsuitable for path integral quantization. Nevertheless, if the second term were absent, one could "solve" for Γ_α by putting

$$\Gamma_\beta = \boldsymbol{\nabla}_\beta W \qquad , \tag{57}$$

and W would be an unconstrained field. ($\boldsymbol{\nabla}_{(\alpha}\boldsymbol{\nabla}_{\beta)} W = \{ \boldsymbol{\nabla}_\alpha , \boldsymbol{\nabla}_\beta \} W = 0$.) However, since the background is arbitrary, one can in this way find an unconstrained parameterization of an infinitesimal neighbourhood of any covariant derivative satisfying the Yang-Mills constraints (14). We need eventually to extend this to a parameterization of a finite neighbourhood, but first we investigate the properties of the infinitesimal parameterization.

We thus examine the structure of small perturbations of N=1 Yang-Mills, $\nabla_\alpha \to \nabla_\alpha + \delta\nabla_\alpha = \nabla_\alpha + \delta\Gamma_\alpha$ where $\delta\Gamma_\alpha$ is a Lie algebra valued superfield. (We prefer to write $\delta\nabla_\alpha$ instead of $\delta\Gamma_\alpha$ because the former is manifestly gauge invariant. However, note that $\delta\nabla_\alpha$, unlike ∇_α, has no derivative piece, and is a superfield.) The following reasoning is given in detail because it is adapted to

N=2 Yang-Mills in chapter 4.

One must only allow those perturbations which preserve the constraints (14), since any others correspond to degrees of freedom outside the N=1 Yang-Mills multiplet. As mentioned, the only constraint that is not just a definition is (14a), so

$$\delta\{\nabla_\alpha, \nabla_\beta\} = \nabla_{(\alpha}\delta\Gamma_{\beta)} = 0$$

characterizes the "allowed" perturbations. One such perturbation is the gauge transformation

$$\delta\Gamma_\alpha = \nabla_\alpha K \equiv [\nabla_\alpha, K] = e^{-K}\nabla_\alpha e^{K} - \nabla_\alpha$$

for an infinitesimal real Lie algebra valued superfield K, which must be allowed because the constraint is a priori covariant.

Explicitly,

$$\nabla_{(\alpha}\delta\nabla_{\beta)} = \nabla_{(\alpha}\nabla_{\beta)}K = \{\nabla_\alpha, \nabla_\beta\}K = 0 \ . \tag{58}$$

The gauge variation suggests another allowed variation $\delta\nabla_\alpha = \nabla_\alpha(iZ)$, where iZ is imaginary and Lie algebra valued. By expanding in components one can then show that this exhausts the degrees of freedom, so the most general allowed variation of ∇_α can be written as

$$\delta\nabla_\alpha = \nabla_\alpha Y_\Delta + \nabla_\alpha iZ_\Delta = \nabla_\alpha(Y_\Delta + iZ_\Delta) \tag{59}$$

for some Y_Δ and Z_Δ. Y_Δ and Z_Δ are called covariant variations, and are unconstrained. For given $\delta\nabla_\alpha$, Y_Δ and Z_Δ are not unique:

$$Y_\Delta + iZ_\Delta \ \rightarrow \ Y_\Delta + iZ_\Delta + i\nabla^2\lambda \tag{60}$$

for arbitrary λ is an invariance of (59). Separating real and imaginary parts gives

$$\delta Y_\Delta = \nabla^2\lambda + \overline{\nabla}^2\overline{\lambda} \qquad \delta Z_\Delta = i(\nabla^2\lambda - \overline{\nabla}^2\overline{\lambda}) \tag{61a,b}$$

The δY_Δ piece is just a special case of the gauge invariance $\delta Y_\Delta = K$, but the δZ_Δ piece is a new gauge invariance arising because we have solved a constraint in terms of an unconstrained field.

One way of checking these ideas is to assume that one is linearizing about empty space, so that $\nabla_\alpha \to D_\alpha$, and interpret Y_Δ and Z_Δ as fields U and V in the theory. One uses the K gauge invariance of the theory to set $U = 0$, so that only V appears. The remaining gauge invariance is just (61a). (61b) is now irrelevant, as it can always be compensated by a K gauge transformation to maintain $U = 0$. Thus the unconstrained prepotential in linearized N=1 Yang-Mills is a single real scalar superfield V, as claimed in sect. (2.7.2). However, the field U also plays a role, absorbing the K gauge transformations and allowing one to use λ transformations instead. It is the mechanism by which the theory converts from the part-constrained, part-gauged form (1) which we forced upon it, into the pure unconstrained form. Any field like U that can be completely gauged away is called a compensating field, and while they are technically redundant, it is often more convenient to leave them in expressions to make both types of gauge invariance manifest.

Returning to the general case, we have a parameterization of an infinitesimal neighbourhood of any covariant derivative by the unconstrained objects Y_Δ and Z_Δ. How do we obtain a parameterization of a finite neighbourhood? The solution to an analogous problem is actually well known: in the theory of Lie groups one wishes to take a parameterization of the tangent space (i.e., the Lie algebra) and produce a coordinate system for a finite region of the group.

Recall how this is done. Suppose we seek to find an unconstrained parametrization of $U(N)$, i.e., all complex $N\times N$ matrices $\mathbf{U}$ satisfying $\mathbf{U}\,\mathbf{U}^\dagger = 1$. Vary this to get a constraint on allowed infinitesimal transformations

$$\delta\mathbf{U}\,\mathbf{U}^\dagger + \mathbf{U}\,\delta\mathbf{U}^\dagger = 0 \quad . \tag{62}$$

This can be satisfied for fixed $\mathbf{U}$ by allowing $\delta\mathbf{U}\,\mathbf{U}^\dagger$ to be any antihermitian matrix A: $\delta\mathbf{U} = A\,\mathbf{U}$ with $A = -A^\dagger$. Choose a fixed $\mathbf{U}_0$ to be the coordinate origin. Next, introduce a parameter t, and for some fixed A apply consecutively many infinitesimal transformations $A\,dt$. We thus produce a curve $\mathbf{U}(t)$ in the space of $N\times N$ complex matrices, such that

$$\mathbf{U}(0) = \mathbf{U}_0 \qquad\qquad d\,\mathbf{U}(t) = A\,\mathbf{U}(t)\,dt \tag{63a,b}$$

Every matrix on this curve satisfies the constraint $\mathbf{U}\,\mathbf{U}^\dagger = 1$ since

$$\mathbf{U}(0)\,\mathbf{U}^\dagger(0) = \mathbf{U}_0\,\mathbf{U}_0^\dagger = 1$$

is given and

$$\frac{d}{dt}\mathbf{U}(\mathbf{t})\,\mathbf{U}^\dagger(t) = \frac{d\,\mathbf{U}}{dt}\mathbf{U}^\dagger + \mathbf{U}\left[\frac{d\,\mathbf{U}}{dt}\right]^\dagger = A\,\mathbf{U}\,\mathbf{U}^\dagger + \mathbf{U}\,\mathbf{U}^\dagger A = A + A^\dagger = 0 \quad .$$

One can solve eq.(63) exactly :

$$\mathbf{U}(t) = e^{tA}\,\mathbf{U}_0 \quad . \tag{64}$$

The parameter t can then be absorbed into the matrix A, and one can write

$$\mathbf{U}(A) = e^{A}\,\mathbf{U}_0 \quad . \tag{65}$$

Thus as A ranges over all antihermitian matrices, $\mathbf{U}(A)$ ranges over all unitary matrices in some neighbourhood of $\mathbf{U}_0$. One knows at least one group element (the identity) explicitly, so one can specialize to $\mathbf{U}_0 = 1$, $\mathbf{U}(A) = e^A$, and obtain a full description of the connected part of the group.

Some details have been glossed over here. First, in writing equation (63a,b), one could allow A to be a function of t. However, this is unnecessary. Since the set of constant A's already parameterizes the neighbourhood, allowing A to be t-dependent would mean that many different functions $A(t)$ led to the same $\mathbf{U}(t=1)$. Thus a large unwanted "gauge invariance"

$$A(t) \to A(t) + \delta A(t)$$

would be introduced. Nevertheless, by allowing restricted types of t dependence, one can find different parameterizations of the neighbourhood. The essential idea is to associate with each antihermitian A an extension $A(t)$, with $A(0) = A$. Solving (63) will then still map each A to a unique $\mathbf{U}(1)$. An elegant way to assign a curve $A(t)$ to each each A is to choose an arbitrary first-order differential equation for A(t), using $A(0) = A$ as the initial condition. As an example, let $\mathbf{B}$ be an arbitrary but fixed antihermitian matrix, and demand

$$\frac{d}{dt} A(t) = [\mathbf{B}, A(t)] \qquad A(0) = A \quad . \tag{66}$$

Then solving the coupled equations (63) and (66) gives

$$A(t) = e^{Bt} A e^{-Bt} \qquad\qquad U(t) = e^{Bt} e^{(A-B)t}\, U_0 \quad . \tag{68}$$

This is another parametrization of the group, not very different from the one in eq. (64).

The second detail is that arbitrariness in the solution (63) exists, since one could have put $\delta \mathbf{U} = \mathbf{U}A$ and obtained $\mathbf{U} = \mathbf{U}_0 e^{At}$. Nevertheless, one always obtains an exponential solution, because (63b) is linear in $\mathbf{U}$.

We do not wish to overemphasise the connection between Lie group theory and super Yang-Mills formalism. The important idea is that one can get from the infinitesimal linear theory to the full nonlinear theory by introducing a parameter t, and noting that all the relevant properties can be written as equations local in t.

One may adapt this method to solving (14) by writing

$$\frac{d}{dt}\nabla_\alpha(t) = \nabla_\alpha(t)(u(t) + iv(t)) \qquad \nabla_\alpha(0) = \boldsymbol{\nabla}_\alpha \quad , \tag{69}$$

$$u(0) + iv(0) = U + iV \quad . \tag{70}$$

$\frac{du}{dt}$ and $\frac{dv}{dt}$ must still be specified, and the most straightforward way is to take u,v independent of t, in which case (69) is soluble in closed form: noting that $\nabla_\alpha(t)(u+iv) = -[(u+iv),\nabla_\alpha(t)]$, so that (69) is just Schrodinger's equation, gives

$$\nabla_\alpha = e^{-(U+iV)}\boldsymbol{\nabla}_\alpha e^{(U+iV)} \tag{71}$$

at $t=1$, where $\boldsymbol{\nabla}_\alpha$ is a background covariant derivative satisfying (14). This solution is familiar as "superspace Yang-Mills in the vector representation"[36]. Another form may be found by taking an arbitrary Lie algebra valued superfield H and setting

$$\frac{d}{dt}(u+iv) = [H,u+iv] \quad , \tag{72}$$

which gives

$$u + iv = e^{Ht}\,(U+iV)\,e^{-Ht} \tag{73}$$

$$\nabla_\alpha(t) = e^{Ht}\,e^{-(U+iV+H)t}\,\nabla_\alpha\,e^{(U+iV+H)t}\,e^{-Ht} \quad . \tag{74}$$

In particular, putting $H = -U$ and absorbing the t gives the useful form

$$\nabla_\alpha = e^{-U}e^{-iV}\nabla_\alpha e^{iV}e^{U} \quad , \tag{75}$$

in which it is manifest that U may be completely gauged away by a K gauge transformation $\nabla_\alpha \to e^{K}\nabla_\alpha e^{-K}$. In this form, the solution has many of the convenient properties of the "chiral representation" [36].

The analogy with canonical coordinates for Lie groups breaks down when we examine "quantum gauge transformations." The fundamental difference between the equations $\frac{d}{dt}u = uA$ and $\frac{d}{dt}\nabla_\alpha = \nabla_\alpha(u+iv)$ is that ∇_α is not an invertible operator. This is the origin of the quantum gauge transformations: covariant variations of the form $Y_\Delta + iZ_\Delta = \nabla^2\lambda$ leave all covariant objects invariant. However, if we wish to quantize the theory and maintain unitarity, it is necessary to know which field variations δU and δV are invariances of the theory, in order to construct a ghost action. These are difficult to specify because the problem is nonlocal in t, unlike the others we examine. V is defined at $t=0$, and we wish to find those δV's equivalent to a K gauge variation at $t=1$. This is a non-trivial problem which is soluble[37] in this case because the explicit solution (75) is available. The solution is

$$\delta V = \frac{i}{2}\,V(\cot\,V)\,(\bar\Lambda - \Lambda) + \frac{1}{2}\,V(\bar\Lambda + \Lambda) \tag{76}$$

where $(\cot V)$ is defined by its power series expansion, and Λ is a background-chiral superfield, $\bar\nabla_{\dot\alpha}\Lambda = 0$. However, no method which can be generalized to the N=2 case is known for obtaining (76).

Let us discuss the easier but also important problem of writing the action in terms of unconstrained fields. First, we take the constrained form of the action, eq. (24), and make a constrained variation $\delta\nabla_\alpha = \nabla_\alpha(Y_\Delta + iZ_\Delta)$, giving

$$\delta S = \tfrac{1}{4}\int d^4x\, d^2\theta\, (\delta W_\alpha) * W^\alpha \quad . \tag{77}$$

$\delta\nabla_{\alpha\dot\beta}$ and δW_α can be evaluated:

$$\delta\nabla_{\alpha\dot\beta} = -i\delta\{\nabla_\alpha, \nabla_{\dot\beta}\} = -i\nabla_\alpha\nabla_{\dot\beta}(Y_\Delta - iZ_\Delta) - i\nabla_{\dot\beta}\nabla_\alpha(Y_\Delta + iZ_\Delta)$$

$$= \nabla_{\alpha\dot\beta}\, Y_\Delta - (\nabla_\alpha\nabla_{\dot\beta} - \nabla_{\dot\beta}\nabla_\alpha)\, Z_\Delta \tag{78}$$

and

$$\delta W_\alpha = W_\alpha \cdot (Y_\Delta - iZ_\Delta) - (\bar\nabla^2 \nabla_\alpha Z_\Delta) \quad . \tag{79}$$

Then some algebra gives

$$\delta S = -\tfrac{1}{4}\int d^4x\, d^2\theta\ (\bar\nabla^2\nabla_\alpha Z_\Delta) * W_\alpha \tag{80}$$

and because $\nabla_{\dot\alpha} W_\beta = 0$ we can turn the $\bar\nabla^2$ into a $d^2\bar\theta$ and obtain

$$\delta S = \tfrac{1}{4}\int d^4x\, d^2\theta\, d^2\bar\theta\ (Z_\Delta) * (\nabla_\alpha W^\alpha) \quad . \tag{81}$$

Unlike the constrained action, this is an integral over the whole superspace. Also, it can be deduced without using eqs. (76-79) by noting that

(1) The field equation $\nabla_\alpha W^\alpha$ occurs as the coefficient of the unconstrained covariant variation Z_Δ.

(2) Y_Δ does not occur in δS because it is a compensating field whose variations amount to gauge transformations, under which S is invariant.

(3) The field equation satisfies a Bianchi identity $\nabla_\alpha W^\alpha = \overline{\nabla}_{\dot\alpha} \overline{W}^{\dot\alpha}$, which is dual to the gauge invariance $Z_\Delta = \nabla^2\lambda + \overline{\nabla}^2\overline{\lambda}$ for arbitrary λ

$$\delta S = \frac{1}{4}\int d^4x\, d^2\theta\, d^2\overline{\theta}\;\; (\nabla^2\lambda + \overline{\nabla}^2\overline{\lambda}) * \nabla_\alpha W^\alpha$$

$$= \frac{1}{4}\int d^4x\, d^2\theta\, d^2\overline{\theta}\; (\lambda * \nabla^2\nabla_\alpha W^\alpha + \text{c.c.}) = 0 \tag{82}$$

The result allows us to write down an expression that determines the action formula in terms of the unconstrained prepotential:

$$\frac{dS}{dt} = \frac{1}{4}\int d^4x d^2\theta\, d^2\overline{\theta}\;\; v * (\nabla_\alpha W^\alpha) \tag{83}$$

by the same process as above. Note that none of the arbitrariness in the exact definition of v, eq. (72) say, enters, and the expression is unique up to an overall constant factor. One can now obtain higher t derivatives and expand S about the background. We show only one more derivative:

$$\frac{d^2S}{dt^2} = \frac{1}{4}\int d^4x\, d^2\theta\, d^2\overline{\theta}\;\; \frac{dv}{dt} *(\nabla_\alpha W^\alpha) + v * \frac{d}{dt}(\nabla_\alpha W^\alpha)$$

$$= \frac{1}{4}\int d^4x\, d^2\theta\, d^2\overline{\theta}\;\; (\frac{dv}{dt} + [u,v]) *(\nabla_\alpha W^\alpha) + v *(\nabla_\alpha \overline{\nabla}^2\nabla^\alpha + 2i W^\alpha\nabla_\alpha)v \tag{84}$$

There is a simplification if one chooses u and v to satisfy $\frac{dv}{dt} + [u,v] = 0$, the same relation that yields the simple form (75) for the covariant derivative. Expanding about $t = 0$ and absorbing t into V gives

$$S = \mathbf{S} + \frac{1}{4}\int d^4x\, d^2\theta\, d^2\overline{\theta}\;\; (V * \boldsymbol{\nabla}_\alpha \mathbf{W}^\alpha$$

$$+ \frac{1}{2}\, V * (\boldsymbol{\nabla}_\alpha \overline{\boldsymbol{\nabla}}^2 \boldsymbol{\nabla}^\alpha + 2i\, \mathbf{W}^\alpha \boldsymbol{\nabla}_\alpha)\, V + \cdots) \tag{85}$$

One generates S without ever using the solution (75) explicitly. For perturbation theory such an expansion is all one needs. Setting $\boldsymbol{\nabla}_\alpha = D_\alpha$ gives the linearized results of the previous chapter. The fact that U does not appear in (85) is verification that S is gauge-invariant. If (75) were substituted into (24), U would cancel, since it has the same effect as a K gauge transformation. The action (83), involving only V, is known as the "chiral" form of the N=1 Yang-Mills action. We have derived it without using solution (71), the "vector representation," explicitly, which is a useful thing to be able to do, the method being applicable to the much more complicated case of N=1 superfield supergravity.

The main ideas of sections (3.4,3.5,3.6) can be summarized:

[1] To find an unconstrained formulation of any gauge theory, it is sufficient to find the linearized unconstrained formulation with arbitrary background fields.

[2] The linearized solution can be extended to a full nonlinear solution by a method analogous to exponentiation of a Lie algebra. The theory is thus formulated in terms of an unconstrained (pre)potential and arbitrary background covariant derivatives and field strengths.

[3] In formulating the theory this way, one obtains simple expressions for derivatives of covariant objects with respect to a parameter t. These expressions are "covariantizations" $\partial, D \rightarrow \nabla$ of those for the linearized theory in empty space.

[4] One can also write down a simple expression for the t-derivative of the action, which in the superfield case is an integral over the whole superspace. The derivative of the action is simpler than the action itself.

[4] These differential equations in t can be solved to obtain the full nonlinear expressions.

[6] Background "classical" gauge invariance is manifest, but the nonlinear "quantum" gauge transformation of the prepotential is difficult to determine beyond lowest order.

Chapter 4

SIX-DIMENSIONAL SUPERSYMMETRY

Until now we have had little to say about extended supersymmetry, mainly because superfield methods have told us little about the subject that could not be derived more easily by standard component methods. One technique employed very successfully in the component approach is dimensional reduction [37], the idea that a field theory in a higher dimensional spacetime defines a class of similar field theories in four dimensions, or more generally in any lower dimension. The approach is particularly fruitful in supersymmetry theory and supergravity theory, because unextended ("simple") supersymmetry in a higher dimension becomes extended supersymmetry in the lower dimension. Indeed, many extended theories were originally obtained by this method.

The approach has had limited success in superspace, mainly because the four-dimensional formulations rely heavily on the concept of chiral superfields, which does not have a convenient analogue in higher dimensions. There have been earlier attempts to study higher-dimensional superspaces [38], but they were not successful enough to warrant preferring these to four-dimensional extended superspace. However, the extended superspace treatments remained at the constrained level in most cases, and no unconstrained interacting theories were known.

For these reasons, it was suggested by Siegel [39] that a program be undertaken to construct a theory of six-dimensional superspace, which would then reduce to an N=2 superspace in four dimensions. The theory is specific to six dimensions, in the same way that any approach based on SL(2,C) must be specific to four dimensions. In this way, one might hope to see which features of the superspace approach were generic. In particular, we hoped to obtain an unconstrained formalism for at least one interacting extended multiplet.

In this chapter I describe the results of my effort. Some of the results were obtained independently by a group at Imperial College and Ecole Normale Supérieure [40].

1. Six-dimensional Supersymmetry Algebras

Superfield calculations in d=4 are simpler in SL(2,C) notation, which obviates having to use Dirac matrices. This is due to the fact that the smallest spinor representation of SO(3,1) is the fundamental representation of SL(2,C), and that the vector can be expressed as a product of two spinor representations (with a suitable reality condition on the vector). This eliminates the need for Dirac matrices, which are Clebsch-Gordon coefficients necessitated by an inexpedient choice of basis for the vector representation.

Actually, for all dimensions up to six an analogous result holds: the smallest spinor representation is the fundamental representation of a convenient classical matrix group, and the vector can be formed by just symmetrizing, antisymmetrizing or tracing products of spinors. For d>6 the product of two spinors needs more than a GL(N) decomposition to produce irreducible SO(d-1,1) representations and one cannot avoid introducing Clebsch-Gordon coefficients.

Thus we look for a spinor notation in six dimensions, which is the analogue of the SL(2,C) notation of d=4.

1.1. Spinors in d=6

Here we recast six-dimensional spinor theory [41] in a form without Clebsch-Gordon coefficients. In any even dimension we can begin with Dirac spinors, which are direct sums of left- and right-handed Weyl spinors

$$\Psi_D = \begin{bmatrix} \Psi_L \\ \Psi_R \end{bmatrix} .$$

In d=4 these turn out to be the defining (fundamental) representation of SL(2,C) and its complex conjugate, denoted by dotted and undotted indices, respectively,

$$\Psi_D^{(4)} = \begin{bmatrix} \psi_\alpha \\ \bar{\xi}_{\dot{\alpha}} \end{bmatrix} .$$

From this one can construct the entire SL(2,C) formalism used in the previous chapters. Alternatively, it is also SL(2,C) covariant to require $\bar{\xi}_{\dot{\alpha}} = (\psi_\alpha)^*$, in which case the spinor is Majorana, and there is also a formalism based on such spinors.

In comparison, in d=6 the left-handed Weyl spinors are in the fundamental representation of the noncompact group $SU^*(4)$, and we can denote such objects by upper-case subscripts Ψ_A. Right-handed Weyl spinors are denoted Ξ^A since this representation is the transpose of the left-handed one, making contraction of upper and lower indices covariant (see the appendix for more details). There is no way to raise or lower Weyl spinor indices in d=6 as we can in d=4. Both these representations are pseudo-real in the sense that while they are equivalent to their complex conjugate representations (i.e., the antiparticle of a left-handed spinor is left-handed), it is inconsistent to demand that they be real. The noncompact group $SU^*(4)$ differs from SU(4) in this respect.

Thus, because ψ_A and ξ^A are equivalent to their complex conjugates $\overline{\psi}_{\dot{A}} = (\psi_A)^*$ and $\overline{\xi}^{\dot{A}} = (\xi^A)^*$, we can form

$$\overline{\psi}_A \equiv C_A{}^{\dot{B}}\, \overline{\psi}_{\dot{B}} \tag{1}$$

$$\overline{\xi}^A \equiv C^A{}_{\dot{B}}\, \overline{\xi}^{\dot{B}} \quad .$$

Pseudo-reality means that $\qquad (C_A{}^{\dot{B}})^* = -(C^{-1})_{\dot{A}}{}^B$

and we can always choose $\qquad C^A{}_{\dot{B}} = (C^{-1})_{\dot{B}}{}^A \quad , \quad \det C = 1 \qquad (2)$

so that $\overline{\psi}_{\dot{A}}\, \overline{\xi}^{\dot{A}} = \overline{\psi}_A\, \overline{\xi}^A$.

A d=6 Lorentz vector is an antisymmetric SU*(4) tensor $V_{AB} = -V_{BA}$, with the reality condition $V_{AB} = \overline{V}_{AB} \equiv C_A{}^{\dot{C}}\, C_B{}^{\dot{D}}\, (V_{CD})^*$ which is now consistent because V has an even number of indices. Note that the number of components is correct: 4×3/ 2 *real* = 6. Thus we can dispense with SO(5,1) notation and use SU*(4) notation throughout. Some of the correspondences are shown in Table 1. SU*(4) is the local covering group of SO(5,1) in the same way that SL(2,C) is the local covering group of SO(3,1).

We can define the operation "bar," which is complex conjugation, followed by multiplication by C to get back to our canonical (undotted) representation. For any SU*(4) tensor $W_{AB\ldots C}$ we define

$$\overline{W}_{AB\ldots C} \equiv C_A{}^{\dot{D}}\ C_B{}^{\dot{E}} \cdots C_C{}^{\dot{F}}\ (W_{DE\ldots F})^* \quad , \tag{3}$$

and similarly for any combination of up and down indices. In general for a tensor with l indices

$$\overline{\overline{W}} = (-1)^l\ W \quad . \tag{4}$$

Finally, note that ε^{ABCD} and ε_{ABCD} are $SU^*(4)$ invariant tensors, so we can raise and lower antisymmetrized indices with them. For convenience we raise from the left and lower from the right, i.e.,

$$W^{AB} = \frac{1}{2}\varepsilon^{ABCD}\,W_{CD} \qquad W_{AB} = W^{CD}\,\frac{1}{2}\varepsilon_{CDAB} \quad , \tag{5}$$

and because of convention (1), raising and lowering indices commutes with barring. There is also a trivial but useful identity

$$\delta_F^{[A}\varepsilon^{BCDE]} = 0 \qquad ;$$

i.e., antisymmetrizing on any 5 $SU^*(4)$ indices gives 0. This is sometimes useful for shuffling indices.

A useful trick [42] in dealing with pseudo-real objects is to form a doublet ψ_{aA} of $\psi_{1A} = \psi_A$ and $\psi_{2A} \propto \overline{\psi}_A$. This definition is in fact SU(2) covariant: if we let

$$\overline{\psi}_{aA} = \overline{\psi}^{b}_{A}\, C_{ba} \qquad (\, C_{ba} = -C_{ab} \,) \quad , \tag{6}$$

it becomes just

$$\overline{\psi}_{aA} = \psi_{aA} \qquad . \tag{7}$$

We have here in effect put two pseudo-real representations together and put a (now consistent) reality condition on the result. The use of such spinors in six dimensions is advantageous for two reasons. First, it obviates having to use complex conjugation in any expression, providing many of the benefits of the d=4 Majorana representation, and secondly, for N=2 at least, it is this SU(2) which becomes the SU(2) chiral symmetry of N=2 d=4 supersymmetry after dimensional reduction. Some previous attempts [43] to use superfields in six dimensions have tried using an analogue of the d=4 chiral constraint, but this was unsuccessful and leads via dimensional reduction to unconventional extended superfields without any SU(2) symmetry. While there is no a priori reason to insist on an SU(2) covariance in d=6, it arises so naturally that we will try to maintain it. In this notation, the actions for spinors ψ_{aA} and ξ^{aA} are

$$\int d^6x \; i\, \psi^a{}_A\, \partial^{AB}\, \psi_{aB} \qquad \text{and} \qquad \int d^6x \; i\, \xi^{aA}\, \partial_{AB}\, \xi_a{}^B \quad , \tag{8a,b}$$

respectively.

1.2. Supersymmetry Algebras

The simple (N=2) supersymmetry algebra in flat six-dimensional superspace can now be written down by matching up indices, once the basic relation $\{Q, Q\} \approx \partial$ is assumed. Thus

$$\{ Q_{aA} \, , Q_{bB} \} = -i\, C_{ab}\, \partial_{AB} \quad , \tag{9}$$

where $Q_{aA} = \overline{Q}_{aA}$ is the supersymmetry generator and $\partial_{AB} = -\partial_{BA} = \overline{\partial}_{AB}$ is the $SU^*(4)$ version of the $SO(5,1)$ ∂_μ.

As in d=4 though, the Q's are never used explicitly, and one can formulate everything in terms of a set of spinorial derivatives D_{aA} satisfying

$$\{ D_{aA} \, , Q_{bB} \} = 0 \qquad \{ D_{aA} \, , D_{bB} \} = i\, C_{ab}\, \partial_{AB} \qquad a, b = 1,2 \tag{10}$$

$$\overline{D}_{aA} = D_{aA} \quad .$$

This follows uniquely up to real factors from Lorentz, SU(2) and complex conjugation covariance, but can also be derived straight from the known Dirac spinor expression. One cannot absorb the factor i into C_{ab} since that changes the definition of $\overline{D}_{aA} = (D^b{}_A)\, C_{ba}$. However, we can fix the sign by absorbing any minus into ∂_{AB}. We must be careful to represent P^0 by a positive definite operator, but this can always be arranged by choosing $P^\mu = \pm i \partial^\mu$ appropriately.

One can generalize this to six-dimensional N-extended supersymmetry, for N even, but it must be noted that off shell one may have left- or right-handed supersymmetry in six dimensions, and that these are not in general equivalent, so one must specify separately the number of left-handed D_A's and right-handed D^A's. Considering first the purely left-handed case, it is easy to see that the above discussion of SU(2) can be generalized to the group USp(N):

$$\overline{D}_{aA} \equiv \overline{D}^b{}_A C_{ba} = D_{aA}$$

$$\{ D_{aA} , D_{bB} \} = i\, C_{ab}\, \partial_{AB} \qquad a,b = 1,\ldots,N \tag{11}$$

where $C_{ab} = -C_{ba} = (C^{ab})^*$ is the invariant antisymmetric tensor of USp(N), $C_a{}^b \equiv \delta_a^b$, and the whole algebra is covariant under this external USp(N) (note USp(2) = SU(2)). Thus one discovers a subgroup of the four-dimensional SU(N) symmetry to be present even in six dimensions.

One can do the same thing for any number of right-handed D^A's, so the most general supersymmetry algebra in d=6 is

$$\{ D_{aA} , D_{bB} \} = i\, C_{ab}\, \partial_{AB} \qquad \{ D_m{}^A , D_n{}^B \} = i\, C_{mn}\, \partial^{AB} \quad . \tag{12}$$

One could also try adding a central charge

$$\{ D_{aA} , D_m{}^B \} = \delta_A^B\, Z_{am} \quad , \tag{13}$$

but there are then non-trivial Jacobi identities to be satisfied, and the left-hand group ('a' indices) can be related to the right-hand one ('m' indices). In the absence of central charges, however, the left- and right-handed D's can be treated independently. There is no analogue of a chiral condition, since for any superfield Φ, $D_m{}^B \Phi = 0$ implies that $\partial^{AB}\Phi = 0$.

Superspace and superfields are introduced just as in four dimensions, and we define superfield components by operating various products of covariant spinorial derivatives on the superfield and evaluating the result at $\theta = 0$.

1.3. Dimensional Reduction

To carry out dimensional reduction, we may choose a representation in which $\psi_{aA} = \begin{pmatrix}\psi_{a\alpha} \\ \overline{\psi}_{a\dot{\alpha}}\end{pmatrix}$ $\alpha, \dot{\alpha} = 1,2$ showing the transformation properties under the SL(2,C) subgroup. SO(2) rotations in the "extra" dimensions are then generated by the matrix $\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$. These are not quite d=4 Majorana spinors, since to produce a Majorana spinor one must first raise the USp(N) index on $\overline{\psi}_{a\dot{\alpha}}$ using C^{ab}. The $\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$ rotations are then a chiral U(1), and if one can choose field definitions and a lagrangian to avoid explicit use of C_{ab}, the result is U(N) covariant. In the formalism presented here it is always at least $USp(N_{left}) \times USp(N_{right}) \times U(1)$ covariant in d=4, but the U(1) need not be associated with the USp(N) indices.

2. The Complete D-Operator Algebra

For two reasons, component field extraction is more complicated in six dimensions than in four. First, in d=4, the D_α's and $\overline{D}_{\dot{\alpha}}$'s each form an SL(2,C) invariant subalgebra in which all anticommutators vanish, and one can use $D_\alpha D_\beta = -\frac{1}{2}C_{\alpha\beta}D^2$. In six dimensions there is no analogous partition into subalgebras invariant under both $SU^*(4)$ and SU(2). A second difficulty is that an arbitrary product of six-dimensional D's operating on a superfield often produces a reducible $USp(N)\times SU^*(4)$ representation. As an illustration consider

$$D_{aA} D_{bB} \Phi = \Phi_{a,b,A,B} = \Phi^{(1)}{}_{(ab)(AB)} + \Phi^{(2)}{}_{(ab)[AB]} + C_{ab} \Phi^{(3)}{}_{(AB)} + C_{ab} \Phi^{(4)}{}_{[AB]}$$

where we have decomposed the term on the left into irreducible representations by symmetrizing and antisymmetrizing indices. Thus there are several component fields at each level (power of θ) in a d=6 superfield. Observe that the anticommutation relation in eq. (10) implies $\Phi^{(1)} = 0$ and $\Phi^{(4)} \propto \partial_{AB}\Phi$, so only $\Phi^{(2)}$ and $\Phi^{(3)}$ are independent.

Thus, in order to find the field content of a superfield in terms of irreducible $\mathrm{USp(N)} \times \mathrm{SU^*(4)}$ components, it is necessary to decompose all possible products of the D's into irreducible representations, avoiding pure spacetime derivative terms. The process described here is a generalization of that used for d=4 chiral superspace [44]. We assume there are no central charges, and consider left and right D's separately.

2.1. Arbitrary Even N

The relation $\{ D_{aA} , D_{bB} \} = i\, C_{ab}\, \partial_{AB}$ is formally the same as that for an SO(4N) Clifford algebra with $g_{aA,bB} = iC_{ab}\, \partial_{AB}$. A convenient basis for this is formed by the 2^{4N} antisymmetrized products $D_{[aA} D_{bB} \cdots D_{cC]}$. Any other ordering differs from this by terms proportional to spacetime derivatives. After forming such a product, one may decompose it into irreducible $\mathrm{USp(N)} \times \mathrm{SU^*(4)}$ representations. Actually, a decomposition into $\mathrm{SU^*(4)}$ irreps also guarantees USp(N) irreducibility (except for the possible need to separate USp(N) traces). To see this, suppose one operates the Young projector for the tableau $\begin{array}{|c|c|}\hline A & B \\ \hline C \\ \cline{1-1} D \\ \cline{1-1}\end{array}$ (say) on an antisymmetric tensor $T_{aA,bB,cC,dD}$; i.e., one symmetrizes on AB, then antisymmetrizes on ACD. Overall antisymmetry implies this is equivalent to symmetrizing on acd then antisymmetrizing on ab, which is the definition of projection by the USp(N) tableau $\begin{array}{|c|c|c|}\hline a & c & d \\ \hline b \\ \cline{1-1}\end{array}$, the transpose of the first tableau. This fact depends critically on the original product being antisymmetric in pairs, and places some severe restrictions on the representations (and hence component fields) one can get: since USp(N) or $SU^*(4)$ columns can have no more than N or 4 blocks respectively, all relevant representations also have a restriction on the row lengths: no more than 4 USp(N) blocks and N $SU^*(4)$ blocks in a row.

It is also necessary to take into account raising and lowering with the totally antisymmetric ε for both groups, and USp(N) traces (i.e., parts proportional to C_{ab}), must be separated, so the number of different irreducible representations is still large. But note however that two standard tableaux with the same "frame" (arrangement of boxes) but different index arrangements (e.g., $\begin{array}{|c|c|}\hline A & B \\ \hline C \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline A & C \\ \hline B \\ \cline{1-1}\end{array}$) project out the same tensor here, because the two are related by a change in the order of the USp(N) indices. As an example, consider some tensor T_{abcABC}. If T has no symmetries, projection by these two tableaux gives rise to two linearly independent tensors $T^{(1)}_{abcAC,B}$ and $T^{(2)}_{abcAB,C}$. However, if T is totally antisymmetric under interchange of the pairs aA,bB,cC, then $T^{(1)}_{abcAC,B} = -T^{(2)}_{acbAC,B}$. Thus each *frame* permitted by the above size restrictions gives rise to one independent USp(N)×$SU^*(4)$ tensor operator on superspace.

2.2. The Complete D-Algebra for N=2

For N=2, we have an SU(2) symmetry, and a useful simplification takes place, since SU(2) Young tableaux have at most two blocks per column, and in addition a two block column is equivalent to a scalar (i.e., the tensor is proportional to the antisymmetric C_{ab}). We are therefore in a position to catalogue all the D-operators on N=2 d=6 superspace. For instance, decomposing $[D_{aA}\, D_{bB}]$ under SU*(4), we must perform a projection for each of the two tableaux $\begin{array}{|c|}\hline A\\\hline B\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline A & B\\\hline\end{array}$ which gives rise to the SU*(4) antisymmetric $D^{(2)}_{abAB}$ and symmetric $\hat{D}^{(2)}_{ab\,AB}$. $D^{(2)}_{abAB}$ is symmetric in ab since it corresponds to a projection by $\begin{array}{|c|c|}\hline a & b\\\hline\end{array}$(the 'transpose' of $\begin{array}{|c|}\hline A\\\hline B\\\hline\end{array}$). $\hat{D}^{(2)}_{ab\,AB}$ is antisymmetric in ab, and may be written as $C_{ab}\, D^{(2)}_{AB}$.

Antisymmetric SU*(4) indices may be raised and lowered with ε^{ABCD} and ε_{ABCD}. The complete list of D operators is given overleaf.

$$\begin{array}{lll}
1 & & \\
D^{(1)}_{aA} & & \\
D^{(2)}_{ab\,AB} & D^{(2)}_{AB} & \\
D^{(3)\,A}_{abc} & D^{(3)}_{a\,AB,C} & \\
D^{(4)}_{abcd} & D^{(4)}_{ab\,A}{}^{B} & D^{(4)\,CD}_{AB} \\
D^{(5)}_{abc\,A} & D^{(5)}_{a\,AB}{}^{C} & \\
D^{(6)}_{ab\,AB} & D^{(6)AB} & \\
D^{(7)A}_{a} & & \\
D^{(8)} & &
\end{array} \tag{14}$$

A numerical superscript (m) means the operator is an irreducible part of the totally antisymmetric product of m single D's. We have removed all factors of C and ε, so that the SU(2)×SU*(4) representations are manifestly irreducible. Thus all terms are symmetric in SU(2) indices. $D^{(3)}_{a[AB,C]} = 0$, and $D^{(2)}_{AB}$ and $D^{(6)AB}$ are symmetric. Other pairs of SU*(4) indices of the same type are antisymmetric, and contraction of any upper index with any lower index is zero.

We may choose factors of i so that all operators are real, $\bar{D} = D$, but at this point the definitions are still arbitrary by a real factor. To complete the definitions, it is necessary to give multiplication rules for the $D^{(n)}$'s, and it is sufficient to specify $D^{(1)} D^{(n)}$ for each n. The product takes the generic form

$$D^{(1)} D^{(n)} \approx D^{(n+1)} + D^{(n-1)} \partial \quad ,$$

as may be seen by the following argument: in a totally antisymmetric product of

$(n+1)$ single D's $D_{[aA} D_{bB} \cdots D_{eE]}$, the anticommutation relation (10) can be used to bring D_{aA} to the left, but terms with $(n-1)$ D's and one ∂ will be produced. A decomposition into irreducible pieces can then be made. To actually perform this calculation would be cumbersome, so one instead uses a consistency argument. For each $D^{(n)}$, one writes out the most general expression of the form (15), with the correct index structure and arbitrary real coefficients on the right-hand side. Then enforcing the relation

$$\{D_{aA}, D_{bB}\} D^{(n)} = i\, C_{ab}\, \partial_{AB}\, D^{(n)}$$

gives a set of relations between the unknown coefficients, which determines most them. The rest can be absorbed into the definitions of the $D^{(n)}$.

The multiplication rules are given the appendix, and are sufficient to determine the components of any superfield, as well as their transformation laws. However, these tasks are simplified if one can identify which products of $D^{(n)}$'s are identically zero. These are called orthogonality relations because they can be used to construct orthogonal projection operators. In four dimensions one finds the relation $D_\alpha D^2 = 0$, and the analogous result in d=6 is $D_{A(a} D^{(4)}_{bcde)} = 0$ as can be seen from the explicit multiplication table. However, a better proof, which will later be generalized, is to note that there are no $D^{(n)}$ operators with five symmetric SU(2) indices, so the product must be zero.

We can turn this into an orthogonality relation by multiplying by D_{fB} and symmetrizing on A, B , then contracting with C^{fa} to give

$$D^{(2)}_{AB}\, D_{abcd} = 0.$$

Similarly $$D_{abcd} D^{(2)}_{AB} = 0. \qquad (15)$$

This result turns out to be fundamental in all of six-dimensional superspace theory, and is the relevant analogue of the d=4 N=1 "chirality" property $D_{\alpha}D^2 = 0$. Previous efforts to work with extended superfields have tried using the relation $D_{i\alpha}D^{2N} = 0$ as a starting point, but this has failed. We contend that chiral superfields are only relevant for d=4 N=1 superfields, because the extended Yang-Mills constraints do not allow a chiral condition.

3. Six-Dimensional Multiplets

As an indication of how the relation (10) is used, we describe some supermultiplets with the new formalism. Most of these were previously known in dimensionally reduced form as d=4 N=2 multiplets, which is how we knew where to look.

3.1. The Linear Multiplet [45]

The linear multiplet is known only in component and constrained superfield forms. Its components are a triplet of scalars L_{ab} , a spinor ψ_{aA} and a conserved vector V_{AB} $(\partial^{AB} V_{AB} = 0)$. V_{AB} can be considered as the dual of the field strength of a fourth-rank antisymmetric tensor (in SO(5,1) language), and describes a single on-shell degree of freedom. To find a constrained superfield formulation, one can apply the ideas presented in the table at the end of chapter 2. The scalar triplet is the lowest dimension component and must occur at the bottom of a superfield which we also call L_{ab}. The spinor must then appear as the next component. However, since $D_{c\,C}\dot{L}_{ab}$ contains both $D_{C(c}\, L_{ab)}$ and $D_{C}{}^{b}\, L_{ab}$ and only the latter is needed, we must set $D_{C(c}\, L_{ab)} = 0$. Use of the multiplication rules then gives

$$D_{aA}\, L_{bc} = C_{a(b}\, \psi_{c)A}$$

$$D_{aA}\psi_{bB} = -i\,\partial_{AB}\,L_{ab} + iC_{ab}\,V_{AB}$$

$$D_{aA}\,V_{BC} = -\partial_{BC}\psi_{aA} - 2\,\partial_{A[B}\psi_{C]a} \quad . \tag{16}$$

The relation (10) on V gives $\partial^{AB}\,V_{AB} = 0$. The linear multiplet is interesting because it has no auxiliary fields: the number of bosonic and fermionic components matches both on and off shell (4+4 or 8+8, respectively). Actually, V_{AB} on shell is equivalent to a scalar by a duality transformation, which one could explicitly perform to obtain an on-shell multiplet with four scalars and a spinor, the "d=6 hypermultiplet." The numbers of bose and fermi components are no longer equal off shell.

To find an action for the linear multiplet, we note that from dimensional analysis it must be of the form $\int d^6x\, D^{(4)}\, L\, L$, since scalars have mass dimension 2 in d=6. Examining the list of D's shows that the only possibility is

$$S = \int d^6x\; D^{(4)abcd}\; L_{ab}\, L_{cd} \equiv \int d^6x\; d^4\theta^{abcd}\; L_{ab}\, L_{cd} \quad . \tag{17}$$

This is not an integral over the full superspace, so one must prove it is supersymmetric. The proof is instructive and follows from known d=4 ideas [46]. Note that the lagrangian $L_{abcd} \equiv \frac{1}{4!}L_{(ab}\,L_{cd)}$ satisfies $D_{E(e}\,L_{abcd)} = 0$, from which we get

$$D_{Ee}\,L_{abcd} = -\frac{1}{30}D_{Ef}\,L^{f}{}_{(abc}\,C_{d)e} \quad .$$

Thus under supersymmetry we get

$$Q_{eE}\,S = D_{eE}\,S = \int d^6x D^{(4)}_{abcd}\,D_{eE}L^{abcd} = \int d^6x D^{(4)}_{abcd}(-\frac{1}{30}D_{Ef}\,L^{f\,(abc}\,C^{d)}{}_{e}$$

$$= \frac{4!}{30}\int d^6x\; D^{(4)}_{eabc}\;D_{dE}L^{abcd} \quad . \tag{18}$$

(The first step is valid because $\partial_{AB}\,S = 0$. Then D's may be reordered as ∂_{aA}'s inside a $\int d^6x$.) On the other hand, since $D^{(4)}_{(abcd} D_{e)E} = 0$ one can cycle the e onto the $D^{(4)}$ and write

$$Q_{eE} S = -4 \int d^6x D^{(4)}_{e\,abc} D_{dE} L^{abcd}$$

in contradiction to (18). Thus $Q_{eE} S = 0$ and the action is supersymmetric.

Explicitly operating the $D^{(4)}$ on L gives the component result

$$S = \int d^6x \, L_{ab} \Box L^{ab} + i\psi_{aA} \partial^{AB} \psi^a{}_B + V^{AB} V_{AB} \tag{19}$$

where we have dropped coefficients.

The same dimensional reasoning shows that no self-interaction term is possible.

3.2. The Yang-Mills Multiplet

Various authors have shown that d=6 super Yang-Mills theory can be treated in the same way as d=4 Yang-Mills, at least in the constrained form [47]. Since some of that analysis was done without using the SU(2) symmetry, we repeat it here in our notation.

Introduce Yang-Mills covariant derivatives $\nabla_{aA} = D_{aA} + \Gamma_{aA}$ and $\nabla_{AB} = \partial_{AB} + \Gamma_{AB}$ transforming under gauge transformations with a real Lie algebra valued superfield K as

$$\nabla_{aA} \to e^K \nabla_{aA} e^{-K} \qquad\qquad \nabla_{AB} \to e^K \nabla'_{AB} e^{-K} \quad . \tag{20}$$

To produce an irreducible theory, we impose the constraint

$$\{\nabla_{aA}, \nabla_{bB}\} = iC_{ab} \nabla_{AB} \tag{21}$$

on the ∇'s. Note this is not just a definition of ∇_{AB}, since a priori the right-hand side could also have a field strength $G_{ab\,AB}$ symmetric in ab and AB. We have set it to zero just as we demanded $\{\nabla_\alpha, \nabla_\beta\} = 0$ in d=4. One must now use the Bianchi identities to show that we get the component fields of d=6 Yang-Mills theory. First note that at θ=0, Γ_{aA} is pure gauge since under infinitesimal K gauge transformations $\delta\Gamma_{aA}|_{\theta=0} = \nabla_{aA}K|_{\theta=0}$, which is arbitrary. Γ_{AB}, however, transforms as $\delta\Gamma_{AB}|_{\theta=0} = (\partial_{AB} + \Gamma_{AB})K|_{\theta=0}$ and is a bona fide component gauge field.

The next lowest dimension covariant object is $[\nabla_{aA}, \nabla_{BC}]$ (dimension $\frac{3}{2}$). The dim. $\frac{3}{2}$ Bianchi identity

$$[\nabla_{aA}, \{\nabla_{b\,B}, \nabla_{c\,C}\}] + [\nabla_{c\,C}, \{\nabla_{aA}, \nabla_{b\,B}\}] + [\nabla_{b\,B}, \{\nabla_{c\,C}, \nabla_{aA}\}] = 0$$

then implies

$$C_{bc}[\nabla_{aA}, \nabla_{BC}] + C_{ca}[\nabla_{b\,B}, \nabla_{CA}] + C_{ab}[\nabla_{c\,C}, \nabla_{AB}] = 0$$

after substituting (21). Projecting out the various SU(2)×SU*(4) representations in this expression by symmetrizing and antisymmetrizing, one finds that $[\nabla_{aA}, \nabla_{BC}]$ is totally antisymmetric in A, B, C. So we can write

$$[\nabla_{aA}, \nabla_{BC}] = F_a{}^D \varepsilon_{DABC} \quad . \tag{22}$$

Thus the lowest dimension covariant superfield is a right-handed spinor, and $F_a{}^A|_{\theta=0}$ can be identified as the physical fermion. (The fact that it should have dimension $\frac{5}{2}$ to be a physical spinor will be remedied below.) The dimension two Bianchi identities give

$$\nabla_{aA} F_b{}^B = i\, F_{ab}\, \delta_A^B + i\, C_{ab}\, M_A{}^B \tag{23}$$

where $M_A{}^B$ is the field strength of a vector (a <u>15</u> of SU*(4)) and F_{ab} is an SU(2) triplet of auxiliary fields. The equation of motion in terms of superfields can now be identified. Since it must give an algebraic equation for the auxiliary field, the only possibility on dimensional grounds is $F_{ab} = 0$. This is verified by the dimension $\frac{5}{2}$ Bianchi identities, which give

$$\nabla_{(aA} F_{bc)} = 0 \qquad \nabla_{aA} F_{bc} = \frac{1}{2} C_{a(b} \nabla_{AB} F_{c)}^{B} \quad . \tag{24}$$

Thus the fermion equations of motion occur at first θ order in F_{ab} as we expect. So we have a suitable Yang-Mills multiplet, consisting of the component fields Γ_{AB} (or its field strength $W_A{}^B$), a spinor $\psi_a{}^A$, and a triplet of auxiliary fields F_{ab}. If this theory is dimensionally reduced to four dimensions, the vector decomposes into a d=4 vector and a complex physical scalar, the latter corresponding to the two "extra" spacial components, as is most easily seen in SO(5,1) notation. This scalar is Yang-Mills covariant, since, when fields do not depend on x^4 and x^5, $\delta\Gamma_4 = (\partial_4 + \Gamma_4)K = \Gamma_4 K$. Thus the multiplet in d=4 consists of a gauge vector, a complex physical scalar, an SU(2) spinor and a triplet of auxiliary fields. We do not perform this reduction explicitly here.

We now need an action, and must introduce a coupling constant g with dimension -1 (since $g\,A_\mu$ for canonical bose field A_μ must have the dimension of ∂_μ). The coupling constant can be absorbed in a redefinition of the fields, leaving $S = \frac{1}{g^2} \int d^6x L$. In this case, canonical bosons and fermions have dimensions 1 and 3/2 as in d=4.

There are then two possible action formulae

$$S = \frac{1}{g^2} \int d^6x \, d^2\theta_{ab\,AB} \;\; F^{aA} * F^{b\,B} \tag{25}$$

and

$$S' = \frac{1}{g^2} \int d^6x d^2\theta_{AB} \; F_a{}^A * F^{b\,B} \quad .$$

Since these are not complete superspace integrals, one needs to check supersymmetry. It turns out that S' is not invariant, but S is. The proof is similar to that in the linear multiplet case above, but messier because of the two types of index, so we do not give it here. It involves using the identity $\nabla_{aA}F^{aA} = 0$ which follows from (23). Expanding out S in components gives:

$$S = \frac{1}{g^2}\int d^6x \;\; W_A{}^B\, {}^*W_B{}^A + 2i\, F^{aA}\, {}^*\nabla_{AB}F_a{}^B + 2\,F^{ab}\, {}^*F_{ab} \quad , \qquad (26)$$

which is exactly what we want: a spinor minimally coupled to a Yang-Mills field, with a triplet of auxiliary fields, all in the adjoint representation. Thus we have the correct constrained description of six-dimensional Yang-Mills theory. In the next section we find an unconstrained formulation.

4. Unconstrained formulation

We come now to the main result, the unconstrained form of d=6 Yang-Mills theory. The analysis is similar to the discussion of the d=4 case in chapter 3.

It is helpful to start with the linearized theory, since D's are easier to manipulate than full ∇'s.

4.1. Linearized theory

Keeping only the linear terms in all expressions, the constraint (1) becomes

$$D_{aA}\Gamma_{bB} + D_{bB}\Gamma_{aA} = iC_{ab}\Gamma_{A\dot{B}} \quad . \qquad (27)$$

Equating parts symmetric and antisymmetric in ab then gives one definition and one constraint:

$$D_{a[A}\Gamma^a{}_{B]} = 2i\Gamma_{AB} \qquad\qquad D_{(a(A}\Gamma_{B)b)} = 0 \quad . \qquad (28a,b)$$

Now note that one solution to this is that Γ be pure gauge

$$\Gamma_{aA} = D_{aA}\, U \qquad \text{for real scalar U .} \tag{29}$$

Explicitly,

$$D_{(a(A}\, \Gamma_{B)b)} = D_{(a(A}\, D_{B)b)} U = 0$$

after using (10). We would now expect that the complete linear solution be some simple extension of this. We cannot, however, just let $U \rightarrow U + iV$ as before, because Γ_{aA} has a reality condition, unlike Γ_{α} in $d=4$. The simplest modification of (29) is

$$\Gamma_{aA} = D_{bA}\, W^{b}{}_{a} \qquad \text{where} \quad W_{ab} = W_{ba} = \overline{W}_{ab} \quad . \tag{30}$$

Substituting into (28b) gives $D_{(a(A}\, D^{c}{}_{B)}\, W_{b)c} = 0$, or more succinctly

$$D^{(2)}_{AB}\, W_{ab} = 0 \quad , \tag{31}$$

after using the multiplication rules in the appendix. This is on the right track, since it involves $D^{(2)}_{AB}$, which satisfies an orthogonality relation

$$D^{(2)}_{AB}\, D^{(4)}_{abcd} = 0 \quad . \tag{32}$$

Thus it suffices to set $W_{ab} = D^{(4)}_{abcd}$(something) , and choose the "something" to absorb the two extra indices. There are three ways to do this

(1) $\quad W^{ab} = D^{(4)}_{cdef}\, V^{abcdef} \qquad V$ totally symmetric.

In this case $\quad \Gamma^{a}{}_{A} = D_{bA} D^{(4)}_{cdef}\, V^{abcdef} = 0 \quad$ since $\quad D_{A(b}\, D^{(4)}_{cdef)} = 0$.

(2) $\quad W_{ab} = D^{(4)cde}{}_{(a}\, V_{b)cde} \qquad V$ totally symmetric.

Then Γ_{aA} can be manipulated into the form

$$\Gamma_{aA} = D^b{}_A D^{(4)}{}_{(a}{}^{cde} V_{b)cde} = -\tfrac{3}{2} D_{aA} (D^{(4)bcde} V_{bcde}) \quad ,$$

and is just a special case of the pure gauge solution.

(3) The only viable solution is

$$W_{ab} = D^{(4)}_{abcd} V^{cd} \qquad V^{cd} = V^{dc} \quad . \tag{33}$$

Thus the full solution in the linearized case is

$$\Gamma_{aA} = D^b{}_A D^{(4)}_{abcd} V^{cd} + D_{aA} U \quad . \tag{34}$$

V^{cd} satisfies the criteria for a prepotential given at the end of chapter 2: it is the same representation as the highest component in the multiplet, F^{ab}, and has the right dimension (-2) for F to occur in it as $\theta^{(8)} F^{ab}$, i.e., at the top of V.

As before, U is a compensating field and can be completely gauged away by K transformations (20), so one can work in the gauge $U = 0$. There is, however, a new gauge invariance now: at next-to-highest order in V_{cd} we have the two components $D^{(7)A}_{(a} V_{cd)}$ and $D^{(7)Ab} V_{ab}$. The latter corresponds to $F_a{}^A$, but the former must be pure gauge. Thus we guess there is a gauge invariance with parameter $\xi^A{}_{abc}$, the same representation as the highest dimension unwanted component of V_{ab}. A variation $\delta V^{bc} = iD_{aA} \xi^{Aabc}$ can be manipulated into the form

$$\delta\Gamma_{aA} = iD^b{}_A D^{(4)}_{abcd} D_{eE} \xi^{Eecd} = -\frac{1}{3} D_{aA} (i D^{(4)}_{bcde} D^b{}_E \xi^{Eecd}) \quad , \tag{35}$$

which is a special case of a K gauge transformation. Equivalently, one can say there is an "internal" gauge invariance

$$\delta V^{bc} = iD_{aA} \xi^{Aabc} \qquad \delta U = -\frac{i}{3} D^{(4)}_{bcde} D^b{}_E \xi^{Eecd} \quad . \tag{36}$$

In the gauge $U = 0$ this reduces to a special class of K gauge transformations.

The decomposition into U and V is not unique, since we could replace

$$U \to U + \beta \partial^{AB} D^{(2)}_{abAB} V^{ab} \qquad \text{for arbitrary } \beta \tag{37}$$

and change the form of the solution. However, (34) is more elegant as it stands, so we ignore this.

Less compact four-dimensional versions of these linearized results are known [48], and may be obtained by dimensional reduction.

The action may be written in the unconstrained form

$$S = \frac{1}{g^2}\int d^6x d^8\theta \;\; V^{ab} F_{ab} \quad , \tag{38}$$

where F_{ab} is expressed in terms of V^{ab} by using the definitions and Bianchi identities of eqs. (23,24,34). This form of S is manifestly supersymmetric. It is also gauge-invariant because, as before, a linearized F is invariant, and

$$\delta\int V^{ab} \; F_{ab} = \int (D_{cC}\,\xi^{Cc\,ab})F_{ab} = \int \xi^{Cc\,ab} D_{Cc}\, F_{ab}) = 0$$

by the Bianchi identity (24a). It gives the right field equation because it can be written $S = \int V\mathbf{O}V$ where $\mathbf{O}$ is the collection of D's such that $F = \mathbf{O}\, V$, and $\mathbf{O}$ can be explicitly shown to satisfy

$$\int V_1 \mathbf{O} V_2 = \int V_2 \mathbf{O} V_1 \quad .$$

Since V is unconstrained, the field equation is $\mathbf{O}V \equiv F=0$. Thus we have a complete superspace theory of the linearized vector multiplet in six dimensions.

4.2. Nonlinear Theory

In chapter 3 we showed how to get from the linearized theory to the full nonlinear one. We just need to find fully covariant analogues of the results (34,38). It is not necessary to find covariant versions of all the D multiplication rules in the appendix, since most are irrelevant for the Yang-Mills multiplet.

The complete list of relevant covariant objects consists of all possible products of covariant derivatives ∇_{aA}. An arbitrary product $\nabla_{aA}\nabla_{bB}\cdots\nabla_{eE}$ can be written as a piece totally antisymmetric in $aA\ bB\ \cdots\ eE$ plus pieces containing vector covariant derivatives. The totally antisymmetric piece can be decomposed into SU(2)×SU*(4) irreps, yielding covariant $\nabla^{(n)}$'s corresponding to the $D^{(n)}$'s listed in eq. (4). The pieces containing vector covariant derivatives can then be rearranged similarly as well, producing new terms containing field strengths, and so on, until all terms are in the canonical form

(product of field strengths) × (product of vector covariant derivatives)
× (irreducible $\nabla^{(n)}$)

all operating to the right. The only available field strengths are $F_a{}^A$, F_{ab}, $W_A{}^B$ and their vector covariant derivatives, with $[F_a{}^A]=\frac{3}{2}$, $[F_{ab}]=[W_A{}^B]=2$.

An immediate result is the covariant analogue of (15)

$$\nabla_{aA}\nabla_{bB}=i\,\nabla^{(2)}_{abAB}+iC_{ab}\nabla^{(2)}_{AB}+\frac{i}{2}C_{ab}\nabla_{AB}\quad . \tag{39}$$

However, the crucial result is that the covariant analogue of (32) also holds:

$$\nabla_{A(a}\nabla^{(4)}_{bcde)}=\nabla^{(4)}_{(abcd}\nabla_{e)A}=0\quad , \tag{40}$$

since there is no dimension $\frac{5}{2}$ object of the canonical form symmetric in 5 SU(2) indices. This would not be true if one removed the constraints on the covariant derivatives and wrote $\{\nabla_{aA},\nabla_{bB}\}=iC_{ab}\nabla_{AB}+G_{abAB}$ (see the remark after eq. (21)), since then one could obtain, say,

$$\nabla_{A(a}\nabla^{(4)}_{bcde)} = \nabla^{(4)}_{(abcd}\nabla_{e)A} = G_{AB(ab}\,\nabla^{(3)}_{cde)}{}^{B} \quad .$$

Thus the Yang-Mills constraints are responsible for (40). This is an extension of the "preservation of representations" idea discussed in chapter 3.

Thus (59) is the N=2 analogue of the d=4 condition $\overline{\nabla}_{\dot\alpha}\overline{\nabla}_{\dot\beta}\overline{\nabla}_{\dot\gamma} = 0$. It implies the orthogonality relations

$$\nabla^{(2)}_{AB}\nabla^{(4)}_{abcd} = \nabla^{(4)}_{abcd}\nabla^{(2)}_{AB} = 0 \quad . \tag{41}$$

Also, since $\nabla_{aA}\nabla^{(4)}_{bcde}\nabla_{f\,F}$ symmetrized on either the first five or the last five SU(2) indices is zero, it must be of the form

$$\nabla_{aA}\nabla^{(4)}_{bcde}\nabla_{f\,F} = C_{a(b}\,\widehat{\nabla}^{(6)}_{cdAB}\,C_{e)f} \quad , \tag{42}$$

where $\widehat{\nabla}^{(6)}$ is symmetric in cd but not necessarily in AB.

These results are sufficient to define the nonlinear theory by following the steps of sect. (3.5). Define a "covariant variation" of ∇_{aA} which preserves the constraint in eq. (21) by

$$\delta\nabla_{aA} = \nabla^{b}{}_{A}\,\nabla^{(4)}_{abcd}\,(V_\Delta)^{cd} + \nabla_{aA}\,(U_\Delta) \quad . \tag{43}$$

There is again an internal gauge invariance here: Putting

$$V_\Delta{}^{cd} = i\nabla_{e\,E}\,\xi^{Eecd} \quad , \tag{44}$$

gives

$$\delta\nabla_{aA} = i\,\nabla^{b}{}_{A}\nabla^{(4)}_{abcd}\nabla_{e\,E}\,\xi^{Eecd} + \nabla_{aA}\,(U_\Delta)$$

$$= \nabla_{aA}\,(-\frac{i}{3}\nabla^{(4)}_{bcde}\,\nabla^{b}{}_{E}\,\xi^{Eecd}) + \nabla_{aA}(U_\Delta) \quad , \tag{45}$$

by using (42), which can be cancelled by choosing

$$U_\Delta = \frac{i}{3} \nabla^{(4)}_{bcde} \nabla^b{}_E \xi^{Eecd} \quad . \tag{46}$$

Now introduce a parameter t and write the system of ordinary differential equations

$$\frac{d}{dt} \nabla_{aA} = \nabla^b{}_A \nabla^{(4)}_{abcd} v^{cd} + \nabla_{aA} u \tag{47}$$

$$\nabla_{aA}(0) = \boldsymbol{\nabla}_{aA} \qquad v^{cd}(0) = V^{cd} \qquad u(0) = U \quad . \tag{48}$$

One must choose equations for $\frac{dv}{dt}$ and $\frac{du}{dt}$, and two convenient choices are

$$\frac{du}{dt} = \frac{dv^{ab}}{dt} = 0 \tag{49}$$

which gave the "vector representation" in N=1, and

$$\frac{du}{dt} = 0 \qquad\qquad \frac{d}{dt} v_{ab} = [\, U , v_{ab} \,] \quad . \tag{50}$$

In the second case, the U dependence splits off neatly as

$$\nabla_{aA} = e^{-U} \tilde{\nabla}_{aA} \, e^{U}$$

where $\tilde{\nabla}$ satisfies eqs. (47,48,49) with $U = 0$. This is an N=2 analogue of (3.75).

Exactly as before, the solution to (47,48) provides an unconstrained parameterization of the theory. Any other choice of $\frac{dv}{dt}$ and $\frac{du}{dt}$ gives a different parameterization of the theory.

It is fairly clear that (47) will not be solved easily in closed form, since it is a nonlinear differential equation in ∇, (i.e., in Γ_{aA}) . In our previous examples, the analogue was linear and soluble, viz.

$\frac{d}{dt}\nabla_\mu = A_\mu$ in d=3 Yang-Mills in ordinary space,

$\frac{du}{dt} = uA$ in Lie group theory,

$\frac{d}{dt}\nabla_\alpha = \nabla_\alpha(u+iv)$ in d=4 superspace Yang-Mills.

Here, however, it is quintic in ∇, with a more complicated index arrangement. This enormous increase in complexity is likely to prove a stumbling block in any practical applications of extended superfields. In principle, though, one can solve (47) by repeatedly differentiating it, i.e.,

$$\nabla_{aA} = \boldsymbol{\nabla}_{aA} + t\,(\boldsymbol{\nabla}^b{}_A\,\boldsymbol{\nabla}^{(4)}_{abcd})\,V^{cd} + t\,\boldsymbol{\nabla}_{aA}\,U + \cdots \tag{51}$$

The action can also be written. If we vary the constrained form (26) using (43) we must eventually get, after absorbing six ∇'s into the superspace measure:

$$\delta S = \frac{1}{g^2}\int d^6x d^8\theta\ (V_\Delta)^{ab}\ {}^*F_{\alpha\beta} \quad , \tag{52}$$

because we know from the constrained approach that $F_{ab} = 0$ is the field equation. Also, $\delta S = 0$ for $U_\Delta =$ anything and $(V_\Delta)^{aa} = \nabla_{e\,E}\xi^{E\,e\,ab}$ since the full superspace integral allows us to integrate $\nabla_{e\,E}$ onto F_{ab} and use $\nabla_{E(e}\,F_{ab)} = 0$. Thus immediately we know that the unconstrained action satisfies

$$\frac{d}{dt}S = \int d^6x\, d^8\theta\ \ v^{ab}\ {}^*F_{ab} \quad , \tag{53}$$

and may be expanded about $t=0$ by repeatedly differentiating with respect to t. In this way, one obtains an expansion of S involving an arbitrary background covariant derivative $\boldsymbol{\nabla}_{aA}$, and the prepotential V^{ab}. With the parameterization defined by (50), this expansion will have no U dependence, and this can be used as a check on the K gauge invariance of (54). Two remarks on eq. (53) must be made.

First, it is more useful than the constrained form (25) because it has a full $d^8\theta$ integral. Thus in any further manipulations, ∇'s can be partially integrated. The exact manipulations by which six ∇'s were absorbed to give (53) are irrelevant, and are not known explicitly in this case (although they can be carried out explicitly on the linearized d=4 action). Our method is practically useful because it circumvents this.

Second, it is important that the action reduces to the correct form when linearized. When using a prepotential as a fundamental field, one is not free to make arbitrary field redefinitions, because these would introduce Jacobians in the path integral measure. However, if the field redefinitions do not affect the linearized lagrangian, the Jacobian is unity in dimensional regularization, and the redefinition has no effect on physical quantities [49]. Moreover, the linearized form of the unconstrained action is known [50] to provide the correct path integral measure (in four dimensions). Thus, the reparameterizations allowed by choosing $\frac{du}{dt}$ and $\frac{dv^{ab}}{dt}$ arbitrarily all lead to correct quantum descriptions of N=2 Yang-Mills, since they all have the same linearized action (38). This is an advantage of our definition of the prepotential V^{ab} as $v^{ab}(0)$.

The background gauge invariance of the theory is understood by noting that eqs. (43,48,49,50,53) are all invariant under the t-independent transformation

$$(\nabla_{\alpha A}, \boldsymbol{\nabla}_{\alpha A}, V^{ab}, U) \rightarrow e^M (\nabla_{\alpha A}, \boldsymbol{\nabla}_{\alpha A}, V^{ab}, U)\, e^{-M} \quad .$$

However, since the prepotential V^{ab} is not a gauge field under this transformation, this symmetry can be preserved by any quantization procedure. The quantum gauge transformation of V^{ab}, resulting from the invariance (45,46), is not yet known beyond lowest order in V^{ab}, where it is just the linear result (36). Determination of the higher-order terms by hand is extremely difficult, and we hope to obtain some of them with a computer since they are essential for quantization. The N=1 analogue, eq. (3.76), is indicative of the complexity of the solution.

5. Minimal Coupling

We now describe the coupling of the matter multiplet to the Yang-Mills supermultiplet.

The linear multiplet of section (3.1) could not be converted to unconstrained form as it stood, because the vector in it is constrained to be divergenceless. The problem was recently solved in d=4 by Howe et al. [51], and the result is quite simple in six-dimensional notation: Relax the constraint $D_{A(a}L_{bc)}$ by requiring only that

$$D_{A(a}L_{bc)} = D_A{}^d L_{dabc} \qquad D_{A(a}L_{bcde)} = 0 \quad , \tag{55}$$

where L_{abcd} is a new totally symmetric superfield. The extra components introduced are then set to zero by a Lagrange multiplier superfield. As a result the conserved vector is automatically turned into a fourth scalar on shell, as is shown in ref. [51] using a component expansion. They have named this multiplet the "relaxed hypermultiplet."

Here we are interested in solving the covariant form of the constraint (55)

$$\nabla_{A(a}L_{bc)} = \nabla_A{}^d L_{dabc} \qquad \nabla_{A(a}L_{bcde)} = 0 \quad .$$

The solution is straightforward:

$$L_{abcd} = \nabla^{(4)}_{abcd} \nabla_{eE} \rho^{eE} \quad ,$$

$$L_{ab} = -\tfrac{1}{2} \nabla^{(4)}_{abcd} \nabla^c{}_E \rho^{dE} \quad . \tag{56}$$

so the prepotential ρ is a spinor with an upper SU*(4) index.

The action[51] is then

$$S = \frac{1}{g^2}\int d^6x\, d^8\theta \left[\rho^{aA}\nabla^b{}_A L_{ab} + L_{abcd}X^{abcd} \right], \qquad (57)$$

where X is a Lagrange multiplier.

The point we wish to make is that, as in d=4, one may couple matter to Yang-Mills by letting $D_{aA} \to \nabla_{aA}$ in all expressions. It is nontrivial that the Yang-Mills constraints allow this.

6. Conclusions and Outlook

One cannot leave this subject without discussing a simple and beautiful result that follows from the existence of the unconstrained formalism of N=2 Yang-Mills, the "non-renormalization theorem" of Grisaru and Siegel [52].

Observe that

[1] The perturbative form of the action is an integral over the whole superspace $\frac{1}{g^2}\int d^6x d^8\theta \;\; L(\nabla, V)$ (or d^4x if reduced to four dimensions).

[2] In the background field method, the only background quantities that appear are the covariant derivatives ∇_{aA}, ∇_{AB} and the background field strengths $\mathbf{F}_a^A$ etc.

Therefore, when a perturbative calculation of the effective action is made, only full superspace integrals of such covariant objects arise. Also, it is a general result that the divergences produced in the effective action are local in x, so they must be of the form

$$\int d^6x d^8\theta_1 \cdots d^8\theta_n \; L^{ct}(\nabla, g) \qquad \text{for integer n >0}$$

where L^{ct} is a background gauge covariant counterterm local in x. (In fact, Grisaru et al. [52] show that n=1, but we do not need that here. It means that the counterterms are also "local in θ".) Note that, since the only dimensional parameter is the $\frac{1}{g^2}$ in front of S, we have propagators $\propto g^2$, vertices $\propto g^{-2}$, and thus any one-particle-irreducible diagram proportional to $g^{2(P-V)} = g^{2(L-1)}$ where P, L, V are the numbers of propagators, vertices and loops, respectively. Thus an L-loop counterterm is of the form

$$(g^2)^{L-1} \int d^6x \{d^8\theta\}^n \; L^{ct}(\nabla) \quad . \tag{58}$$

Dimensional analysis gives

$$2(L-1)[g] - d + 4n + [L^{ct}] = 0 \quad .$$

In four dimensions, $[g] = 0$ so $[L^{ct}] = 4 - 4n \leq 0$. There is no way to make a gauge invariant L^{ct} of dimension ≤ 0 out of dimension $\frac{1}{2}$ $\nabla_{\alpha A}$'s. Thus there are no possible counterterms and thus no divergences.

This simple reasoning breaks down at one loop because we did not consider the effect of Faddeev-Popov ghosts. Siegel and Gates [53] have pointed out that the ghost lagrangian of N=2 Yang-Mills theory has a gauge invariance. This requires gauge-fixing, and a set of higher-order ghosts must thus be introduced. However, these also have a gauge invariance, requiring more ghosts, etc. An infinite "tower" of ghosts is obtained. This is not a serious problem in empty space perturbation theory, because the higher-order ghosts are then free and can be ignored. However, in the background field method, they couple to the background field, and thus occur in one-loop graphs. (In the background field method, background fields only occur as external lines.) Howe et al. [54] have proposed a method for decoupling all but a finite number of higher-order ghosts, but the one-loop counterterms then no longer have the simple form (58). We therefore conclude that *any N=2 theory in four dimensions has no divergences beyond one loop*. In particular, N=4 Yang-Mills, which is finite at one loop, is finite to all orders.

In six dimensions, $[g] = -1$ so $[L^{ct}] = 4 - 4n + 2L$, and counterterms have maximum dimension $2L$. For N=2 Yang-Mills theory, the linearized counterterm

$$L^{ct} = \Gamma_{aA} F^{aA}$$

is available at one loop. However, this is irrelevant, because the argument does not hold there.

Howe et al. [55] have noted that the available two-loop counterterms,

$$F_{ab} F^{ab} \qquad F_a{}^A \nabla_{AB} F^{aB} \qquad W_A{}^B W_B{}^A \quad ,$$

all vanish on shell when integrated $d^8\theta$. Thus, if N=4 Yang-Mills theory is written with N=2 superfields, using a relaxed hypermultiplet coupled to an N=2 Yang-Mills theory, there can be no on-shell counterterms and the S-matrix elements are finite at two loops. Any counterterms in the matter sector would be accompanied by their N=4 supersymmetrizations in the Yang-Mills sector, and none is possible. This result was obtained first by the component calculation of Marcus and Sagnotti [56].

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These counterterm arguments are not rigorous, and no quantum calculations with extended superfields have yet been done. There are at least four potential problems:

[1] To decouple the infinite tower of ghosts, the special precautions outlined in ref. [54] must be taken. This involves introducing a prepotential for the background field.

[2] For perturbative calculations to be practicable, a superspace "Fermi-Feynman" gauge and the associated ghost couplings must be found. This has only been done for the free case [53]. One must thus determine the quantum gauge transformation to higher orders, and no general method for finding this is known.

[3] In a FF gauge, N=2 superspace propagators have $\frac{1}{\Box^2}$ dependence, and graphs therefore contain infrared divergences. These must cancel, because they are not present in the component approach, but it is not known how this happens.

[4] A consistent supersymmetric regularization scheme must be found.

An estimate has shown that a one-loop calculation of the N=2 Yang-Mills vertex in d=4 may be feasible with a computer. A current project is to see whether these problems can be circumvented, and actually do the calculation.

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Appendix A: CONVENTIONS

Conventions for Operators

Our notation avoids unnecessary brackets. For an operator $\mathbf{O}$ and a field φ, "$\mathbf{O}$ acting on φ" is written

$$\mathbf{O}\varphi \quad .$$

The product of two operators $\mathbf{O}_1$ and $\mathbf{O}_2$ is defined by

$$(\mathbf{O}_1\mathbf{O}_2)\varphi \equiv \mathbf{O}_1(\mathbf{O}_2\varphi) \ .$$

Some of the fields used in the text are themselves operators, e.g., Lie algebra valued fields. We avoid ambiguities with the convention that if the rightmost object in a term is a field, then everything to the left acts upon it; e.g., in

$$\mathbf{O}_1\mathbf{O}_2\varphi_1\mathbf{O}_3\varphi_2\varphi_3 \ ,$$

φ_2, $\mathbf{O}_3$,φ_1, $\mathbf{O}_2$, $\mathbf{O}_1$ act consecutively upon φ_3. Numbers act by ordinary multiplication. If the rightmost object is not a field, the term is interpreted as a product of operators; e.g., in

$$\mathbf{O}_1\mathbf{O}_2\varphi_1\mathbf{O}_3\varphi_3\mathbf{O}_4 \ ,$$

$\mathbf{O}_4$, φ_2, $\mathbf{O}_3$,φ_1, $\mathbf{O}_2$, $\mathbf{O}_1$ are all acting to the right. Explicit examples of expressions with these two interpretations are

$$\frac{d}{dx} f(x) \frac{d^2}{dx^2} g(x) \equiv \frac{d}{dx}(f \frac{d^2 g}{dx^2})$$

and the operator

$$\frac{d}{dx} f(x) \frac{d^2}{dx^2} \ ,$$

respectively. Parentheses enable one to write any desired expression; e.g., $(\mathbf{O}_1 \varphi_1)\varphi_2$ means that $\mathbf{O}_1$ acts on $\boldsymbol{\varphi}_1$, producing a new operator that acts on φ_2.

Many of the operators used in the text are "derivations" which satisfy

$$\mathbf{O}(\varphi_1\varphi_2) = (\mathbf{O}\varphi_1)\varphi_2 + \varphi_1(\mathbf{O}\varphi_2) \quad .$$

Deleting the first and third sets of parentheses, which the conventions make redundant, gives

$$\mathbf{O}\varphi_1\varphi_2 - \varphi_1\mathbf{O}\varphi_2 = (\mathbf{O}\varphi_1)\varphi_2 \,.$$

With the further convention that any term in a commutator is regarded as an operator, we have for a derivation $\mathbf{O}$

$$[\mathbf{O}, \varphi] = \mathbf{O}\varphi \quad .$$

In particular, covariant derivatives have this property.

Conventions for Lie Algebras

The generators X_a of a Lie algebra are "abstract" operators which transform any field in a manner depending on the particular representation of the Lie algebra to which the field belongs. For example, the generators $X_{\mu\nu}$ of SL(2,C) act by

$$X_{\mu\nu}\psi_\alpha = i\sigma_{\mu\nu\alpha}{}^{\beta}\psi_\beta$$

$$X_{\mu\nu}\overline{\psi}_{\dot\alpha} = i\overline{\sigma}_{\mu\nu\dot\alpha}{}^{\dot\beta}\overline{\psi}_{\dot\beta}$$

$$X_{\mu\nu}A_{\alpha\dot\beta} = i\sigma_{\mu\nu\alpha}{}^{\gamma}A_{\gamma\dot\beta} + i\overline{\sigma}_{\mu\nu\dot\beta}{}^{\dot\gamma}A_{\alpha\dot\gamma} \quad .$$

They are derivations since

$$X_a(\varphi\psi) = (X_a\varphi)\psi + \varphi(X_a\psi) \quad .$$

They can also be regarded as acting on themselves by

$$X_a \times X_b = [X_a, X_b] = C_{ab}{}^c X_c$$

where $C_{ab}{}^c$ are the structure constants of the Lie algebra.

The invariant metric on the Lie algebra of a compact group is written $X_a * X_b$ ($= \delta_{ab}$ in a suitable basis). "Invariant" means that

$$[A, B] * C = -B * [A, C] \quad .$$

This is like an integration by parts.

All these rules must be modified for fermionic operators and fields by including $(-1)^n$, where n is the number of fermion pairs whose order was changed during the operation. The bracket becomes a commutator if two fermions are involved; e.g., in the above relation, if A and B are now fermionic and C is bosonic we get

$$\{A, B\} * C = B * [A, C] \quad .$$

Our implicit bracket notation does this automatically.

Complex Conjugation

We use superspace complex conjugation defined for operators by

$$(O\Phi)^* = (-1)^{O\Phi} O^* \Phi^* \qquad\qquad (O_1 O_2)^* = (-1)^{O_1 O_2} O_1{}^* O_2{}^* \quad .$$

The operation "bar" is related to complex conjugation, but may include extra signs (e.g., $\overline{\psi}_{\dot\alpha} = -(\psi_\alpha)^*$), or multiplication by a matrix (e.g., $\overline{\psi}_A = C_A{}^{\dot B} (\psi_B)^*$), depending on the type of external indices. For scalars φ, $\overline{\varphi} = \varphi^*$.

For a connection $\Gamma = \Gamma_i X_i$ with the X_i in an abstract Lie Algebra we define

$$\Gamma^{*} = (\Gamma_i)^{*}\, X_i \quad .$$

Note that "*" is not hermitian conjugation, which reverses the order of products. However, the sign is chosen to make it equal hermitian conjugation for Grassmann numbers. Thus an action is still real in this sense.

In chapter 4 all representations of $\mathrm{USp}(2N) \times \mathrm{SU}^{*}(4)$ are taken to be real in the sense

$$\bar{L}_{a \ldots b A \ldots B}{}^{C \ldots D} = L_{a \ldots b A \ldots B}{}^{C \ldots D} \qquad \bar{L}^{a \ldots b}{}_{A \ldots}{}^{C \ldots} = (\, L_{a \ldots b\, A \ldots}{}^{C \ldots}\,)^{*} \quad .$$

Note that there is always an even number of indices in total and that we define $\bar{L}$ with subscript internal indices.

Young Tableaux and Brackets

Our convention for the projection associated with a given tableau is "symmetrize on indices in the same row, then antisymmetrize on indices in the same column." For brackets

$$X_{[A}\, Y_{B]} = X_A\, Y_B - X_B\, Y_A \qquad X_{(A}\, Y_{B)} = X_A\, Y_B + X_B\, Y_A$$

with no factor of $\frac{1}{2}$, sometimes used by other authors.

Appendix B: GROUP THEORY

Pseudo-real Representations

Given any complex matrix representation of a group $g \to \Gamma(g)$ $\Gamma(g)\,\Gamma(h) = \Gamma(gh)$, we are implicitly given three other matrix representations, some of which may be equivalent.

(1) $g \to \Gamma(g)$

(2) $g \to \{\Gamma(g)\}^*$

(3) $g \to \{\Gamma(g)\}^{-1\,T}$

(4) $g \to \{\Gamma(g)\}^{-1\dagger}$

We denote components of objects transforming under these representations by φ^a , $\psi^{\dot{a}}$ ξ_a , $\chi_{\dot{a}}$ respectively, which makes it covariant to contract upper and lower indices of the same type; i.e. $\xi_a \varphi^a$ and $\chi_{\dot{a}} \psi^{\dot{a}}$ are group scalars.

If (1) is equivalent to (3) ("(1) ~ (3)") then also (2) ~ (4) and there exists an invariant metric with which to raise and lower indices:

$$\varphi_a = \varphi^b\, C_{ba} \qquad \varphi^a = C^{ab}\,\varphi_b \qquad C_a{}^b = \delta_a^b \quad .$$

(Note that this definition makes it consistent to raise and lower the indices on C. Also $C^a{}_b$ need not be simply related to $C_b{}^a$.)

If (1) ~ (2) there are two possibilities, and Γ is called a 'real' or 'pseudo-real' representation accordingly. To have a real representation, one needs to be able to put a reality restriction on the vector components: $\xi^* = C\,\xi$, which demands $C^* = C^{-1}$. But (1) ~ (2) $\to$ $\Gamma^* = C\,\Gamma\,C^{-1}$ gives only $C^* = \pm\, C^{-1}$. When the minus sign pertains, the representation is said to be pseudo-real and there is no equivalent representation using purely real matrices; but when the plus sign pertains we can always choose our basis so that $\Gamma = \Gamma^*$, and it makes covariant sense to talk about those vectors with real coefficients.

The following groups are used in the text:

[1] SO(N), fundamental representation: 1~2~3~4 , there is an invariant metric to raise and lower indices, and the matrices are real.

[2] SU(N), fundamental representation: 1~4 and 2~3. Taking the complex conjugate moves the indices up and down.

[3] SL(2,C), fundamental representation: 1~3 and 2~4, 1 $\not\sim$ 2 , and there are two independent representations ξ^α and $\bar{\chi}^{\dot\alpha}$, $\xi_\alpha = \xi^\beta C_{\beta\alpha}$ $\quad \bar{\chi}_{\dot\alpha} = \bar{\chi}^{\dot\beta} C_{\dot\beta\dot\alpha}$.

[4] $SU^*(4)$: 1~2 and 3~4 (both pseudo-real) with 1 $\not\sim$ 3 . There are again two inequivalent representations ξ_A and φ^A .

Note that the direct product of an even (odd) number of pseudo-real representations is always real (pseudo-real).

SL(2, C) Notation

In four dimensions we use SL(2,C) notation where left-handed spinors are represented by objects with a Greek superscript, e.g., ψ^α, and which transform as the fundamental representation of the group SL(2,C). ψ^α thus has two complex components. Right-handed spinors are written $\xi^{\dot\alpha}$, and transform in the complex conjugate representation. The complex conjugate of an undotted spinor is denoted by a bar,

$$\bar{\psi}^{\dot\alpha} = (\psi^\alpha)^* \ .$$

However, since the index also acquires a dot, the bar is often redundant. To streamline notation, it can be omitted, and reintroduced only to distinguish between $\overline{\psi}^2 = \overline{\psi}^{\dot\alpha}\overline{\psi}_{\dot\alpha}$ and $\psi^2 = \psi^\alpha\psi_\alpha$.

The antisymmetric tensor $\varepsilon^{\alpha\beta}$ is an invariant. The metric $C^{\alpha\beta}$ is $\varepsilon^{\alpha\beta}$ times a fixed but unspecified phase. It is a unitary matrix. The other forms of C are defined by

$$C_{\alpha\beta} \equiv (C^{-1})_{\beta\alpha} \qquad C^{\dot\alpha\dot\beta} \equiv -(C^{\alpha\beta})^* \qquad C_{\dot\alpha\dot\beta} \equiv -(C_{\alpha\beta})^* \quad .$$

One raises from the left and lowers from the right:

$$\psi_\alpha = \psi^\beta C_{\beta\alpha} \qquad \psi^\alpha = C^{\alpha\beta}\psi_\beta \qquad \overline{\psi}_{\dot\alpha} = \overline{\psi}^{\dot\beta} C_{\dot\beta\dot\alpha} \qquad \overline{\psi}^{\dot\alpha} = C^{\dot\alpha\dot\beta}\overline{\psi}_{\dot\beta} \quad .$$

Some observations facilitate index manipulations:

(1) $\delta_\alpha^\beta = C_\alpha{}^\beta = -C^\beta{}_\alpha$ and $\delta_{\dot\alpha}^{\dot\beta} = C_{\dot\alpha}{}^{\dot\beta} = -C^{\dot\beta}{}_{\dot\alpha}$, so δ is redundant notation.

(2) Complex conjugation and barring differ by a minus sign if an index is lowered:

$$(\psi_\alpha)^* = (\psi^\beta C_{\beta\alpha})^* = -\overline{\psi}^{\dot\beta} C_{\dot\beta\dot\alpha} = -\overline{\psi}_{\dot\alpha}$$

(3) $\psi^\alpha \xi_\alpha = -\psi_\alpha \xi^\alpha$ and $\overline{\psi}^{\dot\alpha}\overline{\xi}_{\dot\alpha} = -\overline{\psi}_{\dot\alpha}\overline{\xi}^{\dot\alpha}$

(4) $(\psi^\alpha \xi_\alpha)^* = \overline{\psi}^{\dot\alpha}\overline{\xi}_{\dot\alpha}$ and $(\theta^2)^* = \overline{\theta}^2$.

An SO(3,1) vector is a hermitian SL(2,C) matrix

$$A^\mu \to A^\mu(\sigma_\mu)_{\alpha\dot\beta} \equiv A_{\alpha\dot\beta} \text{ with } \overline{A}_{\dot\alpha\beta} \equiv (A_{\alpha\dot\beta})^* = A_{\beta\dot\alpha} \quad .$$

The σ_μ are the Pauli matrices $(1, \sigma_x, \sigma_y, \sigma_z)$, and one can show that

$$A^{\alpha\dot\beta} A_{\gamma\dot\beta} = C^\alpha{}_\gamma (A^\mu A_\mu) = C^\alpha{}_\gamma A^2 \quad \text{and} \quad V^{\alpha\dot\beta} V_{\alpha\dot\gamma} = C^{\dot\beta}{}_{\dot\gamma} V^2 \quad ,$$

which is the SL(2,C) analogue of $\not{A}\not{A} = A^2$. A symmetric bispinor $W_{\alpha\beta}$ corresponds to an SO(3,1) self-dual antisymmetric tensor. One decomposes an antisymmetric tensor into self- and antiself-dual parts as

$$W_{\mu\nu} \rightarrow W_{\alpha\dot{\alpha},\beta\dot{\beta}} = \frac{i}{2} C_{\dot{\alpha}\dot{\beta}} W_{\alpha\beta} + \frac{i}{2} C_{\alpha\beta} \overline{W}_{\dot{\alpha}\dot{\beta}} \quad ,$$

where $\overline{W}_{\dot{\alpha}\dot{\beta}} = (W_{\alpha\beta})^*$. Antisymmetric indices are proportional to $C_{\alpha\beta}$ or $C_{\dot{\alpha}\dot{\beta}}$, so irreducible SL(2,C) representations are symmetric in dotted and in undotted indices. Also, since antisymmetrizing three indices gives zero, one may cycle indices from one object to another in a "Fierz transformation,"

$$C_{\alpha\beta}\psi_\gamma = -C_{\beta\gamma}\psi_\alpha - C_{\gamma\alpha}\psi_\beta \quad .$$

The matrices generating SL(2,C) transformations are denoted $i(\sigma_{\mu\nu})_\alpha{}^\beta$ and spinors transform as

$$\delta\psi_\alpha = i(\sigma_{\mu\nu})_\alpha{}^\beta \psi_\beta \qquad \delta\psi^\alpha = -i\psi^\beta(\sigma_{\mu\nu})_\beta{}^\alpha$$

$$\delta\overline{\psi}_{\dot{\alpha}} = i(\overline{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\psi}_{\dot{\beta}} \qquad \delta\overline{\psi}^{\dot{\alpha}} = -i\overline{\psi}^{\dot{\beta}}(\overline{\sigma}_{\mu\nu})_{\dot{\beta}}{}^{\dot{\alpha}} \quad ,$$

where

$$(\sigma_{\mu\nu})_{\alpha\beta} = -(\sigma_{\nu\mu})_{\alpha\beta} = (\sigma_{\mu\nu})_{\beta\alpha} \quad \text{and} \quad (\overline{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = ((\sigma_{\mu\nu})_{\alpha\beta})^* \quad .$$

Using the fact that $(\sigma_\mu)_{\alpha\dot{\beta}}$ and $(\sigma_{\mu\nu})_{\alpha\beta}$ are invariant if all the indices are transformed, one may obtain multiplication rules for them. Thus, once the Lorentz transformations are given for the vector indices, the SL(2,C) algebra is specified. The algebra is not used explicitly in the text, since the formalism ensures that the symmetry is always manifest.

Appendix C: D-OPERATOR ALGEBRA IN SIX DIMENSIONS

Multiplication rules for N=2 D-operators

The formulae that follow are essentially supersymmetry transformations for component fields. Only left multiplication by $D^{(1)}$ is described since right multiplication is obtained by the rule

$$D^{(1)} D^{(n)} = D^{(n+1)} + D^{(n-1)} \partial$$

$$D^{(n)} D^{(1)} = (-1)^n \, (D^{(n+1)} - D^{(n-1)} \partial)$$

where the index arrangement on the RHS is identical in the two cases.

$$D_{aA} \, D_{bB} = i \, C_{ab} \, D^{(2)}_{AB} + i \, D^{(2)}_{abAB} + \frac{i}{2} \, C_{ab} \, \partial_{AB}$$

$$D_{cC} \, D^{(2)}_{AB} = -\frac{1}{2} \, D^{(3)}_{c(A,B)C} + \frac{1}{4} \, D_{c\,(A} \, \partial_{B)C}$$

$$D_{cC} \, D^{(2)}_{abAB} = D^{(3)}_{abc}{}^{N} \, \varepsilon_{NABC} - \frac{1}{2} \, C_{c\,(a} \, D^{(3)}_{b)C,AB} + \frac{1}{4} \, C_{c\,(a} \, D_{b)[A} \, \partial_{B]C}$$

$$D_{dD} \, D^{(3)}_{abc}{}^{N} = -i D^{(4)}_{abcd} \, \delta^{N}_{D} - \frac{i}{3!} \, C_{d\,(a} \, D^{(4)}_{bc)D}{}^{N} - \frac{i}{4!} \, C_{d\,(a} \, D^{(2)}_{bc)AB} \, \partial_{CD} \, \varepsilon^{NABC}$$

$$D_{dD} \, D^{(3)}_{cC,AB} = -i \, (D^{(4)}_{cdC,ABD} - \frac{1}{3} \, D_{cdD,ABC}) - i \, C_{dc} \, D^{(4)}_{AB,CD} + \frac{i}{2} \, C_{dc} \, D^{(2)}_{C[A} \, \partial_{B]D}$$
$$+ \frac{i}{6} \, D^{(2)}_{dcC[A} \, \partial_{B]D} + \frac{i}{3} \, D^{(2)}_{dcAB} \partial_{DC}$$

$$D_{eE}\, D^{(4)}_{abcd} = -\frac{1}{4!}\, C_{e(a}\, D^{(5)}_{bcd)E} + \frac{1}{2}\,\frac{1}{4!}\, C_{e(a}\, D^{(3)}_{bcd)}{}^{A}\, \partial_{AE}$$

$$\begin{aligned} D_{eE}\, D^{(4)}_{cdB}{}^{F} = {} & (D^{(5)}_{cdeB}\, \delta^{F}_{E} - \frac{1}{4} D^{(5)}_{cdeE}\, \delta^{F}_{B}) - \frac{1}{2} C_{e(c}\, D^{(5)}_{d)BE}{}^{F} \\ & + \frac{1}{2} (D^{(3)}_{cde}{}^{F}\, \partial_{EB} - \frac{1}{4} D^{(3)}_{cde}{}^{G}\, \partial_{EG}\, \delta^{F}_{B}) \\ & + \frac{1}{8}\, \varepsilon^{FACD}\, C_{e(c}\, D^{(3)}_{d)B,AC}\, \partial_{DE} \end{aligned}$$

$$D_{eE}\, D^{(4)}_{AB,CD} = \frac{1}{2} (D^{(5)}_{eEAB,CD} + D^{(5)}_{eECD,AB}) - \frac{1}{4} D^{(3)}_{eCD,[A}\, \partial_{B]E} - \frac{1}{4} D^{(3)}_{eAB,[C}\, \partial_{D]E}$$

$$D_{fF}\, D^{(5)}_{cdeB} = -\frac{i}{3!}\, C_{f(c}\, D^{(6)}_{de)FB} + \frac{i}{2}\, D^{(4)}_{fcde}\, \partial_{FB} - \frac{i}{2.3!}\, C_{f(c}\, D^{(4)}_{de)B}{}^{G}\, \partial_{GF}$$

$$\begin{aligned} D_{fF}\, D^{(5)}_{cDE}{}^{G} = {} & i\, (D^{(6)}_{fcDE}\, \delta^{G}_{F} + \frac{1}{3} D^{(6)}_{fcF[D}\, \delta^{G}_{E]}) + i\, C_{fc}\, D^{(6)G}{}_{,DEF} \\ & - \frac{i}{2} D^{(4)}_{fc[D}{}^{G}\, \partial_{E]F} + \frac{i}{6} D^{(4)}_{fc[D}{}^{C}\, \delta^{G}_{E]}\, \partial_{CF} - \frac{i}{4} C_{cf}\, \varepsilon^{GABC}\, D^{(4)}_{DE,AB}\, \partial_{CF} \end{aligned}$$

$$D_{gG}\, D^{(6)}_{cdEF} = -\frac{1}{2} C_{g(c}\, D^{(7)N}_{)}\, \varepsilon_{NEFG} + \frac{1}{2} D^{(5)}_{cdg[E}\, \partial_{F]G} - \frac{1}{4} C_{g(c}\, D^{(5)}_{d)EF}{}^{N}\, \partial_{NG}$$

$$D_{gG}\, D^{(6)MP} = \frac{1}{2} D^{(7)(M}_{g}\, \delta^{P)}_{G} - \frac{1}{2.4!}\, \partial_{G[A}\, D^{(5)}_{gBC]}{}^{(P}\, \varepsilon^{M)ABC}$$

$$D_{hH}\, D^{(7)N}_{d} = i\, C_{hd}\, D^{(8)}\, \delta^{N}_{H} - \frac{i}{4}\, \varepsilon^{NEFG}\, D^{(6)}_{hdEF}\, \partial_{GH} - \frac{i}{2}\, C_{hd}\, D^{(6)NP}\, \partial_{PH}$$

$$D_L \, D^{(8)} = \frac{1}{2} D_l^{(7)N} \, \partial_{NL}$$

Useful relations for products of derivatives:

(1) $$\partial^{AB} \partial_{CB} = \Box \, \delta^A_C$$

(2) $$\varepsilon^{ABCD} \partial_{GB} \partial_{FC} \partial_{ED} = \partial^{HA} \varepsilon_{HGFE} \Box$$

(3) $$\varepsilon^{CDEF} \partial_{AF} = ½ \delta_A^{[C} \partial^{DE]}$$

(4) $$\varepsilon^{ABCD} \partial_{EC} \partial_{FD} = \delta^A_{[F} \delta^B_{E]} \Box - \partial^{AB} \partial_{FE}$$

Correspondence with SO(5,1) *Notation*

Table 3 shows the relation between our notation and the more usual SO(5,1) notation for some of the lower spins.

Table 3. SO(5,1) and SU*(4) Notation

Dimension of Rep	SU*(4) Object	SO(5,1) Object
1 real	ε^{ABCD} , δ^A_B	$g_{\mu\nu}$, 1
4 complex	ψ_A	$\frac{1}{2}(1-\gamma_7)\psi$
4 complex	ψ^A	$\frac{1}{2}(1+\gamma_7)\psi$
6 real	$V_{AB} = -V_{BA}$	V_μ
10 real	$T^{AB} = T^{BA}$	$T^{\mu\nu\sigma} = \frac{1}{3!}\,\varepsilon^{\mu\nu\sigma\kappa\omega\tau}\,T_{\kappa\omega\tau}$
10 real	$T_{AB} = T_{BA}$	$T^{\mu\nu\sigma} = -\frac{1}{3!}\,\varepsilon^{\mu\nu\sigma\kappa\omega\tau}\,T_{\kappa\omega\tau}$
15 real	$\Sigma_A{}^B$ *traceless*	$\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}$
20 real	$R_{AB}{}^{CD}$ *traceless*	$R_{\mu\nu} = R_{\nu\mu}$ *traceless*

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