Uncertainty with ordinal likelihood information*

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Abstract

We present a model that is closely related to the so-called models of *choice under complete uncertainty*, in which the agent has no information about the probability of the outcomes. There are two approaches within the said models: the *state space-based* approach, which takes into account the possible states of nature and the correspondence between states and outcomes; and the *set-based* approach, which ignores such information, and solves certain difficulties arising from the state space-based approach. Kelsey [**?**] incorporates into a state space-based framework the assumption that the agent has ordinal information about the likelihood of the states. This paper incorporates this same assumption into a set-based framework, thus filling a theoretical gap in the literature. Compared to the set-based models of choice under complete uncertainty we introduce the information about the ordinal likelihood of the outcomes while, compared to Kelsey's approach, we incorporate the advantages of describing uncertainty environments from the set-based perspective. We present an axiomatic study that includes adaptations of some of the axioms found in the related literature and we characterize some rules featuring different combinations of information about the ordinal likelihood of the outcomes and information about their desirability.

Keywords: Complete Uncertainty, Ordinal Likelihood, Leximax, Leximin.
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1 Introduction

In so-called models of choice under *complete uncertainty* or *ignorance*, agents cannot formulate any kind of belief about the probabilities associated to each action, or even their relative likelihood. We find two approaches in these kinds of models. One, the *state space-based* approach, describes each action by means of a vector of outcomes contingent upon the possible states of nature. In this case, *complete uncertainty* concerns the probability or likelihood of the states of nature (see Arrow and Hurwicz [?], Maskin [?], Cohen and Jaffray [?], Barberà and Jackson [?], and Barrett and Pattanaik [?], among others). The other, the *set-based* approach, describes each action exclusively in terms of the outcomes it might generate. That is, *complete uncertainty* in this framework directly concerns the probability or likelihood of the outcomes (see Barberà et al. [?], Kannai and Peleg [?], Nitzan and Pattanaik [?], Pattanaik and Peleg [?], Bossert [?, ?], Bossert et al. [?], Arlegi [?, ?], and, for a survey, see Barberà et al. [?]).

The authors of the set-based approach invoke several relative advantages over the state space-based formulation. One is that the former might be more suitable for the tractability of overly complex problems, where it might be difficult, first, to accurately identify the states of nature and, second, to find the correspondence between states and outcomes. These tasks are sometimes unnecessary or simply impossible, in which case, only the possible outcomes of each action are considered. In some situations, moreover, the states of nature may be arbitrarily partitioned in different ways, making the state space-based approach subject to this arbitrariness. Finally, the set-based approach has also been defended as a more suitable way to represent the Rawlsian problem of choice under the veil of ignorance (a deeper discussion of all these arguments can be found in Pattanaik and Peleg [?], and Bossert et al. [?]).

In this paper, we model a choice situation with uncertainty where the decision maker's beliefs have the structure of an ordinal ranking by likelihood of the outcomes associated with each action. The consideration of only ordinal likelihood information in an uncertainty environment is an approach that has already been made by Kelsey [?] within a state space framework, but the model we propose adopts the set-based perspective. Thus, while in Kelsey [?] actions are described by a function that associates outcomes with states of nature on which a likelihood ordering is defined, each action in our model is described simply by the set of outcomes it might generate, the elements of which appear ordered from most to least likely. In short, compared to the set-based models of choice under complete uncertainty we introduce in the problem the information about the ordinal likelihood of the outcomes while, compared to Kelsey's approach, we incorporate the aforementioned advantages of describing uncertain environments from the set-based perspective.

Our model also relates to Jaffray's ([?], [?]) model of choice among *belief functions*. In fact, our *likelihood-ordered sets* are special cases of what Jaffray ([?], [?]) calls *imprecise risk situations*, i.e., decision problems under uncertainty where there is some imprecision about the probability of their consequences, and for which a *belief function* can be defined. Jaffray ([?], [?]) axiomatically characterizes a *generalized expected utility* function to compare such kind of alternatives. However, his model does not apply to our setting, basically because, while we are interested in a particular class of decision problems (which we represent by means of likelihood-ordered sets) over which we assume transitive and complete comparability, Jaffray ([?], [?]) assumes transitivity and completeness when comparing any pair of belief functions, thus making a stronger assumption. This affects the applicability of Jaffray's model both at the axiomatic level and the results level. In particular, under complete comparability of any kind of belief functions, he imposes axioms that imply dealing with linear combinations of belief functions. However, although likelihood-ordered sets are representable by belief functions, it is not true that any linear combination of belief functions finds a corresponding representation as a likelihood-ordered set, making it impossible to fit his axioms into our framework. Similarly, the *generalized expected utility* function that he characterizes to evaluate imprecise risk situations uses as support *elementary belief functions*, which are a specific kind of belief functions that are also beyond our domain: they are, too, unrepresentable by means of likelihood-ordered sets.

Methodologically, we present an axiomatic study that includes adaptations of some axioms from the related literature and investigate their logical implications in our setting. In particular, we characterize alternative criteria that combine information about the ordinal likelihood of the outcomes with information about their desirability in different ways. Two of the families of rules that we characterize, the *leximax-desirability* rules and the *leximin-desirability* rules, are related to other criteria that appear in the literature on complete uncertainty, the *leximax* and *leximin* rules proposed by Pattanaik and Peleg [**?**], in the sense that they closely reflect extreme types of optimistic and pessimistic behavior in the agent, while also incorporating the ordinal likelihood information. We also characterize another family of criteria, namely, the *leximax-likelihood* rules, which display a type of behavior in which the agent's attention is focused on the most likely outcome of each action. Unlike the previous two, the leximax-likelihood family has no parallel in the complete uncertainty literature, but it does in the state-space approach to the ordinal likelihood information problem formulated by Kelsey [**?**]. Additionally, we present a family of *weighted likelihood* criteria, which evaluate actions by calculating a weighted average of the utilities of all the possible results that the action may generate, using higher weights for the results perceived by the agent as most likely. These criteria can naturally be interpreted from a *subjective expected utility* perspective.

The paper is organized as follows: Section 2 contains the basic notation of the model and presents a preliminary result showing that there is no preorder over the set of actions that at once satisfies three simple properties. This result hints that the number of possible outcomes of each action is relevant in the analysis. This fact will determine the structure of Section 3. In Section 3.1, we restrict the domain to rankings involving only actions with the same cardinality (equal number of possible outcomes). This enables the characterization of some lexicographic criteria. In Section 3.2, we extend these criteria to the general domain, obtaining some families of rankings. In Section 4, we present a discussion on the combinations of axioms that appear in the characterization results of Section 3 and, as a consequence, we present new axioms that characterize the weighted likelihood criteria. We conclude in Section 5 by indicating some possible lines of further research. The Appendix collects the proofs of all the results presented throughout the paper.

2 A model of ordinal uncertainty and a first result

Our agent is equipped with a complete preorder, R, defined over an infinite universal set of outcomes X, which reflects the agent's preferences over this set.[1] We will denote by P and I the asymmetric and the symmetric parts of R, respectively.

We want to derive individual preferences over actions, where an action is a set of possible final outcomes determined by a chance mechanism. The main assumption of our model is that the decision maker is *only* able to assign a likelihood ranking over the possible outcomes of an action. Thus, actions are denoted $\vec{a} = (a_1, \ldots, a_n)$, where the mutually exclusive outcomes $a_1, \ldots, a_n$ (and no others) are perceived to be the possible final outcomes in decreasing order of likelihood, that is, a_i is perceived to be more likely than a_j if $i < j$.[2] The set of all possible actions includes all non-empty finite ordered subsets of X, which we denote by Q.

In particular, by considering only ordinal likelihood information, the possibility is excluded that the agent determines, for example, how much more likely outcomes are with respect to each other. Moreover, it is also excluded the possibility that the agent can compare the likelihood of results across different actions. Our model can thus be seen as an intermediate model between the standard lottery representation and the choice under complete uncertainty. In the former, an action would also include each outcome's exact probability of occurring. In the latter, an action merely describes the set of possible outcomes with no information about their likelihood or probability.[3]

We denote by $\succsim$ the individual preference over actions, where we assume that for all $x, y \in X$, $xRy \Leftrightarrow (x) \succsim (y)$. This natural assumption is usually presented in the literature of choice under complete uncertainty as an independent axiom under the name of *Extension*.

Let us discuss the interest and complexity of the problem at hand by introducing a set of basic properties on how to construct individual preferences over actions. The first property, called *Reordering* (REO), refers to an intuition already established in Kelsey [**?**] under the name of *Interchange*. Assume that a_i and a_j are possible outcomes under a certain action $\vec{a}$. Suppose that

[1] We will say that a binary relation is a *preorder* if it satisfies reflexivity and transitivity, and that it is a *complete preorder* if it is both a preorder and also satisfies completeness.

[2] We assume throughout the paper that the likelihood binary relation takes the form of a linear ordering. The implications of other possible structures are addressed in Section 5.

[3] Obviously, in the choice under complete uncertainty framework, the order of presentation is meaningless, and the same action could be represented by any permutation of the outcomes. In our framework, any permutation of the outcomes within a set would represent a different action, since it would modify the relative likelihood of the outcomes.

a_j is the better outcome of the two, i.e., $a_j P a_i$, but it is also the less likely of the two, i.e., $i < j$. Then, action $\Pi_{(i,j)}(\vec{a})$, which simply consists of permuting the likelihood positions of a_i and a_j in $\vec{a}$, has to be perceived as strictly better than $\vec{a}$.[4]

Reordering: For all $\vec{a} \in Q$ and $i < j$,

$$a_j P a_i \Rightarrow \Pi_{(i,j)}(\vec{a}) \succ \vec{a}.$$

The second property is a plausible adaptation of the *Dominance* axiom (DOM) in the set-based approach to choice under complete uncertainty and is related to Gärdenfors' principle [?], introduced by Kannai and Peleg [?]. Consider an action $\vec{a}$ and another action with the same possible outcomes as $\vec{a}$ and the same relative likelihood ordering plus an additional outcome x that is the least likely outcome in this new action. We will represent this action by $(\vec{a}, x)$. Then, the axiom says that, if the new outcome x is strictly better than all the outcomes of $\vec{a}$, then action $(\vec{a}, x)$ is strictly better than $\vec{a}$. Similarly, if x is strictly worse than all the outcomes of $\vec{a}$, then action $(\vec{a}, x)$ is strictly worse than $\vec{a}$. Finally, if x is indifferent to all the outcomes of $\vec{a}$, then $(\vec{a}, x)$ is indifferent to $\vec{a}$.

Dominance: For all $\vec{a} \in Q$ and all $x \notin \vec{a}$:

$$\begin{aligned} xPa_i \text{ for all } i \in \{1, \ldots, |\vec{a}|\} &\Rightarrow (\vec{a}, x) \succ \vec{a}. \\ xIa_i \text{ for all } i \in \{1, \ldots, |\vec{a}|\} &\Rightarrow (\vec{a}, x) \sim \vec{a}. \\ a_iPx \text{ for all } i \in \{1, \ldots, |\vec{a}|\} &\Rightarrow \vec{a} \succ (\vec{a}, x). \end{aligned}$$

The last property describes a consistency property on the composition of actions, along the lines of the independence conditions in the set-based approach. We will say that the *composition* of actions $\vec{a}$ and $\vec{c}$, with $\vec{a} \cap \vec{c} = \emptyset$, is the action $(\vec{a}, \vec{c}) = (a_1, \ldots, a_{|\vec{a}|}, c_1, \ldots, c_{|\vec{c}|})$.[5] Such a composition considers all possible outcomes in $\vec{a}$ and $\vec{c}$, maintains the internal likelihood orders of the outcomes of $\vec{a}$ and $\vec{c}$, and is such that any outcome in $\vec{a}$ is more likely than any outcome in $\vec{c}$. Now, suppose that action $\vec{a}$ is strictly better than action $\vec{b}$, and similarly, action $\vec{c}$ is strictly better than action $\vec{d}$. The *Composition* axiom (COM) establishes that the composition of $\vec{b}$ and $\vec{d}$ should not be strictly better than the composition of $\vec{a}$ and $\vec{c}$.

[4] Formally, $\Pi_{(i,j)}(\vec{a}) = (a_{\pi(1)}, \ldots, a_{\pi(|\vec{a}|)})$, where π is a permutation on $\{1, \ldots, |\vec{a}|\}$ such that $\pi(i) = j$, $\pi(j) = i$, and $\pi(l) = l$ for all $l \notin \{i, j\}$.

[5] We define the intersection of two ordered sets, $\vec{a}$ and $\vec{b}$, as the non-ordered set $\vec{a} \cap \vec{b} = \{x \in X \mid x \in \vec{a} \text{ and } x \in \vec{b}\}$. The union of ordered sets is defined analogously.

Composition: For all $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in Q$ such that $\vec{a} \succ \vec{b}$, $\vec{c} \succ \vec{d}$ and $\vec{a} \cap \vec{c} = \vec{b} \cap \vec{d} = \emptyset$,

$$(\vec{a}, \vec{c}) \succsim (\vec{b}, \vec{d}).$$

Although the three proposed properties might seem rather natural, they are, in fact, mutually incompatible, as the following result shows.

Result 1 *If there are at least three non-indifferent outcomes in X, there is no preorder $\succsim$ satisfying REO, DOM and COM.*[6]

When investigating the source of the impossibility addressed by Result **??**, we find that COM is controversial when the actions to be compared are of different cardinality. Consider, as in the statement of the axiom, that $\vec{a}$ is better than $\vec{b}$ and $\vec{c}$ is better than $\vec{d}$. Imagine, furthermore, that the cardinality of $\vec{a}$ is greater than that of $\vec{b}$ and that both $\vec{a}$ and $\vec{b}$ are *considerably* worse sets of outcomes than either $\vec{c}$ or $\vec{d}$. Then, the addition of $\vec{c}$ at the end of $\vec{a}$ and the addition of $\vec{d}$ at the end of $\vec{b}$ both have a *positive* effect. However, given that $\vec{a}$ is bigger than $\vec{b}$, the effect of $\vec{c}$ on $\vec{a}$ is smaller than that of $\vec{d}$ on $\vec{b}$, due to the lesser importance of the outcomes of $\vec{c}$ in $(\vec{a}, \vec{c})$ in relation to those of $\vec{d}$ in $(\vec{b}, \vec{d})$. Therefore, $(\vec{b}, \vec{d})$ might become a better action than $(\vec{a}, \vec{c})$. In other words, even though $\vec{a}$ is better than $\vec{b}$ and $\vec{c}$ is better than $\vec{d}$, the relative importance of the outcomes of $\vec{c}$ and $\vec{d}$ in $(\vec{a}, \vec{c})$ and $(\vec{b}, \vec{d})$ is a crucial aspect. Clearly, such a relative importance is going to depend on the cardinality of the involved actions.

Hence, COM makes the implicit assumption that the addition of new outcomes at the end of an action has similar effects on its desirability, regardless of the cardinality of the action. However, and with no detriment to our assumption of only ordinal likelihood information within a given action, it makes sense to recognize that there is information to be deduced from the number of outcomes in an action. For example, the addition of an outstandingly good outcome, x, at the end of two actions makes it the least likely to occur in either case, but it is natural to expect the agent to attach more importance to x when it is added to a singleton than when it is added to a large set of outcomes. As a matter of fact, Result **??** highlights this problem by showing

[6] The result also applies for the case in which X is finite. Additionally, the Extension assumption over $\succsim$ can also be eliminated without affecting the impossibility.

that COM is logically incompatible with REO and DOM, which are rather plausible properties.

This lack of plausibility of COM in the case of unequal cardinality motivates the direction of the rest of the paper. Throughout the following sections we will provide independence properties that constitute an alternative to COM.

3 Lexicographic criteria

In this section, we examine the way in which the combination of an independence property that is an alternative to COM with other axioms leads to characterizations of some lexicographic criteria. These criteria reflect behaviors in which the individual focuses on the most likely outcomes, the best outcomes and the worst outcomes, respectively.

3.1 The equal-cardinality case

Given the apparent relevance of cardinality in comparing actions, we first study the case in which the agent establishes comparisons only between actions with an equal number of outcomes. Formally, a preference on equal-cardinality actions will be a subset of $\bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$, instead of a subset of $Q \times Q$, where Q_k is the set of all ordered subsets of X with cardinality k. In other words, the agent can compare actions with exactly the same number of outcomes, but cannot establish any comparison between actions with different cardinalities. In Section 3.2, we will generalize the preferences derived in this section to the general domain.

We propose a collection of axioms for the comparison of sets with the same number of outcomes. We sort them into four categories on the basis of the ideas they describe: (i) an independence property, (ii) an invariance property (iii) likelihood sensitivity properties and (iv) outcome sensitivity properties. Later, we will provide several characterization results by using a combination of, at most, one property in each of the categories.

An independence property

Independence-like conditions are very common across most of the set-ranking models, including choice under complete uncertainty problems (see

Kannai and Peleg [?], or Pattanaik and Peleg [?], among others). Axiom COM constituted a first crude attempt to reflect this idea. In this section, we propose a property for the case of equal-cardinality comparisons. *Independence* (IND) is a translation to our framework of Savage's [?] *Sure-Thing Principle* in line with Kelsey [?]: Consider two actions, $\vec{a}$ and $\vec{b}$, and a further two new actions that have the same possible outcomes as $\vec{a}$ and $\vec{b}$, respectively, while also maintaining the original relative likelihood ordering of these outcomes. Assume, furthermore, that the new actions have one extra outcome each, x and y, that are mutually indifferent and occupy the same position in the likelihood ordering. Axiom IND says that the two new actions should be compared in the same way as $\vec{a}$ and $\vec{b}$. Formally,

Independence: For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$, all $x \notin \vec{a}$, $y \notin \vec{b}$ such that xIy, and all $m \in \{1, 2, \ldots, k+1\}$,

$$\vec{a} \succsim \vec{b} \Leftrightarrow (a_1, \ldots, a_{m-1}, x, a_m, \ldots, a_k) \succsim (b_1, \ldots, b_{m-1}, y, b_m, \ldots, b_k).$$ [7]

An invariance property

Neutrality (NEU) is a natural adaptation to our informational framework of an axiom with the same name that appears in the literature of choice under complete uncertainty (see Bossert [?], Nitzan and Pattanaik [?] and Pattanaik and Peleg [?], among others), and also of the *Independence of Ranking of Irrelevant Outcomes* property in Kelsey [?]. In words, NEU implies that the criterion $\succsim$ should be immune to changes with no effect either on the likelihood ordering of the outcomes within each action or on the desirability *ordering* of all the outcomes of the two actions to be compared. In particular, NEU implies that the criterion to be constructed disregards any kind of cardinal information in the agent's preferences. NEU therefore has a more natural interpretation in a context where the decision-maker is a social planner who has access only to ordinal information about individual's preferences over the consequences.

Neutrality: For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ and all one-to-one mappings $f : X \to X$ such that for all $x, y \in (\vec{a} \cup \vec{b})$, $xRy \Leftrightarrow f(x)Rf(y)$,

$$\vec{a} \succsim \vec{b} \Leftrightarrow (f(a_1), \ldots, f(a_k)) \succsim (f(b_1), \ldots, f(b_k)).$$

[7] Obviously, when $m = 1$, the results a_{m-1} and b_{m-1} do not exist and the two ordered sets start with outcomes x and y. Similarly, when $m = k+1$, the two ordered sets end with outcomes x and y.

Likelihood sensitivity properties

The consideration of likelihood information is the new feature that differentiates this analysis from the previous literature on choice under complete uncertainty. REO reflects a basic idea about how likelihood information can be considered. The following property, called *Likelihood Sensitivity* (LS), encompasses the idea of the sensitivity of the ranking $\succsim$ with respect to the likelihood information. To introduce this property, let us consider two actions with the same cardinality, $\vec{a}$ and $\vec{b}$, such that action $\vec{a}$ is strictly better than action $\vec{b}$. Furthermore, let us consider two new outcomes, x and y. The property states that we can always find an action $\vec{c}$ such that the composite action $(\vec{a}, \vec{c}, x)$ is preferred to $(\vec{b}, \vec{c}, y)$, regardless of the difference in desirability between x and y. The real scope of the axiom arises when yPx. Then, LS establishes that we can reduce the relative likelihood of outcomes x and y by a degree sufficient to maintain the original preference for $\vec{a}$ over $\vec{b}$. This axiom is specific to our framework and has no direct links with any axiom from the literature on choice under complete uncertainty.

Likelihood Sensitivity: For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $\neg(a_i I b_i)$ for all $i \in \{1, \ldots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow (\vec{a}, \vec{c}, x) \succ (\vec{b}, \vec{c}, y).$$

We now introduce two weaker versions of LS, called *Weak Likelihood Sensitivity 1* (WLS1) and *Weak Likelihood Sensitivity 2* (WLS2). They also reflect related ideas of "robustness" of the strict preference relation between actions. WLS1 (respectively, WLS2) requires the new outcomes x and y to be no better (respectively, worse) than the best (respectively, worst) outcome in the original actions, thus making the argument in LS more plausible. In order to formulate these axioms we need to introduce an additional piece of notation: for all finite $C \subset X$, $\max\{C\} = \{x \in C \mid xRy \text{ for all } y \in C\}$ and $\min\{C\} = \{x \in C \mid yRx \text{ for all } y \in C\}$. With a slight abuse of notation, we define the max and min operators for the elements of Q in the same way. That is, $\max\{\vec{a}\}$ ($\min\{\vec{a}\}$) represents the subset of best (worst) outcomes in $\vec{a}$.[8]

[8] Given that all elements of the best (worst) outcomes of a set belong to the same indifference class according to R, we will apply, with a slight abuse of notation, this binary relation to these sets also.

Weak Likelihood Sensitivity 1: For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $\neg(a_i I b_i)$ for all $i \in \{1, \dots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$ such that $\max\{\vec{a} \cup \vec{b}\} R \max\{x, y\}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow (\vec{a}, \vec{c}, x) \succ (\vec{b}, \vec{c}, y).$$

Weak Likelihood Sensitivity 2: For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $\neg(a_i I b_i)$ for all $i \in \{1, \dots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$ such that $\min\{x, y\} R \min\{\vec{a} \cup \vec{b}\}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow (\vec{a}, \vec{c}, x) \succ (\vec{b}, \vec{c}, y).$$

Outcome sensitivity properties

The Extension assumption made in this model implies that the desirability of the outcomes matters when comparing elementary actions. We now discuss two properties related to the sensitivity of the preferences towards the desirability of the outcomes that go beyond such a basic assumption as Extension. These properties, called *High Outcome Sensitivity* (HOS) and *Low Outcome Sensitivity* (LOS), reflect the idea that there always exist outcomes that are sufficiently good or bad as to reverse a given preference over two actions. To see the implications of HOS, consider two actions, $\vec{a}$ and $\vec{b}$, and suppose that $\vec{b}$ is better than $\vec{a}$. Then, construct another new action that has the same outcomes as $\vec{a}$, except the least likely one, with the same likelihood ordering. Then, HOS states that it is always possible to find a sufficiently good outcome, y, that, when taking last place in the likelihood ordering, will make the new action better than $\vec{b}$. The intuitive idea is that we can always compensate for the difference in the preference between $\vec{a}$ and $\vec{b}$ with an outcome, x, provided it is sufficiently good. LOS is a dual property, which establishes that it is possible for a sufficiently bad outcome to compensate for a difference in the preference between two actions. The purpose of these properties is to establish that the criterion $\succsim$ should be sensitive to the utilities of the outcomes. These properties are also specific to our framework.

High Outcome Sensitivity: For all $k \in \mathbb{N}$ and all $\vec{a}, \vec{b} \in Q_k$ such that there exists $x \in X$ with xPz for any $z \in (\vec{a} \cup \vec{b})$, there exists $y \notin (\vec{a} \cup \vec{b})$ such that

$$(a_1, \dots, a_{k-1}, y) \succ \vec{b}.$$

Low Outcome Sensitivity: For all $k \in \mathbb{N}$ and all $\vec{a}, \vec{b} \in Q_k$ such that there exists $x \in X$ with zPx for any $z \in (\vec{a} \cup \vec{b})$, there exists $y \notin (\vec{a} \cup \vec{b})$ such that

$$\vec{b} \succ (a_1, \ldots, a_{k-1}, y).$$

We have presented four classes of axioms for ranking actions of the same cardinality. We now present characterization results for the equal-cardinality case in which we always make use of the new independence property (IND). In our first result, we will show the great strength of LS when combined with IND so that it determines a very particular way of ranking actions. The combination of these two properties is strong enough to: (i) imply an invariance property like NEU, and (ii) be incompatible with any of the above proposed outcome sensitivity properties.

Theorem 1 *A reflexive binary relation $\succsim \subseteq \bigcup_{k \in \mathbb{N}} (Q_k \times Q_k)$ satisfies IND and LS if, and only if, for all $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$:*

$$\vec{a} \succsim \vec{b} \Leftrightarrow \text{ there is not } j \leq k \text{ such that } b_i I a_i \text{ for all } i < j \text{ and } b_j P a_j.$$

Theorem **??**, in fact, characterizes what we will call the *leximax-likelihood* rule, $\succsim_{LL}$, which proceeds as follows: the agent first looks at the most likely outcome in each action. If one of them is strictly better than the other, then the action with the better most likely outcome is declared strictly preferred. In the event of a tie, the agent looks at the second most likely outcome in $\vec{a}$ and $\vec{b}$ respectively and proceeds analogously. If ties occur successively until both sets are exhausted, they are then declared indifferent. Clearly, this criterion leaves no room for outcome sensitivity properties.

Next, we explore the consequences of weakening LS by means of WLS1 and WLS2. As the next two results show, this will allow for some outcome sensitivity (in the form of HOS and LOS). In order to present the theorems, we need some additional notation. For all $\vec{a} \in Q_k$, $\gamma(\vec{a})$ (respectively, $\beta(\vec{a})$) will denote the permutation of the outcomes in $\vec{a}$ such that $\gamma_i(\vec{a}) R \gamma_{i+1}(\vec{a})$ (respectively, $\beta_{i+1}(\vec{a}) R \beta_i(\vec{a})$) for all $i < k$ and, in case of indifference, the most likely outcome occupies a previous (lower) position after the permutation, where $\gamma_i(\vec{a})$ (respectively, $\beta_i(\vec{a})$) denotes the element of $\vec{a}$ that occupies the i-th position after the permutation. That is, γ (respectively, β) reorders the elements of an action from best to worst (respectively, worst to best), in terms of the preferability of the outcomes, while preserving, in the event of indifference, their relative positions in terms of their likelihood.

Furthermore, $L(\gamma_i(\vec{a}))$ will denote the position in likelihood terms that element $\gamma_i(\vec{a})$ occupies in $\vec{a}$. That is, $L(\gamma_i(\vec{a})) = s$ if $\gamma_i(\vec{a}) = a_s$. The position $L(\beta_i(\vec{a}))$ is defined analogously.

Theorem 2 *A preorder* $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ *satisfies IND, NEU, WLS1 and HOS if, and only if, for all* $\vec{a}, \vec{b} \in Q_k$ *for any* $k \in \mathbb{N}$*:*

$$\vec{a} \succsim \vec{b} \Leftrightarrow \text{ there is not } j \leq k \text{ such that:}$$

$$\text{for all } i < j, \left[\gamma_i(\vec{b}) I \gamma_i(\vec{a}) \text{ and } L(\gamma_i(\vec{b})) = L(\gamma_i(\vec{a}))\right] \text{ and}$$

$$\left[\gamma_j(\vec{b}) P \gamma_j(\vec{a}) \text{ or } \left(\gamma_j(\vec{b}) I \gamma_j(\vec{a}) \text{ and } L(\gamma_j(\vec{b})) < L(\gamma_j(\vec{a}))\right)\right].$$

Theorem **??** characterizes what we will call the *leximax-desirability* rule, $\succsim_{LD}$, which starts by looking, respectively, at the best outcome in each action (if the best outcome is not unique, the criterion focuses on the most likely of these outcomes). If there is a strict preference for one of the outcomes over the other then the action that contains the former is declared strictly better. In the event of indifference between the two outcomes, the rule proceeds to look at their positions in likelihood terms and declares a strict preference for the action whose best outcome takes a lower likelihood position. Only in the event that the respective best outcomes take the same likelihood positions the criterion proceeds to look, respectively, at the best remaining outcome of each action (with the same tie-breaking rule) and proceeds as previously.

Theorem 3 *A preorder* $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ *satisfies IND, NEU, WLS2 and LOS if, and only if, for all* $\vec{a}, \vec{b} \in Q_k$ *for any* $k \in \mathbb{N}$*:*

$$\vec{a} \succsim \vec{b} \Leftrightarrow \text{ there is not } j \leq k \text{ such that:}$$

$$\text{for all } i < j, \left[\beta_i(\vec{b}) I \beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\right] \text{ and}$$

$$\left[\beta_j(\vec{b}) P \beta_j(\vec{a}) \text{ or } \left(\beta_j(\vec{b}) I \beta_j(\vec{a}) \text{ and } L(\beta_j(\vec{b})) > L(\beta_j(\vec{a}))\right)\right].$$

Theorem **??** characterizes what we will call the *leximin-desirability* rule, $\succsim_{ld}$, which is, in a sense, dual with respect to $\succsim_{LD}$. The rule looks at the respective worst outcome in each set (if the worst outcome is not unique, the criterion focuses on the most likely of these outcomes). The set where the worst outcome is better is declared preferred, and in the event of indifference, the rule selects the action in which the worst outcome occupies a less likely position. If the two outcomes occupy the same likelihood position in their sets, then the rule looks at next worst outcome of each action (with the same tie-breaking rule) and proceeds as previously described.

The three rules characterized above are related to other lexicographic rules in Pattanaik and Peleg [**?**] within the set-based approach to problems of choice under complete uncertainty, and in Kelsey [**?**] within the state space-based approach to our problem. The leximax-desirability rule and the leximin-desirability rule reflect, respectively, optimistic and pessimistic attitudes of the agent. Under the former, the agent focuses on the best possible outcome of each action, while, under the latter, the agent tries to maximize the worst possible outcome. In contrast with the *leximax* and *leximin* rules in Pattanaik and Peleg [**?**], our rules assign a relevant role to likelihood information. While in the complete uncertainty case, the leximax (respectively, leximin) rule proceeds to consider the second best (respectively, worst) outcome of each action in the event of indifference between the best (respectively, worst) ones, in our context, the rule takes into account information about the relative likelihood of the best (respectively, worst) outcomes. Only if they are also equivalent in terms of their likelihood, the rule proceeds to consider the second best (respectively, worst) outcome.

Likelihood information plays a more important role in the leximax-likelihood rule, where the agent focuses her attention on the most likely outcome of each action, and it does not appear to matter whether the agent's attitude is pessimistic or optimistic. This rule has the same spirit as the *lex-likelihood* rule of Kelsey [**?**], which evaluates primarily the outcomes that arise in the most likely state of nature. Both rules also proceed in a similar fashion in the event of this first comparison between two actions being inconclusive: that is, looking at the second most likely outcome of each action in the case of our rule, and at the respective outcomes that arise in the second most likely state of nature in Kelsey's case.

We additionally show that the axioms used in each of the previous theorems are independent.

Proposition 1 *The following collections of axioms are independent:*

1. *IND and LS.*
2. *IND, NEU, WLS1 and HOS.*
3. *IND, NEU, WLS2 and LOS.*

3.2 The general case

We now generalize the criteria characterized in Section 3.1 to the general case in which actions may have different cardinality. We will approach the problem solely by adding DOM, which is rather plausible in the general case, to the axioms used in the above characterization theorems. We will also add a technical condition of *richness* of the domain X, which states that, for all $x, y \in X$ such that xPy, there exist $a, b, c \in X$ such that $aPxPbPyPc$. That is, in a *rich* domain, for any two outcomes, there is always another that is better, another that is worse, and another that is between the two. In rich domains, the addition of DOM provides *extensions* of the rules characterized in Section 3.1, in the sense that they compare actions having the same cardinality as their respective particularizations, while also establishing certain comparisons between actions having different cardinality.[9]
We will first introduce some families of rules that extend each of the lexicographic criteria characterized in the previous section.

Definition 1 *A preorder $\succsim \subseteq Q \times Q$ belongs to the family of* extended leximax-likelihood rules*, $\succsim \in \succsim^e_{LL}$, if $\succsim$ extends $\succsim_{LL}$ and, for all $\vec{a}, \vec{b} \in Q$ such that $|\vec{a}| < |\vec{b}|$:*

- *If $(b_1, \ldots, b_{|\vec{a}|}) \succ_{LL} \vec{a}$ or $[(b_1, \ldots, b_{|\vec{a}|}) \sim_{LL} \vec{a}$ and $b_j P b_i$ for all $j > |\vec{a}|$ and $i \leq |\vec{a}|]\}$, then $\vec{b} \succ \vec{a}$.*
- *If $\vec{a} \succ_{LL} (b_1, \ldots, b_{|\vec{a}|})$ or $[\vec{a} \sim_{LL} (b_1, \ldots, b_{|\vec{a}|})$ and $b_i P b_j$ for all $i \leq |\vec{a}|$ and $j > |\vec{a}|]$, then $\vec{a} \succ \vec{b}$.*

[9]Formally, a preference over actions $\succsim \subseteq Q \times Q$ extends a preference on equal-cardinality actions $\succsim^* \subseteq \bigcup_{k \in \mathbb{N}} (Q_k \times Q_k)$ if for all $\vec{a}, \vec{b}$ such that $|\vec{a}| = |\vec{b}|$, $\vec{a} \succsim \vec{b} \Leftrightarrow \vec{a} \succsim^* \vec{b}$.

- *If* $a_i I b_j$ *for all* $i, j \leq |\vec{a}|$ *and* $\gamma_1(\vec{b}) P \beta_1(\vec{b}) I \gamma_1(\vec{a})$, *then* $\vec{b} \succ \vec{a}$.
- *If* $a_i I b_j$ *for all* $i, j \leq |\vec{a}|$ *and* $\gamma_1(\vec{a}) I \gamma_1(\vec{b}) P \beta_1(\vec{b})$, *then* $\vec{a} \succ \vec{b}$.
- *If* $a_i I b_j$ *for all* $a_i \in \vec{a}, b_j \in \vec{b}$, *then* $\vec{a} \sim \vec{b}$.

The extended leximax-likelihood rules coincide with the leximax-likelihood rule when the actions to be compared have the same cardinality. When the cardinality of the actions is different, the extended leximax-likelihood rules proceed as follows: they select the first (more likely) outcomes of the action with the greater cardinality, such as to form a subset (another action) containing the same number of outcomes as in the other set. Then, the extended leximax-likelihood rules compare these sets by the leximax-likelihood rule. If there is a strict preference, they replicate what is established by the leximax-likelihood rule. Otherwise, if the two selected sets are indifferent, they look at the remaining (less likely) outcomes of the larger set. If all of these are weakly better than all the preceding outcomes with at least one strict inequality, then the action with the larger set of outcomes is declared the better of the two. If they are weakly worse with at least one strict inequality, the action with the smaller set of outcomes is the better of the two. If all the outcomes are indifferent, then the two actions are indifferent. The remaining possible comparisons are not univocally determined, which is what distinguishes the different members of the family of extended leximax-likelihood rules.

Definition 2 *A preorder* $\succsim \subseteq Q \times Q$ *belongs to the family of* extended leximax-desirability rules, $\succsim \in \succsim^e_{LD}$, *if* $\succsim$ *extends* $\succsim_{LD}$ *and, for all* $\vec{a}, \vec{b} \in Q$ *such that* $|\vec{a}| < |\vec{b}|$:

- *If* $\{\gamma_1(\vec{b}) P \gamma_1(\vec{a})$ *or* $\beta_1(\vec{b}) I \gamma_1(\vec{a}) P \beta_1(\vec{a})\}$, *then* $\vec{b} \succ \vec{a}$.
- *If there is* $j \leq |\vec{a}|$ *such that:*

$$\textit{for all } i < j,\ \left[\gamma_i(\vec{a}) I \gamma_i(\vec{b}) \textit{ and } L(\gamma_i(\vec{a})) = L(\gamma_i(\vec{b}))\right] \textit{ and}$$

$$\left[\gamma_j(\vec{a}) P \gamma_j(\vec{b}) \textit{ or } \left(\gamma_j(\vec{a}) I \gamma_j(\vec{b}) \textit{ and } L(\gamma_j(\vec{a})) < L(\gamma_j(\vec{b}))\right)\right],$$

  then $\vec{a} \succ \vec{b}$.

- *If $a_i I b_i$ for all $i \leq |\vec{a}|$ and $\{\beta_1(\vec{a}) P \gamma_{|\vec{a}|+1}(\vec{b})$ or $\gamma_1(\vec{a}) I \beta_1(\vec{a}) I \gamma_1(\vec{b})\ P \beta_1(\vec{b})\}$, then $\vec{a} \succ \vec{b}$.*

- *If $a_i I b_j$ for all $a_i \in \vec{a}, b_j \in \vec{b}$, then $\vec{a} \sim \vec{b}$.*

The extended leximax-desirability rules coincide with the leximax-desirability rule when the actions to be compared have the same cardinality. When the actions to be compared have different cardinality, the *intersection* of all the extended leximax-desirability rules does not treat them symmetrically. In order to establish a preference for the smaller set, $\vec{a}$, it follows a lexicographic procedure parallel to the leximax-desirability rule. If this lexicographic procedure leads to an indifference between $\vec{a}$ and $(b_1, \ldots, b_{|\vec{a}|})$, then $\vec{a}$ is preferred when (i): the remaining outcomes of $\vec{b}$ are strictly worse or (ii): they are weakly worse with at least one strict inequality and all the outcomes of $\vec{a}$ are indifferent. The conditions to ensure that a strict preference for the larger set is declared by every rule in the family are more demanding, however. They require either: (i): that the best outcome of the larger action is strictly better than the best outcome of the smaller one or (ii): that all the outcomes of the larger action are weakly better than all the outcomes of the smaller one, with at least one strict preference. The intuition behind these more demanding conditions is that, when the best outcomes of the two sets are indifferent and occupy the same likelihood position, this same likelihood position appears to hold more weight when the number of outcomes is smaller. If all the outcomes of the two actions are indifferent, then the two actions are declared indifferent. In all other cases, comparisons are not univocally determined by all the members of the family.

Definition 3 *A preorder $\succsim \subseteq Q \times Q$ belongs to the family of* extended leximin-desirability rules, *$\succsim \in \succsim_{ld}^{e}$, if $\succsim$ extends $\succsim_{ld}$ and, for all $\vec{a}, \vec{b} \in Q$ such that $|\vec{a}| < |\vec{b}|$:*

- *If $\beta_1(\vec{a}) P \beta_1(\vec{b})$ or $\gamma_1(\vec{a}) P \beta_1(\vec{a}) I \gamma_1(\vec{b})$, then $\vec{a} \succ \vec{b}$.*

- *If there is $j \leq |\vec{a}|$ such that:*

$$\text{for all } i < j,\ \left[\beta_i(\vec{b}) I \beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\right] \text{ and}$$

$$\left[\beta_j(\vec{b}) P \beta_j(\vec{a}) \text{ or } \left(\beta_j(\vec{b}) I \beta_j(\vec{a}) \text{ and } L(\beta_j(\vec{b})) > L(\beta_j(\vec{a}))\right)\right],$$

then $\vec{b} \succ \vec{a}$.

- *If* $a_i I b_i$ *for all* $i \leq |\vec{a}|$ *and* $\{\beta_{|\vec{a}|+1}(\vec{b}) P \gamma_1(\vec{a})$ *or* $\gamma_1(\vec{b}) P \beta_1(\vec{b}) I \gamma_1(\vec{a}) I \beta_1(\vec{a})\}$, *then* $\vec{b} \succ \vec{a}$.

- *If* $a_i I b_j$ *for all* $a_i \in \vec{a}, b_j \in \vec{b}$ *then* $\vec{a} \sim \vec{b}$.

Again, the extended leximin-desirability rules coincide with the leximin-desirability rule when the actions to be compared have the same cardinality. Otherwise, they follow a comparison process that is dual to that of the extended leximax-desirability rules. In particular, a preference for the action with the smaller number of outcomes is now established unanimously by every rule in the family only if its worst outcome is better than the worst outcome of the other action or when all the outcomes of the smaller action are weakly better than all those of the larger one. In turn, in order to ensure a unanimous preference for the action with the larger number of outcomes, the extended leximin-desirability rules apply the leximin-desirability procedure in a way analogous to that in which extended leximax-desirability rules apply the leximax-desirability procedure to establish a preference for the action with the smaller number of outcomes.

The above extensions can be identified by making use of the axiomatic battery from Section 3 and the additional assumptions of DOM and richness of the domain.[10]

Theorem 4 *Let* X *be rich. Then, a preorder* $\succsim \subseteq Q \times Q$ *satisfies IND, LS and DOM if, and only if,* $\succsim \in \succsim^e_{LL}$.

Theorem 5 *Let* X *be rich. Then, a preorder* $\succsim \subseteq Q \times Q$ *satisfies IND, NEU, WLS1, HOS and DOM if, and only if,* $\succsim \in \succsim^e_{LD}$.

Theorem 6 *Let* X *be rich. Then, a preorder* $\succsim \subseteq Q \times Q$ *satisfies IND, NEU, WLS2, LOS and DOM if, and only if,* $\succsim \in \succsim^e_{ld}$.

Remark 1 *Kannai and Peleg [**?**] proved the impossibility of combining certain ideas of Dominance and Independence in the set-based approach to choice*

[10]Although one might think these families of rules are empty due to intransitivities, this is not the case, as shown by the examples included in the Appendix, where we present examples of rules from each of the families.

under complete uncertainty (ranking sets of outcomes with no likelihood information) when X has at least six non-indifferent outcomes. Also, Bossert [?] and Barberà et al. [?] proved that, when adding Neutrality to the said ideas of Dominance and Independence, the impossibility holds for any domain X with at least four non-indifferent outcomes. A remarkable feature of Theorems **??**, **??** *and* **??** *is that, if we admit ordinal likelihood information, our proposed adaptations of the ideas of Dominance, Independence and Neutrality (axioms DOM, IND and NEU) become compatible, even in an infinite domain.*

4 Weighted likelihood criteria

The results of Theorems **??** to **??** contribute towards a discussion of the proposed lexicographic rules based on their axiomatic structure.[11] $\succsim_{LL}$ is characterized in Theorem **??** by the combination of IND and LS. However, IND and LS seem to appeal to logically incompatible ideas. LS appeals to the intuitive idea that the relative likelihood of one outcome of an action with respect to another depends on the number of places between them in the likelihood ordering. Meanwhile, IND argues that the deletion of the same outcome of two actions, when their position is the same, makes no difference to the comparison between the actions, which is an implicit assumption that there is no information to be gained from the number of places (in the likelihood ordering) separating two outcomes of an action. Thus, what Theorem 1 makes clear to us is that there is a way of making the two conditions compatible, but that it forces the rule to proceed lexicographically in the manner of $\succsim_{LL}$.[12]

Axioms IND and LS are not the only ones introducing strong logical tension into the characterization results. IND also conflicts indirectly with NEU because their combination leads to the strong conclusion that there is no possible trade-off between the desirability of the outcomes and their likelihood positions. To see this, let $a, b, c, d, e \in X$ be such that $aPbPcPdPe$. Then, observe that, by IND, we have that $(b, c) \succ (b, d)$ and, by NEU, we have that $(a, e) \sim (b, d) \Leftrightarrow (a, e) \sim (b, c)$. These two facts, together with transitivity, imply that it is not possible for a criterion satisfying IND and NEU to establish that $(a, e) \sim (b, d)$. This is a quite plausible comparison that is precluded

[11] We thank an anonymous referee for drawing our attention to this discussion.

[12] The same applies to the combination of IND and the weaker versions of LS used in Theorems **??** and **??**.

by rules such as $\succsim_{LD}$ and $\succsim_{ld}$, which, as shown in Theorems **??** and **??**, are characterized by the combination of these axioms.[13]

In sum, IND seems to be at the center of many axiomatic tensions. Next, we proceed by relaxing IND to the following implied property, which we call *Responsiveness* (RES): Consider two actions $\vec{a}$ and $\vec{b}$ such that both share $(k-1)$ possible outcomes with the same relative likelihood ordering. The only difference is that $\vec{a}$ and $\vec{b}$ may each generate an additional outcome (a_j and b_j, respectively) at the same relative likelihood position. Then, RES states that the preferred action is the one where the different outcome is better. RES is related to the *Dominance* properties in Kelsey's [**?**] state space-based framework.

Responsiveness: For all $j, k \in \mathbb{N}$ such that $j \leq k$, and all $\vec{a}, \vec{b} \in Q_k$ such that $a_i = b_i$ for all $i \in (\{1, \ldots, k\} \setminus \{j\})$,

$$a_j R b_j \Leftrightarrow \vec{a} \succsim \vec{b}.$$

According to the above discussion, the weakening of IND to RES alleviates the tension with LS, but not with NEU. Therefore, besides relaxing the independence property IND to RES, in this section we proceed by replacing NEU with a new invariance property called *Reversal of Order* (RO). As we will observe, this replacement allows for cardinal information to be embedded in the preference ranking. The new property is inspired by another of the same name that appears in the Anscombe-Aumann subjective expected utility model (see Anscombe and Aumann [**?**] and Hammond [**?**]). Intuitively, the property states that, if the outcome the agent receives is to be determined by both an uncertain situation and a *risk* situation, then it is irrelevant whether the risk is faced before or after the uncertainty. A formal definition requires the extension of our model to cover risk situations.

We denote a standard risk situation (i.e., lottery) by $[p^1, x_1; p^2, x_2; \ldots; p^k, x_k]$, where p^i specifies the objective probability of outcome x_i. In a similar fashion, we can define lotteries in which probabilities are associated to actions rather than outcomes. A lottery over actions is denoted by $[p^1, \vec{a}^1; p^2, \vec{a}^2; \ldots; p^k, \vec{a}^k]$. In this general model, we must also consider actions in which the ordinal likelihood relation applies not to final outcomes but to lotteries. We denote

[13] This fact is not exclusive to our characterizations. Something similar happens with Pattanaik and Peleg's [**?**] characterization of the leximin criterion for the case of complete uncertainty.

such general actions by $([p^1, \vec{a}^1; p^2, \vec{a}^2; \ldots; p^k, \vec{a}^k], \ldots, [q^1, \vec{b}^1; q^2, \vec{b}^2; \ldots; q^l, \vec{b}^l])$. In this extended model, a preference must compare not only uncertainty situations (i.e., actions) but also lotteries, and more generally lotteries over actions and actions over lotteries. For the sake of simplicity, we will maintain the same notation for the preference relation defined on this extended domain.

Reversal of Order: For all $k, m \in \mathbb{N}$, for all probability distribution $(p^1, \ldots, p^m)$ and all $\vec{a^1}, \ldots, \vec{a^m} \in Q_k$,

$$[p^1, \vec{a^1}; \ldots; p^m, \vec{a^m}] \sim ([p^1, a_1^1; \ldots, p^m, a_1^m], \ldots, [p^1, a_k^m; \ldots; p^m, a_k^m]).$$

Finally, we assume that the agent satisfies the standard Expected Utility assumptions on lotteries on outcomes. We call this the *Expected Utility* (EU) axiom.

Expected Utility: $\succsim$ satisfies the Expected Utility assumptions on lotteries on final outcomes.

An additional piece of notation will be useful for formulating our next result. Take any cardinality, $k \in \mathbb{N}$, of actions, any $s \in \{1, \ldots, k\}$ and any $T \subseteq \{1, \ldots, k\}$. Then, we define $T_{-s} = \{j \in T, j < s\}$, $T_{+s} = \{j \in T, j \geq s\}$ and $T_{+s}^* = \{j \in \{2, \ldots, k+1\}, j = p+1, \text{ with } p \in T_{+s}\}$.

T represents any subset of positions of actions of dimension k, T_{-s} and T_{+s} represent the respective subsets of the positions in T which are lower (resp. higher) than a given position s, and T_{+s}^* represents a set consisting of the positions of T_{+s} increased in one unit.

We are now ready to introduce the characterization theorem of the family of *weighted likelihood* criteria, where actions are evaluated by a weighted average of the utilities of their outcomes according to weights that decrease with the likelihood positions.[14]

Theorem 7 *Let X be rich such that all indifference classes are infinite and assume that RO and EU hold. Then, a complete preorder $\succsim \subseteq Q \times Q$ satisfies RES, REO and DOM if, and only if, there exists $u : X \to \mathbb{R}$ and for all $k \in \mathbb{N}$, there exists $\omega^k = (\omega_1^k, \ldots, \omega_k^k) \in \mathbb{R}_{++}^k$ such that (i) $\omega_i^k > \omega_{i+1}^k$ for*

[14] The family of weighted likelihood criteria can, alternatively, be characterized by axioms that deal only with uncertain actions, but Theorem **??** allows us to obtain the characterization on the basis of more primitive assumptions. We thank an anonymous referee for guiding us in this direction.

all $i \in \{1,\ldots,k-1\}$, *(ii)* $\sum_{i=1}^{k} \omega_i^k = 1$, *(iii)* $\sum_{i\in T} \omega_i^k \leq \sum_{i\in T_{-s}\cup T^*_{+s}\cup\{s\}} \omega_i^{k+1}$ *for all* $s \in \{1,\ldots,k\}$ *and all* $T \subseteq \{1,\ldots,k\}$*, such that:*

$$\textit{for all } \vec{a},\vec{b} \in Q,\ \vec{a} \succsim \vec{b} \Leftrightarrow \sum_{i=1}^{|\vec{a}|} \omega_i^{|\vec{a}|} \cdot u(a_i) \geq \sum_{i=1}^{|\vec{b}|} \omega_i^{|\vec{b}|} \cdot u(b_i).$$

The weighted likelihood criteria compare actions with the same cardinality in a *subjective expected utility* manner. However, when comparing actions with different cardinality, it is necessary to determine a vector of weights for each cardinality value. In this case, the axioms imply that, when a new better outcome is inserted into an action, the distribution function for the random utility associated with the new action first order stochastically dominates the distribution function for the random utility associated with the given action. Formally, this is expressed by condition (*iii*), which imposes restrictions on the way weights can vary across different cardinality values, as follows: Consider any action, $\vec{a} \in Q_k$, for any $k \in \mathbb{N}$. The sum of the weights of any subset of outcomes of $\vec{a}$ that occupy a set of positions T should be lower than the sum of the weights of this set of outcomes and the weight of a new outcome that has been inserted into any s-th position of this action. For example, for $k = 5$, if we consider $T = \{1,4,5\}$ and $s = 4$, the restriction says that $\omega_1^5 + \omega_4^5 + \omega_5^5 \leq \omega_1^6 + \omega_4^6 + \omega_5^6 + \omega_6^6$.[15]

5 Conclusions and further research

We have proposed a new formal framework for the analysis of problems of choice under uncertainty in environments where the decision-maker is unable to establish a complete probability distribution among the outcomes of each action, but is able to rank them in terms of their likelihood. We first show that the comparison of actions having different cardinality poses more difficulties than the comparison of actions having equal cardinality. We therefore begin by analyzing the equal-cardinality case, characterizing different rules by imposing intuitive adaptations to our framework of axioms from the related literature. These rules compare actions lexicographically, maintaining the spirit of other

[15] As in Theorems **??**, **??** and **??**, one might think that the weighted likelihood family is empty, in this case due to the incompatibility of all the restrictions imposed on the weights. Again, we show in the Appendix that this is, in fact, not the case.

lexicographic rules proposed in the related literature. Taking these results as a reference, we explore the different-cardinality case by simply introducing the Dominance axiom, thus obtaining the characterization of three families that extend the respective rules of the equal-cardinality case to the general case. Additionally, we characterize a family of rules that evaluate actions by a weighted utility of the possible outcomes, in which the weights retain the ordinal likelihood ordering perceived by the individual.

Regarding further research, it would be of interest to investigate whether additional plausible conditions might constrain the families characterized in the paper. Another line of research would be to relax the linearity assumption about the likelihood relation among the outcomes within each action. It could be the case that certain pairs of possible outcomes within an action are perceived by the agent as being equally likely, in which case the likelihood relation among the outcomes should admit indifferences. It appears that this would affect the model in a nontrivial way, right from the notational stage, because actions could then no longer be described as ordered sets.

As a matter of fact, from a bounded rationality-like perspective, it would be reasonable to relax even further the structure of the binary likelihood relation among the outcomes within each action. A very appealing line of research would be to analyze the consequences of assuming that the likelihood relation is, for example, no more than an interval order, a semi-order, or a partial order.

It is also worth noting that, in our framework, the agent is able to establish comparisons of relative likelihood between consequences of the same action, but not across outcomes that are consequences of different actions.[16] For example, our framework does not allow us to determine whether an outcome x under action A is more or less likely than outcome y under action B. There are numerous situations in which ordinal comparisons of this kind are part of the decision-maker's input of the problem. Consideration of this possibility would constitute a very reasonable extension of our model. An adequate development of this issue is sufficiently complicated as to be beyond the scope of this paper.

[16] We thank an anonymous referee for raising this point.

Appendix

Here we show the proofs of the theorems. We begin with a lemma that will be useful in the proofs of the results.

Lemma 1 *Let $\succsim$ be a binary relation on Q. Then, the following statements hold:*

1. *If $\succsim$ satisfies IND and WLS1, then it also satisfies REO.*
2. *If $\succsim$ satisfies IND and WLS2, then it also satisfies REO.*

Proof: We will prove both statements using the same reasoning. Let $\succsim \subseteq (Q \times Q)$ satisfying IND and WLS1 (or WLS2), and $\vec{a} \in Q_k$ for any $k \in \mathbb{N}$ such that $a_j P a_i$, with $j > i$. Note that $\Pi_{(i,j)}(\vec{a})$ and $\vec{a}$ have the same outcomes in each position, except for positions i and j. Then, we apply IND $(k-2)$-times obtaining

$$\vec{a} \succsim \Pi_{(i,j)}(\vec{a}) \Leftrightarrow (a_i, a_j) \succsim (a_j, a_i)$$

$$\Pi_{(i,j)}(\vec{a}) \succsim \vec{a} \Leftrightarrow (a_j, a_i) \succsim (a_i, a_j)$$

Given that $(a_j) \succ (a_i)$, the application of WLS1 (or WLS2) implies that there exists $\vec{c} \in Q$ such that $(a_j, \vec{c}, a_i) \succ (a_i, \vec{c}, a_j)$. Now, applying IND $|\vec{c}|$-times, we obtain that $(a_j, a_i) \succ (a_i, a_j)$. Consequently, $\Pi_{(i,j)}(\vec{a}) \succ \vec{a}$ and $\succsim$ satisfies REO. $\square$

Proof of Result ??

Let $x, y, z \in X$ be such that $xPyPz$, and consider the set $\vec{a} = (x, z, y)$. Then, if we apply the permutation $\Pi_{(2,3)}$ to $\vec{a}$, we obtain the set $\Pi_{(2,3)}(\vec{a}) = (x, y, z)$. Given that yPz, by REO we have that $(x, y, z) \succ (x, z, y)$. Furthermore, by DOM we can conclude that $(x) \succ (x, y)$ and $(z, y) \succ (z)$. Applying COM, we have that $(x, z, y) \succsim (x, y, z)$, and by transitivity, $(x, y, z) \succ (x, y, z)$, thus contradicting reflexivity.

Proof of Theorem ??

The necessary part can easily be checked. To prove the sufficient part, take $\vec{a}, \vec{b} \in Q_k$. If $a_i I b_i$ for all $i \in \{1, \ldots, k\}$, then, by the Extension assumption on

$\succsim$ we have that $(a_1) \sim (b_1)$. Then, by successive applications of IND we obtain $\vec{a} \sim \vec{b}$. Otherwise, we can assume, by IND, that $\neg(a_i I b_i)$ for all $i \in \{1, \ldots, k\}$ and we need to prove that $\vec{a} \succ \vec{b}$ whenever $a_1 P b_1$. We will proceed by induction on k. Let us start with $k = 1$. Suppose, without loss of generality, that $a_1 P b_1$. Then, by the Extension assumption, we have that $(a_1) \succ (b_1)$ and it is proved for $k = 1$. Now, we will suppose that the statement is true for $k = t$ and we will prove the case $k = t + 1$. We have, by the induction hypothesis, that $(a_1, \ldots, a_t) \succ (b_1, \ldots, b_t)$ when $a_1 P b_1$. Then, LS says that there exists $\vec{c} \in Q$ such that $(a_1, \ldots, a_t, \vec{c}, a_{t+1}) \succ (b_1, \ldots, b_t, \vec{c}, b_{t+1})$. Applying IND $|\vec{c}|$-times, we obtain that $\vec{a} \succ \vec{b}$, thus proving the result. Therefore, $\succsim = \succsim_{LL}$.

Proof of Theorem ??

The necessary part is straightforward. To prove the sufficient part, take $\vec{a}, \vec{b} \in Q_k$. If $a_i I b_i$ for all $i \in \{1, \ldots, k\}$, then, by the Extension assumption on $\succsim$ we have that $(a_1) \sim (b_1)$. Then, by successive applications of IND we obtain $\vec{a} \sim \vec{b}$. In other case, we can assume, by IND, that $\neg(a_i I b_i)$ for all $i \in \{1, \ldots, k\}$. If $k = 1$, we know that $a_1 P b_1 \Rightarrow \vec{a} \succ \vec{b}$. If $k > 1$, we need to prove, without loss of generality, the following two cases:

1. For all $x \in \max\{\vec{a}\}$, $y \in \max\{\vec{b}\}$, xPy. Consider, first, that $|\max\{\vec{a}\}| = 1$ and $L(\gamma_1(\vec{a})) = k$. Select $x \notin \vec{a}$ such that $a_k P x$, the existence of which is guaranteed. Then, we construct the set $\vec{a}' = (a_1, \ldots, a_{k-1}, x)$. Now, by applying HOS to sets $\vec{a}'$ and $\vec{b}$, we find that there exists $y \notin \vec{a}'$ such that $(a_1, \ldots, a_{k-1}, y) \succ \vec{b}$. Now, if $a_k R y$, we can apply RES, which is weaker than IND (see Section 4) and which gives us $\vec{a} \succsim (a_1, \ldots, a_{k-1}, y)$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$. If $y P a_k$, then, by NEU and transitivity, $\vec{a} \succ \vec{b}$. If $|\max\{\vec{a}\}| = 1$ and $L(\gamma_1(\vec{a})) = i < k$, we have, by Lemma **??**, that REO can be applied to give $\vec{a} \succ \Pi_{(i,k)}(\vec{a})$. Now, by applying the previous reasoning to $\Pi_{(i,k)}(\vec{a})$ and $\vec{b}$, we obtain that $\Pi_{(i,k)}(\vec{a}) \succ \vec{b}$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$. If, on the other hand, $|\max\{\vec{a}\}| > 1$, let $j = L(\gamma_1(\vec{a}))$ and let $T \subset \{1, \ldots, k\}$ be such that $i \in T \Leftrightarrow a_i \notin \max\{\vec{a}\}$. Then, we construct the set $\vec{a}''$ such that $a_i = a_i''$ for all $i \in T \cup \{j\}$ and $a_i P a_i''$ for all $i \in \{1, \ldots, k\} \setminus (T \cup \{j\})$. By RES, $\vec{a} \succ \vec{a}''$. Given that $|\max\{\vec{a}''\}| = 1$, we can apply the previous reasoning to obtain $\vec{a}'' \succ \vec{b}$. By transitivity $\vec{a} \succ \vec{b}$.

2. For all $x \in \max\{\vec{a}\}$, $y \in \max\{\vec{b}\}$, xIy, with $L(\gamma_1(\vec{a})) = i < L(\gamma_1(\vec{b}))$. Consider the actions $(a_1, \ldots, a_i), (b_1, \ldots, b_i) \in Q_i$. We can apply Case 1, which gives us $(a_1, \ldots, a_i) \succ (b_1, \ldots, b_i)$. We know, by WLS1, that there exists $\vec{c} \in Q$ such that $(a_1, \ldots, a_i, \vec{c}, a_{i+1}) \succ (b_1, \ldots, b_i, \vec{c}, b_{i+1})$. Then, by applying IND $|\vec{c}|$-times, we find that $(a_1, \ldots, a_i, a_{i+1}) \succ (b_1, \ldots, b_i, b_{i+1})$. By repeating this process $(k - i)$-times, we obtain $\vec{a} \succ \vec{b}$.

Therefore, $\succsim = \succsim_{LD}$.

Proof of Theorem ??

The necessary part is straightforward. To prove the sufficient part, take $\vec{a}, \vec{b} \in Q_k$. If $a_i I b_i$ for all $i \in \{1, \ldots, k\}$, then, by the Extension assumption on $\succsim$ we have that $(a_1) \sim (b_1)$. Then, by successive applications of IND we obtain $\vec{a} \sim \vec{b}$. In other case, we can assume, by IND, that $\neg(a_i I b_i)$ for all $i \in \{1, \ldots, k\}$. If $k = 1$, we know that $a_1 P b_1 \Rightarrow \vec{a} \succ \vec{b}$. If $k > 1$, we need to prove, without loss of generality, the following two cases:

1. For all $x \in \min\{\vec{a}\}$, $y \in \min\{\vec{b}\}$, xPy. Consider, first, that $|\min\{\vec{b}\}| = 1$ and $L(\beta_1(\vec{b})) = k$. Select $x \notin \vec{b}$ such that xPb_k, the existence of which is guaranteed. Then, we construct the action $\vec{b}' = (b_1, \ldots, b_{k-1}, x)$. Now, by applying LOS to actions $\vec{a}$ and $\vec{b}'$, we find that there exists $y \notin \vec{b}'$ such that $\vec{a} \succ (b_1, \ldots, b_{k-1}, y)$. Now, if yRb_k, we can apply RES, which is weaker than IND and which gives us $(b_1, \ldots, b_{k-1}, y) \succsim \vec{b}$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$. If $b_k P y$, then, by NEU and transitivity, $\vec{a} \succ \vec{b}$. If $|\min\{\vec{b}\}| = 1$ and $L(\beta_1(\vec{b})) = i < k$, we have, by Lemma **??**, that REO can be applied to give $\Pi_{(i,k)}(\vec{b}) \succ \vec{b}$. Now, by applying the previous reasoning to $\vec{a}$ and $\Pi_{(i,k)}(\vec{b})$, we obtain that $\vec{a} \succ \Pi_{(i,k)}(\vec{b})$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$. If, on the other hand, $|\min\{\vec{b}\}| > 1$, let $j = L(\beta_1(\vec{b}))$ and let $T \subset \{1, \ldots, k\}$ be such that $i \in T \Leftrightarrow b_i \notin \min\{\vec{b}\}$. Then, we construct the set $\vec{b}''$ such that $b_i = b_i''$ for all $i \in T \cup \{j\}$ and $b_i'' P b_i$ for all $i \in \{1, \ldots, k\} \setminus (T \cup \{j\})$. By RES, $\vec{b}'' \succ \vec{b}$. Given that $|\min\{\vec{b}''\}| = 1$, we can apply the previous reasoning to obtain $\vec{a} \succ \vec{b}''$. By transitivity, $\vec{a} \succ \vec{b}$.

2. For all $x \in \min\{\vec{a}\}$, $y \in \min\{\vec{b}\}$, xIy, with $L(\beta_1(\vec{a})) = i > L(\beta_1(\vec{b}))$. Consider the actions $(a_1, \ldots, a_i), (b_1, \ldots, b_i) \in Q_i$. We can apply Case 1,

which gives us $(a_1, \ldots, a_i) \succ (b_1, \ldots, b_i)$. We know, by WLS2, that there exists $\vec{c} \in Q$ such that $(a_1, \ldots, a_i, \vec{c}, a_{i+1}) \succ (b_1, \ldots, b_i, \vec{c}, b_{i+1})$. Then, by applying IND $|\vec{c}|$-times, we find that $(a_1, \ldots, a_i, a_{i+1}) \succ (b_1, \ldots, b_i, b_{i+1})$. By repeating this process $(k-i)$-times, we obtain $\vec{a} \succ \vec{b}$.

Therefore, $\succsim = \succsim_{ld}$.

Proof of Proposition ??

We will now define some rankings on $\bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ to show the independence of the collections of axioms used in the different characterization theorems of Section 3.1.

1. Independence of IND and LS.
   - Let $\succsim_1 \in \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ be such that for all $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$, $\vec{a} \succsim_1 \vec{b} \Leftrightarrow a_1 R b_1$. Then, $\succsim_1$ satisfies LS, but it does not satisfy IND.
   - $\succsim_{LD}$ satisfies IND, but it does not satisfy LS.
2. Independence of IND, NEU, WLS1 and HOS.
   - Let $\succsim_2 \in \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ be such that for all $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$,

     $$\vec{a} \succsim_2 \vec{b} \Leftrightarrow [\gamma_1(\vec{a}) P \gamma_1(\vec{b}) \text{ or } (\gamma_1(\vec{a}) I \gamma_1(\vec{b}) \text{ and } L(\gamma_1(\vec{a})) \leq L(\gamma_1(\vec{b})))].$$

     Then, $\succsim_2$ satisfies NEU, WLS1 and HOS, but not IND.
   - Consider a utility function u that represents R, any $x, y \in X$, with $x \neq y$, and a non-decreasing real-valued function, f, such that $f(u(x)) = f(u(y))$ and such that it is strictly increasing outside the interval between $u(x)$ and $u(y)$. We denote by $\pi(\vec{a})$ a permutation of the outcomes in $\vec{a}$ such that for all $i \in \{1, \ldots, k-1\}$, $f(u(\pi_i(\vec{a}))) > f(u(\pi_{i+1}(\vec{a})))$ or $f(u(\pi_i(\vec{a}))) = f(u(\pi_{i+1}(\vec{a})))$ and $L(\pi_i(\vec{a})) < L(\pi_{i+1}(\vec{a}))$, where $\pi_i(\vec{a})$ denotes the element of $\vec{a}$ that occupies the i-th position after the permutation.

     Let $\succsim_3 \in \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ be such that for all $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$,

     $$\vec{a} \succsim_3 \vec{b} \Leftrightarrow \text{ there is } j \leq k \text{ such that:}$$

for all $i \leq j$, $\left[f(u(\pi_i(\vec{a}))) = f(u(\pi_i(\vec{b}))) \text{ and } L(\pi_i(\vec{a})) = L(\pi_i(\vec{b}))\right]$ and

– If $j < k$, $\left[f(u(\pi_j(\vec{a}))) > f(u(\pi_j(\vec{b}))) \text{ or } \left(f(u(\pi_j(\vec{a}))) = f(u(\pi_j(\vec{b}))) \text{ and } L(\pi_j(\vec{a})) < L(\pi_j(\vec{b}))\right)\right]$.

– If $j = k$, $\vec{a} \succsim_{LD} \vec{b}$.

Then, $\succsim_3$ satisfies IND, WLS1 and HOS, but not NEU.

- Let $\succsim_4 \in \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ be such that for all $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$,

$$\vec{a} \succsim_4 \vec{b} \Leftrightarrow \text{ there is not } j \leq k \text{ such that for all } i < j, \gamma_i(\vec{b}) I \gamma_i(\vec{a}) \text{ and } \gamma_j(\vec{b}) P \gamma_j(\vec{a}).$$

Then, $\succsim_4$ satisfies IND, NEU and HOS, but not WLS1.

- $\succsim_{LL}$ satisfies IND, NEU and WLS1, but not HOS.

3. Independence of IND, NEU, WLS2 and LOS. The proof is analogous to that of item 2. The kind of rules that enable us to prove the independence of each axiom are dual to those used in item 2.

Proof of Theorem ??

We have that IND and LS imply the desired result for all comparisons when the sets are of the same cardinality (see Theorem **??**). For the remaining comparisons, take $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $(b_1, \ldots, b_k) \succ_{LL} \vec{a}$. Then, the richness assumption enables us to select $y_1, \ldots, y_{m-k} \in X$ such that $y_{m-k} P y_{m-k-1} P \ldots P y_1 P \gamma_1(\vec{a})$. By applying DOM, we find that $(\vec{a}, y_1) \succ \vec{a}$. By another application of DOM we find that $(\vec{a}, y_1, y_2) \succ (\vec{a}, y_1)$. Successive applications of this process and transitivity show that $(\vec{a}, y_1, \ldots, y_{m-k}) \succ \vec{a}$. Note that $(\vec{a}, y_1, \ldots, y_{m-k})$ and $\vec{b}$ have the same cardinality. Then, by Theorem **??**, we know that $\vec{b} \succ (\vec{a}, y_1, \ldots, y_{m-k})$ and, by transitivity, $\vec{b} \succ \vec{a}$.

2. $(b_1, \ldots, b_k) \sim_{LL} \vec{a}$ and $b_j P b_i$ for all $j > k$ and $i \leq k$. Then, the richness assumption enables us to select $z_1, \ldots, z_{m-k} \in X$ such that

$\gamma_1(\vec{b})Pz_{m-k}P\ldots Pz_1P\gamma_1(\vec{a})$. Then, by applying DOM and transitivity, as in item 1, we find that $(\vec{a}, z_1, \ldots z_{m-k}) \succ \vec{a}$. Note that $(\vec{a}, z_1, \ldots, z_{m-k})$ and $\vec{b}$ have the same cardinality. Then, by Theorem **??**, we know that $\vec{b} \succ (\vec{a}, z_1, \ldots, z_{m-k})$ and, by transitivity, $\vec{b} \succ \vec{a}$.

3. $\vec{a} \succ_{LL} (b_1, \ldots, b_k)$. Then, by the richness assumption, we can select $x_1, \ldots, x_{m-k} \in X$ such that $\beta_1(\vec{a})Px_1P\ldots Px_{m-k}$. By applying DOM, we find that $\vec{a} \succ (\vec{a}, x_1)$. By another application of DOM, we find that $(\vec{a}, x_1) \succ (\vec{a}, x_1, x_2)$. Successive applications of this process and transitivity show that $\vec{a} \succ (\vec{a}, x_1, \ldots, x_{m-k})$. Note that $(\vec{a}, x_1, \ldots, x_{m-k})$ and $\vec{b}$ have the same cardinality. By Theorem **??**, we know that $(\vec{a}, x_1, \ldots, x_{m-k}) \succ \vec{b}$, and by transitivity, $\vec{a} \succ \vec{b}$.

4. $(b_1, \ldots, b_k) \sim_{LL} \vec{a}$ and b_iPb_j for all $i \leq k$ and $j > k$. Then, the richness assumption enables us to select $w_1, \ldots, w_{m-k} \in X$ such that $\beta_1(\vec{a})Pw_1P\ldots Pw_{m-k}P\gamma_{k+1}(\vec{b})$. Then, by applying DOM and transitivity, as in item 1, we find that $\vec{a} \succ (\vec{a}, w_1, \ldots w_{m-k})$. Note that $(\vec{a}, w_1, \ldots, w_{m-k})$ and $\vec{b}$ have the same cardinality. Then, by Theorem **??**, we know that $(\vec{a}, w_1, \ldots, w_{m-k}) \succ \vec{b}$ and, by transitivity, $\vec{a} \succ \vec{b}$.

5. a_iIb_j for all $i, j \leq |\vec{a}|$ and $\gamma_1(\vec{b})P\beta_1(\vec{b})I\gamma_1(\vec{a})$. If b_jPb_i for all $j > k$ and $i \leq k$, case 2 applies. If not, let t be the highest integer such that $\beta_1(\vec{b})I\beta_t(\vec{b})$. Observe that $k < t < m$. Consider $t-k$ outcomes $w_1, \ldots, w_{t-k} \in (X \setminus \vec{a})$ such that $w_iI\beta_1(\vec{b})$ for all $i \in \{1, \ldots, t-k\}$, which always exist by definition of t. Then, by DOM and transitivity, we have that $\vec{a} \sim (\vec{a}, w_1, \ldots, w_{t-k})$. On the other hand, consider $\beta(\vec{b}) = (\beta_1(\vec{b}), \ldots, \beta_m(\vec{b}))$. By Lemma **??**, we can apply REO to obtain that $\vec{b} \succ \beta(\vec{b})$. Now, by the richness assumption, consider $v_1, \ldots, v_{m-t} \in X$ such that $\beta_{t+1}(\vec{b})Pv_{m-t}P\ldots Pv_1P\beta_1(\vec{b})$. Successive applications of DOM and transitivity leads to $(\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t}) \succ (\vec{a}, w_1, \ldots, w_{t-k})$. Note that $(\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t})$ and $\beta(\vec{b})$ have the same cardinality and, therefore, Theorem **??** can be applied to obtain that $\beta(\vec{b}) \succ (\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t})$. Transitivity concludes that $\vec{b} \succ \vec{a}$.

6. a_iIb_j for all $i, j \leq |\vec{a}|$ and $\gamma_1(\vec{a})I\gamma_1(\vec{b})P\beta_1(\vec{b})$. If b_iPb_j for all $i \leq k$ and $j > k$, then case 4 applies. Otherwise, let t be the highest integer such that $\gamma_1(\vec{b})I\gamma_t(\vec{b})$. Observe that $k < t < m$. Consider

$t-k$ outcomes $w_1, \ldots, w_{t-k} \in (X \setminus \vec{a})$ such that $w_i I \gamma_1(\vec{b})$ for all $i \in \{1, \ldots, t-k\}$ that always exist by definition of t. Then, by DOM and transitivity, we have that $\vec{a} \sim (\vec{a}, w_1, \ldots, w_{t-k})$. On the other hand, consider $\gamma(\vec{b}) = (\gamma_1(\vec{b}), \ldots, \gamma_m(\vec{b}))$. By Lemma **??**, we can apply REO to obtain that $\gamma(\vec{b}) \succ \vec{b}$. Now, by the richness assumption, consider $v_1, \ldots, v_{m-t} \in X$ such that $\gamma_1(\vec{b}) P v_1 P \ldots P v_{m-t} P \gamma_{t+1}(\vec{b})$. Successive applications of DOM and transitivity leads to $(\vec{a}, w_1, \ldots, w_{t-k}) \succ (\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t})$. Note that $(\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t})$ and $\gamma(\vec{b})$ have the same cardinality and, therefore, Theorem **??** can be applied to obtain that $(\vec{a}, w_1, \ldots, w_{t-k}, v_1, \ldots, v_{m-t}) \succ \gamma(\vec{b})$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

7. $a_i I b_j$ for all $a_i \in \vec{a}$ and $b_j \in \vec{b}$. By Theorem **??**, we know that $(b_1, \ldots, b_k) \sim \vec{a}$. Now, by successive applications of DOM and transitivity, $\vec{b} \sim (b_1, \ldots, b_k)$. Transitivity concludes that $\vec{a} \sim \vec{b}$.

8. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim^e_{LL}$.

Proof of Theorem ??

We have that IND, NEU, WLS1 and HOS imply the result for all comparisons of sets having the same cardinality (see Theorem **??**). For the remaining comparisons, take $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $\gamma_1(\vec{b}) P \gamma_1(\vec{a})$. Then, making use of the richness assumption, take $x_1, \ldots, x_{m-k} \in X$ such that $\gamma_1(\vec{b}) P x_{m-k} P \ldots P x_1 P \gamma_1(\vec{a})$. Application of DOM shows that $(\vec{a}, x_1) \succ \vec{a}$. Repeated application of DOM, as in the proof of Theorem **??**, shows that $(\vec{a}, x_1, \ldots, x_{m-k}) \succ \vec{a}$. Note that $(\vec{a}, x_1, \ldots, x_{m-k})$ and $\vec{b}$ have the same cardinality. Then, by the result of Theorem **??**, we know that $\vec{b} \succ (\vec{a}, x_1, \ldots, x_{m-k})$, and by transitivity, $\vec{b} \succ \vec{a}$.

2. $\beta_1(\vec{b}) I \gamma_1(\vec{a}) P \beta_1(\vec{a})$. If $\gamma_1(\vec{b}) P \beta_1(\vec{b})$, case 1 applies. Then, we only have to prove the case in which $\gamma_1(\vec{b}) I \beta_1(\vec{b})$ and, therefore, $b_i I b_j$ for all $i, j \in$

$\{1, \ldots, m\}$. By Theorem **??**, we know that $(b_1, \ldots, b_k) \succ \vec{a}$. By successive applications of DOM and transitivity, $\vec{b} \sim (b_1, \ldots, b_k)$. Finally, transitivity concludes that $\vec{b} \succ \vec{a}$.

3. there is $j \leq k$ such that for all $i < j$, $\left[\gamma_i(\vec{a}) I \gamma_i(\vec{b}) \text{ and } L(\gamma_i(\vec{a})) = L(\gamma_i(\vec{b}))\right]$ and $\left[\gamma_j^*(\vec{a}) P \gamma_j(\vec{b}) \text{ or } \left(\gamma_j(\vec{a}) I \gamma_j(\vec{b}) \text{ and } L(\gamma_j(\vec{a})) < L(\gamma_j(\vec{b}))\right)\right]$. Then, by the richness assumption, consider $y_1, \ldots, y_{m-k} \in X$ such that $\beta_1(\vec{a}) P y_1 P \ldots P y_{m-k}$. Successive applications of DOM and transitivity, as in item 1, shows that $\vec{a} \succ (\vec{a}, y_1, \ldots, y_{m-k})$. Again, $(\vec{a}, y_1, \ldots, y_{m-k})$ and $\vec{b}$ have the same cardinality and, therefore, Theorem **??** can be applied to obtain that $(\vec{a}, y_1, \ldots, y_{m-k}) \succ \vec{b}$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$.

4. $a_i I b_i$ for all $i \leq k$ and $\beta_1(\vec{a}) P \gamma_{k+1}(\vec{b})$. Then, by the richness assumption, consider $z_1, \ldots, z_{m-k} \in X$ such that $\beta_1(\vec{a}) P z_1 P \ldots P z_{m-k} P \gamma_{k+1}(\vec{b})$. Successive applications of DOM and transitivity, as in item 1, shows that $\vec{a} \succ (\vec{a}, z_1, \ldots, z_{m-k})$. Again, $(\vec{a}, z_1, \ldots, z_{m-k})$ and $\vec{b}$ have the same cardinality and, therefore, Theorem **??** can be applied to show that $(\vec{a}, z_1, \ldots, z_{m-k}) \succ \vec{b}$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$.

5. $a_i I b_i$ for all $i \leq k$ and $\gamma_1(\vec{a}) I \beta_1(\vec{a}) I \gamma_1(\vec{b}) P \beta_1(\vec{b})$. If $\beta_1(\vec{a}) P \gamma_{k+1}(\vec{b})$, then case 4 applies. Otherwise, let t be the highest integer such that $\gamma_1(\vec{b}) I \gamma_t(\vec{b})$. Now, consider $v_1, \ldots, v_{t-k} \in (X \setminus \vec{a})$ such that $v_i I \gamma_1(\vec{b})$, whose existence is guaranteed by the definition of t. Now, by DOM and transitivity, we can conclude that $\vec{a} \sim (\vec{a}, v_1, \ldots, v_{t-k})$. Then, by the richness assumption, consider $w_1, \ldots, w_{m-t} \in X$ such that $\gamma_1(\vec{b}) P w_1 P \ldots P w_{m-t} P \gamma_{t+1}(\vec{b})$. Successive applications of DOM and transitivity leads to $(\vec{a}, v_1, \ldots, v_{t-k}) \succ (\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-t})$. Note that $(\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-t}$ and $\vec{b}$ have the same cardinality and, therefore, Theorem **??** can be applied to obtain that $(\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-k}) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

6. $a_i I b_j$ for all $a_i \in \vec{a}$ and $b_j \in \vec{b}$. By Theorem **??**, we know that $(b_1, \ldots, b_k) \sim \vec{a}$. Now, by successive applications of DOM and transitivity, $\vec{b} \sim (b_1, \ldots, b_k)$. Transitivity concludes that $\vec{a} \sim \vec{b}$.

7. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim^e_{LD}$

Proof of Theorem ??

We have that IND, NEU, WLS2 and LOS imply the result for all comparisons of sets having the same cardinality (see Theorem **??**). For the remaining comparisons, take $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $\beta_1(\vec{a})P\beta_1(\vec{b})$. Then, by the richness assumption, we can select $x_1, \ldots,$ $x_{m-k} \in X$ such that $\beta_1(\vec{a})Px_1P \ldots Px_{m-k}P\beta_1(\vec{b})$. By successive applications of DOM and transitivity, we have that $\vec{a} \succ (\vec{a}, x_1, \ldots, x_{m-k})$. Now, Theorem **??** can be applied to obtain $(\vec{a}, x_1, \ldots, x_{m-k}) \succ \vec{b}$. Transitivity allows us to conclude that $\vec{a} \succ \vec{b}$.

2. $\gamma_1(\vec{a})P\beta_1(\vec{a})I\gamma_1(\vec{b})$. If $\gamma_1(\vec{b})P\beta_1(\vec{b})$, case 1 applies. Then, we only have to prove the case in which $\gamma_1(\vec{b})I\beta_1(\vec{b})$ and, therefore, b_iIb_j for all $i, j \in \{1, \ldots, m\}$. By Theorem **??**, we know that $\vec{a} \succ (b_1, \ldots, b_k)$. By successive applications of DOM and transitivity, $\vec{b} \sim (b_1, \ldots, b_k)$. Finally, transitivity concludes that $\vec{a} \succ \vec{b}$.

3. there is $j \leq k$ such that for all $i < j$, $\left[\beta_i(\vec{b})I\beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\right]$ and $\left[\beta_j(\vec{b})P\beta_j(\vec{a}) \text{ or } \left(\beta_j(\vec{b})I\beta_j(\vec{a}) \text{ and } L(\beta_j(\vec{b})) > L(\beta_j(\vec{a}))\right)\right]$. Then, by the richness assumption, consider $x_1, \ldots, x_{m-k} \in X$ such that $x_{m-k}P \ldots Px_1P\gamma_1(\vec{a})$. As before, the successive application of DOM and transitivity lead to $(\vec{a}, x_1, \ldots, x_{m-k}) \succ \vec{a}$. Now, by Theorem **??**, $\vec{b} \succ (\vec{a}, x_1, \ldots, x_{m-k})$. Transitivity allows us to conclude that $\vec{b} \succ \vec{a}$.

4. If a_iIb_i for all $i \leq k$ and $\beta_{k+1}(\vec{b})P\gamma_1(\vec{a})$. Then, by the richness assumption, consider $z_1, \ldots, z_{m-k} \in X$ such that $\beta_{k+1}(\vec{b})Pz_{m-k}P \ldots Pz_1P\gamma_1(\vec{a})$. Successive applications of DOM and transitivity shows that $(\vec{a}, z_1, \ldots,$ $z_{m-k}) \succ \vec{a}$. Again, $(\vec{a}, z_1, \ldots, z_{m-k})$ and $\vec{b}$ have the same cardinality and, therefore, Theorem **??** can be applied to show that $\vec{b} \succ (\vec{a}, z_1, \ldots, z_{m-k})$. Transitivity allows us to conclude that $\vec{b} \succ \vec{a}$.

5. a_iIb_i for all $i \leq k$ and $\gamma_1(\vec{b})P\beta_1(\vec{b})I\gamma_1(\vec{a})I\beta_1(\vec{a})$. If b_jPb_i for all $j > k$ and $i \leq k$, case 4 applies. Otherwise, let t be the highest integer such that $\beta_1(\vec{b})I\beta_t(\vec{b})$. Now, consider $v_1, \ldots, v_{t-k} \in (X \setminus \vec{a})$ such that

$v_i I \beta_1(\vec{b})$, whose existence is guaranteed by the definition of t. Now, by DOM and transitivity, we can conclude that $\vec{a} \sim (\vec{a}, v_1, \ldots, v_{t-k})$. Then, consider, by the richness assumption, $w_1, \ldots, w_{m-t} \in X$ such that $\beta_{t+1}(\vec{b}) P w_{m-t} P \ldots P w_1 P \beta_1(\vec{b})$. Successive applications of DOM and transitivity leads to $(\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-t}) \succ (\vec{a}, v_1, \ldots, v_{t-k})$. Note that $(\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-t})$ and $\vec{b}$ have the same cardinality and, therefore, Theorem **??** can be applied to obtain that $\vec{b} \succ (\vec{a}, v_1, \ldots, v_{t-k}, w_1, \ldots, w_{m-t})$. Transitivity concludes that $\vec{b} \succ \vec{a}$.

6. $a_i I b_j$ for all $a_i \in \vec{a}$ and $b_j \in \vec{b}$. By Theorem **??**, we know that $(b_1, \ldots, b_k) \sim \vec{a}$. Now, by successive applications of DOM and transitivity, $\vec{b} \sim (b_1, \ldots, b_k)$. Transitivity concludes that $\vec{a} \sim \vec{b}$.

7. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim_{ld}^{e}$.

Proof of Theorem ??

Let us assume that $\succsim$ satisfies RES, REO, RO, EU and DOM. First, we are going to prove that actions with the same cardinality must be compared using a weighted average of the utilities of their possible results. This proof can be carried out following the same steps as described in the proof of Theorem 4.5 in Hammond [**?**], which, in turn, is a development of the proof provided by Anscombe and Aumann [**?**]. It is only necessary to note that we can assimilate the *states of nature* in his model with the likelihood positions in our model. Taking this into account, the parallelisms between Hammond's axiom structure and ours are the following: (*i*) his axiom *Ordering* (O) is equivalent to our assumption that $\succsim$ is a complete preorder; (*ii*) his axioms of *Independence* (I) and *Continuity* (C) are implied by EU; (*iii*) our RO axiom is an exact transfer to our domain of Hammond's axiom of the same name; (*iv*) his axiom of *State Independence* (SI) ensures that the individual preference is the same across all states of nature, while RES implies the same thing in our domain, except that it refers to positions instead of states of nature; and (*v*) his axiom of *Sure Thing Principle* (STP) is implied, as proved in Lemma 4.1 in Hammond [**?**], by O, RO and *Strong Independence* (I*), a property implied by EU.

Then, taking into account our assumption that all indifference classes on X are infinite, we can apply the reasoning presented in the proof of Theorem 4.5 in Hammond [?]. Thus, we find that there exists $u : X \to \mathbb{R}$ and for each $k \in \mathbb{N}$, there exists $\omega^k = (\omega_1^k, \ldots, \omega_k^k) \in \mathbb{R}_{++}^k$ such that for all $\vec{a}, \vec{b} \in Q_k$, $\vec{a} \succsim \vec{b} \Leftrightarrow \sum_{i=1}^{k} \omega_i^k \cdot u(a_i) \geq \sum_{i=1}^{k} \omega_i^k \cdot u(b_i)$.

We find that REO implies that $\omega_i^k > \omega_{i+1}^k$ for all $k \in \mathbb{N}$ and all $i \in \{1, \ldots, k-1\}$ (condition (i) in the statement of the theorem). We now need to prove that the weights also satisfy conditions (ii) and (iii). On the one hand, by DOM we have that any pair of actions having different cardinality are indifferent if all their outcomes are indifferent. Then, the sum of the weights must be the same for different cardinality values. Therefore, condition (ii) is satisfied, given that we can normalize this sum to 1. As for condition (iii), let us suppose that it is not true and, therefore, that the corresponding restrictions on the weights do not hold. Then, consider the weights associated with dimension k, a subset $T \subseteq \{1, \ldots, k\}$ and a value s violating condition (iii). Consider, by the assumption of richness and the assumption that all indifference classes are infinite, two actions $\vec{a} \in Q_k$, $\vec{b} \in Q_{k+1}$ such that: (a) $\vec{b}$ consists of the *insertion* of outcome x into position $s \in \{1, \ldots, k\}$ of action $\vec{a}$, (b) the utility values of the outcomes occupying the positions in T of action $\vec{a}$ are of a specific exact value $\bar{u}$, (c) the utilities of the remaining outcomes of $\vec{a}$ are of a specific exact value $\hat{u}$, with $\bar{u} > \hat{u}$, and (d) $u(x) = \bar{u} + \varepsilon$, with $\varepsilon > 0$ arbitrarily small.

Then, by DOM and REO, we should have that $\vec{b} \succ \vec{a}$. On the other hand, by construction, $\sum_{i=1}^{k} \omega_i^k \cdot u(a_i) = (\sum_{i \in T} \omega_i^k) \cdot \bar{u} + (1 - \sum_{i \in T} \omega_i^k) \cdot \hat{u}$ and $\sum_{i=1}^{k+1} \omega_i^{k+1} \cdot u(b_i)$ is arbitrarily close to $(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1}) \cdot \bar{u} + (1 - \sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1}) \cdot \hat{u}$. However, if $\sum_{i \in T} \omega_i^k > \sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1}$, it is possible that $\sum_{i=1}^{k} \omega_i^k \cdot u(a_i) \geq \sum_{i=1}^{k+1} \omega_i^{k+1} \cdot u(b_i)$ and, therefore, $\vec{a} \succsim \vec{b}$, which leads to a contradiction.

Let us now assume that $\succsim$ is a weighted likelihood criteria with the restriction on the weights stated in the theorem. First, it is easy to see that $\succsim$ satisfies RES and REO. In order to check that it also satisfies DOM, consider any action $\vec{a} \in Q_k$ for any $k \in \mathbb{N}$ and an outcome $x \notin \vec{a}$ such that $u(x) > u(\gamma_1(\vec{a}))$. Consider a new action $\vec{b}$ that consists of the insertion of x in position $s \in \{1, \ldots, k\}$ of action $\vec{a}$. We have to prove that $\vec{b} \succ \vec{a}$.

We will start by proving that the following equation holds for any set of positions $T = \{L(\gamma_1(\vec{a})), \ldots, L(\gamma_t(\vec{a}))\}$ for all $t \in \{1, \ldots, k\}$:

$$\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in T} \omega_i^k \cdot u(a_i) > u(\gamma_t(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big). \tag{1}$$

We will proceed inductively on t. Take $t = 1$. Then, $T = \{L(\gamma_1(\vec{a}))\}$. First, we have that

$$u(\gamma_1(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \omega^k_{L(\gamma_1(\vec{a}))} \big) = \sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(\gamma_1(\vec{a})) - \omega^k_{L(\gamma_1(\vec{a}))} \cdot u(\gamma_1(\vec{a})).$$

On the other hand, given that $u(x) > u(\gamma_1(\vec{a}))$, we have that

$$\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) > \sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(\gamma_1(\vec{a})).$$

Then, (**??**) holds for $t = 1$.

Now suppose we have proved (**??**) for a value p. Let us denote $T = \{L(\gamma_1(\vec{a})), \ldots, L(\gamma_p(\vec{a}))\}$ and $S = (T \cup \{L(\gamma_{p+1}(\vec{a}))\})$. In this case, the left-hand side of Equation **??**

$$\sum_{i \in S_{-s} \cup S^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in S} \omega_i^k \cdot u(a_i)$$

can be broken down into the addition of the following two terms:

$$\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in T} \omega_i^k \cdot u(a_i) \text{ and } u(\gamma_{p+1}(\vec{a})) \cdot \big(\omega^{k+1}_{L(\gamma_{p+1}(\vec{a}))} - \omega^k_{L(\gamma_{p+1}(\vec{a}))} \big).$$

Given the induction hypothesis, we know that the first term satisfies:

$$\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in T} \omega_i^k \cdot u(a_i) > u(\gamma_p(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big).$$

Then, we have that

$$\sum_{i \in S_{-s} \cup S^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in S} \omega_i^k \cdot u(a_i) >$$

$$u(\gamma_p(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big) + u(\gamma_{p+1}(\vec{a})) \cdot \big(\omega^{k+1}_{L(\gamma_{p+1}(\vec{a}))} - \omega^{k}_{L(\gamma_{p+1}(\vec{a}))} \big).$$

We also know by the condition on the weights assumed in the theorem that $(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k) \geq 0$. Then, given that $u(\gamma_p(\vec{a})) > u(\gamma_{p+1}(\vec{a}))$,

$$u(\gamma_p(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big) \geq u(\gamma_{p+1}(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big).$$

Therefore,

$$\sum_{i \in S_{-s} \cup S^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) - \sum_{i \in S} \omega_i^k \cdot u(a_i) >$$
$$u(\gamma_{p+1}(\vec{a})) \cdot \big(\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in T} \omega_i^k \big) + u(\gamma_{p+1}(\vec{a})) \cdot \big(\omega^{k+1}_{L(\gamma_{p+1}(\vec{a}))} - \omega^{k}_{L(\gamma_{p+1}(\vec{a}))} \big) =$$
$$u(\gamma_{p+1}(\vec{a})) \cdot \big(\sum_{i \in S_{-s} \cup S^*_{+s} \cup \{s\}} \omega_i^{k+1} - \sum_{i \in S} \omega_i^k \big).$$

Then, (**??**) holds for every $t \in \{1, \ldots, k\}$. In particular, when $t = k$, $\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} = \sum_{i=1}^{k+1} \omega_i^{k+1} = \sum_{i \in T} \omega_i^k = \sum_{i=1}^{k} \omega_i^k = 1$. Then, the right-hand side of (**??**) equals to 0. Therefore, $\sum_{i \in T_{-s} \cup T^*_{+s} \cup \{s\}} \omega_i^{k+1} \cdot u(b_i) = \sum_{i=1}^{k+1} \omega_i^{k+1} \cdot u(b_i) > \sum_{i \in T} \omega_i^k \cdot u(a_i) = \sum_{i=1}^{k} \omega_i^k \cdot u(a_i)$. Thus, $\vec{b} \succ \vec{a}$.

The proof that when $u(x) < u(\beta_1(\vec{a}))$, then $\vec{a} \succ \vec{b}$ is analogous. Finally, the proof that when $u(x) = u(\gamma_1(\vec{a})) = u(\beta_1(\vec{a}))$, then $\vec{a} \sim \vec{b}$ is straightforward given condition (ii) on the weights. Then, the proof is finished.

Examples of rules in the extended families

We provide four rules that belong to each of the extended families characterized in Theorems **??**, **??**, **??** and **??**. For that purpose we define Q^* as the set of all possible non-empty vectors that can be constructed with the elements of X. (Note that the difference between Q^* and Q is that the former domain allows to repeat elements of X).

- Theorem **??**: Consider $\succsim_1 \in \succsim^e_{LL}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ such that $|\vec{a}| < |\vec{b}|$ as follows:

$$\vec{a} \succsim_1 \vec{b} \Leftrightarrow (\vec{a})^*_{|\vec{b}|} \succsim^*_{LL} \vec{b},$$

where $(\vec{a})^*_{|\vec{b}|} = (\vec{a}, \beta_1(\vec{a}), \overset{|\vec{b}|-|\vec{a}|}{\ldots}, \beta_1(\vec{a}))$ and $\succsim^*_{LL}$ compares the elements of Q^* in the same way as $\succsim_{LL}$ compares the elements of Q.

- Theorem **??**: Consider $\succsim_2 \in \succsim^e_{LD}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ such that $|\vec{a}| < |\vec{b}|$ as follows:

$$\vec{a} \succsim_2 \vec{b} \Leftrightarrow (\vec{a})^*_{|\vec{b}|} \succsim^*_{LD} \vec{b},$$

where $(\vec{a})^*_{|\vec{b}|} = (\vec{a}, \beta_1(\vec{a}), \overset{|\vec{b}|-|\vec{a}|}{\ldots}, \beta_1(\vec{a}))$ and $\succsim^*_{LD}$ compares the elements of Q^* in the same way as $\succsim_{LD}$ compares the elements of Q.

- Theorem **??**:

Consider $\succsim_3 \in \succsim^e_{ld}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ such that $|\vec{a}| < |\vec{b}|$ as follows:

$$\vec{a} \succsim_3 \vec{b} \Leftrightarrow (\vec{a})^*_{|\vec{b}|} \succsim^*_{ld} \vec{b},$$

where $(\vec{a})^*_{|\vec{b}|} = (\vec{a}, \gamma_1(\vec{a}), \overset{|\vec{b}|-|\vec{a}|}{\ldots}, \gamma_1(\vec{a}))$ and $\succsim^*_{ld}$ compares the elements of Q^* in the same way as $\succsim_{ld}$ compares the elements of Q.

- Theorem **??**: We denote by $\lfloor x \rfloor$ the lowest integer part of x. Then, consider the criterion $\succsim_4 \in \Omega$ associated with an arbitrarily high positive number M such that $\omega^k_i = \frac{1}{k} + (\lfloor \frac{k}{2} \rfloor - i + 1) \cdot \varepsilon_k$, when $i < \frac{k+1}{2}$; $\omega^k_i = \frac{1}{k}$, when $i = \frac{k+1}{2}$; and $\omega^k_i = \frac{1}{k} - (i - \lfloor \frac{k+1}{2} \rfloor) \cdot \varepsilon_k$, when $i > \frac{k+1}{2}$, with $\varepsilon_k = \frac{1}{M \cdot k \cdot \lfloor \frac{k+1}{2} \rfloor}$.

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