



# Comparison meaningful operators and ordinal invariant preferences

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## Abstract

The existence of a continuous and order-preserving real-valued function, for the class of continuous and ordinal invariant total preorders, defined on the Banach space of all bounded real-valued functions, which are in turn defined on a given set  $\Omega$ , is characterized. Whenever the total preorder is nontrivial, the type of representation obtained leads to a functional equation that is closely related to the concept of comparison meaningfulness, and is studied in detail in this setting. In particular, when restricted to the space of bounded and measurable real-valued functions, with respect to some algebra of subsets of  $\Omega$ , we prove that, if the total preorder is also weakly Paretian, then it can be represented as a Choquet integral with respect to a  $\{0, 1\}$ -valued capacity. Some interdisciplinary applications to measurement theory and social choice are also considered.

*Keywords:*

Comparison meaningfulness, Ordinal invariant preferences, Ordinal covariant operators, Measurement theory, Social choice theory

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## 1. Introduction

A *meaningful statement* is an expression that comes from measurement theory and follows the so-called Luce's principle of dimensional analysis (also termed as the principle of theory construction). According to this, a statement (usually involving a formula or an equation) is meaningful if admissible transformations of the input variables (the scale by which input variables are measured) lead to admissible transformations of the output variables (the scale by which output variables are measured). Whenever this happens, then it is said that *comparison meaningfulness* is satisfied. For an account of measurement theory, see [13] and [15].

Let  $\Omega$  stand for a given nonempty set. Let  $\mathbb{R}^\Omega$  denote the set of all real-valued functions defined on  $\Omega$ , and let  $B_\Omega$  be the subset of  $\mathbb{R}^\Omega$ , which consists of all bounded functions. Let  $\succsim$  be a total preorder (also known as a preference) on  $B_\Omega$ . In simple terms,  $\succsim$  is *ordinal invariant* if the ranking between any two functions of  $B_\Omega$  does not change whenever the values of these two functions are measured on the same ordinal scale. In formula,  $\succsim$  is ordinal invariant provided that  $f \succsim g \Rightarrow \phi \circ f \succsim \phi \circ g$ , for every  $f, g \in B_\Omega$ ,  $\phi \in \Phi$ , where  $\Phi$  stands for the set that consists of all strictly increasing real-valued functions of a single variable.

A real-valued function  $T : B_\Omega \rightarrow \mathbb{R}$  is said to be an *operator* (or a *functional*). An operator  $T$  is said to be *ordinal covariant* (respectively, *negatively ordinal covariant*) if  $T(\phi \circ f) = \phi(T(f))$  (respectively, if  $T(\phi \circ f) = -\phi(-T(f))$ ) holds true for every  $f \in B_\Omega$ ,  $\phi \in \Phi$ . Note that these two concepts define corresponding functional equations to be satisfied by  $T$ . As will be seen later, a key class of ordinal invariant total preorders; viz., the continuous and representable ones, are naturally linked to these types of operators.

Ordinal covariant operators have some interesting features. On the one hand, they satisfy a property, mentioned above, that plays an important role in the theory of aggregation functions; namely, comparison meaningfulness. On the other hand, in the finite dimensional case, ordinal covariant operators are closely related to lattice polynomial functions. Lattice polynomial functions were introduced by Birkhoff [2]. Recently, they have been completely characterized (see [17], [20]). Roughly speaking, a lattice polynomial function on  $\mathbb{R}^n$  is a Boolean max-min map, i.e., a real-valued function of  $n$  variables that is obtained by computing maxima and minima according to a fixed collection of subsets of variables. In [18], Marichal and Mesiar provide an

excellent survey regarding meaningful aggregation functions mapping ordinal scales into an ordinal scale. These authors study three classes of aggregation functions defined on particular subsets of the Euclidean space, including certain interpretations of the main functional equations involving these classes in the setting of aggregation on finite chains.

Corresponding extensions of these results in the context of  $B_{\Omega}^{\Sigma}$  have been given by Ovchinnikov and Dukhovny (see [21] and [22]). Here,  $\Sigma$  is a nonempty algebra of subsets of  $\Omega$ , and  $B_{\Omega}^{\Sigma} \subseteq B_{\Omega}$  is the space of all bounded measurable real-valued functions on  $\Omega$ . In particular, in the latter articles, it is proven that a continuous functional on  $B_{\Omega}^{\Sigma}$  is invariant (under transformations from the automorphism group of the set of all real numbers) if and only if it can be represented as a Choquet or Sugeno (fuzzy) integral with respect to a  $\{0, 1\}$ -valued capacity<sup>1</sup>. Similar results appear in [8], where the continuity assumption is replaced with a monotonicity condition. In addition, the connection between this type of operator and the so-called probabilistic quantiles, which are of significant prevalence in statistics, is also studied.

The main purpose of the current paper, quite different from the starting-point of the articles just mentioned, is to offer an account of the continuous representation problem for the class of continuous and ordinal invariant preferences defined on  $B_{\Omega}$  ( $B_{\Omega}$  is equipped with the supremum norm topology). Thus, we work in a framework more general than those mentioned above, including the finite-dimensional case. In addition, we show the significance of our characterization results in social sciences by introducing certain applications in measurement theory and social choice theory.

Here is a brief outline of the contents of the paper. In Section 2 we introduce the basics regarding orders and operators on  $B_{\Omega}$ . In Section 3 our main theorem is shown: Continuous and ordinal invariant total preorders defined on  $B_{\Omega}$  are identified with those that admit a continuous and ordinal covariant utility (order-preserving) function, or a continuous and negatively ordinal covariant utility function, or are trivial. When restricted to  $B_{\Omega}^{\Sigma}$ , by taking advantage of the results in [21] and [22], we can offer the following finer statement: If the total preorder is also weakly Paretian, then it can be represented as a Choquet integral with respect to a  $\{0, 1\}$ -valued capacity.

In Section 4, and as a consequence of our main theorem, we present some

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<sup>1</sup>We are grateful to an anonymous referee for bringing our attention to references [18], [21] and [22].

results in the field of measurement theory. In particular, we generalize a result by Marichal and Mathonet [17] (see also [16]) about the characterization of the class of continuous and *comparison meaningful operators*. Moreover, certain monotonicity properties of these operators are also shown. The corresponding results for continuous and ordinal invariant total preorders defined on  $B_\Omega$  are also presented.

As a second application, we also offer a characterization of certain social rules, called *social evaluation functionals*, in the setting of utility theory in social choice. In particular, we study a slight deviation of the two most significant approaches to the social aggregation problem, following Arrow and Sen, respectively (see [1] and [23]). The important fact is that, in our scenario, and in a similar way to what occurs in Sen's setting, ordinal invariant preferences allow for both intra- and inter-personal comparisons of well-being among individuals in society. Thus, we provide a characterization of those social evaluation functionals that satisfy social state separability, continuity, and information invariance, with respect to a single ordinal scale, in terms of continuous and ordinal covariant operators. For thorough accounts of the axiomatic foundations of the various welfarist aspects in social choice theory see [9], [4] and [11]. Section 5 draws the main conclusions of the paper, including a comparison with the existing results in the literature.

## 2. Preliminaries

We begin by recalling some elementary concepts related to orderings.

**Definition 2.1.** Let  $U$  be a nonempty set. A transitive and complete (hence, reflexive) binary relation  $R$  defined on  $U$  is said to be a *total preorder*.<sup>2</sup> A *total order* on  $U$  is an antisymmetric total preorder.

Associated with  $R$  the *asymmetric part*, also called the *strict part*, denoted by  $R_a$ , is defined as the following binary relation on  $U$ :  $a R_a b$  if and only if  $a R b$  and  $\neg(b R a)$  (or, equivalently, since  $R$  is a total preorder,  $a R_a b$  if and only if  $\neg(b R a)$ ). Similarly, its *symmetric part*, denoted by  $R_s$ , is defined by:  $a R_s b$  if and only if  $a R b$  and  $b R a$ . Given two elements  $a, b \in U$ ,  $a$  is said to be *indifferent* to  $b$  if  $a R_s b$ . The (total) preorder  $R$  is said to be *nontrivial* if  $a R_a b$  for some  $a, b \in U$ . Otherwise, it is said to be *trivial*.

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<sup>2</sup>In the economics literature a total preorder is also referred to as a preference, whereas in social choice theory it is known as a social welfare ordering.

The *dual* total preorder associated with  $R$  is defined as  $a R_d b$  if and only if  $b R a$ . A real-valued map  $u : U \rightarrow \mathbb{R}$  is called a *utility function* for  $R$  if, for every  $a, b \in U$ , it holds  $a R b \iff u(a) \leq u(b)$ . (Alternatively, it is also said that  $R$  is *representable*).

From now on, let  $\Omega$  denote a nonempty set. Let  $\mathbb{R}^\Omega$  (respectively,  $B_\Omega$ ) be the set of all real-valued functions (respectively, of all bounded real-valued functions) defined on  $\Omega$ . On  $B_\Omega$  the supremum norm topology will be considered.

**Definition 2.2.** The *supremum norm topology* on  $B_\Omega$  is given by the metric induced by the supremum norm. That is,  $d(f, g) = \|f - g\|_\infty = \sup_{\omega \in \Omega} |f(\omega) - g(\omega)|$ , ( $f, g \in B_\Omega$ ). With this norm, actually,  $B_\Omega$  becomes a Banach space.

Throughout the paper, total preorders defined on  $B_\Omega$  will be considered too. We will use the notation  $\succsim$  to refer to such total preorders while keeping the symbols  $R, S, \dots$ , etc. for a total preorder defined on an abstract set  $U$ . In accordance with this notation, the symbols  $\prec$ ,  $\sim$ , and  $\succsim_d$  for the asymmetric part, the symmetric part, and the dual, respectively, will be used in the sequel when referring to a total preorder  $\succsim$  defined on  $B_\Omega$ .

Given  $f, g \in B_\Omega$  and  $\lambda \in \mathbb{R}$ , we will use the notation  $f + g$  and  $\lambda f$  for the usual binary operations of addition and multiplication by scalars, pointwise defined, respectively. The zero function in  $B_\Omega$  will be denoted by  $0_\Omega$ . The indicator function of a subset  $E \subseteq \Omega$  will be denoted by  $1_E$ .

**Definition 2.3.** Given a natural number  $k \in \mathbb{N}$ , a *simple function*  $s \in B_\Omega$  is one of the form  $s = \sum_{j=1}^k x_j 1_{E_j}$ , where, for each  $j \in \{1, \dots, k\}$ ,  $x_j \in \mathbb{R}$ ,  $E_j \subseteq \Omega$ , and  $(E_j)_{j=1}^k$  is a partition of  $\Omega$ .

Given  $f, g \in B_\Omega$ , we write  $f \ll g$  whenever  $f(\omega) < g(\omega)$  for all  $\omega \in \Omega$ ,  $f < g$  whenever  $f(\omega) \leq g(\omega)$  for all  $\omega \in \Omega$  and, for some  $\bar{\omega} \in \Omega$ ,  $f(\bar{\omega}) < g(\bar{\omega})$ . Finally, we write  $f \leq g$  provided that  $f(\omega) \leq g(\omega)$  for all  $\omega \in \Omega$ .

As usual, for  $f \in B_\Omega$  and  $g \in \mathbb{R}^\mathbb{R}$ ,  $g \circ f$  will be their composition. The *diagonal* of  $B_\Omega$  will be denoted by  $D$ , and it is defined as the subset of all constant real-valued functions defined on  $\Omega$ , that is,  $D = \{\alpha 1_\Omega : \alpha \in \mathbb{R}\}$ .

An important particular case in our analysis will be the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Let  $n$  be a positive integer. Call  $N = \{1, \dots, n\}$ . Note that  $B_\Omega$  can be identified, both algebraically and topologically, with  $\mathbb{R}^n$ , provided that  $\Omega = N$ . A typical element of  $\mathbb{R}^n$  will be denoted by  $x = (x_j)$ ,  $j \in N$ .

Let  $A, B$  be two nonempty sets, and let  $u : A \rightarrow B$  be a map. If  $S \subseteq A$ , then the restriction of  $u$  to  $S$  will be denoted by  $u|_S$ .

We now state a basic definition that collects the main properties, that will be used from now on, about total preorders on  $B_\Omega$ . Let  $f, g \in B_\Omega$  and  $\alpha, \lambda \in \mathbb{R}$ , with  $0 \leq \lambda$ , be arbitrarily given.

**Definition 2.4.** A total preorder  $\preceq$  defined on  $B_\Omega$  is:

- (1) *translatable* if  $f \preceq g$  implies that  $f + \alpha 1_\Omega \preceq g + \alpha 1_\Omega$ ,
- (2) *homothetic* if  $f \preceq g$  implies that  $\lambda f \preceq \lambda g$ ,
- (3) *continuous* if both the lower contour set  $L(f) = \{g \in B_\Omega : g \preceq f\}$  and the upper contour set  $G(f) = \{g \in B_\Omega : f \preceq g\}$  are closed subsets of  $B_\Omega$ ,
- (4) *Paretian* if  $f \leq g$  implies that  $f \preceq g$ ,
- (5) *anti-Paretian* if  $f \leq g$  implies that  $g \preceq f$ ,
- (6) *monotonic* if it is Paretian or anti-Paretian,
- (7) *weakly Paretian* if  $f \ll g$  implies that  $f \prec g$ ,
- (8) *strongly Paretian* if  $f < g$  implies that  $f \prec g$ .

We now introduce some concepts corresponding to real-valued functions defined on  $B_\Omega$ .

**Definition 2.5.** An operator  $T : B_\Omega \rightarrow \mathbb{R}$  is:

- (1) *translative* if  $T(f + \alpha 1_\Omega) = T(f) + \alpha$ ,
- (2) *homogeneous of degree one* if  $T(\lambda f) = \lambda T(f)$ ,
- (3) *continuous* if it is continuous when considering the supremum norm topology on  $B_\Omega$ , and the Euclidean topology on  $\mathbb{R}$ ,
- (4) *idempotent* if  $T(\alpha 1_\Omega) = \alpha$ ,
- (5) *negatively idempotent* if  $T(\alpha 1_\Omega) = -\alpha$ ,
- (6) *increasing* if  $f \leq g$  implies that  $T(f) \leq T(g)$ ,

- (7) *decreasing* if  $f \leq g$  implies that  $T(g) \leq T(f)$ ,
- (8) *monotonic* if it is increasing or decreasing,
- (9) *strictly increasing* if  $f \ll g$  implies that  $T(f) < T(g)$ ,
- (10) *strongly increasing* if  $f < g$  implies that  $T(f) < T(g)$ .

**Remark 2.6.** In a similar fashion the concepts of a weak anti-Paretian, a weak monotonic, a strongly anti-Paretian, and a strongly monotonic total preorder  $\succsim$  on  $B_\Omega$  are defined. Alternatively, the concepts of a strictly decreasing, a strictly monotonic, a strongly decreasing, and a strongly monotonic operator are established.

### 3. Comparison meaningfulness and ordinal invariance

In this section the main result of the paper is established. Basically, it provides a characterization of the continuous and ordinal invariant preferences defined on  $B_\Omega$ , in terms of continuous utility functions that preserve an ordinal invariance property. This will be the contents of Subsection 3.2. Before, throughout Subsection 3.1, we present some basic results about comparison meaningful and ordinal covariant operators that will be used later.

#### 3.1. Some facts about comparison meaningful and ordinal covariant operators

Recall that the set that consists of all strictly increasing real-valued functions of a single variable will be denoted by  $\Phi$ , and a typical element of this set will be denoted by  $\phi$ .

**Definition 3.1.** Let  $f, g \in B_\Omega$  and  $\phi \in \Phi$  be arbitrarily given. An operator  $T : B_\Omega \rightarrow \mathbb{R}$  is said to be:

- (1) *ordinal covariant*<sup>3</sup> if  $T(\phi \circ f) = \phi(T(f))$ ,
- (2) *negatively ordinal covariant* if  $T(\phi \circ f) = -\phi(-T(f))$ ,
- (3) *comparison meaningful* if  $T(f) \leq T(g) \iff T(\phi \circ f) \leq T(\phi \circ g)$ .

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<sup>3</sup>We follow the terminology given in [8]. Alternative names for the same concept are those of *order invariance* or *ordinal stability* (see, e.g., [18], [21], and [19]). As already stated in Section 1, the terminology *ordinal invariance* will be used here when referring to total preorders.

**Remark 3.2.** (i) Note that an ordinal covariant (respectively, a negatively ordinal covariant) operator  $T$  satisfies the functional equation  $T(\phi \circ f) = \phi(T(f))$  (respectively,  $T(\phi \circ f) = -\phi(-T(f))$ ) for every  $f \in B_\Omega$ ,  $\phi \in \Phi$ .

(ii) Suppose that  $T$  is an ordinal covariant operator. Then, obviously,  $T$  cannot be constant. Moreover, for every  $E \subseteq \Omega$ ,  $\alpha \in \mathbb{R}$ , it holds  $T(\alpha 1_E) \in \{0, \alpha\}$ . In particular,  $T(1_\Omega) = 1$  and  $T(0_\Omega) = 0$ . Indeed, assume  $T(\alpha 1_E) = \beta \notin \{0, \alpha\}$ . Consider a strictly increasing function  $\phi \in \Phi$  such that  $\phi(\alpha) = \alpha$ ,  $\phi(0) = 0$  and  $\phi(\beta) = \frac{\alpha+\beta}{2}$ , provided that  $\beta > 0$ , and  $\phi(\beta) = \frac{\beta}{2}$  otherwise. Then  $T(\phi \circ \alpha 1_E) = T(\phi(\alpha) 1_E) = T(\alpha 1_E) = \beta$ , and  $\phi(T(\alpha 1_E)) = \phi(\beta) \neq \beta$ . Thus,  $T(\phi \circ \alpha 1_E) \neq \phi(T(\alpha 1_E))$ , a contradiction. Therefore,  $T(\alpha 1_E) \in \{0, \alpha\}$ .

The next lemma generalizes Theorem 2.1 in [19].

**Lemma 3.3.** *Let  $T : B_\Omega \rightarrow \mathbb{R}$  be a comparison meaningful and idempotent (respectively, negatively idempotent) operator. Then  $T$  is ordinal covariant (respectively, negatively ordinal covariant).*

*Proof.* Let us prove the lemma for the case  $T$  idempotent. The case  $T$  negatively idempotent is similar. Let  $f \in B_\Omega$  and  $\phi \in \Phi$ . Then, since  $T$  is idempotent, it holds that  $T(f) = T(T(f)1_\Omega)$ . Now, by comparison meaningfulness, we have that  $T(\phi \circ f) = T(\phi \circ T(f)1_\Omega)$ . That is,  $T(\phi \circ f) = T(\phi \circ T(f)1_\Omega) = T(\phi(T(f))1_\Omega) = \phi(T(f))$ , where the last equality holds true by idempotency again. This completes the proof.  $\square$

As a part of the folklore, a density result is now stated. Let us denote by  $S$  the subset of  $B_\Omega$  which consists of all simple functions.

**Lemma 3.4.**  *$S$  is dense in  $B_\Omega$ .*

*Proof.* Let  $f \in B_\Omega$  and  $\epsilon > 0$  be fixed. Then there are reals  $m, M$  such that  $m \leq f(\omega) < M$ , for every  $\omega \in \Omega$ . Take a positive integer  $n$  such that  $\frac{M-m}{n} < \epsilon$ . Define, for each  $j = 0, \dots, n-1$ , the numbers  $\alpha_j = m + j\frac{M-m}{n}$ , and consider the subsets of  $\Omega$  defined as follows:  $E_j = \{\omega \in \Omega : \alpha_j \leq f(\omega) < \alpha_{j+1}\}$ , for every  $j$ . Note that  $(E_j)$  is a partition of  $\Omega$ . Finally, define the simple function  $s = \sum_j \alpha_j 1_{E_j}$ . Then,  $d(f, s) = \|f - s\|_\infty = \sup_{\omega \in \Omega} |f(\omega) - s(\omega)| \leq \frac{M-m}{n} < \epsilon$ , which proves Lemma 3.4.  $\square$

A helpful technical result needed to prove Theorem 3.10 below is now included. Note that, in the context of Lemma 3.5,  $\Omega = N$ . In addition, the notation  $1_n := (1, \dots, 1) \in \mathbb{R}^n$  will be used, and the diagonal  $D$  will be  $D = \{(\alpha, \dots, \alpha) \in \mathbb{R}^n : \alpha \in \mathbb{R}\}$ .

**Lemma 3.5.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a comparison meaningful and continuous function such that  $f|_D = c$ , with  $c \in \mathbb{R}$ . Then  $f$  is constant.*

*Proof.* Let  $\pi$  stands for a permutation of  $N$  and consider  $Z_\pi$  to be the subset of  $\mathbb{R}^n$  defined in the following way:  $Z_\pi = \{(x_j) \in \mathbb{R}^n : x_{\pi(1)} \leq \dots \leq x_{\pi(n)}\}$ . Let us call  $Z_\pi$  a rank-ordered set. Let us prove that  $f$  is constant by showing that it is constant over each rank-ordered set of  $\mathbb{R}^n$ . For ease of notation suppose, without loss of generality, that the rank-ordered set is  $Z = Z_{id} = \{x = (x_j) \in \mathbb{R}^n : x_1 \leq \dots \leq x_n\}$ . Let us denote by  $Z^\circ$ , the (topological) interior of  $Z$  and notice that  $D \subseteq \text{Bd}(Z)$ , where  $\text{Bd}(Z)$  stands for the boundary of  $Z$ .

Let  $x, y \in Z^\circ$  and suppose first that  $\max_{j \in N}\{x_j\} < \min_{j \in N}\{y_j\}$ . Assume then, by way of contradiction, that  $f(x) < f(y)$ . If this occurs, then consider the vectors  $\bar{x}, \bar{y} \in D$  defined as follows:  $\bar{x} = \max\{x_j\}1_n$ ,  $\bar{y} = \min\{y_j\}1_n$ . Then, since  $f$  is comparison meaningful, it clearly follows that there are (open) balls  $U_{\bar{x}}, U_{\bar{y}}$ , centered at  $\bar{x}$  and  $\bar{y}$  respectively, such that the inequality  $f(s) < f(t)$  holds true for every  $s \in \mathcal{S} := U_{\bar{x}} \cap Z^\circ$ ,  $t \in \mathcal{T} := U_{\bar{y}} \cap Z^\circ$ . Now, since  $f(\bar{x}) = f(\bar{y})$ , by using a connectedness argument it can be easily seen that the latter inequality is possible if and only if  $f(s) < f(\bar{x}) = f(\bar{y}) < f(t)$ , for every  $s \in \mathcal{S}$ ,  $t \in \mathcal{T}$ . Then, by continuity of  $f$ , we can select an element  $w \in Z \setminus D$  such that  $\bar{x} \ll w$  and  $f(\bar{x}) = f(w)$ . Consider the segment connecting  $\bar{x}$  and  $w$ , denoted by  $[\bar{x}, w]$ . Then, by comparison meaningfulness again, it follows that  $f|_{[\bar{x}, w]}$  is  $f(w)$  ( $= f(\bar{x})$ ). But this contradicts the fact that  $f(s) < f(\bar{x})$ , for every  $s \in \mathcal{S}$ . So,  $f(x) < f(y)$  is not possible. By arguing as above, we can see that the case  $f(y) < f(x)$  leads to a similar contradiction. Therefore, we arrive at  $f(x) = f(y)$ . So, we have shown that, for each  $x \in Z^\circ$ , it holds true that  $f(y) = f(x)$ , for every  $y \in Z^\circ$  such that  $\max_{j \in N}\{x_j\} < \min_{j \in N}\{y_j\}$ . Since  $x$  is an arbitrary element of  $Z^\circ$ , this clearly implies that  $f|_{Z^\circ}$  is a constant function. Continuity of  $f$  again, together with the fact that  $f|_D = c$ , for some  $c \in \mathbb{R}$ , show that  $f|_Z = c$ . Note that the arguments above are independent of the rank-ordered set  $Z_\pi$  chosen. So,  $f$  is a constant function.  $\square$

Another technical and useful construction, that will be subsequently used,

is now presented. This resource allows us first to generalize the contents of Lemma 3.5 to include the case  $B_\Omega$ .

Let  $n \geq 1$  be a natural number and let be fixed a partition of  $\Omega$ ,  $(E_j)_{j \in N}$ . Consider the subset of  $B_\Omega$  defined as follows  $A_{(E_j)} = \{\sum_j x_j 1_{E_j} : (x_j) \in \mathbb{R}^n\}$ . Note that  $A_{(E_j)}$  can be identified, both algebraically and topologically, with  $\mathbb{R}^n$ . Accordingly,  $x = (x_j)$ ,  $x \in \mathbb{R}^n$ , will be used to denote a typical element of  $A_{(E_j)}$ . Moreover, for any strictly increasing real-valued function of one variable  $\phi \in \Phi$ , and any  $x = (x_j) \in A_{(E_j)}$ , it holds  $\phi \circ x = \sum_j \phi(x_j) 1_{E_j} \in A_{(E_j)}$ . In other words,  $A_{(E_j)}$  is a  $\Phi$ -invariant subset of  $B_\Omega$ .

**Corollary 3.6.** *Let  $T : B_\Omega \rightarrow \mathbb{R}$  be a comparison meaningful and continuous operator such that  $T|_D = c$ , with  $c \in \mathbb{R}$ . Then  $T$  is constant.*

*Proof.* Let  $n \geq 1$  be a natural number and let be fixed a partition of  $\Omega$ ,  $(E_j)_{j \in N}$ . Following the notation just introduced above, let  $A_{(E_j)} \subseteq B_\Omega$ . Then, by identifying  $A_{(E_j)}$  with  $\mathbb{R}^n$ ,  $T|_{A_{(E_j)}}$  can be viewed as a continuous and comparison meaningful real-valued function defined on  $\mathbb{R}^n$ . So, by Lemma 3.5,  $T|_{A_{(E_j)}}$  is a constant function. Moreover, since  $D$ , the diagonal of  $B_\Omega$ , is contained in each subset  $A_{(E_j)}$ , it follows that  $T|_S$  is a constant map (i.e., when restricted to the set of simple functions  $S \subseteq B_\Omega$ ,  $T$  turns to be a constant map). To finish the proof note that, by Lemma 3.4,  $S$  is dense in  $B_\Omega$ . Hence, by continuity again, it follows that there is  $c \in \mathbb{R}$  such that  $T(f) = c$ , for all  $f \in B_\Omega$ , and the proof is complete.  $\square$

**Remark 3.7.** (i) If an operator  $T$  is ordinal covariant, or negatively ordinal covariant, or constant, then it is comparison meaningful. If  $T$  is constant then it is obviously comparison meaningful. Suppose now that  $T$  is ordinal covariant and let  $f, g \in B_\Omega$  such that  $T(f) \leq T(g)$ . Then, for any  $\phi \in \Phi$ , we have  $T(\phi \circ f) = \phi(T(f)) \leq \phi(T(g)) = T(\phi \circ g)$ . The converse follows directly by taking the identity map  $\phi(t) = t$ , ( $t \in \mathbb{R}$ ). The case in which  $T$  is negatively ordinal covariant is similar.

(ii) A comparison meaningful operator  $T$  may fail to be ordinal covariant or negatively ordinal covariant. However, as proven in Lemma 3.3 above, if  $T$  is comparison meaningful and idempotent, then it is ordinal covariant (similarly, if  $T$  is comparison meaningful and negatively idempotent, then it is negatively ordinal covariant).

### 3.2. Characterization of continuous ordinal invariant preferences

We recall the definition of an ordinal invariant total preorder on  $B_\Omega$ .

**Definition 3.8.** A total preorder  $\succsim$  defined on  $B_\Omega$  is said to be *ordinal invariant*<sup>4</sup> if  $f \succsim g$  then  $\phi \circ f \succsim \phi \circ g$ , for every  $f, g \in B_\Omega$ ,  $\phi \in \Phi$ <sup>5</sup>.

**Remark 3.9.** (i) Let  $\succsim$  be a total preorder defined on  $B_\Omega$  and  $T$  a utility function for  $\succsim$ . Then  $T$  is comparison meaningful if and only if  $\succsim$  is ordinal invariant.

(ii) Let  $\succsim$  be a total preorder defined on  $B_\Omega$ . If  $\succsim$  is ordinal invariant then so is  $\succsim_d$ . Moreover,  $\succsim$  admits an ordinal covariant utility function if and only if  $\succsim_d$  admits a negatively ordinal covariant utility function. Indeed, it is easy to see that  $T$  is an ordinal covariant utility function for  $\succsim$  if and only if  $-T$  is a negatively ordinal covariant utility function for  $\succsim_d$ .

So, if a total preorder  $\succsim$  defined on  $B_\Omega$  admits a comparison meaningful utility function then, as stated in Remark 3.9 (i) above, it is ordinal invariant. The converse, under the continuity assumption, is discussed in the next Theorem 3.10. Note that, actually, in its statement comparison meaningfulness has been replaced by the stronger property of ordinal covariance (or negatively ordinal covariance).

**Theorem 3.10.** *Let  $\succsim$  be a total preorder defined on  $B_\Omega$ . Then the following statements are equivalent:*

- (i)  $\succsim$  admits a continuous and ordinal covariant utility function, or a continuous and negatively ordinal covariant utility function, or is trivial,
- (ii)  $\succsim$  is continuous and ordinal invariant.

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<sup>4</sup>This terminology comes from the following basic fact: Suppose that  $R$  is a representable total preorder defined on an abstract set  $U$ ,  $u$  a utility function for  $R$ , and  $\phi \in \Phi$ . Then it is easily shown that  $\phi \circ u$  is also a utility function for  $R$ . So, strictly increasing transformations preserve ordinal properties.

<sup>5</sup>The concept of an ordinal invariant total preorder on  $B_\Omega$  could be given for a proper subset of  $\Phi$ . For instance, this notion can be found in the literature when restricted to the subset of  $\Phi$  which consists of continuous functions. It is a remarkable fact that if we do so in all definitions that come later, then the corresponding results stated in the article remain true under this more restrictive and demanding qualification.

*Proof.* (i) implies (ii) is straightforward. So, we will prove (ii) implies (i). First note that, since  $\succsim$  is ordinal invariant, it is homothetic and translatable. To proceed with the proof the following three cases, that are exhaustive and mutually exclusive, should be analyzed:

- (1)  $0_\Omega \prec 1_\Omega$ . By Theorem 3.1 in [6] (see also Remark 3.2 (iii) in [6]) there is a continuous, translative and homogeneous of degree one utility function  $T$  that represents  $\succsim$ . In particular,  $T$  fulfils the functional equation  $T(\lambda f + \alpha 1_\Omega) = \lambda T(f) + \alpha$ , for every  $f \in B_\Omega$ , and  $\alpha, \lambda \in \mathbb{R}$ , with  $0 \leq \lambda$ . Let us see that, actually,  $T$  is ordinal covariant. To that end, let  $f \in B_\Omega$  and  $\phi \in \Phi$  be arbitrarily given. Since  $T$  is idempotent, and therefore  $T(f) = T(T(f)1_\Omega)$ , it holds true that  $f \sim T(f)1_\Omega$ . Thus, by ordinal invariance, it follows that  $\phi \circ f \sim \phi \circ T(f)1_\Omega$ . Therefore,  $T(\phi \circ f) = T(\phi \circ T(f)1_\Omega)$ . But,  $T(\phi \circ T(f)1_\Omega) = T(\phi(T(f))1_\Omega) = \phi(T(f))$ . So,  $T(\phi \circ f) = \phi(T(f))$ , and we are done.
- (2)  $1_\Omega \prec 0_\Omega$ . In this case, let us consider the dual total preorder  $\succsim_d$ . Then  $\succsim_d$  is a continuous and ordinal invariant total preorder on  $B_\Omega$  for which  $0_\Omega \prec_d 1_\Omega$ . So, by case (1) above there is a continuous and ordinal covariant utility function, say  $T$ , that represents  $\succsim_d$ . Therefore, by Remark 3.9 (ii),  $-T$  is a continuous and negatively ordinal covariant utility function for  $\succsim$ , and we are done.
- (3)  $1_\Omega \sim 0_\Omega$ . Let  $\alpha \in \mathbb{R}$ . If  $0 \leq \alpha$ , then, by homotheticity, it follows that  $\alpha 1_\Omega \sim 0_\Omega$ . If  $\alpha < 0$ , then  $-\alpha 1_\Omega \sim 0_\Omega$  and, by translativity,  $-\alpha 1_\Omega + \alpha 1_\Omega \sim 0_\Omega + \alpha 1_\Omega$ . That is,  $0_\Omega \sim \alpha 1_\Omega$ . So, we have proven that  $\alpha 1_\Omega \sim 0_\Omega$ , for every  $\alpha \in \mathbb{R}$ .

Let  $n \geq 1$  be a natural number and consider a partition of  $\Omega$ ,  $(E_j)_{j \in N}$ . Define the following binary relation  $\succsim^*$  on  $\mathbb{R}^n$ :  $x = (x_j) \succsim^* y = (y_j) \iff \sum_j x_j 1_{E_j} \succsim \sum_j y_j 1_{E_j}$ . It is obvious to see that  $\succsim^*$ , so-defined, is actually a total preorder on  $\mathbb{R}^n$ . Moreover since  $\succsim$  is continuous and ordinal invariant, so is  $\succsim^*$ . Now, by Debreu's theorem (see [10] or [5]), there is a continuous utility function, say  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ , for  $\succsim^*$ . By Remark 3.9 (i)  $u$  is comparison meaningful since  $\succsim^*$  is ordinal invariant. In addition observe that, by the argument above,  $u|_D$  is a constant function ( $D$  is the *diagonal* of  $\mathbb{R}^n$ ). Now, by Lemma 3.5, it follows that  $u$  is a constant function. In other words,  $\succsim^*$  turns out to be the trivial total preorder on  $\mathbb{R}^n$ . But this means that, when restricted to the subset  $\{\sum_j x_j 1_{E_j} : (x_j) \in \mathbb{R}^n\}$ ,  $\succsim$  is also trivial. Now, since the partition

$(E_j)_{j \in N}$  is arbitrary, it follows that  $\succsim$  is trivial over  $S$ ,  $S$  being the subset which consists of all simple functions on  $B_\Omega$ . Since, by Lemma 3.4,  $S$  is dense in  $B_\Omega$  it follows, by continuity again, that  $\succsim$  is trivial.

So, the proof is complete<sup>6</sup>.  $\square$

**Remark 3.11.** By Theorem 3.10 we may note that continuous and non-trivial ordinal invariant preferences (i.e., total preorders) can be interpreted as solutions of functional equations involving comparison meaningfulness and/or ordinal covariance of operators (see Remark 3.2).

As a direct consequence of Theorem 3.10 we obtain the following corollary.

**Corollary 3.12.** *Any continuous, ordinal invariant, and weakly Paretian total preorder defined on  $B_\Omega$  admits a continuous and ordinal covariant utility function.*

**Remark 3.13.** (i) The continuity assumption cannot be dispensed with from the statement of Theorem 3.10. Indeed, let  $B_\Omega = \mathbb{R}^n$ . Consider the lexicographic total order on  $\mathbb{R}^n$  which, clearly, is ordinal invariant and nontrivial. Yet, neither an ordinal covariant nor a negatively ordinal covariant utility function can exist that represents it (actually, it is not even representable (see [10])).

(ii) It can be shown that the conclusion of Theorem 3.10 remains valid if the supremum norm topology on  $B_\Omega$  is replaced by the *weak topology* (i.e., the topology of the pointwise convergence of real-valued functions inherited from the product topology on  $\mathbb{R}^\Omega$ ). Also, the conclusion remains true if  $B_\Omega$  is replaced by  $B_\Omega^\Sigma$ , i.e., the subset of  $B_\Omega$  which consists of all bounded measurable real-valued functions defined on  $\Omega$  (here  $\Sigma$

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<sup>6</sup>Case (3) in the proof of Theorem 3.10 can be argued in the following alternative way: Note that, if  $\Omega$  is infinite, then  $B_\Omega$  is a nonseparable topological space. So, in general, neither of the two fundamental results in topological utility theory; namely, the Debreu's Theorem or the Eilenberg's Theorem (see Theorems 3.2.5 and 3.2.6 in [5]) can be used to derive the existence of a utility function for  $\succsim$ . Yet, since  $\succsim$  is homothetic and continuous (and, obviously, the zero function  $0_\Omega$  belong to  $B_\Omega$ ) we can appeal to Corollary in [3] to conclude that there is a continuous and homogeneous of degree one utility function, say  $T$ , for  $\succsim$ . Since  $T$  is homogeneous of degree one and continuous, it follows that  $T(0_\Omega) = 0$ . Also, by the first part of the proof of case (3), it holds that  $T|_D$  is zero. Then, Corollary 3.6 applies to conclude that  $T$  is the zero operator and therefore  $\succsim$  is trivial.

denotes an algebra, or a  $\sigma$ -algebra, of subsets of  $\Omega$ ). These were the function spaces considered in [21], [22], and [8]. Moreover, our proof of Theorem 3.10 can be straightforwardly adapted to other function spaces such as the Banach space of essentially bounded and measurable real-valued functions defined on a measure space  $(\Omega, \Sigma, \mu)$ , usually denoted by  $L^\infty(\Omega)$ , or the Banach space of continuous real-valued functions defined on a compact topological space.

- (iii) By using the results stated in [21] and [22], a finer description of Corollary 3.12 can be presented<sup>7</sup>.

*Let  $\succsim$  be a continuous, ordinal invariant, and weakly Paretian total preorder defined on  $B_\Omega^\Sigma$ . Then, there is a  $\{0, 1\}$ -valued capacity<sup>8</sup>  $\mu$ , defined on  $\Sigma$ , such that  $T(f) = \int f d\mu$ ,  $f \in B_\Omega^\Sigma$ , is a continuous utility function for  $\succsim$ , where the integral on the right-hand side is the Choquet integral with respect to  $\mu$ <sup>9</sup>.*

#### 4. Interdisciplinary applications to measurement theory and social choice

In this section two applications of our main result (namely, Theorem 3.10 above) are included. The first refers to *measurement theory*, and plays an important role in mathematical psychology and aggregation functions (see [13], [17], [16], [18] and [12]). The second has to do with utility theory methods encountered in social choice (see [9], [4] and [11]).

##### 4.1. Applications to measurement theory

In relation to the first application, and following our approach, we are going to characterize the class of continuous and comparison meaningful operators defined on  $B_\Omega$ . This result generalizes the one obtained in the finite-dimensional case by Marichal and Mathonet [17] using other technical tools. Our proof is much shorter than theirs and is based upon order-theoretical principles.

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<sup>7</sup>We are grateful to an anonymous referee for calling our attention to the precise references that have allowed us to reach this result.

<sup>8</sup>A capacity is a set function  $\mu : \Sigma \rightarrow \mathbb{R}$  such that  $\mu(\emptyset) = 0$ ,  $\mu(\Omega) = 1$ , and  $A \subset B$  implies  $\mu(A) \leq \mu(B)$ .

<sup>9</sup>In addition, the integral representation provided in (i) turns out to be a (generalized) lattice polynomial (see Theorems 3 and 4 in [21]).

**Theorem 4.1.** *Let  $T : B_\Omega \rightarrow \mathbb{R}$  be an operator. Then the following statements are equivalent:*

- (i)  *$T$  is comparison meaningful and continuous,*
- (ii) *Either  $T$  is constant or there are both a strictly monotonic continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous and ordinal covariant operator  $U$  such that  $T = g \circ U$ .*

*Proof.* (ii) implies (i) is straightforward. For the converse, consider the binary relation,  $\mathcal{R}_T$ , on  $B_\Omega$  defined as follows:  $f \mathcal{R}_T g \iff T(f) \leq T(g)$ , ( $f, g \in B_\Omega$ ). It is simple to see that  $\mathcal{R}_T$  is a continuous total preorder on  $B_\Omega$ . In accordance with the convention of Section 2, let us denote  $\mathcal{R}_T$  by  $\preceq_T$ . Now, since  $T$  is comparison meaningful it follows, by Remark 3.9(i), that  $\preceq_T$  is ordinal invariant. So, by Theorem 3.10,  $\preceq_T$  admits a continuous and ordinal covariant utility function, or a continuous and negatively ordinal covariant utility function, or is trivial. If  $\preceq_T$  is trivial, then  $T$  is constant and we are done. Suppose that  $\preceq_T$  admits a continuous and ordinal covariant utility function, say  $U$ , that represents  $\preceq_T$ . Then, since  $T$  is also a continuous representation of  $\preceq_T$ , it follows that there is a continuous and strictly increasing function, say  $g : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $T = g \circ U$ <sup>10</sup> and we are done. If  $\preceq_T$  admits a continuous and negatively ordinal covariant utility function, then  $(\preceq_T)_d$  admits a continuous and ordinal covariant utility function. Therefore, by the latter argument,  $-T = g \circ U$ , for some continuous and strictly increasing function  $g : \mathbb{R} \rightarrow \mathbb{R}$  and some continuous and ordinal covariant utility function  $U$ . So,  $T = -g \circ U$  and the proof is finished.  $\square$

We now include some results regarding certain monotonicity properties of operators and total preorders on  $B_\Omega$ . Marichal and Mathonet [17] studied, in the finite dimensional case, the mathematical structure of the set of continuous, comparison meaningful, and idempotent real-valued functions. They showed that this set coincides with the class of the so-called *lattice polynomial* functions. Roughly speaking, a lattice polynomial function defined on  $\mathbb{R}^n$  is a Boolean max-min map, that is, a real-valued function of  $n$  variables that is

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<sup>10</sup>It is a standard result in utility theory (see e.g. [14], p.26) that if both  $h$  and  $p$  represent a total preorder  $\preceq$  on a set  $\Omega$ , then there is an increasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  such that (a)  $\varphi$  is strictly increasing on  $p(\Omega) = \{r : r = p(\omega), \text{ for some } \omega \in \Omega\}$ , (b)  $h = \varphi \circ p$ . Notice that in our case  $p(\Omega) = S(B_\Omega) = \mathbb{R}$ , since  $S(\alpha 1_\Omega) = \alpha$ , for every  $\alpha \in \mathbb{R}$ .

obtained by computing maxima and minima according to a fixed collection of subsets of variables (see also [20]). In particular, any lattice polynomial function is nondecreasing (actually, it is strictly increasing). Based on this fact, we prove the following result.

**Theorem 4.2.** *Any continuous and comparison meaningful operator  $T : B_\Omega \rightarrow \mathbb{R}$  is monotonic.*

*Proof.* Note that, by Theorem 4.1, it is sufficient to show the statement for ordinal covariant operators. Moreover, by the proof of Theorem 3.10 in combination with the conclusion of Theorem 4.1 again, we may assume that this operator is translative and homogeneous of degree one. In particular, it is idempotent. Let then  $T : B_\Omega \rightarrow \mathbb{R}$  be a continuous, idempotent and ordinal covariant operator. Let us prove that  $T|_S$  is nondecreasing, where  $S \subseteq B_\Omega$  consists of all simple functions. The result then would follow by density. Let  $s, r \in S$  such that  $s \leq r$ . Then, by changing the partitions if necessary, we may assume without loss of generality that there are a positive integer  $n$  and a partition of  $\Omega$ ,  $(E_j)_j$ , such that  $s = \sum_j s_j 1_{E_j}$ ,  $r = \sum_j r_j 1_{E_j}$  where  $s_j, r_j \in \mathbb{R}$ , for every  $j \in N$ . Consider the subset of  $B_\Omega$ ,  $A_{(E_j)}$ , defined just before Corollary 3.6. Observe that  $s, r \in A_{(E_j)}$  and also that  $D \subset A_{(E_j)}$ , where  $D$  is the diagonal of  $B_\Omega$ . Consider the restriction of  $T$  to  $A_{(E_j)}$ , i.e.,  $T|_{A_{(E_j)}}$ . Then,  $T|_{A_{(E_j)}}$  is a continuous, comparison meaningful, and idempotent function on  $\mathbb{R}^n$ . By Theorem 3.1 in [17]<sup>11</sup>, it follows that  $T|_{A_{(E_j)}}$  is a lattice polynomial function, whence a nondecreasing function. So,  $T(s) \leq T(r)$ , as desired, and the proof is complete.  $\square$

The following result is a direct consequence of Theorem 4.2.

**Corollary 4.3.** *Any continuous and ordinal invariant total preorder on  $B_\Omega$  is monotonic.*

*Proof.* Indeed, let  $\lesssim$  a continuous and ordinal invariant total preorder on  $B_\Omega$ . Then, by Theorem 3.10 and Remark 3.7(i), there is a continuous and comparison meaningful utility function, say  $T$ , that represents  $\lesssim$ . Now, by

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<sup>11</sup>Theorem 3.1 in [17] is stated for functions defined on an  $n$ -dimensional cube; i.e., for domains of the form  $[a, b]^n$ , for some closed and bounded real interval  $[a, b]$ . Yet, the proof can be directly extended to the whole Euclidean space  $\mathbb{R}^n$  (for that purpose, see also Corollary 4.1 in [16]).

Theorem 4.2,  $T$  is monotonic, hence  $\succsim$  is Paretian or anti-Paretian. So,  $\succsim$  is monotonic.  $\square$

**Remark 4.4.** (i) As a consequence of Corollary 4.3, if a continuous and ordinal invariant total preorder  $\succsim$  on  $B_\Omega$  is such that  $0_\Omega \succsim 1_\Omega$ , then it is Paretian. Nevertheless, note that  $\succsim$  may not be weakly Paretian even though  $0_\Omega \prec 1_\Omega$ . Indeed, let  $\Omega = (0, 1)$  and consider the total preorder on  $B_\Omega$  given by:  $f \succsim g \iff \sup_{x \in (0,1)} f(x) \leq \sup_{x \in (0,1)} g(x)$ . Then,  $\succsim$ , clearly, is continuous, ordinal invariant and Paretian. Moreover,  $0_{(0,1)} \prec 1_{(0,1)}$ . However, it is not weakly Paretian since, for all  $x \in (0, 1)$ ,  $f(x) = x^2 \ll g(x) = x$  and  $f \sim g$ . Also, note that  $T(f) = \sup_{x \in (0,1)} f(x)$ ,  $f \in B_\Omega$ , is a continuous and ordinal covariant utility function for  $\succsim$ . This means that the converse to Corollary 3.12 does not hold in general. Yet, for the case  $\Omega = N$ , Corollary 3.12 is an if and only if statement since any lattice polynomial function on  $\mathbb{R}^n$  is strictly increasing.

(ii) As a consequence of Corollary 4.3, in combination with Remark 3.2(ii), the following strong version of Corollary 3.12 is obtained.

*Let  $\succsim$  be a total preorder defined on  $B_\Omega$ . Then the following statements are equivalent:*

- (i)  $\succsim$  admits a continuous and ordinal covariant utility function,
- (ii)  $\succsim$  is continuous, ordinal invariant, nontrivial, and Paretian.

(iii) Also, the following generalization of the result presented in Remark 3.13(iii) holds.

*Let  $\succsim$  be a total preorder defined on  $B_\Omega^\Sigma$ . Then the following statements are equivalent:*

- (i) There is a  $\{0, 1\}$ -valued capacity  $\mu$ , defined on  $\Sigma$ , such that  $T(f) = \int f d\mu$ ,  $f \in B_\Omega^\Sigma$ , is a continuous utility function for  $\succsim$ , where the integral on the right-hand side is the Choquet integral with respect to  $\mu$ ,
- (ii)  $\succsim$  is continuous, ordinal invariant, nontrivial, and Paretian.

In relation to strong monotonicity we reach the following negative result.

**Theorem 4.5.** *Suppose that  $\Omega$  contains more than one point. Then no continuous operator  $T : B_\Omega \rightarrow \mathbb{R}$  can exist that is comparison meaningful and strongly monotonic.*

*Proof.* It is sufficient to prove the corollary for a strongly increasing operator  $T$  since for the case of a strongly decreasing operator the proof follows directly by considering the operator  $-T$ . Let then  $T : B_\Omega \rightarrow \mathbb{R}$  be a continuous, comparison meaningful, and strongly increasing operator. Note that  $T$  is not a constant operator since it is strongly increasing. So, by Theorem 4.1, there are both a strictly monotonic continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous and ordinal covariant operator  $U$  such that  $T = g \circ U$ . In addition, note that  $U$  is strongly monotonic since  $U = g^{-1} \circ T$ . Assume that  $U$  is strongly increasing, the other case being handled similarly. We will use once again the construction presented just before Corollary 3.6. So, let  $(E_j)_j, j \in N$ , be a nontrivial partition of  $\Omega$  and consider the restriction of  $U$  to  $A_{(E_j)}, U|_{A_{(E_j)}}$ . Since  $U|_{A_{(E_j)}}$  can be viewed as a function from  $\mathbb{R}^n$  into  $\mathbb{R}$  and, by hypothesis, the cardinality of  $\Omega$  is strictly greater than one, it follows that  $n > 1$ . Now, by Theorem 4.1 in [19] (see also Remark 3.2(ii)), it holds that  $U(\sum_j x_j 1_{E_j}) \in \{x_1, \dots, x_n\}$ , for every  $x = (x_j) \in \mathbb{R}^n$ . But the fulfilment of the latter condition contradicts the fact that  $U$  is strongly increasing. Indeed, let  $(1, \dots, n) \in \mathbb{R}^n$ . Then, there is  $i \in N$  such that  $U(\sum_j j 1_{E_j}) = i$ . Let  $p$  be a positive integer and consider the vector  $x^p = (1, \dots, x_k^p = k + \frac{1}{p}, \dots, n) \in \mathbb{R}^n$ , where  $k \in N, k \neq i$ . Note that, for every  $p$ ,  $(1, \dots, n) < x^p$ , which means that  $\sum_j x_j^p 1_{E_j} < \sum_j j 1_{E_j}$ . Then, by continuity, for a large enough  $p$  it follows  $U(\sum_j x_j^p 1_{E_j}) = i = U(\sum_j j 1_{E_j})$ , which violates the fact that  $U$  is strongly increasing. Therefore, whenever the cardinality of  $\Omega$  is strictly greater than one, no continuous, comparison meaningful, and strongly increasing operator  $T : B_\Omega \rightarrow \mathbb{R}$  can exist.  $\square$

**Remark 4.6.** If  $\Omega$  is a singleton, then  $B_\Omega = \mathbb{R}$  and the strongly monotonic operators  $T : B_\Omega \rightarrow \mathbb{R}$  coincide with the set that consists of all strictly increasing and all strictly decreasing real-valued functions. A similar statement to Theorem 4.5 above, for function spaces that include bounded measurable real-valued functions defined on  $\Omega$ , such as  $B_\Omega^\Sigma$  or  $L^\infty(\Omega)$ , holds true provided that some other assumption (stronger than the cardinality of  $\Omega$  is one) on the algebra (or  $\sigma$ -algebra)  $\Sigma$  is added. This assumption is that  $\Sigma$  contains, at least, one nontrivial set. In other words,  $\{\emptyset, \Omega\} \subset \Sigma$ . Indeed, note that if  $\Sigma = \{\emptyset, \Omega\}$ , then the only measurable real-valued functions on  $\Omega$  are the constant ones and, therefore,  $B_\Omega^\Sigma$ , or  $L^\infty(\Omega)$ , equals to the reals.

Arguing as in the proof of Corollary 4.3, in combination with Theorem 4.5, the following result is obtained.

**Corollary 4.7.** *Suppose that  $\Omega$  contains more than one point. Then no continuous total preorder on  $B_\Omega$  can exist that is ordinal invariant and strongly monotonic.*

#### 4.2. An application to social choice

A second application of our approach to the context of utility theory in the social choice framework is now developed. In the spirit of the results that appear in the literature on utility measurability, and intra- and inter-personal comparability (see, e.g., [4]), we provide a characterization of a class of social rules called *social evaluation functionals*.

Before introducing the application some definitions and notations are still needed.

Let  $X$  be a nonempty set, usually called in the social choice literature the set of *social outcomes* (or, alternatively, the set of *alternatives*). Denote by  $\Omega$  the set of agents, or individuals, (that in the model is allowed to be infinite) in society. For a given individual  $\omega \in \Omega$ , a typical function from  $X$  to  $\mathbb{R}$  will be denoted by  $u_\omega$ . Actually,  $u_\omega$  can be understood as a utility function for the agent  $\omega$ . Indeed, consider the total preorder  $R_{u_\omega}$  on  $X$  defined by  $x R_{u_\omega} y \iff u_\omega(x) \leq u_\omega(y)$ , ( $x, y \in X$ ). Then, it is straightforward to see that  $u_\omega$  is a utility function for  $R_{u_\omega}$ . For this reason we will use the term utility function when referring to real-valued functions defined on  $X$ . Let us denote by  $\mathcal{U}$  the set of all real-valued functions defined on  $X$ . For technical reasons, derived from topological considerations, we will assume from now on that the utility functions considered are bounded. So,  $\mathcal{U}_b$  will denote the set of all bounded real-valued functions endowed with the supremum norm topology. From the point of view of applications, limiting the analysis to  $\mathcal{U}_b$  is not a severe restriction. Indeed, it can be easily shown that for any utility function, say  $u$ , there is a bounded utility function, say  $v$ , which represents the same ordering as  $u$  (see footnote (4) above).

A *profile* of (utility) functions will be denoted by  $U = (u_\omega)_{\omega \in \Omega}$  and  $\mathcal{U}^\Omega$  will stand for the set of all possible profiles. Note that each profile  $U = (u_\omega)_{\omega \in \Omega}$  can be interpreted as a *social state* where the individuals provide evaluations of the corresponding social outcomes. For a given  $x \in X$  and  $U = (u_\omega) \in \mathcal{U}^\Omega$ ,  $U(x)$  will denote the map on  $\Omega$ ,  $U(x)(\omega) = u_\omega(x)$ . Finally,  $\mathcal{U}_b^\Omega \subseteq \mathcal{U}^\Omega$  is given by  $\mathcal{U}_b^\Omega := \{U = (u_\omega) \in \mathcal{U}^\Omega : \sup_{x \in X, \omega \in \Omega} |U(x)(\omega)| < \infty\}$ . On  $\mathcal{U}_b^\Omega$  we will consider the topology that is induced by the norm  $\sup_{x \in X, \omega \in \Omega} |U(x)(\omega)|$ .

Following [6], a *social evaluation functional* is a rule  $F : \mathcal{U}_b^\Omega \rightarrow \mathcal{U}_b$  that assigns a real-valued function  $F(U) \in \mathcal{U}_b$ , interpreted as the social utility

function, to any profile  $U$  in the domain  $\mathcal{U}_b^\Omega$ .

**Definition 4.8.** Let  $x, y \in X$  and  $U, V \in \mathcal{U}_b^\Omega$  be arbitrarily given. A social evaluation functional  $F$  is:

- (1) *Paretian* if  $U(x) \leq U(y)$  implies that  $F(U)(x) \leq F(U)(y)$ ,
- (2) *weakly Paretian* if  $U(x) \ll U(y)$  implies  $F(U)(x) < F(U)(y)$ ,
- (3) *strongly Paretian* if  $U(x) < U(y)$  implies that  $F(U)(x) < F(U)(y)$ ,
- (4) *social state separable* if  $U(x) = V(y)$  entails  $F(U)(x) = F(V)(y)$ ,
- (5) *continuous* if it is continuous with respect to the topologies on  $\mathcal{U}_b$  and  $\mathcal{U}_b^\Omega$  introduced above.

For a profile  $U = (u_\omega) \in \mathcal{U}_b^\Omega$  and a real-valued function of a single variable  $\phi$ ,  $\phi \circ U$  stands for the following profile:  $\phi \circ U = (\phi \circ u_\omega) \in \mathcal{U}_b^\Omega$ .

We now translate to our context the standard concept of informational invariance with respect to a single ordinal scale, that is usually encountered in the social choice literature. Note that it allows for both intra and inter personal comparability of welfare among individuals.

**Definition 4.9.** Let  $x, y \in X$ ,  $U \in \mathcal{U}_b^\Omega$  and  $\phi \in \Phi$  be arbitrarily given. A map  $F : \mathcal{U}_b^\Omega \rightarrow \mathcal{U}_b$  satisfies *information invariance with respect to a single ordinal scale* if  $F(U)(x) \leq F(U)(y)$  entails  $F(\phi \circ U)(x) \leq F(\phi \circ U)(y)$ .

By taking advantage of the results obtained earlier, we reach the following characterization.

**Theorem 4.10.** For a map  $F : \mathcal{U}_b^\Omega \rightarrow \mathcal{U}_b$  the following statements are equivalent:

- (i)  $F$  satisfies social state separability, continuity, and information invariance with respect to a single ordinal scale,
- (ii) Either  $F$  is constant or there are both a strictly monotonic continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous and ordinal covariant operator  $T : B_\Omega \rightarrow \mathbb{R}$  such that  $F(U)(x) = (g \circ T)(U(x))$ , for every  $x \in X$ ,  $U \in \mathcal{U}_b^\Omega$ .

*Proof.* The implication (ii)  $\implies$  (i) is straightforward. So, we will focus on the other implication (i)  $\implies$  (ii). Suppose that  $F$  is not constant. Define then the operator  $T : B_\Omega \longrightarrow \mathbb{R}$  as follows: For each  $f \in B_\Omega$ , set  $T(f) = F(U)(x)$ , where  $U \in \mathcal{U}_b^\Omega$  is such that  $U(x) = f$ , for some  $x \in \Omega$ . Then, since  $F$  satisfies social state separability, it follows that  $T$  is well-defined. Moreover, since  $F$  is continuous so is  $T$ . Furthermore,  $T$  is comparison meaningful since  $F$  satisfies information invariance with respect to a single ordinal scale. Hence, by a direct application of Theorem 4.1, the desired conclusion is obtained.  $\square$

**Remark 4.11.** (i) Note that some of the most popular social evaluation functionals in social choice theory appear within the frame of Theorem 4.10. For instance, the dictatorial rules (there is  $\omega_0$  (the dictator)  $\in \Omega$  such that  $F(U)(x) = u_{\omega_0}(x)$ , for every  $U = (u_\omega) \in \mathcal{U}_b^\Omega$ ,  $x \in X$ ), or the Rawlsian rules (which are given by  $F(U)(x) = \inf_{\omega \in \Omega} \{u_\omega(x)\}$ , for every  $U = (u_\omega) \in \mathcal{U}_b^\Omega$ ,  $x \in X$ ).

(ii) If in the statement of Theorem 4.10 it is also required that  $F$  be weakly Paretian, then  $g$  is strictly increasing. In addition, by calling at Corollary 4.7, we can offer the following impossibility result: “Suppose that  $\Omega$  contains more than one individual. Then no strongly Paretian social evaluation functional  $F : \mathcal{U}_b^\Omega \rightarrow \mathcal{U}_b$  can exist that satisfies social state separability, continuity and information invariance with respect to a single ordinal scale”.

## 5. Conclusions

In this paper, we study the continuous representation problem, for the class of ordinal invariant preferences defined on  $B_\Omega$ , the Banach space of all bounded real-valued functions defined on  $\Omega$ . The kind of representation obtained involves the so-called ordinal covariant operators, which are naturally linked to the concept of comparison meaningfulness. Certain interplays between these operators are established, including a monotonicity property. In addition, two applications of our results in behavioral sciences are presented; specifically, one in measurement theory and another in social choice theory.

In relation to the existing literature on the topic of aggregation functions, or functionals, that preserve ordinal invariance properties, we could mention the fundamental papers [17], [16], [18], [19], and [20] to [22]. Since we work in a preferential context, the scope of all these papers substantially differs

from ours. Yet, we can take advantage of certain interesting functional representations, such as those shown in [21] and [22], to obtain some refinements in terms of the existence of a Choquet integral representation.

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