AVERAGING, REDUCTION AND RECONSTRUCTION IN THE SPATIAL THREE-BODY PROBLEM

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Contents

Resumen

El objetivo de esta tesis es el estudio de la dinámica del problema espacial de tres cuerpos. En particular, se establece la existencia de toros KAM asociados a diferentes tipos de movimientos. El problema espacial de tres cuerpos es un sistema hamiltoniano de nueve grados de libertad. La primera parte de la tesis consiste en aplicar técnicas de promedios y reducción con el fin de obtener un sistema reducido de un grado de libertad, es decir, aquel en el que todas las simetrías continuas han sido reducidas.

El estudio, desarrollado a lo largo del presente documento, es válido en las regiones en las cuales el hamiltoniano del problema espacial de tres cuerpos puede ser expresado como suma de dos sistemas keplerianos más un pequeña perturbación.

El proceso de reducción consta de las siguientes etapas:

1.- Reducción de la simetría traslacional.

2.- Reducción kepleriana, introducida en el proceso de normalización.

3.- Reducción de la simetría rotacional.

4.- Reducción de las simetría introducida al truncar el desarrollo del potencial.

En primer lugar, reducimos la simetría traslacional, escribiendo el hamiltoniano en función de las coordenadas de Jacobi. A continuación, utilizamos las variables de Deprit para eliminar los nodos. Posteriormente, normalizamos con respecto de las anomalías medias en una región sin resonancias y truncamos los términos de mayor orden. El sistema obtenido es expresado en términos de los invariantes que definen el espacio reducido, el cual es una variedad simpléctica de dimensión ocho.

En segundo lugar, se reduce la simetría rotacional que viene determinada por el hecho de que el módulo del momento angular total y su proyección en el eje vertical del sistema de referencia inercial son integrales del movimiento. Una vez calculados los invariantes asociados a las simetrías generadas por dichas integrales y el espacio reducido correspondiente, expresamos el hamiltoniano en término de estos invariantes. Ahora el espacio reducido tiene dimensión seis y es singular

para algunos valores de los parámetros. En esta parte del estudio, la teoría de la reducción singular juega un papel clave.

El último paso en el proceso de reducción es el de eliminar la simetría asociada al argumento del pericentro del cuerpo exterior. Dicha simetría aparece al truncar el hamiltoniano, puesto que este resulta ser independiente del argumento del pericentro. Una vez finalizado el proceso de reducción, obtenemos un espacio, que puede ser regular y difeomorfo a S^2 o singular con a lo sumo tres puntos singulares, de dimensión dos parametrizado por medio de tres invariantes. En este espacio estudiamos los equilibrios relativos, su estabilidad y bifurcaciones.

Partiendo del análisis de los equilibrios relativos en el espacio más reducido, llevamos a cabo la reconstrucción de toros KAM alrededor de cada equilibrio de tipo elíptico. Nuestro estudio consiste en una combinación de técnicas de regularización basadas en la construcción de espacios reducidos a diferentes niveles y la determinación explícita de coordenadas simplécticas. Todo esto nos permite calcular las torsiones para todas las posibles combinaciones de movimientos que las tres partículas puede seguir, incluyendo aquellos en los que los cuerpos interiores siguen trayectorias casi rectilíneas. Para probar la existencia de soluciones cuasi-periódicas utilizamos el teorema de Han, Li y Yi para sistemas hamiltonianos con alta degeneración y obtenemos toros KAM, de dimensión cinco, alrededor de equilibrios elípticos que representan diferentes tipos de movimientos.

Centrándonos en los movimientos casi rectilíneos, encontramos soluciones cuasi-periódicas de los tres cuerpos tales que los dos cuerpos interiores describen órbitas cercanas a las de colisión. Los cuerpos interiores no colisionan, siguen órbitas acotadas con excentricidades próximas a uno. Estas soluciones están asociadas a puntos de equilibrio elípticos y o bien están en el plano invariable o son perpendiculares a él. Estas soluciones llenan toros invariantes de dimensión cinco.

Introduction

Aims.

We deal with the dynamics of the three-body problem in the three-dimensional space. The three-body problem has attracted interest of the most notable mathematicians since Newton, giving as result different studies, see [35] and references therein. We restrict to the case of non-zero angular momentum and negative energy, avoiding collisions among the three bodies. Our purpose is the study of the dynamics of the system and particularly the proof of the existence of different families of quasi-periodic solutions. One way to proceed is to use perturbation theory to get a simpler system with the same relevant qualitative information as the original one but with a lower dimension. Then, we study the simpler system and we apply KAM theory to obtain conclusions about the original system. Thus, our first aim is to apply a combination of averaging techniques with reduction theory in order to build a reduced Hamiltonian and a reduced phase space as simple as possible. This reduction process takes into account all possible continuous symmetries of the problem, including the symmetry generated by the two approximate integrals obtained after performing the normalisation with respect to the two fast angles and truncating the higher-order terms.

Reductions of all continuous symmetries.

The spatial three-body problem is a Hamiltonian system of nine degrees of freedom. After reducing the translational and rotational symmetries we obtain a four degrees of freedom system. The reduction by the translational symmetry is usually performed through the introduction of Jacobi coordinates, passing to an equivalent system of six degrees of freedom, after attaching the frame of reference to the centre of mass of the system.

After that, the elimination (or reduction) of the nodes proposed originally by Jacobi is performed using Deprit's elements of the N-body problem introduced by André Deprit in 1983 [26] and used later by Ferrer and Osácar in the stellar three-body problem [34] and very recently by Chierchia and Pinzari to determine invariant tori of the spatial N-body problem through three consecutive outstanding papers [11, 12, 13]. The resulting Hamiltonian defines a system of four degrees of freedom to which we can apply normalisation in order to get rid of the two fast angles, i.e., the mean anomalies of the fictitious inner and outer ellipses. In order to apply perturbation theory we need to establish the possible regimes where the Hamiltonian of the three-body problem can be split into two Hamiltonians: the unperturbed Hamiltonian composed of two Keplerian terms and the perturbation, which is supposed to be small with respect to the principal part. We make this discussion as general as possible in order to include all possible cases where this splitting is properly done. Indeed, the classification of the different zones of the phase space where the splitting is valid has been already done by Féjoz [31] for the planar three-body problem and we export it to the spatial case. Our study is valid in all the regimes defined by Féjoz. The averaging is performed up to terms including the Legendre polynomials of degree two. After truncating the higher-order terms the averaged Hamiltonian defines a system of one degree of freedom since, up to this approximation, this Hamiltonian is also independent of the (planar) argument of the pericentre of the outer ellipse.

Reduction theory is used to pass from the Hamiltonian defined on an open subset of the phase space $\mathbb{R}^{12}$ (e.g., the Hamiltonian written in Jacobi coordinates that describes the motion of the system with the inner and outer bodies) to the fully-reduced space whose dimension is two and which is embedded in $\mathbb{R}^3$. The reduction process is realised using invariant theory which allows to obtain global coordinates in the reduced spaces. For convenience, we have performed the reductions by following the stages given by: (i) We start with the Keplerian reduction that is performed using the Laplace-Runge-Lenz and the angular momentum vectors of each fictitious body. This procedure lies on the regular reduction theory introduced by Meyer [59] and independently by Marsden and Weinstein [57]. It is related to the normalisation of the corresponding two anomalies and allows us to define the associated reduced system as a Hamiltonian of four degrees of freedom in a manifold of dimension eight which is defined by the Cartesian product of four two-spheres. The twelve invariants associated to the reduction plus four relations among them are written explicitly in terms of the Laplace-Runge-Lenz and angular momentum vectors. (ii) The next step consists in reducing the symmetry resulting out of the elimination of the nodes. This reduction is singular in the sense of Arms, Cushman and Gotay [2] (see also [20, 21, 22]) and the new set of invariants are obtained as polynomials in the invariants of the Keplerian reduction. There are six fundamental invariants subject to two constraints relating them. As the computations turn to be very involved we make use of Deprit's coordinates in order to choose the invariants that generate the phase space properly. The reduced phase space has dimension four and is singular for various combinations of the parameters involved in the reduction process. The corresponding Hamiltonian system has two degrees of freedom. (iii) The final step consists in reducing out the symmetry introduced by the modulus of the angular momentum vector of the outer ellipse. The three invariants related to this last reduction and the relation among them define the fully-reduced two-dimensional phase space. This phase space is a surface that depends on three parameters, it is parametrised using Deprit's elements and may have zero, one, two or three singular points. We also obtain the fully-reduced one-degree-of-freedom Hamiltonian. Our approach is global in the sense that we deal with the flow of the fully-reduced system in the whole fully-reduced space.

Study of the simplest system.

Once all the invariants are built, the averaged Hamiltonian is written in terms of them and the right form of the fully-reduced phase space is established, the next step is the discussion of the occurrence of the different relative equilibria of the reduced Hamiltonian system. This is done in terms of the invariants and the fundamental constraint that define the fully-reduced phase space. There are two basic parameters to perform the analysis, namely the modulus of the total angular momentum vector and the modulus of the angular momentum vector of the outer ellipse. They generate the plane of parameters which is divided into six different regions and presents five bifurcation lines. There are also three special points in the bifurcation lines. Each region has a different number of relative equilibria, ranging from two to six. The number and stability of the equilibria change when crossing the different lines.

There are several papers dealing with the spatial three-body problem from the same viewpoint as ours. The usual procedure to perform the elimination of the nodes is by using Delaunay coordinates. However Jacobi's approach applies in a submanifold of the twelve-dimensional phase space of dimension ten, thus its validity is limited. This is pointed out by Biasco *et al.* in [6] (corrigendum) (and see also [8]). Jefferys and Moser in [46], McCord, Meyer and Wang in [58], Lidov and Ziglin in [53] and Zhao in [90] avoid this by taking the invariable plane as the horizontal plane in the inertial frame. Jefferys and Moser obtain a collection of invariant 3-tori encasing near-circular quasiperiodic motions. Harrington [40] deals with the stellar three-body problem, which concerns with the motion of three bodies of arbitrary masses moving such that the distance between two of them is much less than the distance of either from the third. This situation is also called the lunar case of the three-body problem. Harrington applies the elimination of the nodes and von Zeipel method to average the Hamiltonian up to third order. After truncation the resulting system has two degrees of freedom and surfaces of section are computed to analyse some stellar systems. Lidov and Ziglin [53] also use Delaunay coordinates to eliminate the nodes. Then, they apply averaging in order to simplify the Hamiltonian equations. They make a complete discussion of the relative equilibria, their stability and bifurcation lines. Nevertheless, their fully-reduced phase space is not right, hence some of the conclusions they derive from the analysis of the plane of parameters are not correct.

In a remarkable paper Ferrer and Osácar [34] make a comprehensive analysis of the spatial three-body problem following the guidelines of Lidov and Ziglin but using Deprit's instead of Delaunay elements. We have followed their approach. The plane of parameters discussed in [34] is analysed in great detail and represents an improvement to that of [53]. However, the reduction process by stages of [34] is performed in the context of regular reduction, thus some conclusions extracted from the points of the fully-reduced phase space that should be singular are not correct. Hence, one of our aims is to clarify the dynamics of the fully-reduced system related to the singular points of the surface. More recently Farago and Laskar [28] use Ferrer and Osácar's approach to study the so-called Lidov-Kozai mechanism and apply the theory to multiple star systems. Recently, Zhao [90] does a similar work as ours but without taking into account the singular reduction theory which makes the study global.

Related works treating the planar case of the three-body problem using reduction and analysing the relative equilibria, their stable character and bifurcations are due to Lieberman [54] and more recently to Féjoz [30, 31] and Cordani [18]. The last three papers are of great interest as they analyse the reduced problem from a global point of view, as Ferrer and Osácar do in [34]. On the one hand, Féjoz uses the equilibria to reconstruct the invariant tori of different stability type and dimensions, getting a large variety of tori for the problem. On the other hand, Cordani continues with Féjoz's approach but using singular reduction, which allows him to clarify some conclusions obtained in [30, 31]. However, as these authors recognise (p. 328 of [31] and p. 15 of [18]), the spatial case of the three-body problem deserves further research. In this respect our main contribution is the application of the singular reduction theory for the three-body problem in the space.

Flow reconstruction.

Once the study of the relative equilibria, stability and bifurcations is done, the next step is to prove the existence of invariant tori of the full Hamiltonian that appear as circles surrounding the relative equilibria of elliptic type in the fully-reduced space. Thus we reconstruct certain families of 5-tori corresponding to the full Hamiltonian by introducing a pair of action-angle variables needed for the local analysis around the elliptic equilibria in the fully-reduced space. The other actions we use are indeed the four independent integrals of motion that are employed to build the fully-reduced space.

The motions we deal with in our study admit different combinations, for instance, the outer particle may move in a near-circular orbit or the invariable plane may coincide with the horizontal plane and the inner particles may follow a near-rectilinear trajectory lying in the invariable plane or being perpendicular to it. This leads to different situations that have to be analysed in different intermediate reduced spaces. We achieve our study by considering all possible cases, constructing an adequate set of coordinates and computing the corresponding torsion in each case.

Nevertheless other families of KAM tori cannot be reconstructed from the fully-reduced space. The reason is that the action variables (either Deprit's or Delaunay coordinates) are not defined for all kind of bounded motions. For instance if the outer ellipse is near circular and the inner and outer ellipses do not lie in the same plane we cannot use the modulus of the angular momentum vector of the outer ellipse as an action variable. Therefore this type of quasi-periodic motions are studied in the reduced spaces previous to the last reduction process. More specifically there are five families of invariant tori that are reconstructed from the four-dimensional reduced space. In these cases one needs to introduce for each family two pairs of action-angle variables to carry out the KAM theory. Moreover another family is studied in a reduced space whose dimension is six and one more family is studied in the eight-dimensional manifold where the only reduction performed is the Keplerian one. A classification of all type of tori reconstructed from the different reduced spaces appears in Table 5.1.

In all the cases an appropriate KAM theorem is needed. However the spatial three-body problem is a degenerate Hamiltonian system, that is, the perturbation appears at least in three different scales. Therefore we cannot apply the KAM theorems available for the degenerate cases. Recently Han, Li and Yi [36] have proved a KAM theorem that is valid for highly degenerate Hamiltonian systems and applies in our cases. We will introduce it in Chapter 1.

One of the goals of this thesis is to classify different types of motions in the spatial three-body problem. Starting from the analysis of the elliptic relative equilibria of the one-degree-of-freedom reduced system done we reconstruct the flow of the full problem. We show that the different motions of the three bodies have to be studied in the adequate reduced (orbit) spaces accordingly to their level of degeneracy and that specific variables should be designed for achieving this study.

Near-rectilinear motions of the inner particles can be studied properly because we justify the use of the averaged system by means of the regularisation mapping due to Ligon-Schaaf for the Kepler problem. This regularising procedure does not need to change time and can be applied to perturbed Keplerian problems provided the perturbation is regular for collision orbits, which is our case. Next, the averaged (and truncated) Hamiltonian is reduced out by the Keplerian symmetry and the double inner collisions can be analysed in the resulting manifold of the reduction. This feature is maintained through the rest of reductions, therefore the inner ellipses are allowed to become straight lines and rectilinear solutions are taken into consideration. A first issue is that when the (real) inner bodies move on straight lines the outer ellipse lies on the plane perpendicular to the total angular momentum vector. This plane remains fixed in space and is called the invariable plane. In particular there are relative equilibria in the fully-reduced space corresponding to inner motions such that their projections into the three-dimensional coordinate space are parallel to the total angular momentum vector. They are always singular points of the fully-reduced phase space. In addition to that, there is a relative equilibrium that is related to near-rectilinear motions that are in the invariable plane, thus the outer and inner bodies share the same plane, e.g. their motions are coplanar. Moreover, the point of the fully-reduced space that corresponds to any kind of coplanar motions — not necessarily rectilinear or circular — is always an equilibrium of the equations of motion. The point corresponding to circular motions of the inner ellipses is an equilibrium provided that the action related with the mean anomaly of the inner bodies does not exceed the sum of the modulus of the total angular momentum vector and the modulus of the angular momentum vector of the outer ellipse. When these two quantities are equal the point of the fully-reduced phase space becomes singular. If the action conjugate to the mean anomaly of the inner fictitious ellipse is bigger than the sum of the modulus of the two angular momentum vectors aforementioned, then circular motions of the inner bodies are no longer allowed. Other relative equilibria of the fully-reduced space are related with other types of inner ellipses that have different eccentricities and inclinations.

Concerning the near-circular-coplanar motions in the planetary case, where one body dominates the system and the others are small, Robutel [75] extends Arnold's result to the spatial planetary three-body problem. The existence of quasi-periodic motions for almost all values of the ratio of the semi-major axis and almost all values of the mutual inclination up to about one degree is proved. Biasco, Chierchia and Valdinoci [6] deal with the case of lower-dimensional tori, proving the existence of two-dimensional KAM tori in the spatial three-body problem. Féjoz [32] (following Herman) gives a complete proof of 'Arnold's Theorem' on the planetary N-body problem, establishing the existence of a positive measure set of smooth Lagrangian invariant tori. The analytic version of the invariant tori is due to Chierchia and Pusateri [14]. Another direct proof of Arnold's Theorem as well as the existence of elliptic lower dimensional tori are carried out by Chierchia and Pinzari [12, 13].

Only a few results are known outside the near-circular-coplanar regime. Jefferys and Moser [46] prove the existence of two- and three-dimensional invariant tori for the spatial three-body problem. The three bodies move around their centre of mass in quasi-periodic orbits that are nearly circular and inclined. They find these motions in two situations, the planetary case and the lunar case, where the mass ratios are arbitrary but the ratio of the two semimajor axes is small. In the planar case Lieberman [54] analyses the relative equilibria, together with their stable character and bifurcations. More recently Féjoz [30, 31] determines the quasi-periodic motions related to the relative equilibria of elliptic and hyperbolic character obtained after reducing out the symmetries of the problem. These solutions belong to what he calls the perturbing region, where the Hamiltonian splits as the sum of two Keplerian systems plus a smaller perturbation. By using singular reduction, Cordani [18] confirms Féjoz's conjecture on the number of relative equilibria of the two-dimensional reduced system. Recently Zhao in his thesis [90] (see also [91, 93]) uses Herman and Féjoz's ideas on special KAM theorems valid for degenerate cases to obtain a large variety of quasi-periodic solutions, including near-circular-coplanar and almost-collision orbits in the lunar case of the spatial three-body problem.

The reconstruction of the flow goes in the same lines as Zhao's work [90, 91, 92, 93] in the sense that we also obtain quasi-periodic solutions for the spatial three-body problem. Nevertheless, we focus on the classification of all possible bounded motions of the three bodies in the different reduced spaces, introducing adequate action-angle variables. Besides, Zhao uses KAM results due to Herman and Féjoz, whereas we use Han, Li and Yi's Theorem. As we work in the context of singular reduction, our analysis is global. In particular we prove the existence of quasi-periodic motions where the inner particles describe bounded near-rectilinear trajectories whereas the outer particle follows an orbit lying near the invariable plane. These motions fill in five-dimensional invariant tori. Moreover, the inner particles move in orbits either near an axis perpendicular to the invariable plane or near the invariable plane.

In the circular restricted three-body problem, Moser [67] pointed out that there are near-collision periodic motions in the spatial lunar case, both in the plane of the primaries and in the perpendicular axis. Belbruno [5] gave a proof for the existence of the vertical solutions when the mass parameter is small, generalising a previous result by Sitnikov [84]. This was enlarged in [88] for any value of the mass parameter; it was also proved that these periodic solutions are elliptic. Concerning the existence of quasi-periodic solutions and KAM tori, Chenciner and Llibre [9] established in the planar lunar case the existence of quasi-periodic almost-collision solutions filling in KAM 2-tori. For the spatial lunar case it was shown in [64] that there are both vertical and coplanar quasi-periodic almost-collision solutions filling in KAM 3-tori. For the non-restricted planar problem, Féjoz [29, 31] proved the existence of invariant KAM 3-tori filled up by the near-rectilinear quasi-periodic solutions in the asynchronous region (a zone of phase space where the inner bodies revolve quickly when compared to the outer body, this region enlarges the lunar one). This has been generalised recently by Zhao [90, 93] who dealt with the spatial three-body problem in the lunar case concluding the existence of quasi-periodic almost-collision solutions and KAM 5-tori. In all these studies the periodic and quasi-periodic solutions are bounded (although the semimajor axes can be very big). Moreover in the restricted problems the infinitesimal does not collide with the primary it revolves around whereas in the non-restricted case the two inner bodies get arbitrarily close one another an infinite number of times but they do not collide.

The studies accomplished in this thesis are presented in three papers [69, 70, 71]. The first one develops the reduction and the study of the dynamics in the most reduced space. The following two we reconstruct the flow in the intermediate spaces concretely those related with the KAM tori of dimension five of the original system. In the second paper for the non-rectilinear motions and in the third one for the near-rectilinear motions.

Structure of the thesis.

We devote the first chapter to the revision and summary of some basic concepts in perturbed Hamiltonian systems, KAM theory, normalisation, reduction and Gröbner bases. In fact, we consider the three-body problem as a perturbation of an integrable system, i.e., a nearly integrable system. The section devoted to the KAM theory finishes recalling the Han, Li and Yi's Theorem, which is the result that allows us to conclude the existence of KAM tori.

In Chapter 2 we present the Hamiltonian of the problem focusing on the different types of three-body problems that can be dealt with using our approach. Then we present Deprit's variables and the normalisation of the fast angles, i.e. the two mean anomalies of the problem. The successive reductions are also determined in this chapter, obtaining the intermediate phase spaces together with the invariants and the final expression of the fully-reduced Hamiltonian. Finally, we deal with an account of the main features of the reduced phase spaces, the dynamical meaning of the singularities and the location of other classes of motions on the different surfaces. See also [69, 70].

Chapter 3 is devoted to the working out of the equations of motion corresponding to the fully-reduced problem, classifying the relative equilibria, studying their stability and the bifurcations in terms of the two relevant parameters of the problem, see also [69].

In Chapter 4 we account for the passage from the fully-reduced space to the higher-dimensional ones through stages. We analyse the singularities of some intermediate spaces and where the possible motions of the three bodies are located in the different reduced spaces, including the reconstruction concerning the rectilinear motions of the inner particles. Specifically, starting from the one-dimensional compact set that contains all motions which are represented by elliptic relative equilibria in the fully-reduced space, we map this set to three reduced spaces whose dimensions are four, six and eight. This will be needed later on to achieve the construction of adequate action-angle variables to deal with different types of motions in these spaces. The results also appear in [70, 71].

Chapter 5 deals with the proof of the theorem which establishes the existence of invariant 5-tori related with elliptic equilibrium points without taking into account those related with near-rectilinear motions. We achieve this by computing the torsions in the different cases. We choose one representative case of each group in Table 5.1 and develop the proof, see also [70]. One can find the remaining cases in Appendix B.

In Chapter 6 we present the different types of invariant 5-tori related with near-rectilinear motions. The results establish the existence of the invariant 5-tori and quasi-periodic solutions of near-rectilinear type for the inner particles. These solutions correspond to the three relative equilibria in the fully-reduced space. In particular, we conclude the existence of KAM 5-tori for motions such that the inner particles move near the axis perpendicular to the invariable plane while the outer particle moves near the invariable plane in a non-circular orbit. We also prove the existence of KAM 5-tori for the inner bodies but such that the outer particle describes a near-circular motion. At the end of this chapter, we focus on the case where the three particles move near the invariable plane and the inner particles have motions of rectilinear type, ending up with the existence of KAM 5-tori for these solutions. The results are collected in [71].

Finally, the main conclusions and future work are delineated. The formulae that relate Deprit's elements with the invariants of the (regular) Keplerian reduction appear in Appendix A. In Appendix B, one can find the remaining cases which have not been proved in Chapter 5.

Chapter 1

Basic concepts of perturbation theory, symplectic reduction and computer algebra

In this chapter we introduce some basic concepts about the study of perturbed Hamiltonian systems. Particularly, perturbation theory, Lie transformations, normal forms, reduction theory, KAM theory. We also introduce some results related with Gröbner bases which are used to define the invariants associated to the reductions of the continuous symmetries which we apply in Chapter 2.

1.1 Symplectic transformations

The use of symplectic transformations to simplify a Hamiltonian system has been employed widely in Celestial Mechanics. Here we summarise some well known concepts.

1.1.1 Averaging

Perturbation theory studies the problem of the influence of small Hamiltonian perturbations on an integrable Hamiltonian system. Following the book by Arnold, Kozlov and Neishtadt [4] we introduce the concept of averaging in Hamiltonian systems.

Given an unperturbed completely integrable Hamiltonian system $\mathcal{H}_0$ for which some domain of its phase space is foliated into invariant tori, and the action-angle variables $I = (I_1, \ldots, I_n) \in B \subset \mathbb{R}^n$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \bmod 2\pi \in \mathbb{T}^n$. The unperturbed Hamiltonian system depends only on the action variables, i.e., $\mathcal{H}_0(I)$ and this Hamiltonian is subjected to a small perturbation by $\mathcal{H} = \mathcal{H}_0(I) +$

$\varepsilon\mathcal{H}_1(I,\varphi;\varepsilon)$ or equivalently $\dot{I} = -\varepsilon\frac{\partial\mathcal{H}_1}{\partial\varphi}$ and $\dot{\varphi} = \frac{\partial\mathcal{H}_0}{\partial I} + \varepsilon\frac{\partial\mathcal{H}_1}{\partial I}$ where $\mathcal{H}_1(I,\varphi;\varepsilon)$ has period 2π in φ.

By assuming the functions $\mathcal{H}_0$ and $\mathcal{H}_1$ to be analytic and applying averaging we obtain a simpler Hamiltonian which describes the slow motion and is called the *averaged Hamiltonian*:

$$\bar{\mathcal{H}}(J,\varepsilon) = \mathcal{H}_0(J) + \varepsilon\bar{\mathcal{H}}_1(J) + \mathcal{O}(\varepsilon^2)$$

where

$$\bar{\mathcal{H}}_1 = \langle\mathcal{H}_1(J,\varphi;0)\rangle_\varphi = \frac{1}{(2\pi)^n}\iint_{\mathbb{T}^n}\mathcal{H}_1(J,\varphi;0)\mathrm{d}\varphi$$

(here $\mathrm{d}\varphi = (\mathrm{d}\varphi_1,\ldots,\mathrm{d}\varphi_n)$). For us averaging is the same as normalisation.

When the Hamiltonian does not depend on all the action variables (proper degeneracy), correspondingly, some of the unperturbed frequencies are identically equal to zero, i.e., $\mathcal{H} = \mathcal{H}_0(I_1,\ldots,I_r) + \varepsilon\mathcal{H}_1(I,\varphi;\varepsilon)$ with $r < n$, then the phases φ_j, $j > r$, are slow variables. One should average the equations of the perturbed motions over the fast phases, meaning, φ_i, $i \leq r$. Let us note that the variables conjugate to the fast phases are integrals of the averaged system and the averaged Hamiltonian system has $n - r$ degrees of freedom for the slow phases and their conjugate variables. The correspondence between the solutions of the exact and averaged system can be solved by using KAM theory. Summarising, the critical points of the averaged system are in correspondence with periodic orbits of the original one. Particularly, non-degenerate critical points of the averaged system lead to quasi-periodic orbits of the original system with the same type of stability.

We will focus on the averaging in *non-resonant domains*, that is,

$$|k\cdot\omega| \geq \frac{\alpha}{|k|^\tau} \quad \text{for all} \quad k \in \mathbb{Z}^n\setminus\{0\} \text{ and } \alpha,\tau > 0,$$

where $|k| = \sum_{1\leq i\leq n}|k_i|$, $\omega = (\omega_1,\ldots,\omega_n)$ the frequencies' vector and the operator $(\cdot)$ refers to the usual dot product. We shall deal with this type of conditions in the last section of this chapter, in the context of KAM theory. If the domain is non-resonant, the averaging can be applied with an accuracy of order ε, i.e. $I(t) - J(t) = \mathcal{O}(\varepsilon)$ on the time scale $1/\varepsilon$.

Sometimes the averaging at first order is not enough because one needs to consider the dynamics of the higher-order terms. It can be achieved through the normalisation by applying Lie transformations which are introduced in the following subsection.

1.1.2 Lie transformations

The method of Lie transformations, initiated by Deprit [24], is a procedure to define a change of variables. Particularly, a near-identity symplectic change of variables is determined in a system of equations that depends on a small parameter. We introduce Lie transformations following Meyer, Hall, Offin [62].

A symplectic change of variables $\mathbf{x} \equiv \mathbf{X}(\mathbf{y};\varepsilon)$ is called near-identity if it is symplectic for each fixed ε and is of the form $\mathbf{X}(\mathbf{y};\varepsilon) = \mathbf{y} + \mathcal{O}(\varepsilon)$; i.e., $\mathbf{X}(\mathbf{y};0) = \mathbf{y}$. Let $\mathbf{y} \equiv \mathbf{Y}(\mathbf{X}(\mathbf{y};\varepsilon);\varepsilon)$ be the inverse of $\mathbf{x} \equiv \mathbf{X}(\mathbf{Y}(\mathbf{x};\varepsilon);\varepsilon)$, both are symplectic for fixed ε.

The transformation $\mathbf{X}(\mathbf{y};\varepsilon)$ is a near-identity symplectic change of variables if and only if it is a general solution of a Hamiltonian differential equation of the form $\frac{d\mathbf{x}}{d\varepsilon} = J\nabla\mathcal{W}(\mathbf{x};\varepsilon)$ (where $\mathcal{W}$ is smooth and J is the usual skew-symmetric matrix) satisfying the initial condition $\mathbf{x}(0) = \mathbf{y}$.

Let $\mathcal{H}(\mathbf{x};\varepsilon)$ be a Hamiltonian and $\mathcal{G}(\mathbf{y};\varepsilon) \equiv \mathcal{H}(\mathbf{X}(\mathbf{y};\varepsilon);\varepsilon)$ the Hamiltonian in the new coordinates. $\mathcal{G}$ is called the Lie transformation of $\mathcal{H}$ generated by $\mathcal{W}$. We denote $\mathcal{H}$ by $\mathcal{H}_*$ and $\mathcal{G}$ by $\mathcal{H}^*$. Using this notation we introduce the method of *Lie transformations* which is a recursive procedure given by the following formulas

$$\mathcal{H}(\mathbf{x};\varepsilon) = \mathcal{H}_*(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_i^0(\mathbf{x}), \tag{1.1}$$

$$\mathcal{G}(\mathbf{y};\varepsilon) = \mathcal{H}^*(\mathbf{y};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_0^i(\mathbf{y}), \tag{1.2}$$

$$\mathcal{W}(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{W}_{i+1}(\mathbf{x}), \tag{1.3}$$

where $\{\mathcal{H}_j^i\}$ for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots$ verify the recursive identities:

$$\mathcal{H}_j^i = \mathcal{H}_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{\mathcal{H}_{j+k}^{i-1}, \mathcal{W}_{k+1}\}.$$

For example, to compute the series expansion for $\mathcal{H}_*$ through terms of order ε^2, one first determines $\mathcal{H}_0^1$ by the formula $\mathcal{H}_0^1 = \mathcal{H}_1^0 + \{\mathcal{H}_0^0, \mathcal{W}_1\}$ which gives the term of order ε and then one computes $\mathcal{H}_1^1 = \mathcal{H}_2^0 + \{\mathcal{H}_1^0, \mathcal{W}_1\} + \{\mathcal{H}_0^0, \mathcal{W}_2\}$ and $\mathcal{H}_0^2 = \mathcal{H}_1^1 + \{\mathcal{H}_0^1, \mathcal{W}_1\}$ getting $\mathcal{H}^*(\varepsilon, \mathbf{x}) = \mathcal{H}_0^0(\mathbf{x}) + \varepsilon\mathcal{H}_0^1(\mathbf{x}) + \frac{\varepsilon^2}{2}\mathcal{H}_0^2(\mathbf{x})$.

In the reduction process of Chapter 2, the elimination of the fast angles is performed by averaging with respect to the two fast angles only to first order in a small parameter. Our study is valid in a region where no resonances between the fast angles occur. In Chapters 5 and 6, we shall make use of averaging and Lie transformations in order to eliminate the angular dependence from the different Hamiltonians we shall obtain, so that we can apply the KAM theory.

1.1.3 Normal forms

Definition 1.1. *A Hamiltonian system (1.1) Hamiltonian $\mathcal{H}(\mathbf{x};\varepsilon)$ admits an expansion in powers of the small parameter ε is said to be normal if the Poisson bracket $\{\mathcal{H}\,,\,\mathcal{H}_0\} = 0$.*

If the system is not normal, one can normalise it by using a Lie transformations. It means that a Lie transformation $\mathbf{x} \equiv \mathbf{X}(\mathbf{y};\varepsilon)$ is said to normalise it if the transformed Hamiltonian $\mathcal{H}(\mathbf{x};\varepsilon) = \mathcal{K}(\mathbf{y};\varepsilon)$ is normal. Then, $\mathcal{K}$ is called the normal form of $\mathcal{H}$.

Since in Chapters 5 and 6 we shall apply Lie transforms to Hamiltonian systems expanded around equilibria we need to introduce some results about normal forms at an equilibrium, following [62].

Given an analytic Hamiltonian $\mathcal{H}$ which has an equilibrium point at the origin in $\mathbb{R}^{2n}$ and is zero at the origin, then the $\mathcal{H}$ can be expanded in Taylor series by

$$\mathcal{H}(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_i^0(\mathbf{x}),$$

where $\mathcal{H}_i^0$ is a homogeneous polynomial in $\mathbf{x}$ of degree $i+2$. Thus, $\mathcal{H}_0^0 = \frac{1}{2}\mathbf{x}^T S\mathbf{x}$, where S is a $2n\times 2n$ real symmetric matrix, and $A = JS$ is a Hamiltonian matrix. The linearised equations about the critical point $\mathbf{x} = \mathbf{0}$ are $\dot{\mathbf{x}} = A\mathbf{x} = JS\mathbf{x} = J\nabla\mathcal{H}_0^0$, and their general solution is $\phi = \exp(At)\xi$.

The most general result about the existence of the symplectic change which allows us to define a normal form at an equilibrium point, is introduced as follows.

Theorem 1.1. *Let A be a Hamiltonian matrix. Then there exists a formal symplectic change of variables, $\mathbf{x} = \mathbf{X}(\mathbf{y};\varepsilon) = \mathbf{y}+\ldots$, that transforms the Hamiltonian $\mathcal{H}(\mathbf{x};\varepsilon)$ to $\mathcal{H}(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}\mathcal{H}_0^j(\mathbf{y})$, where $\mathcal{H}_0^j$ is a homogeneous polynomial of degree $j+2$ such that $\mathcal{H}_0^j(e^{A^T t}\mathbf{y}) \equiv \mathcal{H}_0^j(\mathbf{y})$, for all $j = 0,1,\ldots$, all $\mathbf{y}\in\mathbb{R}^{2n}$, and all $t\in\mathbb{R}$.*

If the simple component of the decomposition of a matrix A into simple and nilpotent matrices does not vanish, Theorem 1.1 implies that an approximate (formal) integral is built in the process of the normal form computation, after truncating the higher-order terms. Hence a continuous symmetry is introduced in the normalised Hamiltonian, allowing us to apply reduction theory, see for instance [72].

The classical case is the one where the matrix A is simple, that is, A has $2n$ linearly independent eigenvectors that may be real or complex or, in other words it is diagonalisable. We introduce the above theorem particularised for a simple matrix because it is the type of matrices which we deal with in Chapters 5 and 6.

Theorem 1.2. *Let A be simple. Then there exists a formal symplectic change of variables, $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon) = \mathbf{y} + \ldots$, that transforms the Hamiltonian $\mathcal{H}(\mathbf{x}; \varepsilon)$ to $\mathcal{H}(\mathbf{y}; \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{H}_0^i(\mathbf{y})$, where $\mathcal{H}^i$ is a homogeneous polynomial of degree $i+2$ such that $\mathcal{H}_0^i(e^{At}\mathbf{y}) \equiv \mathcal{H}_0^i(\mathbf{y})$ for all $i = 0, 1, \ldots$, all $\mathbf{y} \in \mathbb{R}^{2n}$, and all $t \in \mathbb{R}$.*

The expression of $\mathcal{H}(\mathbf{y})$ given in the theorem is the classical characterisation of normal form for a Hamiltonian near an equilibrium point with a simple linear part. This formula is equivalent to $\{\mathcal{H}_0^i, \mathcal{H}_0^0\} = 0$ for all i.

1.2 Reduction theory

In Chapter 2 we reduce out all the continuous symmetries by using regular and singular reduction theory. The modern regular reduction theory was introduced by Meyer [59] and by Marsden and Weinstein [57]. We introduce a result which characterise the regular reduction. So we introduce some concepts and results related with reduction theory, the reader may also consult the book [22] which we have taken into account to write down this section, see also [85].

Given a compact Lie group G, let $\mathcal{G} = T_eG$ be the Lie algebra of G (e being the identity element and T denoting the tangent space), and let M be a symplectic manifold with symplectic form ω. An *action* ϕ of G on M is a smooth mapping $\phi : G \times M \to M$; $(g, m) \mapsto \phi(g, m) = \phi_g(m)$ such that for all g, $h \in G$ an all $m \in M$, $\phi_{gh}(m) = \phi_g(\phi_h(m))$ and $\phi_e(m) = m$. The action ϕ is called *proper* if the map $G \times M \to M \times M$; $(g, m) \mapsto (m, \phi_g(m))$ is proper, that is, the inverse image of a compact set under this map is compact. If the Hamiltonian flow has no fixed point, the corresponding group action is *free*. If the action ϕ is free and proper, then the quotient M/G is a smooth manifold, which is called *orbit space* and consists of all G-orbits of ϕ on N. For $m \in M$ the *isotropy group* is defined as $G_m = \{g \in G \mid \phi_g(m) = m\}$. Then the action ϕ is free if $G_m = \{e\}$ for all $m \in M$.

In order to state the main result related with regular reduction we need to introduce the concept of *momentum map*. For every $\xi \in \mathcal{G}$, let the vector field X^ξ be defined by $X^\xi : M \to TM$; $m \mapsto X^\xi(m) = \frac{d}{dt}_{|t=0} \phi_m(\exp(t\xi)) = (T_e\phi_m)\xi$. Let $Ad_g : \mathcal{G} \to \mathcal{G}$; $\xi \mapsto \frac{d}{dt}_{|t=0} g \exp(t\xi)\, g^{-1}$ and Ad_g^* its dual.

Definition 1.2. *The action ϕ of a Lie group G on a symplectic manifold M is called a Hamiltonian G-action if:*

(i) For every $\xi \in \mathcal{G}$, X^ξ is a Hamiltonian vector field on (M, ω), that is, there is a smooth function $J^\xi : M \to \mathbb{R}$ such that $X^\xi = X_{J^\xi}$,

(ii) ϕ_g is a symplectic diffeomorphism for every $g \in G$.

The mapping $J : M \to \mathcal{G}$ defined by $J(m)\xi = J^\xi(m)$ is called a momentum map of ϕ provided $\{J^\xi, J^\nu\} = J^{[\xi,\nu]}$ for all ξ, $\nu \in \mathcal{G}$, where $\{\ ,\}$ and $[,]$ stand *for the Poisson brackets in M and $\mathcal{G}$ respectively. A momentum map J is called coadjoint equivariant if $J(\phi_g(m)) = Ad^*_g(J(m))$ holds for all $m \in M$ and $g \in G$. The coadjoint orbit $\mathcal{O}_\eta$ through η is $\{\nu = Ad^*_g(\eta) \in \mathcal{G}^* | g \in G\}$.*

Since we are dealing with the reduction process of a Hamiltonian system, we introduce the definition of *symmetry* of a Hamiltonian system.

Definition 1.3. *Let G be a Lie group and $(M, \omega, \mathcal{H})$ a Hamiltonian system. An action ϕ on M is called a symmetry of this system if ϕ is a Hamiltonian action that preserves $\mathcal{H}$.*

We are ready to formulate the *regular reduction theorem* due to Meyer [59] and Marsden and Weinstein [57].

Theorem 1.3. *Let $(M, \omega, \mathcal{H})$ be a Hamiltonian system and G a Lie group with a free and proper Hamiltonian action ϕ on M that preserves $\mathcal{H}$. Let $J : M \to \mathcal{G}^*$ be a coadjoint equivariant momentum map of ϕ and $\eta \in \mathcal{G}^*$ a regular value of J and let G_η be the isotropy group of η under the coadjoint action of G on $\mathcal{G}^*$. Then $M_\eta = J^{-1}(\eta)/G_\eta$ is a smooth symplectic manifold (reduced phase space). Let*

$$\pi_\eta : J^{-1}(\eta) \to M_\eta$$

be the orbit (reduction map) of the G_η-action $\phi|_{G_\eta \times J^{-1}(\eta)}$ and

$$i : J^{-1}(\eta) \to M$$

be the inclusion. Then the symplectic form ω_η is defined by

$$\omega_\eta \circ \pi_\eta = \omega \circ i$$

and the reduced Hamiltonian $\mathcal{H}_\eta$ on M_η is given by

$$\mathcal{H}_\eta \circ \pi_\eta = \mathcal{H} \circ i.$$

On $J^{-1}(\eta)$ the Hamiltonian vector field $X_\mathcal{H}$ is π_η related to $X_{\mathcal{H}_\eta}$, that is

$$T\pi_\eta \circ X_\mathcal{H} = X_{\mathcal{H}_\eta} \circ \pi_\eta.$$

There is a different type of reduction, that is, singular reduction, which occurs when there is some $m \in M$ whose isotropy group is not trivial. So the action is not free and the reduced phase space is a *symplectic orbifold*. Satake [78] introduced the concept of orbifold with the name of V-manifold, see also [51] for all the definitions related to symplectic orbifolds. This type of points are singular in the reduced phase space whereas the remaining points are transformed into regular ones. For further details on the subject of singular reduction the reader is referred to [2, 50, 22]. Here we state the *singular reduction theorem*.

Theorem 1.4. *Let $(M, \omega, \mathcal{H})$ be a Hamiltonian system and G a Lie group with a proper Hamiltonian action ϕ on M. Let $J : M \to \mathcal{G}^*$ be a coadjoint equivariant momentum map of ϕ. Furthermore, suppose that the coadjoint orbit $\mathcal{O}_\eta$ through $\eta \in \mathcal{G}^*$ is locally closed. Then on the singular reduced space $M_\eta = J^{-1}(\mathcal{O}_\eta)/G$ there is a nondegenerate Poisson algebra $(\mathcal{C}^\infty(M_\eta), \{,\}_\eta, \cdot)$. In addition, M_η is a locally finite union of symplectic manifolds called symplectic pieces. The flow of a Hamiltonian derivation corresponding to a smooth function on M_η preserves the decomposition of M_η into symplectic pieces and the inclusion map of symplectic pieces into M_η is a Poisson map.*

Singular reduction theory plays a key role to accomplish the reduction process correctly. In particular the reduction of the rotational symmetry in the three-body problem will be performed in the frame of singular reduction whereas the Keplerian reduction lies in the setting of regular reduction.

1.3 Aspects of computational algebra

In Chapter 2, we will reduce each continuous symmetry of the problem. The way of carrying out each reduction is by making use of *invariant theory* because we want to parametrise the reduced phase space in terms of the polynomials which are invariant under a certain G-action associated to the symmetry which we are reducing out. As well, we express our Hamiltonian as a function of them. With the use of invariant theory we can find global coordinates for realising the regular or singular reduced space M_η. Indeed, according to Cushman and Bates [22], invariant theory provides an algebraic technique that gives a geometrically faithful model of the reduced phase space, regardless whether we deal with a smooth manifold or not.

If G is a compact group the Lie algebra of all polynomials which are invariant under the G-action is finitely generated. Suppose, $g_1, \ldots, g_t$ are generators of this algebra. It can be shown [22] that the Hilbert map $\omega : N \to \mathbb{R}^t$; $m \mapsto (g_1(m), \ldots, g_t(m))$ separates G-orbits, because G is compact. By a theorem of Schwarz [80] even every smooth invariant function can be expressed as a smooth function of the basic polynomial invariants. However, the symmetry group of the rotational symmetry of the three-body problem is not compact, but still we can get a set of generators that sufficient for the singular reduction because their Poisson structure is closed.

In the three-body problem there are some reductions for which the invariants are given by the geometric features of the problem, for example, the Keplerian reduction, but for other reductions the polynomial invariants will be determined constructively.

Once the invariants are computed, to define the reduced phase space, i.e. the orbifold, we need the independent constraints between these invariants. Theoretically, this is achieved by obtaining the Gröbner basis associated to them which gives us the syzygies. The syzygies are the restrictions which we are looking for. The Gröbner bases routines of MATHEMATICA determine each Gröbner basis but this is not feasible in some cases for which the constraints are determined by using Deprit's variables [26] as one can see in Chapter 2.

So in this section we introduce the basic theory associated to the Gröbner bases following [87, 19]. We start by the *division algorithm* in a polynomial ring $K[x_1, \ldots, x_n]$.

Theorem 1.5. *(Division Algorithm in $K[x_1,\ldots,x_n]$) Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, and let $F = (f_1,\ldots,f_s)$ be an ordered s-tuple of polynomials in $K[x_1,\ldots,x_n]$. Then, every $f \in K[x_1,\ldots,x_n]$ can be written as $f = a_1f_1+\cdots+a_sf_s+r$ where a_i, $r \in K[x_1,\ldots,x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in K, of monomials, none of which is divisible by any of $LT(f_1),\ldots,LT(f_s)$ (being $LT(a_0x^m + \cdots + a_m) = a_0x^m$ the leading term). We will call r a remainder of f on division by F. Furthermore, if $a_if_i \neq 0$, then we have multideg$(f) \geq$ multideg (a_if_i) (multideg $(\sum_\alpha a_\alpha x^\alpha) = max(\alpha \in \mathbb{Z}^n_{\geq 0} : a_\alpha \neq 0)$).*

The division algorithm does not have the same properties as one variable's version. Particularly, in one variable the remainder is uniquely determined but, in general, this is not true for multivariate polynomials. The algorithm achieves its full potential when coupled with Gröbner bases.

First, we introduce some definitions and results which we need to introduce Gröbner bases.

Definition 1.4. *Let $I \subset K[x_1,\ldots,x_n]$ be an ideal other than $\{0\}$. We denote by $LT(I)$ the set of leading terms of elements of I. Thus,*

$$LT(I) \;=\; \left\{cx^\alpha : \text{there exists } f \in I \text{ with } LT(f) = cx^\alpha\right\} \tag{1.4}$$

and by $\langle LT(I)\rangle$ the ideal generated by the elements of $LT(I)$.

As we can see the leading terms play an important role in the division algorithm. Namely, if we are given a finite generating set for I, i.e., $I = \langle f_1,\ldots,f_s\rangle$ then $\langle LT(f_1),\ldots,LT(f_s)\rangle \subset \langle LT(I)\rangle$. $\langle LT(I)\rangle$ can be strictly larger than $\langle LT(f_1),\ldots,LT(f_s)\rangle$.

Proposition 1.6. *Let $I \subset K[x_1,\ldots,x_n]$ be an ideal. Then, $\langle LT(I)\rangle$ is a monomial ideal and there are $g_1,\ldots,g_t \in I$ such that $\langle LT(I)\rangle = \langle LT(g_1),\ldots,LT(g_t)\rangle$.*

Applying Proposition 1.6 and the division algorithm one can prove the existence of a finite generating set for every polynomial ideal.

Theorem 1.7. *(Hilbert Basis Theorem) Every ideal $I \subset K[x_1, \ldots, x_n]$ has a finite generating set, meaning $I = \langle g_1, \ldots, g_t \rangle$ for some $g_1, \ldots, g_t \in I$.*

In the proof of Hilbert Basis Theorem [19] the basis $\{g_1, \ldots, g_t\}$ verifies that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$. As we have said before this is not the behaviour of all bases. The set $\{g_1, \ldots, g_t\}$ is called Hilbert basis.

Definition 1.5. *A basis $\{g_1, \ldots, g_t\}$ which verifies that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$ is called a Gröbner basis. Equivalently, a set $\{g_1, \ldots, g_t\} \subset I$ is a Gröbner basis of I if and only if the leading term of any element is divisible by one of the $LT(g_i)$.*

Corollary 1.8. *Fix a monomial order. Then every ideal $I \subset K[x_1, \ldots, x_n]$ other than $\{0\}$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal I is a basis of I.*

Now, we want to know how to detect when a given basis is a Gröbner basis.

Proposition 1.9. *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ and let $f \in K[x_1, \ldots, x_n]$. Then there is a unique $r \in K[x_1, \ldots, x_n]$ which is not divisible by any of $LT(g_1), \ldots, LT(g_t)$ and there is $g \in I$ such that $f = g + r$. In particular, r is the remainder on division of f by G no matter how the elements of G are listed when using the division algorithm.*

If we list the generators in a different order then the quotients produced by the division algorithm can change. Thus, we introduce the following criterion to know when a polynomial lies in an ideal.

Corollary 1.10. *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ and let $f \in K[x_1, \ldots, x_n]$. Then $f \in I$ if and only if the remainder on division of f by G is zero.*

This property is sometimes taken as the definition of a Gröbner basis. The reason is that this condition is true if and only if $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$.

Given a polynomial ideal different from zero, one can construct a Gröbner basis in a finite number of steps by following the Buchberger's Algorithm, see [19].

In Chapter 2 we shall use Corollary 1.10 to check if any invariant polynomial belongs to the ideal formed by a certain basis. Once a Gröbner basis, which determines the invariants associated to each reduction, is computed we need to find the constraints to define the reduced spaces. Each constraint is given by a syzygy. Given a Gröbner basis, then the syzygies associated can be determined easily. The syzygies are obtained from cofactors of all S-polynomials. S-polynomials play a key role for finding syzygies and for the construction of Gröbner bases, see Buchberger's Algorithm in [19].

Definition 1.6. *Let $F = (f_1, \ldots, f_n)$. A syzygy on the leading terms $LT(f_1), \ldots, LT(f_s)$ of F is an s-tuple of polynomials $S = (h_1, \ldots, h_s) \in (K[x_1, \ldots, x_n])^s$ such that $\sum_{i=1}^{s} h_i.LT(f_i) = 0$. We let $S(F)$ be the subset of $(K[x_1, \ldots, x_n])^s$ consisting of all syzygies on the leading terms of F.*

1.4 Introduction to KAM theory

We want to study the dynamics of a Hamiltonian system with respect to the influence of small Hamiltonian perturbations. This is achieved by applying KAM theory. The reader is addressed to the book by Arnold, Kozlov and Neishtadt [4] to consult about this issue.The classical KAM theory demands two properties of the unperturbed system, namely, the integrability and the non-degeneracy.

Considering perturbed integrable Hamiltonian systems of the form

$$\mathcal{H}(I, \varphi, \varepsilon) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \varphi, \varepsilon), \tag{1.5}$$

where ε is a small parameter. The phase space associated to $\mathcal{H}_0$ is foliated by invariant tori and there are n independent first integrals of motion. That is to say, a level set of the n independent first integrals of motion is diffeomorphic to an n-dimensional torus $T^n = \{\varphi = (\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$, φ_i being angular coordinates for $i = 1, \ldots, n$. The frequencies of the motions are given by $\omega_i = d\varphi_i/dt$. In order to maintain the Hamiltonian structure, action coordinates – $I = (I_1, \ldots, I_n)$ – are defined and together with the angles define the phase space of the system and are called action-angle variables. Action coordinates are related with the frequencies by $\omega_i = \partial\mathcal{H}_0/\partial I_i$ and the trajectories describing these motions are dense in the tori. These motions are known by quasi-periodic motions.

A system is non-degenerate if the determinant $|\partial^2\mathcal{H}_0/\partial I^2| = |\partial\dot{\varphi}/\partial I|$ is not zero in an open domain of the phase space. It means that the frequencies are functionally independent.

Definition 1.7. *The frequencies $\omega = (\omega_1, \ldots, \omega_n)$ are called resonant if they are rationally independent, i.e.*

$$k \cdot \omega \neq 0 \ \text{ for all } \ k \in \mathbb{Z}^n \setminus \{0\},$$

and are non-resonant otherwise.

In the non-resonant case, each orbit is dense on the n-torus and in the resonant case, the torus decomposes into an m-parameter family of invariant $(n - m)$-tori and given an orbit it is dense on a lower-dimensional torus.

Kolmogorov (see for instance the appendix of [1]), Arnold [3] and Moser [66] proved the persistence of those tori, whose frequencies verify the *Diophantine condition*, that is,

$$|k \cdot \omega| \geq \frac{\alpha}{|k|^{\tau}} \quad \text{for all} \quad k \in \mathbb{Z}^n \setminus \{0\} \text{ and } \alpha, \tau > 0.$$

If we ask about the existence of these Diophantine frequencies, this is answered with:

Lemma 1.11. *(Arnold) Let $\Omega \in \mathbb{R}^n$ be a bounded domain and let $\tau > n-1$ be fixed. Almost all vectors $\omega \in \Omega$ satisfy the Diophantine condition.*

The classical KAM theorem states this fact in the following way:

Theorem 1.12. *(Kolmogorov, Arnold and Moser) Consider the system of equations induced by an analytic Hamiltonian $\mathcal{H}_0$ to be non-degenerate, then most of the invariant tori which exist for the unperturbed system ($\varepsilon = 0$) will, slightly deformed, also exist for $\varepsilon \neq 0$ sufficiently small. Moreover, the Lebesgue measure of the complement of the set of tori tends to zero as ε tends to zero.*

There is a variation of the KAM theorem for isoenergetically non-degenerate systems.

Definition 1.8. *An n-dimensional system is isoenergetically non-degenerate if*

$$\begin{vmatrix} \dfrac{\partial^2 \mathcal{H}_0}{\partial I^2} & \dfrac{\partial \mathcal{H}_0}{\partial I} \\ \dfrac{\partial \mathcal{H}_0}{\partial I} & 0 \end{vmatrix} \neq 0.$$

Theorem 1.13. *(Kolmogorov) If $\mathcal{H}_0$ is non-degenerate or isoenergetically non-degenerate, then under a sufficiently small Hamiltonian perturbation most of the non-resonant invariant tori do not disappear but are only slightly deformed, so that in phase space of the perturbed system there also exist invariant tori. In the case of isoenergetic non-degeneracy the invariant tori form a majority on each energy-level manifold.*

There are systems where $\mathcal{H}_0$ does not depend on all the actions, they are the so called properly degenerate or superintegrable framework. One of these systems is the N-body problem. Now a question arise: How can we study the degenerate system?

The perturbation is said to *remove the degeneracy* if the full Hamiltonian can be written as

$$\mathcal{H}(I, \varphi, \varepsilon) \;=\; \mathcal{H}_{00}(I) + \varepsilon \mathcal{H}_{01}(I) + \varepsilon^2 \mathcal{H}_{11}(I, \varphi, \varepsilon), \tag{1.6}$$

where $\mathcal{H}_{00}$ depends only on the first r action variables and is either non-degenerate or isoenergetically non-degenerate with respect to these variables and $\mathcal{H}_{01}$ is non-degenerate with respect to the last $n-r$.

Theorem 1.14. *(Arnold) Suppose that the unperturbed system is degenerate, but the perturbation removes the degeneracy. Then a larger part of the phase space is filled with invariant tori that are close to the invariant tori $I = const$ of the intermediate system. Among these frequencies, r correspond to the fast phases, and $n-r$ to the slow phases. If the unperturbed Hamiltonian is isoenergetically non-degenerate with respect to those r variables on which it depends, then the invariant tori just described form a majority on each energy-level manifold of the perturbed system.*

There are many other results on KAM theory such as Moser's invariant curve Theorem, Arnold's stability Theorem for two degrees-of-freedom Hamiltonians and others, as well as many related results, see for instance [4].

In our case of the spatial three-body problem the perturbation at first order in ε does not remove the degeneracy because the degrees of freedom are added to the dynamics of the system order by order. The unperturbed Hamiltonian depends on two actions and the dependence on the remaining actions appear in the following orders but not all of them in the first one, as we shall see in Chapters 5 and 6.

There are some results on the existence of KAM tori for the spatial N- body problem. Nevertheless they cannot be applied on our context. Thus, we apply Han, Li and Yi's Theorem, designed specifically to deal with highly degenerated Hamiltonians, which turns to be essential to obtain the results in Chapters 5 and 6. This theorem is introduced as follows:

Han, Li and Yi consider in a bounded closed region $Z \times \mathbb{T}^n \times [0, \varepsilon^*] \subset \mathbb{R}^n \times \mathbb{T}^n \times [0, \varepsilon^*]$ for some fixed ε^* with $0 < \varepsilon^* < 1$, a real analytic Hamiltonian of the form

$$\mathcal{H}(I, \varphi, \varepsilon) \;=\; h_0(I^{n_0}) + \varepsilon^{\beta_1} h_1(I^{n_1}) + \ldots + \varepsilon^{\beta_a} h_a(I^{n_a}) + \varepsilon^{\beta_a+1} p(I, \varphi, \varepsilon), \qquad (1.7)$$

where $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables with the standard symplectic structure $dI \wedge d\varphi$, and $\varepsilon > 0$ is a sufficiently small parameter. The parameters a, n_i $(i = 0, 1, \ldots, a)$ and β_j $(j = 1, 2, \ldots, a)$ are positive integers satisfying $n_0 \leq n_1 \leq \ldots \leq n_a = n$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_a = \beta$, $I^{n_i} = (I_1, \ldots, I_{n_i})$, for $i = 1, 2, \ldots, a$, and p depends on ε smoothly.

For each ε the integrable part of $\mathcal{H}$:

$$X_\varepsilon(I) \;=\; h_0(I^{n_0}) + \varepsilon^{\beta_1} h_1(I^{n_1}) + \ldots + \varepsilon^{\beta_a} h_a(I^{n_a})$$

admits a family of invariant n-tori $T^\varepsilon_\zeta = \{\zeta\} \times \mathbb{T}^n$ with linear flows $\{x_0 + \omega^\varepsilon(\zeta)t\}$, where for each $\zeta \in Z$, $\omega^\varepsilon(\zeta) = \nabla X_\varepsilon(\zeta)$ is the frequency vector of the n-torus T^ε_ζ

and ∇ is the gradient operator. When $\omega^{\varepsilon}(\zeta)$ is non-resonant, the flow on the n-torus T^{ε}_{ζ} becomes quasi-periodic with slow and fast frequencies of different scales. We refer the integrable part X_{ε} and its associated tori $\{T^{\varepsilon}_{\zeta}\}$ as the intermediate Hamiltonian and intermediate tori, respectively.

Let $\bar{I}^{n_i} = (I_{n_{i-1}+1}, \ldots, I_{n_i})$, $i = 0, 1, \ldots, a$ (where $n_{-1} = 0$, hence $\bar{I}^{n_0} = I^{n_0}$), and define

$$\Omega = \Big(\nabla_{\bar{I}^{n_0}} h_0(I^{n_0}), \ldots, \nabla_{\bar{I}^{n_a}} h_{n_a}(I^{n_a})\Big) \tag{1.8}$$

such that for each $i = 0, 1, \ldots, a$, $\nabla_{\bar{I}^{n_i}}$ denotes the gradient with respect to $\bar{I}^{n_i}$.

We assume the following high-order degeneracy-removing condition (A): there is a positive integer s such that

$$\mathrm{Rank}\left\{\partial^{\alpha}_{I}\,\Omega(I)\ :\ 0 \leq |\alpha| \leq s\right\} = n \ \ \forall\, I \in Z. \tag{1.9}$$

(A) is the weakest existing condition. This condition is of Bruno-Rüssmann type so named by Han, Li, and Yi [36], giving credit to Bruno and Rüssmann, who provided weak conditions on the frequencies guaranteeing the persistence of the invariant tori, see [7, 76, 77]. KAM type of theorems using Bruno-Rüssmann non-degenerate condition were shown in [81]. Other related references about KAM type of results under Bruno-Rüssmann non-degenerate conditions are [52, 82], see also the survey by Hanßmann [39]. In this context, one of the valuable issues of Han, Li and Yi's Theorem is to provide the weakest condition that the frequencies have to satisfy for high order degenerate systems. The following theorem gives the right setting where the persistence of KAM tori for Hamiltonians like (1.7) can be ensured.

Theorem 1.15. *(Han, Li and Yi, 2010). Assume condition (A) and let δ with $0 < \delta < 1/5$ be given. Then there exists an $\varepsilon^* > 0$ and a family of Cantor sets $Z_{\varepsilon} \subset Z$, $0 < \varepsilon \leq \varepsilon^*$, such that each $\zeta \in Z_{\varepsilon}$ corresponds to a real analytic, invariant, quasi-periodic n-torus $\bar{T}^{\varepsilon}_{\zeta}$ of Hamiltonian (1.7), which is slightly deformed from the intermediate n-torus T^{ε}_{ζ}. The measure of $Z \setminus Z_{\varepsilon}$ is $\mathcal{O}(\varepsilon^{\delta/s})$ and the family $\{\bar{T}^{\varepsilon}_{\zeta} : \zeta \in Z_{\varepsilon},\ 0 < \varepsilon \leq \varepsilon^*\}$ varies Whitney smoothly.*

Chapter 2

Reductions in the spatial three-body problem

2.1 Hamiltonian of the problem

The N-body problem is the study of the motion of N point masses (with $N \geq 2$) interacting only through the mutual Newtonian gravitational attraction. For $N = 2$, the problem was solved by Newton but for $N \geq 3$ despite the efforts many researches and the progress since the times of Laplace, Lagrange and other outstanding mathematicians, there are still many unanswered questions. Our main aim is to study the dynamics of the system for $N = 3$, which is a Hamiltonian system [62].

In Hamiltonian systems, the equations of motion can be described by a Hamiltonian function and the total energy is a constant of motion, particularly a first integral. This integral, which is not the only one for the N-body problem, can be also expressed in terms of the Hamiltonian function. If a first integral is constant along the Hamiltonian, then it said that this integral is in involution.

We consider three point masses moving in a Newtonian reference system, $\mathbb{R}^3$, with the only force acting on them being their mutual gravitational attraction. Let the i-th particle have position vector $\mathbf{q}_i$ and mass $m_i > 0$. Let $\mathbf{Q}_i$ denote the conjugate momentum to $\mathbf{q}_i$, thus, $\mathbf{Q}_i = m_i \dot{\mathbf{q}}_i$ and let $\mathbf{q}_{ij}$ be the distance between m_i and m_j. Let $\mathcal{G}$ denote the universal gravitational constant. The Hamiltonian of the three-body problem accounting for the mutual Newtonian interaction of the three particles (i.e. the three bodies) in the three-dimensional space is:

$$\mathcal{H} = \frac{1}{2} \sum_{i=0}^{2} \frac{\mathbf{Q}_i^2}{m_i} - \frac{\mathcal{G} m_0 m_1}{\mathbf{q}_{01}} - \frac{\mathcal{G} m_0 m_2}{\mathbf{q}_{02}} - \frac{\mathcal{G} m_1 m_2}{\mathbf{q}_{12}}. \tag{2.1}$$

Note that there are eighteen variables, nine of them are the coordinates and the remaining nine are their associate momenta, thus (2.1) represents a problem of nine degrees of freedom. The corresponding phase space is an open set of the cotangent bundle $T^*\mathbb{R}^9$ where all possible collisions among the bodies are ruled out. The system is symmetric under translations and rotations.

We will use the integrals of the three-body problem combined with averaging in order to perform our study. Indeed our aim is to obtain the simplest reduced Hamiltonian in the simplest reduced space after applying normalisation (i.e. averaging) and reducing out all the possible exact and approximate continuous symmetries. The main results are summarised in Theorem 2.1.

As it is well known the N-body problem has ten independent integrals. This allows one the reduction of the Hamiltonian function from dimension $6N$ to dimension $6N-10$, i.e. the passage from a Hamiltonian with $3N$ degrees of freedom to a reduced Hamiltonian with $3N-5$ degrees of freedom. By virtue of the reduction of the translational symmetry, the centre of mass is placed at the origin of the frame and the linear momentum is fixed. This reduces the problem to a linear subspace of dimension $6N-6$. Then one can reduce the rotational symmetry in two steps: (i) Fixing the angular momentum which reduces the problem to a $(6N-9)$-dimensional space. (ii) Identifying configurations that differ by a rotation about the angular momentum vector which reduces the problem to the reduced space of dimension $6N-10$. This last operation is classically called the elimination of the nodes; see more details in [1, 62]. The general results about the symplectic nature of the reduction and the reduced space appear in Meyer [59] and in Marsden and Weinstein [57]. Hence, for $N=3$ the spatial three-body problem can be studied as a Hamiltonian system with four degrees of freedom [58] after reducing by the symmetries mentioned above.

We introduce Jacobi coordinates. As the centre of mass moves uniformly with time, then:

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{q}_0, & \mathbf{x}_1 &= \mathbf{q}_1-\mathbf{q}_0, & \mathbf{x}_2 &= \mathbf{q}_2-\delta_0\mathbf{q}_0-\delta_1\mathbf{q}_1, \\ \mathbf{y}_0 &= \mathbf{Q}_0+\mathbf{Q}_1+\mathbf{Q}_2, & \mathbf{y}_1 &= \mathbf{Q}_1+\delta_1\mathbf{Q}_2, & \mathbf{y}_2 &= \mathbf{Q}_2, \end{aligned} \tag{2.2}$$

where $1/\delta_0 = 1+m_1/m_0$ and $1/\delta_1 = 1+m_0/m_1$.

We apply the linear change (2.2) to the Hamiltonian (2.1) giving the same name to the transformed Hamiltonian. It defines a system of six degrees of freedom. We also change the time unit by setting $\mathcal{G}=1$.

The reference frame is attached to the centre of mass by making $\mathbf{y}_0=\mathbf{0}$, then if $\mathbf{x}_2\neq\mathbf{0}$ we can split $\mathcal{H}$ into two Hamiltonians:

$$\mathcal{H} = \mathcal{H}_{\text{Kep}}+\mathcal{H}_{\text{pert}} \tag{2.3}$$

with

$$\begin{aligned}\mathcal{H}_{\text{Kep}} &= \frac{|\mathbf{y}_1|^2}{2\mu_1}+\frac{|\mathbf{y}_2|^2}{2\mu_2}-\frac{\mu_1 M_1}{|\mathbf{x}_1|}-\frac{\mu_2 M_2}{|\mathbf{x}_2|},\\ \mathcal{H}_{\text{pert}} &= -\frac{m_0m_1-\mu_1M_1}{|\mathbf{x}_1|}+\frac{\mu_2M_2}{|\mathbf{x}_2|}-\frac{m_1m_2}{|\mathbf{x}_2-\delta_0\mathbf{x}_1|}-\frac{m_0m_2}{|\mathbf{x}_2+\delta_1\mathbf{x}_1|},\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\frac{1}{\mu_1} &= \frac{1}{m_0}+\frac{1}{m_1}, \qquad & \frac{1}{\mu_2} &= \frac{1}{m_0+m_1}+\frac{1}{m_2},\\ M_1 &= m_0+m_1, & M_2 &= m_0+m_1+m_2.\end{aligned}$$

This splitting is valid in the domain of bounded motions, i.e. a certain region of phase space that we will define later. Function $\mathcal{H}_{\text{Kep}}$ is the so called Keplerian Hamiltonian, and we will focus on *bounded motions and small perturbations.* Then, $\mathcal{H}_{\text{Kep}}$ is the Hamiltonian of two fictitious bodies of masses μ_1 and μ_2 which revolve along ellipses around a fixed centre of attraction without mutual interaction and it is a completely integrable system. We outline that under the action of $\mathcal{H}_{\text{Kep}}$, the three (real) bodies move on Keplerian ellipses whose foci are the moving centre of mass of m_0 and m_1. The ellipses corresponding to the masses m_0 and m_1 are described by $\delta_1\mathbf{x}_1$ and $-\delta_0\mathbf{x}_1$. They are coplanar, they have the same eccentricity and their pericentres are in opposition. The Hamiltonian $\mathcal{H}_{\text{pert}}$ is called the perturbing function. It is real analytic outside collisions of the bodies and outside collisions of the fictitious body of mass μ_2 with the origin of the frame. This is not a problem as we will suppose along this and next chapters that the ellipse described by μ_2 is the outer ellipse. From now on the subindex 1 accounts for the inner bodies while 2 refers to the outer body. This issue is represented in Fig. 2.1.

Hamiltonian $\mathcal{H}_{\text{pert}}$ may be expanded in terms of the Legendre polynomials if $|\mathbf{x}_1|/|\mathbf{x}_2|<1$. More specifically if we denote by $\hat{\delta}=\max(\delta_0,\delta_1)$ and by ξ the oriented angle $(\widehat{\mathbf{x}_1,\mathbf{x}_2})$, the perturbed Hamiltonian reads as:

$$\mathcal{H}_{\text{pert}} = -\frac{\mu_1 m_2}{|\mathbf{x}_2|}\sum_{n\geq 2}\delta_n P_n(\cos\xi)\left(\frac{|\mathbf{x}_1|}{|\mathbf{x}_2|}\right)^n, \text{ with } \delta_n=\delta_0^{n-1}+(-1)^n\delta_1^{n-1}, \tag{2.5}$$

where $\cos\xi=(\mathbf{x}_1\cdot\mathbf{x}_2)/(|\mathbf{x}_1||\mathbf{x}_2|)$ and P_n is the n-th Legendre polynomial. According to [31] this expansion is convergent in the complex disk

$$\frac{|\mathbf{x}_1|}{|\mathbf{x}_2|}<\frac{1}{\hat{\delta}}\in[1,2].$$

We deal with the relative size of $\mathcal{H}_{\text{pert}}$ with respect to $\mathcal{H}_{\text{Kep}}$ so that this product can be considered small enough in order to apply averaging and reduction techniques in the subsequent sections. We follow the nice discussion proposed by Féjoz

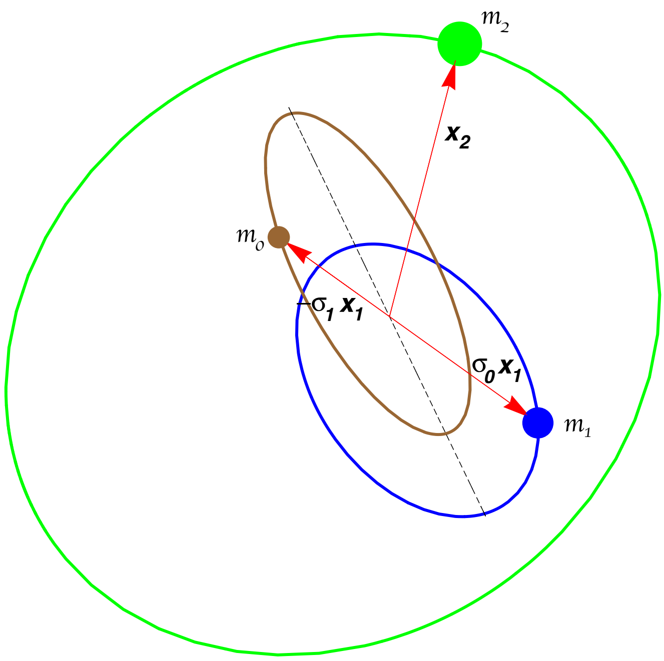

Figure 2.1: Inner and outer ellipses.

[31] for the planar three-body problem that also applies for the spatial case. Let α_1, α_2 be the semimajor axes of the inner and outer fictitious ellipses, respectively, e_1 and e_2 the corresponding eccentricities and let $\eta_k = \sqrt{1-e_k^2}$. We define

$$\Delta = \hat{\delta}\frac{\alpha_1(1+e_1)}{\alpha_2(1-e_2)}, \tag{2.6}$$

which is a measure of how close the outer ellipse is from the inner ellipses when they lie in the same plane. We will assume that $\Delta < 1$, thus the outer ellipse cannot meet the inner ones, and in particular, if the semi-major axes α_1 and α_2 are given, the eccentricity e_2 of the outer ellipse cannot be arbitrarily close to 1.

For $0 < \varepsilon \ll 1$ and $n \in \mathbb{Z}^+$, the perturbing region $\mathcal{P}_{\varepsilon,n}$ is defined as the part of the space $T^*\mathbb{R}^6$ where

$$\mathcal{P}_{\varepsilon,n} = \max\left\{\frac{m_2}{M_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{3/2}, \frac{\mu_1\sqrt{M_2}}{M_1^{3/2}}\left(\frac{\alpha_1}{\alpha_2}\right)^2\right\}\frac{1}{\eta_2^{3(n+2)}(1-\Delta)^{2n+1}} < \varepsilon.$$

Féjoz proved that inside $\mathcal{P}_{\varepsilon,n}$ the perturbation $\mathcal{H}_{\text{pert}}$ and its averaged Hamiltonian with respect to ℓ_1 and ℓ_2 are ε-small in a certain $\mathcal{C}^k$-norm, see [31].

Thus, five different possibilities arise if $0 \leq e_2 < 1$ holds, in all of them $\mathcal{H}_{\rm pert}$ is small compared to $\mathcal{H}_{\rm Kep}$.

(i) The *planetary region*: the eccentricity of the outer ellipse and both semimajor axes are small and two masses out of three, including the outer mass, are ε-small compared to the third mass.

(ii) The *lunar region*: the masses are in a compact set, and the outer body is $1/\varepsilon$-far away from the other two.

(iii) The *anti-planetary region*: to which extent the outer mass may be large provided that the outer ellipse is far from the other two.

(iv) The *anti-lunar region*: to which extent the outer ellipse may be close to the other two provided that one of the two inner bodies has a large mass.

(v) The *asynchronous region*: it is an open subregion of $\mathcal{P}_{\varepsilon,n}$. If $\omega_j = \sqrt{M_j}/\alpha_j^{3/2}$ is the Keplerian frequency of the j-th body, we require the condition $\frac{\omega_2}{\omega_1} < \varepsilon$. This region extends the lunar region.

This provides the most general setting where $\mathcal{H}_{\rm pert}$ can be considered as a small perturbation of $\mathcal{H}_{\rm Kep}$. More details appear in [31]. Alternatively one can define different classes of the N-body problem applying the symplectic scaling techniques by Meyer, see [60] and the different classes of restricted and non-restricted N-body problems [61]. From now on we set

$$\mathcal{H} = \mathcal{H}_{\rm Kep} + \varepsilon\, \mathcal{H}_{\rm pert} \tag{2.7}$$

assuming that $\varepsilon\, \mathcal{H}_{\rm pert}$ is small compared to $\mathcal{H}_{\rm Kep}$, regardless of the nature of ε.

As the total angular momentum vector $\sum_{k=1}^{2} \mathbf{G}_k = \sum_{k=1}^{2} \mathbf{x}_k \times \mathbf{y}_k = \mathbf{C} \neq \mathbf{0}$ is an integral, the plane perpendicular to $\mathbf{C}$ through the centre of mass is invariable. This is the so called *invariable plane* also called the *Laplace plane*, and thus, we can eliminate the nodes. We recall that although the three components of $\mathbf{C}$ are independent integrals they are not in involution. However, we can choose the magnitude of $\mathbf{C}$, that is, $C = |\mathbf{C}|$, and its third component $\mathbf{C} \cdot \mathbf{k}$ (where $\mathbf{k}$ stands for the vertical unit vector of an inertial frame centered at the centre of mass of the system) as they are commuting integrals. Thus, we can reduce the Hamiltonian defined by $\mathcal{H}$ out of the symmetry generated by the two integrals. This is the Jacobi elimination or reduction of the nodes, although strictly speaking the first full reduction of the three-body problem was carried out by Lagrange [48].

The classical approach to achieve this reduction explicitly is by introducing Delaunay coordinates $(\ell_k, g_k, h_k, L_k, G_k, H_k)$, $k = 1, 2$ and applying the Jacobi reduction of the nodes [45]. Here, for the ellipse k, ℓ_k designates the mean anomaly,

g_k the argument of the pericentre, h_k the argument of the node, $L_k = \mu_k\sqrt{M_k\alpha_k}$, G_k is the modulus of the angular momentum vector $\mathbf{G}_k$ and H_k is its third component. The Hamiltonian $\mathcal{H}$ in these coordinates depends on h_k only through the combination $h_1 - h_2$ as a consequence of the symmetry of the system with respect to rotations about the vector $\mathbf{C}$. The conservation of the components of $\mathbf{C}$ requires that:

$$\begin{aligned} h_1 - h_2 &= \pi, \\ G_1^2 - H_1^2 &= G_2^2 - H_2^2, \\ H_1 + H_2 &= \mathbf{C}\cdot\mathbf{k}. \end{aligned} \tag{2.8}$$

Nevertheless, this transformation as it is used in [6], is only obtained through the restriction to the vertical angular momentum manifold defined by the relations (2.8). This manifold has dimension ten and is a submanifold of the manifold $\mathbb{R}^{12}$, i.e. the twelve-dimensional phase space where $\mathcal{H}$ defined in (2.3) lives, see [11]. This drawback can be overcome by placing the invariable plane in the horizontal plane, as is done in [46] or [53]. Instead of Delaunay elements we have preferred to use an adaptation to the three-body problem of Deprit's coordinates [26] devised for eliminating two nodal angles in the N-body problem. By doing so we avoid the drawback inherent to Delaunay coordinates, distinguishing between the horizontal plane from the invariable one.

2.2 Elimination of the nodes and normalisation

2.2.1 Deprit's coordinates

As commented above, the elimination of the nodes is performed properly by using Deprit's elements [26]. We follow the presentation of these variables given by Ferrer and Osácar [34] for the three-body problem. In particular, half of Deprit's variables, namely ℓ_k, L_k, G_k, $k = 1, 2$, coincide with the spatial Delaunay variables. However, as in [34], instead of g_k, h_k and H_k we introduce four new angles and two new actions in the following way.

We choose an inertial frame $\mathcal{F} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$. Assuming $\mathbf{C} \neq \mathbf{0}$ there is a unique polar decomposition $\mathbf{C} = C\mathbf{n}$ with $C > 0$ and $|\mathbf{n}| = 1$. We introduce an angle I such that $\mathbf{k}\cdot\mathbf{n} = \cos I$ with $0 \leq I \leq \pi$. When $I \in (0, \pi)$ there exists a unit vector $\mathbf{l}$ with $\mathbf{k}\times\mathbf{n} = \mathbf{l}\sin I$ and $|\mathbf{l}| = 1$. We define a reference frame $\mathcal{I} = (\mathbf{n}, \mathbf{l}, \mathbf{m})$ with $\mathbf{m} = \mathbf{n}\times\mathbf{l}$. This frame is called the invariable frame. The longitude of $\mathbf{l}$ is an angle ν such that $\mathbf{l} = \mathbf{i}\cos\nu + \mathbf{j}\sin\nu$ with $0 \leq \nu \leq 2\pi$.

Now we suppose that $\mathbf{G}_k \neq \mathbf{0}$ for $k = 1, 2$. There exists a unique polar decomposition $\mathbf{G}_k = G_k\mathbf{n}_k$ with $|\mathbf{n}_k| = 1$. We define the angle I_k such that $\mathbf{n}\cdot\mathbf{n}_k = \cos I_k$ with $0 \leq I_k \leq \pi$. If $I_k \in (0, \pi)$ there exists a unique direction $\mathbf{l}_k$

with $\mathbf{n} \times \mathbf{n}_k = \mathbf{l}_k \sin I_k$ with $|\mathbf{l}_k| = 1$. Physically I_k is the angle between vectors $\mathbf{C}$ and $\mathbf{G}_k$, that is, the inclinations of the ellipses 1 and 2 with respect to the invariable plane.

The longitude $\mathbf{l}_k$ in the invariable plane $(\mathbf{l}, \mathbf{m})$ is defined by $\mathbf{l}_k = \mathbf{l} \cos \nu_k + \mathbf{m} \sin \nu_k$ with $0 \leq \nu_k \leq 2\pi$. The nodal frame $\mathcal{N}_k$ is defined through the three orthonormal directions $(\mathbf{n}_k, \mathbf{l}_k, \mathbf{m}_k)$, where $\mathbf{m}_k = \mathbf{n}_k \times \mathbf{l}_k$.

The computation of the products $\mathbf{C} \times \mathbf{n}$ and $\mathbf{C} \cdot \mathbf{n}$ yields that $\mathbf{l}_2 = -\mathbf{l}_1$, $\nu_1 = \nu_2 + \pi$ and that

$$C = G_1 \cos I_1 + G_2 \cos I_2, \quad G_1 \sin I_1 - G_2 \sin I_2 = 0, \tag{2.9}$$

see details in [34]. These identities relate the inclinations of the outer and inner ellipses with respect to the invariable plane and are valid for I_k in $[0, \pi]$.

We introduce the momentum B as the projection $B = \mathbf{C} \cdot \mathbf{k}$. We also decompose $\mathbf{x}_k$, $\mathbf{y}_k$ into Cartesian coordinates on the plane spanned by $\mathbf{l}_k$ and $\mathbf{m}_k$ as

$$\mathbf{x}_k = x_{k1} \mathbf{l}_k + x_{k2} \mathbf{m}_k, \quad \mathbf{y}_k = y_{k1} \mathbf{l}_k + y_{k2} \mathbf{m}_k.$$

According to [34] the transformation

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) \longrightarrow (x_{11}, x_{12}, x_{21}, x_{22}, \nu_1, \nu, y_{11}, y_{12}, y_{21}, y_{22}, C, B) \tag{2.10}$$

is symplectic. Besides, by construction C and B are the conjugate actions to the angles ν_1 and ν, respectively. Notice that $|B| \leq C$ and that B is related with the spatial Delaunay elements through $B = H_1 + H_2$. The set of variables (x_{11}, x_{12}, x_{21}, x_{22}, ν_1, ν, y_{11}, y_{12}, y_{21}, y_{22}, C, B) is called the Cartesian-nodal set of coordinates.

At this point we introduce polar-symplectic coordinates $(r_1, r_2, \vartheta_1, \vartheta_2, R_1, R_2, \Theta_1, \Theta_2)$ in the following way:

$$\begin{aligned} x_{k1} &= r_k \cos \vartheta_k, & x_{k2} &= r_k \sin \vartheta_k, \\ y_{k1} &= R_k \cos \vartheta_k - \frac{\Theta_k}{r_k} \sin \vartheta_k, & y_{k2} &= R_k \sin \vartheta_k + \frac{\Theta_k}{r_k} \cos \vartheta_k, \end{aligned} \tag{2.11}$$

for $k = 1, 2$. Then, we introduce the (usual) planar Delaunay transformation $(r_k, \vartheta_k, R_k, \Theta_k) \to (\ell_k, \gamma_k, L_k, G_k)$, $k = 1, 2$, see for example [25]. Note that although in the polar-symplectic and in the planar Delaunay coordinates $\Theta_k \equiv G_k$ may be negative, in our approach $G_k \geq 0$ by construction. In particular γ_k corresponds to the argument of the pericentre of the ellipse k in the plane defined by $\mathbf{l}_k$ and $\mathbf{m}_k$, while ℓ_k, L_k and G_k are the same as the spatial Delaunay coordinates, see more details in [11].

By composing the previous changes we construct the following symplectic transformation:

$$\varphi : (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) \longrightarrow (\ell_1, \gamma_1, \nu_1, \ell_2, \gamma_2, \nu, L_1, G_1, C, L_2, G_2, B). \tag{2.12}$$

The set of action-angle coordinates $(\ell_1, \gamma_1, \nu_1, \ell_2, \gamma_2, \nu, L_1, G_1, C, L_2, G_2, B)$ are the so called Deprit's elements which were also used by Chierchia and Pinzari in [11, 12, 13] but they use $\gamma_2 - \pi$ instead of γ_2. They are defined on an open subset of $\mathbb{R}^{12}$. We shall be more explicit in the next sections about the constraints among the actions of these variables. See an illustration of these coordinates in Fig. 2.2.

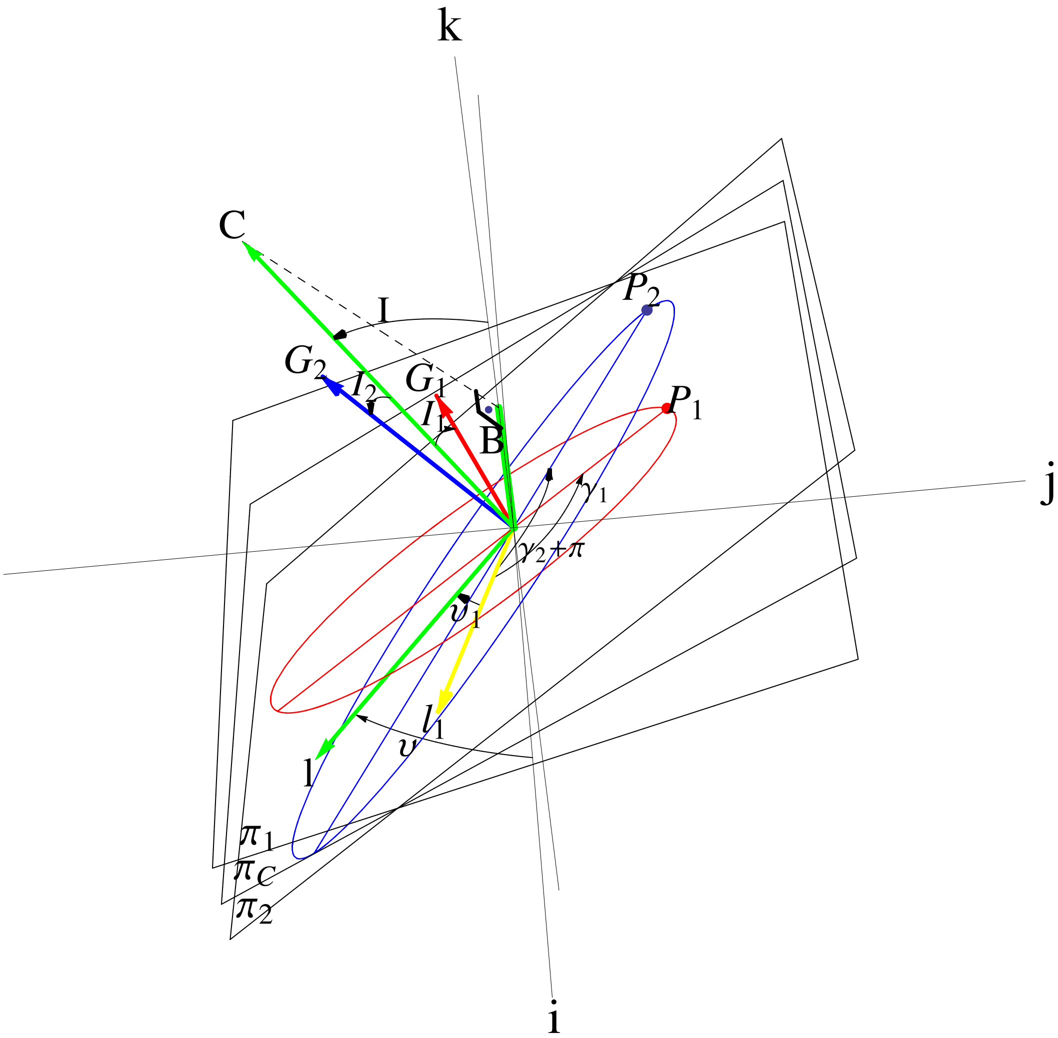

Figure 2.2: Deprit's action-angle variables. π_C is the invariable plane; π_k, with $k = 1, 2$, is the plane determined by the ellipse k, and P_k is its pericentre.

The crucial feature is that the expression of the Hamiltonian $\mathcal{H}$ defined in (2.3) using the set of variables (2.12) leads to a Hamiltonian function which is free of the angles ν and ν_1 and the action B, thence the coordinates B, C and ν are integrals of motion. In particular the nodes ν and ν_1 are eliminated from the Hamiltonian and from the equations of motion, thus the Jacobi elimination of the nodes is performed properly. We remark that for the N-body problem the Jacobi elimination of the nodes can be made in a symplectic context and in the whole phase space only using Deprit's collection of action-angle coordinates, eliminating two angles explicitly [11, 13]. Even in the case $N = 3$ this is the only valid way of executing the Jacobi reduction of the nodes in a right way, and is not attributable to the classical papers by Jacobi [45] or Radau [74].

In particular $\mathcal{H}_{\rm Kep}$ introduced in (2.4) gets transformed into

$$\mathcal{H}_{\rm Kep} = -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}. \tag{2.13}$$

For $\mathcal{H}_{\rm pert}$ we take into account that $|\mathbf{x}_k| = r_k$ and compute

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = -x_{11}x_{21} - x_{12}x_{22}\cos(I_1 + I_2).$$

From (2.9) it is inferred that

$$\cos(I_1 + I_2) = \frac{C^2 - G_1^2 - G_2^2}{2G_1G_2}. \tag{2.14}$$

Hence, the term $P_n(\cos\xi)$ depends on r_k, ϑ_k, Θ_k, C for $n \geq 2$ and $k = 1, 2$, which readily implies that $\mathcal{H}_{\rm pert}$ in (2.5) is independent of ν_1, ν and B. Thus, $\mathcal{H}$ defines a Hamiltonian system of four degrees of freedom on the open subset of $\mathbb{R}^8$ outside collisions. Specifically, in terms of Deprit's variables, $\mathcal{H}$ depends on the four angles ℓ_1, ℓ_2, γ_1, γ_2 and their conjugate momenta L_1, L_2, G_1, G_2. It also depends on the integral C and on the three masses m_i, $i = 0, 1, 2$. We remark that it is also possible to apply the symplectic change φ to the perturbation $\mathcal{H}_{\rm pert}$ of (2.4) and perform the Legendre expansion later. Both approaches lead to the same result.

We also have to take into account some relations involving G_1, G_2 and C. Using relation (2.14) and the fact that $G_1 \geq 0$ and $C, G_2 > 0$ we arrive at:

$$|C - G_2| \leq G_1 \leq C + G_2, \quad |C - G_1| \leq G_2 \leq C + G_1, \tag{2.15}$$

which appears in [34] as Lemma 1. Thus, G_1 is lower-bounded by $|C - G_2|$ and upper-bounded by $\min\{L_1, C + G_2\}$. From (2.15) the case $G_1 = 0$ implies $C = G_2$. Furthermore $G_1 = 0$ implies $\mathbf{C} = \mathbf{G}_2$, as $\mathbf{G_1} = \mathbf{0}$.

2.2.2 Averaging the fast angles

Now the Hamiltonian $\mathcal{H}$ is ready so that $\mathcal{H}_{\rm pert}$ can be normalised over the two mean anomalies. The averaging procedure is made using a Lie transformation [24], introduced in Chapter 1, thus the averaging is performed by constructing a change of variables through a generating function. We exclude possible resonances between ℓ_1 and ℓ_2, that is, we restrict ourselves to a certain subset of the perturbing region where the ratio ω_2/ω_1 is not too close to a rational number or, in other words, the frequencies' vector (ω_1, ω_2) is Diophantine. In the asynchronous subregion of $\mathcal{P}_{\varepsilon,n}$ the normalisation can be carried out to any order as no resonance can occur between ℓ_1 and ℓ_2. The reason is that the two terms of the unperturbed Hamiltonian (2.13) can be arranged at different orders and the average process can be done in two steps, eliminating firstly one of the mean anomalies and then the other one, avoiding therefore the appearance of small denominators, see for instance [90, 91]. In the planetary subregion these resonances are overcome if the semimajor axes are well-spaced, i.e. α_1 and α_2 are functions satisfying the following conditions: there are constants $\bar{\alpha}_1$, $\bar{\alpha}_{12}$ and $\bar{\alpha}_2$ such that $0 < \bar{\alpha}_1 < \alpha_1 < \bar{\alpha}_{12} < \alpha_2 < \bar{\alpha}_2$ for all time. See more details for the N-body problem in [13]. (Note that this well-spaced assumption is compatible with the Legendre expansion of $\mathcal{H}_{\rm pert}$.)

Thus, from now on we assume that Hamiltonian $\mathcal{H}$ belongs to the open subregion of $\mathcal{P}_{\varepsilon,n}$ where no resonances between ℓ_1 and ℓ_2 occur, adding the well-spaced condition of the semimajor axes in the planetary regime. The reader can also check the hypotheses of the Averaging Theorem in Proposition 2.1 of [6], where similar conditions are given in order to avoid this type of resonances in the three-body problem. We also assume that $L_1 < L_2$ which is compatible with the condition on the semimajor axes established before. This allows us to distinguish between the inner and outer ellipses corresponding respectively to the motions of the inner and outer bodies. A related prerequisite that we also require and that is compatible is that in the forthcoming Legendre expansions of the perturbation the quadrupolar terms would be bigger than the rest of the expansion. This is enough to ensure that the terms of the Hamiltonian $\mathcal{H}$ truncated after making one step of the Lie transformation are of order $\mathcal{O}(\varepsilon^2)$ and that we retain only the quadrupolar terms of the perturbation. Finally we fix a maximum value for e_2, i.e. $0 \leq e_2 \leq e_2^{\rm max} < 1$, equivalently $L_2 \geq G_2 \geq G_2^{\rm min} > 0$ to avoid that the outer body can collide with the inner ones. This subregion is denoted by $\mathcal{Q}_{\varepsilon,n}$.

Therefore, we can average the perturbation over the two anomalies to get:

$$\mathcal{K}_0 = \mathcal{H}_{\rm Kep}, \qquad \mathcal{K}_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathcal{H}_{\rm pert}\, d\,\ell_1\, d\,\ell_2,$$

and the generating function satisfies the partial differential equation:

$$\omega_1 \frac{\partial \mathcal{W}_1}{\partial \ell_1} + \omega_2 \frac{\partial \mathcal{W}_1}{\partial \ell_2} = \mathcal{H}_{\rm pert} - \mathcal{K}_1.$$

After truncating the Legendre expansion at the quadrupolar terms, i.e. including the Legendre polynomials up to degree two, we arrive at:

$$\mathcal{H}_{\rm pert} = \frac{\mu_1 m_2 r_1^2}{2 r_2^3}\Big(1 - 3\big(\cos\vartheta_1\cos\vartheta_2 + \cos(I_1+I_2)\sin\vartheta_1\sin\vartheta_2\big)^2\Big), \qquad (2.16)$$

where $\cos(I_1+I_2)$ is taken from (2.14).

In order to make the average with respect to ℓ_1 and ℓ_2 we use the explicit expressions of r_1 and ϑ_1 in terms of the eccentric anomaly of the inner ellipse and r_2 and ϑ_2 in terms of the true anomaly of the outer ellipse, see details in [25, 31, 68]. The formulae applied to put in (2.16) r_1 and ϑ_1 as functions of the eccentric anomaly E_1 and r_2 and ϑ_2 as functions of the true anomaly f_2 are the same regardless if we are in the planar or in the spatial context, thus we can use the standard formulae for handling the averaging over ℓ_1 and ℓ_2 in terms of Deprit's elements. Thus, we get:

$$\begin{aligned}\mathcal{K}_1 = \frac{\mathcal{M}L_1^2}{L_2^3G_1^2G_2^5}\Big(&\big(-3(C^2-G_1^2)^2 + 2(3C^2-G_1^2)G_2^2 - 3G_2^4\big)(5L_1^2-3G_1^2)\\ &+15\big((C-G_2)^2-G_1^2\big)\big((C+G_2)^2-G_1^2\big)(L_1^2-G_1^2)\cos 2\gamma_1\Big),\end{aligned} \qquad (2.17)$$

with

$$\mathcal{M} = \frac{\mu_2^7 M_2^4}{64\mu_1^3 M_1^3}.$$

It is remarkable that, as $G_1 = 0$ implies $C = G_2$, then $\mathcal{K}_1$ is simplified and the term G_1^2 cancels out with the numerator, concluding that the Hamiltonian (2.17) is well defined when the inner ellipses are straight lines; thus $\mathcal{K}_1$ extends analytically to $e_1 = 0$.

The Hamiltonian $\mathcal{K}_1$ coincides with the one calculated in [46, 53, 34, 28, 90, 91] but this should be expected according to [12, 13]. The explicit expression of $\mathcal{W}_1$ is too long to be written down and in general it is obtained using Fourier series in some angles related to ℓ_k but, as well as $\mathcal{K}_1$, it is a function expressed in closed form with respect to the eccentricities e_1 and e_2, making the approach as general as possible for motions in the elliptic domain.

A key feature of Hamiltonian $\mathcal{K}_1$ is that it is independent of the argument of the pericentre γ_2, as we have taken into account only up to the Legendre polynomial P_2. This fact will allow us to reduce the Hamiltonian function with respect to the symmetry generated by the integral G_2. Nevertheless, if the next terms in the ratio α_1/α_2 are taken into account, the resulting system is no longer independent of g_2, a fact that was pointed out in [40].

2.2.3 Regularisation

The singularity related to the Keplerian Hamiltonian of the fictitious body 1 can be removed using the standard regularisation technique of Moser or the one due to Kustaanheimo and Stiefel [47]. This is the so called *regularisation of the double inner collisions*, so that one can study the possible collisions between the particles with masses m_0 and m_1. Specifically Moser [67] showed that the n-dimensional Kepler problem can be regularised in the sense that there is a symplectomorphism that takes the Kepler flow for a fixed negative energy level to the geodesic flow onto the unit cotangent bundle of the punctured n-sphere which is punctured at the north pole. The geodesic flow of the unit sphere over the north pole corresponds to the collision orbits and by adding it back the collisions are incorporated as a regular flow. If $\mathcal{E}$ is the whole negative energy region of the Kepler problem corresponding to the ellipse 1, let $\hat{S}^3$ be the punctured 3-sphere and $T^+\hat{S}^3$ be the cotangent bundle of the punctured 3-sphere minus the zero section. Ligon and Schaaf [55] transform canonically the whole elliptic region $\mathcal{E}$ to the bundle $T^+\hat{S}^3$, with no need to make the process for each energy level and without changing the time. This transformation brings the Kepler problem (i.e. the term $-\mu_1^3 M_1^2/(2L_1^2)$) to a Hamiltonian, say D_1, written in Ligon and Schaaf's coordinates and called Delaunay Hamiltonian, on $T^+\hat{S}^3$. Hamiltonian D_1 extends naturally to T^+S^3 making effective the regularisation of the Kepler problem corresponding to the ellipse 1 for all negative energies. Heckman and de Laat [41] give a simpler approach to the issue showing that Ligon-Schaaf's regularisation map can be understood as an adaptation of the Moser's regularisation map, see a similar approach in [56].

We apply Ligon-Schaaf's regularisation to the fictitious inner orbit for the system (2.7), so the flow is extended to double inner collisions since $\mathcal{H}_{\text{pert}}$ is regular for $G_1 = 0$. The term of $\mathcal{H}_{\text{Kep}}$ corresponding to the ellipse 1 results in the Hamiltonian D_1. Since the time is not changed through the regularising transformation, by Darboux Theorem [1] we may introduce action-angle variables in a neighbourhood of the north pole of T^+S^3 such that one of the actions, say $\bar{L}_1$, is taken as $-\mu_1^3 M_1^2/(2\bar{L}_1^2) = D_1$ whereas its conjugate momentum, say $\bar{\ell}_1$, is essentially ℓ_1, and we normalise with respect to it. Thus, averaging with respect to $\bar{\ell}_1$ is equivalent to averaging with respect to ℓ_1 and our normalisation process performed in Deprit's coordinates extends to deal with double inner collisions.

Zhao [90, 92, 93] uses Kustaanheimo and Stiefel's transformation to regularise double inner collisions. This transformation changes the time and the new one is essentially the eccentric anomaly.

2.3 Reduction by stages

2.3.1 Keplerian reduction

We could have attempted to reduce first the symmetry introduced by eliminating the nodes — e.g. the so called Jacobi reduction of the nodes — and then reduce the T^2-symmetry related to the elimination of the mean anomalies. However, this is a more complicated approach, as the computation of the invariants related with the Keplerian reduction from the invariants associated to the Jacobi reduction of the nodes is highly nontrivial. We have preferred to begin by applying the Keplerian reduction first and then the rest of reductions, making the whole process in three stages.

Associated to the angular momentum vectors $\mathbf{G}_1$ and $\mathbf{G}_2$, the Laplace-Runge-Lenz vectors $\mathbf{A}_k$ are defined as $\mathbf{A}_k = (\mathbf{y}_k \times \mathbf{G}_k)/\mu_k - \mathbf{x}_k/r_k$ for $k = 1, 2$. We introduce the vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ and $\mathbf{d} = (d_1, d_2, d_3)$ through

$$\mathbf{a} = \mathbf{G}_1 + L_1\mathbf{A}_1, \quad \mathbf{b} = \mathbf{G}_1 - L_1\mathbf{A}_1, \quad \mathbf{c} = \mathbf{G}_2 + L_2\mathbf{A}_2, \quad \mathbf{d} = \mathbf{G}_2 - L_2\mathbf{A}_2. \tag{2.18}$$

Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ satisfy

$$\begin{gathered} a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2 = L_1^2, \quad c_1^2 + c_2^2 + c_3^2 = d_1^2 + d_2^2 + d_3^2 = L_2^2, \\ a_i, b_i \in [-L_1, L_1], \quad c_i, d_i \in [-L_2, L_2], \;\; i = 1, 2, 3. \end{gathered} \tag{2.19}$$

For fixed and strictly positive values of L_1 and L_2 the reduced phase space (i.e. the orbit space) related to the normalisation of ℓ_1 and ℓ_2 and the truncation of the corresponding tail is given by

$$\begin{aligned} \mathcal{A}_{L_1,L_2} &= S^2_{L_1} \times S^2_{L_1} \times S^2_{L_2} \times S^2_{L_2} \\ &= \Big\{ (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathbb{R}^{12} \;\;|\;\; a_i,\, b_i,\, c_i \text{ and } d_i \text{ with } i = 1, 2, 3, \text{ satisfy } (2.19) \Big\}. \end{aligned} \tag{2.20}$$

Thus, we reduce from $\mathbb{R}^{12}$ to the space $\mathcal{A}_{L_1,L_2}$, which is a symplectic manifold whose dimension is eight. This space is also obtained by Ferrer and Osácar in [34]. It is parametrised by the twelve invariants $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ subject to the four constraints given in the first line of (2.19). This conclusion is a straightforward generalisation of the Keplerian reduction for one Keplerian ellipse, see [67], as the possible resonances have been excluded in the analysis. The invariants a_i and b_i for the Keplerian reduction are due to Pauli [73] and used by Souriau [86] and Cushman [20]. This reduction lies in the context of Meyer's [59] and Marsden-Weinstein's reduction [57], see also [1], and is regular as $\mathcal{A}_{L_1,L_2}$ does not contain any singular point; in other words $\mathcal{A}_{L_1,L_2}$ is a smooth, see Chapter 1.2. We give to these invariants the name of Keplerian invariants. The reduced Hamiltonian of the three-body problem parametrised by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in the space (2.20) has four degrees of freedom.

Focusing on the double inner collisions, that is, the case $G_1 = 0$ and $G_2 = C$. In $\mathcal{A}_{L_1,L_2}$ these motions are defined by the terms of the form $(\mathbf{a}, -\mathbf{a}, \mathbf{c}, \mathbf{d})$ where $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{d}$ satisfy (2.19). It is a six-dimensional set diffeomorphic to $S^2_{L_1} \times S^2_{L_2} \times$

$S^2_{L_2}$. Thus, these solutions can be studied in $\mathcal{A}_{L_1,L_2}$ and therefore the Keplerian reduction is able to handle the double inner collisions, and the bodies with masses m_0 and m_1 can follow rectilinear and near-rectilinear trajectories, while the motion of the outer body occurs in the invariable plane. However we cannot allow the outer body to follow a rectilinear trajectory. Thus, the reduced Hamiltonian defined in $\mathcal{A}_{L_1,L_2}$ defines a system of four degrees of freedom and is represented by a rational function in the Keplerian invariants which is well defined when $G_1 = 0$. We consider the reduced Hamiltonian in the (compact) subset of $\mathcal{A}_{L_1,L_2}$ where we need to remove the part of the reduced space such that $0 \le G_2 < G_2^{\min}$.

Other types of trajectories that are not well characterised in terms of Deprit's coordinates are the circular motions $G_k = L_k$ for $k = 1$ or $k = 2$ and the motions where the nodes needed to construct the angles ν and ν_1 are not well defined. For example this happens if the inner and outer ellipses lie in the same plane, i.e. the motions of the three bodies are coplanar, the plane of motion being the invariable plane. However, these trajectories are properly covered in the manifold $\mathcal{A}_{L_1,L_2}$ and are also well defined in the next reduced phase spaces. We shall be more specific about this when dealing with the main features of the flow in the fully-reduced space in Chapter 3.

The invariants $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ written in terms of Delaunay coordinates can be found for instance in [20, 68]. Nevertheless, we are interested in the form of these invariants as functions of Deprit's coordinates. We have obtained them in Appendix A. These formulae will be critical to obtain the right set of invariants in the next reduction process. The relations of Appendix A are very useful if one needs to identify some type of motions in $\mathcal{A}_{L_1,L_2}$ — for instance the inner particles follow a circular orbit whereas the outer one moves in the invariable plane — parametrising them with the Keplerian invariants.

The set (2.18) is a system of fundamental invariants and a Hilbert basis that generates $\mathcal{A}_{L_1,L_2}$. By expressing the Deprit variables G_1, G_2, C and $\cos 2\gamma_1$ in terms of the those invariants we put the perturbation (2.17) in terms of these invariants, arriving at a vector-like expression given by

$$\begin{aligned}\mathcal{K}_1 = & -\frac{2\sqrt{2}\mathcal{M}L_1^2}{L_2^3(\mathbf{c}\cdot\mathbf{d}+L_2^2)^{5/2}} \\ & \times \Bigg(\tfrac{3}{4}|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|^4 - 3|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|^2(\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{d}+L_1^2+L_2^2) \\ & + 3\Big((\mathbf{a}\cdot\mathbf{b})^2 - 6(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + (\mathbf{c}\cdot\mathbf{d})^2 - 5\big((\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}+\mathbf{d})\big)^2 \\ & - 2(3\mathbf{a}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{d})L_2^2 + L_1^4 + L_2^4\Big) + 2L_1^2\big(3\mathbf{a}\cdot\mathbf{b}+11(\mathbf{c}\cdot\mathbf{d}+L_2^2)\big)\Bigg).\end{aligned} \tag{2.21}$$

2.3.2 Reduction by the rotational symmetry

It is a well-known fact that the reduction by the rotational symmetry has to be studied in the context of singular reduction, see [22, 27]. In our setting that means that there are some points in the manifold $\mathcal{A}_{L_1,L_2}$ whose isotropy group is not trivial, so that the corresponding action is not free. Therefore, the reduced space is not a manifold but a symplectic orbifold, as we stated in Chapter 1.

In order to achieve the reduction due to the invariance of the angular momentum $\mathbf{C}$ we have to calculate the polynomial invariants associated with the elimination of the angles ν_1 and ν as polynomial combinations of the invariants a_i, b_i, c_i and d_i, $i = 1, 2, 3$. We work constructively, computing the combinations of arbitrary homogeneous polynomials involving the Keplerian invariants such that they are independent of ν and ν_1. In other words, and what is more practical from a computational viewpoint, such that the Poisson brackets of these polynomials with respect to C and B are zero. This yields some conditions on the coefficients of the polynomials.

We start at degree one. An arbitrary polynomial of degree one in the Keplerian invariants is:

$$p_1 = z_1 a_1 + z_2 a_2 + z_3 a_3 + z_4 b_1 + z_5 b_2 + z_6 b_3 + z_7 c_1 + z_8 c_2 + z_9 c_3 + z_{10} d_1 + z_{11} d_2 + z_{12} d_3.$$

The actions C and B in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are:

$$\begin{aligned} C &= \tfrac{1}{2}\sqrt{(a_1+b_1+c_1+d_1)^2 + (a_2+b_2+c_2+d_2)^2 + (a_3+b_3+c_3+d_3)^2}, \\ B &= \tfrac{1}{2}(a_3+b_3+c_3+d_3). \end{aligned}$$

The Poisson structure on $\mathcal{A}_{L_1,L_2}$ of the Keplerian invariants is readily generalised from the case of one single Kepler Hamiltonian, see for instance [21, 68]. It is:

$$\begin{array}{llll} \{a_1\,,\,a_2\} = 2a_3, & \{a_2\,,\,a_3\} = 2a_1, & \{a_3\,,\,a_1\} = 2a_2, & \\ \{b_1\,,\,b_2\} = 2b_3, & \{b_2\,,\,b_3\} = 2b_1, & \{b_3\,,\,b_1\} = 2b_2, & \{a_i\,,\,b_j\} = 0, \\ \{c_1\,,\,c_2\} = 2c_3, & \{c_2\,,\,c_3\} = 2c_1, & \{c_3\,,\,c_1\} = 2c_2, & \\ \{d_1\,,\,d_2\} = 2d_3, & \{d_2\,,\,d_3\} = 2d_1, & \{d_3\,,\,d_1\} = 2d_2, & \{c_i\,,\,d_j\} = 0, \\ \{a_i\,,\,c_j\} = 0, & \{a_i\,,\,d_j\} = 0, & \{b_i\,,\,c_j\} = 0, & \{b_i\,,\,d_j\} = 0. \end{array} \tag{2.22}$$

Thus, we calculate the Poisson brackets $\{p_1\,,\,C^2\}$ and $\{p_1\,,\,B\}$ using (2.22) and force the two brackets to be zero at the same time, obtaining some constraints among the z_i with $i = 1, \ldots, 12$. The reason for computing $\{p_1\,,\,C^2\}$ instead of $\{p_1\,,\,C\}$ is that we get a polynomial. The result yields one valid combination:

$$\pi_1 = a_3 + b_3 + c_3 + d_3.$$

We go on with polynomials of degree two. An arbitrary homogeneous polynomial of degree two in a_i, b_i, c_i and d_i $(i = 1, 2, 3)$ has 78 terms. We call such a polynomial p_2 and calculate $\{p_2\,,\, C^2\}$ and $\{p_2\,,\, B\}$ with the aid of (2.22). Making that these two brackets be zero results in a linear system of 339 equations with 78 unknowns (the unknowns being the coefficients of p_2). We notice that $\{p_2\,,\, C^2\}$ is a polynomial in the Keplerian invariants of degree three, while $\{p_2\,,\, B\}$ yields a polynomial of degree two. Forcing the coefficients of the two Poisson brackets to be zero, the resulting linear system is overdetermined and is solved with MATHEMATICA yielding non-null solutions. Replacing the values of the coefficients of p_2 obtained as solutions of the system we end up with the relevant invariants, namely:

$$
\begin{aligned}
\pi_2 &= a_1b_1 + a_2b_2 + a_3b_3, \;\; \pi_3 = a_1c_1 + a_2c_2 + a_3c_3, \;\; \pi_4 = a_1d_1 + a_2d_2 + a_3d_3,\\
\pi_5 &= b_1c_1 + b_2c_2 + b_3c_3, \;\; \pi_6 = b_1d_1 + b_2d_2 + b_3d_3, \;\; \pi_7 = c_1d_1 + c_2d_2 + c_3d_3,\\
\pi_8 &= (a_1 + b_1 + c_1 + d_1)^2 + (a_2 + b_2 + c_2 + d_2)^2,\\
\pi_9 &= -(a_1 + b_1 + c_1)(a_1 + b_1 + c_1 + 2d_1) - (a_2 + b_2 + c_2)(a_2 + b_2 + c_2 + 2d_2)\\
&\quad + d_3^2.
\end{aligned}
$$

At this point a natural question arises. Do we have to push the computations to degree three? A related question is: how many invariants do we need to calculate? This is equivalent to ask if the invariants π_i, $i = 1, \ldots, 9$ can generate all the invariant functions with respect to the actions C and B from the Keplerian invariants.

From the point of view of computer algebra, this is a typical application of Gröbner bases [87, 19], whose basic ideas are introduced in Chapter 1, and the questions are related to test whether or not a polynomial is in an ideal with a given set of generators. This is achieved as follows. One constructs a Gröbner basis using some of the polynomials π_i $(i = 1, \ldots, 9)$ and applies the multivariate division algorithm with respect to the Gröbner basis as it is explained in Chapter 1.3. In order to decide if a polynomial f belongs to the ideal generated by π_i $(i = 1, \ldots, 9)$ one computes the remainder of the division and it yields 0 if and only if f is in the ideal.

In our context the argument works in the following manner. Out of the nine invariants one chooses the invariants that are intended to generate the symmetry that is reduced and builds with them a Gröbner basis, checking if the rest of invariants of degree one and two belong to the ideal defined by the selected invariants using the multivariate division algorithm. Then, one follows with the invariants of degree three, four and so on. If we conclude that all the invariants of any degree can be expressed in terms of the set of the selected invariants we have solved problem. These invariants form what is called a fundamental set of invariants for the rotational symmetry.

As we are computing the invariants with respect to two independent actions, departing from a space of dimension eight, the reduced space we are determining has to be of dimension four. Moreover the corresponding reduced Hamiltonian in this space has two degrees of freedom, thus we speculate that six invariants are needed subject to two independent relations. However, it is not so evident which six of the nine polynomials we should take to form a fundamental set of invariants, or if we need to take some of the invariants of degree three or higher.

Thus, we change our viewpoint, specifically we express the polynomials π_i, with $i = 1, \ldots, 9$, in terms of Deprit's coordinates and see how they look like. Roughly speaking we need to obtain invariants π_i that written in terms of Deprit's coordinates contain the functions $\cos \gamma_1$, $\sin \gamma_1$, $\cos \gamma_2$, $\sin \gamma_2$, G_1 and G_2.

We easily obtain that

$$\pi_2 = 2G_1^2 - L_1^2, \quad \pi_7 = 2G_2^2 - L_2^2,$$

thus we choose π_2 and π_7 to be incorporated to the set we are looking for. The invariants π_1, π_8 and π_9 are functions of L_1, L_2, C and B but they do not depend on γ_1 or γ_2. Hence, they are of no relevance as we wish to obtain invariants that are functions of sines and cosines of γ_1 and γ_2, thus we discard these invariants. We see that π_3, π_4, π_5 and π_6 are long expressions containing the desired terms but they are not independent, that is, it is not possible to put $\cos \gamma_1$, $\sin \gamma_1$, $\cos \gamma_2$ and $\sin \gamma_2$ as functions of π_3, π_4, π_5 and π_6, L_1, L_2, C, B, G_1 and G_2. More precisely $\sin \gamma_1$, $\sin \gamma_2$ and $\cos \gamma_1, \cos \gamma_2$ can be obtained in this manner but only $\sin \gamma_1$ and $\sin \gamma_2$ can be put in terms of the invariants π_i $(i = 1, \ldots, 9)$ through polynomial expressions, so only two of the four invariants are useful. At least we compute:

$$\begin{aligned}
\pi_3 + \pi_4 - \pi_5 - \pi_6 &= \frac{2}{G_1}\sqrt{\big((C+G_2)^2 - G_1^2\big)\big(G_1^2 - (C-G_2)^2\big)}\sqrt{L_1^2 - G_1^2}\,\sin \gamma_1, \\
\pi_3 - \pi_4 + \pi_5 - \pi_6 &= \frac{2}{G_2}\sqrt{\big((C+G_2)^2 - G_1^2\big)\big(G_1^2 - (C-G_2)^2\big)}\sqrt{L_2^2 - G_2^2}\,\sin \gamma_2.
\end{aligned}$$

Thus, we introduce the following invariants:

$$\begin{aligned}
\sigma_1 &= \pi_2, & \sigma_2 &= \pi_7, \\
\sigma_3 &= \tfrac{1}{2}(\pi_3 + \pi_4 - \pi_5 - \pi_6), & \sigma_4 &= \tfrac{1}{2}(\pi_3 - \pi_4 + \pi_5 - \pi_6).
\end{aligned} \tag{2.23}$$

In terms of the invariants of $\mathcal{A}_{L_1,L_2}$ the σ_i (with $i = 1, \ldots, 4$) are:

$$\begin{aligned}
\sigma_1 &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\
\sigma_2 &= c_1 d_1 + c_2 d_2 + c_3 d_3, \\
\sigma_3 &= \tfrac{1}{2}\big((a_1 - b_1)(c_1 + d_1) + (a_2 - b_2)(c_2 + d_2) + (a_3 - b_3)(c_3 + d_3)\big), \\
\sigma_4 &= \tfrac{1}{2}\big((a_1 + b_1)(c_1 - d_1) + (a_2 + b_2)(c_2 - d_2) + (a_3 + b_3)(c_3 - d_3)\big).
\end{aligned} \tag{2.24}$$

The conclusion is that we cannot obtain a set of fundamental invariants with polynomials of degrees one and two and we have to calculate invariants of degree three, selecting carefully two of them to incorporate to the set of invariants composed by σ_i $(i = 1, \ldots, 4)$.

We compute $\{p_3\,,\, C^2\}$ and $\{p_3\,,\, B\}$ where the arbitrary polynomial p_3 contains all the possible combinations among a_i, b_i, c_i and d_i $(i = 1, 2, 3)$ of degree three. It has 364 monomials. Besides, $\{p_3\,,\, C^2\}$ is homogeneous of degree four and $\{p_3\,,\, B\}$ is homogeneous of degree three. Forcing these two brackets to be null implies to form a linear system of 1533 equations. We have solved it with MATHEMATICA obtaining eight new invariants that are not trivial combinations of the previous invariants π_i. Among these eight invariants we take the combination of two of them that gives the terms cos γ_1 and cos γ_2 without other combinations of sin γ_1 or sin γ_2. We arrive at the following invariants:

$$\begin{aligned}
\sigma_5 \;&=\; \tfrac{1}{2}\Big(a_1\big(b_3(c_2+d_2)-b_2(c_3+d_3)\big)+a_2\big(-b_3(c_1+d_1)+b_1(c_3+d_3)\big)\\
&\qquad +a_3\big(b_2(c_1+d_1)-b_1(c_2+d_2)\big)\Big),\\
\sigma_6 \;&=\; \tfrac{1}{2}\Big(c_1\big(-d_2(a_3+b_3)+d_3(a_2+b_2)\big)+c_2\big(d_1(a_3+b_3)-d_3(a_1+b_1)\big)\\
&\qquad +c_3\big(-d_1(a_2+b_2)+d_2(a_1+b_1)\big)\Big).
\end{aligned} \tag{2.25}$$

We have tried to calculate the Gröbner basis of the σ_i in terms of the Keplerian invariants with MATHEMATICA but without success. In any case we can stop here the calculations with the guarantee that $\{\sigma_1, \ldots, \sigma_6\}$ provides a set of fundamental invariants related to the reduction we are carrying out. In other words, any function that is invariant under the Keplerian symmetry and the symmetry generated by C and B can be put as a function of the σ_i $(i = 1, \ldots, 6)$.

The relations among the invariants and Deprit's action-angle elements is given by

$$\begin{aligned}
\sigma_1 &= 2G_1^2 - L_1^2,\\
\sigma_2 &= 2G_2^2 - L_2^2,\\
\sigma_3 &= \frac{1}{G_1}\sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_1^2-G_1^2}\,\sin\gamma_1,\\
\sigma_4 &= \frac{1}{G_2}\sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_2^2-G_2^2}\,\sin\gamma_2,\\
\sigma_5 &= \sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_1^2-G_1^2}\,\cos\gamma_1,\\
\sigma_6 &= \sqrt{\big((C+G_2)^2-G_1^2\big)\big(G_1^2-(C-G_2)^2\big)}\sqrt{L_2^2-G_2^2}\,\cos\gamma_2.
\end{aligned} \tag{2.26}$$

There are two independent constraints (syzygies) relating the σ_i. They have been obtained from (2.26) expressing $\sin\gamma_k$, $\cos\gamma_k$ in terms of the σ_i and the rest of Deprit's coordinates and using the identity $\sin^2 x + \cos^2 x = 1$. We get:

$$\begin{aligned}(\sigma_1 - L_1^2)\Big((\sigma_2 - \sigma_1 + L_2^2 - L_1^2 + 2C^2)^2 - 8C^2(\sigma_2 + L_2^2)\Big) &= 4(\sigma_1 + L_1^2)\sigma_3^2 + 8\sigma_5^2,\\ (\sigma_2 - L_2^2)\Big((\sigma_1 - \sigma_2 + L_1^2 - L_2^2 + 2C^2)^2 - 8C^2(\sigma_1 + L_1^2)\Big) &= 4(\sigma_2 + L_2^2)\sigma_4^2 + 8\sigma_6^2.\end{aligned} \tag{2.27}$$

Therefore, we arrive at the following set:

$$\mathcal{S}_{L_1,L_2,C} = \Big\{(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6) \in \mathbb{R}^6 \ \mid \ \sigma_i \text{ with } i = 1,\ldots,6 \text{ satisfy (2.27)}\Big\}. \tag{2.28}$$

The reduced space $\mathcal{S}_{L_1,L_2,C}$ is four dimensional. It is a symplectic orbifold that also can be understood as a semialgebraic variety embedded in $\mathbb{R}^6$, i.e. a subset of $\mathbb{R}^6$ that is defined through polynomial equalities and inequalities. It is parametrised by the six invariants σ_i defined through (2.24) and (2.25) that satisfy the two relations given in (2.27). Studying the Jacobian 2×6-matrix formed after calculating the derivatives of the two equations of (2.27) with respect to σ_1, …, σ_6, we may analyse the possibility of singularities which we know they will occur, concluding that singular points can arise when the outer or the inner bodies follows a circular trajectory or the inner ellipses are straight lines. For example, if the two fictitious bodies move on circular trajectories then $\sigma_1 = L_1^2$, $\sigma_2 = L_2^2$, $\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$. In addition to that, if $L_2 = L_1 + C$ the resulting Jacobian matrix has range zero, thus the point $(L_1^2, L_2^2, 0, 0, 0, 0)$ is singular. There are other combinations leading to other singularities which will be studied in Section 2.4.2.

The formulae (2.26) are useful to parametrise $\mathcal{S}_{L_1,L_2,C}$, as we will show later in this chapter and in Chapter 4. We also apply it to put the perturbation in terms of the σ_i. Specifically we solve (2.26) for G_k, $\cos\gamma_k$ and $\sin\gamma_k$ and replace the result in the Hamiltonian (2.17). We get:

$$\begin{aligned}\mathcal{K}_1 = &-\frac{2\sqrt{2}\mathcal{M}L_1^2}{L_2^3(L_2^2+\sigma_2)^{5/2}}\\ &\times\Big(3(\sigma_1^2 - 6\sigma_1\sigma_2 + \sigma_2^2 - 20\sigma_3^2) + 6(L_1^2 - 3L_2^2 - 2C^2)\sigma_1\\ &\quad + 2(11L_1^2 + 3L_2^2 - 6C^2)\sigma_2 + 3(L_1^2 - 2C^2)^2\\ &\quad + 2(11L_1^2 - 6C^2)L_2^2 + 3L_2^4\Big).\end{aligned} \tag{2.29}$$

Note that (2.29) is well defined for $G_1 = 0$, which in terms of the σ_i reads as $\sigma_1 = -L_1^2$.

The reduced Hamiltonian system of the three-body problem in the reduced space $\mathcal{S}_{L_1,L_2,C}$ is a system of two degrees of freedom. When the terms factorised by $(\alpha_1/\alpha_2)^m$ with $m \geq 3$ are included in the averaged Hamiltonian the right space to study the reduced system is $\mathcal{S}_{L_1,L_2,C}$.

The Poisson structure on $\mathcal{S}_{L_1,L_2,C}$ of the invariants σ_i is obtained after computing the Poisson brackets $\{\sigma_i, \sigma_j\}$ using (2.26) in terms of Deprit's coordinates, recalling that these variables are canonical. Then, we return to the invariants expressing G_k, $\cos\gamma_k$ and $\sin\gamma_k$ as functions of σ_i. After some simplifications involving the use of (2.27) we arrive at:

$$
\begin{aligned}
&\{\sigma_1\,,\,\sigma_2\} \;=\; 0,\;\; \{\sigma_1\,,\,\sigma_3\} \;=\; -4\sigma_5,\;\; \{\sigma_1\,,\,\sigma_4\} \;=\; 0,\;\; \{\sigma_1\,,\,\sigma_5\} \;=\; 2(L_1^2+\sigma_1)\sigma_3,\\
&\{\sigma_1\,,\,\sigma_6\} \;=\; 0,\\
&\{\sigma_2\,,\,\sigma_3\} \;=\; 0,\;\; \{\sigma_2\,,\,\sigma_4\} \;=\; -4\sigma_6,\;\; \{\sigma_2\,,\,\sigma_5\} \;=\; 0,\;\; \{\sigma_2\,,\,\sigma_6\} \;=\; 2(L_2^2+\sigma_2)\sigma_4,\\
&\{\sigma_3\,,\,\sigma_4\} \;=\; 4\mathcal{X}^{-1}\left(2C^2(\sigma_3\sigma_6-\sigma_4\sigma_5)+(L_1^2-L_2^2+\sigma_1-\sigma_2)(\sigma_3\sigma_6+\sigma_4\sigma_5)\right),\\
&\{\sigma_3\,,\,\sigma_5\} \;=\; \tfrac{1}{4}\left(4C^4-4C^2(L_2^2+2\sigma_1+\sigma_2)+(L_2^2-3\sigma_1+\sigma_2)(L_2^2-\sigma_1+\sigma_2)\right.\\
&\qquad\qquad\qquad \left.+\,2L_1^2\sigma_1-4\sigma_3^2-L_1^4\right),\\
&\{\sigma_3\,,\,\sigma_6\} \;=\; -2\mathcal{X}^{-1}\left((L_1^2-L_2^2+\sigma_1-\sigma_2)\left(\sigma_3\sigma_4(L_2^2+\sigma_2)-2\sigma_5\sigma_6\right)\right.\\
&\qquad\qquad\qquad\qquad \left.+\,2C^2\left(\sigma_3\sigma_4(L_2^2+\sigma_2)+2\sigma_5\sigma_6\right)\right),\\
&\{\sigma_4\,,\,\sigma_5\} \;=\; -2\mathcal{X}^{-1}\left((L_2^2-L_1^2+\sigma_2-\sigma_1)\left(\sigma_3\sigma_4(L_1^2+\sigma_1)-2\sigma_5\sigma_6\right)\right.\\
&\qquad\qquad\qquad\qquad \left.+\,2C^2\left(\sigma_3\sigma_4(L_1^2+\sigma_1)+2\sigma_5\sigma_6\right)\right),\\
&\{\sigma_4\,,\,\sigma_6\} \;=\; \tfrac{1}{4}\left(4C^4-4C^2(L_1^2+\sigma_1+2\sigma_2)+(L_1^2+\sigma_1-3\sigma_2)(L_1^2+\sigma_1-\sigma_2)\right.\\
&\qquad\qquad\qquad \left.+\,2L_2^2\sigma_2-4\sigma_4^2-L_2^4\right),\\
&\{\sigma_5\,,\,\sigma_6\} \;=\; 2\mathcal{X}^{-1}\left((L_1^2+\sigma_1)(2C^2-L_1^2+L_2^2-\sigma_1+\sigma_2)\sigma_3\sigma_6\right.\\
&\qquad\qquad\qquad\qquad \left.+\,(L_2^2+\sigma_2)(-2C^2-L_1^2+L_2^2-\sigma_1+\sigma_2)\sigma_4\sigma_5)\right),
\end{aligned}
\tag{2.30}
$$

where

$$
\mathcal{X} \;=\; 4C^4-4C^2\left(L_1^2+L_2^2+\sigma_1+\sigma_2\right)+\left(L_1^2-L_2^2+\sigma_1-\sigma_2\right)^2.
$$

As expected the Poisson brackets are closed for the invariants but they do not represent a Hilbert basis since some of the brackets are not polynomials but rational functions. Fortunately it is not a major drawback for the calculations made in the thesis.

2.3.3 Reduction by the symmetry related with G_2

Since γ_2 is not present in (2.17), G_2 becomes a constant of motion and generates another symmetry in the Hamiltonian system so that $\mathcal{S}_{L_1,L_2,C}$ can be reduced. This time we have to reduce out an S^1-symmetry.

The reduction is easily performed as we need to get those invariants from the set of the σ_i ($i = 1, \ldots, 6$) that are related with G_1, sin γ_1 and cos γ_1. The choice looks clear, we take σ_1, σ_3 and σ_5. So we define:

$$\tau_1 = \sigma_1, \quad \tau_2 = \sigma_3, \quad \tau_3 = \sigma_5. \tag{2.31}$$

The constraint relating τ_i ($i = 1, 2, 3$) is derived from the first equation of (2.27), replacing σ_2 by $2G_2^2 - L_2^2$, while the second equation yields a trivial identity. We arrive at:

$$(\tau_1 - L_1^2)\big((\tau_1 + L_1^2 - 2C^2 - 2G_2^2)^2 - 16C^2G_2^2\big) = 4(\tau_1 + L_1^2)\tau_2^2 + 8\tau_3^2. \tag{2.32}$$

The fully-reduced phase space is introduced as follows:

$$\mathcal{T}_{L_1,C,G_2} = \Big\{(\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 \ \mid \ \text{the invariants } \tau_i \text{ with } i = 1, 2, 3 \text{ satisfy } (2.32)\Big\}. \tag{2.33}$$

The set $\mathcal{T}_{L_1,C,G_2}$ is a two-dimensional phase space that can be embedded in $\mathbb{R}^3$. It is parametrised by the three invariants τ_i defined in (2.31), which satisfy the relation (2.32). It is also a symplectic orbifold. The fully-reduced Hamilton function defined in $\mathcal{T}_{L_1,C,G_2}$ is a system of one degree of freedom. The space is singular for some combinations of L_1, C, G_2 concerning specific motions of the inner bodies, in particular, the rectilinear motions such that their projections into the three-dimensional coordinate space are perpendicular to the invariable plane. Leaving apart the combinations among the parameters that lead to these particular motions, $\mathcal{T}_{L_1,C,G_2}$ is a smooth manifold diffeomorphic to S^2. In the next section we shall treat in detail the issue of the singularities and how to deal with the rectilinear, circular and coplanar motions, as well as some other features of the surface $\mathcal{T}_{L_1,C,G_2}$.

It is important to note that our space $\mathcal{T}_{L_1,C,G_2}$ is different from the one obtained by Ferrer and Osácar in [34] that they called $P(L_1, L_2, C, G_2)$, as in this latter reduced space the possible singularities are not taken into account, so their space is diffeomorphic to S^2. However, the fact that a certain reduction is regular or singular is intrinsic to the type of symmetry and does not depend on the way one chooses the set of coordinates. More specifically, if in the process of introducing the action map to make the reduction explicit, this map has fixed points the reduction is singular [22]. Thus, the dynamics of the three-body problem studied in Chapter 3 concerning the singular points of $\mathcal{T}_{L_1,C,G_2}$ is not properly done in the space $P(L_1, L_2, C, G_2)$.

Using (2.29), after putting G_2 in terms of σ_2, it is readily deduced that the

Hamiltonian $\mathcal{K}_1$ in terms of τ_i $(i = 1, 2, 3)$ is

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{\mathcal{M} L_1^2}{2 L_2^3 G_2^5}\Big(&12(C^2 - G_2^2)^2 + 4(11 G_2^2 - 3C^2) L_1^2 + 3 L_1^4 \\ &+ 6(L_1^2 - 2C^2 - 6G_2^2)\tau_1 + 3\tau_1^2 - 60\tau_2^2\Big).\end{aligned} \tag{2.34}$$

Alternatively we have also obtained (2.34) from (2.21) considering the Gröbner basis of the τ_i in terms of the Keplerian invariants and the division algorithm. The result agrees with $\mathcal{K}_1$ in (2.34).

We close the section with the following theorem, summarising the whole reduction process.

Theorem 2.1. *The set $\mathcal{T}_{L_1,C,G_2}$ defined in (2.33) is the fully-reduced phase space obtained after reducing the phase space $\mathbb{R}^{12}$ through three stages:*

(i) The reduction of the Keplerian-symmetry generated by L_1 and L_2.

(ii) The reduction of the $SO(3)$-symmetry generated by C and B.

(iii) The reduction of the S^1-symmetry generated by G_2.

The sets $\mathcal{A}_{L_1,L_2}$ and $\mathcal{S}_{L_1,L_2,C}$ are the intermediate spaces obtained through the reduction process by stages. Concretely $\mathcal{A}_{L_1,L_2}$ corresponds to the space obtained by reducing the Keplerian-symmetry generated by L_1 and L_2 in the context of regular reduction. It is an eight-dimensional manifold defined by the twelve invariants given in (2.18) and the four constraints of (2.19). The set $\mathcal{S}_{L_1,L_2,C}$ is the space resulting after reducing by the $SO(3)$-symmetry generated by C and B. Its dimension is four and it is defined by the six invariants introduced in (2.24) and (2.25) and the two relations given in (2.27). This space has singular points for some combinations of L_1, L_2 and C.

The set $\mathcal{T}_{L_1,C,G_2}$ is a symplectic orbifold (and a semialgebraic variety in $\mathbb{R}^3$) of dimension two (a surface) that may have singular points for some combinations of L_1, C and G_2, which are related with some types of circular and rectilinear motions of the inner bodies. The systems that can be studied in the space $\mathcal{T}_{L_1,C,G_2}$ correspond to Hamiltonian functions of one degree of freedom.

In particular, the spatial three-body problem considered in the perturbing region $\mathcal{Q}_{\varepsilon,n}$ of the phase space $T^*\mathbb{R}^9$ may be analysed in $\mathcal{T}_{L_1,C,G_2}$ after truncating the expansions in the Legendre polynomials at $n = 2$, averaging the Hamiltonian with respect to the mean anomalies ℓ_k at first order of the Lie transformation and applying the reductions outlined above.

2.4 Description of the reduced phase spaces

This section deals with the description of the different reduced spaces. That is, we develop a complete study of the fully-reduced phase space, a study of the singularities in $\mathcal{S}_{L_1,L_2,C}$ and a study of one specific point in $\mathcal{R}_{L_1,L_2,B}$, which is going to be used in the following chapters.

2.4.1 The fully-reduced phase space

We start by parametrising $\mathcal{T}_{L_1,C,G_2}$ in order to have a better understanding of the reduced space. As τ_1, τ_2 and τ_3 are represented in terms of Deprit's coordinates by

$$\begin{aligned} \tau_1 &= 2G_1^2 - L_1^2, \\ \tau_2 &= \frac{1}{G_1}\sqrt{\big((C+G_2)^2 - G_1^2\big)\big(G_1^2 - (C-G_2)^2\big)}\sqrt{L_1^2 - G_1^2}\sin\gamma_1, \\ \tau_3 &= \sqrt{\big((C+G_2)^2 - G_1^2\big)\big(G_1^2 - (C-G_2)^2\big)}\sqrt{L_1^2 - G_1^2}\cos\gamma_1, \end{aligned} \tag{2.35}$$

we can think of G_1 and γ_1 as the coordinates that define the surface (2.32) while L_1, C and G_2 act as parameters. We know that $\gamma_1 \in [0, 2\pi)$ and $G_1 \in [0, L_1]$.

In particular as $G_1 = 0$ implies $G_2 = C$, hence we can compute the values of τ_2 and τ_3 in (2.35) when G_1 vanishes. Changing G_2 by C in (2.35) and simplifying we get

$$\begin{aligned} \tau_1 &= 2G_1^2 - L_1^2, \\ \tau_2 &= \sqrt{4C^2 - G_1^2}\sqrt{L_1^2 - G_1^2}\sin\gamma_1, \\ \tau_3 &= G_1\sqrt{4C^2 - G_1^2}\sqrt{L_1^2 - G_1^2}\cos\gamma_1, \end{aligned} \tag{2.36}$$

that are well defined if $0 < G_1 \leq 2C$, which are the right bounds for G_1 when $G_2 = C$. For $G_1 = 0$ we obtain $\tau_3 = 0$ and $\tau_1 = -L_1^2$ but τ_2 depends on γ_1 and γ_1 is meaningless if $G_1 = 0$. Replacing τ_3 by zero and G_2 by C in (2.32) and taking into account that $\tau_1 \in [-L_1^2, \min\{L_1^2, 8C^2 - L_1^2\}]$ we conclude that $\tau_2 \in [-2L_1C, 2L_1C]$. Thus rectilinear trajectories for the inner bodies are represented properly in $\mathcal{T}_{L_1,C,C}$. We stress that we are excluding the case $G_2 = 0$ as the Hamiltonian of the three-body problem $\mathcal{H}$ is not bounded for rectilinear motions of the outer body, indeed we are assuming that $G_2 \geq G_2^{\min}$. This simplifies the study of $\mathcal{T}_{L_1,C,G_2}$ a bit, however for any other Hamiltonian system that has the same symmetries as the ones appearing in this thesis but that is defined for $G_2 = 0$ — and undefined for $G_1 = 0$ so that we avoid C to be zero — we should take this into consideration.

The fact that G_2 is bounded in the interval $[|C - G_1|, C + G_1]$ also implies that $(C+G_2)^2 - G_1^2$ and $G_1^2 - (C-G_2)^2$ are both non-negative, thus the parametrisation

(2.35) makes sense for the allowed values of the variables and parameters. We remark that (2.6) gives another lower-bound for G_2, so both bounds must be satisfied. Now we observe that, using (2.9) and (2.15), it follows that

$$|G_1 - G_2| \leq C \leq G_1 + G_2. \tag{2.37}$$

We give an account of some special motions concerning the inner bodies, specifically those problematic points of $\mathcal{T}_{L_1,C,G_2}$ for which Deprit's coordinates are singular. These motions are of three types:

(i) *Circular trajectories*, i.e. motions where $G_1 = L_1$ for which the angle γ_1 is undefined. They are represented in $\mathcal{T}_{L_1,C,G_2}$ by the point $(L_1^2, 0, 0)$. As the upper-bound of G_1 is $\min\{L_1, C+G_2\}$, circular solutions are not reachable if $C+G_2 < L_1$. Then if this inequality holds circular motions cannot occur and the point with lowest possible eccentricity is $(2(C+G_2)^2 - L_1^2, 0, 0)$. Besides replacing G_1 by $C+G_2$ in (2.9), we get $I_1 = 0$ and $I_2 = \pi$ and the three bodies move on the same plane which is the invariable plane, but the inner bodies do it in the opposite sense to the outer body. The limit situation is $L_1 = C + G_2$ where $((C+G_2)^2, 0, 0)$ represents the circular motions that are coplanar with respect to the outer fictitious body.

(ii) *Coplanar motions*, i.e. the inner and outer ellipses lie in the same plane, where the node ν_1 does not exist (equivalently $I_1 = 0$ or $I_1 = \pi$). As $G_1 \geq 0$ and $C, G_2 > 0$ we deduce from (2.9) that $I_1 = 0$ implies $C = G_1 \pm G_2$ and $I_2 = 0$ or $I_2 = \pi$ while $I_1 = \pi$ implies $C = G_2 - G_1$ and $I_2 = 0$. Thus ν_1 is undefined for $C = G_1 + G_2$ and $C = |G_1 - G_2|$ and the three ellipses share the same plane. We should add the case where ν is not defined. It occurs for $C = |B|$ but it does not involve any combination among C, L_1, G_1 and G_2, thus we do not to take care of it. This situation can be analysed properly studying first the dynamics of a certain flow in $\mathcal{T}_{L_1,C,G_2}$ and then assuming $C = |B|$. Collecting the two possibilities we substitute in (2.35) G_1 by $|C - G_2|$ leading to the point $(2(C-G_2)^2 - L_1^2, 0, 0)$ which represents the point of $\mathcal{T}_{L_1,C,G_2}$ of coplanar motions with respect to the outer body. This point lies on the same axis of the space spanned by τ_1, τ_2 and τ_3 as the point referring to circular solutions but in the opposite direction to it. Finally, as we said before, the analysis of a certain Hamiltonian in $\mathcal{T}_{L_1,C,G_2}$ cannot take into consideration the relative value of B with respect to the other parameters. We know that B is a constant of motion and satisfies $|B| \leq C$. Then if $C = |B|$ and $C = G_1 + G_2$ or $C = |G_1 - G_2|$ the invariable plane coincides with the horizontal plane of the inertial frame $\mathcal{F}$.

(iii) *Rectilinear motions*, that is, trajectories such that $G_1 = 0$ and $G_2 = C$. Then, none of Deprit's angles are defined and (2.9) does not apply but $\mathbf{G_2} =$

$\mathbf{C}$, thus $I_2 = 0$. The angle I_1 also makes sense. They are represented in $\mathcal{T}_{L_1,C,C}$ by the segment

$$\Big\{(-L_1^2\,,\,\tau_2\,,\,0)\ \mid\ \tau_2\in[-2L_1C,\ 2L_1C]\Big\}, \tag{2.38}$$

As $I_2 = 0$ it implies that the outer ellipse lies in the invariable plane. To better understand what type of rectilinear motions we are dealing with we write the expressions of τ_i in terms of a_i, b_i, c_i and d_i $(i = 1, 2, 3)$ through the various formulae of Section 2.3. We arrive at the following expression

$$\big(-L_1^2, a_1(c_1+d_1)+a_2(c_2+d_2)+a_3(c_3+d_3), 0\big),$$

that is put in terms of the spatial Cartesian coordinates, getting rectilinear motions with all possible types of inclinations. In particular the points $(-L_1^2, \pm 2L_1C, 0)$ of $\mathcal{T}_{L_1,C,C}$ correspond to rectilinear solutions of the inner bodies that are perpendicular to the invariable plane. The negative sign of the second coordinate of the point happens when the vectors $\mathbf{C}$ and $\mathbf{x}_1$ are parallel while the positive sign happens when $\mathbf{C}$ and $\mathbf{x}_1$ are antiparallel. The point $(-L_1^2, 0, 0)$ corresponds with the case where the three bodies are in the invariable plane, that is, their motions are coplanar. Other interesting points are $(-L_1^2, \pm 2L_1|B|, 0)$ where the inner bodies move on the axis $\mathbf{k}$. Finally, when $G_2 = C = |B|$ the invariable plane is the horizontal plane of the inertial frame $\mathcal{F}$ and the points $(-L_1^2, \pm 2L_1C, 0)$ correspond to rectilinear trajectories such that the inner bodies move on the axis $\mathbf{k}$, thus the motions of the two fictitious bodies being perpendicular one each other. Besides, the point $(-L_1^2, 0, 0)$ corresponds to solutions where the three bodies move on the plane spanned by $\mathbf{i}$ and $\mathbf{j}$ and the inner bodies move on straight lines. Note that the motions of the inner bodies and the outer body occur in perpendicular planes only at the point $(-L_1^2, \pm 2L_1C, 0)$, but in general the two planes can form any other angle between 0 and π. This clarifies the comment made in [34] at the bottom of p. 252, which is not correct.

In Figs. 2.3 and 2.4 we show two examples of the fully-reduced space $\mathcal{T}_{L_1,C,G_2}$. The first one has only a singularity at the point referring to the circular motions and the second figure shows a smooth surface.

We deal now with the singularities of the space $\mathcal{T}_{L_1,C,G_2}$. We compute the gradient of (2.32) with respect to τ_1, τ_2 and τ_3, calculating in what points the gradient vanishes and for what values of L_1, C and G_2 that happens. In addition to that, we need to take into account the relation (2.32).

The gradient is:

$$\Big(3\tau_1^2-4\tau_2^2+2\big(L_1^2-4(C^2+G_2^2)\big)\tau_1-L_1^4+4(C^2-G_2^2)^2, -8(\tau_1+L_1^2)\tau_2, -16\tau_3\Big). \tag{2.39}$$

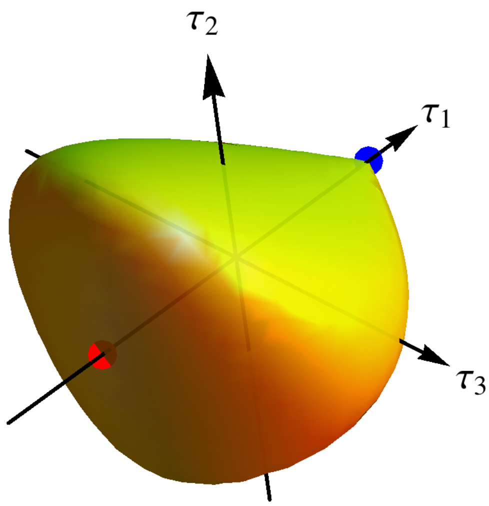

Figure 2.3: The space $\mathcal{T}_{5,3,2}$ has a singularity at the blue point. The red point refers to coplanar solutions with the outer body with $G_1 = |C - G_2|$.

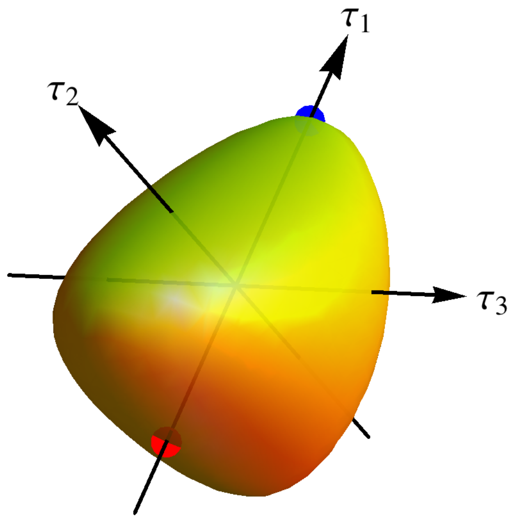

Figure 2.4: View of the regular space $\mathcal{T}_{4,3,2}$. The blue point refers to circular motions whereas the red point means coplanar solutions with the outer body with $G_1 = |C - G_2|$.

A necessary condition to make the gradient vanish is that $\tau_3 = 0$. Moreover, looking at the middle term of (2.39), either $\tau_1 = -L_1^2$ or $\tau_2 = 0$.

We make the replacement $\tau_1 = -L_1^2$, $\tau_3 = 0$ in the first term of (2.39) and in (2.32) and obtain the resultant of the two polynomials with respect to τ_2, obtaining $1024L_1^4(C - G_2)^4(C + G_2)^4$. The only significant case for which the resultant vanishes is when $G_2 = C$. Replacing this condition in the polynomial of the first term of the gradient it yields two values, $\tau_2 = \pm 2L_1C$. The consequence is that the points $(-L_1^2, \pm 2L_1C, 0)$ are singularities of the surface $\mathcal{T}_{L_1,C,C}$. They are the points related to the rectilinear solutions perpendicular to the invariable plane.

On the other hand if $\tau_2 = 0$ the resultant of the first component of (2.39) and (2.32) for $\tau_2 = \tau_3 = 0$ is $-1024C^2G_2^2((L_1-C)^2-G_2^2)^2((L_1+C)^2-G_2^2)^2$. We discard that $L_1 = |C-G_2|$ as the reduced space is a point, thus the only possibility for the resultant to be zero is that $L_1 = C + G_2$. Substituting this value in the gradient and in (2.32) leads to a unique valid solution, namely $\tau_1 = L_1^2$, $\tau_2 = \tau_3 = 0$, which is the point of $\mathcal{T}_{L_1,C,L_1-C}$ accounting for circular motions. It corresponds to the limit case such that if $C + G_2 < L_1$ the circular motions are no longer allowed and so they are not represented in $\mathcal{T}_{L_1,C,G_2}$ as an equilibrium point.

We stress that the singular points are always equilibria of a certain Hamiltonian defined in $\mathcal{T}_{L_1,C,G_2}$.

Summarising the above paragraphs there can be up to three singular points in $\mathcal{T}_{L_1,C,G_2}$. If $G_2 = C$ the points $(-L_1^2, \pm 2L_1C, 0)$ are singular points of $\mathcal{T}_{L_1,C,C}$ representing rectilinear motions parallel to vector $\mathbf{C}$. If $L_1 = C + G_2$ the point $(L_1^2, 0, 0)$, that represents the circular coplanar motions when considering the outer ellipse, is a singular point of $\mathcal{T}_{L_1,C,L_1-C}$. When $G_2 = C$ and $L_1 = 2C$ the three points, namely, $(-L_1^2, \pm L_1^2, 0)$ and $(L_1^2, 0, 0)$, are singular points and the surface $\mathcal{T}_{L_1,L_1/2,L_1/2}$ is a *tricorn*. The rest of combinations among the three parameters leads to regular surfaces.

In Fig. 2.5 we show the fully-reduced space when $G_2 = C$.

The size and shape of $\mathcal{T}_{L_1,C,G_2}$ depend on the relative values of the three parameters. If $L_1 = |C - G_2|$, since $|C - G_2| \leq G_1 \leq C + G_2$ it is readily concluded that $G_1 = L_1 = |C-G_2|$, therefore using (2.35), $\tau_1 = L_1^2$, $\tau_2 = \tau_3 = 0$ and the space gets reduced to a unique point. We discard the analysis of this particular point which corresponds to motions of the inner bodies that are both coplanar with respect to the outer body and circular. Similarly as it is done in [44], these solutions should be analysed in a space of higher dimension, in this case in $\mathcal{S}_{L_1,L_2,C}$.

Concerning the bounds of τ_i $(i = 1, 2, 3)$, it is straightforward to conclude that $\tau_1 \in [2(C - G_2)^2 - L_1^2, 2\min\{L_1^2, (C + G_2)^2\} - L_1^2]$. However the bounds for τ_2 and τ_3 are more complicated to deduce. We have determined them by maximising the expressions of τ_2 and τ_3 in (2.35) in terms of G_1 but they are cumbersome expressions involving L_1, C and G_2. In the particular case $G_2 = C$ the maximum

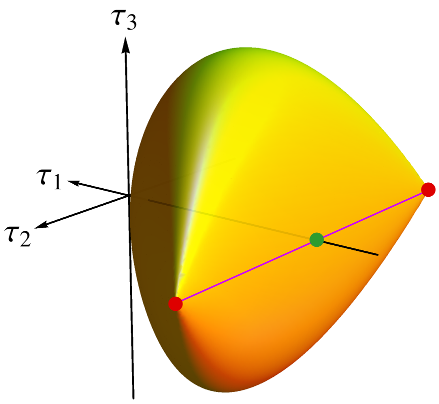

Figure 2.5: Reduced space $\mathcal{T}_{L_1,C,C}$ showing the set of rectilinear motions of the inner particles by the magenta segment. The red points correspond to the singular relative equilibria $(-L_1^2, \pm 2L_1C, 0)$ whereas the green point corresponds to the regular equilibrium $(-L_1^2, 0, 0)$.

values of τ_2 and τ_3 are computed using (2.36) yielding that $|\tau_2| \leq 2L_1C$ and

$$|\tau_3| \leq \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{(L_1^4 - 4L_1^2C^2 + 16C^4)^{3/2} - L_1^6 + 6L_1^4C^2 + 24L_1^2C^4 - 64C^6}.$$

We deal now with the Poisson brackets among the invariants of $\mathcal{T}_{L_1,C,G_2}$. We have to compute $\{\tau_1\,,\,\tau_2\}$, $\{\tau_1\,,\,\tau_3\}$ and $\{\tau_2\,,\,\tau_3\}$. It is possible to make the whole process putting τ_i ($i = 1, 2, 3$) in terms of the invariants of $\mathcal{S}_{L_1,L_2,C}$ using (2.31), calculating the three Poisson brackets through (2.30). Then one has to determine a Gröbner basis of the polynomial set composed by the three τ_i as functions of the Keplerian invariants as well as the four constraints of (2.19). The three expressions giving the Poisson brackets in terms of a_i, b_i, c_i and d_i are divided with respect to the Gröbner basis applying the multivariate division algorithm to obtain the required Poisson brackets as the remainders of the divisions. We have been able to do it with Mathematica. Alternatively one can write each τ_i in terms of Deprit's coordinates and determine the Poisson brackets in terms of this set of action-angles coordinates. Finally we go back to τ_i, arriving straightforwardly at the following Poisson structure on $\mathcal{T}_{L_1,C,G_2}$:

$$\begin{aligned}
\{\tau_1\,,\,\tau_2\} &= -4\tau_3,\\
\{\tau_1\,,\,\tau_3\} &= 2(\tau_1 + L_1^2)\tau_2, \qquad (2.40)\\
\{\tau_2\,,\,\tau_3\} &= \tfrac{3}{4}\tau_1^2 - \tau_2^2 + \tfrac{1}{2}\big(L_1^2 - 4(C^2 + G_2^2)\big)\tau_1 - \tfrac{1}{4}L_1^4 + (C^2 - G_2^2)^2.
\end{aligned}$$

The τ_i form a set of fundamental invariants for the space $\mathcal{T}_{L_1,C,G_2}$ and a Hilbert basis.

The main conclusions of this section are encapsulated in the following theorem.

Theorem 2.2. *The fully-reduced phase space $\mathcal{T}_{L_1,C,G_2}$ is parametrised using (2.35) when $C \neq G_2$ with $\gamma_1 \in [0, 2\pi)$ and $G_1 \in [|C - G_2|, min\{L_1, C + G_2\}]$. If $C = G_2$ we use the equations of (2.36) where γ_1 ranges in the same interval as before and $G_1 \in [0, min\{L_1, 2G_2\}]$.*

Special types of motions that in Deprit's coordinates are undefined are covered with the invariants τ_i in $\mathcal{T}_{L_1,C,G_2}$. In particular the inner ellipses are allowed to become straight lines, that is, the inner bodies can move on straight lines that have any inclination with respect to the invariable plane. Coplanar motions between the inner and outer ellipses are allowed and the inner and outer bodies can move on their ellipses following trajectories with any eccentricity in the elliptic domain (and $0 \leq e_2 < 1$). Moreover the common plane where the three bodies move can have any inclination with respect to the horizontal plane of the inertial frame $\mathcal{F}$. The inner ellipses can be circular provided that $C + G_2 \geq L_1$, otherwise they are not taken into account in the fully-reduced space.

The space $\mathcal{T}_{L_1,C,G_2}$ is a regular surface diffeomorphic to S^2 if $C \neq G_2$ and $L_1 \neq C + G_2$, otherwise it has one, two or three singular points. The singularities are always equilibria of all the Hamiltonian systems that are globally defined in $\mathcal{T}_{L_1,C,G_2}$. In particular, if $L_1 = C + G_2$ and $G_2 \neq C$, the space $\mathcal{T}_{L_1,C,L_1-C}$ has one singular point at $(L_1^2, 0, 0)$ which corresponds to circular motions of the inner ellipses that are coplanar with the outer ellipse. If $G_2 = C$ and $L_1 \neq C + G_2$ the space $\mathcal{T}_{L_1,C,C}$ has two singular points at $(-L_1^2, \pm 2L_1C, 0)$ which corresponds to motions of the inner bodies in a straight line perpendicular to the invariable plane and such that the outer body remains in the invariable plane. If in addition $C = |B|$ then the inner bodies move on the axis $\mathbf{k}$ while the outer ellipse is in the plane spanned by $\mathbf{i}$ and $\mathbf{j}$. If $G_2 = C$ and $L_1 = 2C$ the space $\mathcal{T}_{L_1,L_1/2,L_1/2}$ has three singular points, namely $(L_1^2, 0, 0)$ and $(-L_1^2, \pm L_1^2, 0)$. The first point corresponds to circular motions of the inner bodies while the outer body is in the same plane but moving in the opposite sense. The other two points correspond to rectilinear motions of the inner bodies in a direction parallel to the total angular momentum vector.

If $L_1 = |C - G_2|$, the space $\mathcal{T}_{|C-G_2|,C,G_2}$ gets reduced to a unique point which corresponds to circular motions of the inner bodies that are coplanar with respect to the outer body.

The Poisson structure on $\mathcal{T}_{L_1,C,G_2}$ is given in (2.40). The invariants τ_i form a Hilbert basis for the fully-reduced space.

2.4.2 The space $\mathcal{S}_{L_1,L_2,C}$

The reduced space $\mathcal{S}_{L_1,L_2,C}$ is generically a four-dimensional space parametrised by the six invariants σ_i defined through (2.24) that satisfy the two relations given in (2.27). The relationship between the invariants and Deprit's action-angle elements is obtained in Section 2.3.2 and are given by (2.26).

The coordinates are $G_k \in (0, L_k]$ and $\gamma_k \in [0, 2\pi)$ for $k = 1, 2$, while the parameters satisfy $L_2 > L_1 > 0$ and $C > 0$. Besides, one has $|G_1 - G_2| \leq C \leq G_1 + G_2$. When $C = L_1 + L_2$ the reduced space is a single point that represents circular coplanar motions of the three bodies. When $C > L_1 + L_2$ the reduced space is empty.

The inner bodies can follow bounded straight lines in $\mathcal{S}_{L_1,L_2,C}$. Note that they satisfy $G_1 = 0$ and $G_2 = C$. Indeed these motions are represented by the segment

$$\Big\{(-L_1^2\,,\,2C^2 - L_2^2\,,\,\sigma_3\,,\,0\,,\,0\,,\,0) \;\mid\; \sigma_3 \in [-2L_1C,\;2L_1C]\Big\}. \tag{2.41}$$

The space $\mathcal{S}_{L_1,L_2,C}$ has singular points, as it is obtained from the manifold $\mathcal{A}_{L_1,L_2}$ after reducing out the rotational symmetry. In order to determine the singularities of this space we study the Jacobian 2×6-matrix obtained by calculating the derivatives of the two constraints (2.27) with respect to $\sigma_1, \ldots, \sigma_6$. When the rank of this matrix is not maximum, we have a singularity in $\mathcal{S}_{L_1,L_2,C}$. The Jacobian matrix is given by

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \end{pmatrix}, \tag{2.42}$$

where

$$\begin{aligned}
J_{11} &= L_2^4 - L_1^4 + 4C^4 + 3\sigma_1^2 + \sigma_2^2 - 4\sigma_3^2 + 2L_1^2\sigma_1 + 2L_2^2(\sigma_2 - 2\sigma_1) - 4\sigma_1\sigma_2 \\
&\quad - 4C^2(L_2^2 + 2\sigma_1 + \sigma_2), \\
J_{12} &= 2(L_1^2 - \sigma_1)(L_1^2 - L_2^2 + 2C^2 + \sigma_1 - \sigma_2), \\
J_{13} &= -8(L_1^2 + \sigma_1)\sigma_3, \\
J_{14} &= 0, \\
J_{15} &= -16\sigma_5, \\
J_{16} &= 0, \\
J_{21} &= 2(L_2^2 - \sigma_2)(L_2^2 - L_1^2 + 2C^2 - \sigma_1 + \sigma_2), \\
J_{22} &= L_1^4 - L_2^4 + 4C^4 + 3\sigma_2^2 + \sigma_1^2 - 4\sigma_4^2 + 2L_2^2\sigma_2 + 2L_1^2(\sigma_1 - 2\sigma_2) - 4\sigma_1\sigma_2 \\
&\quad - 4C^2(L_1^2 + 2\sigma_2 + \sigma_1), \\
J_{23} &= 0, \\
J_{24} &= -8(L_2^2 + \sigma_2)\sigma_4, \\
J_{25} &= 0, \\
J_{26} &= -16\sigma_6.
\end{aligned}$$

The space $\mathcal{S}_{L_1,L_2,C}$ has singular points whenever the Jacobian matrix J has rank one or zero. Three possibilities arise: (i) the first row of J is zero; (ii) the second row of J is zero; (iii) the two rows are proportional.

A necessary condition for the first row of the matrix to vanish is that $\sigma_5 = 0$ and a necessary condition for the second row to vanish is that $\sigma_6 = 0$. So, first we replace σ_5 by zero in the first row of J and in the constraints (2.27), we equate to zero all the elements of the matrix's first row and solve the resulting system. The following singular points of $\mathcal{S}_{L_1,L_2,C}$ are obtained:

(a) The points $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1C, 0, 0, 0)$. As $G_1 = 0$ and $G_2 = C$ these points represent rectilinear motions of the inner bodies that are orthogonal to the invariable plane, which is the plane where the outer body's orbit lies.

(b) The points $(L_1^2, 2(L_1 \pm C)^2 - L_2^2, 0, 0, 0, 0)$, which stand for circular motions of the inner bodies. As $G_1 = L_1$ and $\sigma_2 = 2G_2^2 - L_2^2$ then $G_2 = |C \pm G_1|$, thus the inner and outer bodies move in the invariable plane.

(c) When $C = L_1$ the points $(L_1^2, -L_2^2, 0, \sigma_4, 0, 0)$ with $\sigma_4 \in [-2L_2^2, 2L_2^2]$. Since $G_1 = L_1 = C$ and $G_2 = 0$, we infer that the inner bodies move in circular orbits in the invariable plane whereas the outer body follows a rectilinear trajectory with any inclination $I_2 \in [0, \pi]$. In particular when $\sigma_4 = \pm 2L_2^2$ the rectilinear motions are perpendicular to the invariable plane while when $\sigma_4 = 0$ the motions of the three bodies are coplanar.

Now we replace σ_6 by zero in the second row of the Jacobian matrix and in the constraints (2.27). Then, we equate to zero all the elements of the matrix's second row, studying the resulting system. We get the following singularities in $\mathcal{S}_{L_1,L_2,C}$:

(d) The points $(2C^2 - L_1^2, -L_2^2, 0, \pm 2L_2C, 0, 0)$. The inner bodies remain in the invariable plane and the outer body describes a rectilinear trajectory orthogonal to it.

(e) The point $(2(L_2 - C)^2 - L_1^2, L_2^2, 0, 0, 0, 0)$, which represents prograde circular motions of the outer body that are coplanar with the inner bodies' motions. The point $(2(L_2 + C)^2 - L_1^2, L_2^2, 0, 0, 0, 0)$ is discarded because the parameters do not satisfy all the constraints.

(f) When $C = L_2$ the points $(-L_1^2, L_2^2, \sigma_3, 0, 0, 0)$ with $\sigma_3 \in [-2L_1^2, 2L_1^2]$. They represent rectilinear motions of the inner bodies with any inclination $I_1 \in [0, \pi]$ whereas the outer body moves in circular orbits in the invariable plane. When $\sigma_3 = \pm 2L_1^2$ the rectilinear motions are perpendicular to the invariable plane while when $\sigma_3 = 0$ the three bodies move in the invariable plane.

When setting the first row of J to be proportional to the second row of it with constant $\alpha \neq 0$ then σ_5 and σ_6 have to be zero. It leads to four possibilities for $\sigma_1, \ldots, \sigma_4$, namely $\sigma_3 = \sigma_4 = 0$; $\sigma_2 = -L_2^2, \sigma_3 = 0$; $\sigma_1 = -L_1^2$, $\sigma_2 = -L_2^2$; $\sigma_4 = 0$, $\sigma_1 = -L_1^2$. After replacing the corresponding values of the σ_i in J and in (2.27) we discard the last case as the constraints defining $\mathcal{S}_{L_1,L_2,C}$ are not fulfilled. The other three cases lead to the following situations:

(g) The point $(L_1^2, L_2^2, 0, 0, 0, 0)$ with $C = L_2 \pm L_1$, which corresponds to coplanar circular inner and outer motions. When $C = L_1 + L_2$ the point is not properly a singularity as it is the entire space.

(h) The points $(-L_1^2, 2C^2 - L_2^2, \sigma_3, 0, 0, 0)$ with $\sigma_3 \in [-2L_1C, 2L_1C]$. They stand for rectilinear motions of the inner bodies having any inclination I_1 in $[0, \pi]$ while the outer body remains in the invariable plane. These points are the ones given in (2.41).

(i) The points $(2C^2 - L_1^2, -L_2^2, 0, \sigma_4, 0, 0)$ with $\sigma_4 \in [-2L_2C, 2L_2C]$. They represent rectilinear motions of the outer body with an inclination I_2 in $[0, \pi]$ whereas the inner bodies move in the invariable plane.

(j) The points $(L_1^2 - \alpha(L_2^2 - \sigma_2), \sigma_2, 0, 0, 0, 0)$ where the constant α is related with σ_2 through

$$\alpha = -1 - \frac{2(C^2 - L_1^2 + \sigma_2 \pm \sqrt{2}C\sqrt{L_2^2 + \sigma_2})}{L_2^2 - \sigma_2},$$

with $\alpha > 0$ and $-L_2^2 \leq \sigma_2 < L_2^2$. These points account for coplanar motions of the three bodies such that the ellipses have any eccentricity in $(0, 1]$.

Cases (c), (f), (h) and (i) are segments in $\mathcal{S}_{L_1,L_2,C}$ and (j) is a curve, so they are not isolated singularities. Besides, cases (c), (d), (i) and (j) when $\sigma_2 = -L_2^2$ are excluded from our analysis because they represent collisions of the outer body with the centre of mass of the inner bodies. Finally (a) is a particular case of (h) with $\sigma_3 = \pm 2L_1C$ while (d) is obtained from (i) when $\sigma_4 = \pm 2L_2C$. Thus (b), (e) and (g) when $C = L_2 - L_1$ are the only isolated singular points of $\mathcal{S}_{L_1,L_2,C}$ such that the outer body does not follow a straight line.

2.4.3 The space $\mathcal{R}_{L_1,L_2,B}$

For convenience we introduce the reduced space $\mathcal{R}_{L_1,L_2,B}$ that is an intermediate space between $\mathcal{A}_{L_1,L_2}$ and $\mathcal{S}_{L_1,L_2,C}$ whose dimension is six. It is associated to the reduction by the symmetry B. We use it in Chapters 4, 5 and 6 in order to study the equilibrium related to circular and coplanar motions of the inner and outer fictitious bodies such that $C \neq B$ and rectilinear motions of the inner fictitious bodies which are perpendicular to the invariable plane and $C \neq B$. The space is built using polynomial invariants as we do in the construction of the space $S_{L_1,L_2,C}$. Specifically, starting with an arbitrary expression of a polynomial in terms of the Keplerian invariants we determine the polynomials that are invariant with respect to B. The set of invariants of degree one and two in terms of a_i, b_i, c_i and d_i that we choose is:

$$
\begin{aligned}
\rho_1 &= a_3, \quad \rho_2 = b_3, \quad \rho_3 = c_3, \quad \rho_4 = d_3,\\
\rho_5 &= a_2b_1 - a_1b_2, \quad \rho_6 = a_1b_1 + a_2b_2, \quad \rho_7 = a_2c_1 - a_1c_2,\\
\rho_8 &= a_1c_1 + a_2c_2, \quad \rho_9 = a_2d_1 - a_1d_2, \quad \rho_{10} = a_1d_1 + a_2d_2,\\
\rho_{11} &= b_2c_1 - b_1c_2, \quad \rho_{12} = b_1c_1 + b_2c_2, \quad \rho_{13} = b_2d_1 - b_1d_2,\\
\rho_{14} &= b_1d_1 + b_2d_2, \quad \rho_{15} = c_2d_1 - c_1d_2, \quad \rho_{16} = c_1d_1 + c_2d_2.
\end{aligned}
\tag{2.43}
$$

We have constructed a Gröbner basis with the sixteen invariants with respect to the Keplerian invariants and checked that all the invariants computed up to degree four belong to the ideal defined by the selected invariants using the multivariate division algorithm. This fact suggests that the invariants of any degree can be expressed in terms of the sixteen invariants using the computed Gröbner basis. Since we know that the dimension of $\mathcal{R}_{L_1,L_2,B}$ is six, we need ten functionally independent constraints among the ρ_i. A set of independent relations, that is, the syzygies, with lowest possible degree is:

$$
\begin{aligned}
&\rho_1 + \rho_2 + \rho_3 + \rho_4 = 2B,\\
&(\rho_1^2 - L_1^2)(\rho_2^2 - L_1^2) - \rho_5^2 - \rho_6^2 = 0, \quad (\rho_1^2 - L_1^2)(\rho_3^2 - L_2^2) - \rho_7^2 - \rho_8^2 = 0,\\
&(\rho_1^2 - L_1^2)(\rho_4^2 - L_2^2) - \rho_9^2 - \rho_{10}^2 = 0, \quad (\rho_2^2 - L_1^2)(\rho_3^2 - L_2^2) - \rho_{11}^2 - \rho_{12}^2 = 0,\\
&\rho_5\rho_{15} - \rho_8\rho_{14} + \rho_{10}\rho_{12} = 0, \quad \rho_5\rho_{16} + \rho_8\rho_{13} - \rho_9\rho_{12} = 0,\\
&\rho_6\rho_{15} - \rho_8\rho_{13} + \rho_{10}\rho_{11} = 0, \quad \rho_6\rho_{16} - \rho_8\rho_{14} - \rho_9\rho_{11} = 0,\\
&\rho_7\rho_{14} + \rho_8\rho_{13} - \rho_9\rho_{12} - \rho_{10}\rho_{11} = 0.
\end{aligned}
\tag{2.44}
$$

We have checked that this collection of constraints is functionally independent. It implies that the set (2.43) forms a fundamental set of generators which is also a Hilbert basis. Thus we define the reduced space as the set

$$
\mathcal{R}_{L_1,L_2,B} = \Big\{(\rho_1,\ldots,\rho_{16}) \in \mathbb{R}^{16} \;\mid\; \text{the invariants } \rho_i \text{ satisfy (2.44)}\Big\}. \tag{2.45}
$$

This space has singularities. The reason is that it is obtained through an axially-symmetric type of reduction that fixes some points of the phase space. The fixed points become singularities of the reduced space. Thus the set $\mathcal{R}_{L_1,L_2,B}$ is also a symplectic orbifold. We do not study all its singular points since we are interested in a few specific points.

This set of polynomial invariants in terms of a_i, b_i, c_i and d_i is given by

$$\begin{array}{lll} \rho_1 = a_3, \quad \rho_2 = b_3, \quad \rho_3 = c_3, \quad \rho_4 = d_3, \\ \rho_5 = a_2 b_1 - a_1 b_2, \quad \rho_6 = a_1 b_1 + a_2 b_2, \quad \rho_7 = a_2 c_1 - a_1 c_2, \\ \rho_8 = a_1 c_1 + a_2 c_2, \quad \rho_9 = a_2 d_1 - a_1 d_2, \quad \rho_{10} = a_1 d_1 + a_2 d_2, \\ \rho_{11} = b_2 c_1 - b_1 c_2, \quad \rho_{12} = b_1 c_1 + b_2 c_2, \quad \rho_{13} = b_2 d_1 - b_1 d_2, \\ \rho_{14} = b_1 d_1 + b_2 d_2, \quad \rho_{15} = c_2 d_1 - c_1 d_2, \quad \rho_{16} = c_1 d_1 + c_2 d_2, \end{array} \tag{2.46}$$

The space (2.45) can be parametrised in terms of Deprit's action-angle coordinates similarly to the other reduced spaces. This is achieved by using (2.46) and expressions of a_i, b_i, c_i and d_i given in Appendix A in terms of L_1, L_2, B, C, G_1, G_2 and the angles ν_1, γ_1 and γ_2. Note that at this stage L_1, L_2 and B are constants of motion whereas the rest of the coordinates vary. The parametrisation is not valid when $G_1 = 0$ but then it can be arranged in an analogous way as it is done for $\mathcal{T}_{L_1,C,G_2}$ and $\mathcal{S}_{L_1,L_2,C}$.

Chapter 3

Relative equilibria, stability and bifurcations of the fully-reduced system

In the previous chapters we have obtained the perturbation $\mathcal{K}_1$ expressed in terms of the invariants τ_i $(i = 1, 2, 3)$ of the fully-reduced space, that is, the fully-reduced Hamiltonian (2.34). In this chapter we compute the equations of motion corresponding to the fully-reduced problem, classifying the relative equilibria, studying their stability and the bifurcations in terms of the two relevant parameters of the problem. One of our aims is to clarify the dynamics of the fully-reduced system related to the singular points of $\mathcal{T}_{L_1,C,G_2}$, because previous results [53, 34, 28, 90] dealing with the qualitative analysis of the flow in the fully-reduced space do not take into account the singular character of the reduction process.

3.1 Equations of motion

Starting from (2.34), after dropping constant terms and scaling time we arrive at the Hamiltonian function

$$\mathcal{K}_1 \;=\; 2(-L_1^2 + 2C^2 + 6G_2^2)\tau_1 - \tau_1^2 + 20\tau_2^2, \tag{3.1}$$

which is the *fully-reduced Hamiltonian*. The vector field associated to $\mathcal{K}_1$ is obtained as follows:

$$\begin{aligned}
\dot{\tau}_1 &= \{\tau_1\,,\,\mathcal{K}_1\} \;=\; -160\tau_2\tau_3,\\
\dot{\tau}_2 &= \{\tau_2\,,\,\mathcal{K}_1\} \;=\; -8(\tau_1 + L_1^2 - 2C^2 - 6G_2^2)\tau_3,\\
\dot{\tau}_3 &= \{\tau_3\,,\,\mathcal{K}_1\} \;=\; 2\tau_2\Big((\tau_1 + L_1^2)(-13\tau_1 + 7L_1^2) + 20\tau_2^2 + 4C^2(9\tau_1 - L_1^2)\\
&\qquad\qquad + 4G_2^2(7\tau_1 - 3L_1^2 + 10C^2) - 20(C^4 + G_2^4)\Big),
\end{aligned} \tag{3.2}$$

that together with the constraint (2.32)

$$(\tau_1 - L_1^2)\big((\tau_1 + L_1^2 - 2C^2 - 2G_2^2)^2 - 16C^2G_2^2\big) \;=\; 4(\tau_1 + L_1^2)\tau_2^2 + 8\tau_3^2$$

gives the fully-reduced Hamiltonian system of the spatial three-body problem in the space $\mathcal{T}_{L_1,C,G_2}$. It depends on the three parameters L_1, C and G_2 but L_1 can be absorbed as follows. Introduce p and q by $p = C/L_1$, $q = G_2/L_1$ and scale τ_i by defining $\bar{\tau}_1 = \tau_1/L_1^2$, $\bar{\tau}_2 = \tau_2/L_1^2$ and $\bar{\tau}_3 = \tau_3/L_1^3$. After dividing t by L_1^3, we get:

$$\begin{aligned}
\dot{\bar{\tau}}_1 &= -160\bar{\tau}_2\bar{\tau}_3,\\
\dot{\bar{\tau}}_2 &= -8(\bar{\tau}_1 - 2p^2 - 6q^2 + 1)\bar{\tau}_3,\\
\dot{\bar{\tau}}_3 &= 2\bar{\tau}_2\Big((\bar{\tau}_1 + 1)(-13\bar{\tau}_1 + 7) + 20\bar{\tau}_2^2 + 4p^2(9\bar{\tau}_1 - 1)\\
&\qquad + 4q^2(7\bar{\tau}_1 + 10p^2 - 3) - 20(p^4 + q^4)\Big),
\end{aligned} \tag{3.3}$$

and

$$(\bar{\tau}_1 - 1)\big((\bar{\tau}_1 - 2p^2 - 2q^2 + 1)^2 - 16p^2q^2\big) \;=\; 4(\bar{\tau}_1 + 1)\bar{\tau}_2^2 + 8\bar{\tau}_3^2. \tag{3.4}$$

The Hamiltonian associated to (3.3) is

$$\bar{\mathcal{K}}_1 \;=\; 2(2p^2 + 6q^2 - 1)\bar{\tau}_1 - \bar{\tau}_1^2 + 20\bar{\tau}_2^2, \tag{3.5}$$

The parameters p and q — essentially the integrals C and G_2 — become the main constants used to achieve the analysis of (3.3). Because of the last scaling performed, from now on in this chapter we drop the first term in the fully-reduced space, identifying $\mathcal{T}_{1,p,q}$ with $\mathcal{T}_{p,q}$.

From the fact that $|C - G_2| \leq L_1$ one has that $|p - q| \leq 1$. The set $\mathcal{T}_{p,q}$ is a surface if $|p-q| < 1$ and a mere point if $|p-q| = 1$, so we restrict ourselves to the case $|p - q| < 1$. The inequalities $p > 0$ and $q > 0$ also hold. On the other hand as $G_1 \leq \min\{L_1, C + G_2\}$ then $\eta_1 \leq \min\{1, p + q\}$.

3.2 Relative equilibria

Now, the first conclusions are the following:

(a) If $p + q \geq 1$ the point $(1, 0, 0)$ is an equilibrium of the space $\mathcal{T}_{p,q}$ defined through (3.4) since it satisfies the equations obtained by equating the right-hand sides of the equations of (3.3) to zero. This point represents motions of circular type. If $p + q < 1$ the point $(2(p + q)^2 - 1, 0, 0)$ is an equilibrium of the reduced space as it satisfies (3.3). This point represents coplanar motions with the inner bodies having their lowest possible eccentricity.

(b) The point $(2(p-q)^2-1,0,0)$ is always an equilibrium of (3.4) related with the coplanar motions of the three bodies. In this case $\eta_1 = |p-q|$.

(c) Rectilinear motions of the inner bodies are treated in the straight line $p = q$ of the plane of parameters. The points $(-1, \pm 2q, 0)$ and $(-1, 0, 0)$ are always equilibria. On the one hand $(-1, \pm 2q, 0)$ refer to straight lines in the direction of the vector $\mathbf{C}$ whereas $(-1, 0, 0)$ refers to coplanar motions of the three bodies where the inner bodies move on straight lines. No other points of rectilinear type are equilibria of the system (3.3).

(d) In the cases (b) and (c) whenever $C = |B|$ the invariable plane coincides with the plane spanned by $\mathbf{i}$ and $\mathbf{j}$.

(e) The case $C = G_2$, $L_1 = 2G_2$ is reflected in the parameter space by the point $(p,q) = (1/2, 1/2)$. In this point the space $\mathcal{T}_{1/2,1/2}$ has at least three relative equilibria, e.g. the three singularities, i.e. the points $(1,0,0)$, $(-1,\pm 1, 0)$.

There are always at least two equilibria, the points $(2(p-q)^2-1,0,0)$ and either $(1,0,0)$ or $(2(p+q)^2-1,0,0)$. The discussion about the different relative equilibria as functions of p and q is as follows. According to the first equation of (3.3), $\bar{\tau}_2$ or $\bar{\tau}_3$ must vanish. If both are zero at the same time the equilibria are the ones mentioned in (a), (b) and (c). If $\bar{\tau}_2 = 0$ and $\bar{\tau}_3 \neq 0$ then $\bar{\tau}_1 = 2p^2 + 6q^2 - 1$ and it is valid when $(p-q)^2 \leq p^2 + 3q^2 \leq \min\{1, (p+q)^2\}$. The corresponding value of $\bar{\tau}_3$ is deduced from (3.4), giving $\bar{\tau}_3 = \pm 2q\sqrt{(p^2-q^2)(1-p^2-3q^2)}$, obtaining up to two new points. If $\bar{\tau}_3 = 0$ and $\bar{\tau}_2 \neq 0$, the values of $\bar{\tau}_1$ and $\bar{\tau}_2$ are obtained from the third equation of (3.3) and (3.4). Concretely $\bar{\tau}_1$ would be obtained as a root of a polynomial equation of degree three whose coefficients depend on p and q while $\bar{\tau}_2$ would be computed from the value obtained for $\bar{\tau}_1$ from a quadratic equation. Thus, we could get up to six equilibria in the plane $\bar{\tau}_3 = 0$, however the maximum number of equilibria on this plane is four — discounting the points on the principal axes. There are no other equilibria outside the principal planes $\bar{\tau}_2 = 0$ and $\bar{\tau}_3 = 0$. At this point we are not interested in calculating explicitly the expressions of the relative equilibria, claiming that the maximum number of equilibria of the vector field (3.3) is bounded by six. We shall be more precise below.

3.3 Stability and bifurcations

3.3.1 Non-coplanar circular solutions

Now we want to obtain the bifurcation lines. We start with the non-coplanar circular solutions, introducing the symplectic change:

$$x = \sqrt{2(L_1 - G_1)}\cos\gamma_1, \quad y = \sqrt{2(L_1 - G_1)}\sin\gamma_1.$$

This transformation extends analytically to the origin of the xy-plane, provided that x and y are written in terms of γ_1 and G_1 and all the computations that we have to carry out satisfy the d'Alembert characteristic; see details in [42].

Replacing γ_1 and G_1 in (2.17) and expanding the result in powers of x and y we obtain a Taylor series whose 1-jet is zero. After dropping the constant terms and scaling the Hamiltonian the 2-jet gives:

$$\bar{\mathcal{K}}_1^{\text{2-jet},L_1} = 2(p^2+3q^2-1)x^2 + \big(5(p^4+q^4) - 2p^2(5q^2+4) - 4q^2+3\big)y^2. \quad (3.6)$$

When one of the factors of x^2 and y^2 (or the two) vanishes we obtain possible bifurcation lines from the circular solutions. Besides the stability of the equilibrium $(1,0,0)$ is obtained from the signs of the factors of x^2 and y^2. When both signs coincide the equilibrium is a centre otherwise it is a saddle.

3.3.2 Coplanar solutions

Concerning the coplanar motions — with the additional conditions $C \neq G_2$ and $G_1 \neq L_1$, i.e., discarding the coplanar motions that are also rectilinear or circular — we have two options. Either $G_1 = |C - G_2|$ or $G_1 = C + G_2$. When $G_1 = |C - G_2|$ we introduce the symplectic change:

$$x = \sqrt{2(G_1 - |C-G_2|)}\,\sin\gamma_1, \quad y = \sqrt{2(G_1 - |C-G_2|)}\,\cos\gamma_1.$$

As in the circular case, the same considerations about its analyticity hold for this transformation.

Taking into account that $C \neq G_2$ implies $p \neq q$, we can simplify the 2-jet by including a multiplication by $|p-q|$. Then,

$$\bar{\mathcal{K}}_1^{\text{2-jet},|C-G_2|} = \big(-4p^3+9p^2q+q^3-p(6q^2-5)\big)x^2 + (p-q)^2(p+q)y^2. \quad (3.7)$$

From this expression we obtain possible bifurcation lines and determine the stability of the point $(2(p-q)^2-1,0,0)$: if the factors of x^2 and y^2 have the same sign the point is a linear centre, otherwise it is a saddle.

For $G_1 = C + G_2$ we make the symplectic transformation:

$$x = \sqrt{2(C+G_2-G_1)}\,\cos\gamma_1, \quad y = \sqrt{2(C+G_2-G_1)}\,\sin\gamma_1.$$

This change extends analytically to the origin of the xy-plane, as in the previous cases it satisfies the d'Alembert characteristic.

The corresponding 2-jet is:

$$\bar{\mathcal{K}}_1^{\text{2-jet},C+G_2} = (q-p)(p+q)^2x^2 + \big(4p^3+9p^2q+q^3+p(6q^2-5)\big)y^2. \quad (3.8)$$

Thus we get two more possible bifurcation lines. Besides, the point $(2(p+q)^2-1,0,0)$ is stable (a centre) provided that the terms factorising x^2 and y^2 have the same sign, whereas it is a saddle if they have opposite signs.

Examining the curves of the three 2-jets we have computed we discard the lines obtained from the coefficients of x^2 in (3.7) and (3.8) as they do not give any curve in the valid domain for p and q. Furthermore, from the coefficient of y^2 in (3.7) we extract the straight line $p = q$.

3.3.3 Rectilinear solutions

We consider the possibility of bifurcations of the points representing rectilinear motions. As these motions satisfy $G_2 = C$ we set in (3.3) $p = q$. The resulting equations have four solutions for all $q > 0$. The points of $\mathcal{T}_{q,q}$ are $(-1,0,0)$, $(-1,\pm 2q,0)$ and $(8q^2-1,0,0)$, the first three accounting for rectilinear motions, while the last one accounts for circular motions such that the three bodies move on the invariable plane. Therefore the number of equilibria related to rectilinear motions is unaltered when q varies, i.e. no bifurcation of this type is expected.

Now we focus on the stability analysis of the three relative equilibria of rectilinear type in $\mathcal{T}_{q,q}$. We shall see below that the three points are linear centres.

As said before, Hamiltonian (2.17) extends analytically to the case $G_2 = C$ and $G_1 = 0$. Specifically we get

$$\mathcal{K}_1^r = \frac{\mathcal{M}L_1^2}{L_2^3C^5}\Big((4C^2-3G_1^2)(5L_1^2-3G_1^2)-15(4C^2-G_1^2)(L_1^2-G_1^2)\cos 2\gamma_1\Big). \quad (3.9)$$

The angle γ_1 is undefined for rectilinear motions of the inner bodies but by means of the analytical extension we could consider that it makes sense as an angle that measures the inclination of the line described by the inner particles with the invariable plane. Specifically, the points $(-1,\pm 2q,0)$ dealing with rectilinear motions of the inner particles in the direction of the vector $\mathbf{C}$ satisfy either $\gamma_1 = \pi/2$ (prograde motions) or $3\pi/2$ (retrograde motions) whereas the point $(-1,0,0)$ is related with rectilinear motions of the inner bodies which are coplanar with the outer body and in this case $\gamma_1 = 0$. Another issue to take into consideration is that while the point $(-1,0,0)$ represents a regular point of the surface $\mathcal{T}_{q,q}$, $(-1,\pm 2q,0)$ are singular points of this surface, thus for these latter points we need to work with the polynomial invariants τ_i and desingularise locally the surface $\mathcal{T}_{q,q}$ around these two points.

Concerning the point $(-1,0,0)$ we introduce the trivial symplectic change

$$\gamma_1 = x, \quad G_1 = y. \quad (3.10)$$

This transformation may be extended analytically to $(x,y) = (0,0)$ and all the expressions and computations satisfy the d'Alembert characteristic, see [42, 62].

Thence it is well defined and makes sense for rectilinear motions of the inner bodies which are coplanar with the outer body.

Applying (3.10) to (3.9), after dropping constant terms and scaling the time, the 2-jet is

$$\bar{\mathcal{K}}_1^{\text{2-jet},r} = 5L_1^2x^2 + 2y^2. \tag{3.11}$$

As the coefficients of x^2 and y^2 are positive the point $(-1,0,0)$ is a linear centre and does not bifurcate.

To study the stability of the points $(-1,\pm 2q,0)$ in $\mathcal{T}_{q,q}$ we introduce the symplectic changes

$$\gamma_1 = x+\frac{\pi}{2}, \quad G_1 = y, \tag{3.12}$$

for $(-1,2q,0)$ and

$$\gamma_1 = x+\frac{3\pi}{2}, \quad G_1 = y. \tag{3.13}$$

for the point $(-1,-2q,0)$. These transformations can be extended analytically to the point $(x,y)=(0,0)$ but we cannot just apply them in (3.9) because the resulting Hamiltonian is not differentiable at the origin. What we do is to compose the changes (3.12) and (3.13) with the parametrisation (2.36) arriving at:

$$\begin{aligned}
\tau_1 &= 2y^2 - L_1^2,\\
\tau_2 &= \pm\sqrt{4C^2-y^2}\sqrt{L_1^2-y^2}\cos x,\\
\tau_3 &= \mp y\sqrt{4C^2-y^2}\sqrt{L_1^2-y^2}\sin x.
\end{aligned} \tag{3.14}$$

The upper sign applies for $(-1,2q,0)$ whereas the lower one is used for $(-1,-2q,0)$. Both transformations (3.14) are properly defined and make sense for rectilinear (and near-rectilinear) motions occurring in the axis orthogonal to the invariable plane.

The changes (3.14) desingularise the surface $\mathcal{T}_{L_1,C,C}$ locally around the two singular points related to the rectilinear motions, equivalently, they desingularise $\mathcal{T}_{q,q}$ locally around the points $(-1,\pm 2q,0)$. Indeed, for the two transformations, the constraint (2.32) with $G_2=C$ in the $\tau_1\,x\,y$-space reads as

$$(8C^2 - L_1^2 - \tau_1)(L_1^4 - \tau_1^2) + 8y^2(L_1^2-y^2)(y^2-4C^2) = 0,$$

and this transformed surface is smooth around the point $(\tau_1,x,y)=(-L_1^2,0,0)$ since its gradient does not vanish at this point.

We apply (3.14) to Hamiltonian (2.34) with $G_2=C$. After dropping constant terms and scaling the time, the 2-jet yields in both cases

$$\bar{\mathcal{K}}_1^{\text{2-jet},r} = 20L_1^2p^2x^2 + (5+12p^2)y^2. \tag{3.15}$$

The coefficients of x^2 and y^2 are always positive which in turn implies that the equilibria $(-1, \pm 2q, 0)$ are centres, thus elliptic points.

Summarising the previous paragraphs, the points $(-1, \pm 2q, 0)$ and $(-1, 0, 0)$ of $\mathcal{T}_{q,q}$ (equivalently, the points $(-L_1^2, \pm 2L_1C, 0)$ and $(-L_1^2, 0, 0)$ of $\mathcal{T}_{L_1,C,C}$) corresponding to rectilinear motions of the inner bodies are always elliptic and never bifurcate.

3.3.4 Other bifurcations

There is another source from where bifurcation sets can arise. Leaving apart the circular, coplanar and rectilinear motions for the inner particles we may get a bifurcation from the fact that a single equilibrium point of $\mathcal{T}_{p,q}$ can become multiple.

Working in Deprit's coordinates one determines the vector field in the pair γ_1-G_1, that is, $(\dot{\gamma}_1, \dot{G}_1) = (\partial\mathcal{K}_1/\partial G_1, -\partial\mathcal{K}_1/\partial\gamma_1)$. The corresponding relative equilibria are the points (γ_1^0, G_1^0) such that $\dot{\gamma}_1(\gamma_1^0, G_1^0) = \dot{G}_1(\gamma_1^0, G_1^0) = 0$. As we are discarding $G_1 = 0$, $G_1 = |C - G_2|$, $G_1 = C + G_2$, $G_1 = L_1$, then $\dot{\gamma}_1 = 0$ if and only if $\gamma_1 \in \{0, \pi/2, \pi, 3\pi/2\}$. Thence the polynomial that must vanish when evaluated at G_1^0 is obtained from the partial derivative of $\mathcal{K}_1$ with respect to G_1. We end up with:

$$\begin{aligned} s_2(G_1) &= G_1^2 - L_1^2(p^2 + 3q^2) \quad \text{for } \gamma_1 = 0,\ \pi, \\ s_6(G_1) &= 8G_1^6 - L_1^2 G_1^4(8p^2 + 4q^2 + 5) + 5L_1^6(p^2 - q^2)^2 \quad \text{for } \gamma_1 = \pi/2,\ 3\pi/2. \end{aligned}$$

The valid roots of $s_2 = 0$ and $s_6 = 0$ lead to equilibria of $\mathcal{T}_{p,q}$ that are not of rectilinear, coplanar or circular motions. To analyse the possibility that a single root exploits into multiple roots for some combinations of p and q we compute the resultants between s_2 and ds_2/dG_1 and between s_6 and ds_6/dG_1. For s_2 it is $p^2 + 3q^2$ which never vanishes, so we discard the choice $\gamma_1 \in \{0, \pi\}$ to obtain a bifurcation line. For the polynomial s_6 the relevant term of the resultant is the polynomial

$$\begin{aligned} &\mathrm{Res}^*\left(s_6, \frac{ds_6}{dG_1}\right) = \\ &\quad (4p^2 - 5)^2(32p^2 + 5) + 12(64p^4 + 440p^2 + 25)q^2 + 384(p^2 - 5)q^4 + 64q^6. \end{aligned}$$

3.3.5 Plane of bifurcations

Description

Collecting what we have said in the previous paragraphs we know that the bifurcation lines are the following curves (or parts of these curves):

$$
\begin{aligned}
\Gamma_1 &\equiv p^2 + 3q^2 = 1, \\
\Gamma_2 &\equiv 5p^4 + 5q^4 - 2p^2(5q^2 + 4) - 4q^2 + 3 = 0, \\
\Gamma_3 &\equiv p = q, \\
\Gamma_4 &\equiv 4p^3 + 9p^2q + q^3 + p(6q^2 - 5) = 0, \\
\Gamma_5 &\equiv (4p^2 - 5)^2(32p^2 + 5) + 12(64p^4 + 440p^2 + 25)q^2 + 384(p^2 - 5)q^4 \\
&\quad + 64q^6 = 0.
\end{aligned}
\tag{3.16}
$$

The information about the bifurcation lines and different regions is encapsulated in Fig. 3.1. Some of the features about the relative equilibria and stability are based on numerical calculations. The computations are tedious as there are many regions, but we have checked all types of equilibria in each region and their stability character.

The lines $|p - q| = 1$.

Drawn in blue, then do not represent real bifurcation lines as the reduced space gets reduced to a point along them. More precisely, the fully-reduced spaces $\mathcal{T}_{p,q}$ get smaller and smaller when p and q are such that $|p - q|$ approaches to 1 from the permitted values of p and q.

The line $p + q = 1$.

Drawn in light green, it is not a bifurcation line but it separates two different regimes. When $p + q \geq 1$ the circular solutions are allowed as $G_1 \leq L_1 \leq C + G_2$ holds and the corresponding point $(1, 0, 0)$ is an equilibrium of $\mathcal{T}_{p,q}$. If $p + q < 1$ then $C + G_2 < L_1$ and the motion of the inner bodies cannot be circular. In this case the lowest eccentricity reached by these bodies is $\sqrt{1 - (p + q)^2}$, which occurs at the point $(2(p+q)^2 - 1, 0, 0)$. If $p+q = 1$ then $\mathcal{T}_{1-q,q}$ has a singularity at $(1, 0, 0)$.

The line $p = q$ *with* $p + q > 1$.

Also drawn in light green, it is not a bifurcation curve, as the number of relative equilibria of the spaces $\mathcal{T}_{q,q}$ is four and this number does not vary if $p > q$ or $p < q$.

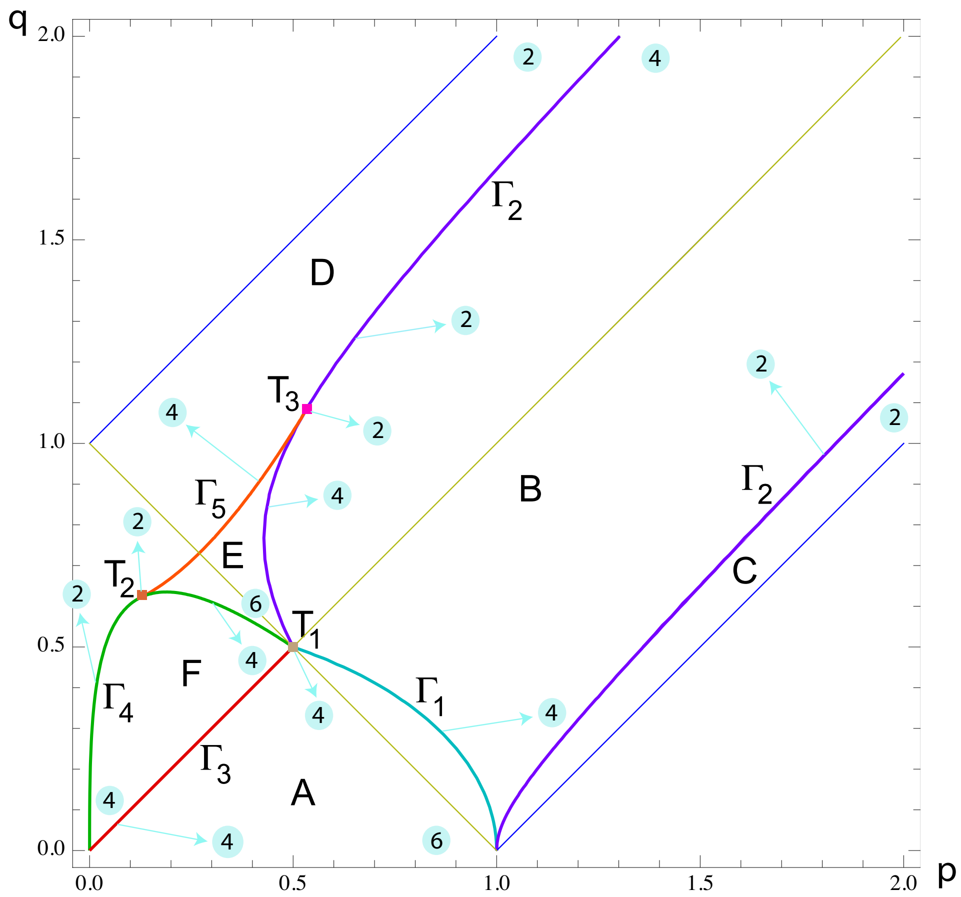

Figure 3.1: Plane of parameters with the bifurcation lines and the number of relative equilibria in each region, bifurcation line and special point. The corresponding fully-reduced spaces $\mathcal{T}_{p,q}$ are diffeomorphic to S^2 outside the lines $p+q=1$ and $p=q$.

Thus, this straight line is part of the region B. Nevertheless if $p=q$, the singular points $(-1, \pm 2q^2, 0)$, referring to motions of the inner bodies on straight lines perpendicular to the invariable plane, are relative equilibria. Besides, when $p=q$ the point $(-1,0,0)$ is also an equilibrium representing rectilinear motions of the inner bodies moving on the invariable plane.

The curve Γ_3, *i.e.* $p=q$ *with* $p+q \leq 1$.

It is a bifurcation line, as it separates the region A (with six equilibria) from the region F, that has four equilibria. The number of equilibria on the line Γ_3 is also four. When crossing from A to F through Γ_3, two points in the plane $\bar{\tau}_2 = 0$

and the point $(2(p+q)^2-1,0,0)$ collapse into this latter point. This is the typical scenario of a Hamiltonian pitchfork bifurcation of equilibrium points related to the coplanar motions with $G_1 = C + G_2$. For a pair of symplectic coordinates, α and β, the normal form is:

$$K_\lambda(\alpha,\beta) \;=\; \tfrac{1}{2}\beta^2 - \tfrac{1}{4}\alpha^4 + \lambda\alpha^2, \tag{3.17}$$

reflecting the fact that a saddle in the region F splits into a centre and two saddles. Thus, the saddle of region F is the point $(2(p+q)^2-1,0,0)$, that becomes a centre once in region A.

Region B.

It has four equilibria, the same as the curve Γ_1. Indeed, Γ_1 is a bifurcation line of the point $(1,0,0)$. This point and two more points in the plane $\bar{\tau}_2 = 0$ collapse into the point $(1,0,0)$ when crossing from A to B through Γ_1. This is again a pitchfork bifurcation of an equilibrium point related with circular motions. The normal form is (3.17) and a saddle in region B splits into a centre and two saddles in region A. The point which changes from a saddle in B to a centre in A is again $(1,0,0)$.

Line Γ_2.

It appears in two pieces. When crossing from region B to C through Γ_2 another bifurcation line is crossed. There are two equilibria in region C and on the line Γ_2, one corresponding to the circular motions $(1,0,0)$ and the other one corresponding to the coplanar motions of the type $(2(p-q)^2-1,0,0)$. When crossing from C to B through Γ_2, the point related to circular motions bifurcates into three points, the bifurcation being of pitchfork type. The same situation occurs when passing from B to D through Γ_2, since D has two relative equilibria of the same type as C, thence a pitchfork bifurcation takes place. Specifically the point $(1,0,0)$ is a centre in regions C and D and it splits into a saddle (the same point) and two centres when crossing Γ_2 to enter region B. The normal form is:

$$K_\lambda(\alpha,\beta) \;=\; \tfrac{1}{2}\beta^2 + \tfrac{1}{4}\alpha^4 + \lambda\alpha^2. \tag{3.18}$$

Nevertheless, the passage from B to E through Γ_2 is different. The point $(1,0,0)$ also experiences a Hamiltonian pitchfork bifurcation, but it is a saddle in region B that splits into a centre (the same point) and two saddles when crossing Γ_2 to enter region E, the normal form being in this case (3.17). There are four equilibria on Γ_2 when the curve is between T_1 and T_3.

Region E.

It contains six equilibria, four of them in the plane $\bar{\tau}_3 = 0$. These four points are obtained from the roots of the polynomial s_6. The transition between E and D through Γ_5 is different from the bifurcations explained so far. We recall that in D there are two equilibria, but on the line Γ_5 the number of equilibria is four. This is a (Hamiltonian) saddle-centre bifurcation of the points in the plane $\bar{\tau}_3 = 0$ occurring in pairs. In region E, each pair of a saddle and a centre in the plane $\bar{\tau}_3 = 0$ collapses on the line Γ_5 and disappear once in D. This situation, already described in [43], is called a double saddle-centre bifurcation and the related normal form is:

$$K_\lambda(\alpha, \beta) \;=\; \tfrac{1}{2}\beta^2 - \tfrac{1}{3}\alpha^3 + \lambda\alpha. \tag{3.19}$$

Line Γ_4.

It represents a pitchfork bifurcation of $(2(p+q)^2 - 1, 0, 0)$, which is the point related to coplanar motions with $G_1 = C + G_2$. In region F there are four equilibria, namely, $(2(p-q)^2 - 1, 0, 0)$, $(2(p+q)^2 - 1, 0, 0)$, and two other points in the plane $\bar{\tau}_3 = 0$. Considering the passage through the piece of Γ_4 between T_1 and T_2, the point $(2(p+q)^2 - 1, 0, 0)$, as said above, is a saddle in region F that splits into a centre (the same point) and two saddles when it enters region E. On this part of the curve Γ_4 there are also four equilibria and the corresponding normal form is the one in (3.17). Nevertheless, the transition between F and D is different. The other two points in F that are in $\bar{\tau}_3 = 0$ are centres and, together with $(2(p+q)^2 - 1, 0, 0)$, merge when crossing Γ_4 between the points T_2 and $(0, 0)$, becoming the resulting point a centre in D. The normal form is (3.18) and on this part of Γ_4 there are two relative equilibria.

The point $T_1 = (1/2, 1/2)$.

It is the intersection of the lines $p + q = 1$ and Γ_3 and corresponds to the case where the fully-reduced space has been coined as a tricorn, that is, the space has three singular points, as already mentioned in (e). Besides, the space has a fourth equilibrium, the point $(-1, 0, 0)$, that corresponds to rectilinear motions of the inner bodies in the invariable plane. It is straightforward to check that the three points representing the rectilinear motions are linear centres, whereas $(1, 0, 0)$ is degenerate but has to be unstable — in order to maintain the Poincaré index to two. It deserves a further analysis, basically one needs to desingularise locally the surface $\mathcal{T}_{1/2,1/2}$ around $(1, 0, 0)$ in order to get an adequate normal form along the lines of the desingularisation technique used in [33].

In Fig. 3.2 we detail a neighborhood of the point T_1 in the parametric plane where rectilinear motions of the inner bodies occur.

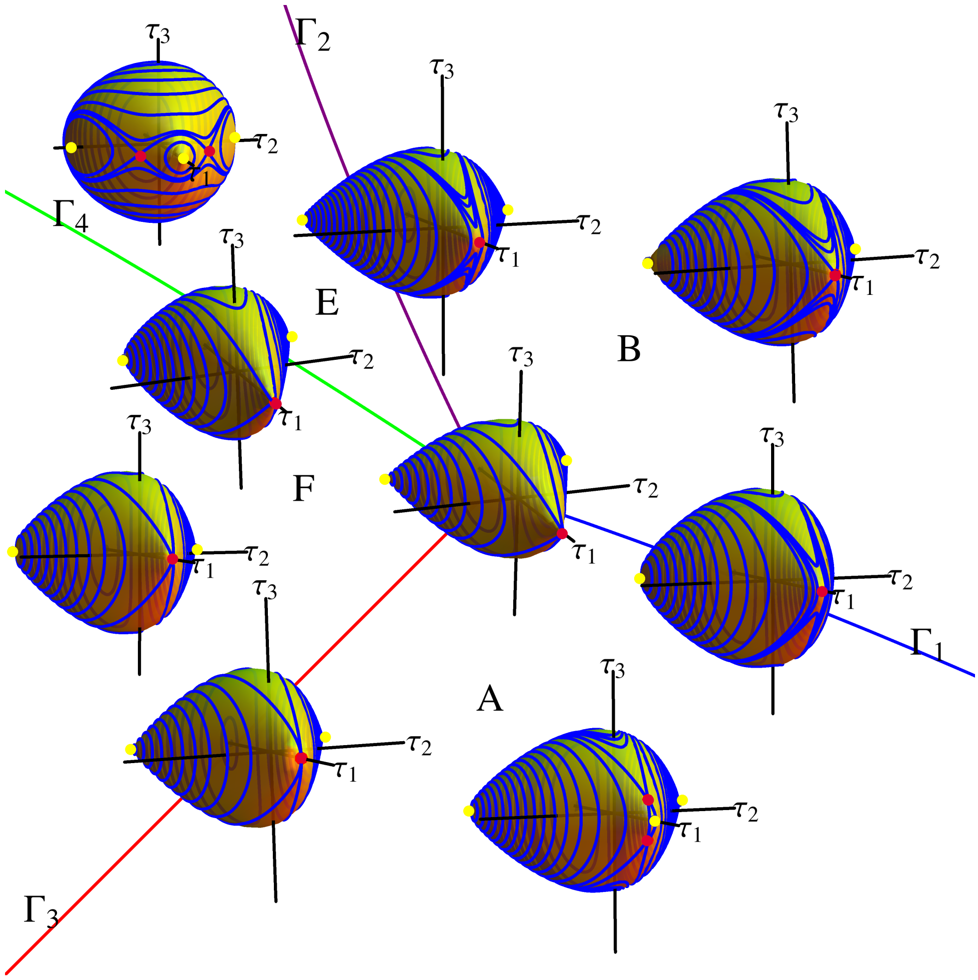

Figure 3.2: A neighborhood of the point T_1 in the plane of parameters. The flow near T_1 with the different regions limited by the bifurcation lines can be seen, all bifurcations being of pitchfork type. The red relative equilibria represent saddles and the yellow ones are centres.

Curve Γ_5.

It is a bifurcation line only between the points T_2 and T_3. In particular, T_2 is obtained as the tangency point between Γ_4 and Γ_5, while T_3 is located at the tangency between Γ_2 and Γ_5. Concretely the coordinates of T_2 and T_3 in the plane of parameters are

$$T_2 = \left(\frac{\sqrt{5}}{18}, \frac{5\sqrt{5}}{18}\right), \qquad T_3 = \left(\frac{1}{2}\sqrt{\frac{35-8\sqrt{5}}{15}}, \sqrt{\frac{35+16\sqrt{5}}{60}}\right).$$

The points T_2 and T_3.

They are typical examples of reversible hyperbolic umbilic bifurcations, described in detail by Hanßmann [37] in a general context. In particular in both points a saddle-centre and a pitchfork bifurcation take place. The associated normal form we have determined is of the type:

$$K_{\lambda,\mu}(\alpha,\beta) = \alpha^2\beta + \tfrac{1}{3}\beta^3 + \lambda(\alpha^2 - \beta^2) + \mu\beta. \tag{3.20}$$

The situation is as follows. At $\mu = \lambda^2$ a centre-saddle bifurcation takes place, giving rise to a centre and a saddle. The latter undergoes at $\mu = -3\lambda^2$ a Hamiltonian pitchfork bifurcation, thereby turning into a centre and giving rise to two saddles. Due to the reversibility the two saddles have the same energy and are connected by heteroclinic solutions, see more details in [37]. See also the theory developed in [38] about bifurcations of equilibria and invariant tori using normal forms theory.

Summary

The number of relative equilibria in each region appears in Fig. 3.1. In region A there are six equilibria, namely, two in the plane $\bar{\tau}_2 = 0$, two in the plane $\bar{\tau}_3 = 0$, $(2(p-q)^2 - 1, 0, 0)$ and $(2(p+q)^2 - 1, 0, 0)$ if $p + q < 1$ or $(1, 0, 0)$ if $p + q \geq 1$. Four of them are centres and the other two are saddles. The saddles are the points located in the plane $\bar{\tau}_2 = 0$ that merge with the centre $(1, 0, 0)$ when the line Γ_1 is crossed. The rest of points are centres. Region B has four equilibria, three of them are centres and the other one is a saddle. The saddle corresponds with the point $(1, 0, 0)$, whereas the centres are $(2(p-q)^2 - 1, 0, 0)$ and the two other points are in the plane $\bar{\tau}_2 = 0$. In region C there are two centres that correspond to the points $(1, 0, 0)$ and $(2(p-q)^2 - 1, 0, 0)$. Region D has also two points (centres), namely, $(2(p-q)^2 - 1, 0, 0)$ and $(1, 0, 0)$ if $p + q \geq 1$ or $(2(p+q)^2 - 1, 0, 0)$ if $p + q < 1$. Region E has six equilibria. Specifically, the points $(2(p-q)^2 - 1, 0, 0)$ and either $(1, 0, 0)$ or $(2(p+q)^2 - 1, 0, 0)$ are centres, whereas the other four equilibria are located in the plane $\bar{\tau}_3 = 0$, two of them being centres and the other two saddles. Region F has four equilibria, the point that bifurcates, i.e. $(2(p+q)^2 - 1, 0, 0)$, is a saddle while the other three equilibria correspond to centres, one point with coordinates $(2(p-q)^2 - 1, 0, 0)$ and the other two, centres in the plane $\bar{\tau}_3 = 0$. The stability character of the relative equilibria obtained in the different regions, considered in the space $\mathcal{T}_{p,q}$, is linear and non-linear for the saddles and for the centres that are not singular points. The linear centres are also non-linear if they correspond to regular points of the fully-reduced space as there are Morse functions given by (3.6), (3.7) and (3.8) — and similarly other Morse functions around the linear centres which are not related to coplanar or rectilinear motions.

We have included in Fig. 3.1 the number of equilibria in the bifurcation lines. The stability character of the equilibria is the same as the equilibria's character of the regions the curves define, excepting those points which give rise to bifurcations, which are indeed degenerate points. The stability of the bifurcating points depends on their normal forms for $\lambda = \mu = 0$ and they are stable in the case (3.18) and unstable in the cases (3.17), (3.19) and (3.20).

Our plane of parameters is very similar to the one obtained by Ferrer and Osácar in [34], but we have amended some of the conclusions of [34], especially those related with the rectilinear motions and the singular points of the fully-reduced phase space. Concretely the north and south pole views of the flow given in Figs. 3 and 4 of [34] (pp. 265 and 266) are distorted for $p = q$ because the singular points of $\mathcal{T}_{p,q}$ are not taken into account in the fully-reduced space of [34]. In addition to that, according to our analysis, the line $p = q$ with $q > 1/2$ is not a bifurcation line although it is in the analysis of Ferrer and Osácar, again the reason is that their space is lack of singular points.

We collect the main features of the bifurcation analysis in the following theorem.

Theorem 3.1. *We consider the spatial three-body problem in the perturbing region defined in Chapter 2 by $\mathcal{Q}_{\varepsilon,n}$ for some $0 < \varepsilon \ll 1$ and $n \in \mathbb{Z}^+$. The fully-reduced Hamiltonian function of the spatial three-body problem is given by (3.1) and their related equations of motions are (3.2) or (3.3). This latter vector field depends on two parameters, p and q, essentially the integrals of motion C and G_2. In the parameter plane (p, q) with $p, q > 0$ and $|p - q| < 1$ there are five bifurcation lines, $\Gamma_1, \ldots, \Gamma_5$ given in (3.16), that divide the plane into six regions. These regions have a number of equilibria ranging from two to six and are either saddles or centres. Γ_i $(i = 1, \ldots, 5)$ are either the typical Hamiltonian pitchfork bifurcation of equilibria related to the circular motions of the inner bodies or the coplanar motions of the three bodies or saddle-centre bifurcations corresponding to elliptic motions of the inner bodies that have an inclination with respect to the invariable plane between 0 and π and an eccentricity between $\sqrt{1-(p-q)^2}$ and $\min\{1, \sqrt{1-(p+q)^2}\}$. In T_2 and T_3 reversible hyperbolic umbilic bifurcations occur. In the point T_1 the fully-reduced space has three singular points and there are four relative equilibria.*

3.3.6 Evolution of the flow

We describe now the evolution of the relative equilibria discussed previously, putting a special emphasis in their stability. We calculate two energy-momentum mappings, i.e. we fix a value of one of the parameters, let us say q, we calculate the value of Hamiltonian (3.1) at each equilibrium for $p \in [0, 1+q)$ and plot the corresponding curve. Stable equilibria are represented by solid lines, whereas unstable ones are shown with dashed lines. We choose two different values of q in such a way that we cover most possible regimes and transitions in the bifurcation plane. The two pictures are encapsulated in Figs. 3.3 and 3.4.

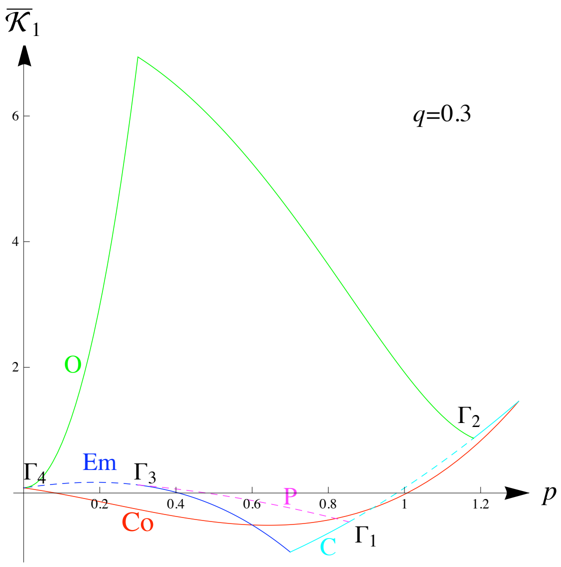

Figure 3.3: Hamiltonian (3.1) evaluated at the equilibria versus p for $q = 0.3$. Solid lines correspond to stable equilibria of centre type and dashed lines are associated to unstable equilibrium points. The red line (the one labelled by "Co") represents the equilibrium $(2(p-q)^2 - 1, 0, 0)$. The blue line (the one labelled by "Em") matches to the equilibrium $(2(p+q)^2 - 1, 0, 0)$. The green line (the one labelled by "O") accounts for two equilibria in the plane $\bar{\tau}_3 = 0$ with the same $\bar{\tau}_1$ and opposite $\bar{\tau}_2$. The magenta line (the one labelled by "P") is associated to two equilibria in the plane $\bar{\tau}_2 = 0$ with the same $\bar{\tau}_1$ and opposite $\bar{\tau}_3$. The cyan line (the one labelled by "C") corresponds to the equilibrium $(1, 0, 0)$.

First, we fix $q = 0.3$, so $p \in [0, 1.3]$. The evolution of the Hamiltonian evaluated at the equilibria for these values of q and p is described in Fig. 3.3. We start in region D in the plane of parameters: there are two elliptic relative equilibria. One is $(2(p-q)^2 - 1, 0, 0)$ (the red one labelled by "Co" in Fig. 3.3), as we

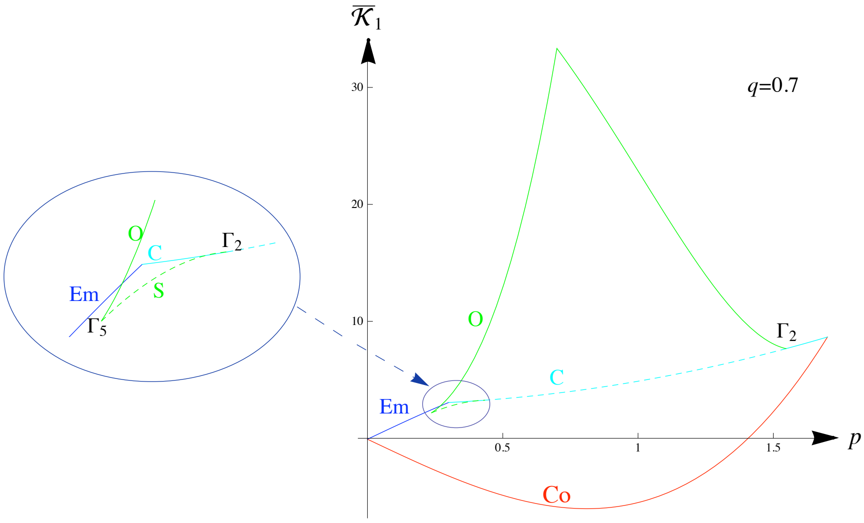

Figure 3.4: Hamiltonian (3.1) evaluated at the equilibria versus p for $q = 0.7$. Solid lines correspond to stable equilibria of centre type and dashed lines are associated to unstable equilibrium points. The left picture is a zoom of the right one in the encircled region. The red line (the one labelled by "Co") represents the equilibrium $(2(p-q)^2-1,0,0)$. The blue line (the one labelled by "Em") matches to the equilibrium $(2(p+q)^2-1,0,0)$. The green lines (the ones labelled by "O" and "S") account for four equilibria in the plane $\bar{\tau}_3 = 0$. The two stable ones share the same $\bar{\tau}_1$ and have opposite $\bar{\tau}_2$. The two unstable ones also share the same $\bar{\tau}_1$ and have opposite $\bar{\tau}_2$. The cyan line labelled by "C" corresponds to the equilibrium $(1,0,0)$.

already know, and the other one is $(2(p+q)^2-1,0,0)$ (the blue one labelled by "Em" in Fig. 3.3), which corresponds to coplanar motions of the three bodies such that the inner orbits have minimum eccentricity and $I_1 = \pi$. These are linear centres (see the proof in [69]) up to the bifurcation line Γ_4, which is a Hamiltonian pitchfork bifurcation such that once in region F the equilibrium $(2(p+q)^2-1,0,0)$ becomes unstable and two stable equilibria appear (the green ones labelled by "O" in Fig. 3.3). These stable equilibria are in the plane $\bar{\tau}_3 = 0$, they have the same $\bar{\tau}_1$ and opposite $\bar{\tau}_2$ (see the details in [69]) and they are linear centres. The value of the Hamiltonian is the same for both, so there is only one line associated to them. The equilibrium $(2(p+q)^2-1,0,0)$ continues to be unstable up to the bifurcation line Γ_3, which is another Hamiltonian pitchfork bifurcation. After crossing Γ_3, once in region A, this equilibrium becomes stable (a linear centre) and two new unstable ones appear (the magenta ones labelled by "P" in Fig. 3.3). They are in the plane $\bar{\tau}_2 = 0$, have the same $\bar{\tau}_1$, opposite $\bar{\tau}_3$ (see the computations in [69]) and thus, the same value of the Hamiltonian. Note that just on Γ_3 the equilibrium that does not bifurcate, i.e. the point $(2(p-q)^2-1,0,0)$, is singular and corresponds to rectilinear inner orbits with $I_1=0$ which are coplanar with the outer one. When $p=0.7$, still in region A, the equilibrium $(2(p+q)^2-1,0,0)$ changes to $(1,0,0)$, which is associated to circular orbits of the inner bodies (the cyan line labelled by "C" in Fig. 3.3). The stability does not change, so it is also a linear centre up to the Hamiltonian pitchfork bifurcation Γ_1. At this value the two unstable orbits in the plane $\bar{\tau}_2=0$ collide with $(1,0,0)$, they disappear and $(1,0,0)$ becomes unstable. It remains so in the whole region B up to the pitchfork bifurcation Γ_2. At this value the two stable orbits in the plane $\bar{\tau}_3=0$ collide with $(1,0,0)$, that becomes stable once in region C.

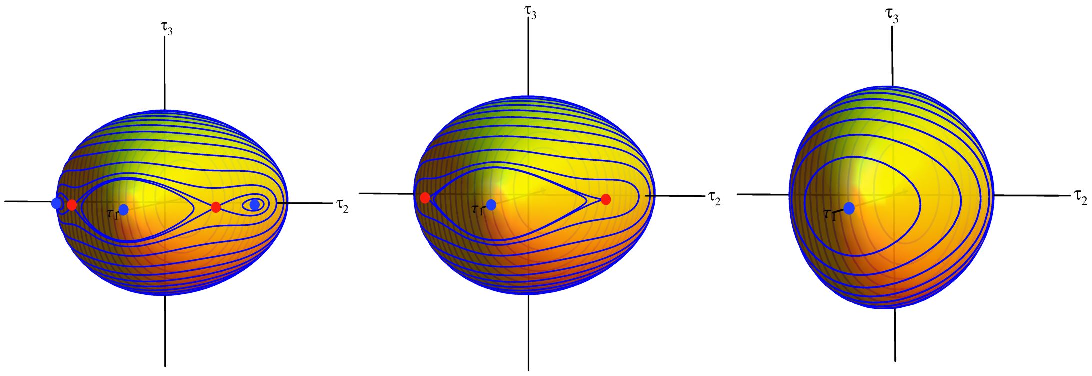

Figure 3.5: Double saddle-centre bifurcation Γ_5. The figure on the left represents the flow in region D of the bifurcation plane, just before the bifurcation takes place. The central picture corresponds to the flow on the bifurcation line Γ_5. The picture on the right accounts for the flow in region E.

Now we fix $q=0.7$, so $p\in[0,1.7]$. The evolution of the Hamiltonian evaluated at the equilibria for these values of q and p is described in Fig. 3.4. We start again in region D of the plane of parameters. Thus, we have two linear centres: $(2(p-q)^2-1,0,0)$ and $(2(p+q)^2-1,0,0)$. Still in region D this second equilibrium changes to $(1,0,0)$ but it maintains its stability. At Γ_5 a double saddle-centre bifurcation takes place and two stable equilibria (the green ones labelled by "O" in Fig. 3.4) and two unstable ones (the green ones labelled by "S" in Fig. 3.4) appear once in region E. They are in the plane $\bar{\tau}_3=0$. The two stable ones share the same $\bar{\tau}_1$ and have opposite $\bar{\tau}_2$. The two unstable ones also share the same $\bar{\tau}_1$ and have opposite $\bar{\tau}_2$ (see Fig. 3.5). They remain so up to the pitchfork bifurcation Γ_2, where the two unstable equilibria in the plane $\bar{\tau}_3=0$ collide with $(1,0,0)$, they disappear and $(1,0,0)$ becomes unstable once in region B. Then, at the other branch of the pitchfork bifurcation Γ_2, the two stable equilibria in the plane $\bar{\tau}_3=0$

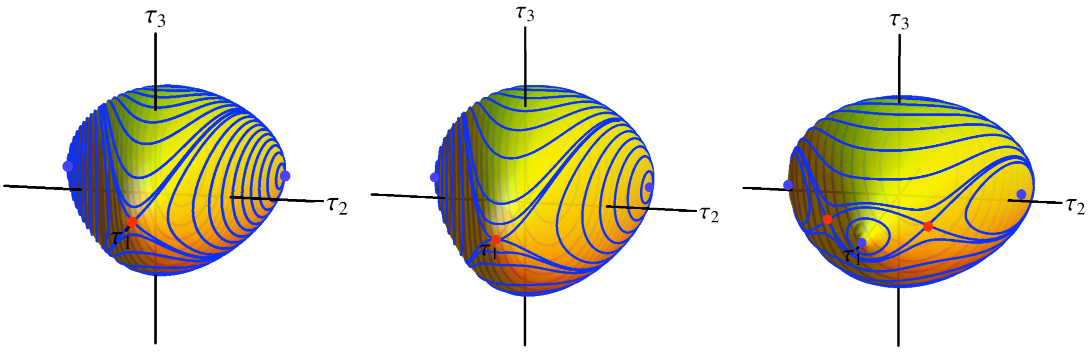

Figure 3.6: A Hamiltonian pitchfork bifurcation occurring when crossing Γ_4 between T_1 and T_2. The flow on the left corresponds to region F. In the middle the bifurcation takes place. On the right the flow corresponds to region E on the line $p + q = 1$, just after the bifurcation has taken place, so the fully reduced space is singular at the point $(1, 0, 0)$.

collide with $(1, 0, 0)$ and, once in region C they disappear and $(1, 0, 0)$ becomes a linear centre. Another sequence of portraits is shown in Fig. 3.6.

Chapter 4

Reconstruction from the reduced spaces

We plan to establish the existence of invariant 5-tori of Hamiltonian (2.3) in the region $\mathcal{Q}_{\varepsilon,n}$ from the elliptic relative equilibria of the fully-reduced space. However not all of the tori can be obtained directly from the analysis in $\mathcal{T}_{L_1,C,G_2}$ thus we need to describe the passage from $\mathcal{T}_{L_1,C,G_2}$ to $\mathcal{A}_{L_1,L_2}$ through the intermediate reduced spaces. In this section the motions related with elliptic equilibria in the fully reduced space are studied in the upper reduced spaces which is going to be useful to establish the existence of invariant tori in Chapters 5 and 6. In Fig. 4.1 an account of the reduced spaces is presented.

4.1 Reconstruction from $\mathcal{T}_{L_1,C,G_2}$ to $\mathcal{S}_{L_1,L_2,C}$

We depart from every point in $\mathcal{T}_{L_1,C,G_2}$, undo the reduction by G_2 and determine the corresponding set in $\mathcal{S}_{L_1,L_2,C}$.

Proposition 4.1. *When $\mathcal{T}_{L_1,C,G_2}$ is a regular surface its points are reconstructed into two-dimensional surfaces in $\mathcal{S}_{L_1,L_2,C}$ of the type (2.33) excepting for the points representing coplanar motions that are reconstructed into simple open curves of $\mathcal{S}_{L_1,L_2,C}$. When $\mathcal{T}_{L_1,C,G_2}$ has singularities, its regular points are reconstructed to either circles or (regular or singular) points of $\mathcal{S}_{L_1,L_2,C}$. The singular points of $\mathcal{T}_{L_1,C,G_2}$ are reconstructed as singular points of $\mathcal{S}_{L_1,L_2,C}$.*

Proof. We assume that $C \leq L_1 + L_2$ and $|C - G_2| \leq L_1$ so that $\mathcal{T}_{L_1,C,G_2}$ and $\mathcal{S}_{L_1,L_2,C}$ are not empty sets.

We start by taking $C = L_1 + L_2$, then the only chance for $\mathcal{T}_{L_1,C,G_2}$ to be non-empty is that $|C - G_2| = L_1$. Moreover one has $G_1 = L_1$ and $G_2 = L_2 = C - L_1$. Hence $\mathcal{T}_{L_1,C,G_2}$ and $\mathcal{S}_{L_1,L_2,C}$ are sets with only one point. Concretely the point

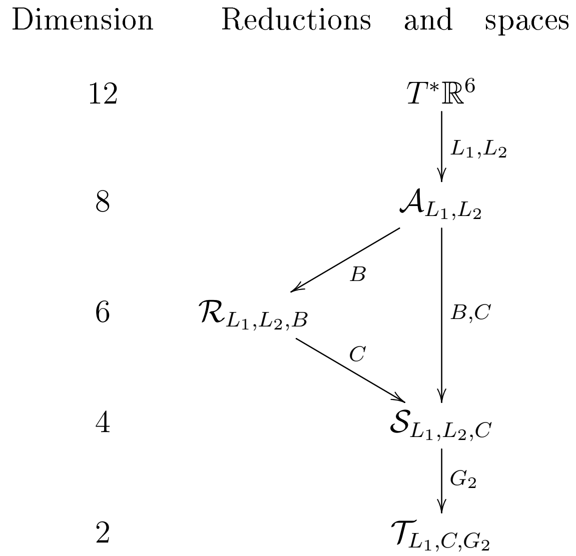

Figure 4.1: Scheme of reductions with the corresponding reduced spaces and integrals. The dimension of each space is shown in the left column.

$(L_1^2, 0, 0)$ gets transformed into $(L_1^2, L_2^2, 0, 0, 0, 0)$, representing circular coplanar motions of the three bodies. From now on we restrict ourselves to $C < L_1 + L_2$.

When $|C - G_2| = L_1$ then $G_1 = L_1$. Thus, $\mathcal{T}_{L_1,C,L_1 \pm C}$ is the point $(L_1^2, 0, 0)$ which is transformed into $(L_1^2, 2(L_1 \pm C)^2 - L_2^2, 0, 0, 0, 0)$, that is the two singularities labelled by (b). Henceforth we assume that $|C - G_2| < L_1$ and consider four different situations:

(i) We consider $G_2 \neq C$ and $L_1 \neq C + G_2$ thus $\mathcal{T}_{L_1,C,G_2}$ is a regular surface. Fixing a point on it, say $(\tau_1^*, \tau_2^*, \tau_3^*)$ with the τ_i^* satisfying (2.32), we take the first equation of (2.27) where we put σ_2 in terms of G_2 and replace σ_1, σ_3 and σ_5 respectively by τ_1^*, τ_2^* and τ_3^*. This equation holds trivially. However in the second equation of (2.27) a relationship among σ_2, σ_4 and σ_6 is established after writing down σ_1 in terms of G_1^* (note that G_1^* is fixed since the τ_i^* are given). Concretely the resulting constraint is:

$$(\sigma_2 - L_2^2)\Big((\sigma_2 + L_2^2 - 2C^2 - 2G_1^{*2})^2 - 16C^2 G_1^{*2}\Big) = 4(\sigma_2 + L_2^2)\sigma_4^2 + 8\sigma_6^2. \quad (4.1)$$

Together with the fixed values $\sigma_1^* = \tau_1^*$, $\sigma_3^* = \tau_2^*$ and $\sigma_5^* = \tau_3^*$, equation (4.1) defines the image of the point $(\tau_1^*, \tau_2^*, \tau_3^*)$ as a subset of $\mathcal{S}_{L_1,L_2,C}$. The constraint (4.1) is the same as (2.32) after interchanging L_2 with L_1, G_1^* with G_2, σ_2 with τ_1, σ_4 with τ_2 and σ_6 with τ_3. Equation (2.32) defines $\mathcal{T}_{L_1,C,G_2}$ and this space is studied in detail in Chapter 2.

When the two fictitious bodies move in different planes then $G_1^* \neq |C \pm G_2|$ and the image of a point of $\mathcal{T}_{L_1,C,G_2}$ is a two-dimensional surface embedded in $\mathcal{S}_{L_1,L_2,C}$ provided that $|C - G_1^*| < L_2$ and a single point when $|C - G_1^*| = L_2$. In addition to it when $|C - G_1^*| < L_2$ the surface is regular if $L_2 \neq C + G_1^*$ and $G_1^* \neq C$ while it has a singularity for $G_2 = L_2$ when $L_2 = C + G_1^*$. However the two singularities of the case $G_1^* = C$ are avoided as G_2 cannot vanish. Indeed we should subtract from the image the segment defined by $\sigma_1 = 2C^2 - L_1^2$, $\sigma_2 = L_2^2$, $\sigma_4 \in [-2L_2C, 2L_2C]$ and $\sigma_3 = \sigma_5 = \sigma_6 = 0$.

When $G_1^* = C \pm G_2$ or $G_1^* = G_2 - C$ then $\sigma_4 = \sigma_6 = 0$ (and $\sigma_3^* = \sigma_5^* = 0$). Using the mapping (2.26) it is readily concluded that for $G_1^* = C + G_2$ the image of the point $(\tau_1^*, \tau_2^*, \tau_3^*) = (2(C+G_2)^2 - L_1^2, 0, 0)$ is $(2(C+G_2)^2 - L_1^2, 2G_2^2 - L_2^2, 0, 0, 0, 0)$ while for $G_1^* = |C - G_2|$ the image of $(\tau_1^*, \tau_2^*, \tau_3^*) = (2(C-G_2)^2 - L_1^2, 0, 0)$ is $(2(C-G_2)^2 - L_1^2, 2G_2^2 - L_2^2, 0, 0, 0, 0)$. Both images are one-dimensional subsets of $S_{L_1,L_2,C}$ parametrised by $G_2 \in (0, L_2]$, indeed they are simple open curves in $S_{L_1,L_2,C}$.

(ii) When $G_2 \neq C$ and $L_1 = C + G_2$ then $\mathcal{T}_{L_1,C,L_1-C}$ has one singularity at $(L_1^2, 0, 0)$. Given a regular point of $\mathcal{T}_{L_1,C,L_1-C}$ we fix values for G_1 and γ_1, say $G_1^* < L_1$ and γ_1^* (note that fixing G_1 and γ_1 is equivalent to fixing τ_i, $i = 1, 2, 3$) and replace G_2 by $L_1 - C$ in (2.26). The point is transformed into a subset of $\mathcal{S}_{L_1,L_2,C}$ with the following coordinates:

$$\begin{aligned}
\sigma_1 &= 2G_1^{*2} - L_1^2,\\
\sigma_2 &= 2(L_1 - C)^2 - L_2^2,\\
\sigma_3 &= \frac{L_1^2 - G_1^{*2}}{G_1^*}\sqrt{G_1^{*2} - (L_1 - 2C)^2}\,\sin\gamma_1^*,\\
\sigma_4 &= \frac{\sqrt{(L_1^2 - G_1^{*2})\,(G_1^{*2} - (L_1 - 2C)^2)\,(L_2^2 - (L_1 - C)^2)}}{L_1 - C}\,\sin\gamma_2,\\
\sigma_5 &= (L_1^2 - G_1^{*2})\sqrt{G_1^{*2} - (L_1 - 2C)^2}\,\cos\gamma_1^*,\\
\sigma_6 &= \sqrt{(L_1^2 - G_1^{*2})\,(G_1^{*2} - (L_1 - 2C)^2)\,(L_2^2 - (L_1 - C)^2)}\,\cos\gamma_2.
\end{aligned}$$

Thus the image of a regular point of $\mathcal{T}_{L_1,C,L_1-C}$ is a circle in $\mathcal{S}_{L_1,L_2,C}$ provided that $G_1^* \neq |L_1 - 2C|$ and $L_2 \neq |L_1 - C|$. Since $|C - G_2| \leq G_1$ and $L_1 = C + G_2$ then $G_1 \geq |L_1 - 2C|$ but $C < L_1 + L_2$ and $0 < L_1 < L_2$ implies $L_2 > |L_1 - C|$. So the only regular point of $\mathcal{T}_{L_1,C,L_1-C}$ that is not transformed into a circle is the one such that $G_1^* = |L_1 - 2C|$. Its image is $(L_1^2 + 8C^2 - 8L_1C, 2(L_1 - C)^2 - L_2^2, 0, 0, 0, 0)$. The Jacobian matrix J evaluated at it has rank one, concretely it is a singular point of $\mathcal{S}_{L_1,L_2,C}$ corresponding to the situation (j).

On the other hand the point $(L_1^2, 0, 0)$ of $\mathcal{T}_{L_1,C,L_1-C}$ corresponds to the case

$G_1^* = L_1$, and its image in $\mathcal{S}_{L_1,L_2,C}$ is $(L_1^2, 2(L_1-C)^2 - L_2^2, 0, 0, 0, 0)$ which is singular, specifically one of the two points (b) studied in Subsection 2.4.2.

(iii) When $G_2 = C$ and $L_1 \neq C + G_2$ the surface $\mathcal{T}_{L_1,C,C}$ has two singular points with coordinates $(-L_1^2, \pm 2L_1 C, 0)$. After picking specific values γ_1^* and G_1^* (or $G_1^* = L_1$) the regular points of $\mathcal{T}_{L_1,C,C}$ get transformed through (2.26) into

$$\begin{aligned}
\sigma_1 &= 2G_1^{*2} - L_1^2,\\
\sigma_2 &= 2C^2 - L_2^2,\\
\sigma_3 &= \sqrt{(L_1^2 - G_1^{*2})(4C^2 - G_1^{*2})}\sin\gamma_1^*,\\
\sigma_4 &= \frac{G_1^*}{C}\sqrt{(L_2^2 - C^2)(4C^2 - G_1^{*2})}\sin\gamma_2,\\
\sigma_5 &= G_1^*\sqrt{(L_1^2 - G_1^{*2})(4C^2 - G_1^{*2})}\cos\gamma_1^*,\\
\sigma_6 &= G_1^*\sqrt{(L_2^2 - C^2)(4C^2 - G_1^{*2})}\cos\gamma_2.
\end{aligned}$$

Therefore the image of a regular point in $\mathcal{T}_{L_1,C,C}$ is a circle in $\mathcal{S}_{L_1,L_2,C}$ parametrised by γ_2 provided that $G_1^* \neq 2C$ and $C \neq L_2$. When $G_1^* = 2C$ the point gets transformed into $(8C^2 - L_1^2, 2C^2 - L_2^2, 0, 0, 0, 0)$ which is a singularity of $\mathcal{S}_{L_1,L_2,C}$ of the type (j). When $C = L_2$ the point is transformed into the regular point $(2G_1^{*2} - L_1^2, L_2^2, \sqrt{(L_1^2 - G_1^{*2})(4L_1^2 - G_1^{*2})}\sin\gamma_1^*, 0, G_1^* \sqrt{(L_1^2 - G_1^{*2})(4L_1^2 - G_1^{*2})}\cos\gamma_1^*, 0)$.

Using (2.38) and (2.41) the singular points $(-L_1^2, \pm 2L_1 C, 0)$ are transformed into $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1 C, 0, 0, 0, 0)$ which are the singular points labelled above by (a).

(iv) When $G_2 = C$ and $L_1 = C + G_2$ there are three singularities in the space $\mathcal{T}_{L_1,L_1/2,L_1/2}$, namely $(L_1^2, 0, 0)$ and $(-L_1^2, \pm L_1^2, 0)$. The regular points of the fully-reduced space are converted into

$$\begin{aligned}
\sigma_1 &= 2G_1^{*2} - L_1^2,\\
\sigma_2 &= \tfrac{1}{2}L_1^2 - L_2^2,\\
\sigma_3 &= (L_1^2 - G_1^{*2})\sin\gamma_1^*,\\
\sigma_4 &= \frac{G_1^*}{L_1}\sqrt{(L_1^2 - G_1^{*2})(4L_2^2 - L_1^2)}\sin\gamma_2,\\
\sigma_5 &= G_1^*(L_1^2 - G_1^{*2})\cos\gamma_1^*,\\
\sigma_6 &= \tfrac{1}{2}G_1^*\sqrt{(L_1^2 - G_1^{*2})(4L_2^2 - L_1^2)}\cos\gamma_2,
\end{aligned}$$

which are circles in $\mathcal{S}_{L_1,L_2,L_1/2}$ parametrised by γ_2 since $L_1 > G_1^*$ and $2L_2 > L_1$.

Concerning the singularities, $(L_1^2, 0, 0)$ corresponds to the case $G_1^* = L_1$, and it is converted into $(L_1^2, \frac{1}{2}L_1^2 - L_2^2, 0, 0, 0, 0)$ which is the singular point (b) with $C = L_1/2$ whereas the points $(-L_1^2, \pm L_1^2, 0)$ are transformed into $(-L_1^2, \frac{1}{2}L_1^2 - L_2^2, \pm L_1^2, 0, 0, 0)$, i.e. the singular points (a) when $C = L_1/2$.

□

4.2 Reconstruction from $\mathcal{S}_{L_1,L_2,C}$ to $\mathcal{R}_{L_1,L_2,B}$

We only reconstruct the point of $\mathcal{S}_{L_1,L_2,C}$ related to the motions that are studied in the spaces $\mathcal{R}_{L_1,L_2,B}$ and $\mathcal{A}_{L_1,L_2}$ in the following chapters.

Proposition 4.2. *(a) The point $(L_1^2, L_2^2, 0, 0, 0, 0)$ in $\mathcal{S}_{L_1,L_2,L_2 \pm L_1}$ corresponding to circular coplanar motions of the three bodies reconstructs to a regular or singular point of $\mathcal{R}_{L_1,L_2,B}$.*

(b) The points of $\mathcal{S}_{L_1,L_2,L_2}$ with coordinates $(-L_1^2, L_2^2, \pm 2L_1L_2,\ 0, 0, 0)$ that stand for prograde or retrograde rectilinear motions of the fictitious inner particle orthogonal to the invariable plane and circular motion for the outer body in the invariable plane, reconstruct to regular or singular points of $\mathcal{R}_{L_1,L_2,B}$.

Proof. (a) It is a singular point (case (g) of Subsection 2.4.2) in $\mathcal{S}_{L_1,L_2,L_2 \pm L_1}$ that corresponds to circular coplanar motions of the three bodies.

Using the coordinates of $\mathcal{R}_{L_1,L_2,B}$ appearing in (2.45) we put the invariants ρ_i in terms of Deprit's action-angle variables, doing $G_1 = L_1$, $G_2 = L_2$ and $C = L_2 \pm L_1$. We arrive at the point $(\rho_1, \ldots, \rho_{16})$ such that

$$\begin{aligned}
\rho_1 &= \rho_2 = \pm\frac{L_1 B}{L_2 \pm L_1}, \\
\rho_3 &= \rho_4 = \frac{L_2 B}{L_2 \pm L_1}, \\
\rho_5 &= \rho_7 = \rho_9 = \rho_{11} = \rho_{13} = \rho_{15} = 0, \\
\rho_6 &= L_1^2\left(1 - \frac{B^2}{(L_2 \pm L_1)^2}\right), \\
\rho_8 &= \rho_{10} = \rho_{12} = \rho_{14} = \pm L_1 L_2\left(1 - \frac{B^2}{(L_2 \pm L_1)^2}\right), \\
\rho_{16} &= L_2^2\left(1 - \frac{B^2}{(L_2 \pm L_1)^2}\right),
\end{aligned} \tag{4.2}$$

which is a point in $\mathcal{R}_{L_1,L_2,B}$ as it satisfies the constraints (2.44).

When $|B| = L_1 + L_2$ then $C = L_1 + L_2$, $G_1 = L_1$ and $G_2 = L_2$ and the space $\mathcal{R}_{L_1,L_2,\pm(L_1+L_2)}$ is merely a point. This is the only combination among L_1, L_2 and B such that $\mathcal{R}_{L_1,L_2,B}$ consists in a point.

We compute the Jacobian 10×16-matrix of the constraints (2.44) with respect to the invariants ρ_i and evaluate it at the equilibrium point with coordinates (4.2). When $|B| \neq L_2 \pm L_1$ the rank is ten, hence the point is regular. However when $|B| = L_2 \pm L_1$ the Jacobian matrix has rank one, thus the point is singular for $|B| = L_2 - L_1$. When $|B| = L_1 + L_2$ the point is not properly a singularity.

(b) The invariants ρ_i are related with Deprit's coordinates through the change (2.46) and the explicit expressions of the Keplerian invariants in terms of γ_1, γ_2, ν_1, ν, G_1, G_2, C and B, see Appendix A of [69]. These formulas make sense even for $G_1 = 0$ as in this case $G_2 = C$ and we can use an argument of analytic extension of Deprit's action-angle coordinates for $G_1 = 0$. So we make $G_1 = 0$, $G_2 = C = L_2$ and $\gamma_1 = \pi/2$ (for $\sigma_3 = 2L_1L_2$) or $\gamma_1 = 3\pi/2$ (for $\sigma_3 = -2L_1L_2$) in the expressions relating the ρ_i with Deprit's coordinates arriving at

$$
\begin{gathered}
\rho_1 = \pm\frac{L_1 B}{L_2}, \quad \rho_2 = \mp\frac{L_1 B}{L_2}, \quad \rho_3 = B, \quad \rho_4 = B, \\
\rho_5 = 0, \quad \rho_6 = -\frac{L_1^2}{L_2^2}\left(L_2^2 - B^2\right), \quad \rho_7 = 0, \quad \rho_8 = \pm\frac{L_1}{L_2}\left(L_2^2 - B^2\right), \\
\rho_9 = 0, \quad \rho_{10} = \pm\frac{L_1}{L_2}\left(L_2^2 - B^2\right), \quad \rho_{11} = 0, \quad \rho_{12} = \mp\frac{L_1}{L_2}\left(L_2^2 - B^2\right), \\
\rho_{13} = 0, \quad \rho_{14} = \mp\frac{L_1}{L_2}\left(L_2^2 - B^2\right), \quad \rho_{15} = 0, \quad \rho_{16} = L_2^2 - B^2,
\end{gathered}
\tag{4.3}
$$

where the upper signs apply for $\sigma_3 = 2L_1L_2$ and the lower ones for $\sigma_3 = -2L_1L_2$.

In order to establish the regular or singular character of the points (4.2) we determine the Jacobian 10×16-matrix of the constraints (2.44) with respect to the invariants ρ_i and evaluate it at the equilibrium points with coordinates (4.2). We conclude that the rank is ten provided that $|B| \neq L_2$, otherwise the rank decreases to one. Thus the points (4.2) are regular points of the set $\mathcal{R}_{L_1,L_2,B}$ provided $|B| \neq L_2$, whereas they become singular points when $|B| = L_2$.

□

4.3 Reconstruction from $\mathcal{S}_{L_1,L_2,C}$ to $\mathcal{A}_{L_1,L_2}$

We only reconstruct the point of $\mathcal{S}_{L_1,L_2,C}$ related to the motions that are studied in the spaces $\mathcal{R}_{L_1,L_2,B}$ and $\mathcal{A}_{L_1,L_2}$ in the following chapters.

Proposition 4.3. *(a) The point $(L_1^2, L_2^2, 0, 0, 0, 0)$ in $\mathcal{S}_{L_1,L_2,L_2\pm L_1}$ corresponding to circular coplanar motions of the three bodies reconstructs to an S^2 in $\mathcal{A}_{L_1,L_2}$.*

(b) The points of $\mathcal{S}_{L_1,L_2,L_2}$ with coordinates $(-L_1^2, L_2^2, \pm 2L_1L_2,\ 0, 0, 0)$ that stand for prograde or retrograde rectilinear motions of the fictitious inner body orthogonal to the invariable plane and circular motion for the outer body in the invariable plane, and such that the invariable plane is the horizontal plane of a fixed reference frame, reconstruct to points of $\mathcal{A}_{L_1,L_2}$.

Proof. (a) It is a singular point (case (g) of Subsection 2.4.2) in $\mathcal{S}_{L_1,L_2,L_2\pm L_1}$ that corresponds to circular coplanar motions of the three bodies.

From (2.19) we infer that

$$\mathbf{a} + \mathbf{b} = 2\mathbf{G}_1, \qquad \mathbf{c} + \mathbf{d} = 2\mathbf{G}_2, \tag{4.4}$$

and that

$$\mathbf{a} \cdot \mathbf{b} = G_1^2 - L_1^2 A_1^2, \qquad \mathbf{c} \cdot \mathbf{d} = G_2^2 - L_2^2 A_2^2. \tag{4.5}$$

As in this case $G_1 = L_1 > 0$ and $G_2 = L_2 > 0$ and considering the relations (2.18) one gets

$$4L_1^2 = 4G_1^2 = |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = 2L_1^2 + 2(L_1^2 - L_1^2 A_1^2),$$
$$4L_2^2 = 4G_2^2 = |\mathbf{c} + \mathbf{d}|^2 = |\mathbf{c}|^2 + |\mathbf{d}|^2 + 2\mathbf{c} \cdot \mathbf{d} = 2L_2^2 + 2(L_2^2 - L_2^2 A_2^2),$$

where $A_k = |\mathbf{A}_k|$. Thus, $A_1 = A_2 = 0$ and so $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{0}$. Applying these equalities in (2.19) we get

$$\mathbf{a} = \mathbf{b} = \mathbf{G}_1, \qquad \mathbf{c} = \mathbf{d} = \mathbf{G}_2. \tag{4.6}$$

Taking into account that $\mathbf{C} = \mathbf{G}_1 + \mathbf{G}_2$ then $\mathbf{C} = \mathbf{a} + \mathbf{c}$. Now, as the orbits are also coplanar thus $G_2 = |C \pm G_1|$, hence $L_2 = |C \pm L_1|$. We discard the cases $C = -L_1 - L_2$ and $L_2 = L_1 - C$ as $L_2 > L_1$ and do not consider the case $L_2 = L_1 + C$ as it is studied in $\mathcal{R}_{L_1,L_2,B}$. Therefore, $L_2 = C - L_1$ and $|\mathbf{c}| = |\mathbf{a} + \mathbf{c}| - |\mathbf{a}|$ from which we deduce that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}||\mathbf{c}| = L_1 L_2$ and $L_2 \mathbf{a} = L_1 \mathbf{c}$. Thus, the point $(L_1^2, L_2^2, 0, 0, 0, 0)$ reconstructs to the following two-dimensional set in $\mathcal{A}_{L_1,L_2}$:

$$\left\{ \left(\mathbf{a}\,,\, \mathbf{a}\,,\, \frac{L_2}{L_1}\mathbf{a}\,,\, \frac{L_2}{L_1}\mathbf{a} \right) \in \mathbb{R}^{12} \ \mid\ |\mathbf{a}| = L_1 \right\}, \tag{4.7}$$

which is diffeomorphic to S^2.

(b) With the same argument as in the previous subsection but setting in addition $|B| = L_2$, we end up with the following points in $\mathcal{A}_{L_1,L_2}$ for $\sigma_3 = 2L_1L_2$:

$$(0\,,\,0\,,\,\pm L_1\,,\,0\,,\,0\,,\,\mp L_1\,,\,0\,,\,0\,,\,\pm L_2\,,\,0\,,\,0\,,\,\pm L_2)\,,$$

such that the upper signs apply to $B = L_2$ whereas the lower ones apply for $B = -L_2$.

When $\sigma_3 = -2L_1L_2$ we get the points

$$(0\,,\,0\,,\,\mp L_1\,,\,0\,,\,0\,,\,\pm L_1\,,\,0\,,\,0\,,\,\pm L_2\,,\,0\,,\,0\,,\,\pm L_2)\,,$$

where the upper signs are used for $B = L_2$ and the lower ones for $B = -L_2$.

Let us remark that all the points in $\mathcal{A}_{L_1,L_2}$ are regular.

□

Chapter 5

Invariant tori associated to non-rectilinear motions

5.1 Main result

In this chapter we reconstruct the elliptic relative equilibria of the fully-reduced space with the aim of establishing the existence of KAM tori in the spatial three-body problem. We reconstruct the elliptic equilibria given in Fig. 3.1 discarding the ones associated to rectilinear motions of the inner bodies as their study deserves a separate chapter. We plan to apply KAM theory, however our system is written as the sum of a Keplerian part plus a perturbation that appears scaled at different orders, so it is very degenerate. Therefore, we cannot conclude the existence of invariant tori using the standard KAM theorems [4] or even some specific results dealing with Hamiltonians with a proper degeneracy. Indeed it is well known that in many cases of perturbed Kepler problems, the leading order of the perturbed Hamiltonian is insufficient to remove the degeneracy, thus one needs to resort to a theorem particularly designed to remove such degeneracy. We apply a theorem by Han, Li and Yi [36] that works in the case of Hamiltonian systems with high-order proper degeneracy and has been applied successfully in other contexts [63]. One can find more details about this issue in Chapter 1. Han, Li and Yi's Theorem can be applied to Hamiltonian systems with finite smoothness using standard arguments of KAM theory, thus we shall use it in the next section for the study of the cases that are summarised in Table 5.1. Our goal is to get invariant 5-tori for Hamiltonian (2.3) in $\mathcal{Q}_{\varepsilon,n}$, the subset of $\mathcal{P}_{\varepsilon,n} \subseteq T^*\mathbb{R}^6$ we are performing the analysis. These tori are related to the elliptic equilibria in the fully-reduced space. We do not reconstruct KAM 6-tori because they would be resonant since B and ν are cyclic coordinates, see [13].

We apply Theorem 1.15 to the Hamiltonian of the three-body problem given

Space	Dimension	Cases (inner / outer ellipses)
$\mathcal{T}_{L_1,C,G_2}$	2	non-circular / non-circular - non-coplanar non-circular / non-circular - coplanar circular / non-circular - non-coplanar
$\mathcal{S}_{L_1,L_2,C}$	4	circular / non-circular - coplanar circular / circular - non-coplanar non-circular / circular - non-coplanar non-circular / circular - coplanar
$\mathcal{R}_{L_1,L_2,B}$	6	circular / circular - coplanar with $C \approx L_2 - L_1 \not\approx \lvert B\rvert$ or $C \approx L_1 + L_2 \not\approx \lvert B\rvert$
$\mathcal{A}_{L_1,L_2}$	8	circular / circular - coplanar with $C \approx L_2 - L_1 \approx \lvert B\rvert$ or $C \approx L_1 + L_2 \approx \lvert B\rvert$

Table 5.1: Reduced spaces where we have carried out the analysis of the different relative equilibria. There are KAM 5-tori of the full system associated with each type of motion on the right column.

in (2.3), or equivalently in (2.7). This Hamiltonian has been reduced out by the translation symmetry and is defined in $\mathcal{Q}_{\varepsilon,n}$. It is also expressed in terms of Deprit's action-angle coordinates after making the normalisation over the mean anomalies and the Legendre expansion. It is given by:

$$\mathcal{H} = \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2) \tag{5.1}$$

where $\mathcal{H}_{\mathrm{Kep}}$ is the Keplerian Hamiltonian and $\mathcal{K}_1$ is the first-order perturbation given in (2.17). The higher-order terms of the perturbation are included in $\mathcal{O}(\varepsilon^2)$. They come from terms of order higher than two in the Legendre expansion of the potential and from the orders higher than one in the Lie transformation performed to average Hamiltonian (2.3). Note that Hamiltonian (5.1) keeps the same name as the one in (2.3). This is because they both represent the same system and (5.1) is obtained from (2.3) after some manipulations.

The cases of Table 5.1 are expanded in the following section. Indeed excepting for $\mathcal{A}_{L_1,L_2}$, in the cases of the other reduced spaces where the orbits of the bodies are coplanar, we need to distinguish among the different types of coplanarity.

The reason is that the combinations of Deprit's action-angle coordinates built to handle the different types of coplanarity depend on the linear combinations of the angles that are well defined. This fact leads to different collections of symplectic coordinates x_i, y_i that are introduced in order to represent all the subcases. These local coordinates are provided in terms of Deprit's variables and are given explicitly in the tables of the next section.

Remark 1. To achieve the reconstruction process we choose a specific relative equilibrium in $\mathcal{T}_{L_1,C,G_2}$ and use the actions L_1, L_2, C and G_2 in order to get the KAM 5-tori for the full Hamiltonian in $T^*\mathbb{R}^6$. The fifth action is built from a pair of rectangular symplectic coordinates, say x_1 and y_1, that are well suited variables defined in a neighbourhood of the equilibrium point in $\mathcal{T}_{L_1,C,G_2}$. However, depending on the relative equilibrium's type, when one or several angles are not properly defined, in principle, we could not use their conjugate actions. Nevertheless, when an angle is undetermined, certain linear combinations of it with the other angles are determined, see for instance [42]. Then we should define its conjugate action as an adequate linear combination of Deprit's actions L_1, L_2, C and G_2. This new action should be used when checking the hypotheses of Theorem 1.15 to compute the matrix containing the partial derivatives of the required orders of the Hamiltonians h_i $(i = 0, \ldots, a)$ with respect to the actions. However, by applying the following reasoning we avoid the use of these new actions and can always use L_1, L_2, C and G_2. In the fully-reduced space all the bounded motions of the fictitious inner body are allowed. This body can even follow straight lines as the flow on the reduced space is regularised with respect to inner double collisions, so the flow is smooth on the whole $\mathcal{T}_{L_1,C,G_2}$ regardless of where ℓ_1, ℓ_2, ν_1 and γ_2 are defined. Thus, we can change from one set of coordinates to the other and, when checking the hypotheses of Theorem 1.15, the matrix containing the partial derivatives of h_i $(i = 0, \ldots, a)$ with respect to the new actions has the same rank as the matrix containing the partial derivatives of h_i $(i = 0, \ldots, a)$ with respect to L_1, L_2, C and G_2. Hence, it is enough to apply Han, Li and Yi's Theorem taking the actions L_1, L_2, C and G_2 where the intermediate Hamiltonians depend on the specific relative equilibrium of $\mathcal{T}_{L_1,C,G_2}$ we reconstruct.

Remark 2. In some situations we cannot make the reconstruction from the fully-reduced space. For instance, when the outer body moves in a near-circular orbit $(G_2 \approx L_2)$ we shall be able to use at most three of the four actions. In such case the analysis has to be performed in a higher-dimensional space. Indeed if the motions of the two bodies are not near coplanar the right space to study the relative equilibrium is $\mathcal{S}_{L_1,L_2,C}$. Then as this space has dimension four we need two pairs of rectangular coordinates, say x_1, y_1 and x_2, y_2, to deal with the point in that space. So, we construct two (local) actions and take three of Deprit's actions, namely L_1, L_2 and C. In conclusion, the relative equilibrium of the fully-reduced space accounting for non-coplanar motions of the two bodies and such that the outer body describes a near-circular trajectory whereas the motion of the inner body is not circular, is studied in $\mathcal{S}_{L_1,L_2,C}$. We also require that this equilibrium be isolated in $\mathcal{S}_{L_1,L_2,C}$. Finally, when at least one of the angles ℓ_1, ℓ_2 or ν_1 is undetermined, the same explanation as the one given in Remark 1 works so that we can use L_1, L_2 and C as actions in order to apply Theorem 1.15.

Remark 3. There are other cases that have to be studied in a higher-dimensional space. When the two fictitious bodies follow near-circular trajectories that are nearly in the same plane then $G_1 \approx L_1$, $G_2 \approx L_2$ and $G_1 \approx |C - G_2|$ (we have discarded the condition $G_1 \approx C + G_2$ because it would lead to $C = L_1 - L_2$). So we introduce three pairs of local symplectic coordinates x_i, y_i if we may make the analysis in $\mathcal{R}_{L_1,L_2,B}$ or four pairs if the study is made in $\mathcal{A}_{L_1,L_2}$. In order to apply Theorem 1.15 we use the actions L_1 and L_2 and three actions I_i obtained from x_i, y_i. Note that in the cases studied in $\mathcal{R}_{L_1,L_2,B}$ and $\mathcal{A}_{L_1,L_2}$ the mean anomalies are not well defined but Remark 1 applies and we can use the actions L_1 and L_2.

To formulate the main result of this chapter we need to take into account the restrictions we have considered before so that our analysis is valid in $\mathcal{Q}_{\varepsilon,n}$. We then have the following result.

Theorem 5.1. *The Hamiltonian system of the spatial three-body problem (2.3) (or, equivalently, Hamiltonian (2.7)) reduced by the symmetry of translations defined in $\mathcal{Q}_{\varepsilon,n} \subseteq T^*\mathbb{R}^6$ has invariant KAM 5-tori densely filled with quasi-periodic trajectories of the fictitious inner and outer bodies of the following types:*

(1) *Motions reconstructed from relative equilibria of $\mathcal{T}_{L_1,C,G_2}$:*

  (i) *near-non-circular solutions of the inner and outer bodies moving in different planes;*

  (ii) *near-non-circular coplanar solutions of the inner and outer bodies;*

  (iii) *near-circular solutions of the inner bodies and non-circular solutions of the outer body moving in different planes.*

(2) *Motions reconstructed from relative equilibria of $\mathcal{S}_{L_1,L_2,C}$:*

  (i) *near-circular solutions of the inner bodies and non-circular solutions of the outer body moving in the same plane;*

  (ii) *near-circular solutions of the inner and outer bodies moving in different planes;*

  (iii) *near-non-circular solutions of the inner bodies and circular solutions of the outer body moving in different planes;*

(iv) near-non-circular solutions of the inner bodies and circular solutions of the outer body moving in the same plane.

(3) Motions reconstructed from relative equilibria of $\mathcal{R}_{L_1,L_2,B}$: near-circular-co-planar solutions of the inner and outer bodies such that $C \approx L_2 - L_1 \not\approx |B|$ or $C \approx L_1 + L_2 \not\approx |B|$.

(4) Motions reconstructed from a relative equilibrium of $\mathcal{A}_{L_1,L_2}$: near-circular solutions of the inner and outer bodies such that $C \approx L_2 - L_1 \approx |B|$ or $C \approx L_1 + L_2 \approx |B|$, that is, the motions of the three bodies nearly occur in the horizontal plane, i.e. the plane perpendicular to the axis $\mathbf{k}$.

Let δ with $0 < \delta < 1/5$ be given, then the excluding measure for the existence of quasi-periodic invariant tori in the four cases is of order $\mathcal{O}(\varepsilon^{\delta/4})$.

The proof is elaborated in the next section using Han, Li and Yi's Theorem. We have chosen a representative case of each reduced space, providing the explicit computations of the torsions. The calculations of the remaining cases have been also performed and they are presented in Appendix B.

5.2 Proof of Theorem 5.1

5.2.1 Study in $\mathcal{T}_{L_1,C,G_2}$

Our aim is to prove item (1) of Theorem 5.1. In Table 5.2 we show all the possible cases (excepting for rectilinear motions of the inner bodies) that are studied in the space $\mathcal{T}_{L_1,C,G_2}$. For each case we give the linear combinations of Deprit's angles that are properly defined as well as the corresponding combinations of the actions. However, we recall that by Remark 1 we check the conditions of Han, Li and Yi's Theorem by using the actions L_1, L_2, G_2 and C. The fifth action is obtained from the symplectic rectangular pair x_1/y_1 that is conveniently introduced in each case and appears in the last column of Table 5.2. In case (a), which deals with motions of the three bodies that are of non-circular and non-coplanar type, γ_1^* and G_1^* stand for the concrete values taken at the relative equilibrium on $\mathcal{T}_{L_1,C,G_2}$.

Although here we shall prove the existence of invariant 5-tori around circular solutions of the inner bodies (case (e) of Table 5.2), we remark that the proofs of the remaining cases in Table 5.2 appear in Appendix B.

The coordinates of the equilibrium point of case (e) in the space $\mathcal{T}_{L_1,C,G_2}$ are $(L_1^2, 0, 0)$. In this case it is assumed that $G_1 \approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \not\approx L_2$ and the motions of the two fictitious bodies are not coplanar, so $G_1 \not\approx |C \pm G_2|$.

	Well defined angles / actions	Variables in $\mathcal{T}_{L_1,C,G_2}$
(a) non-circular / non-circular non-coplanar	$C \not\approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2/G_2, \nu_1/C$ $C \approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2/G_2, \nu_1 \pm \nu/C$	$x_1 = \gamma_1 - \gamma_1^*$ $y_1 = G_1 - G_1^*$
(b) non-circular / non-circular coplanar with $G_1 \approx C + G_2$	$C \not\approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 - \nu_1/G_2,$ $\gamma_1 + \nu_1/C + G_2$ $C \approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 - \nu_1 \mp \nu/G_2,$ $\gamma_1 + \nu_1 \pm \nu/C + G_2$	$x_1 = \sqrt{2(C + G_2 - G_1)}\cos\gamma_1$ $y_1 = \sqrt{2(C + G_2 - G_1)}\sin\gamma_1$
(c) non-circular / non-circular coplanar with $G_1 \approx G_2 - C$	$C \not\approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 + \nu_1/G_2,$ $\gamma_1 - \nu_1/G_2 - C$ $C \approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 + \nu_1 \pm \nu/G_2,$ $\gamma_1 - \nu_1 \mp \nu/G_2 - C$	$x_1 = \sqrt{2(C + G_1 - G_2)}\cos\gamma_1$ $y_1 = -\sqrt{2(C + G_1 - G_2)}\sin\gamma_1$
(d) non-circular / non-circular coplanar with $G_1 \approx C - G_2$	$C \not\approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 + \nu_1/G_2,$ $\gamma_1 + \nu_1/C - G_2$ $C \approx \lvert B\rvert$: $\ell_1/L_1, \ell_2/L_2, \gamma_2 + \nu_1 \pm \nu/G_2,$ $\gamma_1 + \nu_1 \pm \nu/C - G_2$	$x_1 = \sqrt{2(G_1 + G_2 - C)}\cos\gamma_1$ $y_1 = -\sqrt{2(G_1 + G_2 - C)}\sin\gamma_1$
(e) circular / non-circular non-coplanar	$C \not\approx \lvert B\rvert$: $\ell_1 + \gamma_1/L_1, \ell_2/L_2, \gamma_2/G_2, \nu_1/C$ $C \approx \lvert B\rvert$: $\ell_1 + \gamma_1/L_1, \ell_2/L_2, \gamma_2/G_2,$ $\nu_1 \pm \nu/C$	$x_1 = \sqrt{2(L_1 - G_1)}\cos\gamma_1$ $y_1 = \sqrt{2(L_1 - G_1)}\sin\gamma_1$

Table 5.2: Cases studied in $\mathcal{T}_{L_1,C,G_2}$. In the first column we show the types of motions corresponding to elliptic relative equilibria. The second column accounts for the angles that are properly defined in each case, together with the corresponding actions. The upper sign of the expressions for the angles and actions is used for $C \approx B$ (prograde motions) whereas the lower one is used for $C \approx -B$ (retrograde motions). The last column presents the local variables for each case. The rectangular coordinates x_1 and y_1 satisfy $\{x_1, y_1\} = 1$ and are zero in the equilibrium point. All the motions are characterised in the fully-reduced space by isolated points.

First we introduce the symplectic change of coordinates given in Table 5.2(e) to deal with near-circular motions of the inner bodies and non-circular motions of the outer body that are non-coplanar with the inner ones. When $L_1 = G_1$, γ_1 is not properly defined but in this case $x_1 = y_1 = 0$. Thus, the transformation can be extended analytically to the origin of the $x_1 y_1$-plane provided that all the computations that we have to carry out satisfy the d'Alembert characteristic; see details in [42, 62]. As this characteristic is maintained, one can conclude that circular motions of the inner bodies can be analysed properly with these Poincaré-Deprit-like coordinates and that all the expressions are valid in a neighborhood of the circular trajectories of the inner bodies.

Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}\mathcal{K}_1 \;=\; & \frac{\mathcal{M} L_1^2}{16 L_2^3 G_2^5 (x_1^2 + y_1^2 - 2L_1)^2} \Big(15 (y_1^2 - x_1^2)(x_1^2 + y_1^2 - 4L_1) \\ & \times \Big((x_1^2 + y_1^2 - 2L_1 - 2G_2)^2 - 4C^2 \Big) \Big((x_1^2 + y_1^2 - 2L_1 + 2G_2)^2 - 4C^2 \Big) \\ & + 8 \Big(3 (x_1^2 + y_1^2 - 2L_1)^2 - 20 L_1^2 \Big) \Big(6 G_2^4 + G_2^2 \Big((x_1^2 + y_1^2 - 2L_1)^2 - 12 C^2 \Big) \\ & + 3 \Big((x_1^2 + y_1^2 - 2L_1)^2 - 4C^2 \Big)^2 \Big) \Big).\end{aligned}$$

We need to linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_1 \;=\; \varepsilon^{1/4} \bar{x}_1 + x_1^*, \quad y_1 \;=\; \varepsilon^{1/4} \bar{y}_1 + y_1^*,$$

where x_1^* and y_1^* are the values of x_1 and y_1 at the equilibrium, i.e. $(x_1^*, y_1^*) = (0, 0)$. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ introduced in (2.7) we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\mathrm{Kep}} + \varepsilon\, \mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{5.2}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{2 \mathcal{M} L_1}{L_2^3 G_2^5} \Big(& L_1 \Big(3 L_1^4 - 2 L_1^2 (3C^2 - G_2^2) + 3 (C^2 - G_2^2)^2 \Big) \\ & + 3 \varepsilon^{1/2} \Big(2 L_1^2 (C^2 + 3 G_2^2 - L_1^2) \bar{x}_1^2 \\ & + \Big(3 L_1^4 - 4 L_1^2 (2C^2 + G_2^2) + 5 (C^2 - G_2^2)^2 \Big) \bar{y}_1^2 \Big) \Big).\end{aligned} \tag{5.3}$$

The equilibrium in $\mathcal{T}_{L_1,C,G_2}$ associated with the motions we analyse is elliptic when the coefficients of $\bar{x}_1^2$ and $\bar{y}_1^2$ have the same sign. This happens in regions C, D and A and E when $p+q>1$ in Fig. 3.1. In particular the signs are all negative excepting for region A (and $p+q>1$) where they are positive. Moreover, the essential factors of the coefficients of $\bar{x}_1^2$ and $\bar{y}_1^2$ correspond respectively to the bifurcation lines Γ_1 and Γ_2.

Now, in order to apply Theorem 1.15 we introduce action-angle coordinates I_1, ϕ_1 such that $\{\phi_1, I_1\} = 1$ as follows:

$$\begin{aligned}\bar{x}_1 &= 2^{1/2} L_1^{-1/2} \left(\frac{3L_1^4 - 4L_1^2(2C^2+G_2^2) + 5(C^2-G_2^2)^2}{C^2+3G_2^2-L_1^2} \right)^{1/4} I_1^{1/2} \sin\phi_1,\\ \bar{y}_1 &= 2^{3/4} L_1^{1/2} \left(\frac{C^2+3G_2^2-L_1^2}{3L_1^4 - 4L_1^2(2C^2+G_2^2) + 5(C^2-G_2^2)^2} \right)^{1/4} I_1^{1/2} \cos\phi_1.\end{aligned}$$

The fact that the signs of the coefficients of $\bar{x}_1^2$ and $\bar{y}_1^2$ in (5.3) are the same guarantees that I_1 and ϕ_1 are well defined. After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}\mathcal{K}_1 &= -\frac{2\mathcal{M}L_1}{L_2^3 G_2^5} \Bigg(L_1\Big(3L_1^4 - 2L_1^2(3C^2-G_2^2) + 3(C^2-G_2^2)^2\Big)\\ &\quad + 6\varepsilon^{1/2} L_1 I_1 \sqrt{2(C^2+3G_2^2-L_1^2)\Big(3L_1^4 - 4L_1^2(2C^2+G_2^2) + 5(C^2-G_2^2)^2\Big)} \Bigg).\end{aligned}$$

Due to the fact that the perturbation appears scaled at two different orders we cannot apply the standard KAM theorems for degenerate Hamiltonians [4]. That is why we resort to Han, Li and Yi's Theorem. In order to get the Hamiltonian expressed in the form of this theorem we introduce a new parameter $\eta^2 = \varepsilon$. It leads to

$$\mathcal{H} = h_0 + \eta^2 h_1 + \eta^3 h_2 + \mathcal{O}(\eta^4),$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2},\\ h_1 &= -\frac{2\mathcal{M}L_1^2}{L_2^3 G_2^5}\Big(3L_1^4 - 2L_1^2(3C^2-G_2^2) + 3(C^2-G_2^2)^2\Big),\\ h_2 &= -\frac{12\mathcal{M}L_1^2}{L_2^3 G_2^5} I_1 \sqrt{2(C^2+3G_2^2-L_1^2)\big(3L_1^4 - 4L_1^2(2C^2+G_2^2) + 5(C^2-G_2^2)^2\big)}.\end{aligned} \tag{5.4}$$

At this point we easily identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_1}{\partial G_2}, \frac{\partial h_2}{\partial I_1} \right).$$

Now we form the matrix

$$\partial_I^1 \Omega(I) = \begin{pmatrix} \Omega_1 & \dfrac{\partial\Omega_1}{\partial L_1} & \dfrac{\partial\Omega_1}{\partial L_2} & \dfrac{\partial\Omega_1}{\partial C} & \dfrac{\partial\Omega_1}{\partial G_2} & \dfrac{\partial\Omega_1}{\partial I_1} \\ \Omega_2 & \dfrac{\partial\Omega_2}{\partial L_1} & \dfrac{\partial\Omega_2}{\partial L_2} & \dfrac{\partial\Omega_2}{\partial C} & \dfrac{\partial\Omega_2}{\partial G_2} & \dfrac{\partial\Omega_2}{\partial I_1} \\ \Omega_3 & \dfrac{\partial\Omega_3}{\partial L_1} & \dfrac{\partial\Omega_3}{\partial L_2} & \dfrac{\partial\Omega_3}{\partial C} & \dfrac{\partial\Omega_3}{\partial G_2} & \dfrac{\partial\Omega_3}{\partial I_1} \\ \Omega_4 & \dfrac{\partial\Omega_4}{\partial L_1} & \dfrac{\partial\Omega_4}{\partial L_2} & \dfrac{\partial\Omega_4}{\partial C} & \dfrac{\partial\Omega_4}{\partial G_2} & \dfrac{\partial\Omega_4}{\partial I_1} \\ \Omega_5 & \dfrac{\partial\Omega_5}{\partial L_1} & \dfrac{\partial\Omega_5}{\partial L_2} & \dfrac{\partial\Omega_5}{\partial C} & \dfrac{\partial\Omega_5}{\partial G_2} & \dfrac{\partial\Omega_5}{\partial I_1} \end{pmatrix}.$$

After replacing (5.4) in the frequency vector Ω, we deduce that the rank of the previous matrix is four, which is not enough. We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1 \Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions of the inner bodies.

According to Theorem 1.15 the excluding measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^{\delta/2})$ or $\mathcal{O}(\varepsilon^{\delta/4})$ with $0 < \delta < 1/5$. Calculating $b = \sum_{i=1}^{a} \beta_i (n_i - n_{i-1})$ we obtain $b = 7$. So, we cannot apply Remark 2 of [36] p. 1422 because $\eta^{sb+\delta} = \eta^{14+\delta} = \varepsilon^{(14+\delta)/2}$ and the perturbation in (5.2) is of a lower order (it is of order two). Thus, we cannot improve the measure for the existence of invariant tori.

5.2.2 Study in $\mathcal{S}_{L_1,L_2,C}$

Now we deal with the second item in Theorem 5.1. These are the cases collected in Table 5.3, that we study in the reduced space $\mathcal{S}_{L_1,L_2,C}$. As an example, here we develop the proof for case (g) in Table 5.3, which deals with circular motions of the outer body that are coplanar with the inner bodies' motion. The remaining cases are handled in Appendix B. As we know, coplanar motions satisfy $G_1 = |C - G_2|$ or $G_1 = C + G_2$, but $G_1 = C + G_2$ is not possible when $G_2 = L_2$. Here we have chosen the case where $G_2 \approx L_2$ and $G_1 \approx C - G_2$, i.e. the inner and outer bodies follow prograde orbits, that is $I_1 = I_2 = 0$. These motions are represented by an isolated singular point in $\mathcal{S}_{L_1,L_2,C}$, the point (e) in Subsection 2.4.2.

The coordinates of the equilibrium point of case (g) in $\mathcal{S}_{L_1,L_2,C}$ are $(2(L_2 - C)^2 - L_1^2, L_2^2, 0, 0, 0, 0)$. We start by introducing the symplectic change of Poincaré-Deprit-like variables appearing in Table 5.3(g). This set of coordinates desingu-

	Well defined angles / actions	Variables in $\mathcal{S}_{L_1,L_2,C}$
(a) circular / circular non-coplanar	$C \not\approx \|B\|$ $\ell_1+\gamma_1/L_1, \ell_2+\gamma_2/L_2, \nu_1/C$ $C \approx \|B\|$ $\ell_1+\gamma_1/L_1, \ell_2+\gamma_2/L_2, \nu_1 \pm \nu/C$	$x_1 = \sqrt{2(L_1-G_1)}\cos\gamma_1$ $y_1 = \sqrt{2(L_1-G_1)}\sin\gamma_1$ $x_2 = \sqrt{2(L_2-G_2)}\cos\gamma_2$ $y_2 = \sqrt{2(L_2-G_2)}\sin\gamma_2$
(b) circular / non-circular coplanar with $G_1 \approx C+G_2$	$C \not\approx \|B\|$ $\ell_1+\gamma_1+\nu_1/L_1, \ell_2/L_2, \gamma_2-\nu_1/L_1-C$ $C \approx \|B\|$ $\ell_1+\gamma_1+\nu_1 \pm \nu/L_1, \ell_2/L_2,$ $\gamma_2-\nu_1 \mp \nu/L_1-C$	$x_1 = \sqrt{2(L_1-G_1)}\cos(\gamma_1+\gamma_2)$ $y_1 = \sqrt{2(L_1-G_1)}\sin(\gamma_1+\gamma_2)$ $x_2 = \sqrt{2(C+G_2-G_1)}\cos\gamma_2$ $y_2 = -\sqrt{2(C+G_2-G_1)}\sin\gamma_2$
(c) circular / non-circular coplanar with $G_1 \approx G_2-C$	$C \not\approx \|B\|$ $\ell_1+\gamma_1-\nu_1/L_1, \ell_2/L_2, \gamma_2+\nu_1/C+L_1$ $C \approx \|B\|$ $\ell_1+\gamma_1-\nu_1 \mp \nu/L_1, \ell_2/L_2,$ $\gamma_2+\nu_1 \pm \nu/C+L_1$	$x_1 = \sqrt{2(L_1-G_1)}\cos(\gamma_1+\gamma_2)$ $y_1 = \sqrt{2(L_1-G_1)}\sin(\gamma_1+\gamma_2)$ $x_2 = \sqrt{2(C+G_1-G_2)}\cos\gamma_2$ $y_2 = \sqrt{2(C+G_1-G_2)}\sin\gamma_2$
(d) circular / non-circular coplanar with $G_1 \approx C-G_2$	$C \not\approx \|B\|$ $\ell_1+\gamma_1+\nu_1/L_1, \ell_2/L_2, \gamma_2+\nu_1/C-L_1$ $C \approx \|B\|$ $\ell_1+\gamma_1+\nu_1 \pm \nu/L_1, \ell_2/L_2,$ $\gamma_2+\nu_1 \pm \nu/C-L_1$	$x_1 = \sqrt{2(L_1-G_1)}\cos(\gamma_1-\gamma_2)$ $y_1 = \sqrt{2(L_1-G_1)}\sin(\gamma_1-\gamma_2)$ $x_2 = \sqrt{2(G_1+G_2-C)}\cos\gamma_2$ $y_2 = -\sqrt{2(G_1+G_2-C)}\sin\gamma_2$
(e) non-circular / circular non-coplanar	$C \not\approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2/L_2, \nu_1/C$ $C \approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2/L_2, \nu_1 \pm \nu/C$	$x_1 = \gamma_1-\gamma_1^*$ $y_1 = G_1-G_1^*$ $x_2 = \sqrt{2(L_2-G_2)}\cos\gamma_2$ $y_2 = \sqrt{2(L_2-G_2)}\sin\gamma_2$
(f) non-circular / circular coplanar with $G_1 \approx G_2-C$	$C \not\approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2+\nu_1/L_2, \gamma_1-\nu_1/L_2-C$ $C \approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2+\nu_1 \pm \nu/L_2,$ $\gamma_1-\nu_1 \mp \nu/L_2-C$	$x_1 = \sqrt{2(C+G_1-G_2)}\cos\gamma_1$ $y_1 = -\sqrt{2(C+G_1-G_2)}\sin\gamma_1$ $x_2 = \sqrt{2(L_2-G_2)}\cos(\gamma_1+\gamma_2)$ $y_2 = \sqrt{2(L_2-G_2)}\sin(\gamma_1+\gamma_2)$
(g) non-circular / circular coplanar with $G_1 \approx C-G_2$	$C \not\approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2+\nu_1/L_2, \gamma_1+\nu_1/C-L_2$ $C \approx \|B\|$ $\ell_1/L_1, \ell_2+\gamma_2+\nu_1 \pm \nu/L_2,$ $\gamma_1+\nu_1 \pm \nu/C-L_2$	$x_1 = \sqrt{2(G_1+G_2-C)}\cos\gamma_1$ $y_1 = -\sqrt{2(G_1+G_2-C)}\sin\gamma_1$ $x_2 = \sqrt{2(L_2-G_2)}\cos(\gamma_1-\gamma_2)$ $y_2 = -\sqrt{2(L_2-G_2)}\sin(\gamma_1-\gamma_2)$

Table 5.3: Cases studied in $\mathcal{S}_{L_1,L_2,C}$. The types of motions corresponding to elliptic relative equilibria are given in the first column. The second column accounts for the angles and their conjugate actions that are properly defined in each case. The variables x_i, y_i are zero in the equilibrium point and satisfy $\{x_i, y_i\} = -\{y_i, x_i\} = 1$ while the rest of Poisson brackets vanish. The upper sign of the expressions is used for $C \approx B$ while the lower one for $C \approx -B$. In the third column the local rectangular symplectic variables for each case are written down. All the motions are characterised in $\mathcal{S}_{L_1,L_2,C}$ by isolated points. Cases (a) and (e) correspond with regular points on the reduced space whereas the other points are singular.

larises $\mathcal{S}_{L_1,L_2,C}$ locally around the relative equilibrium and is well defined for circular motions of the outer body that are coplanar with the inner bodies' motions. This desingularisation process happens in all the cases where the relative equilibria are singular with the choices of the coordinates x_i, y_i indicated in Table 5.3. The angle γ_2 is not well defined when $G_2 = L_2$, but then $x_2 = y_2 = 0$. Besides, nor γ_1 nor γ_2 are defined when $G_1 = C - G_2$, but then $x_1 = y_1 = 0$. Moreover, the angle $\gamma_2 - \gamma_1$ is properly defined when $G_1 = C - G_2$ and when $G_2 = L_2$. All the computations satisfy the d'Alembert characteristic, so the transformation appearing in Table 5.3(g) can be extended analytically to the subset $x_1 = y_1 = x_2 = y_2 = 0$, see [42].

The expression of $\mathcal{K}_1$ in terms of x_1, x_2, y_1 and y_2 is

$$\begin{aligned}\mathcal{K}_1 \;=\; &-\frac{2\mathcal{M}L_1^2}{L_2^3\big(x_2^2+y_2^2-2L_2\big)^5\big(x_1^2+x_2^2+y_1^2+y_2^2+2C-2L_2\big)^2}\\ &\times\Bigg(15(y_1^2-x_1^2)(x_1^2+y_1^2+4C)\Big(\big(x_1^2+x_2^2+y_1^2+y_2^2+2C-2L_2\big)^2-4L_1^2\Big)\\ &\quad\times\big(x_1^2+2x_2^2+y_1^2+2y_2^2-4L_2\big)\big(x_1^2+2x_2^2+y_1^2+2y_2^2+4C-4L_2\big)\\ &\quad+\Big(3\big(x_1^2+x_2^2+y_1^2+y_2^2+2C-2L_2\big)^2-20L_1^2\Big)\\ &\quad\times\Big(3\big(x_2^2+y_2^2-2L_2\big)^4+3\Big(\big(x_1^2+x_2^2+y_1^2+y_2^2+2C-2L_2\big)^2-4C^2\Big)^2\\ &\qquad+2\big(x_2^2+y_2^2-2L_2\big)^2\Big(\big(x_1^2+x_2^2+y_1^2+y_2^2+2C-2L_2\big)^2-12C^2\Big)\Big)\Bigg).\end{aligned}$$

We linearise $\mathcal{K}_1$ around the equilibrium by introducing the symplectic change with multiplier $\varepsilon^{-1/2}$:

$$\begin{aligned} x_1 &= \varepsilon^{1/4}\bar{x}_1, & x_2 &= \varepsilon^{1/4}\bar{x}_2,\\ y_1 &= \varepsilon^{1/4}\bar{y}_1, & y_2 &= \varepsilon^{1/4}\bar{y}_2.\end{aligned}$$

After applying the transformation to $\mathcal{H}$ introduced in (2.7) we rescale time, ending up with the Hamiltonian

$$\mathcal{H} \;=\; \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{5.5}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; \frac{2\mathcal{M}L_1^2}{L_2^8(C-L_2)}\Bigg(&-4L_2^2(C-L_2)\big(5L_1^2-3(C-L_2)^2\big)\\ &+6\varepsilon^{1/2}L_2\Big(2(C-L_2)^2(L_2+C)\bar{x}_1^2\\ &\qquad+2\big(5L_1^2C+(C-L_2)^2(L_2-4C)\big)\bar{y}_1^2\\ &\qquad-(C-L_2)\big(5L_1^2+C^2-(L_2-2C)^2\big)(\bar{x}_2^2+\bar{y}_2^2)\Big)\Bigg).\end{aligned}$$

Let us note that the coefficients of $\bar{x}_1$ and $\bar{y}_1$ have negative sign for all the allowed values of the parameters whereas for $\bar{x}_2$ and $\bar{y}_2$ the coefficients are the same. Besides the eigenvectors of the associated linear vector field always form a basis of $\mathbb{R}^4$, therefore the relative equilibrium in $\mathcal{S}_{L_1,L_2,C}$ is linearly and parametrically stable for all the combinations of the parameters L_1, L_2 and C, even when possible resonances between the two degrees of freedom are allowed. This is compatible with the fact that the corresponding relative equilibrium is stable in $\mathcal{T}_{L_1,C,G_2}$. In fact, the equilibrium in $\mathcal{T}_{L_1,C,G_2}$ is $(2(L_2-C)^2-L_1^2,0,0)$ or $(2(p-q)^2-1,0,0)$ in $\mathcal{T}_{p,q}$, which is the red one in Figs. 3.3 and 3.4, where we note that it is always stable.

The next step is the introduction of the following symplectic set of action-angle coordinates:

$$\begin{aligned}
\bar{x}_1 &= \frac{\sqrt{2}}{\sqrt{C-L_2}}\left(\frac{5L_1^2C+(C-L_2)^2(L_2-4C)}{L_2+C}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\
\bar{y}_1 &= \sqrt{2(C-L_2)}\left(\frac{L_2+C}{5L_1^2C+(C-L_2)^2(L_2-4C)}\right)^{1/4} I_1^{1/2}\cos\phi_1,\\
\bar{x}_2 &= \sqrt{2I_2}\sin\phi_2,\\
\bar{y}_2 &= \sqrt{2I_2}\cos\phi_2.
\end{aligned}$$

The actions and angles satisfy $\{\phi_i, I_i\} = -\{I_i,\phi_i\} = 1$ and $\{\phi_i, I_j\} = 0$ if $i \neq j$. The radicands have constant positive sign. Now we apply the transformation to $\mathcal{K}_1$ getting

$$\begin{aligned}
\mathcal{K}_1 = \frac{8\mathcal{M}L_1^2}{L_2^7}\Big(&L_2\big(3(C-L_2)^2-5L_1^2\big)\\
&+3\varepsilon^{1/2}\Big(2I_1\sqrt{(L_2+C)\big(5L_1^2C+(C-L_2)^2(L_2-4C)\big)}\\
&+I_2\big((L_2-2C)^2-5L_1^2-C^2\big)\Big)\Big).
\end{aligned}$$

Now we express Hamiltonian $\mathcal{H}$ in the same form as Hamiltonian (1.15). It is achieved by introducing a new parameter $\eta^2=\varepsilon$, arriving at

$$\mathcal{H} = h_0+\eta^2 h_1+\eta^3 h_2+\mathcal{O}(\eta^4),$$

where

$$
\begin{aligned}
h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2},\\
h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^6}\big(3(C-L_2)^2 - 5L_1^2\big),\\
h_2 &= \frac{24\mathcal{M}L_1^2}{L_2^7}\Big(2I_1\sqrt{(L_2+C)\big(5L_1^2C + (C-L_2)^2(L_2-4C)\big)}\\
&\qquad + I_2\big((L_2-2C)^2 - 5L_1^2 - C^2\big)\Big).
\end{aligned}
\tag{5.6}
$$

We stress that the parameters L_1, L_2 and C can vary provided that $0 < C \leq L_1 + L_2$, $0 < L_1 < L_2$ so that there can be resonances between the degrees of freedom represented by ϕ_1/I_1 and ϕ_2/I_2. However when L_1, L_2 and C take values so that it leads to a resonant Hamiltonian h_2, the approach is valid since we do not need to average with respect to any angle ϕ_i.

One can identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 3$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$, then

$$
\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}\right).
$$

Now we build the matrix

$$
\partial_I^1 \Omega(I) = \begin{pmatrix}
\Omega_1 & \frac{\partial \Omega_1}{\partial L_1} & \frac{\partial \Omega_1}{\partial L_2} & \frac{\partial \Omega_1}{\partial C} & \frac{\partial \Omega_1}{\partial I_1} & \frac{\partial \Omega_1}{\partial I_2}\\
\Omega_2 & \frac{\partial \Omega_2}{\partial L_1} & \frac{\partial \Omega_2}{\partial L_2} & \frac{\partial \Omega_2}{\partial C} & \frac{\partial \Omega_2}{\partial I_1} & \frac{\partial \Omega_2}{\partial I_2}\\
\Omega_3 & \frac{\partial \Omega_3}{\partial L_1} & \frac{\partial \Omega_3}{\partial L_2} & \frac{\partial \Omega_3}{\partial C} & \frac{\partial \Omega_3}{\partial I_1} & \frac{\partial \Omega_3}{\partial I_2}\\
\Omega_4 & \frac{\partial \Omega_4}{\partial L_1} & \frac{\partial \Omega_4}{\partial L_2} & \frac{\partial \Omega_4}{\partial C} & \frac{\partial \Omega_4}{\partial I_1} & \frac{\partial \Omega_4}{\partial I_2}\\
\Omega_5 & \frac{\partial \Omega_5}{\partial L_1} & \frac{\partial \Omega_5}{\partial L_2} & \frac{\partial \Omega_5}{\partial C} & \frac{\partial \Omega_5}{\partial I_1} & \frac{\partial \Omega_5}{\partial I_2}
\end{pmatrix}.
$$

After replacing the concrete values of Hamiltonian (5.6) and its partial derivatives we deduce that the rank of this matrix is three, but it is not enough to conclude that there are invariant 5-tori. So, we add to this matrix the columns composed by the partials of second order and calculate the rank of this 5×31-matrix and get the desirable rank five. Therefore, there are KAM 5-tori related with the equilibrium point that represents circular motions of the outer body which are also coplanar with the inner bodies' motions.

In this case $b = 8$ and $s = 2$. So, the excluded measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^{\delta/2})$ (or $\mathcal{O}(\varepsilon^{\delta/4})$) with $0 < \delta < 1/5$ and we cannot improve this measure.

	Well defined angles / actions	Variables in $\mathcal{R}_{L_1,L_2,B}$
(a) circular / circular coplanar with $G_1 \approx G_2 - C$ and $C \not\approx \lvert B\rvert$	$\ell_1+\gamma_1-\nu_1/L_1, \ell_2+\gamma_2+\nu_1/L_2$	$x_1=\sqrt{2(L_1-G_1)}\cos(\gamma_1-\nu_1)$ $y_1=\sqrt{2(L_1-G_1)}\sin(\gamma_1-\nu_1)$ $x_2=\sqrt{2(L_2-G_2)}\cos(\gamma_2+\nu_1)$ $y_2=\sqrt{2(L_2-G_2)}\sin(\gamma_2+\nu_1)$ $x_3=\sqrt{2(C+G_1-G_2)}\cos\nu_1$ $y_3=-\sqrt{2(C+G_1-G_2)}\sin\nu_1$
(b) circular / circular coplanar with $G_1 \approx C - G_2$ and $C \not\approx \lvert B\rvert$	$\ell_1+\gamma_1+\nu_1/L_1, \ell_2+\gamma_2+\nu_1/L_2$	$x_1=\sqrt{2(L_1-G_1)}\cos(\gamma_1+\nu_1)$ $y_1=\sqrt{2(L_1-G_1)}\sin(\gamma_1+\nu_1)$ $x_2=\sqrt{2(L_2-G_2)}\cos(\gamma_2+\nu_1)$ $y_2=\sqrt{2(L_2-G_2)}\sin(\gamma_2+\nu_1)$ $x_3=\sqrt{2(G_1+G_2-C)}\cos\nu_1$ $y_3=\sqrt{2(G_1+G_2-C)}\sin\nu_1$

Table 5.4: Cases studied in $\mathcal{R}_{L_1,L_2,B}$. The types of motions corresponding to elliptic relative equilibria are given in the first column. In the second column the angles and their conjugate actions that are properly defined in each case are given. The local rectangular symplectic coordinates are written down in the third column and they satisfy $\{x_i, y_i\} = -\{y_i, x_i\} = 1$ and $\{x_i, y_j\} = 0$ if $i \neq j$. All the motions are characterised in $\mathcal{R}_{L_1,L_2,B}$ by isolated points. The two points are regular.

5.2.3 Study in $\mathcal{R}_{L_1,L_2,B}$

We show in Table 5.4 the cases – which correspond to the third item of Theorem 5.1 – where we have proved the existence of KAM 5-tori related to elliptic equilibrium solutions in the space $\mathcal{R}_{L_1,L_2,B}$. Specifically we deal with circular motions of the inner and outer bodies all of them moving in the same plane, which is not the horizontal plane. We choose the coplanar case that satisfies $G_1 \approx C - G_2$ and $C \not\approx |B|$ to develop our study. The remaining case has been achieved analogously (see Appendix B).

The equilibrium point in $\mathcal{R}_{L_1,L_2,B}$ that we study has coordinates $(\rho_1, \ldots, \rho_{16})$ with

$$\begin{aligned}
\rho_1 &= \rho_2 = \frac{L_1 B}{L_1+L_2},\\
\rho_3 &= \rho_4 = \frac{L_2 B}{L_1+L_2},\\
\rho_5 &= \rho_7 = \rho_9 = \rho_{11} = \rho_{13} = \rho_{15} = 0,
\end{aligned}$$

$$
\begin{aligned}
\rho_6 &= L_1^2\left(1-\frac{B^2}{(L_1+L_2)^2}\right),\\
\rho_8 = \rho_{10} = \rho_{12} = \rho_{14} &= L_1L_2\left(1-\frac{B^2}{(L_1+L_2)^2}\right),\\
\rho_{16} &= L_2^2\left(1-\frac{B^2}{(L_1+L_2)^2}\right).
\end{aligned}
$$

It is one of the points of (4.2) and is regular since $C \approx L_1 + L_2$ but $B \not\approx C$, so it is impossible that $B = L_1 + L_2$.

In order to analyse the dynamics in a neighborhood of the equilibrium point we define the Poincaré-Deprit-like coordinates appearing in Table 5.4(b). These coordinates are properly defined because, similarly to the previous cases, when the angles are not well defined the respective x_i and y_i are equal to zero. Moreover, all the functions and computations satisfy the d'Alembert characteristic. Hence, these coordinates make sense for circular coplanar motions of the inner and outer bodies. Thus, the transformation introduced in Table 5.4(b) can be extended analytically to the subset $x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = 0$.

The perturbation $\mathcal{K}_1$ in terms of x_i and y_i is

$$
\begin{aligned}
\mathcal{K}_1 = {} & \frac{2\mathcal{M}L_1^2}{L_2^3(x_1^2+y_1^2-2L_1)^2(x_2^2+y_2^2-2L_2)^5}\\
& \times\Bigg(15\big(x_1^2+y_1^2-4L_1\big)\big(2x_1^2+x_3^2+2y_1^2+y_3^2-4L_1\big)\\
& \quad\times\big(2x_2^2+x_3^2+2y_2^2+y_3^2-4L_2\big)\Big(4x_1x_3y_1y_3-(y_1^2-x_1^2)(y_3^2-x_3^2)\Big)\\
& \quad\times\Big(2x_1^2+2x_2^2+x_3^2+2y_1^2+2y_2^2+y_3^2-4(L_1+L_2)\Big)\\
& \quad-\Big(3\big(x_1^2+y_1^2-2L_1\big)^2-20L_1^2\Big)\\
& \qquad\times\Bigg(3\big(x_2^2+y_2^2-2L_2\big)^4\\
& \qquad\quad+3\Big(\big(x_1^2+y_1^2-2L_1\big)^2-\big(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-2(L_1+L_2)\big)^2\Big)^2\\
& \qquad\quad+2\big(x_2^2+y_2^2-2L_2\big)^2\\
& \qquad\qquad\times\Big(\big(x_1^2+y_1^2-2L_1\big)^2\\
& \qquad\qquad\quad-3\big(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-2(L_1+L_2)\big)^2\Big)\Bigg)\Bigg).
\end{aligned}
\tag{5.7}
$$

We linearise $\mathcal{K}_1$ around the point $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0$ by a symplectic change with multiplier $\varepsilon^{-1/4}$, given by

$$\begin{aligned} x_1 &= \varepsilon^{1/8}\bar{x}_1, \quad x_2 = \varepsilon^{1/8}\bar{x}_2, \quad x_3 = \varepsilon^{1/8}\bar{x}_3, \\ y_1 &= \varepsilon^{1/8}\bar{y}_1, \quad y_2 = \varepsilon^{1/8}\bar{y}_2, \quad y_3 = \varepsilon^{1/8}\bar{y}_3. \end{aligned} \tag{5.8}$$

This linear transformation is applied to Hamiltonian (2.7), multiplying by $\varepsilon^{1/4}$ to rescale time and expanding the resulting Hamiltonian in powers of ε, getting a Hamiltonian of the form:

$$\mathcal{H} = \mathcal{H}_{\mathrm{Kep}} + \varepsilon\mathcal{K}_1 + \mathcal{O}(\varepsilon^{7/4}), \tag{5.9}$$

where

$$\begin{aligned} \mathcal{K}_1 = -\frac{16\mathcal{M}L_1^4}{L_2^6}\Bigg(1 &+ \frac{3\varepsilon^{1/4}}{2L_1L_2}\Big(L_1(\bar{x}_2^2 - \bar{x}_3^2 + \bar{y}_2^2 - \bar{y}_3^2) + L_2(\bar{x}_1^2 - \bar{x}_3^2 + \bar{y}_1^2 - \bar{y}_3^2)\Big) \\ &+ \frac{3\varepsilon^{1/2}}{8L_1^2L_2^2}\bigg(L_1^2\Big(4(\bar{x}_2^2 + \bar{y}_2^2)^2 + (\bar{x}_3^2 + \bar{y}_3^2)^2 - 8(\bar{x}_2^2 + \bar{y}_2^2)(\bar{x}_3^2 + \bar{y}_3^2)\Big) \\ &\quad - L_2^2\Big((\bar{x}_1^2 + \bar{y}_1^2)^2 - (\bar{x}_3^2 + \bar{y}_3^2)^2 + 2\big(\bar{x}_1^2(9\bar{x}_3^2 - \bar{y}_3^2) - \bar{y}_1^2(\bar{x}_3^2 - 9\bar{y}_3^2)\big)\Big) \\ &\quad + L_1L_2\Big(3(\bar{x}_3^2 + \bar{y}_3^2)^2 + 6(\bar{x}_1^2 - \bar{x}_3^2 + \bar{y}_1^2 - \bar{y}_3^2)(\bar{x}_2^2 + \bar{y}_2^2) \\ &\qquad + 4\big(\bar{x}_1^2(\bar{y}_3^2 - 4\bar{x}_3^2) + \bar{y}_1^2(\bar{x}_3^2 - 4\bar{y}_3^2)\big)\Big) \\ &\quad + 40(L_1 + L_2)L_2\bar{x}_1\bar{x}_3\bar{y}_1\bar{y}_3\bigg)\Bigg). \end{aligned} \tag{5.10}$$

We introduce a symplectic transformation that allows us to express the Hamiltonian in the form required by Theorem 1.15. The change reads as follows:

$$\begin{aligned} \bar{x}_1 &= \sqrt{2I_1}\sin\phi_1, \qquad \bar{y}_1 = \sqrt{2I_1}\cos\phi_1, \\ \bar{x}_2 &= \sqrt{2I_2}\sin\phi_2, \qquad \bar{y}_2 = \sqrt{2I_2}\cos\phi_2, \\ \bar{x}_3 &= \sqrt{2I_3}\sin\phi_3, \qquad \bar{y}_3 = \sqrt{2I_3}\cos\phi_3. \end{aligned}$$

Hamiltonian $\mathcal{K}_1$ is expressed in the variables ϕ_i, I_i arriving at

$$\begin{aligned} \mathcal{K}_1 = -\frac{16\mathcal{M}L_1^2}{L_2^6}\Bigg(L_1^2 &+ \frac{3\varepsilon^{1/4}L_1}{L_2}\Big(L_2I_1 + L_1I_2 - (L_1 + L_2)I_3\Big) \\ &- \frac{3\varepsilon^{1/2}}{2L_2^2}\bigg(L_2^2I_1^2 - 4L_1^2I_2^2 - (L_1^2 + 3L_1L_2 + L_2^2)I_3^2 - 6L_1L_2I_1I_2 \\ &\quad + 2L_2\Big(3L_1 + 4L_2 - 5(L_1 + L_2)\cos\big(2(\phi_1 - \phi_3)\big)\Big)I_1I_3 \\ &\quad + 2L_1(4L_1 + 3L_2)I_2I_3\bigg)\Bigg). \end{aligned} \tag{5.11}$$

We average $\mathcal{K}_1$ with respect to $\phi_1 - \phi_3$ at first order (i.e. taking only one step in the Lie transformation) checking that no resonances between the angles occur as the generating function is always well defined. This averaging process is standard and has been explained in Chapter 1. The last step before the application of Theorem 1.15 is the introduction of a new parameter $\eta^4 = \varepsilon$, so that we get

$$\mathcal{H} = h_0 + \eta^4 h_1 + \eta^5 h_2 + \eta^6 h_3 + \mathcal{O}(\eta^7), \tag{5.12}$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}, \\ h_1 &= -\frac{16\mathcal{M}L_1^4}{L_2^6}, \\ h_2 &= -\frac{48\mathcal{M}L_1^3}{L_2^7}\Big(L_2 I_1 + L_1 I_2 - (L_1 + L_2) I_3\Big), \\ h_3 &= \frac{24\mathcal{M}L_1^2}{L_2^8}\Big(L_2^2 I_1^2 - 4L_1^2 I_2^2 - (L_1^2 + 3L_1L_2 + L_2^2) I_3^2 - 6L_1L_2I_1I_2 \\ &\qquad + 2L_2(3L_1 + 4L_2) I_1 I_3 + 2L_1(4L_1 + 3L_2) I_2 I_3\Big). \end{aligned} \tag{5.13}$$

The numbers in Theorem 1.15 are: $n_0 = 2$, $n_1 = 2$, $n_2 = 5$, $n_3 = 5$, $\beta_1 = 4$, $\beta_2 = 5$, $\beta_3 = 6$ and $a = 3$, then

$$\begin{aligned} \Omega \equiv\ & (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8,\ \Omega_9,\ \Omega_{10}) = \\ & \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial L_1}, \frac{\partial h_1}{\partial L_2}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_2}{\partial I_3}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}, \frac{\partial h_3}{\partial I_3}\right). \end{aligned} \tag{5.14}$$

Then, we form the 10×6-matrix

$$\partial_I^1 \Omega(I) = \begin{pmatrix} \Omega_1 & \frac{\partial \Omega_1}{\partial L_1} & \frac{\partial \Omega_1}{\partial L_2} & \frac{\partial \Omega_1}{\partial I_1} & \frac{\partial \Omega_1}{\partial I_2} & \frac{\partial \Omega_1}{\partial I_3} \\ \Omega_2 & \frac{\partial \Omega_2}{\partial L_1} & \frac{\partial \Omega_2}{\partial L_2} & \frac{\partial \Omega_2}{\partial I_1} & \frac{\partial \Omega_2}{\partial I_2} & \frac{\partial \Omega_2}{\partial I_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Omega_{10} & \frac{\partial \Omega_{10}}{\partial L_1} & \frac{\partial \Omega_{10}}{\partial L_2} & \frac{\partial \Omega_{10}}{\partial I_1} & \frac{\partial \Omega_{10}}{\partial I_2} & \frac{\partial \Omega_{10}}{\partial I_3} \end{pmatrix} \tag{5.15}$$

and replace (5.13) in Ω and $\partial_I^1 \Omega(I)$. We get that the rank of this matrix is five, so we conclude that there are KAM 5-tori related with circular motions of the inner and outer bodies all of them moving in the same plane, which is not the horizontal plane. Moreover, in this case $b = 15$ and $s = 1$ then, the excluded measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^\delta)$ (or $\mathcal{O}(\varepsilon^{\delta/4})$) with $0 < \delta < 1/5$ and as in the previous cases we cannot improve this measure.

5.2.4 Study in $\mathcal{A}_{L_1,L_2}$

We deal with the motions that have to be studied in the manifold $\mathcal{A}_{L_1,L_2}$. These cases are presented in Table 5.5 and they correspond to the fourth item of Theorem 5.1. In particular the equilibrium points of $\mathcal{A}_{L_1,L_2}$ are related with circular motions of the inner and outer bodies, all of them are nearly moving in the horizontal plane. We choose the case $G_1 \approx L_1$, $G_2 \approx L_2$, $G_1 \approx C - G_2$ and $C \approx |B|$, i.e. case (b) of Table 5.5. The invariant tori that we have found correspond to the ones determined by Féjoz [32] and by Chierchia and Pinzari in [12, 13] for the planetary N-body problem. Case (a) is achieved similarly and the proof appears in Appendix B.

	Well defined angles / actions	Variables in $\mathcal{A}_{L_1,L_2}$
(a) circular / circular coplanar with $G_1 \approx G_2 - C$ and $C \approx \lvert B\rvert$	$\ell_1 + \gamma_1 - \nu_1 \mp \nu / L_1$, $\ell_2 + \gamma_2 + \nu_1 \pm \nu / L_2$	$x_1 = \sqrt{2(L_1 - G_1)} \cos(\gamma_1 - \nu_1 \mp \nu)$ $y_1 = \sqrt{2(L_1 - G_1)} \sin(\gamma_1 - \nu_1 \mp \nu)$ $x_2 = \sqrt{2(L_2 - G_2)} \cos(\gamma_2 + \nu_1 \pm \nu)$ $y_2 = \sqrt{2(L_2 - G_2)} \sin(\gamma_2 + \nu_1 \pm \nu)$ $x_3 = \sqrt{2(C + G_1 - G_2)} \cos(\nu_1 \pm \nu)$ $y_3 = -\sqrt{2(C + G_1 - G_2)} \sin(\nu_1 \pm \nu)$ $x_4 = \sqrt{2(C - \lvert B\rvert)} \cos \nu$ $y_4 = \pm\sqrt{2(C - \lvert B\rvert)} \sin \nu$
(b) circular / circular coplanar with $G_1 \approx C - G_2$ and $C \approx \lvert B\rvert$	$\ell_1 + \gamma_1 + \nu_1 \pm \nu / L_1$, $\ell_2 + \gamma_2 + \nu_1 \pm \nu / L_2$	$x_1 = \sqrt{2(L_1 - G_1)} \cos(\gamma_1 + \nu_1 \pm \nu)$ $y_1 = \sqrt{2(L_1 - G_1)} \sin(\gamma_1 + \nu_1 \pm \nu)$ $x_2 = \sqrt{2(L_2 - G_2)} \cos(\gamma_2 + \nu_1 \pm \nu)$ $y_2 = \sqrt{2(L_2 - G_2)} \sin(\gamma_2 + \nu_1 \pm \nu)$ $x_3 = \sqrt{2(G_1 + G_2 - C)} \cos(\nu_1 \pm \nu)$ $y_3 = \sqrt{2(G_1 + G_2 - C)} \sin(\nu_1 \pm \nu)$ $x_4 = \sqrt{2(C - \lvert B\rvert)} \cos \nu$ $y_4 = \pm\sqrt{2(C - \lvert B\rvert)} \sin \nu$

Table 5.5: Cases studied in $\mathcal{A}_{L_1,L_2}$. The first column contains the types of motions corresponding to elliptic relative equilibria. The angles and actions that are well defined in each case are given in the second column. The local coordinates introduced to study these motions in $\mathcal{A}_{L_1,L_2}$ appear in the third column. These coordinates are symplectic and satisfy $\{x_i, y_i\} = -\{y_i, x_i\} = 1$ and $\{x_i, y_j\} = 0$ if $i \neq j$. The upper sign of the expressions is used for $C \approx B$ while the lower one for $C \approx -B$. All the motions are characterised in the reduced space by isolated points.

The coordinates of the relative equilibrium of case (b) in $\mathcal{A}_{L_1,L_2}$ are:

$$(0,\ 0,\ \pm L_1,\ 0,\ 0,\ \pm L_1,\ 0,\ 0,\ \pm L_2,\ 0,\ 0,\ \pm L_2)\,.$$

The local symplectic variables x_i, y_i are the ones given in the third column of case (b) in Table 5.5. They are Poincaré-Deprit-like coordinates. In the prograde case they correspond to the RPS coordinates (regularised planetary symplectic coordinates) introduced by Chierchia and Pinzari [12, 13]. The coordinates x_i, y_i for $i = 1, \ldots, 4$ are well defined because as in the previous cases when the angles are undetermined the respective x_i and y_i are equal to zero. In addition to it all the functions and computations satisfy the d'Alembert characteristic. So, they are properly defined and make sense for circular coplanar motions of the three bodies when the motion nearly occurs in the horizontal plane. Thus, the transformation introduced in Table 5.5(b) may be extended analytically to $x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = x_4 = y_4 = 0$.

The perturbation $\mathcal{K}_1$ in the coordinates x_i and y_i is the same as Hamiltonian (5.7) where instead of the term $4x_1x_3y_1y_3$ we put the term $\pm 4x_1x_3y_1y_3$ (the upper sign applies for prograde motions and the inner one for retrograde motions). We note that it is natural that $\mathcal{K}_1$ is independent of x_4 and y_4 as they are cyclic coordinates, i.e., integrals of motion. This is equivalent to saying that ν and B are integrals.

We linearise $\mathcal{K}_1$ around the origin through the change (5.8) and consider the full Hamiltonian $\mathcal{H}$ given by (2.7). We end up with a Hamiltonian of the form (5.9) where $\mathcal{K}_1$ is the same as the one of (5.10) with the term $40(L_1 + L_2)L_2\bar{x}_1\bar{x}_3\bar{y}_1\bar{y}_3$ has to be replaced by $\pm 40(L_1 + L_2)L_2\bar{x}_1\bar{x}_3\bar{y}_1\bar{y}_3$.

Following the same steps as in Section 5.2.3 we obtain the Hamiltonian (5.11) where the term $\cos\big(2(\phi_1 - \phi_3)\big)$ is $\cos\big(2(\phi_1 \mp \phi_3)\big)$. After averaging over $\phi_1 \mp \phi_3$ we end up with the Hamiltonian $\mathcal{H}$ given in (5.12) where the h_i are the ones appearing in (5.13).

At this point it is apparent the resonances of the N-body problem in the planetary regime pointed out by Herman and Féjoz [32] and Chierchia and Pinzari [10, 11, 12, 13]. In fact looking at h_2 in (5.13) it is clear that the frequency of the degree of freedom x_4/y_4 is zero and the sum of the frequencies related to x_1/y_1, x_2/y_2 and x_3/y_3 is also zero. (Note that these frequencies are indeed the coefficients of I_1, I_2 and I_3 in h_2.) These resonances do not affect the conclusions of our study as the average with respect to the linear combination $\phi_1 \mp \phi_3$ can be achieved straightforwardly.

Next, Han, Li and Yi's Theorem is applied to Hamiltonian (5.12) with the same numbers n_i, β_i and a as in the previous subsection. Thus the 10×6-matrix is of rank 5 with $s = 1$. Hence there are KAM 5-tori related with the equilibrium point that represents circular motions of the inner and outer bodies which are also coplanar motions when the invariable plane is the horizontal plane.

In this case $b = 15$ (as in Section 5.2.3), thus the excluded measure for the existence of quasi-periodic invariant tori cannot be improved and it is of order $\mathcal{O}(\eta^{\delta})$, i.e. of order $\mathcal{O}(\varepsilon^{\delta/4})$ with $0 < \delta < 1/5$. This concludes the proof of Theorem 5.1.

The proofs of the remaining cases of Tables 5.2, 5.3, 5.4 and 5.5 appear in Appendix B. There one can see that the values of the constants n_i, β_i, a and s are the same as the ones of the representative cases obtained in this chapter. Moreover the estimates on the excluding measure for the existence of quasi-periodic invariant tori agree with the estimates of the representative cases of each reduced space.

Finally we stress that according to Remark 1 we have not used the well defined actions of the second columns of Tables 5.2, 5.3, 5.4 and 5.5, as wee have shown through the representative cases written down in this chapter. Specifically the actions L_1, L_2, C and G_2 are used for the cases studied in the space $\mathcal{T}_{L_1,C,G_2}$, the actions L_1, L_2 and C for the cases studied in the space $\mathcal{S}_{L_1,L_2,C}$ and the actions L_1 and L_2 for the cases analysed in $\mathcal{R}_{L_1,L_2,B}$ and in $\mathcal{A}_{L_1,L_2}$.

Chapter 6

Invariant tori associated to rectilinear motions

In this chapter we reconstruct the rectilinear motions of the inner particles which are represented by elliptic relative equilibria of the fully-reduced space, with the aim of establishing the existence of KAM tori in the spatial three-body problem. The invariant tori corresponding to the Hamiltonian (2.3) are essentially of two types, accordingly to the nature of the elliptic relative equilibria studied in Chapter 2, either the inner particles follow trajectories perpendicular to the outer particle or all the particles move approximately in the same plane. Moreover we have to make other distinctions on the motion of the outer particle, basically if it follows a near-circular solution or not and if the total angular momentum vector is perpendicular to the horizontal plane or not. Similar cases are established in Chapter 5 or in [70] for non-rectilinear type of invariant tori. In Table 6.1 we give an account of all possible cases that we analyse within this chapter.

The analysis of rectilinear-type 5-invariant tori cannot be performed directly starting in $\mathcal{T}_{L_1,C,G_2}$. The reason is that working in the fully-reduced space we build a pair of action-angle coordinates getting an action, say I_1, while the other four actions are L_1, L_2, C and G_2. However for rectilinear motions, as they satisfy $G_2 = C$, we do not have five free actions to apply KAM theory.

When the motion of the fictitious inner body is orthogonal to the motion of the outer body we shall start our reconstruction of the quasi-periodic solutions in $\mathcal{S}_{L_1,L_2,C}$. When the outer body follows a near-circular trajectory we shall pass to the higher-dimensional space $\mathcal{R}_{L_1,L_2,B}$ provided $C \not\approx |B|$, and when the outer body has a near-circular trajectory and $C \approx |B|$ we shall make our analysis starting in $\mathcal{A}_{L_1,L_2}$.

In order to carry out the analysis of rectilinear-type invariant tori where the motions of all the bodies are almost coplanar we shall use an argument similar to the one used in the circular restricted three-body problem [64]. There the existence of near-rectilinear-equatorial quasi-periodic solutions filling in KAM 3-tori was established using an indirect reasoning. The idea was to study the invariant tori of equatorial type valid for all values of the third component of the angular momentum vector, say H, and then to make the limit $H \to 0$. Here we shall proceed in a similar way, working in $\mathcal{T}_{L_1,C,G_2}$ we shall establish the existence of invariant 5-tori for coplanar motions with $G_1 \approx |C - G_2|$ and $G_2 \not\approx L_2$, building a pair (I, ϕ) of action-angle coordinates. Then we shall calculate the limit $G_2 \to C$ checking that the corresponding torsion does not vanish in the limit process. We studied this type of invariant tori in [70], however we cannot take advantage of the approach we followed there since the resulting expressions were unbounded for G_2 tending to C. Thus we need to construct a different set of variables valid for this specific case. When the orbit of the outer body is circular we cannot apply a similar technique working in the space $\mathcal{S}_{L_1,L_2,C}$ because the equilibrium related with these kind of motions is not isolated.

Space	Dimension	Cases (inner / outer ellipses / relative inclination)
$\mathcal{T}_{L_1,C,G_2}$	2	rectilinear / non-circular / coplanar
$\mathcal{S}_{L_1,L_2,C}$	4	rectilinear / non-circular / orthogonal
$\mathcal{R}_{L_1,L_2,B}$	6	rectilinear / circular with $C \not\approx |B|$ / orthogonal
$\mathcal{A}_{L_1,L_2}$	8	rectilinear / circular with $C \approx |B|$ / orthogonal

Table 6.1: Reduced spaces where we have carried out the analysis of the different relative equilibria of rectilinear character. There are KAM 5-tori of the full system associated with each type of motion on the right column

6.1 Invariant 5-tori reconstructed from $\mathcal{S}_{L_1,L_2,C}$

6.1.1 Construction of symplectic coordinates

We are interested in the points of $\mathcal{T}_{L_1,C,G_2}$ with coordinates $(-L_1^2, \pm 2L_1C, 0)$ representing rectilinear motions of the fictitious inner body which are orthogonal to the invariable plane while the outer body follows a non-circular trajectory. In Deprit's coordinates these points are given by $\gamma_1 = \pi/2$ or $3\pi/2$, $G_1 = 0$, $G_2 = C$ and $\gamma_2 \in [0, 2\pi)$ whereas in the space $\mathcal{S}_{L_1,L_2,C}$ the corresponding coordinates are $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1C, 0, 0, 0)$, see Section 4.1. As we have seen in Chapter 3 the relative equilibria are isolated points of $\mathcal{T}_{L_1,C,G_2}$ and the corresponding points of $\mathcal{S}_{L_1,L_2,C}$ are also isolated, see for example [70].

We introduce local rectangular coordinates by means of Deprit's variables γ_1, γ_2, G_1 and G_2, making use of some formulas derived in the context of the restricted three-body problem. It is achieved through the relation between Delaunay coordinates with the invariants $\bar{a}_i$ and $\bar{b}_i$, $i = 1, 2, 3$ of the Kepler reduction of the circular restricted three-body problem, see [20] and [68]:

$$
\begin{aligned}
G &= \tfrac{1}{\sqrt{2}}\sqrt{L^2 + \bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \bar{a}_3\bar{b}_3},\\
\sin g &= \frac{(\bar{a}_3 - \bar{b}_3)\sqrt{L^2 + \bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \bar{a}_3\bar{b}_3}}{\sqrt{\bar{a}_1^2 + \bar{a}_2^2 + \bar{b}_1^2 + \bar{b}_2^2 + 2(\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2)}\sqrt{L^2 - \bar{a}_1\bar{b}_1 - \bar{a}_2\bar{b}_2 - \bar{a}_3\bar{b}_3}},\\
\cos g &= \frac{\sqrt{2}(\bar{a}_2\bar{b}_1 - \bar{a}_1\bar{b}_2)}{\sqrt{\bar{a}_1^2 + \bar{a}_2^2 + \bar{b}_1^2 + \bar{b}_2^2 + 2(\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2)}\sqrt{L^2 - \bar{a}_1\bar{b}_1 - \bar{a}_2\bar{b}_2 - \bar{a}_3\bar{b}_3}}.
\end{aligned}
\tag{6.1}
$$

The expressions of H, $\sin h$ and $\cos h$ in terms of the invariants $\bar{a}_i$ and $\bar{b}_i$ have been calculated in [20, 68]. They are

$$
\begin{aligned}
H &= \tfrac{1}{2}(\bar{a}_3 + \bar{b}_3),\\
\sin h &= \frac{\bar{a}_1 + \bar{b}_1}{\sqrt{\bar{a}_1^2 + \bar{a}_2^2 + \bar{b}_1^2 + \bar{b}_2^2 + 2(\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2)}},\\
\cos h &= -\frac{\bar{a}_2 + \bar{b}_2}{\sqrt{\bar{a}_1^2 + \bar{a}_2^2 + \bar{b}_1^2 + \bar{b}_2^2 + 2(\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2)}}.
\end{aligned}
\tag{6.2}
$$

Now, the canonical coordinates Q_i, P_i with symplectic structure $dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2$ are introduced in terms of the invariants $\bar{a}_i$ and $\bar{b}_i$. We use the expression given in [88] for the case of rectilinear motions orthogonal to the invariable place, i.e. $g = \pi/2$ or $3\pi/2$ and $G = 0$, namely,

$$
\begin{aligned}
Q_1 &= \frac{\bar{a}_2}{\sqrt{L \pm \bar{a}_3}}, \quad Q_2 = \frac{\bar{b}_2}{\sqrt{L \pm \bar{b}_3}},\\
P_1 &= \mp\frac{\bar{a}_1}{\sqrt{L \pm \bar{a}_3}}, \quad P_2 = \pm\frac{\bar{b}_1}{\sqrt{L \pm \bar{b}_3}}.
\end{aligned}
\tag{6.3}
$$

Its inverse is given by

$$\begin{aligned}\bar{a}_1 &= \mp P_1\sqrt{2L-Q_1^2-P_1^2}, \ \ \bar{a}_2 = Q_1\sqrt{2L-Q_1^2-P_1^2}, \ \ \bar{a}_3 = \mp(L-Q_1^2-P_1^2),\\ \bar{b}_1 &= \pm P_2\sqrt{2L-Q_2^2-P_2^2}, \ \ \bar{b}_2 = Q_2\sqrt{2L-Q_2^2-P_2^2}, \ \ \bar{b}_3 = \pm(L-Q_2^2-P_2^2).\end{aligned} \tag{6.4}$$

We are in the position of constructing the required change of coordinates. The idea is to combine formulas (2.26) with (6.1), (6.2) and (6.4), replacing L by L_1, G by G_1, H by $G_2 - C$, g by γ_1 and h by γ_2. After some manipulations and simplifications the final form of the change of coordinates is

$$\begin{aligned}\sigma_1 &= -L_1^2 + L_1(Q_1^2+Q_2^2+P_1^2+P_2^2) - (Q_1^2+P_1^2)(Q_2^2+P_2^2)\\ &\quad + (Q_1Q_2 - P_1P_2)\sqrt{(2L_1-Q_1^2-P_1^2)(2L_1-Q_2^2-P_2^2)},\\ \sigma_2 &= -L_2^2 + \tfrac{1}{2}(2C - Q_1^2 + Q_2^2 - P_1^2 + P_2^2)^2,\\ \sigma_3 &= \pm\tfrac{1}{4}(2L_1 - Q_1^2 - Q_2^2 - P_1^2 - P_2^2)\sqrt{\mathcal{F}},\\ \sigma_4 &= \frac{P_2\sqrt{2L_1-Q_2^2-P_2^2} - P_1\sqrt{2L_1-Q_1^2-P_1^2}}{4(2C-Q_1^2+Q_2^2-P_1^2+P_2^2)}\\ &\quad \times\sqrt{\big(4L_2^2-(2C-Q_1^2+Q_2^2-P_1^2+P_2^2)^2\big)\mathcal{F}},\\ \sigma_5 &= \pm\tfrac{1}{4}(Q_1P_2+Q_2P_1)\sqrt{(2L_1-Q_1^2-P_1^2)(2L_1-Q_2^2-P_2^2)\mathcal{F}},\\ \sigma_6 &= -\tfrac{1}{8}\big(Q_1\sqrt{2L_1-Q_1^2-P_1^2}+Q_2\sqrt{2L_1-Q_2^2-P_2^2}\big)\\ &\quad \times\sqrt{\big(4L_2^2-(2C-Q_1^2+Q_2^2-P_1^2+P_2^2)^2\big)\mathcal{F}},\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}\mathcal{F} &= 16C^2 - 2L_1(Q_1^2+Q_2^2+P_1^2+P_2^2) - 8C(Q_1^2-Q_2^2+P_1^2-P_2^2) + (Q_1^2+P_1^2)^2\\ &\quad + (Q_2^2+P_2^2)^2 - 2(Q_1Q_2-P_1P_2)\sqrt{(2L_1-Q_1^2-P_1^2)(2L_1-Q_2^2-P_2^2)}.\end{aligned}$$

The upper signs correspond to the prograde case and the lower to the retrograde one. When $Q_i = P_i = 0$ for $i = 1, 2$ then $\sigma_1 = -L_1^2$, $\sigma_2 = 2C^2 - L_2^2$, $\sigma_3 = \pm 2L_1C$, $\sigma_4 = \sigma_5 = \sigma_6 = 0$, which correspond to the relative equilibria we are analysing.

The previous transformations are all symplectic by construction, see [88] and Delaunay elements are symplectic as well so the final transformation (6.5) is symplectic. We have verified that it is true by computing the Poisson structure on $S_{L_1,L_2,C}$ in the σ_i i.e., the Poisson brackets $\{\sigma_i, \sigma_j\}$, $i, j = 1, \ldots, 6$. The procedure is carried out by using the formulas (2.21) together with the Poisson structure on $\mathcal{A}_{L_1,L_2}$ in the Keplerian invariants, which is already known, see [68]. So far the Poisson brackets $\{\sigma_i, \sigma_j\}$ are written in terms of a_i, b_i, c_i and d_i and we determine a Gröbner basis of the Keplerian invariants and the σ_i and apply the division algorithm for multivariate polynomials with respect to the Gröbner basis, in order to write down the expressions of $\{\sigma_i, \sigma_j\}$ in terms of the σ_i. The resulting formulas are rational functions of the σ_i. Alternatively we have computed the Poisson brackets among the σ_i in (6.5) by setting that the Q_i, P_i are symplectic. The Poisson brackets obtained through both approaches agree. We have checked that the two constraints that define $S_{L_1,L_2,C}$ are trivially satisfied after replacing the σ_i in terms of Q_i and P_i. From a computer algebra point of view our procedure is an application of Gröbner bases theory and the issue of writing down a polynomial in an ideal using a given set of generators, see [87].

6.1.2 Expansion in the Q_i and P_i variables and normal form computations

We apply the change (6.5) to Hamiltonian $\mathcal{K}_1$ given in (2.29) and introduce the stretching $Q_i = \varepsilon^{1/8}\bar{Q}_i$, $P_i = \varepsilon^{1/8}\bar{P}_i$, $i = 1, 2$, which is a canonical transformation with multiplier $\varepsilon^{-1/4}$. After rescaling time and expanding in powers of ε, Hamiltonian $\mathcal{H}$ introduced in (2.7) results in

$$\mathcal{H} = \mathcal{H}_{\text{Kep}} + \varepsilon\,\mathcal{K}_{10} + \varepsilon^{5/4}\,\mathcal{K}_{12} + \varepsilon^{3/2}\,\mathcal{K}_{14} + \mathcal{O}(\varepsilon^2), \tag{6.6}$$

where

$$\begin{aligned}
\mathcal{K}_{10} &= \frac{80\mathcal{M}L_1^4}{L_2^3C^3},\\
\mathcal{K}_{12} &= -\frac{3\mathcal{M}L_1^3}{L_2^3C^5}\Big((5L_1^2 - 60L_1C + 32C^2)(\bar{Q}_1^2 + \bar{P}_1^2)\\
&\qquad + (5L_1^2 + 60L_1C + 32C^2)(\bar{Q}_2^2 + \bar{P}_2^2)\\
&\qquad + 2(5L_1^2 - 8C^2)(\bar{Q}_1\bar{Q}_2 - \bar{P}_1\bar{P}_2)\Big),
\end{aligned}$$

and $\mathcal{K}_{14}$ is a homogeneous polynomial of degree four in $\bar{Q}_i$ and $\bar{P}_i$ that we do not write down explicitly.

Excepting for the constant factor in $\mathcal{K}_{12}$ that we will incorporate later, the eigenvalues associated to $\mathcal{K}_{12}$ are

$$\begin{aligned}
\pm 8\sqrt{10}C\sqrt{6C^2 + L_1(25L_1 + 3\sqrt{5(5L_1^2 + 12C^2)})}\,\imath &= \pm\omega_1\imath,\\
\pm 8\sqrt{10}C\sqrt{6C^2 + L_1(25L_1 - 3\sqrt{5(5L_1^2 + 12C^2)})}\,\imath &= \pm\omega_2\imath.
\end{aligned} \tag{6.7}$$

Taking into account that $L_2 > L_1$ and $0 < C \leq L_1 + L_2$ we have that $\omega_1 > \omega_2 \geq 0$ and $\omega_2 = 0$ if and only if $L_1 = \sqrt{3/10}C$. The expressions of C and L_1 in terms of

ω_1 and ω_2 are

$$L_1 \;=\; \frac{\omega_1-\omega_2}{4\cdot 3^{1/4}\cdot 5^{3/4}(2\omega_1+\omega_2)^{1/4}(\omega_1+2\omega_2)^{1/4}}, \quad C \;=\; \frac{(2\omega_1+\omega_2)^{1/4}(\omega_1+2\omega_2)^{1/4}}{4\cdot 3^{3/4}\cdot 5^{1/4}}. \tag{6.8}$$

The next step is the diagonalisation of $\mathcal{K}_{12}$ using the eigenvalues and eigenvectors of the matrix associated to it, see for instance [49, 15]. This process is carried out by constructing a transformation matrix whose columns are the eigenvectors of the linearised vector field of $\mathcal{K}_{12}$ multiplied by some constants which make the change symplectic. The quadratic part of the Hamiltonian after diagonalisation is essentially

$$\mp\omega_1\imath\tilde{Q}_1\tilde{P}_1 \pm \omega_2\tilde{Q}_2\tilde{P}_2 \tag{6.9}$$

and higher-order terms, e.g. the terms $\mathcal{K}_{14}$, are transformed accordingly.

In order to eliminate the unessential terms from the Hamiltonian written in the $\tilde{Q}_i$, $\tilde{P}_i$ we normalise it using a single step of a Lie transformation. The procedure lies in the setting of normal form for simple equilibrium points, briefly outlined in Chapter 1, and it is indeed the Birkhoff normal form approach, see for example [72]. To carry out this transformation we examine the possible resonances occurring between ω_1 and ω_2 since the resonant terms have to be kept in the normal form Hamiltonian. Given a monomial of $\mathcal{K}_{14}$, say $\beta\tilde{Q}_1^i\tilde{Q}_2^j\tilde{P}_1^k\tilde{P}_2^l$ with $i+j+k+l=4$, we have checked that, regardless if ω_1/ω_2 is rational or not, it must be retained in the normalised Hamiltonian if and only if $i=k$ and $j=l$, with the only exception that $L_1 \neq \sqrt{3/10}C$. In other words the combination $-\omega_1(i-k)+\omega_2(j-l)=0$ if and only if $i=k$, $j=l$ and $L_1\neq\sqrt{3/10}C$. We exclude the case $L_1=\sqrt{3/10}C$.

6.1.3 Quasi-periodic solutions related to the points $(-L_1^2, \pm 2L_1C, 0)$

Our aim is to apply KAM theory, so we introduce the following symplectic set of action-angle coordinates:

$$\tilde{Q}_i \;=\; \sqrt{I_i}\,(\cos\phi_i-\imath\sin\phi_i)\,, \quad \tilde{P}_i \;=\; \sqrt{I_i}\,(\sin\phi_i-\imath\cos\phi_i)\,, \quad 1\le i\le 2. \tag{6.10}$$

The symplectic structure of the action-angle coordinates is $dI_1\wedge d\phi_1+dI_2\wedge d\phi_2$. After applying this transformation to the Hamiltonian and introducing a new parameter $\eta^4=\varepsilon$, the full Hamiltonian (2.7) reads as

$$\mathcal{H} \;=\; h_0+\eta^4h_1+\eta^5h_2+\eta^6h_3+\mathcal{O}(\eta^8), \tag{6.11}$$

with

$$
\begin{aligned}
h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}, \\
h_1 &= \frac{80\mathcal{M}L_1^4}{L_2^3 C^3}, \\
h_2 &= -\frac{1296\mathcal{M}(\omega_1-\omega_2)^3}{5L_2^3(2\omega_1+\omega_2)^2(\omega_1+2\omega_2)^2}(\omega_1 I_1 - \omega_2 I_2), \\
h_3 &= \frac{32(3/5)^{3/4}\mathcal{M}(\omega_1-\omega_2)^2}{L_2^3(\omega_1+\omega_2)^2(2\omega_1+\omega_2)^{9/4}(\omega_1+2\omega_2)^{9/4}} \\
&\quad \times \Big((650\omega_1^4 + 997\omega_1^3\omega_2 + 21\omega_1^2\omega_2^2 - 341\omega_1\omega_2^3 - 31\omega_2^4)I_1^2 \\
&\qquad + (-31\omega_1^4 - 341\omega_1^3\omega_2 + 21\omega_1^2\omega_2^2 + 997\omega_1\omega_2^3 + 650\omega_2^4)I_2^2 \\
&\qquad + 8(32\omega_1^4 - 83\omega_1^3\omega_2 - 222\omega_1^2\omega_2^2 - 83\omega_1\omega_2^3 + 32\omega_2^4)I_1 I_2\Big),
\end{aligned}
\tag{6.12}
$$

where h_2 and h_3 are, respectively, the transformed Hamiltonians of the normalised Hamiltonian through (6.10). We remark that h_3 does not depend on any combination of ϕ_1 and ϕ_2 because the only terms retained in the normal form Hamiltonian are those whose exponents satisfy $i = k$ and $j = l$. The expressions h_i are the same for the prograde and the retrograde situations.

We are ready to use KAM theory in order to conclude the existence of KAM 5-tori related to near-rectilinear motions of the inner particles moving nearly in the axis perpendicular to the invariable plane. Two of the five actions we choose are I_1 and I_2 with conjugate angles ϕ_1 and ϕ_2 respectively. Another action is C with conjugate angle ν_1 when $C \not\approx |B|$ or the combination $\nu_1 \pm \nu$, when $C \approx \pm B$, so that we do not need to care whether $\mathbf{C}$ is orthogonal to the horizontal plane or not. The fourth and fifth actions are L_1 and L_2 with conjugate angles ℓ_1 and ℓ_2. Indeed we can choose ℓ_1 even when this anomaly is not well defined for rectilinear motions of the fictitious inner body. The reason is that, accordingly to our regularisation process made in Chapter 2, if we consider the unperturbed part of the Hamiltonian function as $D_1 - \mu_2^3 M_2^2/(2L_2^2)$, for the flow defined by the Hamiltonian D_1 the period T is constant on energy levels h and $dT/dh \neq 0$ even at the regularised collision orbits. A similar idea in the context of the restricted three-body problem is employed in [64] (Section 2.4). See also Subsection 2.2.3 and Remark 1 of Chapter 5.

We use Theorem 1.15 as it has been done in Chapter 5, which works in the case of Hamiltonian systems with high-order proper degeneracy. One can identify the following numbers in Han, Li, Yi's Theorem [36]: $n_0 = 2$, $n_1 = 3$, $n_2 = 5$, $n_3 = 5$,

$\beta_1 = 4$, $\beta_2 = 5$, $\beta_3 = 6$ and $a = 3$, and define $I = (L_1, L_2, C, I_1, I_2)$ and

$$\Omega \equiv \left(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7\right) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}\right).$$

Now we build the 7×6-matrix

$$\partial_I^1 \Omega(I) = \left(\Omega_k \, , \frac{\partial \Omega_k}{\partial L_1} \, , \frac{\partial \Omega_k}{\partial L_2} \, , \frac{\partial \Omega_k}{\partial C} \, , \frac{\partial \Omega_k}{\partial I_1} \, , \frac{\partial \Omega_k}{\partial I_2}\right), \quad 1 \le k \le 7.$$

After replacing the concrete values of Hamiltonian (6.12) and its partial derivatives and expressing the ω_i in terms of L_i, C and I_i we have verified that the rank of $\partial_I^1 \Omega(I)$ is five. Therefore there are KAM 5-tori related with the equilibria that represents rectilinear motions of the inner particles orthogonal to the invariable plane.

According to Remark 2 of [36] p. 1422 the excluding measure for the existence of these invariant tori is of order $\mathcal{O}(\eta^\delta)$ or $\mathcal{O}(\varepsilon^{\delta/4})$ with $0 < \delta < 1/5$. It cannot be improved because $b = \sum_{i=1}^{a} \beta_i (n_i - n_{i-1}) = 14$ and $s = 1$ (where s denotes the highest order of derivation), thus $\eta^{sb+\delta} = \eta^{14+\delta} = \varepsilon^{(14+\delta)/4}$ and the perturbation in (6.6) is of a lower order, it is indeed of order $7/4$.

We finish the section with the following result.

Theorem 6.1. *The Hamiltonian system of the spatial three-body problem (2.3) (or, equivalently, Hamiltonian (2.7)), reduced by the symmetry of translations and defined in* $\mathcal{Q}_{\varepsilon,n} \subseteq T^*\mathbb{R}^6$*, has invariant KAM* 5*-tori densely filled with quasi-periodic trajectories provided* $L_1 \not\approx \sqrt{3/10}C$*. In these quasi-periodic solutions the fictitious inner body moves in orbits that are nearly rectilinear, bounded and perpendicular to the invariable plane whereas the outer body moves in a non-circular orbit lying near the invariable plane. For a given* δ *such that* $0 < \delta < 1/5$*, the excluding measure for the existence of invariant* 5*-tori is of order* $\mathcal{O}(\varepsilon^{\delta/4})$.

6.1.4 Stability of the points $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1C, 0, 0, 0)$ in $\mathcal{S}_{L_1,L_2,C}$

We can use the analysis of the previous subsections to study the stability of the points representing rectilinear motions of the inner particles that are perpendicular to the invariable plane, on the reduced space $\mathcal{S}_{L_1,L_2,C}$. The analysis is performed for $C \neq \sqrt{10/3}L_1$. Specifically, looking at (6.9) or at h_2 in (6.12) where the frequencies ω_i are introduced in (6.7), it is straightforward to deduce that the points $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1C, 0, 0, 0)$ are parametrically stable in $\mathcal{S}_{L_1,L_2,C}$ because the Hamiltonian h_2 is not in $1:-1$ resonance; see a characterisation of parametric stability in [62, 88].

Since $\mathcal{S}_{L_1,L_2,C}$ is four-dimensional and (6.11) defines a Hamiltonian with two degrees of freedom, we can study the non-linear stability of these points, using Arnold's Theorem [83, 65]. To achieve this we need to compute $h_3(\omega_2, \omega_1)$ where h_3 is taken from (6.12) and check that it does not vanish. After some simplifications we get

$$h_3(\omega_2, \omega_1) \; = \; \frac{\mathcal{C}(\omega_1 - \omega_2)^4(31\omega_1^4 + 147\omega_1^3\omega_2 + 256\omega_1^2\omega_2^2 + 147\omega_1\omega_2^3 + 31\omega_2^4)}{L_2^3(\omega_1 + \omega_2)^2(2\omega_1 + \omega_2)^{9/4}(\omega_1 + 2\omega_2)^{9/4}},$$

where $\mathcal{C} = -32(3/5)^{3/4}\mathcal{M}$ and h_3 is not null as $\omega_1 > \omega_2 > 0$. Thus,

Proposition 6.2. *The equilibrium points* $(-L_1^2, 2C^2 - L_2^2, \pm 2L_1C, 0, 0, 0)$ *are stable on the orbit space* $\mathcal{S}_{L_1,L_2,C}$ *provided* $C \neq \sqrt{10/3}L_1$.

6.2 Invariant 5-tori reconstructed from $\mathcal{R}_{L_1,L_2,B}$

To develop the study of the quasi-periodic rectilinear motions perpendicular to the invariable while the outer body follows a circular trajectory, we distinguish two different situations, the first one deals with the case that the invariable plane is not the horizontal one and the second one when both planes are the same. This first case is studied in $\mathcal{R}_{L_1,L_2,B}$.

6.2.1 Construction of symplectic coordinates

We make use of the averaged Hamiltonian written in terms of the Keplerian invariants (2.21) and the formulas (2.46) that put the invariants ρ_i as functions of the a_i, b_i, c_i and d_i. The points of $\mathcal{R}_{L_1,L_2,B}$ given in (4.2) stand for relative equilibria of the flow defined by the averaged Hamiltonian (2.17) in the reduced space $\mathcal{R}_{L_1,L_2,B}$. Specifically they represent rectilinear motions of the fictitious inner body such that it moves along the axis defined by $\mathbf{C}$, while the outer body describes a circular trajectory in the invariable plane. These points are isolated in $\mathcal{R}_{L_1,L_2,B}$ and in Deprit's action-angle coordinates are defined by $\gamma_1 = \pi/2$ or $3\pi/2$, $G_1 = 0$, $G_2 = C = L_2$. We remark that as we saw in Section 4.2, the point in (4.2) with the upper signs is reconstructed from the point $(-L_1^2, 2L_1C, 0)$ of $\mathcal{T}_{L_1,C,G_2}$ while the one with the lower signs is reconstructed from the point $(-L_1^2, -2L_1C, 0)$.

We need to construct symplectic rectangular coordinates in $\mathcal{R}_{L_1,L_2,B}$, say Q_i, P_i $i = 1, 3$, so that we can study the flow in a neighbourhood of the equilibria, establishing the existence of KAM tori. We plan to introduce the Q_i, P_i starting from the Keplerian invariants and using a set of canonical action-angle coordinates defined by André Deprit to deal with the dynamics of the rigid body [23]. These variables have been used recently in [16, 17, 79] in the context of construction of canonical coordinates for the reduced space of a cotangent bundle with a free Lie group action.

Setting $\mathcal{V} = \sqrt{L_2^2 - B^2}$ the coordinates a_i, b_i, c_i and d_i for the points (4.2) are

$$\begin{array}{c} a_1 = -b_1 = \pm\dfrac{L_1\mathcal{V}\sin\nu}{L_2}, \; a_2 = -b_2 = \mp\dfrac{L_1\mathcal{V}\cos\nu}{L_2}, \; a_3 = -b_3 = \pm\dfrac{L_1 B}{L_2}, \\ c_1 = d_1 = \mathcal{V}\sin\nu, \quad c_2 = d_2 = -\mathcal{V}\cos\nu, \quad c_3 = d_3 = B. \end{array} \tag{6.13}$$

The idea is to introduce four pairs of rectangular coordinates, say Q_i/P_i, $i = 1, \ldots, 4$, related with **a**, **b**, **c** and **d**. We detail our procedure for **a**. Since the a_i are coordinates in the sphere $|\mathbf{a}| = L_1$, Deprit's variables Q, P for the rigid body are of the type

$$a_1 = \alpha_1\sqrt{\beta L_1^2 - \gamma P^2}\sin Q, \quad a_2 = \alpha_2\sqrt{\beta L_1^2 - \gamma P^2}\cos Q, \quad a_3 = \delta P, \tag{6.14}$$

with α_1, α_2, β, γ and δ constants satisfying $\alpha_1^2 = \alpha_2^2$, $\beta = 1/\alpha_2^2$ and $\gamma = (\delta/\alpha_2)^2$. Specifying a_1, a_2 and a_3 in the points (6.13) we obtain concrete values for Q and P accounting for the two equilibria. We call them $(Q^{k,(0)}, P^{k,(0)})$ with $k = 1$ for the prograde point and $k = -1$ for the retrograde one, and make in (6.14) the replacement $Q = Q_1 + Q^{k,(0)}$, $P = P_1 + P^{k,(0)}$. The values of the constants α_1, α_2, β, γ and δ are determined using the constraints written above and implying the whole change to be symplectic, i.e. if we make $\{Q_1, P_1\} = 1$, the Poisson brackets among the a_i have to be satisfied. Similarly we construct the transformations for the invariants b_i, c_i and d_i. After simplifying the whole expression a bit we end up with

$$\begin{aligned}
a_1 &= \pm L_2^{-1}\sqrt{L_1^2L_2^2 - (L_1B \mp 2L_2P_1)^2}\sin(\nu - Q_1), \\
a_2 &= \mp L_2^{-1}\sqrt{L_1^2L_2^2 - (L_1B \mp 2L_2P_1)^2}\cos(\nu - Q_1), \quad a_3 = \pm\frac{L_1B}{L_2} - 2P_1, \\
b_1 &= \mp L_2^{-1}\sqrt{L_1^2L_2^2 - (L_1B \pm 2L_2P_2)^2}\sin(\nu - Q_2), \\
b_2 &= \pm L_2^{-1}\sqrt{L_1^2L_2^2 - (L_1B \pm 2L_2P_2)^2}\cos(\nu - Q_2), \quad b_3 = \mp\frac{L_1B}{L_2} - 2P_2, \\
c_1 &= \sqrt{L_2^2 - (B - 2P_3)^2}\sin(\nu - Q_3), \\
c_2 &= -\sqrt{L_2^2 - (B - 2P_3)^2}\cos(\nu - Q_3), \quad c_3 = B - 2P_3, \\
d_1 &= \sqrt{L_2^2 - (B - 2P_4)^2}\sin(\nu - Q_4), \\
d_2 &= -\sqrt{L_2^2 - (B - 2P_4)^2}\cos(\nu - Q_4), \quad d_3 = B - 2P_4.
\end{aligned} \tag{6.15}$$

The upper signs apply for the prograde point of (4.2) and the lower ones for the retrograde one. This change is symplectic with Poisson structure $dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2 + dQ_3 \wedge dP_3 + dQ_4 \wedge dP_4$. We have checked that the Poisson structure of the a_i, b_i, c_i and d_i is correct.

Next we have to perform our study in $\mathcal{R}_{L_1,L_2,B}$ and get the explicit expression of the invariants ρ_i in terms of Q_i and P_i. This is achieved by using (2.46) and taking into account that $\frac{1}{2}(a_3 + b_3 + c_3 + d_3) = B$, therefore $P_1 + P_2 + P_3 + P_4 = 0$ and we make in (6.15) the change $P_4 = -P_1 - P_2 - P_3$. We also fix the value of $\nu = 0$ (because ν is an ignorable angle in $\mathcal{R}_{L_1,L_2,B}$ and we can give it a value to get the most simple expression of the canonical coordinates). Finally as we need to fix a value for Q_4 we write down the Hamiltonian (2.21) in terms of Q_i and P_i using the change (6.15) (where we have made $P_4 = -P_1 - P_2 - P_3$ and $\nu = 0$). Expanding the resulting Hamiltonian in Taylor series around $Q_i = 0$, $P_i = 0$, $i = 1, 2, 3$ we find out that after setting $Q_4 = 0$ the points (4.2) are equilibria for the computed Hamilton function. The final expression of the transformation reads as

$$
\begin{aligned}
\rho_1 &= \pm\frac{L_1 B}{L_2} - 2P_1, \qquad \rho_2 = \mp\frac{L_1 B}{L_2} - 2P_2,\\
\rho_3 &= B - 2P_3, \qquad \rho_4 = B + 2(P_1 + P_2 + P_3),\\
\rho_5 &= \mathcal{C}_1\mathcal{C}_2 \sin(Q_1 - Q_2), \qquad \rho_6 = -\mathcal{C}_1\mathcal{C}_2 \cos(Q_1 - Q_2),\\
\rho_7 &= \mp\mathcal{C}_1\sqrt{L_2^2 - (B - 2P_3)^2}\,\sin(Q_1 - Q_3),\\
\rho_8 &= \pm\mathcal{C}_1\sqrt{L_2^2 - (B - 2P_3)^2}\,\cos(Q_1 - Q_3),\\
\rho_9 &= \mp\mathcal{C}_1\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\,\sin(Q_1),\\
\rho_{10} &= \pm\mathcal{C}_1\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\,\cos(Q_1),\\
\rho_{11} &= \pm\mathcal{C}_1\sqrt{L_2^2 - (B - 2P_3)^2}\,\sin(Q_2 - Q_3),\\
\rho_{12} &= \mp\mathcal{C}_2\sqrt{L_2^2 - (B - 2P_3)^2}\,\cos(Q_2 - Q_3),\\
\rho_{13} &= \pm\mathcal{C}_2\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\,\sin(Q_2),\\
\rho_{14} &= \mp\mathcal{C}_2\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\,\cos(Q_2),\\
\rho_{15} &= -\sqrt{L_2^2 - \big(B + 2P_3\big)^2}\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\,\sin(Q_3),
\end{aligned}
\tag{6.16}
$$

$$\rho_{16} = \sqrt{L_2^2 - \big(B + 2P_3\big)^2}\sqrt{L_2^2 - \big(B + 2(P_1 + P_2 + P_3)\big)^2}\cos(Q_3),$$

where

$$\begin{aligned}\mathcal{C}_1 &= L_2^{-1}\sqrt{\big(L_1(L_2 - B) - 2L_2P_1\big)\big(L_1(L_2 + B) + 2L_2P_1\big)},\\ \mathcal{C}_2 &= L_2^{-1}\sqrt{\big(L_1(L_2 + B) - 2L_2P_2\big)\big(L_1(L_2 - B) + 2L_2P_2\big)}.\end{aligned}$$

The upper signs are used for prograde motions and the lower ones for retrograde motions. We observe that setting in (6.16) $Q_i = P_i = 0$ for $i = 1, 2, 3$, the coordinates of ρ_i correspond to the equilibrium points (4.2). By construction the change (6.16) is symplectic as the transformation provided by Deprit [23] is symplectic and all the changes we have performed are also symplectic. The Poisson structure of the new coordinates is $dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2 + dQ_3 \wedge dP_3$. We have checked that the constraints (2.44) that define the space $\mathcal{R}_{L_1,L_2,B}$ are satisfied when the ρ_i are replaced by their expressions in terms of the Q_i and P_i, $i = 1, 2, 3$. The transformation (6.16) is valid provided $|B| < L_2$ which is true since in this section $|B| \approx L_2$ is avoided.

6.2.2 Expansion in the Q_i and P_i variables and normal form computations

After applying the change (2.46) to the Hamiltonian $\mathcal{K}_1$ given in (2.21) we use the transformation (6.16) to write down $\mathcal{K}_1$ in terms of Q_i and P_i. We stretch coordinates by the following canonical transformation with multiplier $\varepsilon^{-1/4}$ by means of $Q_i = \varepsilon^{1/8}\bar{Q}_i$, $P_i = \varepsilon^{1/8}\bar{P}_i$ $i = 1, 2, 3$.

Next we rescale time in the full system (2.7) and expand the resulting Hamiltonian in powers of ε, ending up with

$$\mathcal{H} = \mathcal{H}_{\rm Kep} + \varepsilon\,\mathcal{K}_{10} + \varepsilon^{5/4}\,\mathcal{K}_{12} + \varepsilon^{11/8}\,\mathcal{K}_{13} + \varepsilon^{3/2}\,\mathcal{K}_{14} + \mathcal{O}(\varepsilon^2), \tag{6.17}$$

where

$$\begin{aligned}\mathcal{K}_{10} &= \frac{80\mathcal{M}L_1^4}{L_2^6},\\ \mathcal{K}_{12} &= -\frac{12\mathcal{M}L_1^2}{L_2^8(L_2^2 - B^2)}\\ &\quad\times\Big(L_1^2(L_2^2 - B^2)^2\big(4\bar{Q}_1^2 + 2\bar{Q}_1\bar{Q}_2 + 4\bar{Q}_2^2 - 5(\bar{Q}_1 + \bar{Q}_2)\bar{Q}_3\\ &\qquad - 40L_1^2L_2^2\bar{P}_3(\bar{P}_1 + \bar{P}_2 + \bar{P}_3)\\ &\qquad \pm 20L_1L_2^3(\bar{P}_1^2 - \bar{P}_2^2) + 8L_2^4(2\bar{P}_1^2 - \bar{P}_1\bar{P}_2 - 2\bar{P}_2)\big)\Big),\end{aligned} \tag{6.18}$$

and $\mathcal{K}_{13}$ and $\mathcal{K}_{14}$ are homogeneous polynomials of degree three and four respectively in $\bar{Q}_i$ and $\bar{P}_i$. The upper sign in $\mathcal{K}_{12}$ refers to prograde motions whereas the lower sign is related to retrograde ones.

We calculate the eigenvalues associated to $\mathcal{K}_{12}$ yielding

$$\begin{aligned}
&\pm\frac{60\sqrt{2}L_1^2}{L_2^4}\imath && = \pm\omega_1\imath,\\
&\pm\frac{12\sqrt{5}L_1}{L_2^4}\sqrt{6L_2^2+L_1\big(5L_1-\sqrt{5}\sqrt{5L_1^2+12L_2^2}\big)}\,\imath && = \pm\omega_2\imath,\\
&\pm\frac{12\sqrt{5}L_1}{L_2^4}\sqrt{6L_2^2+L_1\big(5L_1+\sqrt{5}\sqrt{5L_1^2+12L_2^2}\big)}\,\imath && = \pm\omega_3\imath.
\end{aligned}$$

We stress that since $0 < L_1 < L_2$ then $\omega_i > 0$. Moreover the three frequencies are related by $\omega_3 = \omega_1 + \omega_2$. We also get

$$L_1 = \frac{6\cdot 2^{1/4}\cdot\sqrt{3}\omega_1^{3/2}}{\sqrt{5}\omega_2(\omega_1+\omega_2)}, \quad L_2 = \frac{6\cdot 2^{1/4}\omega_1^{1/2}}{\sqrt{2}\omega_2^{1/2}(\omega_1+\omega_2)^{1/2}}. \tag{6.19}$$

Using the eigenvalues and eigenvectors similarly to what we did in the previous section, we build a linear symplectic change introducing new coordinates $\tilde{Q}_i$, $\tilde{P}_i$ so that the quadratic part of the Hamiltonian is diagonalised. It takes the form

$$-\omega_1\imath\tilde{Q}_1\tilde{P}_1 + \omega_2\imath\tilde{Q}_2\tilde{P}_2 + (\omega_1+\omega_2)\imath\tilde{Q}_3\tilde{P}_3.$$

At this point it is apparent the resonances of the N-body problem in the planetary regime pointed out by Herman and Féjoz [32] and Chierchia and Pinzari [13]. Terms of degree three and four in Q_i, P_i are transformed with the linear change that diagonalises $\mathcal{K}_{12}$.

The next step is the transformation of $\mathcal{H}$ to a non-linear normal form up to terms of degree four in $\tilde{Q}_i$, $\tilde{P}_i$ applying a Lie transformation. We need to take two steps in the Lie transformation because the result of the first step is zero, that is, the normal form Hamiltonian composed by terms of degree three vanishes. Given a monomial of $\mathcal{K}_{13}$ or of $\mathcal{K}_{14}$, say $\beta\tilde{Q}_1^i\tilde{Q}_2^j\tilde{Q}_3^k\tilde{P}_1^l\tilde{P}_2^m\tilde{P}_3^n$ with $i+j+k+l+m+n = 3$ or 4, we know that it must be retained in the normalised Hamiltonian if and only if the combination $-\omega_1(i-l)+\omega_2(j-m)+(\omega_1+\omega_2)(k-n)$ is null. This happens trivially for $i = l$, $j = m$ and $k = n$ but when dealing with the terms of degree four, there are other combinations leading to resonant situations, namely, $\omega_1/\omega_2 = 1/3$ (or $L_1/L_2 = 1/(2\sqrt{5})$), $\omega_1/\omega_2 = 1/2$ (or $L_1/L_2 = 1/\sqrt{10}$), $\omega_1/\omega_2 = 1$ (or $L_1/L_2 = \sqrt{3/10}$), $\omega_1/\omega_2 = 2$ (or $L_1/L_2 = 2/\sqrt{5}$) and $\omega_1/\omega_2 = 3$ (or $L_1/L_2 = 3\sqrt{3/5}/2$). For the resonant cases we have proved that the normal form transformation with non-specific ratios ω_1/ω_2 can be used. Concretely, we have calculated the normal forms together with the generating functions for the five resonances, verifying that in each case the result coincides with the normalised Hamiltonian and generating function with non-specific ratios after replacing the values of ω_1 and ω_2 for the given resonance. In summary, we arrive at a unique expression for the normalised Hamiltonian valid for resonant and non-resonant values of the ω_i.

6.2.3 Quasi-periodic solutions related to the points $(-L_1^2, \pm 2L_1C, 0)$

The next step is the introduction of the following symplectic set of action-angle coordinates

$$\tilde{Q}_i = \sqrt{I_i}(\cos\phi_i - \imath\sin\phi_i), \quad \tilde{P}_i = \sqrt{I_i}(\sin\phi_i - \imath\cos\phi_i), \quad 1 \le i \le 3.$$

The actions and angles satisfy $\{I_i, \phi_i\} = 1$ and $\{I_i, \phi_j\} = 0$ if $i \neq j$. Now we apply this transformation to the Hamiltonian and introduce a new parameter $\eta^4 = \varepsilon$ getting the following Hamiltonian

$$\mathcal{H} = h_0 + \eta^4 h_1 + \eta^5 h_2 + \eta^6 h_3 + \mathcal{O}(\eta^8), \tag{6.20}$$

where

$$\begin{aligned}
h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2},\\
h_1 &= \frac{80\mathcal{M}L_1^4}{L_2^6},\\
h_2 &= \frac{2^{1/4}\mathcal{M}\omega_1^{3/2}}{5\omega_2^{1/2}(\omega_1+\omega_2)^{1/2}}\big(\omega_1 I_1 - \omega_2 I_2 - (\omega_1+\omega_2)I_3\big),\\
h_3 &= \frac{\mathcal{M}}{180(\omega_1+2\omega_2)^2}\\
&\quad\times\Big(12\omega_1^2(\omega_1+2\omega_2)^2 I_1^2\\
&\qquad + \omega_2(\omega_1+4\omega_2)(-3\omega_1^2+4\omega_2^2)I_2^2\\
&\qquad + (\omega_1+\omega_2)(3\omega_1+4\omega_2)(\omega_1^2+8\omega_1\omega_2+4\omega_2^2)I_3^2\\
&\qquad - 12\omega_1\omega_2(\omega_1+2\omega_2)(\omega_1+3\omega_2)I_1I_2\\
&\qquad - 12\omega_1(\omega_1+\omega_2)(\omega_1+2\omega_2)(2\omega_1+3\omega_2)I_1I_3\\
&\qquad + 8\omega_2(\omega_1+\omega_2)(3\omega_1^2+4\omega_1\omega_2+4\omega_2^2)I_2I_3\Big).
\end{aligned} \tag{6.21}$$

The Hamiltonians h_i are the same for the two points in (4.2) and for all possible values of ω_1 and ω_2. The fact that h_3 is independent of the angles ϕ_i is outstanding since we do not need to discard any resonant relationship between the frequencies so that we may apply Han, Li and Yi's Theorem in all the cases. Of course, resonant terms would appear when computing higher orders, but it is not relevant in our study.

In order to apply KAM theory we select the three actions I_j with conjugate angles ϕ_j and take L_2 with conjugate angle ℓ_2. As the fifth action we choose L_1 with conjugate angle ℓ_1 since the reasoning made in Section 6.1 applies.

One can identify the following numbers in Han, Li and Yi's Theorem in [36], namely, $n_0 = 2$, $n_1 = 2$, $n_2 = 5$, $n_3 = 5$, $\beta_1 = 4$, $\beta_2 = 5$, $\beta_3 = 6$ and $a = 3$. Thus we introduce the vector Ω by

$$\begin{aligned}\Omega &\equiv \Big(\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8\ \Omega_9\ \Omega_{10}\Big)\\ &= \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial L_1}, \frac{\partial h_1}{\partial L_2}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_2}{\partial I_3}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}, \frac{\partial h_3}{\partial I_3}\right).\end{aligned}$$

We define $I = (L_1, L_2, I_1, I_2, I_3)$ and build the 10×6-matrix

$$\partial_I^1\,\Omega(I) = \left(\Omega_k, \frac{\partial \Omega_k}{\partial L_1}, \frac{\partial \Omega_k}{\partial L_2}, \frac{\partial \Omega_k}{\partial I_1}, \frac{\partial \Omega_k}{\partial I_2}, \frac{\partial \Omega_k}{\partial I_3}\right), \quad 1 \le k \le 10,$$

which has rank five.

In this case $b = 15$ and $s = 1$. So, the excluded measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^\delta)$ (or $\mathcal{O}(\varepsilon^{\delta/4})$) with $0 < \delta < 1/5$ and we cannot improve this measure.

We close this section stating the main result obtained in it.

Theorem 6.3. *The Hamiltonian system of the spatial three-body problem (2.3) (or, equivalently, Hamiltonian (2.7)), reduced by the symmetry of translations and defined in $\mathcal{Q}_{\varepsilon,n}$, has invariant KAM 5-tori densely filled with quasi-periodic trajectories. In these quasi-periodic solutions the fictitious inner body moves in orbits that are nearly rectilinear, bounded and perpendicular to the invariable plane whereas the outer body moves in a near-circular orbit lying near the invariable plane and such that $C \approx L_2 \not\approx |B|$. For a given δ such that $0 < \delta < 1/5$, the excluding measure for the existence of invariant 5-tori is of order $\mathcal{O}(\varepsilon^{\delta/4})$.*

We stress that it is possible to avoid the computation of h_3 in (6.21), obtaining rank five for the matrix composed with the partial derivatives of h_0, h_1 and h_2 with respect to L_1, L_2, I_1, I_2 and I_3. However in this case we should arrive at order four in the derivatives, so $s = 4$ and since if it would be enough to define $\eta^2 = \varepsilon$ the excluding measure for the existence of the invariant tori would be of order $\mathcal{O}(\varepsilon^{\delta/8})$, thus we have preferred to calculate the non-linear terms of the normal form Hamiltonian, getting a lower estimate of the excluding measure for the existence of invariant 5-tori.

6.3 Invariant 5-tori reconstructed from $\mathcal{A}_{L_1,L_2}$

6.3.1 Construction of symplectic coordinates

Our aim is to prove the existence of KAM tori of dimension five associated to the elliptic equilibria in $\mathcal{A}_{L_1,L_2}$ that deal with rectilinear trajectories of the inner bodies parallel to the vector $\mathbf{C}$ and circular motions of the outer body in the invariable plane when it coincides with the horizontal plane. In Deprit's coordinates such motions are defined by $\gamma_1 = \pi/2$ or $3\pi/2$, $G_1 = 0$, $G_2 = C = L_2$ and $C = |B|$. In the manifold $\mathcal{A}_{L_1,L_2}$, accordingly to what we studied in Section 4.3, these equilibria have coordinates

$$\begin{aligned}&(0,\,0,\,\pm L_1,\,0,\,0,\,\mp L_1,\,0,\,0,\,\pm L_2,\,0,\,0,\,\pm L_2)\ \text{ for } \gamma_1 = \pi/2\ \text{ (prograde)},\\ &(0,\,0,\,\mp L_1,\,0,\,0,\,\pm L_1,\,0,\,0,\,\pm L_2,\,0,\,0,\,\pm L_2)\ \text{ for } \gamma_1 = 3\pi/2\ \text{ (retrograde)}.\end{aligned} \tag{6.22}$$

These relative equilibria are isolated points in $\mathcal{A}_{L_1,L_2}$. Note that the points of the first row in (6.22) are reconstructed from the point $(-L_1^2, 2L_1C, 0)$ and the ones in the second row are reconstructed from $(-L_1^2, -2L_1C, 0)$.

We proceed similarly to what we did in Section 6.2.1, introducing a pair of rigid-body-like coordinates [23] for the Keplerian invariants a_i, another pair for the b_i, a third one for the c_i and a fourth pair for d_i. Adjusting the constants in (6.14) in such a way that when $Q_i = P_i = 0$ for $i = 1, \ldots, 4$ the values of the Keplerian invariants correspond to the equilibria (6.22) of $\mathcal{A}_{L_1,L_2}$, we end up with

$$\begin{aligned}
a_1 &= \mp 2P_1, & a_2 &= \sqrt{L_1^2 - 4P_1^2}\sin Q_1, & a_3 &= \pm\sqrt{L_1^2 - 4P_1^2}\cos Q_1,\\
b_1 &= \mp 2P_2, & b_2 &= -\sqrt{L_1^2 - 4P_2^2}\sin Q_2, & b_3 &= \mp\sqrt{L_1^2 - 4P_2^2}\cos Q_2,\\
c_1 &= \mp 2P_3, & c_2 &= \sqrt{L_2^2 - 4P_3^2}\sin Q_3, & c_3 &= \pm\sqrt{L_2^2 - 4P_3^2}\cos Q_3,\\
d_1 &= \mp 2P_4, & d_2 &= \sqrt{L_2^2 - 4P_4^2}\sin Q_4, & d_3 &= \pm\sqrt{L_2^2 - 4P_4^2}\cos Q_4,
\end{aligned} \tag{6.23}$$

for $\gamma_1 = \pi/2$, and

$$\begin{aligned}
a_1 &= \pm 2P_1, & a_2 &= \sqrt{L_1^2 - 4P_1^2}\sin Q_1, & a_3 &= \mp\sqrt{L_1^2 - 4P_1^2}\cos Q_1,\\
b_1 &= \pm 2P_2, & b_2 &= -\sqrt{L_1^2 - 4P_2^2}\sin Q_2, & b_3 &= \pm\sqrt{L_1^2 - 4P_2^2}\cos Q_2,\\
c_1 &= \pm 2P_3, & c_2 &= -\sqrt{L_2^2 - 4P_3^2}\sin Q_3, & c_3 &= \pm\sqrt{L_2^2 - 4P_3^2}\cos Q_3,\\
d_1 &= \pm 2P_4, & d_2 &= -\sqrt{L_2^2 - 4P_4^2}\sin Q_4, & d_3 &= \pm\sqrt{L_2^2 - 4P_4^2}\cos Q_4,
\end{aligned} \tag{6.24}$$

for $\gamma_1 = 3\pi/2$. The upper signs apply for $B = C$ whereas the lower ones apply for $B = -C$. We have checked that the constraints $|\mathbf{a}| = |\mathbf{b}| = L_1$ and $|\mathbf{c}| =$

$|\mathbf{d}| = L_2$ are satisfied when we replace the invariants a_i, b_i, c_i and d_i in terms the Q_i and P_i. Using the Poisson brackets in $\mathcal{A}_{L_1,L_2}$ we have also verified that the transformations (6.23) and (6.24) are symplectic with Poisson structure $dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2 + dQ_3 \wedge dP_3 + dQ_4 \wedge dP_4$.

6.3.2 Expansion in the Q_i and P_i variables and normal form computations

Now we apply (6.23) and (6.24) to the Hamiltonian (2.21) and the stretching by $Q_i = \varepsilon^{1/8}\bar{Q}_i$, $P_i = \varepsilon^{1/8}\bar{P}_i$, $i = 1, \ldots, 4$, which is canonical with multiplier $\varepsilon^{-1/4}$.

Then we apply the transformation to $\mathcal{H}$ given in (2.7), rescale time and expand in powers of ε, ending up with the Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_{10} + \varepsilon^{5/4}\,\mathcal{K}_{12} + \varepsilon^{3/2}\,\mathcal{K}_{14} + \mathcal{O}(\varepsilon^2), \tag{6.25}$$

where

$$\begin{aligned}
\mathcal{K}_{10} &= \frac{80\mathcal{M}L_1^4}{L_2^6},\\
\mathcal{K}_{12} &= -\frac{12\mathcal{M}L_1^2}{L_2^8}\\
&\quad\times\Big(4L_1^2L_2^2\big(4\bar{Q}_1^2 + 2\bar{Q}_1\bar{Q}_2 + 4\bar{Q}_2^2 + 10\bar{Q}_3\bar{Q}_4 - 5(\bar{Q}_1+\bar{Q}_2)(\bar{Q}_3+\bar{Q}_4)\big)\\
&\qquad + 40L_1^2\bar{P}_3\bar{P}_4 + 8L_2^2(2\bar{P}_1^2 - \bar{P}_1\bar{P}_2 + 2\bar{P}_2^2)\\
&\qquad \mp 20L_1L_2(\bar{P}_1 - \bar{P}_2)(\bar{P}_3 + \bar{P}_4)\Big),
\end{aligned}$$

and the upper sign in $\mathcal{K}_{12}$ is used for the prograde motions while the lower sign is used for the retrograde ones. We do not write down $\mathcal{K}_{14}$ which is a homogeneous polynomial of degree four in $\bar{Q}_i$ and $\bar{P}_i$.

Hamiltonian $\mathcal{K}_{12}$ is diagonalised by using the eigenvalues and the eigenvectors associated to the linearised equations of motion. In the new coordinates $\tilde{Q}_i$, $\tilde{P}_i$ we get

$$-\omega_1\imath\tilde{Q}_1\tilde{P}_1 + \omega_2\imath\tilde{Q}_2\tilde{P}_2 + (\omega_1+\omega_2)\imath\tilde{Q}_3\tilde{P}_3,$$

where the ω_i correspond to the eigenvalues

$$\begin{aligned}
&\pm\frac{60\sqrt{2}L_1^2}{L_2^4}\imath &&= \pm\omega_1\imath,\\
&\pm\frac{12\sqrt{5}L_1}{L_2^4}\sqrt{6L_2^2 + L_1(5L_1 - \sqrt{25L_1^2+60L_2^2})}\,\imath &&= \pm\omega_2\imath,\\
&\pm\frac{12\sqrt{5}L_1}{L_2^4}\sqrt{6L_2^2 + L_1(5L_1 + \sqrt{25L_1^2+60L_2^2})}\,\imath &&= \pm\omega_3\imath,\\
&0\imath &&= \pm\omega_4\imath.
\end{aligned} \tag{6.26}$$

The transformation to diagonal coordinates $\tilde{Q}_i$, $\tilde{P}_i$ is valid excepting for $\omega_1/\omega_2 = 2/3$ (or $L_1/L_2 = 2/5$).

As expected from the treatment of the invariant 5-tori for non-rectilinear motions of the spatial three-body problem made in [70], the frequencies ω_1, ω_2 and ω_3 are the same as in (6.18) and moreover $\omega_4 = 0$, reflecting the fact that B and its conjugate angle ν_1 are ignorable coordinates in all the process, see [13]. We proceed as in Section 6.2 computing the non-linear normal form. This time as there are not cubic terms in the Hamiltonians it is enough to make only one step of the Lie transformation. The resonant combinations between ω_1 and ω_2 are the same as in the treatment made in the space $\mathcal{R}_{L_1,L_2,B}$, but we get a unique expression for the normalised Hamiltonian up to terms of degree four in $\tilde{Q}_i, \tilde{P}_i$.

The next step is the introduction of action-angles coordinates through

$$\tilde{Q}_i = \sqrt{I_i}(\cos\phi_i - \imath\sin\phi_i), \quad \tilde{P}_i = \sqrt{I_i}(\sin\phi_i - \imath\cos\phi_i), \quad 1 \le i \le 4,$$

which is a symplectic change of variables where the actions and angles have symplectic structure $dI_1 \wedge d\phi_1 + dI_2 \wedge d\phi_2 + dI_3 \wedge d\phi_3 + dI_4 \wedge d\phi_4$. We also define a new small parameter $\eta^4 = \varepsilon$ and the resulting Hamiltonian is

$$\mathcal{H} = h_0 + \eta^4 h_1 + \eta^5 h_2 + \eta^6 h_3 + \mathcal{O}(\eta^8), \tag{6.27}$$

where h_0, h_1, h_2 and h_3 are exactly the same as in (6.21), a feature that was also true for the non-rectilinear tori dealt with in [70]. Note that I_4 is not present in the h_i. This expression of $\mathcal{H}$ is valid for the four points in (6.22).

We treat the pending case $\omega_1/\omega_2 = 2/3$ separately, starting with the diagonalisation of the quadratic terms and then changing the non-linear terms by means of a Lie transformation. Then we use the usual passage to action-angle coordinates. Similarly to the non-rectilinear type of solutions of Chapter 5, the final normalised Hamiltonian coincides with $\mathcal{H}$ in (6.27), where the h_i are taken from (6.21), after replacing ω_1 by $2\omega_2/3$. So we also use the Hamiltonians (6.21) when $\omega_1/\omega_2 = 2/3$.

Next, Han, Li and Yi's Theorem is applied to Hamiltonian (6.27) with the same numbers n_i, β_i and a as in Section 6.2.3. Thus the 10×6-matrix is of rank 5 with $s = 1$. Hence there are KAM 5-tori related with each equilibrium point of (6.22).

In this case $b = 15$ (as in Section 6.2), thus the excluded measure for the existence of quasi-periodic invariant tori cannot be improved and it is of order $\mathcal{O}(\eta^\delta)$, i.e. of order $\mathcal{O}(\varepsilon^{\delta/4})$ with $0 < \delta < 1/5$.

6.3.3 Quasi-periodic solutions related to the points $(-L_1^2, \pm 2L_1C, 0)$

The main result of this section is the following.

Theorem 6.4. *The Hamiltonian system of the spatial three-body problem (2.3) (or, equivalently, Hamiltonian (2.7)), reduced by the symmetry of translations and defined in $\mathcal{Q}_{\varepsilon,n}$, has invariant KAM 5-tori densely filled with quasi-periodic trajectories. In these quasi-periodic solutions the fictitious inner body moves in orbits that are nearly rectilinear, bounded and perpendicular to the invariable plane which is near the horizontal plane. The outer body moves in a near-circular orbit lying near the invariable plane and such that $C \approx L_2 \approx |B|$. For a given δ such that $0 < \delta < 1/5$, the excluding measure for the existence of invariant 5-tori is of order $\mathcal{O}(\varepsilon^{\delta/4})$.*

As in Section 6.2.3 we can avoid the calculation of the higher-order terms of the normalised Hamiltonian, but since we would obtain a bigger estimate of the excluding measure for the existence of the invariant tori, we have decided to use the non-linear part of the Hamiltonian in normal form.

6.4 Invariant 5-tori related with rectilinear coplanar motions

6.4.1 Construction of symplectic coordinates

Our goal is to establish the existence of KAM tori related to the equilibrium point of $\mathcal{T}_{L_1,C,G_2}$ whose coordinates are $(-L_1^2, 0, 0)$. Intending to proceed similarly to the study made in the previous section for the points $(-L_1^2, \pm 2CL_1, 0)$, we first note that $(-L_1^2, 0, 0)$ is mapped into the point of $\mathcal{S}_{L_1,L_2,C}$ with coordinates $(-L_1^2, 2C^2 - L_2^2, 0, 0, 0, 0)$, as we saw in Section 4.1. However this point is not isolated in $\mathcal{S}_{L_1,L_2,C}$ and we cannot follow an analogous approach to the one made in the previous section.

Thus we use an alternative procedure. In particular we focus on the near-coplanar motions when $G_1 \approx |C - G_2|$, which in the space $\mathcal{T}_{L_1,C,G_2}$ are defined by the point $(-L_1^2 + 2(C - G_2)^2, 0, 0)$. Following a similar reasoning as in S 5.3.3 of [64], we introduce a canonical change of coordinates (Q, P) that when we make $G_2 \to C$, the transformation is also valid in the limit point $(-L_1^2, 0, 0)$. Our aim is to apply Han, Li and Yi's Theorem [36] to the normal form Hamiltonian we will determine, requiring that this approach will be valid for the $G_2 \to C$ in order to conclude the existence of invariant tori of dimension five related to rectilinear coplanar motions. We exclude the case that the outer body moves in an orbit of circular type. In Chapter 5 we also dealt with the existence of KAM tori related to the points $(-L_1^2+2(C-G_2)^2, 0, 0)$, constructing a pair of action-angles coordinates, say I and ϕ. The action I, together with the momenta L_1, L_2, C and G_2, was used to prove the existence of the invariant 5-tori. However we cannot use those calculations here because of the presence of the factor $C - G_2$ in the denominators of the intermediate Hamiltonians.

In Fig. 6.1 we depict the flow of (2.34) on the space $\mathcal{T}_{L_1,C,G_2}$ when $G_2 \approx C$.

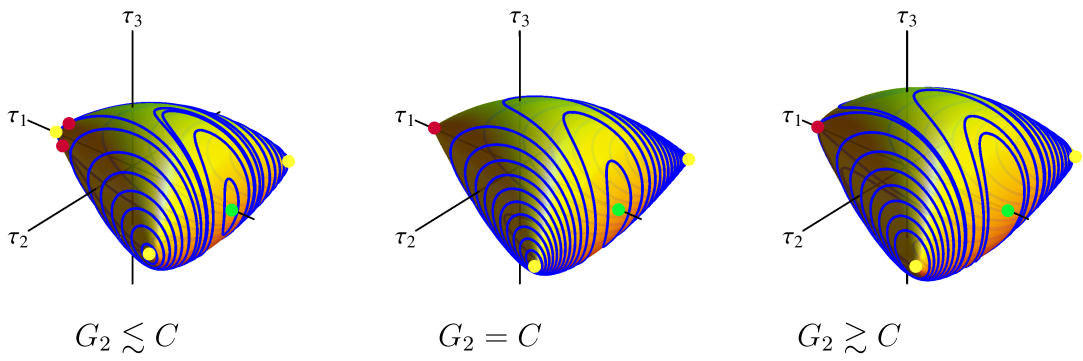

Figure 6.1: Flow on the space $\mathcal{T}_{L_1,C,C}$ for $G_2 \approx C$. The green points correspond to coplanar motions with coordinates $(-L_1^2 + 2(C - G_2)^2, 0, 0)$, the yellow points are elliptic and the red ones hyperbolic. When $G_2 = C$ the two yellow points account for the rectilinear motions of the fictitious inner body and have coordinates $(-L_1^2, \pm 2CL_1, 0)$ and in this particular case as we have also chosen $L_1 = 2C$ the red point accounting for circular coplanar motions is also a singularity of $\mathcal{T}_{L_1,C,G_2}$

Zhao uses a similar argument in [93] making use of an iso-energetic proper-degenerate KAM theorem in the near-collision set, computing the torsion near the set $\{C \equiv C_{\min} = |C - G_2| > 0\}$, i.e. near-coplanar motions, and proving that this torsion does not vanish when $C - G_2 \to 0$. However, he develops his study working in a space without reducing the symmetry related with G_2 whereas we work in the fully-reduced space $\mathcal{T}_{L_1,C,G_2}$.

We look for a symplectic change of the form

$$
\begin{aligned}
\tau_1 &= f_1(Q,P) = f_{1,0}(Q,P) + \beta f_{1,1}(Q,P) + \beta^2 f_{1,2}(Q,P) + \beta^3 f_{1,3}(Q,P),\\
\tau_2 &= f_2(Q,P) = f_{2,0}(Q,P) + \beta f_{2,1}(Q,P) + \beta^2 f_{2,2}(Q,P) + \beta^3 f_{2,3}(Q,P),\\
\tau_3 &= f_3(Q,P) = f_{3,0}(Q,P) + \beta f_{3,1}(Q,P) + \beta^2 f_{3,2}(Q,P) + \beta^3 f_{3,3}(Q,P),
\end{aligned}
\tag{6.28}
$$

where β is a small parameter given by $C = G_2(1 + \beta)$, thus $\beta \to 0$ when $G_2 \to C$. We want to determine $f_{i,j}$ using Taylor expansions in β.

First we express $f_3(Q,P)$ in terms of τ_1 and τ_2 by using (2.32) so that we shall calculate $f_{3,k}$ after having obtained $f_{1,k}$, $f_{2,k}$. In order to build a symplectic change we need to take into account the Poisson structure on $\mathcal{T}_{L_1,C,G_2}$ computed in [69], Eq. (5.5). We proceed beginning at order zero in β, setting $f_{1,0} = -L_1^2 + 2P^2$ and solving a partial differential equation to obtain $f_{2,0}$ so that the Poisson brackets between $f_{1,0}$ and $f_{2,0}$ satisfy the Poisson structure of the τ_i. At order one in β, we make $f_{1,1} = 0$, obtaining $f_{2,1}$ similarly to $f_{2,0}$. Then we continue at order two in β setting $f_{1,2} = 2C^2 \sec^2 Q$ and solving the corresponding differential equation to obtain $f_{2,2}$. At order three in β we make again $f_{1,3} = 0$ and obtain $f_{2,3}$ similarly to the previous orders. Finally we use (2.32) to get $f_{3,k}$. Taking into account that $\beta = (C - G_2)/G_2$ and simplifying the resulting expressions we get the following change of variables

$$
\begin{aligned}
\tau_1 &= -L_1^2 + 2P^2 + 2(C-G_2)^2 \sec^2 Q, \\
\tau_2 &= \frac{(L_1^2 - P^2)^{1/2}}{(4C^2 - P^2)^{1/2}} \big(4C^2 - P^2 - 2C(C-G_2)\big) \sin Q \\
&\quad - \frac{(C-G_2)^2 \sec Q \tan Q}{4(L_1^2 - 4P^2)^{1/2}(4C^2 - P^2)^{5/2}} \\
&\quad\quad \times \Big((4C^2 - P^2)\big(32C^4 + (8C^2 - P^2)(L_1^2 - 3P^2) + (L_1^2 - P^2)P^2 \cos(2Q)\big) \\
&\quad\quad\quad - 2C(C-G_2)\big(32C^4 - (8C^2 - P^2)(L_1^2 + P^2) \\
&\quad\quad\quad\quad - (L_1^2 - P^2)P^2 \cos(2Q)\big)\Big), \\
\tau_3 &= \frac{(L_1^2 - P^2)^{1/2} P}{(4C^2 - P^2)^{1/2}} \big(4C^2 - P^2 - 2C(C-G_2)\big) \cos Q \\
&\quad - \frac{(C-G_2)^2 P \sec Q}{4(L_1^2 - P^2)^{1/2}(4C^2 - P^2)^{5/2}} \\
&\quad\quad \times \Big((4C^2 - P^2)\big(32C^4 + (8C^2 - P^2)(L_1^2 - 3P^2) + (L_1^2 - P^2)P^2 \cos(2Q)\big) \\
&\quad\quad\quad - 2C(C-G_2)\big(32C^4 - (8C^2 - P^2)(L_1^2 + P^2) \\
&\quad\quad\quad\quad - (L_1^2 - P^2)P^2 \cos(2Q)\big)\Big),
\end{aligned}
\tag{6.29}
$$

The Poisson structure of the τ_i is preserved including terms factorised by β^3 so that $\{Q, P\} = 1 + \mathcal{O}(\beta^4)$. The constraint (2.32) in terms of Q and P is also true up order three in β. Setting $Q = P = 0$ we get $\tau_1 = -L_1^2 + 2(C - G_2)^2$, $\tau_2 = \tau_3 = 0$, thus (6.29) may be used to deal with study coplanar solutions such that $G_1 = |C - G_2|$ in the orbit space $\mathcal{T}_{L_1,C,G_2}$.

When G_2 tends to C in (6.29), the transformation reads as

$$
\tau_1 = -L_1^2 + 2(C-G_2)^2 + 2P^2,
$$

$$
\tau_2 = \sqrt{(L_1^2 - P^2)(4C^2 - P^2)} \sin Q, \qquad \tau_3 = \sqrt{(L_1^2 - P^2)(4C^2 - P^2)} \cos Q,
$$

which is also a symplectic change because the Poisson structure on $\mathcal{T}_{L_1,C,G_2}$ is verified. Thus the transformation (6.29) is canonical in a neighbourhood of $G_2 \approx C$ in $\mathcal{T}_{L_1,C,G_2}$.

6.4.2 Expansion in Q and P variables and normal form computations

After applying the change (6.29) and the stretching $Q = \varepsilon^{1/4}\bar{Q}$, $P = \varepsilon^{1/4}\bar{P}$ – which is canonical with multiplier $\varepsilon^{-1/2}$ – to $\mathcal{H}$ given in (2.7) we rescale time and expand the resulting system in powers of ε getting the Hamiltonian in the form

$$\mathcal{H} = \mathcal{H}_{\text{Kep}} + \varepsilon\,\mathcal{K}_{10} + \varepsilon^{3/2}\,\mathcal{K}_{12} + \mathcal{O}(\varepsilon^2), \tag{6.30}$$

where

$$\begin{aligned}\mathcal{K}_{10} &= \frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\left(3(C-G_2)^2 - 5L_1^2\right),\\ \mathcal{K}_{12} &= \frac{3\mathcal{M}}{32L_2^3C^4G_2^5}\left(\mathcal{B}\bar{Q}^2 + 256L_1^2C^4G_2(C+G_2)\bar{P}^2\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B} = {} & 80C^4(C-G_2)^4(C+G_2)^2\\ & - 8L_1^2C^2(C-G_2)^2(C+G_2)(25C^3 + 43C^2G_2 - 25CG_2^2 + 5G_2^3)\\ & + 5L_1^4(5C^3 + 15C^2G_2 - 5CG_2^2 + G_2^3)^2,\end{aligned}$$

which is a positive constant because L_1, C and G_2 are positive.

6.4.3 Quasi-periodic solutions related to the point $(-L_1^2, 0, 0)$

The next step is the introduction of action-angle variables

$$\begin{aligned}Q &= 4\sqrt{2L_1}C\left(\frac{(C+G_2)G_2}{\mathcal{B}}\right)^{1/4}\sqrt{I}\sin\phi,\\ P &= \frac{1}{2\sqrt{2L_1}C}\left(\frac{\mathcal{B}}{(C+G_2)G_2}\right)^{1/4}\sqrt{I}\cos\phi,\end{aligned} \tag{6.31}$$

which is canonical with symplectic structure $d\phi\wedge dI$. The change (6.31) transforms (6.30) into

$$\mathcal{H} = h_0 + \eta^2\,h_1 + \eta^3\,h_2 + \mathcal{O}(\eta^4), \tag{6.32}$$

where

$$\begin{aligned}h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2} - \frac{\mu_2^3M_2^2}{2L_2^2},\\ h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\left(3(C-G_2)^2 - 5L_1^2\right),\\ h_2 &= \frac{3\mathcal{M}L_1}{L_2^3C^2G_2^5}\left((C+G_2)G_2\mathcal{B}\right)^{1/2} I.\end{aligned}$$

At this point one can identify the following numbers in Han, Li and Yi's Theorem in [36]: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$. Thus we define the frequency's vector Ω by

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C},\ \frac{\partial h_1}{\partial G_2},\ \frac{\partial h_2}{\partial I}\right).$$

We build the 5×6-matrix

$$\partial_I^1 \Omega(I) = \left(\Omega_k,\ \frac{\partial \Omega_k}{\partial L_1},\ \frac{\partial \Omega_k}{\partial L_2},\ \frac{\partial \Omega_k}{\partial C},\ \frac{\partial \Omega_k}{\partial G_2},\ \frac{\partial \Omega_k}{\partial I}\right), \quad 1 \leq k \leq 5.$$

After replacing (6.32) in the frequency vector Ω, we deduce that the rank of the previous matrix is four, which is not enough. We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1 \Omega(I)$ the columns corresponding to the partials of second order. (There is a total of 25 second order partial derivatives). This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point of $\mathcal{T}_{L_1,C,G_2}$ that represents coplanar motions when $G_1 \approx |C - G_2|$.

The computations carried out are valid for all possible values of G_2 and C such that $G_1 \approx |C - G_2|$. In particular computing the limit $G_2 \to C$ in the 5×31-matrix, its rank is also five thus we can conclude that the KAM 5-tori also exist for $G_2 = C$.

In this case $b = 7$. So, the excluded measure for the existence of quasi-periodic invariant tori is of order $\eta^{\delta/2} = \varepsilon^{\delta/4}$ with $0 < \delta < 1/5$ and we cannot improve it. We close this section stating the main result obtained in it.

Theorem 6.5. *The Hamiltonian system of the spatial three-body problem (2.3) (or, equivalently, Hamiltonian (2.7)), reduced by the symmetry of translations and defined in $\mathcal{Q}_{\varepsilon,n} \subseteq T^*\mathbb{R}^6$, has invariant KAM 5-tori densely filled with quasi-periodic trajectories. In these quasi-periodic solutions the fictitious inner body moves in orbits that are nearly rectilinear, bounded and lying near the invariable plane whereas the outer body moves in a non-circular orbit that lies near the invariable plane. For a given δ such that $0 < \delta < 1/5$, the excluding measure for the existence of invariant 5-tori is of order $\mathcal{O}(\varepsilon^{\delta/4})$.*

We do not consider the case of rectilinear coplanar motions where the outer particle follows a circular trajectory. As it is said in Section 6.4.1 if we try to follow an approach similar to the one used in this section, we should work in the space $\mathcal{S}_{L_1,L_2,C}$ but in this space this type of motions is a non-isolated equilibrium of the vector field related to (2.29). On the other hand if we try to use the techniques of this chapter, we ought to work either in $\mathcal{R}_{L_1,L_2,B}$ when $|B| \not\approx C$ or in $\mathcal{A}_{L_1,L_2}$ when

$|B| \approx C$. However in both cases the point of $\mathcal{T}_{L_1,C,G_2}$ with coordinates $(-L_1^2, 0, 0)$ does not reconstruct into points of $\mathcal{R}_{L_1,L_2,B}$ and $\mathcal{A}_{L_1,L_2}$ but in higher-dimensional objects, thus we cannot make a usual treatment based on normal forms around equilibrium points.

Conclusions and future work

The spatial three-body problem is studied when the Hamiltonian written in terms of Jacobi coordinates can be decomposed as the sum of two Keplerian Hamiltonians plus a small perturbation. We use averaging and reduction theory in order to reduce out the exact and approximate symmetries of the problem. Hence we obtain a system of one degree of freedom which is defined on a surface that has singular points for some combinations of the integrals of motion. Based on the analysis of the relative equilibria and bifurcations made for the fully-reduced Hamiltonian, we reconstruct the different motions of the three bodies that correspond to the elliptic points in the fully-reduced space, including the equilibria related to near-rectilinear motions of the inner bodies. We obtain KAM 5-tori of the three-body problem in an open subset $\mathcal{Q}_{\varepsilon,n}$ of $\mathcal{P}_{\varepsilon,n} \subseteq T^*\mathbb{R}^6$. Due to the degeneracy of our system we use a theorem by Han, Li and Yi [36] that works in the case of Hamiltonian systems with high-order proper degeneracy. However in order to obtain adequate action-angle variables for each motion we analyse, we need to use the intermediate reduced spaces where the pairs of actions and angles can be constructed properly. This leads to analyse these spaces and classify all possible motions in the elliptic domain of the spatial three-body problem.

The basic achievements are:

(i) We have used singular reduction theory to perform the analysis and get the right fully-reduced phase space. The reduction has been made through three stages reducing out all the continuous symmetries of the system and computing the invariants and reduced spaces of the intermediate steps.

(ii) Following [34] and [11, 12, 13] we have used Deprit's coordinates [26] to perform the Jacobi reduction of the nodes previously to any reduction process. Deprit's variables have also been crucial to identify the fundamental polynomial invariants and the relations defining the reduced spaces $\mathcal{S}_{L_1,L_2,C}$ and $\mathcal{T}_{L_1,C,G_2}$. These sets of invariants are very hard to determine by using Gröbner bases and techniques from computer algebra as the computations involve polynomials of degrees one, two and three in twelve variables.

(iii) We have performed the analysis of the fully-reduced Hamiltonian corresponding to the spatial three-body problem in the fully-reduced space obtained in Chapter 3. Our analysis is made in the same style as those of [53] and [34], studying the number of equilibria, bifurcation lines and stability character of the equilibria. We have clarified some conclusions obtained in [34], those related to the singular points of the fully-reduced space and the rectilinear motions of the inner bodies.

(iv) All possible motions of the spatial three-body problem in the elliptic domain, including the near-rectilinear motions of the inner bodies, have been studied and classified. The analysis is complete in the sense that it takes into account all the possible motions since we construct in all the cases five pairs of action-angles coordinates (in the spaces $\mathcal{T}_{L_1,C,G_2}$, $\mathcal{S}_{L_1,L_2,C}$ and $\mathcal{R}_{L_1,L_2,B}$) and six pairs in the space $\mathcal{A}_{L_1,L_2}$. Specifically the action-angle coordinates are built from the local rectangular variables introduced in each case.

(v) The action-angle coordinates introduced in each case together with the rectangular variables can be used to analyse specific motions of the three-body problem. For instance, for the analysis of the behaviour of three bodies in space such that the inner particles move in circular orbits in a certain plane whereas the outer particle moves in a different plane, one should use the coordinates of Table 5.3(a).

(vi) By applying a theorem by Han, Li and Yi [36] that works in the case of Hamiltonian systems with high-order proper degeneracy, we obtain KAM 5-tori of the three-body problem in the set $\mathcal{Q}_{\varepsilon,n}$. In all the cases considered in the thesis we provide the transformations explicitly, computing the normalised Hamiltonians as well as the torsions needed to verify Han, Li and Yi's Theorem.

(vii) The application of singular reduction theory is crucial as it allows us to reduce out the symmetries properly, arriving at the singular space $\mathcal{T}_{L_1,C,G_2}$ where we could analyse the flow, fixing the deficiencies of previous studies. This fact implies that the reconstruction process is done correctly.

(viii) It is hard to improve the excluded measure for the existence of quasi-periodic invariant tori obtained in this thesis, specifically in the planetary case as the perturbation introduced in (2.3) appears at first order with respect to the small parameter ε. However this measure could be improved in the asynchronous region as in these situations the two Keplerian Hamiltonians are placed at different orders and one may average with respect to the two mean anomalies up to high order, incorporating more terms in the perturbation apart from the quadrupolar ones. In this context, the structure of the Hamiltonian system would allow us to average the Hamiltonian also with respect to the argument of the pericentre γ_2 as Zhao does [90, 93]. Then the remainder of our normal form Hamiltonians would be much smaller allowing us to get a measure of the order $\mathcal{O}(\varepsilon^b)$ for some positive integer b.

(ix) We focus on the study of all possible combinations of motions provided the inner bodies describe bounded near-rectilinear quasi-periodic motions. For achieving this we have used an argument based on the regularisation of the Kepler problem due to Ligon and Schaaf. This procedure does not carry out a change of time and applies to perturbed Keplerian systems provided the perturbation is well defined for collision orbits. We can apply it in our particular setting, and since the transformed Keplerian Hamiltonian related to the inner bodies by the Ligon-Schaaf mapping has the same form as the Keplerian system previous to the transformation, the averaging process performed in Chapter 2 applies for inner collisions. After normalising and truncating, the regular reduction to $\mathcal{A}_{L_1,L_2}$ incorporates the possibility of rectilinear motions for the inner particles, and the same happens for the subsequent reductions.

(x) We characterise properly all type of bounded motions of the three particles, excluding triple collisions. In this sense our analysis extends Zhao's results [91, 93].

Future work:

In order to continue the study carried out in this thesis there is a lot of work to do. We enumerate some of the guidelines that we can follow:

(i) Use a similar scheme to study the N-body problem with the aim of finding families of KAM tori, generalising the analysis done for the three-body problem to the N-body problem. First we could achieve the existence of circular coplanar invariant $(3N-4)$-tori which has been studied by Féjoz [32] and Chierchia and Pinzari [13] but by making use of our techniques. The next step may be the proof of the existence of invariant tori associated to the motion such that the innermost body follows a near-rectilinear trajectory which is perpendicular to the invariable plane where the rest of the bodies move near the invariable plane describing near-circular trajectories. The Keplerian reduction could be applied and the invariants associated could be generalised straightforwardly, although the remaining reductions are not going to be a trivial task. However, there are evidences that seem to suggest that it can be accomplished by taking advantage of the Cartesian heliocentric coordinates and Delaunay variables instead of Deprit's ones, the overall process in the frame of singular reduction theory. This would allow us to compute the normal form associated to each elliptic equilibrium in a quite compact and explicit way to make use of Han, Li and Yi's Theorem.

(ii) Apply our study to some particular examples. We could choose two typical realistic applications, namely, a non-resonant situation of the solar system in the planetary regime choosing a model where mean-motion resonances do not play a role and the Sun-Earth-Moon system as the prototype of the lunar regime. We would look for explicit bounds of the perturbing region $\mathcal{P}_{\varepsilon,n}$ and its open subset $\mathcal{Q}_{\varepsilon,n}$. In addition, another task would be to estimate the excluding measure for the existence of some of the quasi-periodic invariant tori. These two examples have been usually studied in the circular coplanar case. Thence it might be interesting to consider the dynamics in other situations, dealing with the existence of other types of invariant tori.

(iii) Give insight about the following questions:

   (a) Can we ensure the existence of lower-dimensional invariant tori for the full Hamiltonian?

   (b) Do the dynamics of the quasi-periodic motions and related KAM tori whose existence has been established in Chapters 5 and 6 follow a similar pattern to that of the relative equilibria in $\mathcal{T}_{L_1,C,G_2}$ obtained in Chapter 3? In particular, do these tori bifurcate through Hamiltonian saddle-centre, pitchfork or the other bifurcations obtained in Chapter 3?

In both cases, due to the fact that we are dealing with a high-order degenerate system, it is not so evident that one can use the current results available about the existence of lower-dimensional tori and bifurcations of invariant tori, and new theoretical results would be needed.

According to [89], the generalisation of Theorem 1.15 to prove the existence of lower-dimensional tori is not straightforward, mainly because of the resonances occurring at lower-order terms.

Concerning the dynamics of the full system, it looks plausible that the qualitative behaviour of the fully-reduced system is going to be transferred to the spatial three-body problem, at least partially. That is, the relative equilibria would become invariant 4-tori of the full system with the same stability character and it seems that these tori might bifurcate following similar patterns as the ones of the relative equilibria. Moreover, in the case of elliptic equilibria, the reconstructed 4-tori would be surrounded by the 5-tori that we have established. However, since the Hamiltonian $\mathcal{H}_{\mathrm{Kep}}$ is a maximally superintegrable system and the perturbation does not remove the degeneracy, it is not expected that most invariant tori of the integrable approximation survive the perturbation and are only slightly deformed, see [38]. Thence, it is necessary that new theorems appear in this direction.

Appendix A

Invariants of the Keplerian reduction in terms of Deprit's coordinates

The invariants **a**, **b**, **c** and **d** have to be expressed in terms of Deprit's coordinates.

We start with the definitions of the invariants given in (2.18), putting the a_i, b_i, c_i and d_i in terms of the spatial Cartesian coordinates. We construct the frames $\mathcal{I}$, $\mathcal{N}_1$ and $\mathcal{N}_2$ in terms of the Cartesian-nodal coordinates of (2.10), following the steps of Subsection 2.2.1 or the detailed appendix of [34]. Consequently the spatial Cartesian coordinates and hence the invariants are readily written explicitly in terms of the Cartesian-nodal coordinates.

Then we use the change to polar-symplectic coordinates (2.11) expressing the invariants in terms of r_k, ϑ_k, R_k, Θ_k, ν, ν_1, C and B. The resulting expressions have to be independent of ℓ_1 and ℓ_2 as the variables a_i, b_i, c_i and d_i are the invariants of the Keplerian reduction. Thus, the formulae obtained do not depend explicitly on r_k, ϑ_k, R_k and ℓ_k, $k = 1, 2$.

After simplifying considerably the large intermediate expressions using the classical relations among the eccentric, the true, the mean anomalies and the polar-symplectic coordinates, the final form of the invariants in terms of Deprit's coordinates is derived.

Introducing W as

$$W = \sqrt{\big((C + G_2)^2 - G_1^2\big)\big(G_1^2 - (C - G_2)^2\big)},$$

we get:

$$a_1 = \frac{1}{2CG_1}\Bigg(\sin\nu\Big(B\cos\nu_1\big(G_1W-(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) + \sqrt{C^2-B^2}\big(G_1(C^2+G_1^2-G_2^2)+W\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) - 2CG_1B\sqrt{L_1^2-G_1^2}\sin\nu_1\cos\gamma_1\Big) + \cos\nu\Big(2C^2G_1\sqrt{L_1^2-G_1^2}\cos\nu_1\cos\gamma_1 + C\sin\nu_1\big(G_1W-(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big)\Big)\Bigg), \tag{A.1}$$

$$a_2 = \frac{1}{2CG_1}\Bigg(\cos\nu\Big(-B\cos\nu_1\big(G_1W-(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) - \sqrt{L_1^2-G_1^2}\big(W\sqrt{C^2-B^2}\sin\gamma_1-2CG_1B\sin\nu_1\cos\gamma_1\big) - G_1(C^2+G_1^2-G_2^2)\sqrt{C^2-B^2}\Big) + C\sin\nu\Big(\sin\nu_1\big(G_1W-(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) + 2CG_1\sqrt{L_1^2-G_1^2}\cos\nu_1\cos\gamma_1\big)\Big)\Bigg), \tag{A.2}$$

$$a_3 = \frac{1}{2CG_1}\Bigg(\sqrt{C^2-B^2}\cos\nu_1\Big((C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1-G_1W\Big) + \sqrt{L_1^2-G_1^2}\Big(2CG_1\sqrt{C^2-B^2}\sin\nu_1\cos\gamma_1+BW\sin\gamma_1\Big) + G_1B(C^2+G_1^2-G_2^2)\Bigg), \tag{A.3}$$

$$b_1 = \frac{1}{2CG_1}\Bigg(\sin\nu\Big(B\cos\nu_1\big(G_1W+(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) + \sqrt{C^2-B^2}\big(G_1(C^2+G_1^2-G_2^2)-W\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) + 2CG_1B\sqrt{L_1^2-G_1^2}\sin\nu_1\cos\gamma_1\Big) + \cos\nu\Big(-2C^2G_1\sqrt{L_1^2-G_1^2}\cos\nu_1\cos\gamma_1 + C\sin\nu_1\big(G_1W+(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big)\Big)\Bigg), \tag{A.4}$$

$$
\begin{aligned}
b_2 = \frac{1}{2CG_1}\Bigg(& \cos\nu\Big(-B\cos\nu_1\big(G_1W+(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) \\
& \quad +\sqrt{L_1^2-G_1^2}\big(W\sqrt{C^2-B^2}\sin\gamma_1-2CG_1B\sin\nu_1\cos\gamma_1\big) \\
& \quad -G_1(C^2+G_1^2-G_2^2)\sqrt{C^2-B^2}\Big) \\
& +C\sin\nu\Big(\sin\nu_1\big(G_1W+(C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1\big) \\
& \quad -2CG_1\sqrt{L_1^2-G_1^2}\cos\nu_1\cos\gamma_1\Big)\Bigg),
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
b_3 = \frac{1}{2CG_1}\Bigg(& -\sqrt{C^2-B^2}\cos\nu_1\Big((C^2+G_1^2-G_2^2)\sqrt{L_1^2-G_1^2}\sin\gamma_1+G_1W\Big) \\
& -\sqrt{L_1^2-G_1^2}\Big(2CG_1\sqrt{C^2-B^2}\sin\nu_1\cos\gamma_1+BW\sin\gamma_1\Big) \\
& +G_1B(C^2+G_1^2-G_2^2)\Bigg),
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
c_1 = \frac{1}{2CG_2}\Bigg(& \sin\nu\Big(B\cos\nu_1\big(-G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\gamma_2\big) \\
& \quad +\sqrt{C^2-B^2}\big(G_2(C^2-G_1^2+G_2^2)+W\sqrt{L_2^2-G_2^2}\sin\gamma_2\big) \\
& \quad +2CG_2B\sqrt{L_2^2-G_2^2}\sin\nu_1\cos\gamma_2\Big) \\
& +\cos\nu\Big(-2C^2G_2\sqrt{L_2^2-G_2^2}\cos\nu_1\cos\gamma_2 \\
& \quad -C\sin\nu_1\big(G_2W-(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\gamma_2\big)\Big)\Bigg),
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
c_2 = \frac{1}{2CG_2}\Bigg(& \cos\nu\Big(B\cos\nu_1\big(G_2W-(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\gamma_2\big) \\
& \quad -\sqrt{L_2^2-G_2^2}\big(W\sqrt{C^2-B^2}\sin\gamma_2+2CG_2B\sin\nu_1\cos\gamma_2\big) \\
& \quad -G_2(C^2-G_1^2+G_2^2)\sqrt{C^2-B^2}\Big) \\
& +C\sin\nu\Big(\sin\nu_1\big(-G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\gamma_2\big) \\
& \quad -2CG_2\sqrt{L_2^2-G_2^2}\cos\nu_1\cos\gamma_2\Big)\Bigg),
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
c_3 \;=\; \frac{1}{2CG_2}\bigg(&\sqrt{C^2-B^2}\cos\,\nu_1\Big(-(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2+G_2W\Big)\\
&+\sqrt{L_2^2-G_2^2}\Big(-2CG_2\sqrt{C^2-B^2}\sin\,\nu_1\cos\,\gamma_2+BW\sin\,\gamma_2\Big)\\
&+G_2B(C^2-G_1^2+G_2^2)\bigg),
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
d_1 \;=\; \frac{1}{2CG_2}\bigg(&\sin\nu\Big(B\cos\,\nu_1\big(G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2\big)\\
&\quad+\sqrt{C^2-B^2}\big(G_2(C^2-G_1^2+G_2^2)-W\sqrt{L_2^2-G_2^2}\sin\,\gamma_2\big)\\
&\quad+2CG_2B\sqrt{L_2^2-G_2^2}\sin\,\nu_1\cos\,\gamma_2\Big)\\
&+\cos\,\nu\Big(-2C^2G_2\sqrt{L_2^2-G_2^2}\cos\,\nu_1\cos\,\gamma_2\\
&\quad+C\sin\,\nu_1\big(G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2\big)\Big)\bigg),
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
d_2 \;=\; \frac{1}{2CG_2}\bigg(&\cos\,\nu\Big(-B\cos\,\nu_1\big(G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2\big)\\
&\quad+\sqrt{L_2^2-G_2^2}\big(W\sqrt{C^2-B^2}\sin\,\gamma_2-2CG_2B\sin\,\nu_1\cos\,\gamma_2\big)\\
&\quad-G_2(C^2-G_1^2+G_2^2)\sqrt{C^2-B^2}\Big)\\
&+C\sin\,\nu\Big(\sin\,\nu_1\big(G_2W+(C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2\big)\\
&\quad-2CG_2\sqrt{L_2^2-G_2^2}\cos\,\nu_1\cos\,\gamma_2\Big)\bigg),
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
d_3 \;=\; \frac{1}{2CG_2}\bigg(&-\sqrt{C^2-B^2}\cos\,\nu_1\Big((C^2-G_1^2+G_2^2)\sqrt{L_2^2-G_2^2}\sin\,\gamma_2+G_2W\Big)\\
&-\sqrt{L_2^2-G_2^2}\Big(2CG_2\sqrt{C^2-B^2}\sin\,\nu_1\cos\,\gamma_2+BW\sin\,\gamma_2\Big)\\
&+G_2B(C^2-G_1^2+G_2^2)\bigg).
\end{aligned}
\tag{A.12}
$$

We remark that when $G_1 = 0$ then $C = G_2$ and $W = 0$. Then the invariants a_i and b_i can be analytically extended to the case $G_1 = 0$.

Appendix B

Proof of Theorem 5.1 for the remaining cases

B.1 Study in $\mathcal{T}_{L_1,C,G_2}$

B.1.1 Case (a)

In case (a) of Table 5.2, which deals with motions of the three bodies that are of non-circular and non-coplanar type, γ_1^* and G_1^* stand for the concrete values taken at the relative equilibrium on $\mathcal{T}_{L_1,C,G_2}$. We make the study for $\gamma_1^* = 0$ and $G_1^* = \sqrt{C^2 + 3G_2^2}$.

The coordinates of the equilibrium point of case (a) in the space $\mathcal{T}_{L_1,C,G_2}$ are

$$\Big(2C^2 + 6G_2^2 - L_1^2, 0, 2G_2\sqrt{(C^2 - G_2^2)(L_1^2 - C^2 - 3G_2^2)}\Big).$$

In this case it is assumed that $G_1 \not\approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \not\approx L_2$ and the motions of the two fictitious bodies are not coplanar, so $G_1 \not\approx |C \pm G_2|$.

First we introduce the symplectic change of coordinates given in Table 5.2(a). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}\mathcal{K}_1 = \frac{\mathcal{M}L_1^2}{L_2^3 G_2^5 (y_1 + G_1^*)^2}\Bigg(&\Big(-5L_1^2 + 3(G_1^* + y_1)^2\Big)\\ &\times\Big(3G_2^4 + 3\big(C^2 - (G_1^* + y_1)^2\big)^2 + 2G_2^2\big(-3C^2 + (G_1^* + y_1)^2\big)\Big)\\ &- 15\Big((C - G_1^* - y_1)^2 - G_2^2\Big)\Big((C + G_1^* + y_1)^2 - G_2^2\Big)\\ &\times\Big((G_1^* + y_1)^2 - L_1^2\Big)\cos\big(2(\gamma_1^* + x_1)\big)\Bigg).\end{aligned}$$

Then we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_1 \;=\; \varepsilon^{1/4}\bar{x}_1 + x_1^*, \quad y_1 \;=\; \varepsilon^{1/4}\bar{y}_1 + y_1^*,$$

where x_1^* and y_1^* are the values of x_1 and y_1 at the equilibrium, i.e. $(x_1^*, y_1^*) = (0,0)$. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{B.1}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Bigg(&6(C^2+G_2^2)-5L_1^2\\ &-3\varepsilon^{1/2}\bigg(\frac{5}{C^2+3G_2^2}(C^2-G_2^2)(C^2+3G_2^2-L_1^2)\bar{x}_1^2\\ &+\frac{1}{G_2^2}(C^2+3G_2^2)\bar{y}_1^2\bigg)\Bigg).\end{aligned} \tag{B.2}$$

We construct action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}\bar{x}_1 \;&=\; 2^{1/2}G_2^{-1/2}\left(\frac{(C^2+3G_2^2)^2}{5(C^2-G_2^2)(C^2+3G_2^2-L_1^2)}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\ \bar{y}_1 \;&=\; 2^{1/2}G_2^{1/2}\left(\frac{5(C^2-G_2^2)(C^2+3G_2^2-L_1^2)}{(C^2+3G_2^2)^2}\right)^{1/4} I_1^{1/2}\cos\phi_1.\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}\mathcal{K}_1 \;=\; \frac{8\mathcal{M}L_1^2}{L_2^3G_2^4}\Bigg(&\Big(6G_2(C^2+G_2^2)-5L_1^2\Big)\\ &-6\varepsilon^{1/2}I_1\sqrt{5(C^2-G_2^2)(C^2+3G_2^2-L_1^2)}\Bigg).\end{aligned}$$

We introduce a new parameter $\eta^2 = \varepsilon$, leading to

$$\mathcal{H} \;=\; h_0 + \eta^2 h_1 + \eta^3 h_2 + \mathcal{O}(\eta^4),$$

where

$$\begin{aligned}h_0 \;&=\; -\frac{\mu_1^3M_1^2}{2L_1^2} - \frac{\mu_2^3M_2^2}{2L_2^2},\\ h_1 \;&=\; \frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Big(6(C^2+G_2^2)-5L_1^2\Big),\\ h_2 \;&=\; -\frac{48\mathcal{M}L_1^2}{L_2^3G_2^4}I_1\sqrt{5(C^2-G_2^2)(C^2+3G_2^2-L_1^2)}.\end{aligned} \tag{B.3}$$

At this point we easily identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C}, \frac{\partial h_1}{\partial G_2},\ \frac{\partial h_2}{\partial I_1}\right).$$

Now we form the matrix

$$\partial_I^1 \Omega(I) = \left(\Omega_k,\ \frac{\partial \Omega_k}{\partial L_1},\ \frac{\partial \Omega_k}{\partial L_2},\ \frac{\partial \Omega_k}{\partial C},\ \frac{\partial \Omega_k}{\partial G_2},\ \frac{\partial \Omega_k}{\partial I_1}\right), \quad 1 \leq k \leq 5. \tag{B.4}$$

We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1 \Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that we deal with.

According to Theorem 1.15 the excluding measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^{\delta/2})$ or $\mathcal{O}(\varepsilon^{\delta/4})$ with $0 < \delta < 1/5$. Calculating $b = \sum_{i=1}^{a} \beta_i (n_i - n_{i-1})$ we obtain $b = 7$. So, we cannot apply Remark 2 of [36] p. 1422 because $\eta^{sb+\delta} = \eta^{14+\delta} = \varepsilon^{(14+\delta)/2}$ and the perturbation in (5.2) is of a lower order (it is of order two). Thus, we cannot improve the measure for the existence of invariant tori.

B.1.2 Case (b)

Case (b) of Table 5.2 deals with motions of the three bodies that are coplanar.

The coordinates of the equilibrium point of case (b) in the space $\mathcal{T}_{L_1,C,G_2}$ are $(2(C+G_2)^2 - L_1, 0, 0)$. We assume that $G_1 \not\approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \not\approx L_2$ and the motions of the two fictitious bodies are coplanar, so $G_1 \approx C + G_2$.

First we introduce the symplectic change of coordinates given in Table 5.2(b).

Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}
\mathcal{K}_1 \;=\; & \frac{4\mathcal{M}L_1^2}{L_2^3G_2^5(x_1^2+y_1^2-2C-2G_2)^2} \\
& \times \Bigg(\frac{15}{64}\Big(x_1^2-y_1^2\Big)\Big(x_1^2+y_1^2-4C\Big)\Big(x_1^2+y_1^2-4C-4G_2\Big) \\
& \quad \times \Big(x_1^2+y_1^2-4G_2\Big)\Big((x_1^2+y_1^2-2G_2-2C)^2-4L_1^2\Big) \\
& \quad + \Big(-5L_1^2+\frac{3}{4}(x_1^2+y_1^2-2C-2G_2)^2\Big) \\
& \quad \times \Big(3G_2^4+3\big(C^2-\frac{1}{4}(x_1^2+y_1^2-2C-2G_2)^2\big)^2\Big) \\
& \quad + 2G_2^2\Big(-3C^2+\frac{1}{4}(x_1^2+y_1^2-2C-2G_2)^2\Big)\Bigg).
\end{aligned}$$

Then, we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_1 \;=\; \varepsilon^{1/4}\bar{x}_1 + x_1^*, \quad y_1 \;=\; \varepsilon^{1/4}\bar{y}_1 + y_1^*,$$

where x_1^* and y_1^* are the values of x_1 and y_1 at the equilibrium, i.e. $(x_1^*, y_1^*) = (0,0)$. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\text{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{B.5}$$

where

$$\begin{aligned}
\mathcal{K}_1 \;=\; -\frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Bigg(& 3(C+G_2)^2-5L_1^2 \\
& + 3\varepsilon^{1/2}\Big(\frac{1}{G_2(C+G_2)}\big(5CL_1^2-(C+G_2)^2(4C+G_2)\big)\bar{x}_1^2 \\
& \quad + \frac{1}{G_2}(C^2-G_2^2)\bar{y}_1^2\Big)\Bigg).
\end{aligned} \tag{B.6}$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}
\bar{x}_1 \;&=\; 2^{1/2}\left(\frac{(C-G_2)(C+G_2)^2}{5CL_1^2-(C+G_2)^2(4C+G_2)}\right)^{1/4} I_1^{1/2}\sin\phi_1, \\
\bar{y}_1 \;&=\; 2^{1/2}\left(\frac{5CL_1^2-(C+G_2)^2(4C+G_2)}{(C-G_2)(C+G_2)^2}\right)^{1/4} I_1^{1/2}\cos\phi_1.
\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\mathcal{K}_1 = \frac{8\mathcal{M}L_1^2}{L_2^3G_2^4}\Bigg(G_2\big(3(C+G_2)^2-5L_1^2\big) + 6\varepsilon^{1/2}I_1\sqrt{(C-G_2)\big(5CL_1^2-(C+G_2)^2(4C+G_2)\big)}\Bigg).$$

We define a new parameter $\eta^2 = \varepsilon$, ending up with

$$\mathcal{H} = h_0 + \eta^2 h_1 + \eta^3 h_2 + \mathcal{O}(\eta^4),$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2} - \frac{\mu_2^3M_2^2}{2L_2^2},\\ h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Big(3(C+G_2)^2-5L_1^2\Big),\\ h_2 &= \frac{48\mathcal{M}L_1^2}{L_2^3G_2^4}I_1\sqrt{(C-G_2)\big(5CL_1^2-(C+G_2)^2(4C+G_2)\big)}. \end{aligned} \tag{B.7}$$

We can identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_1}{\partial G_2}, \frac{\partial h_2}{\partial I_1}\right).$$

Next we build the matrix (B.4). We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1 \Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents coplanar motions such that $G_1 \approx C + G_2$.

B.1.3 Case (c)

Case (c) of Table 5.2 deals with motions of the three bodies that are coplanar.

The coordinates of the equilibrium point of case (c) in the space $\mathcal{T}_{L_1,C,G_2}$ are $(2(C-G_2)^2 - L_1, 0, 0)$. In this case it is assumed that $G_1 \not\approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \not\approx L_2$ and the motions of the two fictitious bodies are coplanar, so $G_1 \approx G_2 - C$.

First we introduce the symplectic change of coordinates given in Table 5.2(c). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}\mathcal{K}_1 \;=\; & \frac{4\mathcal{M}L_1^2}{L_2^3G_2^5(x_1^2+y_1^2-2C+2G_2)^2} \\ & \times\Bigg(\frac{15}{64}\Big(x_1^2-y_1^2\Big)\Big(x_1^2+y_1^2-4C\Big)\Big(x_1^2+y_1^2-4C+4G_2\Big) \\ & \qquad\times\Big(x_1^2+y_1^2+4G_2\Big)\Big((x_1^2+y_1^2+2G_2-2C)^2-4L_1^2\Big) \\ & \quad+\Big(-5L_1^2+\frac{3}{4}(x_1^2+y_1^2-2C+2G_2)^2\Big) \\ & \qquad\times\Big(3G_2^4+3\big(C^2-\frac{1}{4}(x_1^2+y_1^2-2C+2G_2)^2\big)^2\Big) \\ & \quad+2G_2^2\Big(-3C^2+\frac{1}{4}(x_1^2+y_1^2-2C+2G_2)^2\Big)\Bigg).\end{aligned}$$

Then we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_1 \;=\; \varepsilon^{1/4}\bar{x}_1+x_1^*,\quad y_1 \;=\; \varepsilon^{1/4}\bar{y}_1+y_1^*,$$

where x_1^* and y_1^* are the values of x_1 and y_1 at the equilibrium, i.e. $(x_1^*,y_1^*)=(0,0)$. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\text{Kep}}+\varepsilon\,\mathcal{K}_1+\mathcal{O}(\varepsilon^2), \tag{B.8}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Bigg(& 3(C-G_2)^2-5L_1^2 \\ & -3\varepsilon^{1/2}\Bigg(\frac{1}{G_2(C-G_2)}\big(5CL_1^2-(C-G_2)^2(4C-G_2)\big)\bar{x}_1^2 \\ & \qquad+\frac{1}{G_2}(C^2-G_2^2)\bar{y}_1^2\Bigg)\Bigg).\end{aligned} \tag{B.9}$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}\bar{x}_1 \;&=\; 2^{1/2}\left(\frac{(C-G_2)^2(C+G_2)}{5CL_1^2-(C-G_2)^2(4C-G_2)}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\ \bar{y}_1 \;&=\; 2^{1/2}\left(\frac{5CL_1^2-(C-G_2)^2(4C-G_2)}{(C-G_2)^2(C+G_2)}\right)^{1/4} I_1^{1/2}\cos\phi_1.\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\mathcal{K}_1 = \frac{8\mathcal{M}L_1^2}{L_2^3 G_2^4}\Bigg(G_2\big(3(C-G_2)^2 - 5L_1^2\big) - 6\varepsilon^{1/2} I_1 \sqrt{(C+G_2)\big(5CL_1^2 - (C-G_2)^2(4C-G_2)\big)}\Bigg).$$

We introduce a new parameter $\eta^2 = \varepsilon$. It leads to

$$\mathcal{H} = h_0 + \eta^2 h_1 + \eta^3 h_2 + \mathcal{O}(\eta^4),$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}, \\ h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^3 G_2^3}\Big(3(C-G_2)^2 - 5L_1^2\Big), \\ h_2 &= -\frac{48\mathcal{M}L_1^2}{L_2^3 G_2^4} I_1 \sqrt{(C+G_2)\big(5CL_1^2 - (C-G_2)^2(4C-G_2)\big)}. \end{aligned} \tag{B.10}$$

At this point we easily identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C}, \frac{\partial h_1}{\partial G_2},\ \frac{\partial h_2}{\partial I_1}\right).$$

Next we form the matrix (B.4). We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1 \Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents coplanar motions such that $G_1 \approx G_2 - C$.

B.1.4 Case (d)

Case (d) of Table 5.2 deals with motions of the three bodies that are coplanar.

The coordinates of the equilibrium point of case (d) in the space $\mathcal{T}_{L_1,C,G_2}$ are $(2(C-G_2)^2 - L_1, 0, 0)$.

In this case it is assumed that $G_1 \not\approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \not\approx L_2$ and the motions of the two fictitious bodies are coplanar, so $G_1 \approx C - G_2$.

First we introduce the symplectic change of coordinates given in Table 5.2(d). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}\mathcal{K}_1 \;=\; & \frac{4\mathcal{M}L_1^2}{L_2^3G_2^5(x_1^2+y_1^2+2C-2G_2)^2}\\ & \times\Bigg(\frac{15}{64}\Big(x_1^2-y_1^2\Big)\Big(x_1^2+y_1^2+4C\Big)\Big(x_1^2+y_1^2+4C+4G_2\Big)\\ & \quad\times\Big(x_1^2+y_1^2-4G_2\Big)\Big((x_1^2+y_1^2-2G_2+2C)^2-4L_1^2\Big)\\ & \quad+\Big(-5L_1^2+\frac{3}{4}(x_1^2+y_1^2+2C-2G_2)^2\Big)\\ & \quad\times\Big(3G_2^4+3\big(C^2-\frac{1}{4}(x_1^2+y_1^2+2C-2G_2)^2\big)^2\Big)\\ & \quad+2G_2^2\Big(-3C^2+\frac{1}{4}(x_1^2+y_1^2+2C-2G_2)^2\Big)\Bigg).\end{aligned}$$

Then we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_1 \;=\; \varepsilon^{1/4}\bar{x}_1+x_1^*,\quad y_1 \;=\; \varepsilon^{1/4}\bar{y}_1+y_1^*,$$

where x_1^* and y_1^* are the values of x_1 and y_1 at the equilibrium, i.e. $(x_1^*,y_1^*)=(0,0)$. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\rm Kep}+\varepsilon\,\mathcal{K}_1+\mathcal{O}(\varepsilon^2), \tag{B.11}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Bigg(& 3(C-G_2)^2-5L_1^2\\ & +3\varepsilon^{1/2}\bigg(\frac{1}{G_2(C-G_2)}\big(5CL_1^2-(C-G_2)^2(4C-G_2)\big)\bar{x}_1^2\\ & \qquad+\frac{1}{G_2}(C^2-G_2^2)\bar{y}_1^2\bigg)\Bigg).\end{aligned} \tag{B.12}$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}\bar{x}_1 \;&=\; 2^{1/2}\left(\frac{(C-G_2)^2(C+G_2)}{5CL_1^2-(C-G_2)^2(4C-G_2)}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\ \bar{y}_1 \;&=\; 2^{1/2}\left(\frac{5CL_1^2-(C-G_2)^2(4C-G_2)}{(C-G_2)^2(C+G_2)}\right)^{1/4} I_1^{1/2}\cos\phi_1.\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\mathcal{K}_1 = \frac{8\mathcal{M}L_1^2}{L_2^3G_2^4}\Big(G_2\big(3(C-G_2)^2-5L_1^2\big) + 6\varepsilon^{1/2}I_1\sqrt{(C+G_2)\big(5CL_1^2-(C-G_2)^2(4C-G_2)\big)}\Big).$$

We introduce a new parameter $\eta^2 = \varepsilon$. It leads to

$$\mathcal{H} = h_0 + \eta^2 h_1 + \eta^3 h_2 + \mathcal{O}(\eta^4),$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2} - \frac{\mu_2^3M_2^2}{2L_2^2}, \\ h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^3G_2^3}\Big(3(C-G_2)^2-5L_1^2\Big), \\ h_2 &= \frac{48\mathcal{M}L_1^2}{L_2^3G_2^4}I_1\sqrt{(C+G_2)\big(5CL_1^2-(C-G_2)^2(4C-G_2)\big)}. \end{aligned} \tag{B.13}$$

At this point we easily identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 4$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C}, \frac{\partial h_1}{\partial G_2},\ \frac{\partial h_2}{\partial I_1}\right).$$

Now we build the matrix (B.4). We need rank five because we are looking for KAM 5-tori. Then, we construct the 5×31-matrix that results from adding to $\partial_I^1\Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s = 2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents coplanar motions such that $G_1 \approx G_2 - C$.

B.2 Study in $\mathcal{S}_{L_1,L_2,C}$

B.2.1 Case (a)

Case (a) of Table 5.3 deals with motions of the three bodies that are circular for the inner and outer bodies.

The coordinates of the equilibrium point of case (a) in the space $\mathcal{S}_{L_1,L_2,C}$ are $(L_1^2, L_2^2, 0, 0, 0, 0)$. We have that $G_1 \approx L_1$ and the outer body is not moving in a near-circular orbit, thus $G_2 \approx L_2$.

First we introduce the symplectic change of coordinates given in Table 5.3(a). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}\mathcal{K}_1 \;=\; & -\frac{2\mathcal{M}L_1^2}{L_2^3\big(x_1^2+y_1^2-2L_1\big)^2\big(x_2^2+y_2^2-2L_2\big)^5}\\ & \times\Bigg(15(x_1^2-y_1^2)(x_1^2+y_1^2-4L_1)\Big(\big(x_1^2-x_2^2+y_1^2-y_2^2-2L_1+2L_2\big)^2-4C^2\Big)\\ &\quad\times\Big(\big(x_1^2+x_2^2+y_1^2+2y_2^2-2L_1+2L_2\big)-4C^2\Big)\\ &\quad-4\Big(\frac{3}{4}\big(x_1^2+y_1^2-2L_1\big)^2-5L_1^2\Big)\\ &\quad\times\Big(3\big((x_1^2+y_1^2-2L_1)^2-4C^2\big)^2-2\big(12C^2-(x_1^2+y_1^2-2L_1)^2\big)\\ &\quad\times\big(x_2^2+y_2^2-2L_2\big)^2+3\big(x_2^2+y_2^2-2L_2\big)^4\Big)\Bigg).\end{aligned}$$

Then we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_i \;=\; \varepsilon^{1/4}\bar{x}_i+x_i^*,\quad y_i \;=\; \varepsilon^{1/4}\bar{y}_i+y_i^*,$$

where x_i^* and y_i^*, $i=1,2$ are the values of x_i and y_i at the equilibrium. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\text{Kep}}+\varepsilon\,\mathcal{K}_1+\mathcal{O}(\varepsilon^2), \tag{B.14}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; -\frac{\mathcal{M}L_1}{L_2^9}\Bigg(& 2L_1L_2\Big(3(C^2-L_1^2)^2+2(-3C^2+L_1^2)L_2^2+3L_2^4\Big)\\ & -3\varepsilon^{1/2}\Big(4L_1^2L_2\Big(-C^2+L_1^2-3L_2^2\Big)\bar{x}_1^2\\ &\qquad +2L_2\Big(-5C^4-3L_1^4+4L_1^2L_2^2-5L_2^4+2C^2(4L_1^2+5L_2^2)\Big)\bar{y}_1^2\\ &\qquad +L_1\Big(-5(C^2+L_1^2)^2+2(3C^2-L_1^2)L_2^2-L_2^4\Big)(\bar{x}_2^2+\bar{y}_2^2)\Big)\Bigg).\end{aligned} \tag{B.15}$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}
\bar{x}_1 &= 2^{1/4}\left(\frac{5C^4+3L_1^4-4L_1^2L_2^2+5L_2^4-2C^2(4L_1^2+5L_2^2)}{L_1^2(C^2-L_1^2+3L_2^2)}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\
\bar{y}_1 &= 2^{1/2}\left(\frac{L_1^2(C^2-L_1^2+3L_2^2)}{5C^4+3L_1^4-4L_1^2L_2^2+5L_2^4-2C^2(4L_1^2+5L_2^2)}\right)^{1/4} I_1^{1/2}\cos\phi_1,\\
\bar{x}_2 &= \sqrt{2I_2}\sin\phi_2,\\
\bar{y}_2 &= \sqrt{2I_2}\cos\phi_2.
\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}
\mathcal{K}_1 = -\frac{2\mathcal{M}L_1^2}{L_2^8}\Bigg(&\Big(3(C^2-L_1^2)^2+2(-3C^2+L_1^2)L_2^2+3L_2^4\Big)\\
&+3\varepsilon^{1/2}\Big(2^{3/2}I_1\sqrt{C^2-L_1^2+3L_2^2}\\
&\qquad\times\sqrt{5C^4+3L_1^4-4L_1^2L_2^2+5L_2^4-2C^2(4L_1^2+5L_2^2)}\\
&\qquad+\frac{I_2}{L_2}\big(5(C^2-L_1^2)^2-6C^2L_2^2+2L_1^2L_2^2-L_2^4\big)\Big)\Bigg).
\end{aligned}$$

We define a new parameter $\eta^2=\varepsilon$ for the Hamiltonian $\mathcal{H}$. It leads to

$$\mathcal{H} = h_0+\eta^2h_1+\eta^3h_2+\mathcal{O}(\eta^4),$$

where

$$\begin{aligned}
h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2}-\frac{\mu_2^3M_2^2}{2L_2^2},\\
h_1 &= \frac{2\mathcal{M}L_1^2}{L_2^8}\Big(3(C^2-L_1^2)^2+2(-3C^2+L_1^2)L_2^2+3L_2^4\Big),\\
h_2 &= \frac{48\mathcal{M}L_1^2}{L_2^3G_2^4}\Bigg(2^{3/2}L_2I_1\sqrt{C^2-L_1^2+3L_2^2}\\
&\qquad\times\sqrt{5C^4+3L_1^4-4L_1^2L_2^2+5L_2^4-2C^2(4L_1^2+5L_2^2)}\\
&\qquad-I_2\big(5(C^2-L_1^2)^2+2(3C^2-L_1^2)L_2^2-L_2^4\big)\Bigg).
\end{aligned} \tag{B.16}$$

We obtain the following numbers in Theorem 1.15: $n_0=2$, $n_1=3$, $n_2=5$, $\beta_1=2$, $\beta_2=3$ and $a=2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C},\frac{\partial h_2}{\partial I_1},\ \frac{\partial h_2}{\partial I_2}\right).$$

We form the matrix

$$\partial_I^1\,\Omega(I) \;=\; \left(\Omega_k,\; \frac{\partial\Omega_k}{\partial L_1},\; \frac{\partial\Omega_k}{\partial L_2},\; \frac{\partial\Omega_k}{\partial C},\; \frac{\partial\Omega_k}{\partial I_1},\; \frac{\partial\Omega_k}{\partial I_2}\right), \quad 1\le k\le 5. \tag{B.17}$$

We need rank five because we are looking for KAM 5-tori. Then we construct the 5×31-matrix that results from adding to $\partial_I^1\,\Omega(I)$ the columns corresponding to the partials of second order. This time the rank of the matrix is five and $s=2$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions for the inner and outer bodies.

B.2.2 Case (b)

Case (b) of Table 5.3 deals with motions of the three bodies that circular for the inner and outer bodies.

The coordinates of the equilibrium point of case (b) in the space $\mathcal{S}_{L_1,L_2,C}$ are $(L_1^2, 2(C-L_1)^2 - L_2^2, 0, 0, 0, 0)$. In this case it is assumed that $G_2\approx L_2$ and $G_1\approx C+G_2$.

First we introduce the symplectic change of coordinates given in Table 5.3(b). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}
\mathcal{K}_1 \;=\; &-\frac{2\mathcal{M}L_1^2}{L_2^3\big(x_1^2+y_1^2-2L_1\big)^2\big(-x_1^2+x_2^2-y_1^2+y_2^2-2C+2L_1\big)^5}\\
&\times\Bigg(15\Big(-x_1^2-y_1^2-4L_1\Big)\Big(x_2^2+y_2^2-4C\Big)\\
&\quad\times\Big(-2x_1^2+x_2^2-2y_1^2+y_2^2+4L_1\Big)\\
&\quad\times\Big(-2x_1^2+x_2^2-2y_1^2+y_2^2-4C+4L_1\Big)\\
&\quad\times\Big(y_1(y_2-x_2)+x_1(x_2+y_2)\Big)\\
&\quad\times\Big(x_1(x_2-y_2)+y_1(x_2+y_2)\Big)\\
&\quad+\Big(\frac{3}{4}\big(x_1^2+y_1^2-2L_1\big)^2-5L_1^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg(3\Big((x_1^2+y_1^2-2L_1)^2-4C^2\Big)^2\\
&\quad -2\Big(12C^2-(x_1^2+y_1^2-2L_1)^2\Big)\\
&\quad \times\Big(2C-2L_1+x_1^2-x_2^2+y_1^2-y_2^2\Big)^2\\
&\quad \times 3\Big(2C-2L_1+x_1^2-x_2^2+y_1^2-y_2^2\Big)^4\Bigg)\Bigg).
\end{aligned}
$$

Next we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_i \;=\; \varepsilon^{1/4}\bar{x}_i + x_i^*, \quad y_i \;=\; \varepsilon^{1/4}\bar{y}_i + y_i^*,$$

where x_i^* and y_i^*, $i = 1, 2$ are the values of x_i and y_i at the equilibrium. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{B.18}$$

where

$$
\begin{aligned}
\mathcal{K}_1 \;=\; \frac{16\mathcal{M}L_1^2}{(C-L_1)^5L_2^3}\Bigg(&L_1^2\Big(C-L_1\Big)^2\\
&+\frac{3}{2}\varepsilon^{1/2}(C-L_1)L_1\Bigg(L_1\Big(-2\bar{x}_1^2+\bar{x}_2^2-2\bar{y}_1^2+\bar{y}_2^2\Big)\\
&\qquad +C\Big(\bar{x}_1^2+\bar{x}_2^2+\bar{y}_1^2+\bar{y}_2^2\Big)\Bigg)\\
&+\frac{3}{8}\varepsilon\Bigg(C^2\Big(-x_1^4-2x_1^2\left(x_2^2+y_1^2-9y_2^2\right)-40x_1x_2y_1y_2+x_2^4\\
&\qquad +18x_2^2y_1^2+2y_2^2(x_2-y_1)(x_2+y_1)-y_1^4+y_2^4\Big)\\
&\quad +CL_1\Big(2y_2^2\left(7x_2^2-10x_1^2\right)\\
&\qquad -4\Big(x_1^4+y_1^2\left(2x_1^2+5x_2^2\right)+y_1^4\Big)\\
&\qquad +40x_1x_2y_1y_2+7x_2^4+7y_2^4\Big)\\
&\quad +L_1^2\left(x_1^2-x_2^2+y_1^2-y_2^2\right)\left(9x_1^2-5x_2^2+9y_1^2-5y_2^2\right)\Bigg)\Bigg).
\end{aligned}
\tag{B.19}
$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}\bar{x}_1 &= \sqrt{2I_1}\sin\phi_1, & \bar{y}_1 &= \sqrt{2I_1}\cos\phi_1,\\ \bar{x}_2 &= \sqrt{2I_2}\sin\phi_2, & \bar{y}_2 &= \sqrt{2I_2}\cos\phi_2.\end{aligned}$$

After applying this transformation Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}\mathcal{K}_1 = \frac{16\mathcal{M}L_1^2}{(C-L_1)^5L_2^3}\Bigg(& L_1^2(C-L_1)^2 \\ & + 3\varepsilon^{1/2}L_1(C-L_1)\Big(I_1\big(C-2L_1\big)+I_2\big(C+L_1\big)\Big) \\ & + \frac{3}{2}\varepsilon\Bigg(C^2\left(-I_1^2+8I_1I_2+I_2^2\right)+CL_1\left(-4I_1^2-10I_1I_2+7I_2^2\right) \\ & \qquad + 10CI_1I_2(L_1-C)\cos(2(\phi_1-\phi_2)) \\ & \qquad + L_1^2(I_1-I_2)(9I_1-5I_2)\Bigg)\Bigg).\end{aligned} \tag{B.20}$$

Next we average with respect to $\phi_1-\phi_2$ at first order and for the full Hamiltonian $\mathcal{H}$ we introduce the small parameter η such that $\eta^2=\varepsilon$, arriving at:

$$\mathcal{H} = h_0+\eta^2h_1+\eta^3h_2+\eta^4h_3+\mathcal{O}(\eta^5),$$

where

$$\begin{aligned}h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2}-\frac{\mu_2^3M_2^2}{2L_2^2},\\ h_1 &= \frac{16\mathcal{M}L_1^4}{(C-L_1)^3L_2^3},\\ h_2 &= \frac{48\mathcal{M}L_1^3}{(C-L_1)^4L_2^3}\Big(I_1\big(C-2L_1\big)+I_2\big(C+L_1\big)\Big),\\ h_3 &= \frac{24\mathcal{M}L_1^2}{(C-L_1)^5L_2^3}\Big(I_1^2\big(-C^2-4CL_1+9L_1^2\big)+I_2^2\big(C^2+7CL_1+5L_1^2\big) \\ &\qquad + I_1I_2\big(8C^2-10CL_1-14L_1^2\big)\Big).\end{aligned} \tag{B.21}$$

We can identify the following numbers in Theorem 1.15: $n_0=2$, $n_1=3$, $n_2=5$, $n_3=5$, $\beta_1=2$, $\beta_2=3$, $\beta_3=4$ and $a=3$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C},\frac{\partial h_2}{\partial I_1},\ \frac{\partial h_2}{\partial I_2},\frac{\partial h_3}{\partial I_1},\ \frac{\partial h_3}{\partial I_2}\right).$$

Next we build the matrix

$$\partial_I^1 \Omega(I) = \left(\Omega_k \, , \, \frac{\partial \Omega_k}{\partial L_1} \, , \, \frac{\partial \Omega_k}{\partial L_2} \, , \, \frac{\partial \Omega_k}{\partial C} \, , \, \frac{\partial \Omega_k}{\partial I_1} \, , \, \frac{\partial \Omega_k}{\partial I_2}\right), \quad 1 \leq k \leq 7. \tag{B.22}$$

The rank of the matrix is five and $s = 1$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions for inner bodies and they are coplanar with the outer body such that $G_1 \approx C + G_2$.

B.2.3 Case (c)

Case (c) of Table 5.3 deals with motions of the three bodies that are circular for the inner and outer bodies. In this case it is assumed that $G_2 \approx L_2$ and $G_1 \approx G_2 - C$. The coordinates of the equilibrium point of case (c) in the space $\mathcal{S}_{L_1,L_2,C}$ are $(L_1^2, 2(C + L_1)^2 - L_2^2, 0, 0, 0, 0)$.

First we introduce the symplectic change of coordinates given in Table 5.3(c). Hamiltonian (2.17) in terms of x_1 and y_1 is:

$$\begin{aligned}
\mathcal{K}_1 = & -\frac{2\mathcal{M}L_1^2}{L_2^3\big(x_1^2 + y_1^2 - 2L_1\big)^2\big(x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2C - 2L_1\big)^5} \\
& \times\Bigg(15\Big(-x_1^2 - y_1^2 + 4L_1\Big)\Big(x_2^2 + y_2^2 - 4C\Big) \\
& \qquad \times\Big(-2x_1^2 - x_2^2 - 2y_1^2 - y_2^2 + 4L_1\Big) \\
& \qquad \times\Big(-2x_1^2 - x_2^2 - 2y_1^2 - y_2^2 + 4C + 4L_1\Big) \\
& \qquad \times\Big(y_1(y_2 - x_2) + x_1(x_2 + y_2)\Big) \\
& \qquad \times\Big(x_1(x_2 - y_2) + y_1(x_2 + y_2)\Big) \\
& \quad -\Big(\frac{3}{4}\big(x_1^2 + y_1^2 - 2L_1\big)^2 - 5L_1^2\Big) \\
& \quad \times\Big(3\Big((x_1^2 + y_1^2 - 2L_1)^2 - 4C^2\Big)^2 \\
& \qquad -2\Big(12C^2 - (x_1^2 + y_1^2 - 2L_1)^2\Big) \\
& \qquad \times\Big(-2C - 2L_1 + x_1^2 + x_2^2 + y_1^2 + y_2^2\Big)^2 \\
& \qquad \times 3\Big(-2C - 2L_1 + x_1^2 + x_2^2 + y_1^2 + y_2^2\Big)^4\Big)\Bigg).
\end{aligned}$$

Now we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the change

$$x_i = \varepsilon^{1/4}\bar{x}_i + x_i^*, \quad y_i = \varepsilon^{1/4}\bar{y}_i + y_i^*,$$

where x_i^* and y_i^*, $i = 1, 2$ are the values of x_i and y_i at the equilibrium. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε.

Next we introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}\bar{x}_1 &= \sqrt{2I_1}\sin\phi_1, \quad \bar{y}_1 = \sqrt{2I_1}\cos\phi_1,\\ \bar{x}_2 &= \sqrt{2I_2}\sin\phi_2, \quad \bar{y}_2 = \sqrt{2I_2}\cos\phi_2.\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}\mathcal{K}_1 = -\frac{16\mathcal{M}L_1^2}{(C+L_1)^5L_2^3}\Bigg(&L_1^2(C+L_1)^2\\ &+3\varepsilon^{1/2}L_1(C+L_1)\Big(I_1\big(C+2L_1\big)-I_2\big(C-L_1\big)\Big)\\ &-\frac{3}{2}\varepsilon\Big(C^2\left(I_1^2+8I_1I_2-I_2^2\right)\\ &\qquad+CL_1\left(-4I_1^2+10I_1I_2+7I_2^2\right)\\ &\qquad-10CI_1I_2(C+L_1)\cos(2(\phi_1-\phi_2))\\ &\qquad-L_1^2(I_1+I_2)(9I_1+5I_2)\Big)\Bigg).\end{aligned} \tag{B.23}$$

Next we average with respect to $\phi_1-\phi_2$ at first order. Considering the full Hamiltonian $\mathcal{H}$, we introduce the small parameter η such that $\eta^2=\varepsilon$, getting:

$$\mathcal{H} = h_0 + \eta^2 h_1 + \eta^3 h_2 + \eta^4 h_3 + \mathcal{O}(\eta^5),$$

where

$$\begin{aligned}h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2}-\frac{\mu_2^3M_2^2}{2L_2^2},\\ h_1 &= -\frac{16\mathcal{M}L_1^4}{(C+L_1)^3L_2^3},\\ h_2 &= -\frac{48\mathcal{M}L_1^3}{(C+L_1)^4L_2^3}\Big(I_1\big(C+2L_1\big)-I_2\big(C-L_1\big)\Big),\\ h_3 &= -\frac{24\mathcal{M}L_1^2}{(C+L_1)^5L_2^3}\Big(I_1^2\big(-C^2+4CL_1+9L_1^2\big)+I_2^2\big(C^2-7CL_1+5L_1^2\big)\\ &\qquad+I_1I_2\big(-8C^2-10CL_1+14L_1^2\big)\Big).\end{aligned} \tag{B.24}$$

Now we readily identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 3$, $n_2 = 5$, $n_3 = 5$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$ and $a = 3$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}\right).$$

We build the matrix (B.22). The rank of the matrix is five and $s = 1$. Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions for inner bodies and they are coplanar with the outer body such that $G_1 \approx G_2 - C$.

B.2.4 Case (d)

Case (d) of Table 5.3 deals with motions of the three bodies that are circular for the inner and outer bodies.

The coordinates of the equilibrium point of case (d) in the space $\mathcal{S}_{L_1,L_2,C}$ are $(L_1^2, 2(C - L_1)^2 - L_2^2, 0, 0, 0, 0)$ which is the same point studied in case (b).

Thus, we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions for inner bodies and they are coplanar with the outer body and $G_1 \approx C - G_2$.

B.2.5 Case (e)

Case (e) of Table 5.3 deals with circular motions of the outer body. In particular we develop this case when $\gamma_1^* = 0$ and $G_1^* = \sqrt{C^2 + 3G_2^2}$. The equilibrium point is given by:

$$\left(2C^2 - L_1^2 + 6L_2^2, L_2^2, 2L_2\sqrt{C^2 - L_2^2}\sqrt{-C^2 + L_1^2 - 3L_2^2}, 0, 0, 0\right).$$

In this case it is assumed that $G_2 \approx L_2$.

First we introduce the symplectic change of coordinates given in Table 5.3(e).

Hamiltonian (2.17) in terms of x_i and y_i is:

$$\begin{aligned}\mathcal{K}_1 \;=\; & -\frac{32\mathcal{M}L_1^2}{L_2^3\big(G_1^*+y_1\big)^2\big(-2L_2+x_2^2+y_2^2\big)^5}\\ & \times\Bigg(\Big(-5L_1^2+3(G_1^*+y_1)^2\Big)\\ & \quad\times\Big(3\big(C^2-(G_1^*+y_1)^2\big)^2-\frac{1}{2}\big(3C^2-(G_1^*+y_1)^2\big)\\ & \qquad\times\big(-2L_2+x_2^2+y_2^2\big)^2+\frac{3}{16}\big(-2L_2+x_2^2+y_2^2\big)^4\Big)\\ & \quad-\frac{15}{16}\Big(\big(G_1^*+y_1\big)^2-L_1^2\Big)\Big(\big(2G_1^*+2L_2-x_2^2+2y_1-y_2^2\big)^2-4C^2\Big)\\ & \quad\times\Big(\big(2G_1^*-2L_2+x_2^2+2y_1+y_2^2\big)^2-4C^2\Big)\cos\big(2(\gamma_1^*+x_1)\big)\Bigg).\end{aligned}$$

Then we linearise $\mathcal{K}_1$ around the equilibrium point. This is achieved by introducing the transformation

$$x_i \;=\; \varepsilon^{1/4}\bar{x}_i+x_i^*,\qquad y_i \;=\; \varepsilon^{1/4}\bar{y}_i+y_i^*,$$

where x_i^* and y_i^*, $i=1,2$ are the values of x_i and y_i at the equilibrium. The change is symplectic with multiplier $\varepsilon^{-1/2}$. After applying it to $\mathcal{H}$ we need to rescale time in order to adjust the resulting Hamiltonian and to expand it in powers of ε. We arrive at a Hamiltonian of the form:

$$\mathcal{H} \;=\; \mathcal{H}_{\text{Kep}}+\varepsilon\,\mathcal{K}_1+\mathcal{O}(\varepsilon^2), \tag{B.25}$$

where

$$\begin{aligned}\mathcal{K}_1 \;=\; & \frac{4\mathcal{M}L_1^2}{(C^2+3L_2^2)L_2^8}\Bigg(2L_2^2\Big(C^2+3L_2^2\Big)\Big(6C^2-5L_1^2+6L_2^2\Big)\\ & +3\varepsilon^{1/2}\Big(10L_2^2\Big(-C^2+L_2^2\Big)\Big(C^2-L_1^2+3L_2^2\Big)\bar{x}_1^2\\ & \qquad -2\Big(C^2+3L_2^2\Big)^2\bar{y}_1^2\\ & \qquad +L_2(6C^2-5L_1^2+2L_2^2)(C^2+3L_2^2)(\bar{x}_2^2+\bar{y}_2^2)\Big)\Bigg).\end{aligned} \tag{B.26}$$

We introduce action-angle coordinates I_1, ϕ_1 as follows:

$$\begin{aligned}
\bar{x}_1 &= 2^{1/2}\left(\frac{(C^2+3L_2^2)^2}{5L_2^2(C^2-L_2^2)(C^2-L_1^2+3L_2^2)}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\
\bar{y}_1 &= 2^{1/2}\left(\frac{5L_2^2(C^2-L_2^2)(C^2-L_1^2+3L_2^2)}{(C^2+3L_2^2)^2}\right)^{1/4} I_1^{1/2}\cos\phi_1,\\
\bar{x}_2 &= \sqrt{2I_2}\sin\phi_2, \quad \bar{y}_2 = \sqrt{2I_2}\cos\phi_2.
\end{aligned}$$

After applying this transformation, Hamiltonian $\mathcal{K}_1$ is transformed into:

$$\begin{aligned}
\mathcal{K}_1 = \frac{8\mathcal{M}L_1^2}{L_2^7}\Bigg(& L_2(6C^2-5L_1^2+6L_2^2)\\
&+3\varepsilon^{1/2}\Big(I_1\big(-2\sqrt{5}\sqrt{(C^2-L_2^2)(C^2-L_1^2+3L_2^2)}\big)\\
&\qquad + I_2\big(6C^2-5L_1^2+2L_2^2\big)\Big)\Bigg).
\end{aligned}$$

We introduce the small parameter η where $\eta^2=\varepsilon$. Thus the full Hamiltonian $\mathcal{H}$ is:

$$\mathcal{H} = h_0+\eta^2 h_1+\eta^3 h_2+\mathcal{O}(\eta^4),$$

where

$$\begin{aligned}
h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2}-\frac{\mu_2^3 M_2^2}{2L_2^2},\\
h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^6}\Big(6C^2-5L_1^2+6L_2^2\Big),\\
h_2 &= \frac{24\mathcal{M}L_1^2}{L_2^7}\Big(-2\sqrt{5}I_1\sqrt{(C^2-L_2^2)(C^2-L_1^2+3L_2^2)}\\
&\qquad + I_2\big(6C^2-5L_1^2+2L_2^2\big)\Big).
\end{aligned} \tag{B.27}$$

We readily identify the following numbers in Theorem 1.15: $n_0=2$, $n_1=3$, $n_2=5$, $\beta_1=2$, $\beta_2=3$ and $a=2$ and construct

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial C}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}\right).$$

At this point we construct the matrix (B.17). The rank of the matrix is five and $s=2$. Thus we conclude that there are KAM 5-tori related with the equilibrium point that represents circular motions for the outer body.

B.2.6 Case (f)

Here we carry out the proof for case (f) in Table 5.3, which deals with circular motions of the outer body that are coplanar with the inner bodies' motion. We have chosen the case where $G_2 \approx L_2$ and $G_1 \approx G_2 - C$.

The coordinates of the equilibrium point of case (f) in $\mathcal{S}_{L_1,L_2,C}$ are $(2(C - L_2)^2 - L_1^2, L_2^2, 0, 0, 0, 0)$. We start by introducing the symplectic change of Poincaré-Deprit-like variables appearing in Table 5.3(f).

The expression of $\mathcal{K}_1$ in terms of x_1, x_2, y_1 and y_2 is

$$
\begin{aligned}
\mathcal{K}_1 \;=\; & -\frac{2\mathcal{M}L_1^2}{L_2^3\big(x_2^2+y_2^2-2L_2\big)^5\big(-x_1^2+x_2^2-y_1^2+y_2^2+2C-2L_2\big)^2} \\
& \times\Bigg(\frac{15}{64}(y_1^2-x_1^2)(x_1^2+y_1^2-4C)\Big(\big(x_1^2-x_2^2+y_1^2-y_2^2-2C+2L_2\big)^2-4L_1^2\Big) \\
& \quad \times\big(x_1^2-2x_2^2+y_1^2+2y_2^2+4L_2\big)\big(x_1^2-2x_2^2+y_1^2+2y_2^2+4C-4L_2\big) \\
& \quad -\Big(\frac{3}{4}\big(-x_1^2+x_2^2-y_1^2+y_2^2+2C-2L_2\big)^2-5L_1^2\Big) \\
& \quad \times\Bigg(\frac{3}{16}\big(x_2^2+y_2^2-2L_2\big)^4 \\
& \quad -\frac{3}{4}\Big(\big(-x_1^2+x_2^2-y_1^2+y_2^2+2C-2L_2\big)^2-4C^2\Big)^2 \\
& \quad +\frac{1}{2}\big(x_2^2+y_2^2-2L_2\big)^2 \\
& \quad \times\Big(\big(-x_1^2+x_2^2-y_1^2+y_2^2+2C-2L_2\big)^2-3C^2\Big)\Bigg)\Bigg).
\end{aligned}
$$

We linearise $\mathcal{K}_1$ around the equilibrium by introducing the symplectic change with multiplier $\varepsilon^{-1/2}$:

$$
\begin{aligned}
x_1 &= \varepsilon^{1/4}\bar{x}_1, & x_2 &= \varepsilon^{1/4}\bar{x}_2, \\
y_1 &= \varepsilon^{1/4}\bar{y}_1, & y_2 &= \varepsilon^{1/4}\bar{y}_2.
\end{aligned}
$$

After applying the transformation to $\mathcal{H}$ we rescale time, ending up with the Hamiltonian

$$
\mathcal{H} \;=\; \mathcal{H}_{\mathrm{Kep}} + \varepsilon\,\mathcal{K}_1 + \mathcal{O}(\varepsilon^2), \tag{B.28}
$$

where

$$
\begin{aligned}
\mathcal{K}_1 \;=\; \frac{4\mathcal{M}L_1^2}{L_2^7(C-L_2)}\Big(& L_2\big(C-L_2\big)\big(3(C-L_2)^2-5L_1^2\big)\\
&-3\varepsilon^{1/2}\Big(2\big(-4C^3+5CL_1^2+9C^2L_2-6CL_2^2+L_2^3\big)\bar{x}_1^2\\
&\qquad+2\big(C-L_2\big)^2\big(C+L_2\big)\bar{y}_1^2\\
&\qquad-\big(C-L_2\big)\big(3C^2-5L_1^2-4CL_2+L_2^2\big)(\bar{x}_2^2+\bar{y}_2^2)\Big)\Big).
\end{aligned}
$$

The next step is the introduction of the following symplectic set of action-angle coordinates:

$$
\begin{aligned}
\bar{x}_1 &= \frac{\sqrt{2}}{\sqrt{C-L_2}}\left(\frac{5L_1^2C+(C-L_2)^2(L_2-4C)}{C+L_2}\right)^{1/4} I_1^{1/2}\sin\phi_1,\\
\bar{y}_1 &= \sqrt{2(C-L_2)}\left(\frac{C+L_2}{5L_1^2C+(C-L_2)^2(L_2-4C)}\right)^{1/4} I_1^{1/2}\cos\phi_1,\\
\bar{x}_2 &= \sqrt{2I_2}\sin\phi_2,\\
\bar{y}_2 &= \sqrt{2I_2}\cos\phi_2.
\end{aligned}
$$

Now we apply the transformation to $\mathcal{K}_1$ getting

$$
\begin{aligned}
\mathcal{K}_1 \;=\; \frac{8\mathcal{M}L_1^2}{L_2^7}\Big(& L_2\big(3(C-L_2)^2-5L_1^2\big)\\
&-3\varepsilon^{1/2}\Big(2I_1\sqrt{(L_2+C)\big(5L_1^2C+(C-L_2)^2(L_2-4C)\big)}\\
&\qquad-I_2\big(3C^2-5L_1^2-4CL_2+L_2^2\big)\Big)\Big).
\end{aligned}
$$

Considering the full Hamiltonian $\mathcal{H}$, we introduce a new parameter η such that $\eta^2=\varepsilon$, arriving at

$$\mathcal{H} \;=\; h_0+\eta^2 h_1+\eta^3 h_2+\mathcal{O}(\eta^4),$$

where

$$
\begin{aligned}
h_0 &= -\frac{\mu_1^3M_1^2}{2L_1^2}-\frac{\mu_2^3M_2^2}{2L_2^2},\\
h_1 &= \frac{8\mathcal{M}L_1^2}{L_2^6}\big(3(C-L_2)^2-5L_1^2\big),\\
h_2 &= -\frac{24\mathcal{M}L_1^2}{L_2^7}\Big(2I_1\sqrt{(L_2+C)\big(5L_1^2C+(C-L_2)^2(L_2-4C)\big)}\\
&\qquad -I_2\big((L_2-2C)^2-5L_1^2-C^2\big)\Big).
\end{aligned}
\tag{B.29}
$$

One can identify the following numbers in Theorem 1.15: $n_0 = 2$, $n_1 = 3$, $n_2 = 5$, $\beta_1 = 2$, $\beta_2 = 3$ and $a = 2$, then

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial L_1},\ \frac{\partial h_0}{\partial L_2},\ \frac{\partial h_1}{\partial C},\ \frac{\partial h_2}{\partial I_1},\ \frac{\partial h_2}{\partial I_2}\right).$$

We build the matrix (B.17). Since the corresponding rank is four we add to this matrix the columns composed by the partials of second order and calculate the rank of this 5×31-matrix and get the desirable rank five. Therefore, there are KAM 5-tori related with the equilibrium point that represents circular motions of the outer body which are also coplanar with the inner bodies' motions.

In this case $b = 8$ and $s = 2$. So, the excluded measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^{\delta/2})$ (or $\mathcal{O}(\varepsilon^{\delta/4})$) with $0 < \delta < 1/5$ and we cannot improve this measure.

B.3 Study in $\mathcal{R}_{L_1,L_2,B}$

B.3.1 Case (a)

We deal with circular motions of the inner and outer bodies all of them moving in the same plane, which is not the horizontal plane. We consider the coplanar case that satisfies $G_1 \approx G_2 - C$ and $C \not\approx |B|$ to carry out our study. This situation corresponds to case (a) of Table 5.4.

The equilibrium point in $\mathcal{R}_{L_1,L_2,B}$ that we study has coordinates $(\rho_1, \ldots, \rho_{16})$ with

$$\begin{aligned}
\rho_1 &= \rho_2 = \frac{L_1 B}{L_1 - L_2},\\
\rho_3 &= \rho_4 = -\frac{L_2 B}{L_1 - L_2},\\
\rho_5 &= \rho_7 = \rho_9 = \rho_{11} = \rho_{13} = \rho_{15} = 0,\\
\rho_6 &= L_1^2\left(1 - \frac{B^2}{(L_1 - L_2)^2}\right),\\
\rho_8 &= \rho_{10} = \rho_{12} = \rho_{14} = -L_1 L_2\left(1 - \frac{B^2}{(L_1 - L_2)^2}\right),\\
\rho_{16} &= L_2^2\left(1 - \frac{B^2}{(L_1 - L_2)^2}\right).
\end{aligned}$$

In order to analyse the dynamics in a neighborhood of the equilibrium point we define the Poincaré-Deprit-like coordinates appearing in Table 5.4(a). The perturbation $\mathcal{K}_1$ in terms of x_i and y_i is

$$
\begin{aligned}
\mathcal{K}_1 \;=\; & \frac{2\mathcal{M}L_1^2}{L_2^3(x_1^2+y_1^2-2L_1)^2(x_2^2+y_2^2-2L_2)^5} \\
& \times \Bigg(15\big(x_1^2+y_1^2-4L_1\big)\big(2x_1^2+2x_2^2+x_3^2+2y_1^2-2y_2^2+y_3^2-4L_1+4L_2\big) \\
& \quad \times \big(2x_2^2-x_3^2+2y_2^2-y_3^2-4L_2\big)\Big(4x_1x_3y_1y_3-(y_1^2-x_1^2)(y_3^2-x_3^2)\Big) \\
& \quad + \Big(3\big(x_1^2+y_1^2-2L_1\big)^2-20L_1^2\Big) \\
& \qquad \times \Bigg(3\big(x_2^2+y_2^2-2L_2\big)^4 \\
& \qquad\quad + 3\Big(\big(x_1^2+y_1^2-2L_1\big)^2-\big(x_1^2-x_2^2+x_3^2+y_1^2-y_2^2+y_3^2-2(L_1-L_2)\big)^2\Big)^2 \\
& \qquad\quad + 2\big(x_2^2+y_2^2-2L_2\big)^2 \\
& \qquad\qquad \times \Big(\big(x_1^2+y_1^2-2L_1\big)^2 \\
& \qquad\qquad\quad - 3\big(x_1^2-x_2^2+x_3^2+y_1^2-y_2^2+y_3^2-2(L_1-L_2)\big)^2\Big)\Bigg)\Bigg).
\end{aligned}
\tag{B.30}
$$

We linearise $\mathcal{H}$ around the point $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0$ by a symplectic change with multiplier $\varepsilon^{-1/4}$ given by

$$
\begin{aligned}
x_1 &= \varepsilon^{1/8}\bar{x}_1, \quad x_2 = \varepsilon^{1/8}\bar{x}_2, \quad x_3 = \varepsilon^{1/8}\bar{x}_3, \\
y_1 &= \varepsilon^{1/8}\bar{y}_1, \quad y_2 = \varepsilon^{1/8}\bar{y}_2, \quad y_3 = \varepsilon^{1/8}\bar{y}_3.
\end{aligned}
\tag{B.31}
$$

After applying the linear change to $\mathcal{H}$ and multiplying by $\varepsilon^{1/4}$ to rescale time, we expand the resulting Hamiltonian in powers of ε getting a Hamiltonian of the form:

$$
\mathcal{H} = \mathcal{H}_{\text{Kep}} + \varepsilon\mathcal{K}_1 + \mathcal{O}(\varepsilon^{7/4}),
\tag{B.32}
$$

where

$$\begin{aligned}
\mathcal{K}_1 \;=\; -\frac{16\mathcal{M}L_1^4}{L_2^6}\Bigg(1 &+ \frac{3\varepsilon^{1/4}}{2L_1L_2}\Big(L_1(\bar{x}_2^2+\bar{x}_3^2+\bar{y}_2^2+\bar{y}_3^2)+L_2(\bar{x}_1^2-\bar{x}_3^2+\bar{y}_1^2-\bar{y}_3^2)\Big)\\
&+\frac{3\varepsilon^{1/2}}{8L_1^2L_2^2}\Bigg(L_1^2\Big(4(\bar{x}_2^2+\bar{y}_2^2)^2+(\bar{x}_3^2+\bar{y}_3^2)^2+8(\bar{x}_2^2+\bar{y}_2^2)(\bar{x}_3^2+\bar{y}_3^2)\Big)\\
&\quad -L_2^2\Big((\bar{x}_1^2+\bar{y}_1^2)^2+(\bar{x}_3^2+\bar{y}_3^2)^2-2\big(\bar{x}_1^2+\bar{y}_1^2\big)^2\\
&\qquad +2\big(\bar{x}_1^2(\bar{x}_3^2-3\bar{y}_3^2)-\bar{y}_1^2(3\bar{x}_3^2+\bar{y}_3^2)\big)\\
&\qquad\quad \times\big(\bar{x}_3^2(\bar{x}_1^2+3\bar{y}_1^2)+\bar{y}_3^2(3\bar{x}_1^2-\bar{y}_1^2)\big)\Big)\\
&\quad +L_1L_2\Big(3(\bar{x}_3^2+\bar{y}_3^2)^2+6(\bar{x}_1^2-\bar{x}_3^2+\bar{y}_1^2-\bar{y}_3^2)(\bar{x}_2^2+\bar{y}_2^2)\\
&\qquad +4\big(\bar{x}_1^2(\bar{y}_3^2-4\bar{x}_3^2)+\bar{y}_1^2(\bar{x}_3^2-4\bar{y}_3^2)\big)\\
&\qquad -6\big(\bar{x}_3^2+\bar{y}_3^2\big)+12\big((\bar{x}_2^2+\bar{y}_2^2)(\bar{x}_3^2+\bar{y}_3^2)\big)+40\bar{x}_1\bar{x}_3\bar{y}_1\bar{y}_3\Big)\Bigg)\Bigg).
\end{aligned} \tag{B.33}$$

Next we introduce a symplectic transformation that allows us to express the Hamiltonian in the form required by Theorem 1.15. The change reads as follows:

$$\begin{aligned}
\bar{x}_1 &= \sqrt{2I_1}\sin\phi_1, & \bar{y}_1 &= \sqrt{2I_1}\cos\phi_1,\\
\bar{x}_2 &= \sqrt{2I_2}\sin\phi_2, & \bar{y}_2 &= \sqrt{2I_2}\cos\phi_2,\\
\bar{x}_3 &= \sqrt{2I_3}\sin\phi_3, & \bar{y}_3 &= \sqrt{2I_3}\cos\phi_3.
\end{aligned}$$

After putting the Hamiltonian $\mathcal{K}_1$ in terms of ϕ_i and I_i, we arrive at:

$$\begin{aligned}
\mathcal{K}_1 \;=\; -\frac{16\mathcal{M}L_1^2}{L_2^6}\Bigg(L_1^2 &+ \frac{3\varepsilon^{1/4}L_1}{L_2}\Big(L_2I_1+L_1I_2-(L_1-L_2)I_3\Big)\\
&+\frac{3\varepsilon^{1/2}}{2L_2^2}\Bigg(-L_2^2I_1^2+4L_1^2I_2^2+(L_1^2-3L_1L_2+L_2^2)I_3^2+6L_1L_2I_1I_2\\
&\quad +2L_2\Big(3L_1-4L_2-5(L_1-L_2)\cos\big(2(\phi_1+\phi_3)\big)\Big)I_1I_3\\
&\quad +2\Big(4L_1^2-3L_1L_2\Big)I_2I_3\Bigg)\Bigg).
\end{aligned} \tag{B.34}$$

Next we average the resulting system with respect to $\phi_1+\phi_3$ at first order, i.e. taking only one step in the Lie transformation, checking that no resonances between the angles occur as the generating function is always well defined.

The last step before the application of Theorem 1.15 is the introduction of a new parameter $\eta^4=\varepsilon$, so that we get

$$\mathcal{H} \;=\; h_0+\eta^4h_1+\eta^5h_2+\eta^6h_3+\mathcal{O}(\eta^7), \tag{B.35}$$

where

$$\begin{aligned} h_0 &= -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}, \\ h_1 &= -\frac{16\mathcal{M}L_1^4}{L_2^6}, \\ h_2 &= -\frac{48\mathcal{M}L_1^3}{L_2^7}\Big(L_2 I_1 + L_1 I_2 - (L_1 - L_2) I_3\Big), \\ h_3 &= \frac{24\mathcal{M}L_1^2}{L_2^8}\Big(L_2^2 I_1^2 - 4L_1^2 I_2^2 - (L_1^2 - 3L_1 L_2 + L_2^2) I_3^2 - 6L_1 L_2 I_1 I_2 \\ &\qquad + 2L_2(3L_1 - 4L_2) I_1 I_3 + 2L_1(4L_1 - 3L_2) I_2 I_3\Big). \end{aligned} \tag{B.36}$$

The numbers in Theorem 1.15 are: $n_0 = 2$, $n_1 = 2$, $n_2 = 5$, $n_3 = 5$, $\beta_1 = 4$, $\beta_2 = 5$, $\beta_3 = 6$ and $a = 3$, then

$$\begin{aligned} \Omega \equiv &\ (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5,\ \Omega_6,\ \Omega_7,\ \Omega_8,\ \Omega_9,\ \Omega_{10}) = \\ &\left(\frac{\partial h_0}{\partial L_1}, \frac{\partial h_0}{\partial L_2}, \frac{\partial h_1}{\partial L_1}, \frac{\partial h_1}{\partial L_2}, \frac{\partial h_2}{\partial I_1}, \frac{\partial h_2}{\partial I_2}, \frac{\partial h_2}{\partial I_3}, \frac{\partial h_3}{\partial I_1}, \frac{\partial h_3}{\partial I_2}, \frac{\partial h_3}{\partial I_3}\right). \end{aligned} \tag{B.37}$$

Then, we build the 10×6-matrix

$$\partial_I^1 \Omega(I) = \left(\Omega_k, \frac{\partial \Omega_k}{\partial L_1}, \frac{\partial \Omega_k}{\partial L_2}, \frac{\partial \Omega_k}{\partial I_1}, \frac{\partial \Omega_k}{\partial I_2}, \frac{\partial \Omega_k}{\partial I_3}\right), \quad 1 \le k \le 10.$$

We get that the rank of this matrix is five, so we conclude that there are KAM 5-tori related with circular motions of the inner and outer bodies all of them moving in the same plane, which is not the horizontal plane. Moreover, in this case $b = 15$ and $s = 1$ then, the excluded measure for the existence of quasi-periodic invariant tori is of order $\mathcal{O}(\eta^\delta)$ (or $\mathcal{O}(\varepsilon^{\delta/4})$) with $0 < \delta < 1/5$ and as in the previous cases we cannot improve this measure.

B.4 Study in $\mathcal{A}_{L_1,L_2}$

B.4.1 Case (a)

We deal with the case (a) of Table 5.5. In particular the equilibrium points of $\mathcal{A}_{L_1,L_2}$ are related with circular motions of the inner and outer bodies, all of them are nearly moving in the horizontal plane. We choose the case $G_1 \approx L_1$, $G_2 \approx L_2$, $G_1 \approx G_2 - C$ and $C \approx |B|$.

The coordinates of the relative equilibrium of case (a) in $\mathcal{A}_{L_1,L_2}$ are:

$$(0,\ 0,\ \mp L_1,\ 0,\ 0,\ \mp L_1,\ 0,\ 0,\ \pm L_2,\ 0,\ 0,\ \pm L_2)\,.$$

The local symplectic variables x_i, y_i are the ones given in the third column of case (a) in Table 5.5.

The perturbation $\mathcal{K}_1$ in the coordinates x_i and y_i is the same as Hamiltonian (B.30) where instead of the term $4x_1x_3y_1y_3$ we put the term $\pm 4x_1x_3y_1y_3$ (the upper sign applies for prograde motions and the inner one for retrograde motions).

Following the same reasoning as in Section 5.2.4 and taking into account the result in the previous section one can conclude there are KAM 5-tori related with the equilibrium point that represents circular motions of the inner and outer bodies which are also coplanar motions when the invariable plane is the horizontal plane.

Bibliography

[1] R. Abraham, J. E. Marsden, *Foundations of Mechanics*, Addison-Wesley, Reading, MA, 1985.

[2] J. M. Arms, R. H. Cushman, M. J. Gotay, *A universal reduction procedure for Hamiltonian group actions*, in The Geometry of Hamiltonian Systems, T. Ratiu, ed., Springer-Verlag, Berlin, 1991, pp. 33–51.

[3] V. I. Arnold, *Proof of a theorem of A.N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian*, Uspekhi Mat. Nauk, 18 (1963), pp 13–40. English translation in: Russ. Math. Surv., 18 (1963), pp 9–36.

[4] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt, *Dynamical Systems III, 3^{rd} Ed.*, Encyclopaedia of Mathematical Sciences 3, Springer-Verlag, New York, 2003.

[5] E. A. Belbruno, *A new family of periodic orbits for the restricted problem*, Celestial Mech., 25 (1981), pp. 195–217.

[6] L. Biasco, L. Chierchia, E. Valdinoci, *Elliptic two-dimensional invariant tori for the planetary three-body problem*, Arch. Rational Mech. Anal., 170 (2003), pp. 91–135. Corrigendum. Arch. Ration. Mech. Anal., 180 (2006), pp. 507–509.

[7] A. D. Bruno, *Analytic form of differential equations*, Trans. Moscow Math. Soc., 25 (1971), pp. 131–288.

[8] A. Celletti, L. Chierchia, *KAM tori for N-body problems: a brief history*, Celestial Mech. Dynam. Astronom., 95 (2006), pp. 117–139.

[9] A. Chenciner, J. Llibre, *A note on the existence of invariant punctured tori in the planar circular restricted three-body problem*, Ergodic Theory Dynam. Systems, 8 (1988), pp. 63–72.

[10] L. Chierchia, G. Pinzari, *Properly-degenerate KAM theory (following V. I. Arnold)*, Discrete Contin. Dyn. Syst. Ser. S., 3 (2010), pp. 545–578.

[11] L. Chierchia, G. Pinzari, *Deprit's reduction of the nodes revisited*, Celestial Mech. Dynam. Astronom., 109 (2011), pp. 285–301.

[12] L. Chierchia, G. Pinzari, *Planetary Birkhoff normal forms*, J. Mod. Dyn., 5 (2011), pp. 623–664.

[13] L. Chierchia, G. Pinzari, *The planetary N-body problem: Symplectic foliation, reductions and invariant tori*, Invent. Math., 186 (2011), pp. 1–77.

[14] L. Chierchia, F. Pusateri, *Analytic Lagrangian tori for the planetary many-body problem*, Ergod. Theory Dyn. Syst., 29 (2009), pp. 849–873.

[15] R. C. Churchill, M. Kummer, *A unified approach to linear and nonlinear normal forms for Hamiltonian systems*, J. Symbolic Comput., 27 (1999), pp. 49–131.

[16] Ü. Çiftçi, H. Waalkens, *Phase space structures governing reaction dynamics in rotating molecules*, Nonlinearity, 25 (2012), pp. 791–812.

[17] Ü. Çiftçi, H. Waalkens, H. W. Broer, *Cotangent bundle reduction and Poincaré-Birkhoff normal forms*, Phys. D, 268 (2014), pp. 1–13.

[18] B. Cordani, *Global study of the 2D secular* 3*-body problem*, Regul. Chaotic Dyn., 9 (2004), pp. 113–128.

[19] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer-Verlag, (2000).

[20] R. H. Cushman, *Reduction, Brouwer's Hamiltonian, and the critical inclination*, Celestial Mech., 31 (1983), pp. 401–429. Correction. Celestial Mech., 33 (1984), p. 297.

[21] R. H. Cushman, *A survey of normalization techniques applied to perturbed Keplerian systems*, in Dynamics Reported Vol. 1, New Series, K. Jones, ed., Springer-Verlag, New York, 1991, pp. 54–112.

[22] R. H. Cushman, L. M. Bates, *Global Aspects of Classical Integrable Systems*, Birkhäuser, Basel, 1997.

[23] A. Deprit, *Free rotation of a rigid body studied in the phase plane*, Amer. J. Phys., 35 (1967), pp. 424–428.

[24] A. Deprit, *Canonical transformations depending on a small parameter*, Celestial Mech., 1 (1969), pp. 12–30.

[25] A. Deprit, *Delaunay normalisations*, Celestial Mech., 26 (1982), pp. 9–21.

[26] A. Deprit, *Elimination of the nodes in problems of N bodies*, Celestial Mech., 30 (1983), pp. 181–195.

[27] H. R. Dullin, *The Lie-Poisson structure of the reduced n-body problem*, Nonlinearity, 26 (2013), pp. 1565–1579.

[28] F. Farago, J. Laskar, *High-inclination orbits in the secular quadrupolar three-body problem*, Mon. Not. R. Astron. Soc., 401 (2010), pp. 1189–1198.

[29] J. Féjoz, *Averaging the planar three-body problem in the neighborhood of double inner collisions*, J. Differential Equations, 175 (2001), pp. 175–187.

[30] J. Féjoz, *Global secular dynamics in the planar three-body problem*, Celestial Mech. Dynam. Astronom., 84 (2002), pp. 159–195.

[31] J. Féjoz, *Quasi periodic motion in the planar three-body problem*, J. Differential Equations, 183 (2002), pp. 303–341.

[32] J. Féjoz, *Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman)*, Ergod. Theory Dyn. Syst., 24(5) (2004), pp. 1521–1582. Version révisée de l'article paru dans le Michael Herman Memorial Issue, Ergod. Theory Dyn. Syst., 24(5) (2004), pp. 1521–1582, available at http://people.math.jussieu.fr/fejoz/articles.html.

[33] S. Ferrer, H. Hanßmann, J. F. Palacián, P. Yanguas, *On perturbed oscillators in 1-1-1 resonance: The case of axially symmetric cubic potentials*, J. Geom. Phys., 40 (2002), pp. 320–369.

[34] S. Ferrer, C. Osácar, *Harrington's Hamiltonian in the stellar problem of three bodies: Reductions, relative equilibria and bifurcations*, Celestial Mech. Dynam. Astronom., 58 (1994), pp. 245–275. Correction. Celestial Mech. Dynam. Astronom., 60 (1994), p. 187.

[35] M. C. Gutzwiller, *Moon-Earth-Sun: The oldest three-body problem*, Rev. Mod. Phys., 70 (1998), pp. 589–639.

[36] Y. Han, Y. Li, Y. Yi, *Invariant tori in Hamiltonian systems with high order proper degeneracy*, Ann. Henri Poincaré, 10 (2010), pp. 1419–1436.

[37] H. Hanßmann, *The reversible umbilic bifurcation*, Phys. D., 112 (1998), pp. 81–94.

[38] H. Hanßmann, *Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems – Results and Examples*, Lecture Notes in Math. 1893, Springer-Verlag, 2007.

[39] H. Hanßmann, *Non-degeneracy conditions in KAM theory*, Indag. Math. (N.S.), 22 (2011), pp. 241–256.

[40] R. S. Harrington, *The stellar three-body problem*, Celestial Mech., 1 (1969), pp. 200–209.

[41] G. Heckman, T. de Laat, *On the regularization of the Kepler problem*, J. Symplect. Geom., 10 (2012), pp. 463–473.

[42] J. Henrard, *Virtual singularities in the artificial satellite theory*, Celestial Mech. 10 (1974), pp. 437–449.

[43] M. Iñarrea, V. Lanchares, J. F. Palacián, A. I. Pascual, J. P. Salas, P. Yanguas, *The Keplerian regime of charged particles in planetary magnetospheres*, Phys. D., 197 (2004), pp. 242–268.

[44] M. Iñarrea, V. Lanchares, J. F. Palacián, A. I. Pascual, J. P. Salas, P. Yanguas, *Symplectic coordinates on $S^2 \times S^2$ for perturbed Keplerian problems: Application to the dynamics of a generalised Størmer problem*, J. Differential Equations, 250 (2011), pp. 1386–1407.

[45] C. G. J. Jacobi, *Sur l'élimination des noeuds dans le problème des trois corps*, Astronom. Nachr., XX (1842), pp. 81–102.

[46] W. H. Jefferys, J. Moser, *Quasi-periodic solutions for the three-body problem*, Astronom. J., 71 (1966), pp. 568–578.

[47] P. Kustaanheimo, E. Stiefel, *Perturbation theory of Kepler motion based on spinor regularization*, J. Reine Angew. Math., 218 (1965), pp. 204–219.

[48] J. L. Lagrange, *Essai sur le Problème des Trois Corps*, Euvres, Tome VI. Gauthler-Villars (Paris), 1867–1892 edition, 1772.

[49] A. J. Laub, K. Meyer, *Canonical forms for symplectic and Hamiltonian matrices*, Celestial Mech., 9 (1974), pp. 213–238.

[50] E. Lerman, R. Montgomery, R. Sjamaar, *Examples of singular reduction*, in Symplectic Geometry, D. Salamon, ed., London Math. Soc. Lecture Note Ser., 192, Cambridge University Press, Cambridge, 1993, pp. 127–155.

[51] E. Lerman, S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc., 349 (1997), pp. 4201–4230.

[52] Y. Li, Y. Yi, *Persistence of invariant tori in generalized Hamiltonian systems*, Ergodic Theory Dynam. Systems, 22 (2002), pp. 1233-1261.

[53] M. L. Lidov, S. L. Ziglin, *Non-restricted double-averaged three body problem in Hill's case*, Celestial Mech., 13 (1976), pp. 471–489.

[54] B. B. Lieberman, *Existence of quasiperiodic solutions in the three-body problem*, Celestial Mech., 3 (1971), pp. 408–426.

[55] T. Ligon, M. Schaaf, *On the global symmetry of the classical Kepler problem*, Rep. Math. Phys., 9 (1976), pp. 281–300.

[56] C. M. Marle, *A property of conformally Hamiltonian vector fields; application to the Kepler problem*, J. Geom. Mech., 4 (2012), pp. 181–206.

[57] J. Marsden, A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys., 5 (1974), pp. 121–130.

[58] C. K. McCord, K. R. Meyer, Q. Wang, *The integral manifolds of the three body problem*, Mem. Amer. Math. Soc., 132 (1998), pp. 1–91.

[59] K. R. Meyer, *Symmetries and integrals in Mechanics*, Dynamical Systems, M. M. Peixoto, ed., Academic Press, New York, 1973, pp. 259–272.

[60] K. R. Meyer, *Counter-examples in dynamical systems via normal form theory*, SIAM Review, 28 (1986), pp. 41–51.

[61] K. R. Meyer, *Periodic Solutions of the N-Body Problem*, Lecture Notes in Math. 1719, Springer-Verlag, 1999.

[62] K. R. Meyer, G. R. Hall, D. Offin, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 2nd ed., Springer-Verlag, New York, 2009.

[63] K. R. Meyer, J. F. Palacián, P. Yanguas, *Geometric averaging of Hamiltonian systems: periodic solutions, stability, and KAM tori*, SIAM J. Appl. Dyn. Syst., 10 (2011), pp. 817–856.

[64] K. R. Meyer, J. F. Palacián, P. Yanguas, *Invariant tori in the lunar problem*, Publ. Mat., Extra (2014), pp. 353–394.

[65] K. R. Meyer, D. S. Schmidt, *The stability of the Lagrange triangular point and a theorem of Arnol'd*, J. Differential Equations, 62 (1986), pp. 222–236.

[66] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen II, Math. Phys. Kl., (1962), pp. 1–20.

[67] J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math., 23 (1970), pp. 609–636.

[68] J. F. Palacián, *Normal forms for perturbed Keplerian systems*, J. Differential Equations, 180 (2002), pp. 471–519.

[69] J. F. Palacián, F. Sayas, P. Yanguas, *Regular and singular reductions in the spatial three-body problem*, Qual. Theory Dyn. Syst., 12 (2013), pp. 143–182.

[70] J. F. Palacián, F. Sayas, P. Yanguas, *Flow reconstruction and invariant tori in the spatial three-body problem*, submitted, 2013.

[71] J. F. Palacián, F. Sayas, P. Yanguas, *Invariant tori of the spatial three-body problem reconstructed from relative equilibria of rectilinear type*, submitted, 2014.

[72] J. F. Palacián, P. Yanguas, *Reduction of polynomial Hamiltonians by the construction of formal integrals*, Nonlinearity, 13 (2000), pp. 1021–1054.

[73] W. Pauli, *Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik*, Zeitschrift für Physik, 36 (1926), pp. 336–363.

[74] R. Radau, *Sur une transformation des équations différentielles de la dynamique*, Ann. Sci. École Norm. Sup. 5 (1868), pp. 311–375.

[75] P. Robutel, *Stability of the planetary three-body problem II. KAM theory and existence of quasiperiodic motions*, Celestial Mech. Dynam. Astronom., 62 (1995), pp. 219–261. Erratum, Celestial Mech. Dynam. Astronom., 84 (2002), p. 317.

[76] H. Rüssmann, *Nondegeneracy in the perturbation theory of integrable dynamical systems*, in Number Theory and Dynamical Systems (York, 1987), London Math. Soc Lecture Note Ser., 134, Cambridge University Press, Cambridge, UK, 1989, pp. 5–18.

[77] H. Rüssmann, *Invariant tori in non-degenerate nearly integrable Hamiltonian systems*, Regul. Chaotic Dyn., 6 (2001), pp. 119–204.

[78] I. Satake, *On a generalization of the notion of a manifold*, Proc. Natl. Acad. Sci. USA, 42 (1956), pp. 359–363.

[79] T. Schmah, C. Stoica, *Normal forms for Lie symmetric cotangent bundle systems with free and proper actions*, to appear in Fields Institute Communications series, Vol. "Geometry, Mechanics and Dynamics: The Legacy of Jerry Marsden," 2014.

[80] G. Schwarz, Smooth functions invariant under the action of a compact group, Topology, 14 (1975), pp 63–68.

[81] M. B. Sevryuk, *KAM-stable Hamiltonians*, J. Dynam. Control Systems, 1 (1995), pp. 351–366.

[82] M. B. Sevryuk, *Partial preservation of frequencies in KAM theory*, Nonlinearity, 19 (2006), pp. 1099–1140.

[83] C. L. Siegel, J. K. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, New York, 1971.

[84] K. Sitnikov, *The existence of oscillatory motions in the three-body problem*, Sov. Phys. Dokl., 5 (1961), pp. 647–650.

[85] B. S. Sommer, *A KAM Theorem for the Spatial Lunar Problem*, Thesis, 2003, Aachen University.

[86] J. M. Souriau, *Sur la varieté de Kepler*, Sympos. Math., 14 (1974), pp. 343–360.

[87] B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, New York, 1993.

[88] P. Yanguas, J. F. Palacián, K. R. Meyer, H. S. Dumas, *Periodic solutions in Hamiltonian systems, averaging, and the lunar problem*, SIAM J. Appl. Dyn. Syst., 7 (2008), pp. 311–340.

[89] Y. Yi, *Private communication*, 2014.

[90] L. Zhao, *Solutions quasi-périodiques et solutions de quasi-collision du problème spatial des trois corps*, Thèse de l'Université Paris Diderot, 2013.

[91] L. Zhao, *Quasi-periodic solutions of the spatial lunar three-body problem*, Celestial Mech. Dynam. Astronom., 119 (2014), pp. 91–118.

[92] L. Zhao, *The Kustaanheimo-Stiefel regularization and the quadrupolar conjugacy*, to appear in Regul. Chaotic Dyn., 2014.

[93] L. Zhao, *Quasi-periodic almost-collision orbits in the spatial three-body problem*, to appear in Comm. Pure Appl. Math., doi: 10.1002/cpa.21539, 2014.