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Mutually Best Matches

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Mutually Best Matches*

Abstract

We study iterated formation of mutually best matches (IMB) in college admissions problems. When IMB produces a maximal individually rational matching, the matching has many good properties like Pareto optimality and stability. If preferences satisfy a single peakedness condition, or have a single crossing property, then IMB produces a maximal individually rational matching. These properties guarantee also that the student proposing Deferred Acceptance algorithm (DA) and the Top Trading Cycles algorithm (TTC) produce the same matching as IMB. We compare these results with some well-known results about when DA is Pareto optimal, or when DA and TTC produce the same matching.

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1. Introduction

Gale and Shapley (1962) introduce the college admissions problem where no money is used to match agents in two disjoint sets with each other. These sets consist of students and colleges (or "schools") having strict preferences over the other set. In this paper we describe iterated formation of mutually best matches (IMB) algorithm as a way to solve the college admission problem.

When IMB produces a maximal individually rational matching, the matching has many good properties like Pareto optimality and stability. Also strategy proofness for students holds at such a matching. A maximal individually rational matching is such that 1) all matched student - school pairs are mutually acceptable, and 2) there does not exist a mutually acceptable student - school pair s, c such that s is unmatched and c has free capacity.

If preferences satisfy a single peakedness condition, or have a single crossing property with setwise increasing acceptability relations, then IMB produces a maximal individually rational matching.[1] It follows that in these cases DA produces the same Pareto optimal matching as IMB. Further, these properties guarantee that the student proposing DA and TTC produce the same matching as IMB.

Ergin (2002) shows that a necessary and sufficient condition for DA to be Pareto optimal is that the priority structure of schools satisfies an acyclicity condition. Kesten (2006) introduces a slightly different acyclicity condition and shows that a necessary and sufficient condition for DA and TTC to produce the same matching is that the priority structure of schools satisfies his acyclicity condition.

We don't need acyclicity conditions for our results. The reason is that our setup is different than that of Ergin or Kesten. Ergin and Kesten fix the priorities of schools, and ask when DA is Pareto optimal (Ergin) or when

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[1]See Gabszewicz *et al.* (2012) and Milgrom and Shannon (1994) for versions of single peakedness and single crossing properties.

DA = TTC (Kesten), for *all* possible preferences of students. And they find that the priority structure must be acyclic.

We assume that single peakedness or single crossing property hold for both sides of the market. Then if we look at a particular priority structure of schools, there is only a subset of possible preferences for students such that single peakedness or single crossing assumptions remain valid.

A third example when IMB produces a maximal individually rational matching is when one side of the market has identical preferences. In this case DA = TTC as well. Although this is a very restrictive assumption from theoretical viewpoint, in some real world applications it may hold approximatively.

When IMB does not produce a maximal individually rational matching, one may try to fix it by continuing the matching process with some other algorithm. We study a variant called IMB* in which one round of TTC is applied whenever IMB halts without producing a maximal individually rational matching. After that, IMB is applied again, *e.t.c.* When schools have capacity of one ("marriage market"), then IMB* = TTC. It follows that IMB* is not stable. Moreover, if some schools have larger capacities IMB* no longer satisfies strategy proofness.

When IMB produces a maximal individually rational matching, it has practically all the nice properties one may wish for. But when it fails it seems very difficult to fix it so that at least some of the good properties of IMB would still hold. Needless to say, we have only explored a small sample of possible cures for IMB.

The paper is organized in the following way. In Section 2 we introduce the notation, axioms, and the used matching algorithms. Section 3 contains the main results. In Section 4 we study the possibilities to fix IMB when it does not produce a maximal individually rational matching. Section 5 concludes.

2. Preliminaries

Let us denote by S the nonempty finite set of students and by C the nonempty finite set of schools. A *matching* is function $\mu : S \longrightarrow C \cup S$ such that $\mu(s) \notin C$ iff $\mu(s) = s$. We denote by $\mu^{-1}(c)$ the set of students that are matched with school c.

Student $s \in S$ has a strict preference order $\prec_s$ over acceptable schools $c \prec_s c'$ means that student s strictly prefers school c' to school c. Notation $c \precsim_s c'$ means that $c \prec_s c'$ or $c = c'$. We may denote preferences by ordered lists like $\prec_s = c_1 c_2 \cdots c_{k_s}$ where c_1 is the best school for s and c_{k_s} is the worst school s finds acceptable. Student s a) strictly prefers being unmatched to being matched with an unacceptable school, b) strictly prefers any acceptable school to being unmatched.

A *preference profile* $(\prec_s)_s$ specifies a preference relation to each $s \in S$. Notation $(\prec'_s, \prec_{-s})_s$ means that student s has preferences $\prec'_s$ while the other students have the same preferences as in the profile $(\prec_s)_s$.

School $c \in C$ has a strict priority order $\prec_c$ over students. Notation $s \precsim_c s'$ means that $s \prec_c s'$ or $s = s'$. We may denote priorities by ordered lists like $\prec_c = s_1 s_2 \cdots s_{t_c}$ where s_1 is the student with top priority in c's list, and s_{t_c} is an acceptable student with the lowest priority. (We call $\prec_c$ priorities in order to distinguish them more clearly from student preferences. In this paper there is no real difference between schools' preferences or priorities.)

Schools order subsets of students as well. We make the common but strong assumption that priorities are *responsive* (see Roth and Sotomayor 1992, p. 128). By this assumption we don't have to represent priorities over subsets of students explicitly.

A *priority profile* $(\prec_c)_c$ specifies a priority relation to each $c \in C$. Notation $(\prec'_c, \prec_{-c})_c$ means that school c has priorities $\prec'_c$ while the other schools have the same priorities as in the profile $(\prec_c)_c$. A school c has capacity $q_c > 0$ which tells the greatest number of students that a school c can accept. We denote by $q = (q_c)_c$ the vector of capacities.

A *matching problem* is $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$. Problems in which

schools may deem some applicants unacceptable are often called college admission problems. School choice problems often refer to matching problems such that all schools have to accept students up to their capacity constraints, in the order given by their priority lists. We consider both kind of problems and indicate, when necessary, whether or not schools have to accept all students.

A *mechanism* is a rule M that to each school choice problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ assigns a matching $M(P) : S \longrightarrow C \cup S$. In this paper mechanisms are sometimes called algorithms, since the mechanisms studied here are given in algorithmic form.

2.1. Properties of mechanisms and matchings

Given a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, we say that a pair s, c is *mutually acceptable*, if s is an acceptable student for school c and c is an acceptable school for student s. A matching μ is *individually rational*, if $c = \mu(s)$ implies that s, c is a mutually acceptable pair.

Axiom 1 (Individual rationality). A mechanism M is *individually rational*, if $M(P)$ is individually rational for all matching problems P.

A matching μ is a *maximal individually rational* matching, if μ is individually rational and there does not exists a mutually acceptable pair s, c such that s is unmatched and c has free capacity.

A matching μ is *stable*, if 1) μ is a maximal individually rational matching; 2) there does not exist any mutually acceptable pair s, c such that a) $\mu(s) \neq c$, and b) $\mu(s) \prec_s c$ and $s' \prec_c s$ for some $s' \in \mu^{-1}(c)$. If there exists a pair s, c satisfying condition 2), then s, c is called a *blocking pair*.

Axiom 2 (Stability). A mechanism M is *stable*, if $M(P)$ is a stable matching for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

Axiom 3 (Strategy proofness). A mechanism M is *strategy proof*, if for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ and $P' = \{S, C, (\prec'_s, \prec_{-s})_s, (\prec_c)_c, q\}$ it holds that $\mu'(s) \precsim_s \mu(s)$, where $\mu = M(P)$ and $\mu' = M(P')$.

A matching μ is *Pareto optimal*, if there does not exist an alternative matching μ' such that $\mu(s) \precsim_s \mu'(s)$ for all $s \in S$, and $\mu(s) \prec_s \mu'(s)$ for some $s \in S$. In this paper we study strategy proofness and efficiency from the viewpoint of students only.

Axiom 4 (Pareto optimality)**.** A mechanism M is *Pareto optimal*, if $M(P)$ is a Pareto optimal matching for all $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$.

2.2. School choice mechanisms

If the choice is restricted to subsets $S' \subset S$ and $C' \subset C$, the preference and priority orders on S' and C' are the original orders restricted to these subsets. We may denote by $A_s(C')$ and $A_c(S')$ the schools and students that are acceptable in subsets C' and and S' for student s and school c, respectively. Note that $A_s(C') = A_s(C) \cap C'$ and $A_c(S') = A_c(S) \cap S'$. In school choice problems $A_c(S') = S'$, for all $S' \subset S$.

Student $s \in S$ and school $c \in C$ are mutually acceptable if $s \in A_c(S)$ and $c \in A_s(C)$. Matching problems without mutually acceptable pairs are not interesting from the viewpoint of matching theory.

The best known mechanisms are the *deferred acceptance* algorithm (Gale and Shapley 1962) and the *top trading cycles* mechanism (Shapley and Scarf 1974, Abdulcadiroğlu and Sönmez 2003).

The deferred acceptance algorithm (hereafter DA), more precisely a student proposing version of it, is defined by the following steps.

Step 1. Any student s names his best school. Any school c tentatively accept the best q_c of those students that named c, and permanently reject the rest of the students that named c.

Step t. Any student s rejected in the previous step names his best school among the schools that have not yet rejected him. A student who has been tentatively accepted cannot name any school. Any school c compares the new applicants to the ones she already has, and tentatively accepts the best q_c of them.

DA ends when any student who is not tentatively accepted by some school does not have any acceptable schools left. The students tentatively accepted by schools are permanently matched with these schools.

The top trading cycle mechanism (hereafter TTC), is defined by the following steps.

1. Each student s names the best school in C. Each school c names the best student in S. If there is no cycle, go to step 3. If there are cycles $(s \to c \to \ldots \to s)$, match each student in a cycle permanently with the school that the student named. Go to step 2.
2. Repeat step 1 with the sets of students S' and schools C' that are still in the market.
3. End.

By *iterated formation of mutually best matches* (IMB) we mean the following process.

1. All schools $c \in C$ tentatively accept those students $s \in S$ that find c acceptable, up to the current free capacity q_c of c. Go to 2.
2. Check if there are any students $s \in S$ that have been tentatively accepted by their best school $c \in C$. If yes, then go to 3. If no, then go to 4.
3. A student $s \in S$ who has been tentatively accepted by his best school c is permanently matched with that school. The capacities of schools are reduced by the number of matched students. Then permanently matched students and those schools whose capacity is full leave the market.

   After that, those schools and students leave the market for whom there are no acceptable students or schools left. If after the removal of these schools and students new schools or students appear without acceptable matches, repeat the procedure as many times as needed. The remaining schools and students are denoted by C' and S', respectively.

If both S' and C' are nonempty, then go to 1, and apply the steps to S' and C'. Otherwise go to 4.

4. End.

Each round of IMB consists of the steps 1. – 3., and the last round ends at step 4.

3. Results

The first result says that when IMB produces a maximal individually rational matching for $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, then the matching is stable and Pareto optimal, and IMB is strategy proof at P.

Proposition 1. *If IMB produces a maximal individually rational matching μ for the problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, then μ is stable and Pareto optimal. Hence DA produces μ as well. Moreover, IMB is strategy proof at P.*

Proof. Stability. If μ is not stable, then there exists a blocking pair s, c because μ is a maximal individually rational matching. Since μ is a maximal individually rational matching, c cannot have free capacity, and $s' \prec_c s$ must hold for some s' that is matched with c. Under IMB, it is not possible that s is matched with some school c' before or at the same time s' is matched with c. This holds since s is above s' in the priority order of c, so c' would then be a better school for s than c, a contradiction. But then it follows that s will remain a tentatively accepted student for c even after s' is matched with c. The worst thing that can happen to s is that he will be eventually matched with c, a contradiction again. Hence μ is stable.

Pareto optimality. Let S_k be the set of students that are matched in round k of IMB. Those matched in round 1 are matched with their best schools. Those matched in round $k = 2$ are matched with the best schools still in the market, and so on. So it is impossible to find a Pareto improving reallocation among the students that are matched with some school. Hence if μ' is a Pareto improving matching, μ' matches all students in subsets

S_k the same way as μ. Hence μ' can be different only if some s that was unmatched under μ is placed in some school under μ'. By the definition of IMB, any school c that has free capacity under μ finds s unacceptable, or else c is unacceptable to s. Hence μ is Pareto optimal.

Since the matching μ' produced by DA Pareto dominates any other stable matching (Roth and Sotomayor 1992), $\mu' = \mu$.

Strategy proofness. Let S_k be the set of students that are matched in round k of IMB. If s is matched in round $k = 1$, then he is matched with his best school and therefore untruthful reporting of preferences cannot be beneficial. If $k = 2$, then untruthful reporting would benefit s only if he would be matched with some school c' that is better than $c = \mu(s)$, and that got its capacity filled already in round $k = 1$. But that is possible only if s was among the $q_{c'}$ best students for school c' in round $k = 1$. If this holds, then c' has still free capacity in round $k = 2$, a contradiction with the assumption that s is matched with a worse school c. Apply induction on k and conclude that no student that is matched with a school can benefit from misreporting preferences. Finally note that a similar argument holds also for students that are unmatched under μ. $\qquad\square$

Let us first show by an example that although IMB produces a Pareto optimal matching μ such that IMB is strategy proof at μ, TTC may produce a different matching μ'.

Example 1. Let $S = \{s_1, s_2, s_3\}$ and $C = \{c_1, c_2\}$. All schools are acceptable to all students and all students are acceptable to all schools. Preferences are: $\prec_{s_1} = c_2 c_1$, $\prec_{s_2} = c_1 c_2$, $\prec_{s_3} = c_2 c_1$. Priorities are $\prec_{c_1} = s_1 s_2 s_3$, $\prec_{c_2} = s_2 s_3 s_1$. School 1 has capacity of two, the other schools have capacity of one.

IMB and DA both produce a matching $\mu = \{(s_1, c_1), (s_2, c_1), (s_3, c_2)\}$. TTC produces a matching $\mu' = \{(s_1, c_2), (s_2, c_1), (s_3, c_1)\}$. $\qquad\triangleleft$

Example 1 shows that TTC can produce a different matching than IMB even when IMB produces a maximal individually rational matching. The following result shows that TTC is equivalent to IMB when the capacities of the schools are restricted to one and IMB produces a maximal individually

rational matching.

Corollary 1. *If IMB produces a maximal individually rational matching μ for the problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, and each school has capacity of one, then IMB generates the same matching as DA and TTC.*

Proof. Suppose TTC produces a cycle $s \to c \to \ldots \to c' \to s$. By construction IMB only has $s \to c \to s$ cycles. Because each school has capacity of one, it follows that IMB did not produce a maximal individually rational matching μ or schools c, c' did not point to their best students under TTC, a contradiction. □

For the remaining part of this section we concentrate on studying situations where restricted preference domains allow IMB to produce a maximal individually rational matching.

Proposition 2. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that all students have identical preferences, then IMB produces a maximal individually rational matching μ for P. This holds also if all schools have identical priorities but may have different capacities.*

Proof. Identical preferences. Index the schools so that the common preference of students is $\prec_s = c_1 c_2 \cdots c_k$, where c_k is the last acceptable school. In the first round of IMB, school c_1 gets the top q_{c_1} students according to its priority $\prec_{c_1}$ or less if c_1 has less acceptable students. After that school c_2 gets the top q_{c_2} students according to its priority $\prec_{c_2}$ or less if c_2 has less acceptable students. Continue this way as long as there are acceptable students for some $c_t, t \leq k$. Clearly μ is a maximal individually rational matching.

Identical priorities. Index the students so that the common priority of schools is $\prec_c = s_1 s_2 \cdots s_k$, where s_k is the last acceptable student. Schools c_t and c_h order all acceptable students s_i, s_j the same way if both of these students accept both schools. At each round r IMB matches student s_r with the best school still in the market, where $r = 1, ..., t$, and $t \leq k$. Note that if there are no acceptable schools left for a student they are removed from the

market as well. When IMB ends, it is impossible that there is a mutually acceptable pair s, c such that s is unmatched and c has free capacity. Hence μ is a maximal individually rational matching. $\qquad\square$

Proposition 3. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that all students have identical preferences, then IMB generates the same matching as DA and TTC. This holds also if all schools have identical priorities but may have different capacities.*

Proof. Identical preferences. When TTC is applied, school c_1 gets the top q_{c_1} students according to its priority $\prec_{c_1}$ or less if c_1 has less acceptable students. After that school c_2 gets the top q_{c_2} students according to its priority $\prec_{c_2}$ or less if c_2 has less acceptable students, and so on. Hence TTC produces the same matching μ as IMB. TTC produces a Pareto optimal matching and IMB produces a stable matching. By Proposition 1, μ is also produced by DA.

Identical priorities. When TTC is applied, in the first round only s_1 is matched, and he is matched with his best school. By the same argument, during each round $t \le k$ only student s_t is matched, and he is matched with the best school still in the market. The resulting allocation is the same as the one produced by IMB. Hence DA produces the same allocation by the same argument as in the identical preferences case. $\qquad\square$

Next we relax the identical preference and priority structure with a *single peakedness* property. To define single peakedness for matching problems $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, we assume that sets S and C of students and schools are subsets of some finite dimensional real space $\mathbb{R}^p$ which is equipped with a norm $||\cdot||$.[2] The following holds for agents $s \in S$, $c, c' \in C$:

$$c \prec_s c' \iff ||s - c|| > ||s - c'||, \tag{1}$$

and similarly for $c \in C$, $s, s' \in S$:

$$s \prec_c s' \iff ||c - s|| > ||c - s'||. \tag{2}$$

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[2]For example, $||\cdot||$ could be the Euclidean norm $||x|| = \sqrt{\sum_i x_i^2}$, but any norm on $\mathbb{R}^p$ will do.

Single peaked preferences are based on the notion of distance between the ideal points s and c of the agents. Since preferences are strict, student s will have the same preferences as student s' if s' is sufficiently close to s, and similarly for schools.

We define the acceptability relations A_s of students to be *distance consistent*, if

$$c \in A_s(C) \text{ and } c' \notin A_s \implies ||s - c'|| \geq p_s > ||s - c||, \tag{3}$$

where $p_s \geq 0$ is some distance threshold for s. The distance consistent acceptability relations A_c for school c is defined analogously. Single peakedness guarantees distance consistency but p_s may be different for different agents.

Single peaked preferences and priorities guarantee that IMB produces a maximal individually rational matching. This is shown in the next proposition.

Proposition 4. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences and priorities satisfy the single peakedness property, and the acceptability relations are distance consistent, then IMB produces a maximal individually rational matching μ for P.*

Proof. Remove all students s (schools c) who have no acceptable schools or are not acceptable to any schools c' (students s'). For any school c, search for student s such that $||c - s|| \equiv m(c)$ is minimized and s, c is a mutually acceptable pair. Since preferences and priorities are strict, such an s exists uniquely. Then go through all schools c and pick the school c' such that $m(c)$ is minimized. Such a school exists, and there may be several schools that minimize $m(c)$. Let s' be such that $m(c') = ||c' - s'||$. Since the norm $|| \cdot ||$ is symmetric, $||c' - s'|| = ||s' - c'||$, and school c' must be the best school for student s'. Hence s', c' is a mutually best match. There may be many mutually best matches but since preferences are strict, any student (school) can belong to at most one such pair. Remove all students that are members of such pairs and schools whose capacity is full.

If there are any mutually acceptable pairs left, then IMB can be continued and the step above can be repeated. If there are no mutually acceptable pairs we are done. $\qquad\square$

Whenever IMB produces a maximal individually rational matching μ, by Proposition 1 DA produces matching μ. Our next result shows that single peaked preferences and priorities guarantee that TTC produces μ as well regardless of school capacities.

Proposition 5. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences and priorities satisfy the single peakedness property, and the acceptability relations are distance consistent, then IMB generates the same matching μ as DA and TTC. Furthermore, matching μ is stable, Pareto optimal, and strategy proof at P.*

Proof. IMB = DA, stability, Pareto-optimality, and strategy proofness follow from Propositions 4 and 1. By definition, IMB produces only short cycles $s \to c \to s$. For TTC = IMB to hold, we need to show that there are no cycles of the form $s \to c \to s' \to c' \to s$, where $s \neq s'$. For such cycles the following inequalities have to hold for $s, s' \in S$ and $c, c' \in C$:

$$s : ||s - c'|| > ||s - c||,$$
$$c : ||c - s|| > ||c - s'||,$$
$$s' : ||s' - c|| > ||s' - c'||,$$
$$c' : ||c' - s'|| > ||c' - s||.$$

It follows that $||s - c|| > ||c - s'|| > ||s' - c'|| > ||c' - s|| > ||s - c||$ has to hold by symmetry of $|| \cdot ||$, a contradiction since $||s - c|| \not> ||s - c||$. Same reasoning can be applied to all cycles $s \to c \to s' \to \cdots \to c' \to s$ since the inequalities $||s - c|| > ||c - s'|| > \cdots > ||c' - s|| > ||s - c||$ cannot hold. $\quad\square$

Example 2. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, all schools are acceptable to all students and all students are acceptable to all schools. Let the capacities of the schools be 1 except for school c_2 let $q_{c_2} = 2$. Let the used norm be the taxicab norm so that for student s_1 and school c_2 the

following holds $||s_1 - c_2|| = |s_1^x - c_2^x| + |s_1^y - c_2^y| = |0 - 4| + |2 - 4| = 6$. The distances can be easily calculated from Figure 1 where each agent is represented by coordinates on $x - y$ plane.

Figure 1: Taxicab norm single peaked preferences.
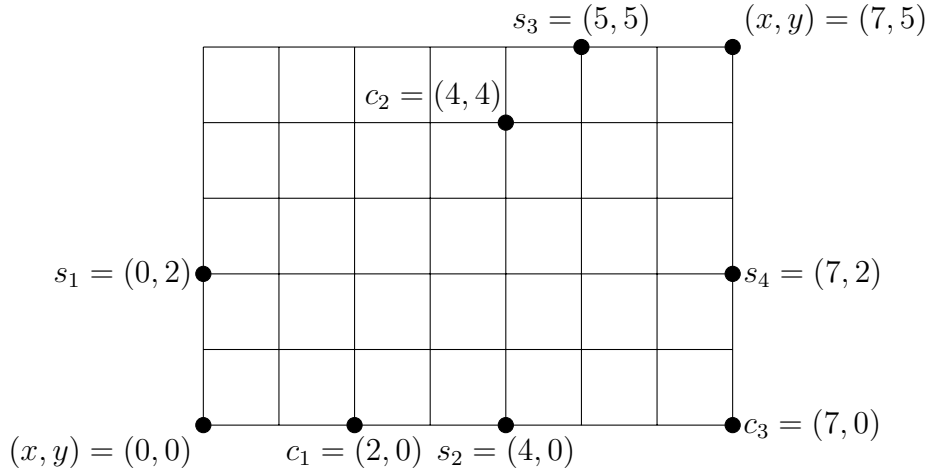
Now the preferences are as follows: $\prec_{s_1} = c_1 c_2 c_3$, $\prec_{s_2} = c_1 c_3 c_2$, $\prec_{s_3} = c_2 c_3 c_1$, $\prec_{s_4} = c_3 c_2 c_1$. Similarly the priorities are: $\prec_{c_1} = s_2 s_1 s_4 s_3$, $\prec_{c_2} = s_3 s_2 s_4 s_1$, $\prec_{c_3} = s_4 s_2 s_3 s_1$. Clearly the preferences and the priorities are strict and satisfy single peakedness.

Applying IMB, the first mutually best matches are (s_2, c_1), (s_3, c_2), and (s_4, c_3). After that, the only mutually best match is (s_1, c_2). IMB produces the matching $\mu = (s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_3)$. DA and TTC produce μ as suggested by Proposition 5.

$\triangleleft$

If students' preferences and schools' priorities satisfy the "single crossing" property, then again IMB produces a maximal individually rational matching for the problem.

Let us assume that the single crossing property holds at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$. Then the students $S = \{s_1, \ldots, s_n\}$ and schools $C = \{c_1, \ldots, c_m\}$ can be thought as being real numbers and indexed so that $s_i <$

s_{i+1} and $c_k < c_{k+1}$. Further, the following holds for $s_i, s_j \in S$, $c_k, c_p \in C$:

$$c_k \prec_{s_i} c_p \text{ and } c_k < c_p \Longrightarrow c_k \prec_{s_j} c_p, \text{ if } s_i < s_j \qquad (4)$$

$$s_i \prec_{c_k} s_j \text{ and } s_i < s_j \Longrightarrow s_i \prec_{c_p} s_j, \text{ if } c_k < c_p. \qquad (5)$$

If all students would accept the same schools and all schools would accept all students, then single crossing alone would be sufficient for IMB to generate a maximal individually rational matching. We give next a condition that allows us to prove a more general result.

Acceptability relation $A_s(C)$ for student s is an *interval*, if $c, c' \in A_s(C)$, $c \leq c'$, implies $c'' \in A_s(C)$, for all schools c'' such that $c \leq c'' \leq c'$. Interval acceptability relation for a school c is defined in the same manner.

Given a nonempty subset A of S, we denote by $\max A$ and $\min A$ the greatest and least elements of A, respectively. Notation $\max B$ and $\min B$ is defined analogously for nonempty subsets B of C.

Acceptability relations A_s and A_c are *setwise increasing intervals*, if

$$s < s' \Longrightarrow \min A_s(C) \leq \min A_{s'}(C), \text{ and } \max A_s(C) \leq \max A_{s'}(C) \qquad (6)$$

$$c < c' \Longrightarrow \min A_c(S) \leq \min A_{c'}(S), \text{ and } \max A_c(S) \leq \max A_{c'}(S). \qquad (7)$$

Constant relations $A_s(C) = A_{s'}(C)$, $A_c(S) = A_{c'}(S)$ for all s, s' and c, c' are special cases of setwise increasing relations.

Given $s \in S$, let us denote by $A_C^{-1}(s)$ the subset of schools that find s acceptable. If $C' \subset C$ is a nonempty subset, then let $A_{C'}^{-1}(s) \equiv A_C^{-1}(s) \cap C'$ denote the schools in C' that find s acceptable. Inverse relations $A_S^{-1}(c)$ and $A_{S'}^{-1}(c)$ are defined analogously for schools c and nonempty subsets $S' \subset S$. We have the following simple result.

Lemma 1. *If acceptability relations are setwise increasing intervals, then their inverses are also setwise increasing intervals.*

Proof. Take any s, s' such that $A_C^{-1}(s)$ and $A_C^{-1}(s')$ are nonempty and $s < s'$. If $c, c' \in A_C^{-1}(s)$ and $c < c'$, then by definition $s \in A_c(S) \cap A_{c'}(S)$. If c'' is such that $c \leq c'' \leq c'$, then $s \in A_{c''}(S)$, since acceptability relations are increasing intervals. Hence $A_C^{-1}(s)$ is an interval because $c'' \in A_C^{-1}(s)$.

Take any $c \in A_C^{-1}(s)$. We want to show that there exists $c' \geq c$ such that $c' \in A_C^{-1}(s')$. Apply equation 6 to the acceptability relations of schools: there must exist some c', $c \leq c'$ such that $s' \in A_{c'}(S)$. Hence $c' \in A_C^{-1}(s')$, and inverse relations for schools are setwise increasing. The proof for students' inverse acceptability relations is the same up to notation. $\square$

Corollary 2. *Suppose $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that for every student s there is some school c such that s, c is a mutually acceptable pair, and for every school c there is some student s such that s, c is a mutually acceptable pair. Then the relations $A_s(C) \cap A_C^{-1}(s)$ for students and $A_c(S) \cap A_S^{-1}(c)$ for schools are nonempty setwise increasing intervals.*

Proof. By assumption $A_s(C) \cap A_C^{-1}(s)$ is nonempty for every student s. It is an interval as an intersection of two intervals. Since both $A_s(C)$ an $A_C^{-1}(s)$ are setwise increasing, their intersection is also setwise increasing. The proof for schools' relations is similar. $\square$

The following lemma is a version of some well-known results from the literature of strategic complements (see *e.g.* Milgrom and Shannon 1994), but we give a proof here for the sake of completeness. For any $s \in S$ that has acceptable schools, let $c(s)$ denote the best school for s among the schools that accept s. That is, $c(s)$ is the best school in the set $A_s(C) \cap A_C^{-1}(s)$. For any $c \in C$ that has acceptable students, let $s(c)$ denote the best student for c among the students that accept c.

Lemma 2. *Suppose that the single crossing property holds at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ and that the acceptability relations of schools and students are setwise increasing intervals.*
Then $s < s'$ implies $c(s) \leq c(s')$ and $c < c'$ implies $s(c) \leq s(c')$.

Proof. It suffices to prove the lemma for students only because the proof for schools is identical.
Take any $s, s' \in S$ such that $s < s'$ and both $c(s)$ and $c(s')$ exist. We want to show $c(s) \leq c(s')$. If this does not hold then $c(s') < c(s)$. Since acceptability relations are setwise increasing intervals, we have that

$c(s'), c(s) \in A_s(C) \cap A_{s'}(C)$ because $s < s'$. But then single crossing property implies that student s' strictly prefers $c(s)$ to $c(s')$, since $s < s'$, $c(s') < c(s)$, and $c(s') \prec_s c(s)$. A contradiction with the definition of $c(s')$. Hence $c(s) \le c(s')$. □

When acceptability relations $A_c(S)$ and $A_s(C)$ on S and C are setwise increasing intervals, then their restrictions $A_c(S')$ and $A_s(C')$ to subsets S', C' are also setwise increasing intervals. The same holds also for the inverse acceptability relations by Lemma 1. The next lemma states that when acceptability relations are setwise increasing intervals, the single crossing property is also a "hereditary property" in the sense that it holds for subsets of S and C.

Lemma 3. *Suppose that the single crossing property holds at a problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$, all acceptability relations are setwise increasing intervals, and $S' \subset S$ and $C' \subset C$ are nonempty subsets. Then the preferences restricted to C' and priorities restricted to S' satisfy the single crossing property, and nonempty acceptability relations are setwise increasing intervals of S' and C'. In particular, Lemma 2 holds for preferences and priorities restricted to C' and S'.*

Proof. It suffices to give a proof for students only. Let $S' \subset S$ and $C' \subset C$ be nonempty subsets.

Preferences and priorities restricted to subsets C' and S' satisfy the single crossing property since they have this property on C and S. Acceptability relations need not be setwise increasing intervals for this result.

Take any $s, s' \in S$ such that $s < s'$. Since $A_s(C)$ is an interval of C, also $A_s(C') = C' \cap A_s(C)$ is an interval (possibly empty) of C'. If both $A_s(C')$ and $A_{s'}(C')$ are nonempty, then it follows immediately that these relations are setwise increasing, and so are their inverses.

If $A_s(C')$ contains a school that accepts s, *i.e.* $A_s(C') \cap A_{C'}^{-1}(s) \ne \emptyset$, then $A_s(C') \cap A_{C'}^{-1}(s)$ contains a school $c'(s)$ that is best school in C' for s that accepts s. Denote also by $c'(s')$ the best school in C' for s' that accepts s', when $A_{s'}(C') \cap A_{C'}^{-1}(s') \ne \emptyset$.

16

If $c'(s) \leq c'(s')$ does not hold, then $c'(s') < c'(s)$ must hold. Now $A_s(C') \cap A_{C'}^{-1}(s)$ and $A_{s'}(C') \cap A_{C'}^{-1}(s')$ are nonempty intervals containing $c'(s)$ and $c'(s')$, respectively. Since acceptability relations and their inverses are setwise increasing intervals we have that $c'(s') \in A_s(C') \cap A_{C'}^{-1}(s)$ and $c(s') \in A_{s'}(C') \cap A_{C'}^{-1}(s')$. But then the single crossing property implies that $c'(s') \prec_{s'} c(s')$ because $c'(s') < c(s')$ and $c'(s') \prec_s c(s')$, a contradiction. Hence Lemma 2 holds for preferences and priorities restricted to C' and S'. $\qquad\square$

We have the following.

Proposition 6. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences and priorities satisfy the single crossing property, and the acceptability relations are setwise increasing intervals, then IMB produces a maximal individually rational matching μ for P.*

Proof. We may assume that each $s \in S$ has at least one $c \in C$ such that s, c is a mutually acceptable, and similarly for all schools $c \in C$. This is so because if s has no school c such that s, c is a mutually acceptable pair, we can delete s from S (C) without affecting the matching generated by IMB, and similarly for all schools $c \in C$.

Round 1 of IMB. Form a sequence $s^1, s^2, \ldots$ of students and a sequence $c^1, c^2, \ldots$ of schools by the following rule: $s^1 = s_n$, and $c^1 = c(s_n)$, given s^t, let $c^t = c(s^t)$ and $s^{t+1} = s(c^t)$. In words, we start from the highest indexed student s_n, and choose the best school c^1 for him. Then we choose the best student s^2 for school c^1, and after that the best school c^2 for s^2, and so on.

Since S and C are nonempty and finite, the choices $s(c^t)$ and $c(s^t)$ can always be made by Corollary 1. By Lemma 2. sequences $\{s^t\}$ and $\{c^t\}$ are decreasing. By finiteness of S and C, these sequence have limits s^* and c^*. The limits satisfy $c^* = c(s^*)$ and $s^* = s(c^*)$, so s^*, c^* is a mutually best pair.

Match all mutually best pairs, remove matched students from S, and remove those schools from C whose capacity is full. If the remaining sets of students and schools contain at least one mutually acceptable pair, then

remove those students and schools for whom there exist no mutually acceptable matches. Update preferences, priorities and capacities of the remaining students and schools, and move to Round 2. If there are no mutually acceptable pairs, then the process ends.

Round k of IMB. Apply Round 1 to those students S' and schools C' that are still in the market, starting the process from the highest indexed student $s^1 = \max S'$. Since there exists a mutually acceptable pair, at least one mutually best pair can be found and matched by Lemma 3.

There must be some round $t \geq 1$ such that after mutually best matches are formed, there are no mutually acceptable pairs left. But then the matching μ produced by IMB is a maximal individually rational matching. $\qquad \square$

In fact, when single crossing property holds, the IMB produces the same matching as TTC. It follows that DA also produces this matching.

Proposition 7. *If $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ is such that preferences and priorities satisfy the single crossing property, and the acceptability relations are setwise increasing intervals, then IMB and TTC produce the same matching μ for P. It follows that the student proposing DA also produces μ for P.*

Proof. When single crossing property holds, it is impossible that TTC generates a cycle that contains at least two students and schools. To see this, assume that there is cycle $s \to c \to s' \to c' \to s$. If $s < s'$, then $c \leq c'$ by single crossing property, and $c = c'$ would imply that $s = s'$. Hence $c < c'$, and therefore $s' \leq s$ a contradiction. In the same manner all longer cycles are impossible under TTC.

Therefore under TTC only cycles $s \to c \to s$ will be formed. In each round of IMB, there is at least one cycle $s \to c \to s$ and s will be matched with c. Every cycle $s \to c \to s$ and s that is formed during the first round of IMB, will be formed also under TTC, so at least these matchings are the same. In the first round of IMB, there could also be matchings s, c such that c is the best school for s, and s is among the best q_c students for c although not the best one. Under each round of TTC, c is either matched with the

18

best student for c, or c gets no students at all during this round. This implies that s will eventually be matched with c also under TTC. Hence all students that are matched in the first round of IMB will eventually be matched with the same schools under TTC as well.

Continuing this way we can conclude that any match s, c that is formed during round k of IMB will eventually be formed under TTC as well. Since IMB produces a maximal individually rational matching μ, TTC must produce this matching as well.

By Proposition 1, μ is a stable matching. Since TTC produces μ, it is also Pareto optimal. The matching μ' produced by the student proposing DA Pareto dominates all other stable matchings. Hence $\mu = \mu'$. $\qquad\square$

In the next example a matching problem is presented that satisfies the single crossing property. However, the acyclicity conditions of Ergin (2002) and Kesten (2006) are violated. We show that DA produces a Pareto optimal matching although Ergin's condition is violated and that DA = TTC at this example although Kesten's condition is violated. A similar situation for single peaked preferences was shown in Example 2.

As explained in the Introduction, this is due to the fact that our setup is different than theirs. They seek a condition for a *fixed* priority structure such that DA is Pareto optimal or TTC = DA, no matter what the students' preferences are. In our setup, preferences cannot vary totally independently of priorities when the single crossing property has to hold.

Example 3. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, all schools are acceptable to all students and all students are acceptable to all schools. Each school has capacity of one. Let the preferences be as follows: $\prec_{s_1} = c_1 c_2 c_3$, $\prec_{s_2} = c_1 c_3 c_2$, $\prec_{s_3} = c_3 c_1 c_2$, $\prec_{s_4} = c_3 c_2 c_1$. Let the priorities be as follows: $\prec_{c_1} = s_1 s_2 s_3 s_4$, $\prec_{c_2} = s_3 s_4 s_1 s_2$, $\prec_{c_3} = s_4 s_3 s_2 s_1$. Now all agents have different preferences and priorities.

Order students in the order given by their indices: $s_1 < s_2 < s_3 < s_4$. Order schools in the order given by their indices: $c_1 < c_2 < c_3$. Student s_3 prefers c_1 to c_2, and so do also students s_2 and s_1. On the other hand student s_2 prefers c_3 to c_2, and so do also students s_3 and s_4.

School c_2 orders s_3 and s_4 higher than s_1 and s_2, and so does school c_3 who is on top in the linear order of schools. On the other hand, c_2 ranks s_3 higher than s_4 and s_1 higher than s_2, and so does c_1 who is the least school in the linear order of schools.

In this way one can check that single crossing property holds for preferences and priorities.

Applying IMB, the first mutually best matches are (s_1, c_1) and (s_4, c_3). After that, the only mutually best match is (s_3, c_2), and student s_2 is left unmatched. Hence the matching is $\mu = \{(s_1, c_1), (s_3, c_2), (s_4, c_2)\}$ and s_2 is left unmatched. By Proposition 7 DA and TTC also produce this Pareto optimal matching.

Both Ergin's and Kesten's acyclicity conditions are violated, since $s_3 \prec_{c_1} s_2 \prec_{c_1} s_1$ and $s_1 \prec_{c_3} s_2 \prec_{c_3} s_3$, and the Scarcity condition of both author's is satisfied since all schools have capacity of one (for details, see Ergin 2002 and Kesten 2006). $\triangleleft$

Single crossing works nicely for IMB when the student and school sets are totally ordered, or "one dimensional". If student and school sets have more complicated lattice structure then IMB may fail as shown in the next example.

Example 4. Consider the subset $\{(0,0), (0,1), (1,0), (1,1)\}$ of $\mathbb{R}^2$. When equipped with the usual order of $\mathbb{R}^2$ this set becomes a lattice. Let $S = \{s_{00}, s_{01}, s_{10}, s_{11}\}$ and $C = \{c_{00}, c_{01}, c_{10}, c_{11}\}$ and order the students and schools by their indices. So $s_{00} \leq s_{01}, s_{10} \leq s_{11}$, and s_{01} and s_{10} are incomparable, and analogously for schools. All schools have capacity of one.

Preferences: $\prec_{s_{00}} = c_{00}c_{10}c_{01}c_{11}$, $\prec_{s_{10}} = c_{11}c_{01}c_{10}c_{00}$, $\prec_{s_{01}} = c_{11}c_{10}c_{01}c_{00}$, $\prec_{s_{11}} = c_{11}c_{01}c_{10}c_{00}$. All schools are acceptable to all students.

Priorities: $\prec_{c_{00}} = s_{00}s_{01}s_{10}c_{11}$, $\prec_{c_{10}} = s_{11}s_{10}s_{01}s_{00}$, $\prec_{c_{01}} = s_{11}s_{01}s_{10}s_{00}$, $\prec_{c_{11}} = s_{11}s_{10}s_{01}c_{00}$. All students are acceptable to all schools.

Note that preferences and priorities satisfy the single crossing property. They satisfy also a condition called "quasi supermodularity". This condition says for student s_{00} that c_{00} must be better than c_{10} *because* c_{01} is better than c_{11} (for details, see Milgrom and Shannon 1994).

IMB matches first s_{00} with c_{00} and s_{11} with c_{11}. But then IMB halts: s_{10} prefers c_{01} to c_{10}, but c_{01} has tentatively accepted only s_{01}. Further, s_{01} prefers c_{10} to c_{01}, but c_{10} has tentatively accepted only s_{10}. $\triangleleft$

4. Fixing IMB?

IMB may halt before it has generated a maximal individually rational matching. Sometimes it may fail to match any pairs of students and schools. In this section we look at possibilities to modify IMB in such a way that it always produces a maximal individually rational matching.

The first modification is called IMB*. The definition is simple: if IMB halts and the matching is not maximal individually rational, then apply one round of TTC to the sets of students S' and C' still in the market. There must be at least one cycle. Remove the matched students from S'. Remove the schools whose capacity is full from C'. Update the preferences, priorities and capacities of the remaining students S'' and schools C'', and try to apply the usual IMB again. And so on. It is clear that the outcome will be a maximal individually rational matching.

The next result states that in marriage markets IMB* is actually TTC.

Proposition 8. *If each school has capacity one, then IMB* = TTC.*

Proof. Let $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ be a matching problem such that all schools have capacity of one. If IMB produces a maximal individually matching for P, then IMB*(P) = IMB(P), and TTC(P) = IMB*(P) by Corollary 1.

Suppose then that IMB does not produce a maximal individually rational matching for P, and that it takes $k \geq 0$ rounds before IMB halts the first time.

If $k = 0$ then IMB cannot make any matches. In such a case IMB* and TTC produce exactly the same cycles and matches during the first round.

If $k > 0$, then look at all cycles containing at least two students that could be formed under IMB in rounds $1, \ldots, k-1$ if TTC had been applied. Such cycles are disjoint. The schools and students in these cycles cannot

21

be matched under IMB. But all these students and schools are matched in round k of IMB*, because then TTC is applied once. In addition, there is at least one mutually best match that is formed in each round $t = 1, \ldots, k-1$ of IMB. Let μ^1 be the set of all matches that IMB* is able to form during the rounds $t = 1, \ldots, k$.

TTC would produce exactly the same matchings μ^1 during rounds $t = 1, \ldots, k$. The only difference with IMB* is that when a cycle containing at least two students is formed under TTC, the students and schools in that cycle are matched immediately. All mutually best matches that can be during periods $t = 1, \ldots, k-1$ are formed by both TTC and IMB*.

If there are still mutually acceptable pairs left after the matches in round k have been formed, then let S' and C' be the sets of agents that are still in the market. Let $P' = \{S', C', (\prec'_s)_s, (\prec'_c)_c, q'\}$ be the remaining matching problem. Then repeat the steps above, and note that IMB* and TTC produce the matchings up to and including the point when IMB halts the first time. □

It follows from Proposition 8 that IMB* is not stable. However IMB* is Pareto optimal.

Proposition 9. *IMB* is Pareto optimal.*

Proof. Let IMB* generate a matching μ for problem P. Then μ is a maximal individually rational matching. Suppose that μ is not Pareto optimal, and that μ' Pareto dominates it.

When IMB* is applied to P, let k be the first round such that some student gets a better match than in μ. Now $k = 1$ is impossible since all students matched in the first round of IMB* get their best match in P. Hence the students matched in the first round are matched with the same schools in μ and in μ'.

Let S', C' be the sets of agents left in the beginning of the second round, and update capacities, preferences, and priorities. Let P^1 be the problem corresponding to this situation. Then it must still hold that IMB* applied to P^1 generates a matching μ^1 that is Pareto dominated by a matching μ'^1.

22

Again, it holds that students matched in the first round are matched the same way in μ^1 and μ'^1. Hence $k = 2$ is impossible.

The proof is completed by applying induction on k. □

A drawback of IMB* as compared to ordinary TTC is that it is not strategy proof. The following is a slightly modified Example 4 in Morrill (2015), we have only changed the notation and specified some preferences in more detail. This example reveals also that IMB* may match a larger number of students with schools than TTC.

Example 5. $S = \{s_1, \ldots, s_5\}$, $C = \{c_1, \ldots, c_4\}$. School c_1 has capacity of two, the other schools have capacity of one. All students find all schools acceptable. Preferences are as follows, unlisted schools could be in any order after the top schools: $\prec_{s_1} = c_3 c_2 c_1$, $\prec_{s_2} = c_1$, $\prec_{s_3} = c_2 c_1$, $\prec_{s_4} = c_4 c_1 c_3$, $\prec_{s_5} = c_3$. All schools find all students acceptable. Priorities are as follows, unlisted students could be in any order after the top students: $\prec_{c_1} = s_1 s_4 s_2 s_3$, $\prec_{c_2} = s_2 s_3 s_1$, $\prec_{c_3} = s_4 s_5$, $\prec_{c_4} = s_5 s_4$.

There are no mutually best matches to eliminate. By the definition of IMB* we must apply one round of TTC to S and C. The only cycle is $s_4 \to c_4 \to s_5 \to c_3 \to s_4$. So the first matches are (s_4, c_4), (s_5, c_3) and these agents are removed. In the second round (s_2, c_1) is a mutually best match. In the third round (s_3, c_2) is a mutually best match. In the fourth round (s_1, c_1) is a mutually best match. So the matching μ generated by IMB* is

$$\mu^* = \{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_4), (s_5, c_3)\}.$$

Suppose s_1 reports that c_2 is his best school. Applying IMB*, we get first a cycle $s_1 \to c_2 \to s_2 \to c_1 \to s_1$. Hence s_1 will be matched with c_2, and therefore IMB* is not strategy proof.

Note that the ordinary TTC with true preferences generates the same matching μ' as IMB* with the false reporting by s_1:

$$\mu' = \{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_4), (s_5, c_3)\}.$$

Now change the preferences of student s_3 so that he accepts only c_2. That will not change the matching μ^* produced by IMB* when all students

report true preferences. But TTC will now produce the matching

$$\mu' = \{(s_1, c_2), (s_2, c_1), (s_4, c_4), (s_5, c_3)\}.$$

Hence in this case IMB* matches more students with schools than TTC. ◁

We saw from the last example that IMB* might produce a different matching compared to TTC when capacities of the schools are larger than one. However, we can guarantee that TTC = IMB* in all college admissions problems when we transform it to a *related marriage market* (see Roth and Sotomayor 1992, p 131). That is, whenever a school c has a capacity larger than one we create $|q_c|$ copies of the school, each copy maintaining the preferences of the original c with capacity one, and replace c by the string $c^1, \ldots c^{|q_c|}$ on preferences of the students. We assume that student s strictly prefers lower number indexed copies of school c to higher indexed copies. Thus for student s with preferences $\prec_s$: cc' we form new cloned preferences as $\prec_s$: $c^1 \ldots c^i c'^1 \ldots c'^j$, where $|q_c| = i$, $|q_{c'}| = j$, and $i, j \geq 1$. We can now present our final result.

Corollary 3. *For every college admissions problem $P = \{S, C, (\prec_s)_s, (\prec_c)_c, q\}$ transformed to a related marriage market, we have IMB*=TTC.*

Proof. As a related marriage market only requires us to make assumptions about preferences of the students and now for all schools we have a quota of one, it immediately follows from Proposition 8 that IMB* = TTC. □

There are of course many ways to solve the deadlock when IMB terminates without producing a maximal individually rational matching. One possibility is to temporarily increase schools' capacities by one, and check if there are any mutually best matches now. If there are no mutually best matches, then increase the capacities by one again, and so on. Eventually at least one mutually best match will be found. Then make these matches, return the true capacities of schools and try the usual IMB with the remaining students and schools. This mechanism produces a maximal individually rational matching.

This mechanism is not strategy proof. Look at Example 5. Expand the capacities of schools temporarily by one, and note that then there are three mutually best matches: (s_3, c_2), (s_4, c_4), and (s_5, c_3). After that, the usual IMB makes matches (s_1, c_1) and (s_2, c_1). But this is the same matching μ^* that IMB* produced in Example 5. and therefore strategy proofness is violated.

The mechanism is not stable either. Look again at Example 5, and set $q_{c_1} = 1$ so all schools have capacity of one. Again, the first matches will be (s_3, c_2), (s_4, c_4), and (s_5, c_3). After that the usual IMB makes the match (s_1, c_1) and so s_2 is left unmatched. But since s_2 is acceptable to c_2, and c_2 prefers s_2 to s_3, stability is violated.

Still another mechanism that we may consider when IMB halts is the following. There are no mutually best pairs, but take the second best school c of any student s, and check if s is among the q_c best students in school c. If this holds then make these matches, and return to the usual IMB. But if no matches are found then look at the third best schools c' of any student, and check whether s is among the $q_{c'}$ students of c'. Continuing this way down the preference lists of students, eventually at least one match can be made. After that, try the usual IMB with the remaining students and schools. This mechanism produces a maximal individually rational matching.

However, this mechanism does not satisfy strategy proofness, stability, or Pareto optimality. We leave the proofs for interested readers.

5. Conclusions

In this paper we have concentrated on college admission problems with tractable restrictions on preferences. We introduced an algorithm based on iterative formation of mutually best matches (IMB) and compared it to the deferred acceptance (DA) algorithm and top trading cycles (TTC). It turns out that in many cases it makes little difference which matching algorithm is used. Most notably if the preferences and priorities are single peaked or satisfy the single crossing property, IMB generates the same maximal individually rational matching as DA and TTC.

Future research is needed on modified version of IMB and other situations with realistic preference restrictions.

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