DYADIC ANALYSIS OF INTEGRAL OPERATORS:
MEDIAN OSCILLATION DECOMPOSITION
AND TESTING CONDITIONS

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Academic dissertation

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LIST OF INCLUDED ARTICLES

This thesis consists of an introductory part and the following five articles:

[A] Hänninen, TS, Hytönen, TP **2014**, 'The A_2 theorem and the local oscillation decomposition for Banach space valued functions', Journal of Operator Theory, vol 72, no. 1, pp. 193-218., 10.7900/jot.2012nov21.1972.

[B] Hänninen, TS, Hytönen, TP **2014**, 'Operator-valued dyadic shifts and the $T(1)$ theorem', submitted to Monatshefte für Mathematik.

[C] Hänninen, TS, Hytönen, TP, Li, K **2014**, 'Two-weight L^p-L^q bounds for positive dyadic operators: unified approach to $p \leq q$ and $p > q$', submitted to Potential Analysis.

[D] Hänninen, TS **2015**, 'Two weight inequality for vector-valued positive dyadic operators by parallel stopping cubes', accepted for publication in the Israel Journal of Mathematics.

[E] Hänninen, TS **2015**, 'Remark on dyadic pointwise domination and median oscillation decomposition', accepted for publication in the Houston Journal of Mathematics.

In the introductory part, these articles are referred to as [A],[B],[C],[D],[E], and the other references as [1],[2],....

AUTHOR'S CONTRIBUTION

The author has had a central role in the research leading to the joint works [A] and [B]. The article [C] is a joint work of three collaborators, where each collaborator has done an essentially equal share of the research; in particular, the author has had a key role in formulating an abstract Wolff potential, and proving the sufficiency of sequential testing conditions for bilinear positive dyadic operators and linearized dyadic maximal operators, results which compose Sections 3.C., 4.B., and 5.A. of the article. The author has written the articles [A] and [B], and a part of the article [C]. The works [D] and [E] consist of the author's independent research.

1. Overview

Dyadic analysis plays an important role in harmonic analysis, both as a method and in its own right: Many non-dyadic problems can be converted into dyadic problems, and dyadic problems can serve as a laboratory for studying new phenomena.

A central object of dyadic analysis is the system of dyadic cubes. Its key property is nestedness: Two dyadic cubes are either disjoint or one is contained in the other. This property is implicitly behind powerful dyadic techniques, such as stopping cubes, and it is also exploited explicitly in many combinatorial or covering arguments.

A crucial observation behind the passage between dyadic and non-dyadic problems is that dyadic cubes behave similarly to geometric cubes, once several dyadic systems are used: In finitely many adjacent dyadic systems, each cube is contained in a dyadic cube of comparable size. With a positive probability in a randomized dyadic system, the dyadic expansion of each cube expands similarly to its geometric expansion.

Using this observation, many non-dyadic operators of harmonic analysis can be proven to be comparable to dyadic model operators: For example, both singular and positive integral operators, such as fractional integral operators (see Cruz-Uribe and Moen [12]) and Calderón–Zygmund operators (see Hytönen, Lacey, and Pérez [30] and Lerner [46] combined with Conde-Alonso and Rey [9], Lerner and Nazarov [47], or Lacey [37]), are pointwise dominated by *positive dyadic operators* of the form

$$f \mapsto \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, \mathrm{d}x 1_Q.$$

We can use this comparison to deduce properties for a non-dyadic operator from the properties of its dyadic model operator. A striking example of this is the proof of the A_2 theorem for Calderón–Zygmund operators; see Hytönen's survey [27].

This thesis is about two themes: The first is domination and representation of integral operators by dyadic model operators, and the second is testing conditions for dyadic operators. The latter refers to characterizing the $L^p \to L^q$ boundedness of an operator by its action on a restricted class of functions.

The contributions of this thesis to domination and representation of integral operators by dyadic model operators are:

- The median oscillation decomposition by Lerner [43] can be used as a method to pointwise dominate operators by positive dyadic operators. In the articles [A] and [E], this decomposition is extended to Banach space valued functions and to non-doubling measures.
- The dyadic representation theorem by Hytönen [26] states that each Calderón–Zygmund operator can be represented as a series of dyadic shifts and paraproducts averaged over randomized dyadic systems. In the article [B], this theorem is extended to Calderón–Zygmund operators with operator-valued kernels.

Next, we discuss the contributions of this thesis to testing conditions for dyadic operators. Let σ and ω be locally finite Borel measures. A recurring testing condition is *the Sawyer testing condition*: An operator $T(\cdot \sigma) : L^p(\sigma) \to L^q(\omega)$ is said to satisfy the Sawyer testing condition if and only if

$$(1.1) \qquad \|1_Q T(1_Q \sigma)\|_{L^q(\omega)} \lesssim \sigma(Q)^{1/p} \quad \text{for every } Q \in \mathcal{D}.$$

Using such testing condition and its dual, the $L^p(\sigma) \to L^q(\omega)$ boundedness for the exponents $1 < p \leq q < \infty$ of many positive integral operators has been characterized. Among these are: A large class of positive integral operators, in particular, fractional integrals and Poisson integrals, by Sawyer [58]; dyadic maximal operators by Sawyer [57]; and positive dyadic operators by Nazarov, Treil, and Volberg [51] and Lacey, Sawyer, and Uriarte-Tuero [41]. Such testing condition and its dual has also been used to characterize the boundedness of singular integral operators, for example: The unweighted $L^2 \to L^2$ boundedness of Calderón–Zygmund operators by David and Journé [14]; and the two-weight $L^2(\sigma) \to L^2(\omega)$ boundedness of the Hilbert transform by Lacey, Sawyer, Shen, and Uriarte-Tuero [36, 40].

However, in certain cases, the Sawyer testing condition can be proven to be insufficient for the boundedness. For example, this is the case for the $L^p \to L^p$ with $p \neq 2$ boundedness of martingale transforms, as shown by Nazarov in an unpublished manuscript.

In the article [C], we study testing conditions in the upper triangular case $T(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ with $1 < q < p < \infty$. We show that in this case the Sawyer testing condition is insufficient even for the boundedness of positive dyadic operators. Instead, we use *the sequential testing condition*: A positive linear operator $T(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ with $1 < q < p < \infty$ satisfies the sequential testing condition if and only if, for the auxiliary exponent $r \in (1, \infty)$ defined by $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, we have

$$\Big(\sum_{F \in \mathcal{F}} \Big(\frac{\|1_F T(1_F \sigma)\|_{L^q(\omega)}}{\sigma(F)^{1/p}} \Big)^r \Big)^{1/r} \lesssim 1 \quad \text{for every } \sigma\text{-sparse } \mathcal{F} \subseteq \mathcal{D}.$$

By definition, a collection $\mathcal{F}$ is *σ-sparse* if and only if for each $F \in \mathcal{F}$ there exists $E(F) \subseteq F$ such that $\sigma(E(F)) \gtrsim \sigma(F)$ and such that the collection $\{E(F)\}_{F \in \mathcal{F}}$ is pairwise disjoint. The particular operators that we study are *positive dyadic operators* $A_\lambda(\cdot\sigma)$ defined by

$$A_\lambda(f\sigma) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q,$$

linearized dyadic maximal operators $M_{\mathcal{E}, \lambda}(\cdot\sigma)$ defined by

$$(1.2) \qquad M_{\mathcal{E}, \lambda}(f\sigma) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_{E(Q)},$$

and their bilinear analogues $A_\lambda(\cdot\sigma_1, \cdot\sigma_2)$ and $M_{\mathcal{E}, \lambda}(\cdot\sigma_1, \cdot\sigma_2)$ defined by

$$A_\lambda(f_1\sigma_1, f_2\sigma_2) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f_1 \, d\sigma_1 \int_Q f_2 \, d\sigma_2 1_Q,$$

$$M_{\mathcal{E}, \lambda}(f_1\sigma_1, f_2\sigma_2) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f_1 \, d\sigma_1 \int_Q f_2 \, d\sigma_2 1_{E(Q)}.$$

By means of sequential testing conditions, we obtain alternative characterizations for the $L^p(\sigma) \to L^q(\omega)$ boundedness of these linear operators in the range of the exponents $\frac{1}{p} < \frac{1}{q}$. Furthermore, we are able to characterize the $L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$ boundedness of these bilinear operators in the range of the exponents $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{q}$. In this range, no characterization for either of these bilinear operators was available until now.

In the article [B] and the article [C], we study testing condition in the vector valued case $T(\cdot\sigma) : L^p_C(\sigma) \to L^q_D(\omega)$ with Banach spaces C and D and $1 < p \leq q < \infty$.

We use *the L^∞ testing condition*: An operator $T(\cdot\sigma) : L_C^p(\sigma) \to L_D^q(\omega)$ satisfies the L^∞ testing condition if and only if

$$\|1_Q T(f1_Q\sigma)\|_{L_D^q(\omega)} \lesssim \|f\|_{L_C^\infty(Q,\sigma)}\sigma(Q)^{1/p} \quad \text{for every } Q \in \mathcal{D} \text{ and } f \in L_C^\infty(Q,\sigma).$$

By using this testing condition, we are able to obtain the following characterizations:

- Let C and D be Banach lattices with the Hardy–Littlewood property. (This property refers to the boundedness of a certain maximal operator; see Definition 2.14 for a precise definition.) Assume that $\{\lambda_Q\}_{Q\in\mathcal{D}}$ are positive linear operators from the Banach lattice C to the Banach lattice D. In the article [D], the boundedness of $A_\lambda(\cdot\sigma) : L_C^p(\sigma) \to L_D^q(\omega)$ is characterized by means of the direct and the dual L^∞ testing condition. This extends Scurry's [59] characterization for the $L^p(\sigma) \to L^q(\omega)$ boundedness of the sequence-valued operator

$$f \mapsto \Big(\sum_{Q\in\mathcal{D}} \big(\beta_Q \int_Q f \, d\sigma\big)^s 1_Q \Big)^{1/s}$$

  associated with non-negative real numbers $\{\beta_Q\}$.
- Let C and D be Banach spaces. Assume that b is a function whose value at each point is an operator from the Banach space C to the Banach space D. In the article [B], the boundedness of the dyadic paraproduct $\Pi_b : L_C^p \to L_D^q$, defined by

$$\Pi_b f := \sum_{Q\in\mathcal{D}} D_Q b\langle f\rangle_Q 1_Q,$$

  is characterized by means of the direct L^∞ testing condition. This extends the classical scalar-valued result that the paraproduct $\Pi_b : L^p \to L^p$ is bounded if and only if $b \in BMO^p$.

We remark that both the vector-valued and two-weight settings are similar in one technical aspect: We need to work directly with the L^p space. This is because interpolation in these settings is of limited use: In the two weight setting, an operator $T : L^p(\sigma) \to L^p(\omega)$ is typically bounded for exactly one exponent $p \in (1,\infty)$, and, in the vector-valued setting, the space L_E^2 is no more tractable than any other L_E^p with $p \in (1,\infty)$. In the article [B], we give alternative (in our opinion simple) proofs for the L^p tools that we use: the L^p variant of Pythagoras' theorem, and a decoupling inequality of martingale differences.

Among the conceptual highlights are the vector-valued median introduced in the article [A], and the abstract Wolff potential associated with any positive linear operator introduced in the article [C].

In the next section, we summarize those aspects of dyadic and vector-valued analysis that are relevant for this work. In the subsequent sections, we discuss each article: What is new and what is its relation to what was known. Besides stating what is proven, the author also aims at giving a flavour of how it is done.

2. Preliminaries

2.1. Dyadic analysis.

Dyadic cubes. The *standard dyadic system* $\mathcal{D}$ on $\mathbb{R}^d$ is the collection of cubes defined by
$$\mathcal{D} := \{2^{-k}([0,1)^d + j) : k \in \mathbb{Z}, j \in \mathbb{Z}^d\}.$$
The *dyadic children* $\operatorname{ch}_{\mathcal{D}}(Q)$ of a dyadic cube $Q \in \mathcal{D}$ is the collection of the maximal (with respect to the set containment) $R \in \mathcal{D}$ such that $R \subsetneq Q$. The *dyadic parent* $\hat{Q}$ of a dyadic cube $Q \in \mathcal{D}$ is the minimal $R \in \mathcal{D}$ such that $R \supsetneq Q$.

For a cube, its dyadic children are determined by bisecting each side, whereas its dyadic parent can be chosen: Each side can be extended either to the right or to the left. Starting from the cube $[0,1)^d$ and extending each side of each ancestor to the right determines the standard dyadic system. We could also start from the translated cube $[0,1)^d + s$ with $s \in [0,1)^d$ and choose for each side of each ancestor whether to extend to the right or to the left.

This can be parameterized as follows: Let $\omega := (\omega_j)_{j \in \mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} =: \Omega$. The *shifted dyadic cube* $Q \dotplus \omega$ is defined by $Q \dotplus \omega := Q + \sum_{j: 2^{-j} < \ell(Q)} 2^{-j} \omega_j$. The *shifted dyadic system* $\mathcal{D}^\omega$ is defined by
$$\mathcal{D}^\omega := \{Q \dotplus \omega : Q \in \mathcal{D}\}.$$
The shifted dyadic systems $\mathcal{D}^\omega$ can be randomized by equipping the parameter set with the natural probability measure: Each component $\omega_j \in \{0,1\}^d$ has an equal probability 2^{-d} of taking any of the 2^d values, and all components are stochastically independent. We can also use finitely many choices of these shifted dyadic systems, such as the *adjacent dyadic systems*
$$\mathcal{D}^u := \{2^{-k}([0,1)^d + j + (-1)^k u : k \in \mathbb{Z}, j \in \mathbb{Z}^d\} \quad \text{for } u \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\}^d.$$

A dyadic system $\mathcal{D}$ can be viewed as a sequence $\{\mathcal{D}_k\}_{k=-\infty}^{\infty}$ of refining partitions of the Euclidean space $(\mathbb{R}^d, |\cdot|)$, where each partition $\mathcal{D}_k := \{Q \in \mathcal{D} : \ell(Q) = 2^{-k}\}$ consists of sets of diameter approximately 2^{-k}. Taking these as the defining properties, dyadic systems can be extended to geometrically doubling metric spaces; for more about that, see, for example, Hytönen and Kairema's article [22].

Dyadic operators. Among the central tools in dyadic analysis is the *dyadic Hardy–Littlewood maximal operator* M^μ defined by
$$M^\mu f := \sup_{Q \in \mathcal{D}} |\langle f \rangle_Q^\mu| 1_Q,$$
and the *dyadic martingale transform* T_ϵ^μ associated with the signs $\{\epsilon_Q\}_{Q \in \mathcal{D}}$ defined by
$$T_\epsilon^\mu f := \sum_{Q \in \mathcal{D}} \epsilon_Q D_Q^\mu f.$$
Whenever we are implicitly assuming that the function f is non-negative, we omit the absolute value in the formula for the Hardy–Littlewood maximal function, and write $M^\mu f = \sup_{Q \in \mathcal{D}} \langle f \rangle_Q^\mu 1_Q$. Here, $\langle f \rangle_Q^\mu$ denotes the average $\langle f \rangle_Q^\mu := \frac{1}{\mu(Q)} \int_Q f \, d\mu$, and $D_Q^\mu f$ the difference of averages $D_Q^\mu f := \sum_{Q' \in \operatorname{ch}_{\mathcal{D}}(Q)} \langle f \rangle_{Q'}^\mu 1_{Q'} - \langle f \rangle_Q^\mu 1_Q$. The central estimates for the dyadic Hardy–Littlewood maximal operator and dyadic martingale transforms are:

Theorem 2.1 (Boundedness of the dyadic Hardy–Littlewood maximal operator)**.** *Let $p \in (1, \infty]$. We have*
$$\|M^\mu\|_{L^p(\mu) \to L^p(\mu)} \le p'.$$

Theorem 2.2 (Boundedness of the dyadic martingale transform). *Let $p \in (1, \infty)$. We have*

$$\|T_\epsilon^\mu\|_{L^p(\mu) \to L^p(\mu)} \lesssim \max\{p, p'\} - 1.$$

By using the boundedness of the dyadic Hardy–Littlewood maximal operator, one can prove the dyadic Carleson embedding theorem:

Theorem 2.3 (Dyadic Carleson embedding theorem). *Let $\{\lambda_Q\}$ be non-negative real numbers. Then*

$$\Big(\sum_{Q \in \mathcal{D}} \lambda_Q \big(\langle f \rangle_Q^\mu \big)^p \Big)^{1/p} \lesssim \|f\|_{L^p(\mu)}$$

if and only if $\sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} \lambda_Q \lesssim \mu(R)$ for every $R \in \mathcal{D}$.

Stopping time techniques. The basic idea of stopping time is that something has not happened until the first moment it happens.

For example, consider moments $t_0 < t_1 < \cdots < t_N$. We can start from a moment t_0, and single out the first later moment t_{n_1} at which something bad happens. By iteration, we obtain bad moments $t_0 < t_{n_1} < t_{n_2} < \ldots < t_N$ and good moments $t_1, \ldots, t_{n_1-1}, t_{n_1+1}, \ldots, t_{n_2-1}, \ldots$. Now, since the good moments are good, we can deal with them, and, if the bad moments are few enough, we can deal with them, too.

In dyadic analysis, this means starting from a dyadic cube F and choosing the maximal (with respect to the set inclusion) dyadic subcubes $F' \subseteq F$ satisfying a certain condition. This condition may depend on the cube F and other relevant quantities. This condition is called the *stopping condition*, and the cubes F' are called the *stopping children* of F. Now, by maximality, if a dyadic cube $Q \subseteq F$ is such that $Q \subseteq F'$ for no F', then Q satisfies the opposite of the condition. The choice of stopping children can be iterated: Assume that for each $F \in \mathcal{D}$ we have chosen a collection $\mathrm{ch}(F) = \{F' \subseteq F\}$ of dyadic subcubes of F. Let $\mathcal{F}_0 = \{F_0\}$ be an initial cube. Define recursively $\mathcal{F}_{k+1} := \bigcup_{F \in \mathcal{F}_k} \mathrm{ch}(F)$. Let $\mathcal{F} := \bigcup_{k=0}^\infty \mathcal{F}_0$. This collection $\mathcal{F}$ is called the *family of stopping cubes starting from F_0*.

Typically, the stopping cubes are few in the sense that the stopping family is sparse:

Definition 2.4 (Sparse collection). A collection $\mathcal{S}$ of sets is μ-*sparse* if for every $S \in \mathcal{S}$ there exists $E(S) \subseteq S$ such that $\mu(E(S)) \gtrsim \mu(S)$ and the collection $\{E(S)\}_{S \in \mathcal{S}}$ is pairwise disjoint.

Sparseness of a collection is almost as good as pairwise disjointness: For example, an L^p variant of Pythagoras' theorem holds for such collections:

Lemma 2.5 (L^p variant of Pythagoras' theorem). Let $1 \leq p < \infty$. Let $\mathcal{S}$ be a collection of dyadic cubes, and $\{f_S\}_{S \in \mathcal{S}}$ a family of locally integrable functions. Assume that, for every $S \in \mathcal{S}$, we have that

- f_S is supported on S,
- f_S is constant on each $S' \in \mathrm{ch}_{\mathcal{S}}(S)$.

Furthermore, assume that the collection $\mathcal{S}$ is μ-sparse. Then

$$\|\sum_S f_S\|_{L^p(\mu)} \leq 3p \Big(\sum_S \|f_S\|_{L^p(\mu)}^p \Big)^{1/p}.$$

Moreover, the reverse estimate

$$\Big(\sum_S \|f_S\|_{L^p(\mu)}^p\Big)^{1/p} \le 6p'\|\sum_S f_S\|_{L^p(\mu)}$$

holds if, in addition, one of the following conditions holds: For every $S \in \mathcal{S}$, we have that

- $\int_S f_S \, \mathrm{d}\mu = 0$, or
- $f_S \ge 0$.

This theorem holds also for Banach space (where the cancellative condition is assumed for the reverse estimate) or Banach lattice (where the positivity condition is assumed for the reverse estimate) valued functions. This L^p variant of Pythagoras' theorem was proven by Katz and Pereyra [34] by using a multilinear estimate. In the article [B], we give an alternative (in our opinion simple) proof for the theorem.

Martingale techniques.

General martingales. We merely summarize the definition of martingales and martingale difference sequences, and state Doob's and Burkholder's inequalities. For more about martingales, see, for example, Williams's [70] textbook 'Probability with martingales'. A *filtration* on a measure space $(X, \mathcal{F}, \mu)$ is a refining sequence $\{\mathcal{F}_k\}_{k=-\infty}^{\infty}$ of σ-finite σ-algebras. A sequence $\{f_k\}_{k=-\infty}^{\infty}$ of *locally integrable* functions *adapted* to the filtration $\{\mathcal{F}_k\}_{k=-\infty}^{\infty}$ is a sequence of functions such that every f_k is $\mathcal{F}_k$ measurable and such that $1_{F_k} f_k$ is integrable for every $F_k \in \mathcal{F}_k$ with $\mu(F_k) < \infty$.

Definition 2.6 (Locally integrable martingale, martingale difference sequence, and a predictable sequence)**.** A sequence $\{f_k\}_{k=-\infty}^{\infty}$ of locally integrable functions adapted to a filtration $\{\mathcal{F}_k\}_{k=-\infty}^{\infty}$ is a *martingale* if

$$E[f_{k+1}|\mathcal{F}_k] = f_k.$$

A sequence $\{d_k\}_{k=-\infty}^{\infty}$ of locally integrable functions adapted to a filtration $\{\mathcal{F}_k\}_{k=-\infty}^{\infty}$ is a *martingale difference sequence* if

$$E[d_{k+1}|\mathcal{F}_k] = 0.$$

A sequence of functions $\{v_k\}_{k=-\infty}^{\infty}$ is *predictable* with respect to the filtration $\{\mathcal{F}_k\}_{k=-\infty}^{\infty}$ if each v_k is $\mathcal{F}_{k-1}$ measurable.

For our purposes, the central tools in the martingale tool box are:

Theorem 2.7 (Doob's inequality)**.** *Let* $\{f_k\}_{1 \le k \le K}$ *be a locally integrable martingale. Then*

$$\| \sup_{1 \le k \le K} |f_k| \|_{L^p} \le p' \|f_K\|_{L^p}.$$

The expression $\sup_{1 \le k \le K} f_k$ is called *Doob's maximal function.*

Theorem 2.8 (Burkholder's inequality)**.** *Let* $\{d_k\}_{1 \le k \le K}$ *be a martingale difference sequence. Let* $\{v_k\}_{1 \le k \le K}$ *be a predictable sequence. Then*

$$\|\sum_{k=1}^{K} v_k d_k\|_{L^p} \le (\max\{p, p'\} - 1) \sup_{1 \le k \le K} \|v_k\|_{L^\infty} \|\sum_{k=1}^{K} d_k\|_{L^p}.$$

The expression $\sum_{k=1}^{K} v_k d_k$ is called the *martingale transform* associated with predictable multipliers $\{v_k\}_{k=-\infty}^{\infty}$ of the martingale $\{f_k := \sum_{i=1}^{k} d_i\}_{1 \le k \le K}$.

Typical martingales in dyadic analysis. The basic idea is to recognize something as a martingale and then apply martingale inequalities. In dyadic analysis, a typical martingale and martingale difference sequence have the following form:

Observation 2.9 (Typical martingale difference sequences and martingales in dyadic analysis). *Let $\{\mathcal{S}_k\}_{k=-\infty}^{\infty}$ be a sequence of collections of pairwise disjoint dyadic cubes. Assume that the sequence $\{\mathcal{S}_k\}_{k=-\infty}^{\infty}$ is nested in the sense that for every $S' \in \mathcal{S}_k$, there is $S \in \mathcal{S}_{k-1}$ such that $S' \subseteq S$. For each $S \in \mathcal{S}_k$, let $\mathrm{ch}_{\mathcal{S}}(S) \coloneqq \{S' \in \mathcal{S}_{k+1} : S' \subseteq S\}$. Let $\mathcal{S} \coloneqq \bigcup_{k=-\infty}^{\infty} \mathcal{S}_k$.*

A family $\{d_S\}_{S \in \mathcal{S}}$ of locally integrable functions is a martingale difference sequence (with respect to the filtration generated by $\{\mathcal{S}_k\}_{k=-\infty}^{\infty}$), if and only if, for every $S \in \mathcal{S}$, we have that

- *d_S is supported on S;*
- *d_S is constant on each $S' \in \mathrm{ch}_{\mathcal{S}}(S)$;*
- *$\int_S d_S \, \mathrm{d}\mu = 0$.*

A family $\{f_S\}_{S \in \mathcal{S}}$ of locally integrable functions is a martingale (with respect to the filtration generated by $\{\mathcal{S}_k\}_{k=-\infty}^{\infty}$), if and only if, for every $S \in \mathcal{S}$, we have that

- *f_S is supported on S;*
- *f_S is constant on S;*
- *$\sum_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} \mu(S') f_{S'} = \mu(S) f_S$.*

Example 2.10. a) The sequence $\{\mathcal{D}_k\}_{k=-\infty}^{\infty}$ with $\mathcal{D}_k \coloneqq \{Q \in \mathcal{D} : \ell(Q) = 2^{-k}\}$ is a nested sequence of collections of pairwise disjoint dyadic cubes. The family $\{\langle f \rangle_Q^{\mu} 1_Q\}_{Q \in \mathcal{D}}$ is a martingale, and the Doob maximal function corresponding to it is the dyadic Hardy–Littlewood maximal function. The family $\{D_Q^{\mu} f\}_{Q \in \mathcal{D}}$ with $D_Q^{\mu} f \coloneqq -\langle f \rangle_Q^{\mu} 1_Q + \sum_{Q' \in \mathrm{ch}(Q)} \langle f \rangle_{Q'}^{\mu} 1_{Q'}$ is a martingale difference sequence, and the martingale transform corresponding to it is the dyadic martingale transform.

b) Let $\mathcal{S}$ be a collection of dyadic cubes that contains a maximal cube S_0. Let $\mathrm{ch}_{\mathcal{S}}(S) \coloneqq \{S' \in \mathcal{S} : S' \text{ maximal with } S' \subsetneq S\}$. Define recursively $\mathcal{S}_0 \coloneqq \{S_0\}$, and $\mathcal{S}_{k+1} \coloneqq \bigcup_{S \in \mathcal{S}_k} \mathrm{ch}(S)$. Then, the sequence $\{\mathcal{S}_k\}_{k=0}^{\infty}$ is a nested sequence of collections of pairwise disjoint dyadic cubes.

Note that from the Lebesgue differentiation theorem together with the observation that the sum is a telescoping sum of averages, it follows that every function $f \in L^p(\mu)$ can be decomposed as

$$1_R f = \langle f \rangle_R^{\mu} 1_R + \sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} D_Q^{\mu} f \quad \text{both in } L^p(\mu) \text{ and pointwise almost everywhere.}$$

Martingale differences enter to many problems via this decomposition.

Decoupling. Roughly speaking, decoupling means introducing more independence to the problem at hand. The following decoupling inequality is available for typical martingales in dyadic analysis:

Theorem 2.11 (Decoupling of martingale differences). *Let $1 < p < \infty$. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $\{\mathcal{A}_k\}_{k=-\infty}^{\infty}$ be a refining sequence of countable partitions of X into measurable sets of finite positive measure. For each $A \in \mathcal{A}_k$, let $\mathrm{ch}_{\mathcal{A}}(A) \coloneqq \{A' \in \mathcal{A}_{k+1} : A' \subseteq A\}$. Let $\mathcal{A} \coloneqq \bigcup_{k=-\infty}^{\infty} \mathcal{A}_k$.*

Equip each set $A \in \mathcal{A}$ with the σ-algebra generated by $\{A\} \cup \mathrm{ch}_{\mathcal{A}}(A)$ and with the normalized measure $\frac{1}{\mu(A)} \mu|_A$. Consider the product measure space of the measure

spaces $\{A, \sigma(A \cup \mathrm{ch}_{\mathcal{A}}(A)), \frac{1}{\mu(A)}\mu|_A\}_{A\in\mathcal{A}}$. Let

$$\mathrm{d}\bar{\mu}(y) := \prod_{A\in\mathcal{A}} \frac{1}{\mu(A)} \mathrm{d}\mu|_A(y_A) \quad \text{for } y = \{y_A\}_{A\in\mathcal{A}} \in \prod_{A\in\mathcal{A}} A$$

denote the product measure on it.

Let $\{d_A\}_{A\in\mathcal{A}}$ be a family of functions such that, for every $A \in \mathcal{A}$, the following conditions are satisfied:

- d_A is supported on A;
- d_A is constant on each $A' \in \mathrm{ch}_{\mathcal{A}}(A)$;
- $\int_A d_A \, \mathrm{d}\mu = 0$.

In probabilistic language, this is to say that $\{d_A\}_{A\in\mathcal{A}}$ is a family martingale differences adapted to the family $\mathcal{A}$. Then, we have

$$\frac{1}{\beta_p} \Big(\mathbb{E} \Big\| \sum_{A\in\mathcal{A}} \varepsilon_A 1_A(x) d_A(y_A) \Big\|^p_{L^p(\mathrm{d}\mu(x)\times\mathrm{d}\bar{\mu}(y))} \Big)^{1/p}$$

$$\leq \Big\| \sum_{A\in\mathcal{A}} d_A(x) \Big\|_{L^p(\mathrm{d}\mu(x))}$$

$$\leq \beta_p \Big(\mathbb{E} \Big\| \sum_{A\in\mathcal{A}} \varepsilon_A 1_A(x) d_A(y_A) \Big\|^p_{L^p(\mathrm{d}\mu(x)\times\mathrm{d}\bar{\mu}(y))} \Big)^{1/p}.$$

Here, the expectation $\mathbb{E}$ is taken over independent, unbiased random signs $(\varepsilon_n)_{n=1}^N$.

This decoupling inequality holds also for UMD space valued functions. A variant of it was proven by Hytönen [28] as a corollary of McConnell's [49] decoupling inequality for UMD-valued martingale difference sequences. In the article [B], we give an alternative proof for the decoupling: We define the auxiliary martingale differences $u_A(x, y_A)$ and $v_A(x, y_A)$ by the pair of equations

$$1_A(x) d_A(x) 1_A(y_A) = u_A(x, y_A) + v_A(x, y_A),$$
$$1_A(x) d_A(y_A) 1_A(y_A) = u_A(x, y_A) - v_A(x, y_A),$$

from which the decoupling equality follows by Burkholder's inequality.

2.2. **Vector-valued analysis.** The by-now-usual paradigm of doing Banach-space valued harmonic analysis beyond Hilbert space is replacing the orthogonality of vectors by the unconditionality of martingale differences, and the uniform boundedness of operators by the R-boundedness. This paradigm was pionereed by Burkholder's [3] and Bourgain's [1] characterization stating that the Hilbert transform is bounded on L_E^p if and only if the Banach space E has the UMD property, and by Weis's [69] operator-valued Fourier multiplier theorems.

Definition 2.12 (UMD property). A Banach space $(E, |\cdot|_E)$ is said to have the *UMD (unconditional martingale difference) property* if for some $p \in (1, \infty)$ there exists a constant $\beta_p(E)$ such that

$$\Big\| \sum_{n=1}^N \epsilon_n d_n \Big\|_{L_E^p} \leq \beta_p(E) \Big\| \sum_{n=1}^N d_n \Big\|_{L_E^p}$$

for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ and for all choices of signs $(\epsilon_n)_{n=1}^N \in \{-1, +1\}^N$.

Definition 2.13 (R-boundedness). A family of operators $\mathcal{T} \subseteq \mathcal{L}(E, F)$ from a Banach space $(E, |\cdot|_E)$ to a Banach space $(F, |\cdot|_F)$ is said to be *R-bounded* if for some $p \in (1, \infty)$ there exists a constant $\mathcal{R}_p(\mathcal{T})$ such that

$$\Big(\mathbb{E}|\sum_{n=1}^{N} \varepsilon_n T_n e_n|_F^p\Big)^{1/p} \leq \mathcal{R}_p(\mathcal{T})\Big(\mathbb{E}|\sum_{n=1}^{N} \varepsilon_n e_n|_E^p\Big)^{1/p}$$

for all choices of operators $(T_n)_{n=1}^{N} \subseteq \mathcal{T}$ and vectors $(e_n)_{n=1}^{N} \subseteq E$. Here, the expectation is taken over independent, unbiased random signs $(\varepsilon_n)_{n=1}^{N}$.

A close relative of the UMD property is the Hardy–Littlewood property: Bourgain, and Rubio de Francia (see [2], and [56]) proved that a Köthe function space X with the Fatou property has the UMD property if and only if both X and its function space dual X' have the Hardy–Littlewood property.

Definition 2.14 (Hardy–Littlewood property). A Banach lattice $(E, |\cdot|_E, \leq)$ is said to have the *Hardy–Littlewood property* if for some $p \in (1, \infty)$ there exists a constant $C_{p,E}$ such that

$$\|\sup_{Q \in \mathcal{D}} \langle f \rangle_Q 1_Q\|_{L_E^p} \leq C_{p,E} \|f\|_{L_E^p}$$

for every $f \in L_E^p$ and every finite collection $\mathcal{D}$ of dyadic cubes. Here, the supremum is taken in the lattice order.

The UMD property, the Hardy–Littlewood property, and R-boundedness are independent (up to the involved constants) of the exponent $p \in (1, \infty)$. For an exposition on Banach-space-valued martingales, UMD spaces, and R-boundedness, among other things, see Neerven's lecture notes [66]. The Hardy–Littlewood property is studied by García-Cuerva, Macías, and Torrea in [18] and [19], where, among other things, they obtain various characterizations of the property.

2.3. **Calderón–Zygmund operators.** A *singular integral operator* T is an operator that has an integral representation outside the diagonal of the kernel: For every compactly supported $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, \mathrm{d}y$$

for every x that lies outside the support of f. (Depending on the operator at hand, it may be required that the function f satisfies certain integrability or continuity conditions so that the integration is defined.) A kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the *standard estimates* if it satisfies the following decay and regularity conditions: There exists a constant $\|K\|_{\mathrm{CZ}}$ such that

$$(2.1) \qquad \sup_{\substack{x,y \in \mathbb{R}^d: \\ x \neq y}} |K(x, y)| \cdot |x - y|^d \leq \|K\|_{\mathrm{CZ}},$$

and there exist a Hölder exponent $\alpha \in (0, 1]$ and a constant $\|K\|_{\mathrm{CZ}_\alpha}$ such that

$$(2.2a) \qquad \sup_{\substack{x,x',y \in \mathbb{R}^d: \\ |x-x'| < \frac{1}{2}|x-y|}} |K(x, y) - K(x', y)| \Big(\frac{|x - y|}{|x - x'|}\Big)^\alpha |x - y|^d \leq \|K\|_{\mathrm{CZ}_\alpha}$$

$$(2.2b) \qquad \sup_{\substack{x,y,y' \in \mathbb{R}^d: \\ |y-y'| < \frac{1}{2}|x-y|}} |K(x, y) - K(x, y')| \Big(\frac{|x - y|}{|y - y'|}\Big)^\alpha |x - y|^d \leq \|K\|_{\mathrm{CZ}_\alpha}.$$

An operator $T : L^p \to L^p$ satisfies the *weak boundedness property* if there exists a constant $\|T\|_{\mathrm{WBP}}$ such that

$$(2.3) \qquad \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int 1_Q T(1_Q) \, \mathrm{d}x \le \|T\|_{\mathrm{WBP}}.$$

Definition 2.15 (Calderón–Zygmund operator)**.** A *Calderón–Zygmund operator*, abbreviated as *CZO*, is a singular integral operator whose kernel satisfies the standard estimates and that is bounded from L^p to L^p.

As mentioned in Section 2.2, in the setting of an operator-valued kernel, the suprema of the real numbers in the standard estimates (2.1) and (2.2), and in the weak boundedness property (2.3) are replaced by R-boundedness of the operator family.

3. Median oscillation decomposition

3.1. **Background.**

Median and mean oscillation. The basic idea is to approximate a function locally by a constant. The discrepancy between the function and any constant is quantified and the approximating constant is taken to be any constant that minimizes this discrepancy.

Definition 3.1 (Median and median oscillation)**.** The *relative median oscillation about zero* $(f1_Q)^*(\lambda\mu(Q))$ of a function f on a set Q is defined by

$$(3.1) \qquad (f1_Q)^*(\lambda\mu(Q)) := \min\{r \ge 0 : \mu(Q \cap \{|f| > r\}) \le \lambda\mu(Q)\}.$$

The *median* $m(f; Q)$ of a function f on a set Q is defined as any real number such that

$$(3.2) \quad \mu(Q \cap \{f > m(f, Q)\}) \le \frac{1}{2}\mu(Q) \quad \text{and} \quad \mu(Q \cap \{f < m(f, Q)\}) \le \frac{1}{2}\mu(Q).$$

The *median oscillation* $\omega_\lambda(f; Q)$ of a function f on a set Q is defined by

$$(3.3) \qquad \omega_\lambda(f; Q) := \inf_{c \in \mathbb{R}}((f - c)1_Q)^*(\lambda\mu(Q)).$$

Remark. The *decreasing rearrangement* $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$f^*(t) := \min\{r \ge 0 : \mu(\{|f| > r\}) \le t\}.$$

Thus, by definition, the quantity $(f1_Q)^*(\lambda\mu(Q))$ is the value of the decreasing rearrangement $(1_Q f)^*$ at the point $\lambda\mu(Q)$. In this exposition, we refer to this quantity as 'relative median oscillation about zero'. We also use the term 'quasiminimal' to mean 'minimal up to a constant'. In this terminology, the fact

$$(3.4) \qquad ((f - m(f; Q))1_Q)^*(\lambda\mu(Q)) \approx \inf_{c \in \mathbb{R}}((f - c)1_Q)^*(\lambda\mu(Q))$$

can be phrased as 'each median is a constant about which the relative median oscillation is quasiminimal'.

A median, the relative median oscillation about zero, and the median oscillation are analogous to the *mean* $\langle f \rangle_Q^\mu$, the *relative mean oscillation about zero* $\frac{1}{\mu(Q)} \int_Q |f| \, \mathrm{d}\mu$, and the *mean oscillation* $\psi(f; Q) := \inf_{c \in \mathbb{R}} \frac{1}{\mu(Q)} \int_Q |f - c| \, \mathrm{d}\mu$. In this exposition, we use the term 'relative mean oscillation about zero' to emphasize

TABLE 1. Analogy between median and mean oscillation. It is assumed that $\lambda \in (0, 1/2)$. The properties of median and mean oscillation are well-known.

Relative oscillation (about zero)	$(f1_Q)^*(\lambda\mu(Q))$	$\frac{1}{\mu(Q)} \int_Q \lvert f \rvert \, \mathrm{d}\mu$
Minimal oscillation	$\omega_\lambda(f; Q)$ $:= \inf_{c \in \mathbb{R}} ((f - c)1_Q)^*(\lambda\mu(Q))$	$\psi(f; Q)$ $:= \inf_{c \in \mathbb{R}} \frac{1}{\mu(Q)} \int_Q \lvert f - c \rvert \, \mathrm{d}\mu$
Quasiminimizing center of oscillation	$m(f; Q)$	$\langle f \rangle_Q^\mu$
Quasiminimization	$((f - m(f;Q))1_Q)^*(\lambda\mu(Q))$ $\approx \omega_\lambda(f; Q)$	$\frac{1}{\mu(Q)} \int_Q \lvert f - \langle f \rangle_Q \rvert \, \mathrm{d}\mu$ $\approx \psi(f; Q)$
Control	$\lvert m(f; Q) \rvert \lesssim (f1_Q)^*(\lambda\mu(Q))$	$\lvert \langle f \rangle_Q \rvert \le \frac{1}{\mu(Q)} \int_Q \lvert f \rvert \, \mathrm{d}\mu$
Linearity	$m(f + c; Q) = m(f; Q) + c$	$\langle f + c \rangle_Q^\mu = \langle f \rangle_Q^\mu + c$
Approximation	$\lim_{\substack{Q \in \mathcal{D}: Q \ni x, \\ \ell(Q) \to 0}} m(f; Q) = f(x)$ for μ-a.e $x \in \mathbb{R}^d$	$\lim_{\substack{Q \in \mathcal{D}: Q \ni x, \\ \ell(Q) \to 0}} \langle f \rangle_Q^\mu = f(x)$ for μ-a.e $x \in \mathbb{R}^d$
Triangle inequality	$\omega_\lambda(f + g; Q)$ $\le \omega_{\lambda/2}(f; Q) + \omega_{\lambda/2}(g; Q)$	$\psi(f; Q)$ $\le \psi(f; Q) + \psi(g; Q)$

the analogy between median oscillation and mean oscillation. This analogy is illustrated in Table 3.1. However, median oscillation is more accurate than mean oscillation: Firstly, the median oscillation is controlled by the mean oscillation,

$$(3.5) \qquad (f1_Q)^*(\lambda\mu(Q)) \le \frac{1}{\lambda} \cdot \frac{1}{\mu(Q)} \int_Q \lvert f \rvert \, \mathrm{d}\mu,$$

and, secondly, the median oscillation is controlled by the weak L^1 norm, whereas the mean oscillation is controlled by the strong L^1 norm:

$$(3.6) \quad (f1_Q)^*(\lambda\mu(Q)) \le \frac{1}{\lambda} \frac{\lVert 1_Q f \rVert_{L^{1,\infty}}}{\mu(Q)} \quad \text{whereas} \quad \frac{1}{\mu(Q)} \int_Q \lvert f \rvert \, \mathrm{d}\mu \le \frac{\lVert 1_Q f \rVert_{L^1}}{\mu(Q)}.$$

The realisation of a close relation between median and mean oscillation goes back to John [33] and Strömberg's [61] work. They proved that, for the Lebesgue

measure, the uniform bounds for mean oscillations and median oscillations are comparable: We have

$$\|f\|_{\mathrm{BMO}^1} := \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |f - \langle f \rangle_Q| \, \mathrm{d}x \approx_\lambda \sup_{Q \in \mathcal{D}} \omega_\lambda(f; Q) =: \|f\|_{\mathrm{BMO}_{0,\lambda}}$$

for every $\lambda \in (0, 1/2)$.

Median and mean oscillation decomposition. Lerner [43, 45] obtained the following median oscillation decomposition:

Theorem 3.2 (Median oscillation decomposition, [43, 45])**.** *Assume that μ is doubling. Let S_0 be an initial cube. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Then, there exists a sparse collection $\mathcal{S}$ of dyadic subcubes of S_0 such that*

$$(3.7) \qquad |f - m(f; S_0)| 1_{S_0} \lesssim \sum_{S \in \mathcal{S}} ((f - m(f; S)) 1_S)^* (\lambda \mu(S)) 1_S$$

μ-almost everywhere. The collection $\mathcal{S}$ depends on the measure μ, the initial cube S_0, and the function f. The parameter λ depends on the doubling constant of the measure μ.

The original decomposition by Lerner [43, 45] contains an additional term (a median oscillation maximal function), which was removed by Hytönen [27]. Furthermore, the localization on an initial cube can be removed, as shown by Lerner and Nazarov [47].

By using the estimate $\omega_\lambda(f; Q) \le \frac{1}{\lambda} \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q^\mu| \, \mathrm{d}\mu$ (or by using Lerner's proof with median replaced by mean), the median oscillation decomposition implies the mean oscillation decomposition:

$$(3.8) \qquad |f - \langle f \rangle_{S_0}| 1_{S_0} \lesssim \sum_{S \in \mathcal{S}} \frac{1}{\mu(S)} \int_Q |f - \langle f \rangle_S| \, \mathrm{d}\mu 1_S.$$

A precursor (which has a less sharp dependence on the oscillations) of the decompositions (3.7) and (3.8) was obtained by Fujii [17]. This was based on Garnett and Jones's [20] dyadic reformulation of Carleson's [4] representation theorem for BMO functions.

Domination via median oscillation decomposition. Many operators of harmonic analysis are pointwise dominated by positive averaging operators. By using the median oscillation decomposition, such a domination has been obtained, for example, for:

- dyadic shifts (see [10], [11], [35], [29]),
- square functions (see [44]),
- and Calderón–Zygmund operators (see [30], [46]).

The by-now-standard procedure is as follows. Let T be an operator. Fix a cube S_0. By the median oscillation decomposition,

$$1_{S_0} |T(f 1_{S_0})| \lesssim (T(f 1_{S_0}) 1_{S_0})^* (\lambda \mu(S_0)) + \sum_{S \in \mathcal{S}} \omega_\lambda(1_S T f; S).$$

The oscillation $\omega_\lambda(1_S T f; S)$ is split into a localized (in both the range and the domain side) and a tail part: Decompose $1_S T f = 1_S T(f 1_S) + 1_S T(f 1_{S^c})$ and use the triangle inequality

$$\omega_\lambda(1_S T f; S) \le \omega_{\frac{\lambda}{2}}(1_S T(f 1_S); S) + \omega_{\frac{\lambda}{2}}(1_S T(f 1_{S^c}); S).$$

The localized part is controlled by using the weak L^1 norm as a black box,

$$\omega_{\frac{\lambda}{2}}(1_S T(f 1_S); S) \lesssim_\lambda \|T\|_{L^1 \to L^{1,\infty}} \langle f \rangle_S,$$

whereas the tail part $\omega_{\frac{\lambda}{2}}(1_S T(f 1_{S^c}); S)$ is controlled by exploiting the specific structure of the operator hands-on.

Example 3.3. As an example, we apply this procedure to the dyadic Hardy–Littlewood maximal function Mf. Recall that $Mf \coloneqq \sup_{R \in \mathcal{D}} \langle f \rangle_R 1_R$. Observe that the tail part $1_S M(f 1_{S^c})$ is constant on S, because $1_S M(f 1_{S^c}) = 1_S \sup_{R \in \mathcal{D}: R \supsetneq S} \langle f \rangle_R$, which implies that $\omega_{\lambda/2}(1_S M(f 1_{S^c}); S) = 0$. Thus,

$$\omega_\lambda(1_S Mf; S) \leq \omega_{\lambda/2}(1_S M(1_S f); S) + \omega_{\lambda/2}(1_S M(f 1_{S^c}); S)$$
$$\lesssim_\lambda \|M\|_{L^1 \to L^{1,\infty}} \langle f \rangle_S + 0 = \langle f \rangle_S.$$

Therefore, we have $Mf \lesssim_\lambda \sum_{S \in \mathcal{S}} \langle f \rangle_S 1_S$ almost everywhere. (In passing, we remark that this domination can also be proven by using the principal cubes.)

Pointwise dyadic domination for CZOs. The following pointwise dyadic domination for CZOs was proven by Hytönen, Lacey, and Pérez [30] and Lerner [46]:

Theorem 3.4 (Pointwise dyadic domination for CZOs; [30, 46]). *Let T be a Calderón–Zygmund operator. Let α denote the Hölder exponent in the Hölder condition for the kernel. Let $f : \mathbb{R}^d \to \mathbb{R}$. Fix a cube S_0. Then, there exists a sparse collection $\mathcal{S}_k^u$ in each shifted dyadic system $\mathcal{D}^u$ and for each complexity k such that*

$$1_{S_0} |T(1_{S_0} f)| \lesssim_T \sum_{u \in \{0, \frac{1}{2}, \frac{2}{3}\}^d} \sum_{k=0}^\infty 2^{-\alpha k} \Big(\sum_{S \in \mathcal{S}_k^u} \langle f \rangle_{S^{(k)}} 1_S \Big).$$

The outline of their proof is as follows. First, they use the median oscillation decomposition together with Jawerth and Torchinsky's [32] oscillation estimate,

$$\omega_\lambda(Tf; S) \lesssim (\|T\|_{L^1 \to L^{1,\infty}} + \|K\|_{\mathrm{CZ}_\alpha}) \sum_{k=0}^\infty 2^{-\alpha k} \langle f \rangle_{2^k S},$$

to yield

$$1_{S_0} |T(1_{S_0} f)| \lesssim_T \sum_{k=0}^\infty 2^{-\alpha k} \sum_{S \in \mathcal{S}} \langle f \rangle_{2^k S} 1_S.$$

Then, the geometric averages are reduced to dyadic averages, by using Hytönen, Lacey, and Pérez's [30] observation that for each cube Q, there is a shifted dyadic cube $R \in \mathcal{D}^u$ for some $u \in \{0, \frac{1}{2}, \frac{2}{3}\}^d$ such that $Q \subseteq R$, $2^k Q \subseteq R^{(k)}$, and $\ell(Q) \approx \ell(R)$. Therefore,

$$1_{S_0} |T(1_{S_0} f)| \lesssim_T \sum_{u \in \{0, \frac{1}{2}, \frac{2}{3}\}^d} \sum_{k=0}^\infty 2^{-\alpha k} \sum_{R \in \mathcal{R}_k^u} \langle f \rangle_{R^{(k)}} 1_R.$$

3.2. **Articles [A] and [E].** One by-now-standard application of the median oscillation decomposition is to dominate an operator by positive averaging operators, as explained in Section 3.1. Thus, it is of interest to extend the median oscillation decomposition to more general settings. The purpose of the article [A] is to extend the median oscillation decomposition to Banach space valued functions, and the purpose of the article [E] to non-doubling measures. The combination of these extensions reads as follows:

Theorem 3.5 (Median oscillation decomposition for Banach space valued functions and non-doubling measures). *Let $(E, |\cdot|_E)$ be a Banach space, and μ be a locally finite Borel measure. Let $0 < \lambda < \kappa < 1/2$. Let S_0 be an initial cube. Let $f : \mathbb{R}^d \to E$ be a strongly measurable function. Then, there exists a sparse collection $\mathcal{S}$ of dyadic subcubes of S_0 such that*

$$|f - c_\kappa(f; \hat{S}_0)| 1_{S_0} \lesssim \sum_{S \in \mathcal{S}} \left(((f - c_\kappa(f; S)) 1_S)^* (\lambda \mu(S)) + |c_\kappa(f; S) - c_\kappa(f; \hat{S})| \right) 1_S$$

μ-almost everywhere. The collection $\mathcal{S}$ depends on the measure μ, the initial cube S_0, and the function f. Here, the vector $c_\kappa(f; S)$ is any vector-valued median (notion which is defined in the next paragraph) of f on a cube S, and the dyadic cube $\hat{S}$ is the dyadic parent of the cube S.

First, we consider the vector-valued extension. Many definitions and computations in the real-valued setting translate verbatim to the vector-valued setting, with only the typographical change of replacing the absolute value $|\cdot|$ by the Banach space norm $|\cdot|_E$. This is the case for the definition of median oscillation, which is given by (3.3). However, the definition of median, given by (3.1), makes explicit reference to the order of the real numbers. We recall that any real-valued median quasiminimizes the median oscillation in the sense of the equation (3.4). The key point of the article [A] is the following observation:

Observation 3.6. *Any constant that quasiminimizes the median oscillation has the properties of median that are summarized in Table 3.1.*

Now, a *vector-valued median* $c_\lambda(f; Q)$ of $f : \mathbb{R}^d \to E$ on Q is defined as any vector $c_\lambda(f; Q) \in E$ that quasiminimizes the median oscillation in the sense of the equation (3.4), that is:

$$((f - c_\lambda(f; Q)) 1_Q)^* (\lambda \mu(Q)) \approx \omega_\lambda(f; Q).$$

Once we are equipped with vector-valued median, we can adapt Lerner's original proof to extend the median oscillation decomposition to Banach space valued functions.

Next, we consider the non-doubling extension. The decomposition (3.8) can not hold for non-doubling measures: If it held, then it would imply the John–Nirenberg inequality for non-doubling measures,

$$\|f\|_{\mathrm{BMO}(\mu)} \approx_p \|f\|_{\mathrm{BMO}^p(\mu)} \quad \text{for all } f \in L^p(\mu) \text{ and all measures } \mu,$$

which is known to be false. Hence, we are forced to modify the decomposition. The key point of the article [E] is the following observation:

Observation 3.7. *The quantity $((f - m(f; \hat{Q})) 1_Q)^* (\lambda \mu(Q))$ is analogous to the quantity $\frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_{\hat{Q}}^\mu| \, \mathrm{d}\mu$, which appears in the definition of the (martingale) dyadic BMO norm for non-doubling measures.*

Once we use the quantity $((f - m(f; \hat{Q})) 1_Q)^* (\lambda \mu(Q))$ in place of the quantity $((f - m(f; Q)) 1_Q)^* (\lambda \mu(Q))$, we can adapt Lerner's original proof to extend the decomposition to non-doubling measures μ.

3.3. **Applications.** Equipped with the vector-valued median oscillation decomposition, we can verbatim run through the proof of Hytönen, Lacey, and Pérez's [30] and Lerner's [46] domination theorem (see Theorem 3.4). Combining this with Conde-Alonso and Rey's [9] domination theorem yields:

Theorem 3.8 (Pointwise dyadic domination for CZOs)**.** *Let T be a vector-valued Calderón–Zygmund operator. Let α denote the Hölder exponent in the Hölder condition for the kernel. Let $f : \mathbb{R}^d \to E$ be supported on a cube S_0. Then, there exists a sparse collection $\mathcal{S}^u$ in each shifted dyadic system $\mathcal{D}^u$ such that*

$$1_{S_0}|T(1_{S_0}f)|_E \lesssim_T \sum_{u \in \{0, \frac{1}{2}, \frac{2}{3}\}^d} \sum_{S \in \mathcal{S}^u} \langle f \rangle_S 1_S$$

almost everywhere.

As in the real-valued setting, this together with the A_2 estimate for the operator $f \mapsto \sum_{S \in \mathcal{S}} \langle f \rangle_S 1_S$, which was proven in three lines by Cruz-Uribe, Martell, and Pérez [10], implies:

Corollary 3.9 (A_2 theorem for vector-valued CZOs, in [A])**.** *Let T be a vector-valued CZO. Then,*

$$\|T\|_{L^2_E(w) \to L^2_E(w)} \lesssim_T [w]_{A_2}$$

for all weights $w \in A_2$.

We can use the non-doubling median oscillation decomposition together with Conde-Alonso and Rey's [9] domination theorem to yield an alternative proof for the following domination theorem by Lacey [37]:

Corollary 3.10 (Alternative proof for Lacey's [37] domination theorem)**.** *Let μ be a (possibly non-doubling) locally finite Borel measure. Let T_ϵ^μ be a dyadic martingale transform associated with the coefficients $\{\epsilon_Q\}_{Q \in \mathcal{D}}$ satisfying $|\epsilon_Q| \leq 1$ for every $Q \in \mathcal{D}$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be supported on a cube S_0. Then, there exists a μ-sparse collection $\mathcal{S}$ of dyadic subcubes of S_0 such that*

$$1_{S_0}|T_\epsilon^\mu(f 1_{S_0})| \lesssim_T \sum_{S \in \mathcal{S}} \langle f \rangle_S^\mu 1_S$$

μ-almost everywhere.

3.4. **Recent developments.** Hytönen, Lacey, and Pérez [30] and Lerner [46] proved that CZOs are pointwise dominated by sparse dyadic operators of complexity k (see Theorem 3.4), which are operators of the form

$$f \mapsto \sum_{S \in \mathcal{S}} \langle f \rangle_{S^{(k)}} 1_S$$

associated with a non-negative integer k (which is called *complexity*) and a sparse collection $\mathcal{S}$. Further, Lerner [45] proved that CZOs are dominated in any Banach function space norm (in particular, in the L^p norm) by sparse dyadic operators of complexity zero, and, at present, it is known that this domination holds even pointwise:

Theorem 3.11 (Pointwise dyadic domination for CZOs)**.** *Let T be a Calderón–Zygmund operator. Let α denote the Hölder exponent in the Hölder condition for*

the kernel. Let $f : \mathbb{R}^d \to \mathbb{R}$ be supported on a cube S_0. Then, there exists a sparse collection $\mathcal{S}^u$ in each shifted dyadic system $\mathcal{D}^u$ such that

$$1_{S_0}|T(1_{S_0}f)| \lesssim_T \sum_{u \in \{0, \frac{1}{2}, \frac{2}{3}\}^d} \sum_{S \in \mathcal{S}^u} \langle f \rangle_S 1_S.$$

By now, there are three proofs for this theorem:

- by Conde-Alonso and Rey [9]. They proved that each positive dyadic operator of arbitrary complexity is pointwise dominated by an operator of complexity zero, with a linear dependence in the complexity. (Combining this with the result of Hytönen, Lacey, and Pérez [30] and Lerner [46] yields the theorem.)
- by Lerner and Nazarov [47];
- by Lacey [37]. This approach works also for kernels such that the modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ in the regularity estimate

$$|K(x,y) - K(x',y)| \le \|K\|_{\mathrm{CZ}_\omega} \omega\Big(\frac{|x-x'|}{|x-y|}\Big)\frac{1}{|x-y|^d} \quad \text{for } \frac{|x-x'|}{|x-y|} < \frac{1}{2}$$

  satisfies the Dini condition $\int_0^\infty \omega(t)\frac{\mathrm{d}t}{t} < \infty$, whereas the other approaches work only under the stronger regularity condition $\int_0^\infty \omega(t)\log\frac{1}{t}\frac{\mathrm{d}t}{t} < \infty$, which originates from the summability of the series $\sum_{k=0}^\infty \omega(2^{-k})k$.

We remark that Lerner and Nazarov's [47] and Lacey's [37] proof each work for Banach space valued functions. Thus, these methods, which have appeared after the article [A], can be alternatively used to prove pointwise dyadic domination theorems in the vector-valued setting, without using vector-valued median.

The key observation behind all the pointwise domination results discussed in this section is that the weak $L^1 \to L^{1,\infty}$ estimate implies that the operator $1_Q T(f 1_Q)$ is dominated by the average $\langle f \rangle_Q$, except for a small portion of the cube Q:

$$\Big|\big\{|1_Q(Tf1_Q)| > 2\|T\|_{L^1 \to L^{1,\infty}}\langle|f|\rangle_Q\big\}\Big| \le \frac{1}{2}|Q|.$$

This is implicit in the estimate $\omega_\lambda(1_Q T(f1_Q); Q) \lesssim_\lambda \|T\|_{L^1 \to L^{1,\infty}}\langle f \rangle_S$, which is used to yield pointwise domination via the median oscillation decomposition.

However, the median oscillation decomposition is tailored for functions, not for operators. Thus, applying it to the function Tf takes into account the range side (the function Tf) but ignores the domain side (the function f). Roughly speaking, the improved domination results stem from exploiting the structure of the operator hands-on so that both the domain and the range side are controlled.

4. Testing conditions

The basic idea is to characterize the boundedness of an operator $T : L^p \to L^q$ by its action on a restricted class of functions. We study quantitative norm inequalities: The aim is to understand how the operator norm $\|T\|_{L^p \to L^q}$ depends on certain relevant quantities, such as the constants in the testing conditions, or the constant in the estimates for the kernel of an integral operator.

4.1. Testing conditions for CZOs. For CZOs, the pioneering testing condition was obtained by David and Journé [14]:

Theorem 4.1 (Global $T1$ theorem for CZOs, [14]). *Let T be a singular integral operator whose kernel satisfies the standard estimates. Then the operator $T : L^2 \to L^2$ is bounded if and only if it satisfies the weak boundedness property,*

(4.1) $$\langle 1_Q, T1_Q \rangle \lesssim |Q| \quad \text{for every } Q \in \mathcal{D},$$

and the testing conditions

(4.2)
$$T1 \in BMO,$$
$$T^*1 \in BMO.$$

As shown by Stein [60], the global conditions (4.2) together with the weak boundedness property (4.1) are equivalent to the local conditions (4.3). Therefore:

Theorem 4.2 (Local $T1$ theorem for CZOs). *Let T be a singular integral operator whose kernel satisfies the standard estimates. Then the operator $T : L^2 \to L^2$ is bounded if and only if it satisfies the testing conditions:*

(4.3)
$$\|1_Q T(1_Q)\|_{L^2} \lesssim \mathfrak{T}|Q|^{1/2},$$
$$\|1_Q T^*(1_Q)\|_{L^2} \lesssim \mathfrak{T}^*|Q|^{1/2}$$

for every $Q \in \mathcal{D}$.

The idea of *a global Tb theorem* was introduced by David, Journé and Semmes [13], and McIntosh and Meyer [50]: Instead of testing the operators T and T^* against the function 1 as in Theorem 4.1, they are tested against non-degenerate functions b and b^*. The idea of *a local Tb theorem* was introduced by Christ [8]: Instead of testing the operators T and T^* against the family of function $\{1_Q\}_{Q \in \mathcal{D}}$ as in Theorem 4.2, they are tested against family of functions $\{b_Q\}_{Q \in \mathcal{D}}$ and $\{b_Q^*\}_{Q \in \mathcal{D}}$, such that each function b_Q satisfies certain non-degeneracy and integrability conditions on its cube. One advantage of more flexible testing conditions is that they are easier to check.

The subsequent development of the $T1$ theorems for CZOs includes, among other things, proceeding along (or combining) the following lines:

- Weakening the integrability conditions $b_Q, b_Q^*, Tb_Q, T^*b_Q^*$ in local Tb theorems.
- Extending $T1/Tb$ theorems to non-homogeneous setting, pioneered by Tolsa, and by Nazarov, and Treil, and Volberg [52].
- Extending $T1/Tb$ theorems to vector-valued functions, pioneered by Figiel [16], and to operator-valued kernels, pioneered by Hytönen and Weis [24].

For the current state of art, see Hytönen and Nazarov [23] for a local Tb theorem with the weakest integrability conditions, Hytönen and Vähäkangas [31] for a non-homogeneous, operator-valued, local Tb theorem, and Lacey and Martikainen [38] for a non-homogeneous, local Tb theorem with the L^2 integrability conditions.

4.2. **Two weight testing conditions for positive operators.** For an operator $T(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ and its adjoint $T^*(\cdot\omega) : L^{q'}(\omega) \to L^{p'}(\sigma)$, the direct and the dual testing condition (4.3) reads

(4.4)
$$\|1_Q T(1_Q \sigma)\|_{L^q(\omega)} \lesssim \sigma(Q)^{1/p}$$
$$\|1_Q T^*(1_Q \omega)\|_{L^{p'}(\sigma)} \lesssim \omega(Q)^{1/q'}.$$

Characterizing the $L^p(\sigma) \to L^q(\omega)$ boundedness by means of such testing conditions was pioneered by Sawyer [58, 57]. He used them to characterize the boundedness of a large class of integral operators with non-negative kernels, in particular, fractional integrals and Poisson integrals, and of dyadic maximal operators.

We consider the following positive dyadic operators:

Definition 4.3 (Dyadic maximal operator). The *dyadic maximal operator* $M_\lambda(\cdot \sigma)$ is defined by

$$M_\lambda(f\sigma) := \sup_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q.$$

Definition 4.4 (Positive dyadic operator). The *positive dyadic operator* $A_\lambda(\cdot \sigma)$ is defined by

$$A_\lambda(f\sigma) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q,$$

We use the convention that, for a dyadic operator T defined by a formula that involves an indexing (typically a summation or supremum) over the collection $\mathcal{D}$ of dyadic cubes, the localized operator T_R is the operator defined by the same formula, except that the indexing is now over the localized collection $\mathcal{D}(R) := \{Q \in \mathcal{D} : Q \subseteq R\}$. Following this convention, the localizations $M_{\lambda,R}(\cdot \sigma)$ and $A_{\lambda,R}(\cdot \sigma)$ are defined by

$$M_{\lambda,R}(f\sigma) := \sup_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} \lambda_Q \int_Q f \, d\sigma 1_Q \quad \text{and} \quad A_{\lambda,R}(f\sigma) := \sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} \lambda_Q \int_Q f \, d\sigma 1_Q.$$

For dyadic maximal operators, the $L^p(\sigma) \to L^q(\omega)$ boundedness was characterized by Sawyer [57]:

Theorem 4.5 (Two-weight testing condition for dyadic maximal operators, [57]). *Let $1 < p \le q < \infty$. Let $\{\lambda_Q\}_{Q \in \mathcal{D}}$ be positive real-numbers. Then*

$$\|M_\lambda(\cdot \sigma)\|_{L^p(\omega) \to L^q(\omega)} \approx_{p,q} \mathfrak{M},$$

where $\mathfrak{M}$ is the least constant in the direct Sawyer testing condition:

$$\|M_{\lambda,R}(1_R\sigma)\|_{L^q(\omega)} \le \mathfrak{M}\sigma(R)^{1/p}$$

for every $R \in \mathcal{D}$.

For positive dyadic operators, the $L^p(\sigma) \to L^q(\omega)$ boundedness was first characterized for $p = q = 2$ by Nazarov, Treil, and Volberg [51] by the Bellman function technique, and for $1 < p \le q < \infty$ by Lacey, Sawyer, and Uriarte-Tuero [41] by techniques that are similar to the ones used by Sawyer [58]:

Theorem 4.6 (Two-weight testing condition for positive dyadic operators, [51, 41]). *Let $1 < p \le q < \infty$. Let $\{\lambda_Q\}_{Q \in \mathcal{D}}$ be positive real numbers. Then,*

$$\|A_\lambda(\cdot \sigma)\|_{L^p(\sigma) \to L^q(\omega)} \approx_{q,p} \mathfrak{A} + \mathfrak{A}^*,$$

where $\mathfrak{A}$ and $\mathfrak{A}^*$ are the least constants in the direct and the dual Sawyer testing condition:

$$\|A_{\lambda,R}(1_R\sigma)\|_{L^q(\omega)} \le \mathfrak{A}\sigma(R)^{1/p},$$

$$\|A_{\lambda,R}^*(1_R\omega)\|_{L^{p'}(\sigma)} \le \mathfrak{A}^*\omega(R)^{1/q'}$$

for every $R \in \mathcal{D}$.

Alternative proofs were obtained by Treil [65] by splitting the summation over dyadic cubes in the dual pairing by the condition '$\sigma(Q)(\langle f \rangle_Q^\sigma)^p > \omega(Q)(\langle g \rangle_Q^\omega)^{q'}$', and by Hytönen [27] by splitting the summation by using parallel stopping cubes. The technique of parallel stopping cubes originates from the work of Lacey, Sawyer, Shen, and Uriarte-Tuero [40] on the two-weight boundedness of the Hilbert transform.

Flavour of the proof by the technique of stopping cubes. In the unweighted case, the characterization for the boundedness of positive dyadic operators can be proven by the technique of stopping cubes as follows. For $f \in L^p(\sigma)$, let $\mathcal{F}$ be the stopping family $\mathcal{F}$ defined by the stopping children

$$\mathrm{ch}_{\mathcal{F}}(F) := \{\ F' \subseteq F \text{ maximal with } \langle f \rangle_{F'}^\mu > 2\langle f \rangle_F^\mu\ \}.$$

Now,

$$\|A_\lambda(f\mu)\|_{L^q(\mu)}$$

$$= \|\sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q) = F}} \lambda_Q \int_Q f\,\mathrm{d}\mu\|_{L^q(\mu)} \qquad \sum_{Q \in \mathcal{D}} = \sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q) = F}}$$

$$\approx_q \Big(\sum_{F \in \mathcal{F}} \|\sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q) = F}} \lambda_Q \int_Q f\,\mathrm{d}\mu\|_{L^q(\mu)}^q\Big)^{1/q} \quad \text{the } L^p \text{ variant of Pythagoras' theorem}$$

$$\lesssim_p \Big(\sum_{F \in \mathcal{F}} \big(\langle f \rangle_F^\mu\big)^q \|A_{\lambda,F}(1_F\sigma)\|_{L^q(\mu)}^q\Big)^{1/q} \quad \langle f \rangle_Q^\mu \lesssim \langle f \rangle_{\pi_{\mathcal{F}}(Q)}^\mu$$

$$\lesssim \mathfrak{A}\Big(\sum_{F \in \mathcal{F}} \big(\langle f \rangle_F^\mu\big)^q \mu(F)^{q/p}\Big)^{1/q} \quad \text{the direct testing condition}$$

$$\leq \mathfrak{A}\Big(\sum_{F \in \mathcal{F}} \big(\langle f \rangle_F^\mu\big)^p \mu(F)\Big)^{1/p} \quad |\cdot|_{\ell^q} \leq |\cdot|_{\ell^p} \text{ whenever } q \geq p$$

$$\lesssim_{p,C} \mathfrak{A}\|f\|_{L^p(\mu)}. \quad \text{the Carleson embedding theorem}$$

In the two weight case, the obstacle to applying the L^p variant of Pythagoras' theorem is that a σ-sparse collection (resulting from the stopping condition related to $f \in L^p(\sigma)$) fails in general to be ω-sparse. This obstacle can be circumvented by using parallel stopping cubes: The norm inequality is dualized, a stopping family $\mathcal{F}$ related to $f \in L^p(\sigma)$ and a stopping family $\mathcal{G}$ related to $g \in L^{q'}(\omega)$ are introduced, and the sum over dyadic cubes Q is splitted by the condition '$\pi_{\mathcal{F}}(Q) \subseteq \pi_{\mathcal{G}}(Q)$'.

5. Dyadic representation for CZOs and the $T1$ theorem

The basic idea is to represent a complicated operator by means of simple operators and deduce properties for the complicated operator from the properties of the simple operators.

5.1. **Background.**

Dyadic shifts and paraproducts. The *shifted Haar projection* D_K^i of *complexity* $(i+1)$ associated with a dyadic cube $K \in \mathcal{D}$ is defined by

$$D_K^i f := \sum_{\substack{I \in \mathcal{D}: I \subseteq K, \\ \ell(I) = 2^{-i}\ell(K)}} D_I f.$$

The *averaging operator* A_K is defined by

$$A_K f(x) := \frac{1_K(x)}{|K|} \int_K a_K(x,y) f(y) \, dy.$$

Definition 5.1. A *(cancellative) dyadic shift* S_a^{ji} associated with parameters (j,i) and kernels $\{a_K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\}_{K \in \mathcal{D}}$ is defined by

$$S_a^{ji} f := \sum_{K \in \mathcal{D}} D_K^j A_K D_K^i f.$$

In the definition, we assume that the family $\{a_K(x,x') : K \in \mathcal{D}, x \in K, x' \in K\}$ of real-numbers is bounded. We use the shorthand $\sup\{|a|\} := \sup_{x,x' \in K, K \in \mathcal{D}} |a_K(x,x')|$.

Definition 5.2. The *dyadic paraproduct* Π_b associated with a function $b : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\Pi_b f := \sum_{Q \in \mathcal{D}} D_Q b \langle f \rangle_Q.$$

Dyadic representation for CZOs. Dyadic shifts and paraproducts are dyadic model operators for CZOs: Petermichl [54] proved that the Hilbert transform can be represented as a particular dyadic shift averaged over randomized dyadic systems, and Hytönen [26] that every CZO can be represented as a series of dyadic shifts and paraproducts averaged over randomized dyadic systems:

Theorem 5.3 (Dyadic representation theorem, [26, 25])**.** *Let T be a Calderón–Zygmund operator. Then, for some dyadic shifts $S_{a_\omega^{ij}, \mathcal{D}^\omega}^{ij}$ and for the dyadic paraproducts $\Pi_{T1, \mathcal{D}^\omega}$ and $\Pi_{T^*1, \mathcal{D}^\omega}$, we have*

$$\langle g, Tf \rangle \approx \mathbb{E}_\omega \big\langle g, \big(\sum_{i \geq 0, j \geq 0} S_{a_\omega^{ij}, \mathcal{D}^\omega}^{ij} + \Pi_{T1, \mathcal{D}^\omega} + (\Pi_{T^*1, \mathcal{D}^\omega})^* \big) f \big\rangle$$

for all $g \in C_0^1$ and $f \in C_0^1$. Moreover,

$$\sup\{|a_\omega^{ij}|\} \lesssim_{d,\alpha} (\|K\|_{CZ_0} + \|K\|_{CZ_\alpha} + \|T\|_{WBP}) P_{d,\alpha}(\max\{i,j\}) 2^{-\alpha \max\{i,j\}}$$

for some polynomial $P_{d,\alpha}$.

For a detailed proof of the dyadic representation theorem, see Hytönen's [25] lecture notes on the A_2 theorem. The dyadic representation is closely related to $T1$ theorems: In fact, the dyadic shifts originate from singling out certain terms as model operators in the decomposition involved in the proof of $T1$ theorems.

Flavour of the proof of the dyadic representation theorem. A rough outline of the proof is as follows. (We assume that $T1 = 0$ and $T^*1 = 0$, or, in other words, we ignore the extraction of paraproducts.) First, we single out the dyadic shifts. By using the Haar decompositions $g = \sum_J D_J g$ and $f = \sum_I D_I f$, we have $\langle g, Tf \rangle = \sum_{I,J} \langle D_J g, TD_I f \rangle$. We use a dyadic system in which every pair I and J of dyadic cubes has a common dyadic ancestor. Now, we can rearrange the summation according to the least common dyadic ancestor $K = I \vee J$ of I and J, and according to the side lengths of I and J relative to K:

$$\sum_{I,J} = \sum_{i \geq 0, j \geq 0} \sum_K \sum_{\substack{I,J: I \vee J = K \\ \ell(I) = 2^{-i}\ell(K) \\ \ell(J) = 2^{-j}\ell(K)}}.$$

By using this, we can write

$$\langle g, Tf \rangle = \sum_{I,J} \langle D_J g, T D_I f \rangle = \sum_{i \geq 0, j \geq 0} \langle g, \sum_K D_K^j A_K^{ji} D_K^i f \rangle = \sum_{i \geq 0, j \geq 0} \langle g, S_{a^{ij}}^{ij} f \rangle,$$

from which we can read out the kernels

(5.1) $$a_K^{ij}(x', x) := |K| \sum_{\substack{I,J:I \vee J = K \\ \ell(I) = 2^{-i}\ell(K) \\ \ell(J) = 2^{-j}\ell(K)}} h_J(x') h_I(x) \langle h_J, T h_I \rangle.$$

Next, we estimate the kernels of the dyadic shifts. A randomization of dyadic systems is introduced in order to force that every dyadic cube lies far away from the boundary of its dyadic ancestors. This together with the standard estimates implies an exponential decay for the kernels of dyadic shifts:

(5.2) $$\sup\{|a_K^{ij}(x', x)| : K \in \mathcal{D}, x' \in K, x \in K\} \lesssim_T P_{d,\alpha}(\max\{i, j\}) 2^{-\alpha \max\{i, j\}}$$

for some polynomial $P_{d,\alpha}$.

Boundedness of dyadic shifts. Dyadic shifts are bounded on L^p for $1 < p < \infty$. Indeed, they are bounded on L^2, which follows from Pythagoras' theorem together with the orthogonality of Haar projections,

$$\|D^{ji} f\|_{L^2} = \Big(\sum_{K \in \mathcal{D}} \|D_K^j A_K D_K^i f\|_{L^2}^2 \Big)^{1/2} \lesssim \Big(\sum_{K \in \mathcal{D}} \|A_K D_K^i f\|_{L^2}^2 \Big)^{1/2}$$

$$\leq \sup\{|a|\} \Big(\sum_{K \in \mathcal{D}} \|D_K^i f\|_{L^2}^2 \Big)^{1/2} = \sup\{|a|\} \| \sum_{K \in \mathcal{D}} D_K^i f \|_{L^2} = \sup\{|a|\} \|f\|_{L^2}.$$

Their boundedness from L^1 to $L^{1,\infty}$ can be proven by using the Calderón–Zygmund decomposition together with their boundedness on L^2. Now, from the Marcinkiewicz interpolation theorem, it follows that dyadic shifts are bounded on L^p for $1 < p \leq 2$, and hence, by duality, on L^p for $2 \leq p < \infty$.

Note that, in order to sum the series in the representation, it is required that the bound for dyadic shifts does not grow too rapidly with the complexity. The weak-L^1 bound with an exponentially increasing dependence on the complexity was proven by Lacey, Petermichl, and Reguera [39], and with linearly increasing dependence by Hytönen [26]:

Theorem 5.4 (Dyadic shifts are bounded with linear depence on complexity, [26]). *Let S_a^k be a dyadic shift of complexity k and associated with the kernels $\{a_K\}_{K \in \mathcal{D}}$. Then*

$$\|S_a^k\|_{L^p \to L^p} \lesssim_p k \cdot \sup\{|a|\}.$$

Testing conditions for dyadic paraproducts. The characterization of the $L^p \to L^p$ boundedness of dyadic paraproducts is classical:

Theorem 5.5 (Testing conditions for real-valued dyadic paraproducts). *We have*

$$\|\Pi_b\|_{L^p \to L^p} \approx \|b\|_{BMO^p}.$$

T1 theorem via the dyadic representation. Combining Theorem 5.3 and Theorem 5.4 yields:

Corollary 5.6 (Quantitative $T1$ theorem for CZOs, [25]). *Let T be a Calderón–Zygmund operator. Then*

$$\|T\|_{L^p \to L^p} \lesssim_{d,\alpha,p} \|K\|_{CZ_0} + \|K\|_{CZ_\alpha} + \|T\|_{WBP} + \|T1\|_{BMO} + \|T^*1\|_{BMO}.$$

5.2. **Article [B].** In the article [B], we study operator-valued CZOs. The operator-valued setting follows the by-now-usual paradigm of doing Banach-space valued harmonic analysis beyond Hilbert space: Orthogonality of vectors is replaced with unconditionality of martingale differences, and uniform boundedness of operators with R-boundedness. The proof of the dyadic representation theorem for real-valued CZOs works verbatim for operator-valued CZOs:

Theorem 5.7 (Operator-valued dyadic representation theorem, [B]). *Let E be a Banach space. Let T be an operator-valued CZO. Then, for some operator-valued dyadic shifts $S^{ij}_{a^{ij}_\omega, \mathcal{D}^\omega}$ and for the operator-valued dyadic paraproducts $\Pi_{T1, \mathcal{D}^\omega}$ and $\Pi_{T^*1, \mathcal{D}^\omega}$, we have*

$$\langle g, Tf \rangle \eqsim \mathbb{E}_\omega \Big\langle g, \Big(\sum_{i \geq 0, j \geq 0} S^{ij}_{a^{ij}_\omega, \mathcal{D}^\omega} + \Pi_{T1, \mathcal{D}^\omega} + (\Pi_{T^*1, \mathcal{D}^\omega})^* \Big) f \Big\rangle.$$

for all $g \in C^1_0(\mathbb{R}^d; \mathbb{R}) \otimes E^*$ and $f \in C^1_0(\mathbb{R}^d; \mathbb{R}) \otimes E$. Moreover,

$$\mathcal{R}\{a^{ij}_\omega\} \lesssim_{d,\alpha} \big(\mathcal{R}_{CZ_0} + \mathcal{R}_{CZ_\alpha} + \mathcal{R}_{WBP} \big) 2^{(1/\epsilon)} 2^{-(1-\epsilon)\alpha \max\{i,j\}}$$

for any auxiliary parameter ϵ with $0 < \epsilon < 1$.

Remark. For computational simplicity, in the operator-valued case, the result is proven with the decay factor $2^{(1/\epsilon)} 2^{-(1-\epsilon)\alpha \max\{i,j\}}$. Nevertheless, as in the scalar-valued case, the result also holds with the decay factor $P_{d,\alpha}(\max\{i,j\}) 2^{-\alpha \max\{i,j\}}$ for some polynomial $P_{d,\alpha}$. The specific decay factor originates from a specific choice of the boundary function in the definition of goodness.

However, as seen from the equality (5.1), the family $\{a_K\}_{K \in \mathcal{D}}$ of kernels involved in a dyadic shift S_a and also the function $T1$ involved in the paraproduct Π_{T1} are now operator-valued. This leads to the following operators:

Definition 5.8 (Operator-valued dyadic shifts). Let E be a Banach space. A *dyadic shift S^{ji}_a* associated with parameters (j,i) and operator-valued kernels $\{a_K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(E)\}_{K \in \mathcal{D}}$ is defined by

$$S^{ji}_a f := \sum_{K \in \mathcal{D}} D^j_K A_K D^i_K f.$$

It is assumed that the $\mathcal{R}$-bound for the family of operators,

$$\mathcal{R}\{a\} := \mathcal{R}\{a_K(x, x') \in \mathcal{L}(E) : K \in \mathcal{D}, x \in K, x' \in K\},$$

is finite.

Definition 5.9 (Operator-valued dyadic paraproducts). Let E be a Banach space. The *dyadic paraproduct* associated with an operator-valued function $b : \mathbb{R}^d \to \mathcal{L}(E)$ is defined by

$$\Pi_b f := \sum_{Q \in \mathcal{D}} D_Q b \langle f \rangle_Q.$$

The main results of the article [B] are about the boundedness of dyadic shifts and paraproducts:

Theorem 5.10 (Operator-valued dyadic shifts are bounded, [B]). *Let E be a UMD space. Let S^k_a be an operator-valued dyadic shift associated with a family of kernels $\{a_K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(E)\}_{K \in \mathcal{D}}$ and having the complexity k. Then*

$$\|S^k_a\|_{L^p_E \to L^p_E} \lesssim_p k \beta_p(E)^2 \mathcal{R}\{a\}.$$

Theorem 5.11 (Boundedness of operator-valued dyadic paraproduct is characterized by the direct L^∞ testing condition, [B]). *Let E be a Banach space. Let Π_b be an operator-valued paraproduct associated with a function $b \colon \mathbb{R}^d \to \mathcal{L}(E)$. Then*

$$\|\Pi_b\|_{L^p_E \to L^p_E} \approx_p \mathfrak{P},$$

where $\mathfrak{P}$ is the least constant in the direct L^∞ testing condition:

$$\|\Pi_{b,R} f\|_{L^p_E} \le \mathfrak{P}\|f\|_{L^\infty_E(R)}|R|^{1/p}$$

for every $R \in \mathcal{D}$ and $f \in L^\infty_E(R)$.

Note that, in the real-valued case, this reduces to the characterization by the BMO condition (see Theorem 5.5) because in that case Burkholder's inequality implies that

$$\Big\| \sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} D_Q b \langle f \rangle_Q \Big\|_{L^p} \le \beta_p \|f\|_{L^\infty(R)} \Big\| \sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} D_Q b \Big\|_{L^p}$$

$$= \beta_p \|f\|_{L^\infty(R)} \|1_R (b - \langle b \rangle_R)\|_{L^p} \le \beta_p \|b\|_{BMO^p} \|f\|_{L^\infty(R)} |R|^{1/p}.$$

Flavour of the proof of the boundedness of dyadic shifts by decoupling of martingale differences. The outline of a typical proof involving $\mathcal{R}$-boundedness is to randomize, use $\mathcal{R}$-boundedness, and remove the randomization. The UMD property is typically used in introducing or removing the randomization.

The twist in our proof is rewriting the integration by means of an auxiliary probability measure, which has the advantage that it can be integrated out: By using the normalized infinite product measure $\mathrm{d}\bar\mu(x) = \prod_{K \in \mathcal{D}} \frac{1}{\mu(K)} \mathrm{d}\mu|_K(x_K)$ with $x = (x_K)_{K \in \mathcal{D}}$, we rewrite

$$\langle g, S^{ji}_a f \rangle = \sum_K \int g D^j_K A_a D^i_K f \, \mathrm{d}\mu = \sum_K \int D^j_K g A_a D^i_K f \, \mathrm{d}\mu$$

$$= \sum_K |K| \int D^j_K g(y_K) \int a_K(y_K, x_K) \frac{1}{|K|} \mathrm{d}\mu|_K(x_K) D^i_K f(x_K) \frac{1}{|K|} \mathrm{d}\mu|_K(y_K)$$

$$= \int \int \int \Big(\sum_K 1_K(z) D^j_K g(y_K) a_K(y_K, x_K) D^i_K f(x_K) \Big) \mathrm{d}\bar\mu(y) \, \mathrm{d}\bar\mu(x) \, \mathrm{d}\mu(z).$$

By introducing random signs using the identity $\sum_K = \sum_{K,L} \delta_{KL} = \sum_{K,L} \mathbb{E}(\epsilon_L \epsilon_K)$, applying Hölder's inequality, and applying the definition of R-boundedness, we obtain

$$\langle g, S^{ji}_a f \rangle \le \Big(\mathbb{E} \Big\| \sum_L \epsilon_L 1_L(z) D^j_K g(y_K) \Big\|_{L^p_E(\mathrm{d}\bar\mu(y)\,\mathrm{d}\mu(z))} \Big)^{1/p'}$$

$$\times \mathcal{R}\{a\} \times \Big(\mathbb{E} \Big\| \sum_K \epsilon_K 1_K(z) D^i_K f(x_K) \Big\|^p_{L^p_E(\mathrm{d}\bar\mu(x)\,\mathrm{d}\mu(z))} \Big)^{1/p}.$$

Now, the key observation is that $\{D^i_K f\}_{K \in \mathcal{D}}$ becomes a martingale difference sequence (see Observation 2.9), once the dyadic scales in the collection $\mathcal{D}$ are separated by the integer $(i + 1)$. Then, we can remove the randomization by using the decoupling of martingale differences (see Theorem 2.11).

5.3. **Corollaries.** By using the direct L^∞ testing condition, we can give a new proof for the following sufficient condition:

Corollary 5.12 (A new proof of a sufficient condition for the boundedness of operator-valued paraproducts, [28])**.** *Let $p \in (1, \infty)$. Let E be a UMD space. Let $b : \mathbb{R}^d \to \mathcal{L}(E)$. Assume that b takes values in a UMD subspace $\mathcal{U}$ of $\mathcal{L}(E)$. Then*

$$\|\Pi_b f\|_{L^p_E \to L^p_E} \lesssim_p \beta_p(E)^2 \beta_p(\mathcal{U}) \|b\|_{BMO^p_\mathcal{U}}.$$

This sufficient condition for the boundedness of the paraproduct $\Pi_b : L^p_E \to L^p_E$ associated with an operator-valued function $b : \mathbb{R}^d \to \mathcal{L}(E)$ was proven by Hytönen [28] by using interpolation and decoupling of martingale differences. Precursors of this operator-valued result (under stronger assumptions) were obtained by Hytönen and Weis [24], based on unpublished ideas of Bourgain recorded by Figiel and Wojtaszczyk [15] in the case of a scalar-valued function $b : \mathbb{R}^d \to \mathbb{R}$.

Similarly as in the real valued case, the dyadic representation theorem together with the boundedness of dyadic shifts and paraproduct gives a new proof for an operator-valued global $T1$ theorem:

Corollary 5.13 (A new proof of a $T1$ theorem for operator-valued kernels)**.** *Let T be an operator-valued CZO. Assume that $T1 \in BMO^p_\mathcal{U}$ and $T^*1 \in BMO^{p'}_{\mathcal{U}^*}$ for some UMD subspaces $\mathcal{U} \subset \mathcal{L}(E)$ and $\mathcal{U}^* \subset \mathcal{L}(E^*)$. Then*

$$\|T\|_{L^p_E \to L^p_E} \lesssim_{d,p,\alpha,\mathcal{U},\mathcal{U}^*} (\mathcal{R}_{CZ_0} + \mathcal{R}_{CZ_\alpha} + \mathcal{R}_{WBP} + \|T1\|_{BMO^p_\mathcal{U}} + \|T^*1\|_{BMO^{p'}_{\mathcal{U}^*}}) \beta_p(E)^2.$$

This $T1$ theorem is a particular case of Hytönen's operator-valued, nonhomogeneous, global Tb theorem [28]. Earlier results of this type include the first vector-valued $T1$ theorem by Figiel [16], and the first operator-valued $T1$ theorem by Hytönen and Weis [24].

5.4. **Related developments.** Pott and Stoica [55] study the dependence of the operator norm $\|T\|_{L^p_E \to L^p_E}$ of vector-valued Calderón–Zygmund operators on the UMD constant $\beta_p(E)$. First, by using the Bellman function technique, they prove:

Theorem 5.14 (Self-adjoint vector-valued dyadic shifts depend linearly on the UMD constant [55])**.** *Let $1 < p < \infty$. Let E be a UMD space. Let S^k be a self-adjoint dyadic shift with the complexity k. Then*

$$\|S^k\|_{L^p_E(\mathbb{R}) \to L^p_E(\mathbb{R})} \lesssim k 2^{k/2} \beta_p(E).$$

Then, by using the dyadic representation theorem, they pass the estimate to CZOs: $\|T\|_{L^p_E(\mathbb{R}) \to L^p_E(\mathbb{R})} \lesssim_T \beta_p(E)$ whenever $T1 = T^*1 = 0$ and the kernel is even and satisfies the Hölder condition with the Hölder exponent $\alpha > 1/2$. (The condition $T1 = T^*1 = 0$ implies that the paraproducts vanish, the condition that the kernel is even implies that only self-adjoint dyadic shifts appear, and the condition $\alpha > 1/2$ implies that the series converges.)

Both we (Theorem 5.10) and Pott and Stoica (Theorem 5.14) prove an estimate for vector-valued dyadic shifts. However, the goals are different: Our goal is an estimate with a linear dependence in complexity, whereas theirs is an estimate with a linear dependence in the UMD constant. It is interesting that in the statements there is a tradeoff between complexity and UMD constant: On the one hand, our estimate depends linearly on the complexity, whereas theirs exponentially; on the other hand, their estimate depends linearly on the UMD constant, whereas ours depends quadratically.

5.5. **Open questions.** A well-known open question is whether the operator norm of vector-valued Calderón–Zygmund operators depends linearly on the UMD constant? Especially, does the operator norm of the Hilbert transform depend linearly on the UMD constant? As in Pott and Stoica's proof strategy, an affirmative answer to this question would follow via the dyadic representation theorem from an affirmative answer to the following open question:

Question 5.15. *Does the operator norm of vector-valued dyadic shifts satisfy an estimate that is linear in the UMD constant and polynomial in the complexity? Does the operator norm of vector-valued paraproducts satisfy an estimate that is linear in the UMD constant?*

In the article [B], the $L_E^p(\mu) \to L_E^p(\mu)$ boundedness of operator-valued dyadic paraproducts Π_b^μ is characterized by the direct L^∞ testing condition, in the case of an arbitrary (possibly non-UMD) Banach space E but a doubling measure μ. (This characterization is stated in the case of the Legesgue measure as Theorem 5.11, but the proof of the characterization works verbatim also in the case of any doubling measure.) Does the characterization by means of the L^∞ testing condition hold even for an arbitrary (possibly non-doubling) measure? In particular:

Question 5.16. *Let E be a Banach space with the UMD property. Let μ be a locally finite (possibly non-doubling) measure. Let $b : \mathbb{R}^d \to \mathcal{L}(E)$ be a locally integrable function. Then, is the boundedness of the dyadic paraproduct $\Pi_b^\mu : L_E^p(\mu) \to L_E^p(\mu)$, which is defined by*

$$\Pi_b^\mu f := \sum_{Q \in \mathcal{D}} D_Q^\mu b\langle f\rangle_Q^\mu 1_Q,$$

characterized by the direct L^∞ testing condition?

6. Testing condition for positive operators: Case $L^p \to L^q$ with
$$1 < q < p$$

In this section, the coefficients $\{\lambda_Q\}_{Q \in \mathcal{D}}$ are non-negative real numbers.

6.1. **Background.**

Linear case. Generalizing Hedberg and Wolff's [21] dyadic non-linear potential associated with the fractional integral operator, Cascante, Ortega, and Verbitsky [6] introduce *the discrete Wolff potential*

$$W_{\lambda,\sigma}^{p'}[\omega] := \sum_{Q \in \mathcal{D}} \lambda_Q \sigma(Q)\Big(\frac{1}{\sigma(Q)}\sum_{R \subseteq Q} \lambda_R \sigma(R)\omega(R)\Big)^{p'-1} 1_Q$$

associated with the dyadic positive operator $A_\lambda(\cdot\,\sigma) : L^p(\sigma) \to L^q(\omega)$ defined by

$$A_\lambda(f\sigma) := \sum_{Q \in \mathcal{D}} \lambda_Q \int f \, d\sigma 1_Q.$$

In another paper [7], they characterize the boundedness of $A_\lambda(\cdot\,\sigma) : L^p(\sigma) \to L^q(\sigma)$ for $0 < q < p < \infty$ and $p > 1$ by means of the discrete Wolff potential $W_{\lambda,\sigma}^{p'}[\omega]$, under the additional assumption that the pair (σ, λ) satisfies the *dyadic logarithmic bounded oscillation condition*. Tanaka [62] characterizes the boundedness without this additional assumption, but for the more restricted range of exponents $1 < q < p < \infty$, which is easier in that duality can be used:

Theorem 6.1 (Characterization by means of discrete Wolff potentials, [62])**.** *Let* $1 < q < p < \infty$. *Define* $r \in (1, \infty)$ *by* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. *Then,*

$$\|A_\lambda(\cdot\sigma)\| \approx_p \|\big(W_{\lambda,\sigma}^{p'}[\omega]\big)^{1/p'}\|_{L^r(\omega)} + \|\big(W_{\lambda,\omega}^{q}[\sigma]\big)^{1/q}\|_{L^r(\sigma)}.$$

From the proof of Tanaka's characterization [62], we can single out the following characterization by sequential testing conditions, which can be viewed as a generalization of the Sawyer testing conditions (in fact, the Sawyer testing conditions correspond to the sequential testing conditions with the value $r = \infty$ of the auxiliary exponent):

Theorem 6.2 (Characterization by sequential testing conditions, [C])**.** *Let* $1 < q < p < \infty$. *Define an auxiliary exponent* $r \in (1, \infty)$ *by* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. *Then,*

$$\|A_\lambda(\cdot\sigma)\| \approx_p \mathfrak{A} + \mathfrak{A}^*,$$

where $\mathfrak{A}$ *and* $\mathfrak{A}$ *are the least constants in the sequential testing conditions:*

$$\left\|\left\{\frac{\|A_F(1_F\sigma)\|_{L^q(\omega)}}{\sigma(F)^{1/p}}\right\}_{F\in\mathcal{F}}\right\|_{\ell^r(\mathcal{F})} \leq \mathfrak{A}$$

$$\left\|\left\{\frac{\|A_G(1_G\omega)\|_{L^{p'}(\sigma)}}{\omega(G)^{1/q'}}\right\}_{G\in\mathcal{G}}\right\|_{\ell^r(\mathcal{G})} \leq \mathfrak{A}^*$$

for every σ*-sparse collection* $\mathcal{F} \subseteq \mathcal{D}$, *and* ω*-sparse collection* $\mathcal{G} \subseteq \mathcal{D}$.

The dyadic maximal operator $M_\lambda(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ is defined by

$$M_\lambda(f\sigma) := \sup_{Q\in\mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q.$$

Its $L^p(\sigma) \to L^q(\omega)$ boundedness for the exponents satisfying $0 < q < p < \infty$ and $p > 1$ is characterized by Verbitsky [67]:

Theorem 6.3 ([67])**.** *Let* $0 < q < p < \infty$ *and* $p > 1$. *For each finite subcollection* $\mathcal{Q} \subseteq \mathcal{D}$ *of dyadic cubes, define the auxiliary function* $M_{\lambda,\mathcal{Q}}(\sigma)$ *by*

$$M_{\lambda,\mathcal{Q}}(\sigma)(x) := \inf_{\substack{Q\in\mathcal{Q}:\\Q\ni x}} \sup_{\substack{R\in\mathcal{D}:\\R\subseteq Q\\R\ni x}} \lambda_R\sigma(R).$$

Then the maximal function $M_\lambda(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ *is bounded if and only if*

$$(6.1) \qquad \int \sup_{Q\in\mathcal{Q}} \Big(\frac{1_Q}{\sigma(Q)} \int_Q M_{\lambda,\mathcal{Q}}(\sigma)^q \, d\omega\Big)^{\frac{q}{p-q}} M_{\lambda,\mathcal{Q}}(\sigma)^q \, d\omega \lesssim 1$$

for all finite subcollections $\mathcal{Q} \subseteq \mathcal{D}$ *of dyadic cubes.*

For each collection $\mathcal{E} = \{E(Q) \subseteq Q\}$ of pairwise disjoint sets, the linearized maximal operator $M_{\mathcal{E},\lambda}(\cdot\sigma)$ is defined by

$$M_{\mathcal{E},\lambda}(f\sigma) := \sum_{Q\in\mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_{E(Q)}.$$

The linearized maximal operator is related to the maximal operator via the observation that for each function f there exists pairwise disjoint sets $\mathcal{E}_f = \{E(Q) \subseteq Q\}$ such that $M_\lambda(f\sigma) = M_{\mathcal{E}_f,\lambda}(f\sigma)$. Therefore,

$$\|M_\lambda(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)} = \sup_{\mathcal{E}}\|M_{\mathcal{E},\lambda}(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)}.$$

Flavour of the proof of the sufficiency of sequential testing conditions for positive dyadic operators in the unweighted case. For $f \in L^p(\mu)$, let $\mathcal{F}$ be the stopping family $\mathcal{F}$ defined by the stopping children

$$\operatorname{ch}_{\mathcal{F}}(F) := \{ \; F' \subseteq F \text{ maximal with } \langle f\rangle_{F'}^\mu > 2\langle f\rangle_F^\mu \; \}.$$

Now,

$$\|A_\lambda(f\mu)\|_{L^q(\mu)}$$

$$= \Big\| \sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q)=F}} \lambda_Q \int_Q f \, \mathrm{d}\mu \Big\|_{L^q(\mu)} \qquad \sum_{Q \in \mathcal{D}} = \sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q)=F}}$$

$$\approx_q \Big(\sum_{F \in \mathcal{F}} \Big\| \sum_{\substack{Q \in \mathcal{D}: \\ \pi_{\mathcal{F}}(Q)=F}} \lambda_Q \int_Q f \, \mathrm{d}\mu \Big\|_{L^q(\mu)}^q \Big)^{1/q} \qquad \text{the } L^p \text{ variant of Pythagoras'}$$

$$\lesssim_p \Big(\sum_{F \in \mathcal{F}} \big(\langle f\rangle_F^\mu\big)^q \|A_{\lambda,F}(1_F\mu)\|_{L^q(\mu)}^q \Big)^{1/q} \qquad \langle f\rangle_Q^\mu \lesssim \langle f\rangle_{\pi_{\mathcal{F}}(Q)}^\mu$$

$$\lesssim \Big(\sum_{F \in \mathcal{F}} \big(\langle f\rangle_F^\mu\big)^p \mu(F) \Big)^{1/p}$$

$$\times \Big(\sum_{F \in \mathcal{F}} \Big(\frac{\|A_{\lambda,F}(1_F\mu)\|_{L^q(\mu)}}{\mu(F)^{1/p}} \Big)^r \Big)^{1/r} \qquad \text{Hölder's inequality}$$

$$\lesssim \mathfrak{A}\Big(\sum_{F \in \mathcal{F}} \big(\langle f\rangle_F^\mu\big)^p \mu(F) \Big)^{1/p} \qquad \text{the direct sequential testing condition}$$

$$\lesssim_{p,C} \mathfrak{A}\|f\|_{L^p(\mu)}. \qquad \text{the Carleson embedding theorem}$$

Multilinear case. The *bilinear dyadic positive operator* $A_\lambda(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$ is defined by

$$A_\lambda(f_1\sigma_1, f_2\sigma_2) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f_1 \, \mathrm{d}\sigma_1 \int_Q f_2 \, \mathrm{d}\sigma_2 1_Q.$$

The *bi(sub)linear maximal operator* $M_\lambda(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$ is defined by

$$M_\lambda(f_1\sigma_1, f_2\sigma_2) := \sup_{Q \in \mathcal{D}} \lambda_Q \int_Q f_1 \, \mathrm{d}\sigma_1 \int_Q f_2 \, \mathrm{d}\sigma_2 1_Q.$$

The *bilinear linearized maximal operator* $M_{\mathcal{E},\lambda}(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$ associated with pairwise disjoint sets $\mathcal{E} = \{E(Q) \subseteq Q\}_{Q \in \mathcal{D}}$ is defined by

$$M_{\mathcal{E},\lambda}(f_1\sigma_1, f_2\sigma_2) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f_1 \, \mathrm{d}\sigma_1 \int_Q f_2 \, \mathrm{d}\sigma_2 1_{E(Q)}.$$

Note that, similarly to the linear case, for each pair of functions (f_1, f_2) there exists pairwise disjoint sets $\mathcal{E}_{f_1,f_2}$ such that

$$M_\lambda(f_1\sigma_1, f_2\sigma_2) = M_{\mathcal{E}_{f_1,f_2},\lambda}(f_1\sigma_1, f_2\sigma_2),$$

from which it follows that

$$\|M_\lambda(\cdot\sigma_1, \cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^q(\omega)} = \sup_{\mathcal{E}} \|M_{\mathcal{E},\lambda}(\cdot\sigma_1, \cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^q(\omega)}.$$

Li and Sun [48] characterized the boundedness of $A_\lambda(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^{p_3'}(\omega)$ for the exponents $p_1, p_2, p_3 \in (1, \infty)$ with the restriction that $\frac{1}{p_i} + \frac{1}{p_j} \geq 1$ for every $i \neq j$, and Tanaka [63] with the weaker restriction $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$.

The idea is to reduce the bilinear case to the linear case. A bilinear operator can be reduced to a linear operator by fixing one of the arguments: From the bilinear operator $A_\lambda(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$, we obtain the localized linear operator $A_{\lambda,R}(\frac{1_R}{\sigma_1(R)^{1/p_1}}\sigma_1, \cdot\sigma_2) : L^{p_2}(\sigma) \to L^q(\omega)$ given by

$$A_{\lambda,R}\Big(\frac{1_R}{\sigma_1(R)^{1/p_1}}\sigma_1, f_2\sigma_2\Big) := \frac{1}{\sigma_1(R)^{1/p_1}} \sum_{Q \in \mathcal{D} : Q \subseteq R} \lambda_Q \sigma_1(Q) \int_Q f_2 \, d\sigma_2 1_Q.$$

The linear operator is then characterized by the Sawyer testing conditions or the discrete Wolff potential depending on the exponents (Theorem 4.6 for $1 < p \le q < \infty$ or Theorem 6.1 for $1 < q < p < \infty$). In this way, Li and Sun, and Tanaka prove:

Theorem 6.4 (Characterization for bilinear positive dyadic operators; [48], [63])**.** *Let $p_1, p_2, p_3 \in (1, \infty)$. Assume that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \ge 1$. Then, the bilinear operator*

$$A_\lambda(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^{p_3'}(\sigma_3)$$

is bounded if and only if the linear operator

$$A_{\lambda,R}\Big(\frac{1_R}{\sigma_i(R)^{1/p_i}}\sigma_i, \cdot\sigma_j\Big) : L^{p_j}(\sigma_j) \to L^{p_k'}(\sigma_k)$$

is bounded for every $R \in \mathcal{D}$ and every permutation (i, j, k) of the indices $(1, 2, 3)$.

For bilinear linearized maximal operators $M_{\mathcal{E},\lambda}(\cdot\sigma_1, \cdot\sigma_2) : L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^q(\omega)$, a similar characterization was proven by Li and Sun [48] for the exponents $p_1, p_2, q \in (1, \infty)$ under the restriction $\frac{1}{p_1} \ge \frac{1}{q}$ and $\frac{1}{p_2} \ge \frac{1}{q}$.

6.2. **Article [C].** The discrete Wolff potential, defined by (6.1), relies explicitly on the structure of positive dyadic operators. Also, in Tanaka's [62] work, the necessity of the sequential testing conditions for the boundedness of dyadic positive operators is proven by using this discrete Wolff potential.

In the article [C], we realise that the sequential testing condition and the Wolff potential can be used for any positive linear operator:

Proposition 6.5 (Sequential testing condition and a Wolff potential for any positive linear operator, [C])**.** *Let $1 < q < p < \infty$. Define the auxiliary exponent $r \in (1, \infty)$ by $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. Let $T(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ be any positive linear operator. Let $T_R(\cdot\sigma)$ denote the localization $1_R T(\cdot 1_R\sigma)$ or the localization $T(\cdot 1_R\sigma)$. Define the sequential testing constant $\mathfrak{T}_r$ by*

$$(6.2) \qquad \mathfrak{T}_r := \sup_{\substack{\mathcal{F} \subseteq \mathcal{D}: \\ \mathcal{F} \ \sigma\text{-sparse}}} \left| \left\{ \frac{\|T_F(1_F\sigma)\|_{L^q(\omega)}}{\sigma(F)^{1/p}} \right\}_{F \in \mathcal{F}} \right|_{\ell^r(\mathcal{F})},$$

and the abstract Wolff potential $W^q_{T,\sigma}[\omega]$ by

$$(6.3) \qquad W^q_{T,\sigma}[\omega] := \sup_{Q \in \mathcal{D}} \frac{1_Q}{\sigma(Q)} \|T_Q(1_Q\sigma)\|^q_{L^q(\omega)}.$$

Then, the sequential testing constant (6.2) and a norm bound for the abstract Wolff potential (6.3) are equivalent,

$$\mathfrak{T}_r \approx_{p,q} \|\big(W^q_{T,\sigma}[\omega]\big)^{1/q}\|_{L^r(\sigma)}.$$

Futhermore, either of them is necessary for the boundedness of the operator $T(\cdot\sigma):$
$L^p(\sigma) \to L^q(\omega)$,

$$\|\left(W^q_{T,\sigma}[\omega]\right)^{1/q}\|_{L^r(\sigma)} \eqsim_{q,r} \mathfrak{T}_r \lesssim_{p,q} \|T(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)}.$$

Then, in the article [C], we use sequential testing conditions systematically to characterize the $L^p(\sigma) \to L^q(\omega)$ boundedness of linearized maximal operators and positive dyadic operators and their bilinear analogues. In the linear case, our characterizations offer alternatives to Tanaka's [62] characterization for linear positive dyadic operators by means of discrete Wolff potentials, and Verbitsky's [67] characterization for dyadic maximal operators. For example, for linearized maximal operators, our characterization reads as follows:

Theorem 6.6 (Boundedness of linearized maximal operators is characterized by the direct sequential testing condition, [C])**.** *Let $1 < q < p < \infty$. Define the auxiliary exponent $r \in (1,\infty)$ by $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. Then*

$$\|M_{\mathcal{E},\lambda}(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)} \eqsim_p \mathfrak{T},$$

where $\mathfrak{T}$ is the least constant in the sequential testing condition:

$$\left|\left\{\frac{\|M_{\mathcal{E},\lambda,F}(1_F\sigma)\|_{L^q(\omega)}}{\sigma(F)^{1/p}}\right\}_{F\in\mathcal{F}}\right|_{\ell^r(\mathcal{F})} \leq \mathfrak{T}$$

for every σ-sparse collection $\mathcal{F}$.

This yields an alternative characterization for the boundedness of dyadic maximal operators via the identity

$$\|M_\lambda(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)} = \sup_{\mathcal{E}}\|M_{\mathcal{E},\lambda}(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)}.$$

We remark that Verbitsky's condition (see Theorem 6.3) for the boundedness of dyadic maximal operators is more tractable in that it does not involve linearizing sets $\mathcal{E}$.

In the bilinear case, we obtain characterizations for linearized dyadic maximal operators and positive dyadic operators for a range of exponents for which there was no characterization until now. For positive dyadic operators, our characterization reads as follows:

Theorem 6.7 (Boundedness of bilinear positive dyadic operators is characterized by sequential testing conditions, [C])**.** *Let $p_1, p_2, p_3 \in (1,\infty)$. Assume that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. We have*

$$\|A(\cdot\sigma_1, \cdot\sigma_2)\|_{L^{p_1}(\sigma_1)\times L^{p_2}(\sigma_2)\to L^{p'_3}(\sigma_3)} \eqsim \sum_{\phi\in S_3} \mathfrak{T}^\phi$$

where S_3 is the set of all permutations ϕ of $(1,2,3)$, and the sequential testing constant $\mathfrak{T}^\phi$ is defined by

$$\mathfrak{T}^\phi := \sup_{\mathcal{F}_{\phi(2)},\mathcal{F}_{\phi(3)}}\left|\left\{\left|\left\{\frac{\|A_{F_{\phi(2)}}(\sigma_{\phi(2)},\sigma_{\phi(3)})\|_{L^{p'_{\phi(1)}}(\sigma_{\phi(1)})}}{\sigma_{\phi(2)}(F_{\phi(2)})^{1/p_{\phi(2)}}\sigma_{\phi(3)}(F_{\phi(3)})^{1/p_{\phi(3)}}}\right\}_{\substack{F_{\phi(2)}\in\mathcal{F}_{\phi(2)}:\\F_{\phi(2)}\subseteq F_{\phi(3)}}}\right|_{\ell^{r^\phi_2}}\right\}_{F_{\phi(3)}\in\mathcal{F}_{\phi(3)}}\right|_{\ell^r},$$

where

- *the supremum is over all $\sigma_{\phi(2)}$-sparse collections $\mathcal{F}_{\phi(2)}$ and all $\sigma_{\phi(3)}$-sparse collections $\mathcal{F}_{\phi(3)}$;*

- *the exponents are defined by*

$$\frac{1}{r_2^\phi} + \frac{1}{p_{\phi(1)}} + \frac{1}{p_{\phi(2)}} = 1, \quad and \quad \frac{1}{r} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.$$

Tanaka [63] suggests that some kind of three-weight generalization of the discrete Wolff potential is needed to characterize the $L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^{p'_3}(\sigma_3)$ boundedness in the case $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$. However, it is not obvious how to generalize the discrete Wolff potential (6.1), and, as far as we know, no such generalization was found until now. By exploiting sequential testing conditions, we are able to generalize the discrete Wolff potential in the article [C]. Later, Tanaka [64] extends the pattern to the n-linear case, with $n \geq 3$, from where the notation used in the statement of the following theorem is taken.

Theorem 6.8 (Characterization by a discrete Wolff potential in the bilinear case, [C]). *Let $p_1, p_2, p_3 \in (1, \infty)$. Assume that $\frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{p_1} < 1$. Let ϕ be a permutation of $(1, 2, 3)$. Define the auxiliary exponents by*

$$\frac{1}{r_0} = 1,$$
$$\frac{1}{r_1^\phi} + \frac{1}{p_{\phi(1)}} = 1,$$
$$\frac{1}{r_2^\phi} + \frac{1}{p_{\phi(1)}} + \frac{1}{p_{\phi(2)}} = 1,$$
$$\frac{1}{r_3} + \frac{1}{p_{\phi(1)}} + \frac{1}{p_{\phi(2)}} + \frac{1}{p_{\phi(3)}} = 1.$$

Define the auxiliary coefficients recursively by

$$(\lambda_0^\phi)_Q = \lambda_Q,$$
$$(\lambda_1^\phi)_Q = (\lambda_0^\phi)_Q \sigma_{\phi(1)}(Q) \Big(\frac{1}{\sigma_{\phi(1)}(Q)} \sum_{R:R\subseteq Q} (\lambda_0^\phi)_R \prod_{i=1}^{3} \sigma_{\phi(i)}(R) \Big)^{r_1^\phi/r_0 - 1},$$
$$\vdots$$
$$(\lambda_3^\phi)_Q = (\lambda_2^\phi)_Q \sigma_{\phi(3)}(Q) \Big(\frac{1}{\sigma_{\phi(3)}(Q)} \sum_{R:R\subseteq Q} (\lambda_2^\phi)_R \sigma_{\phi(3)}(R) \Big)^{r_3/r_2^\phi - 1}.$$

Then, the discrete Wolff potential $W_{\sigma_{\phi(1)}, \sigma_{\phi(2)}}^\phi [\sigma_{\phi(3)}]$ is defined by

$$W_{\sigma_{\phi(1)}, \sigma_{\phi(2)}}^\phi [\sigma_{\phi(3)}] := \sum_Q (\lambda_2^\phi)_Q 1_Q.$$

We have

$$\|A_\lambda(\cdot \sigma_1, \cdot \sigma_2)\|_{L^{p_1}(\sigma_1) \times L^{p_2}(\sigma_2) \to L^{p'_3}(\sigma_3)} \approx_p \sum_{\phi \in S_3} \|\big(W_{\sigma_{\phi(1)}, \sigma_{\phi(2)}}^\phi [\sigma_{\phi(3)}] \big)^{1/r_2^\phi} \|_{L^{r_3}(\sigma_{\phi(3)})}.$$

6.3. **Recent developments.** Tanaka [64] extends his Theorem 6.4 and our Theorem 6.8 to multilinear positive dyadic operators $A_\lambda(\cdot \sigma_1, \ldots, \cdot \sigma_k) : L_{\sigma_1}^{p_1} \times \cdots \times L_{\sigma_k}^{p_k} \to L^q(\omega)$ with $k \geq 3$. Vuorinen [68] introduces an alternative testing condition, *a randomized testing condition*, to characterize the $L^p(\sigma) \to L^q(\omega)$ boundedness of positive dyadic operators for any $p, q \in (1, \infty)$.

7. Testing conditions for positive operators: Operator-valued kernels

7.1. Background.

Vector-valued positive dyadic operators have been studied little. Let $\{\lambda_Q\}_{Q \in \mathcal{D}}$ be non-negative real numbers. In the two weight case, Scurry [59] considers the sequence-valued operator $S_{\lambda,s}(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ defined by

$$(7.1) \qquad S_{\lambda,s} := \Big(\sum_{Q \in \mathcal{D}} \big(\lambda_Q \int_Q f \, \mathrm{d}\sigma\big)^s 1_Q \Big)^{1/s}.$$

and its localization $S_{\lambda,s,R}(\cdot\sigma)$ defined by $S_{\lambda,r,R}(f\sigma) := \Big(\sum_{\substack{Q \in \mathcal{D}: \\ Q \subseteq R}} \big(\lambda_Q \int_Q f \, \mathrm{d}\sigma\big)^s 1_Q \Big)^{1/s}$.

Note that the endpoints correspond to maximal operators and positive dyadic operators: $S_{\lambda,s=\infty}(\cdot\sigma) = M_\lambda(\cdot\sigma)$ and $S_{\lambda,s=1}(\cdot\sigma) = A_\lambda(\cdot\sigma)$. Scurry views this operator as an operator $L^p(\sigma) \to L^q_{\ell^s(\mathcal{D})}(\omega)$. Thus, the adjoint operator $U_\lambda(\cdot\omega) : L^{q'}_{\ell^{s'}(\mathcal{D})}(\omega) \to L^{p'}(\sigma)$ is given by

$$U_\lambda(\cdot\omega) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q g_Q \, \mathrm{d}\omega 1_Q.$$

By adapting Lacey, Sawyer, and Uriarte-Tuero's [41] proof of the characterization for positive dyadic operators, he obtains:

Theorem 7.1 (Sequence-valued two weight testing conditions, [59])**.** *Let $1 < p \leq q < \infty$. Let $s \in (1, \infty)$. Let $S_{\lambda,s}(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ be defined as in (7.1). Then,*

$$\|S_{\lambda,s}(\cdot\sigma)\|_{L^p(\sigma) \to L^q(\omega)} \eqsim_{p,q} \mathfrak{A} + \mathfrak{A}^*,$$

where $\mathfrak{A}$ and $\mathfrak{A}^*$ are the least constants in the testing conditions:

$$\|S_{\lambda,s,R}(1_R\sigma)\|_{L^q(\omega)} \leq \mathfrak{A}\sigma(R)^{1/p},$$

$$\|U_{\lambda,R}(g\omega)\|_{L^{p'}(\sigma)} \leq \mathfrak{A}^* \|g\|_{L^\infty_{\ell^{s'}(\mathcal{D})}(R,\omega)} \omega(R)^{1/q'}$$

for every $R \in \mathcal{D}$, and $g = (g_Q)_{Q \in \mathcal{D}} \in L^\infty_{\ell^{s'}(\mathcal{D})}(R, \omega)$.

In the unweighted case, Nazarov, Treil, and Volberg [52] consider the operator $S_{\lambda,s}(\cdot\mu) : L^p(\mu) \to L^q(\mu)$ defined by

$$(7.2) \qquad S_{\lambda,s}(\cdot\mu) := \Big(\sum_{Q \in \mathcal{D}} \lambda_Q \big(\langle f \rangle_Q^\mu\big)^s 1_Q \Big)^{1/s}.$$

Note that the operators (7.1) and (7.2) are reparameterizations of each other: Scurry's operator with the coefficients $\{\lambda_Q\}_{Q \in \mathcal{D}}$ corresponds to Nazarov, Treil, and Volberg's operator with the coefficients $\{\lambda_Q^s \mu(Q)^s\}_{Q \in \mathcal{D}}$. By using the Bellman function technique, Nazarov, Treil, and Volberg prove:

Theorem 7.2 (Sequence-valued unweighted testing condition, [52])**.** *Let $p \in (1, \infty)$. Let $s \in (1, \infty)$, and let $\{\lambda_Q\}$ be non-negative real numbers. Then the following assertions are equivalent:*

- *The operator $S_{\lambda,s}(\cdot\mu) : L^p(\mu) \to L^p(\mu)$ is bounded.*
- *The operator $S_{\lambda,s}(\cdot\mu) : L^p(\mu) \to L^p(\mu)$ satisfies the direct testing condition:*

$$\|S_{\lambda,s,R}(1_R\mu)\|_{L^p_{\ell^s(\mathcal{D},\beta)}(\mu)} \leq \mathfrak{A}\mu(R)^{1/p}$$

for every $R \in \mathcal{D}$.

- *The coefficients $\{\lambda_Q\}$ satisfy the Carleson condition:*

$$\sum_{Q\in\mathcal{D}:Q\subseteq R} \lambda_Q\mu(Q) \leq \mathfrak{C}\mu(R)$$

for every $R\in\mathcal{D}$.

Moreover,

$$\|S_{\lambda,s}(\cdot\mu)\|_{L^p(\mu)\to L^p(\mu)} \eqsim_{p,s} \mathfrak{A} \eqsim_{p,s} \mathfrak{C}^{1/p}.$$

7.2. Article [D]. Positive dyadic operators $A_\lambda(\cdot\sigma)$ are readily interpreted in the operator-valued setting: Let $(C,|\cdot|_C,\leq)$ and $(D,|\cdot|_D,\leq)$ be Banach lattices. Let $\{\lambda_Q : C \to D\}_{Q\in\mathcal{D}}$ be positive linear operators. Then, the operator $A_\lambda(\cdot\sigma)$ and its localization $A_{\lambda,R}(\cdot\sigma)$ are defined by the same formulas as before,

$$(7.3) \qquad A_\lambda(f\sigma) := \sum_{Q\in\mathcal{D}} \lambda_Q \int_Q f \, \mathrm{d}\sigma 1_Q \quad \text{and} \quad A_{\lambda,R}(f\sigma) := \sum_{Q\in\mathcal{D}} \lambda_Q \int_Q f \, \mathrm{d}\sigma 1_Q.$$

The adjoint $A_\lambda^*(\cdot\omega)$ and its localization $A_{\lambda,R}^*(\cdot\omega)$ are given similarly,

$$A_\lambda^*(g\omega) := \sum_{Q\in\mathcal{D}} \lambda_Q^* \int_Q g \, \mathrm{d}\omega 1_Q \quad \text{and} \quad A_{\lambda,R}^*(g\omega) := \sum_{Q\in\mathcal{D}} \lambda_Q^* \int_Q g \, \mathrm{d}\omega 1_Q.$$

To the author's knowledge, characterizations of two-weight norm inequalities have not been studied in an abstract operator-valued setting until the article [D]. Note that this setting includes the sequence-valued operators (7.1) and (7.2) studied by Scurry; and Nazarov, Treil, and Volberg.

We emphasize that the crux of the problem of characterizing the $L_C^p(\sigma) \to L_D^q(\omega)$ boundedness of the operator $A_\lambda(\cdot\sigma)$ defined by (7.3) is the operator-valuedness of the coefficients $\{\lambda_Q : C \to D\}$. This is because, in the case that the coefficients $\{\lambda_Q\}$ are just non-negative real numbers, the vector-valued operator $A_\lambda(\cdot\sigma) : L_C^p(\sigma) \to L_C^q(\omega)$ and the scalar-valued operator $A_\lambda(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ have an equal norm,

$$\|A_\lambda(\cdot\sigma)\|_{L_C^p(\sigma)\to L_C^q(\omega)} = \|A_\lambda(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\omega)},$$

and, in this case, the problem of characterizing the boundedness of the vector-valued operator reduces to characterizing the boundedness of the scalar-valued operator. This equality of the norms follows from realising that, in the case that the coefficients $\{\lambda_Q\}$ are just non-negative real numbers, the vector-valued operator $A_\lambda(\cdot\sigma) : L_C^p(\sigma) \to L_C^q(\omega)$ can be viewed as the tensor extension of the (positive and linear) scalar-valued operator $A_\lambda(\cdot\sigma) : L^p(\sigma) \to L^q(\omega)$ and applying the following elementary fact:

Proposition 7.3 (Tensor extension of a positive linear operator; see, for example, [66]). *Let E be a Banach space. Let $T : L^p(\sigma) \to L^q(\omega)$ be a bounded operator. Assume that T is linear and positive. Then, the tensor extension $T \otimes I_E$ of the operator T, which is defined by setting*

$$(T \otimes I_E)\left(\sum_n f_n e_n\right) := \sum_n T(f_n)e_n$$

for every finite collection $\{f_n\} \subseteq L^p(\sigma)$ and $\{e_n\} \subseteq E$, extends to a bounded linear operator $T \otimes I_E : L_E^p(\sigma) \to L_E^q(\omega)$. Furthermore,

$$\|T \otimes I_E\|_{L_E^p(\sigma)\to L_E^q(\omega)} = \|T\|_{L^p(\sigma)\to L^q(\omega)}.$$

Proof. Note that the tensor extension $T \otimes I_E$ is defined on simple functions, which are functions of the form $\sum_{n=1}^{N} 1_{A_n} e_n$ with $e_n \in E$ and $A_n \subseteq \mathbb{R}^d$ for $n = 1, \ldots, N$, $N \in \mathbb{N}$. Observe that for every simple function f, we have

$$|(T \otimes I_E)f|_E \le |T(|f|_E)|,$$

which implies the norm inequality

$$\|(T \otimes I_E)f\|_{L_E^q(\omega)} \le \|T(|f|_E)\|_{L^q(\omega)} \le \|T\|_{L^p(\sigma) \to L^q(\omega)} \|f\|_{L_E^p(\sigma)}.$$

The proof is completed by extending the operator $T \otimes I_E$ from simple functions to the whole space $L_E^p(\sigma)$ by appealing to the fact that simple functions are dense in $L_E^p(\sigma)$. For complete details, see, for example, Neerven's lecture notes [66]. $\qquad \square$

Our purpose is to characterize the $L_C^p(\sigma) \to L_D^q(\omega)$ boundedness of positive dyadic operators $A_\lambda(\cdot\sigma)$ in the case that the coefficients $\{\lambda_Q : C \to D\}$ are positive linear operators. The main results of the article [D] are that the two-weight boundedness of operator-valued dyadic positive operators is characterized by the direct and the dual L^∞ testing conditions, and the unweighted boundedness by the endpoint direct L^∞ testing condition:

Theorem 7.4 (Two-weight boundedness of operator-valued dyadic positive operators is characterized by the direct and the dual L^∞ testing conditions, [D])**.** *Let $1 < p \le q < \infty$. Let C and D be Banach lattices. Assume that C and D^* each have the dyadic Hardy–Littlewood property. Let $\{\lambda_Q : C \to D\}_{Q \in \mathcal{D}}$ be positive linear operators. Let the operator $A_\lambda(\cdot\sigma)$ and its localization $A_{\lambda,R}(\cdot\sigma)$ be defined as in (7.3), and the localization $A_{\lambda,R}^*(\cdot\omega)$ of the adjoint $A_\lambda^*(\cdot\omega)$ similarly. Then*

$$\max\{\mathfrak{A}, \mathfrak{A}^*\} \le \|A_\lambda(\cdot\sigma)\|_{L_C^p(\sigma) \to L_D^q(\omega)} \lesssim_{q,p} \|\bar{M}\|_{L_C^p \to L_C^p} \mathfrak{A} + \|\bar{M}\|_{L_{D^*}^{q'} \to L_{D^*}^{q'}} \mathfrak{A}^*,$$

where the testing constants $\mathfrak{A}$ and $\mathfrak{A}^*$ are the least constants in the direct and the dual L^∞ testing conditions:

$$\|A_{\lambda,R}(f\sigma)\|_{L_D^q(\omega)} \le \mathfrak{A}\|f\|_{L_C^\infty(R,\sigma)} \sigma(R)^{1/p},$$
$$\|A_{\lambda,R}^*(g\omega)\|_{L_{C^*}^{p'}(\sigma)} \le \mathfrak{A}^*\|g\|_{L_{D^*}^\infty(R,\omega)} \omega(R)^{1/q'},$$

for every $R \in \mathcal{D}$, every $f \in L_C^\infty(R,\sigma)$, and every $g \in L_{D^*}^\infty(R,\omega)$.

Theorem 7.5 (Unweighted boundedness of operator-valued dyadic positive operators is characterized by the endpoint direct L^∞ testing condition, [D])**.** *Let $1 < p \le q < \infty$. Let C and D be Banach lattices. Assume that C has the Hardy–Littlewood property. Then, we have*

$$\mathfrak{A} \le \|A_\lambda(\cdot\mu)\|_{L_C^p(\mu) \to L_C^q(\mu)} \lesssim_{q,p} \|\bar{M}\|_{L_C^p \to L_C^p} \mathfrak{A},$$

where the testing constants $\mathfrak{A}$ is the least constant in the end-point direct L^∞ testing condition:

$$\|A_{\lambda,R}(f\mu)\|_{L_D^1(\mu)} \le \mathfrak{A}\|f\|_{L_C^\infty(R,\sigma)} \mu(R)^{1/p+1/q'}$$

for every $R \in \mathcal{D}$ and every $f \in L_C^\infty(R,\mu)$.

Flavour of the proof. The proof is based on the technique of stopping cubes. The proof boils down to cooking up a stopping condition $\mathrm{ch}(F)$ and a function f_F such that the replacement rule

$$\int_Q f \,\mathrm{d}\mu \le \int_Q f_F \,\mathrm{d}\mu \quad \text{whenever } \pi_{\mathcal{F}}(Q) = F$$

and the summability condition

$$\Big(\sum_{F\in\mathcal{F}} \|f_F\|_{L_C^\infty(F,\mu)}^p \mu(F) \Big)^{1/p} \lesssim_{p,C} \|f\|_{L_C^p(\mu)}$$

hold. Then, the crux of the proof is captured by the following computation.

$$\|A_\lambda(f\mu)\|_{L_D^q(\mu)}$$

$$= \| \sum_{F\in\mathcal{F}} \sum_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{F}}(Q)=F}} \lambda_Q \int_Q f \,\mathrm{d}\mu \|_{L_D^q(\mu)} \qquad \sum_{Q\in\mathcal{D}} = \sum_{F\in\mathcal{F}} \sum_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{F}}(Q)=F}}$$

$$\approx_q \Big(\sum_{F\in\mathcal{F}} \| \sum_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{F}}(Q)=F}} \lambda_Q \int_Q f \,\mathrm{d}\mu \|_{L_D^q(\mu)}^q \Big)^{1/q} \qquad \text{the } L^p \text{ variant of Pythagoras' theorem}$$

$$\lesssim_p \Big(\sum_{F\in\mathcal{F}} \|A_{\lambda,F}(f_F\sigma)\|_{L_D^q(\mu)}^q \Big)^{1/q} \qquad \text{the replacement rule}$$

$$\lesssim \mathfrak{A} \Big(\sum_{F\in\mathcal{F}} \|f_F\|_{L_C^\infty(F,\mu)}^q \mu(F)^{q/p} \Big)^{1/q} \qquad \text{the direct } L^\infty \text{ testing condition}$$

$$\le \mathfrak{A} \Big(\sum_{F\in\mathcal{F}} \|f_F\|_{L_C^\infty(F,\mu)}^p \mu(F) \Big)^{1/p} \qquad |\cdot|_{\ell^q} \le |\cdot|_{\ell^p} \text{ whenever } q \ge p$$

$$\lesssim_{p,C} \mathfrak{A} \|f\|_{L^p(\mu)}. \qquad \text{the summability condition}$$

In the two-weight case, the obstacle to immediately applying the L^p variant of Pythagoras' theorem is that a σ-sparse collection (resulting from the stopping condition related to $f \in L_C^p(\sigma)$) fails in general to be ω-sparse. This obstacle is circumvented by using parallel stopping cubes, similarly as in Hytönen's [27] proof of the real-valued case.

7.3. Corollaries.

Corollary 7.6 (Sufficient condition for the boundedness of operator-valued positive dyadic operators)**.** *Let C and D be Banach lattices. Assume that C and D^* each have the Hardy–Littlewood property. Assume that $\{\lambda_Q \in \mathcal{L}(C \to D)\}_{Q\in\mathcal{D}}$ are positive linear operators. Define the operator P_λ^μ by*

$$P_\lambda^\mu f := \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q^\mu 1_Q.$$

Then

$$\|P_\lambda^\mu\|_{L_C^p(\mu)\to L_D^p(\mu)} \lesssim_p \|\bar{M}\|_{L_C^p\to L_C^p} \|\bar{M}\|_{L_{D^*}^{p'}\to L_{D^*}^{p'}}$$

$$\times \sup_{Q\in\mathcal{D}} \frac{1}{\mu(R)^{1/p}} \| \sum_{Q:Q\subseteq R} \lambda_Q 1_Q \|_{L_{\mathcal{L}(C\to D)}^p(\mu)}.$$

Proof. By Theorem 7.5 (which is from the article [D]), the direct L^∞ testing condition,

$$\|P^\mu_{\lambda,R}f\|_{L^p_D(\mu)} \lesssim \mathfrak{P}\|f\|_{L^\infty_C(R,\mu)}\mu(R)^{1/p} \quad \text{for every } R \in \mathcal{D} \text{ and } f \in L^\infty_C(R,\mu),$$

implies that

$$\|P^\mu_\lambda\|_{L^p_C(\mu)\to L^p_D(\mu)} \lesssim_p \|\bar{M}\|_{L^p_C\to L^p_C}\mathfrak{B}.$$

Now, we check the direct L^∞ testing condition by using duality. Let $R \in \mathcal{D}$, $f \in L^\infty_C(R,\mu)$, and $g \in L^{p'}_{D*}$. By positivity and Hölder's inequality, we have

$$\int gP^\mu_{\lambda,R}f\,\mathrm{d}\mu = \int \sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \langle g\rangle^\mu_Q \lambda_Q 1_Q f\,\mathrm{d}\mu \le \int \Big(\sup_{Q\in\mathcal{D}}\langle g\rangle^\mu_Q 1_Q\Big)\Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q 1_Q\Big)f\,\mathrm{d}\mu$$

$$\le \Big\|\Big(\sup_{Q\in\mathcal{D}}\langle g\rangle^\mu_Q 1_Q\Big)\Big\|_{L^p_{D*}(\mu)}\Big\|\Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q 1_Q\Big)f|_D\Big\|_{L^p(\mu)}$$

$$\le \|\bar{M}^\mu\|_{L^{p'}_{D*}(\mu)\to L^{p'}_{D*}(\mu)}\|g\|_{L^{p'}_{D*}(\mu)}\Big\|\Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q 1_Q\Big)|_{C\to D}|f|_C\Big\|_{L^p(\mu)}$$

$$\lesssim \|\bar{M}\|_{L^{p'}_{D*}\to L^{p'}_{D*}}\|g\|_{L^{p'}_{D*}(\mu)}\Big\|\Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q 1_Q\Big)\Big\|_{L^p_{\mathcal{L}(C\to D)}(\mu)}\|f\|_{L^\infty_C(R,\mu)}.$$

$$\square$$

Remark. The operator $P^\mu_\lambda f$ can be viewed as a positive analogue of the (cancellative) paraproduct $\Pi_b f := \sum_{Q\in\mathcal{D}} D_Q b\langle f\rangle^\mu_Q 1_Q$. In this light, this sufficient condition for operator-valued positive dyadic operators is analogous to Hytönen's [28] sufficient condition for operator-valued dyadic paraproducts (see Theorem 5.12).

7.4. **Recent developments.** Lai [42] proves a new characterization for the $L^p(\sigma) \to L^p(\omega)$ boundedness of Scurry's [59] sequence-valued positive operator $S_q(\cdot\sigma)$ defined in (7.1). His proof is based on a scaling trick, which can be stated as follows:

Proposition 7.7 (Scaling trick, [42])**.** *Let $0 < q < p < \infty$ and $p > 1$. Let μ be a locally finite Borel measure. Let $F : \mathbb{R}^\mathcal{D}_+ \to \mathbb{R}_+$ be a function. Assume that the function F is monotone in the sense that*

$$F(\{a_Q\}_{Q\in\mathcal{D}}) \le F(\{b_Q\}_{Q\in\mathcal{D}}) \quad \text{whenever } \{a_Q\}_{Q\in\mathcal{D}} \le \{b_Q\}_{Q\in\mathcal{D}},$$

where the notation $\{a_Q\}_{Q\in\mathcal{D}} \le \{b_Q\}_{Q\in\mathcal{D}}$ means that $a_Q \le b_Q$ for every $Q \in \mathcal{D}$. Then, the following assertions are equivalent:

i) *We have that*

$$F(\{(\langle f\rangle^\mu_Q)^q\}_{Q\in\mathcal{D}})^{1/q} \lesssim_{p,q} \|f\|_{L^p(\mu)} \quad \text{for every } f \ge 0.$$

ii) *We have that*

$$F(\{\langle f\rangle^\mu_Q\}_{Q\in\mathcal{D}}) \lesssim_{p,q} \|f\|_{L^{p/q}(\mu)} \quad \text{for every } f \ge 0.$$

Proof. We suppress μ and $\mathcal{D}$ in the notation. We prove the equivalence by cases.

Case '$1 \le q < \infty$'. First, we prove that i) implies ii). We observe that $\langle f\rangle_Q = \langle\langle f\rangle^{1/q}_Q\rangle^q_Q \le \langle(Mf)^{1/q}\rangle^q_Q$. Using the monotonicity together with this observation, the

assumed estimate, and the boundedness of the dyadic Hardy–Littlewood maximal operator, we obtain

$$F(\{\langle f\rangle_Q\}) \leq F(\{\langle (Mf)^{1/q}\rangle_Q^q\}) \lesssim \|(Mf)^{1/q}\|_{L^p}^q = \|Mf\|_{L^{p/q}} \lesssim_{p,q} \|f\|_{L^{p/q}}.$$

Next, we prove that ii) implies i). We observe that, by Jensen's inequality, $\langle f\rangle_Q^q \leq \langle f^q\rangle_Q$. Using the monotonicity together with this observation, and the assumed estimate, we obtain

$$F(\{\langle f\rangle_Q^q\})^{1/q} \leq F(\{\langle f^q\rangle_Q\})^{1/q} \lesssim \|f^q\|_{L^{p/q}}^{1/q} = \|f\|_{L^p}.$$

Case '$0 < q < 1$'. First, we prove that ii) implies i). We observe that $\langle f\rangle_Q^q = \langle\langle f\rangle_Q^q\rangle_Q \leq \langle (Mf)^q\rangle_Q$. Again, using the monotonicity together with this observation, the assumed estimate, and the boundedness of the dyadic Hardy–Littlewood maximal operator, we obtain

$$F(\{\langle f\rangle_Q^q\})^{1/q} \leq F(\{\langle (Mf)^q\rangle_Q\})^{1/q} \lesssim \|(Mf)^q\|_{L^{p/q}}^{1/q} = \|Mf\|_{L^p} \lesssim_p \|f\|_{L^p}.$$

Finally, we prove that i) implies ii). We observe that, by Jensen's inequality, $\langle f\rangle_Q \leq \langle f^{1/q}\rangle_Q^q$. Again, using the monotonicity together with this observation, and the assumed estimate, we obtain

$$F(\{\langle f\rangle_Q\}) \leq F(\{\langle f^{1/q}\rangle_Q^q\}) \lesssim \|f^{1/q}\|_{L^p}^q = \|f\|_{L^{p/q}}.$$

$\square$

In the article [C], we prove the potential-type characterization

(7.4) $$\|M_{\mathcal{E},\lambda}(\cdot\,\sigma)\|_{L^p(\sigma)\to L^q(\omega)} \approx \|W_{M,\sigma}^q[\omega]\|_{L^{r/q}(\sigma)}^{1/q}$$

for a linearized maximal operator $M_{\mathcal{E},\lambda}(\cdot\,\sigma)$ defined by

$$M_{\mathcal{E},\lambda}(f\sigma) := \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q^\sigma 1_{E(Q)}.$$

For this operator, the abstract Wolff potential $W_{M,\sigma}^q[\omega]$ is written out as

$$W_{M,\sigma}^q[\omega] := \sup_{R\in\mathcal{D}} \frac{1_R}{\sigma(R)}\|M_{\mathcal{E},\lambda,R}\|_{L^q(\omega)}^q = \sup_{R\in\mathcal{D}} \frac{1_R}{\sigma(R)} \sum_{Q\in\mathcal{D}:Q\subseteq R} \lambda_Q^q \frac{\omega(E(Q))}{\sigma(Q)}\sigma(Q).$$

By using Lai's scaling trick, we can prove the following variant of this characterization:

Theorem 7.8. *Let $0 < q < p < \infty$ and $p > 1$. Define the auxiliary exponent $r \in (0,\infty)$ by $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. Let $\{E(Q) \subseteq Q\}_{Q\in\mathcal{D}}$ be pairwise disjoint sets. Let $\{\lambda_Q\}_{Q\in\mathcal{D}}$ be positive real numbers. Define the linearized maximal operator $M_{\mathcal{E},\lambda}(\cdot\,\sigma)$ by*

$$M_{\mathcal{E},\lambda}(f\sigma) := \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q^\sigma 1_{E(Q)}.$$

Then, we have

$$\|M_{\mathcal{E},\lambda}(\cdot\,\sigma)\|_{L^p(\sigma)\to L^q(\omega)} \approx \Big\|\sum_{Q\in\mathcal{D}} \lambda_Q^q \frac{\omega(E(Q))}{\sigma(Q)} 1_Q\Big\|_{L^{r/q}(\sigma)}^{1/q}.$$

Remark. This potential-type characterization also follows from the potential-type characterization (7.4). Indeed, Cascante, Ortega, and Verbitsky's lemma [5] states that

$$\|\sum_{Q\in\mathcal{D}} a_Q 1_Q\|_{L^s(\mu)} \approx_s \|\sup_{R\in\mathcal{D}} \frac{1_R}{\mu(R)} \sum_{\substack{Q\in\mathcal{D}:\\Q\subseteq R}} a_Q\mu(Q)\|_{L^s(\mu)}$$

for any locally finite Borel measure μ, exponent $s\in(1,\infty)$, and non-negative coefficients $\{a_Q\}_{Q\in\mathcal{D}}$. Applying this lemma, we obtain

$$\|W_{M,\sigma}^q[\omega]\|_{L^{r/q}(\sigma)}^{1/q} = \|\sup_{R\in\mathcal{D}} \frac{1_R}{\sigma(R)} \sum_{Q\in\mathcal{D}:Q\subseteq R} \lambda_Q^q \frac{\omega(E(Q))}{\sigma(Q)}\sigma(Q)\|_{L^{r/q}(\sigma)}^{1/q}$$

$$\approx_{r,q} \|\sum_{Q\in\mathcal{D}} \lambda_Q^q \frac{\omega(E(Q))}{\sigma(Q)} 1_Q\|_{L^{r/q}(\sigma)}^{1/q}.$$

Proof of the alternative potential-type characterization using Lai's scaling trick. We observe that

$$\|\sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q^\sigma 1_{E(Q)}\|_{L^q(\omega)} = \|\sum_{Q\in\mathcal{D}} \lambda_Q^q (\langle f\rangle_Q^\sigma)^q 1_{E(Q)}\|_{L^1(\omega)}^{1/q}.$$

Therefore, by Lai's scaling trick (see Proposition 7.7), the norm inequality

$$\|\sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q^\sigma 1_{E(Q)}\|_{L^q(\omega)} \lesssim C\|f\|_{L^p(\sigma)} \quad \text{for every } f\in L^p(\sigma)$$

is equivalent to the norm inequality

$$\|\sum_{Q\in\mathcal{D}} \lambda_Q^q \langle f\rangle_Q^\sigma 1_{E(Q)}\|_{L^1(\omega)} \lesssim_{p,q} C^q\|f\|_{L^{p/q}(\sigma)} \quad \text{for every } f\in L^{p/q}(\sigma).$$

Now, by duality, the latter norm inequality is equivalent to the norm inequality

$$\|\sum_{Q\in\mathcal{D}} \lambda_Q^q \frac{1}{\sigma(Q)} \int_{E(Q)} g\,d\omega 1_Q\|_{L^{(p/q)'}(\sigma)} \lesssim_{p,q} C^q\|g\|_{L^\infty(\omega)} \quad \text{for every } g\in L^\infty(\omega).$$

We observe that, by homogeneity and positivity, this norm inequality holds for every $g\in L^\infty(\omega)$ if and only if it holds for the particular function $g=1$. Moreover, we notice that $(p/q)'=r/q$. This completes the proof. $\square$

7.5. **Open question.** The maximal operator $M_\lambda(\cdot\sigma)$ is readily interpreted in the operator-valued setting $\bar{M}_\lambda(\cdot\sigma): L_C^p(\sigma) \to L_D^q(\omega)$: Let C and D be Banach lattices, and $\{\lambda_Q : C \to D\}$ be positive linear operators. We define $\bar{M}_\lambda(f\sigma)$ by

$$\bar{M}_\lambda(f\sigma) \coloneqq \sup_{Q\in\mathcal{D}} \lambda_Q \int_Q f\,d\sigma 1_Q,$$

where the supremum is now taken with respect to the lattice order. In particular, the *Fefferman–Stein vector-valued maximal function* $M_{\mathrm{FS}}(f\sigma): L_{\ell^s}^p(\sigma) \to L_{\ell^s}^p(\omega)$, which is defined componentwise by

$$(M_{\mathrm{FS}}(f\sigma))_k = \sup_Q \langle f_k\sigma\rangle_Q 1_Q,$$

is included in this setting.

The two-weight boundedness of the Fefferman–Stein vector-valued maximal operator was characterized by Peréz [53]. Typically, the norm inequality for the Lebesgue norms such that in the range side there appears a strictly smaller Lebesgue exponent than in the domain side is more difficult than the opposite case. For the

Fefferman–Stein vector-valued maximal operator, in this difficult case $1 < s < p < \infty$, the characterizations reads as follows:

Theorem 7.9 ([53]). *Let $1 < s < p < \infty$. Let $r := p/s$. Then $M_{FS}(\,\cdot\,\sigma) : L^p_{\ell^s}(\sigma) \to L^p_{\ell^s}(\omega)$ is bounded if and only if the following condition holds: There exists a constant C such that for each $h \in L^{r'}$ there exists $H \in L^{r'}$ such that $\|H\|_{L^{r'}} \le \|h\|_{L^{r'}}$ and*

$$\int 1_R M_{FS}(1_R \sigma)^s w^{1/r} h \, \mathrm{d}x \le C \int_R \sigma^{1/r} H \, \mathrm{d}x$$

for every $R \in \mathcal{D}$.

This criterion has a different flavour than the L^∞ testing condition. Nevertheless, from the proof technique of the article [D], it follows that the unweighted (case $\sigma = \omega = \mu$) boundedness of operator-valued maximal operators is characterized by the direct L^∞ testing condition. Is their two weight boundedness also characterized by the direct L^∞ testing condition?

Question 7.10. *Let $p \in (1, \infty)$. Assume that C is a Banach lattice with the Hardy–Littlewood property. Assume that $\{\lambda_Q : C \to C\}$ are positive linear operators. Then, is it true that*

$$\|\bar{M}_\lambda(\,\cdot\,\sigma)\|_{L^p_C(\sigma) \to L^p_C(\omega)} \approx_p \mathfrak{M}$$

holds? Here $\mathfrak{M}$ denotes the least constant in the direct L^∞ testing condition:

$$\|\bar{M}_{\lambda,R}(f\sigma)\|_{L^p_C(\omega)} \le \mathfrak{M}\|f\|_{L^\infty_C(R,\sigma)} \sigma(R)^{1/p}$$

for every $R \in \mathcal{D}$ and $f \in L^\infty_C(R, \sigma)$.

Notation and definitions

In the symbols standing for operators and spaces, we use the convention that the underlying measure μ is suppressed in the notation if and only if it is the Lebesgue measure $\mathrm{d}x$. Thus, for example, we write $BMO^p := BMO^p(\mathrm{d}x)$.

Measures

μ An arbitrary locally finite Borel measure on $\mathbb{R}^d$.

$\mathrm{d}x$ The Lebesgue measure.

Dyadic cubes

$\mathcal{D}$ An arbitrary collection of dyadic cubes.

$\hat{Q}$ The dyadic parent of a dyadic cube Q,

defined as the minimal $R \in \mathcal{D}$ such that $R \supsetneq Q$.

Stopping cubes

$\mathrm{ch}_{\mathcal{F}}(Q)$ The $\mathcal{F}$-*children* of a dyadic cube Q,

defined by $\mathrm{ch}_{\mathcal{F}}(Q) := \{F' \in \mathcal{D} : F' \text{ maximal such that } F' \subsetneq Q\}$.

$\pi_{\mathcal{F}}(Q)$ The $\mathcal{F}$-*parent* of a dyadic cube,

defined as the minimal $F \in \mathcal{F}$ such that $Q \subseteq F$.

$E_{\mathcal{F}}(F)$ $E_{\mathcal{F}}(F) := F \smallsetminus \bigcup_{F' \in \mathrm{ch}_{\mathcal{F}}(F)} F'$.

Mean

$\langle f \rangle_Q^\mu$ The average of f on Q with respect to μ, $\langle f \rangle_Q^\mu := \frac{1}{\mu(Q)} \int_Q f \, \mathrm{d}\mu$.

$\langle f \rangle_Q$ $\langle f \rangle_Q := \langle f \rangle_Q^{\mathrm{d}x}$.

Median and median oscillation

$m(f, Q)$ A real-valued median of f on Q: Any real number $m(f, Q)$ such that $\mu(Q \cap \{f > m(f, Q)\}) \leq \frac{1}{2}\mu(Q)$ and $\mu(Q \cap \{f < m(f, Q)\}) \leq \frac{1}{2}\mu(Q)$.

$c_\lambda(f; Q)$ A vector-valued median of f on Q: Any vector $c_\lambda(f; Q) \in E$ such that $((f - c_\lambda(f; Q))1_Q)^*(\lambda\mu(Q)) \approx \omega_\lambda(f; Q)$.

$(1_Q f)^*(\lambda\mu(Q))$ The relative median oscillation of f (about zero) on Q,

$$(f1_Q)^*(\lambda\mu(Q)) := \min\{r \geq 0 : \mu(Q \cap \{|f| > r\}) \leq \lambda\mu(Q)\}.$$

$\omega_\lambda(f, Q)$ The median oscillation of f on Q,

$$\omega_\lambda(f; Q) := \inf_{c \in \mathbb{R}}((f - c)1_Q)^*(\lambda\mu(Q)).$$

Operators

M^μ Hardy–Littlewood maximal operator, $M^\mu f := \sup_{Q \in \mathcal{D}} \langle f \rangle_Q^\mu 1_Q$.

$\bar{M}^\mu$ Hardy–Littlewood lattice maximal operator,

$\bar{M}^\mu f := \sup_{Q \in \mathcal{D}} \langle f \rangle_Q^\mu 1_Q$ with the supremum in the lattice order.

$M_\lambda(\cdot\sigma)$ Dyadic maximal operator, $M_\lambda(f\sigma) := \sup_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q$

$A_\lambda(\cdot\sigma)$ Dyadic positive operator, $A_\lambda(f\sigma) := \sum_{Q \in \mathcal{D}} \lambda_Q \int_Q f \, d\sigma 1_Q$

D_Q^μ Haar projection, $D_Q^\mu f := -\langle f \rangle_Q^\mu 1_Q + \sum_{Q' \in \mathrm{ch}_\mathcal{D}(Q)} \langle f \rangle_{Q'}^\mu 1_{Q'}$.

D_Q^i Shifted Haar projection, $D_Q^i f := \sum_{\substack{R \in \mathcal{D}: \\ R \subseteq Q, \ell(R) = 2^{-i}\ell(Q)}} D_R f$.

S_a^{ji} Dyadic shift, $S_a^{ji} f := \sum_{Q \in \mathcal{D}} D_Q^j A_Q D_Q^i f$

with $A_Q f(x) := \frac{1}{|Q|} \int_Q a_Q(x, y) f(y) \, dy$.

Π_b^μ Dyadic paraproduct, $\Pi_b^\mu f := \sum_{Q \in \mathcal{D}} D_Q^\mu b \langle f \rangle_Q^\mu 1_Q$

Spaces

$(E, |\cdot|_E)$ An arbitrary Banach space.

$\beta_p(E)$ The UMD$_p$ constant of an UMD space E.

$\mathcal{L}(E, F)$ The space of bounded, linear operators from a Banach space E to a Banach space F.

$\mathcal{B}(E, F)$ The space of bounded operators from a Banach space E to a Banach space F.

$L^p(\mu)$ The Lebesgue space, equipped with the norm

$$\|f\|_{L^p(\mu)} := \int_{\mathbb{R}^d} |f|^p \, d\mu \text{ for } p \in [1, \infty).$$

$L_E^p(\mu)$ The Bochner–Lebesgue space, equipped with the norm

$$\|f\|_{L^p_E(\mu)} \coloneqq \||f|_E\|_{L^p(\mu)} \text{ for } p \in [1, \infty).$$

p' — The Hölder conjugate exponent of an exponent $p \in [1, \infty]$, defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

$\mathrm{BMO}^p(\mu)$ — The space of functions of bounded mean oscillation, $\|b\|_{\mathrm{BMO}^p(\mu)} \coloneqq \sup_{R \in \mathcal{D}} \frac{1}{\mu(R)^{1/p}} \|1_R(b - \langle b \rangle^\mu_R)\|_{L^p(\mu)}$.

C^1_0 — The space of continuously differentiable, compactly supported real-valued functions defined on $\mathbb{R}^d$.

$C^1_0 \otimes E$ — The space consisting of all the functions $f : \mathbb{R}^d \to E$ that can be written as a linear combination of the following form: $f = \sum_{n=1}^{N} h_n e_n$ with $\{h_n\}_{n=1}^{N} \subseteq C^1_0, \{e_n\}_{n=1}^{N} \subseteq E$, and $N \in \mathbb{N}$.

Definitions

μ-*sparse* — A collection $\mathcal{S}$ of sets is μ-*sparse* if for every $S \in \mathcal{S}$ there exists $E(S) \subseteq S$ such that $\mu(E(S)) \gtrsim \mu(S)$ and the collection $\{E(S)\}_{S \in \mathcal{S}}$ is pairwise disjoint.

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