Novel Regularization Methods for ill-posed Problems in Hilbert and Banach Spaces

Publicações Matemáticas

Novel Regularization Methods for ill-posed Problems in Hilbert and Banach Spaces

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30º Colóquio Brasileiro de Matemática

Impresso no Brasil / Printed in Brazil
Capa: Noni Geiger / Sérgio R. Vaz

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ISBN: 978-85-244-0410-8

Distribuição: IMPA
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Preface

The demands of natural science and technology have brought to the fore many mathematical problems that are inverse to the classical direct problems, i.e., problems which may be interpreted as finding the cause of a given effect. Inverse problems are characterized by the fact that they are usually much harder to solve than their direct counterparts (the direct problems) since they are usually associated to ill-posedness effects. As a result an exiting and important area of research has been developed over the last decades. The combination of classical analysis, linear algebra, applied functional and numerical analysis is one of the fascinating features of this relatively new research area.

This monograph does not aim to give an extensive survey of books and papers on inverse problems. Our goal is to present some successful recent ideas in treating inverse problems and to make clear the progress of the theory of ill-posed problems.

These notes arose from the PhD thesis [5], articles [6, 7, 56, 33, 31], as well as from courses and lectures delivered by the authors. The presentation is intended to be accessible to students whose mathematical background include basic courses in advanced calculus, linear algebra and functional analysis.

The text is organized as follows: In Chapter 1 the research area of inverse and ill-posed problems is introduced by means of examples. Moreover, the basic concepts of regularization theory are presented. In Chapter 2 we investigate Tikhonov type regularization methods. Chapter 3 is devoted to Landweber type methods (which illustrate the iterative regularization techniques). Chapter 4 deals with total least square regularization methods. In particular, a novel technique called double regularization is considered.

The authors would like to thank the SBM to make possible these notes. Moreover, we are grateful for the support during the preparation of the manuscript and acknowledge the financial assistance received from the Brazilian research agencies CNPq and CAPES.

May 2015 Ismael Rodrigo Bleyer, Helsinki

Antonio Leitão, Rio de Janeiro

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Chapter 1

Introduction

In this chapter we introduce a wide range of technological applications modelled by inverse problems. We also give an introductory insight into the techniques for modelling and classifying these particular problems. Moreover, we present the most difficult challenge in the inverse problems theory, namely the ill-posedness.

1.1 Inverse problems

The problem which may be considered as one of the oldest inverse problem is the computation of the diameter of the earth by Eratosthenes in 200 b. Chr.. For many centuries people are searching for hiding places by tapping walls and analyzing echo; this is a particular case of an inverse problem. It was Heisenberg who conjectured that quantum interaction was totally characterized by its scattering matrix which collects information of the interaction at infinity. The discovery of neutrinos by measuring consequences of its existence is in the spirit of inverse problems too.

Over the past 50 years, the number of publications on inverse problems has grown rapidly. The following list of inverse problems gives a good impression of the wide variety of applications:

1. X-ray computed tomography (R-ray CT); the oldest tomographic medical imaging technique, that uses computer-processed

X-rays to produce images of specific parts (slices) of the human body; (wikipedia.org/wiki/X-ray_computed_tomography)

2. Magnetic resonance imaging (MRI); medical imaging technique that uses strong magnetic fields and radio waves to form images of the body (Figure 1.2); (wikipedia.org/wiki/Magnetic_resonance_imaging)

3. Electrical impedance tomography (EIT); medical imaging technique in which an image of the conductivity (Figure 1.2) of a part of the body is inferred from surface electrode measurements (also used for land mine detection and nondestructive industrial tomography, e.g., crack detection); (wikipedia.org/wiki/Electrical_impedance_tomography)

4. Positron emission tomography (PET); medical imaging technique that produces a three-dimensional image of functional processes in the body; (wikipedia.org/wiki/Positron_emission_tomography)

5. Positron emission tomography – Computed tomography (PET-CT); medical imaging technique using a device which combines both a PET scanner and CT scanner; (wikipedia.org/wiki/PET-CT);

6. Single photon emission computed tomography (SPECT); a nuclear medicine tomographic imaging technique based on gamma rays; it is able to provide true 3D information; (wikipedia.org/wiki/Single-photon_emission_computed_tomography)

7. Thermoacoustic imaging; technique for studying the absorption properties of human tissue using virtually any kind of electromagnetic radiation; (wikipedia.org/wiki/Thermoacoustic_imaging)

8. Electrocardiography (ECG); a process of recording the electrical activity of the heart over a period of time using electrodes (placed on a patient's body), which detect tiny electrical changes on the skin that arise from the heart muscle depolarizing during each heartbeat. ECG is the graph of voltage

versus time produced by this noninvasive medical procedure; (wikipedia.org/wiki/Electrocardiography)

9. Magneto-cardiography (MCG); a technique to measure the magnetic fields produced by electrical activity in the heart. Once a map of the magnetic field is obtained over the chest, mathematical algorithms (which take into account the conductivity structure of the torso) allow the location of the source of the electrical activity, e.g., sources of abnormal rhythms or arrhythmia); (wikipedia.org/wiki/Magnetocardiography)

10. Optical coherence tomography (OCT); a medical imaging technique that uses light to capture (micrometer-resolution) 3D-images from within optical scattering media, e.g., biological tissue; (wikipedia.org/wiki/Optical_coherence_tomography)

11. Electron tomography (ET); a tomography technique for obtaining detailed 3D structures of sub-cellular macro-molecular objects. A beam of electrons is passed through the sample at incremental degrees of rotation around the center of the target sample (a transmission electron microscope is used to collect the data). This information is used to produce a 3D-image of the subject. (wikipedia.org/wiki/Electron_tomography)

12. Ocean acoustic tomography (OAT); a technique used to measure temperatures and currents over large regions of the ocean (Figure 1.3); (wikipedia.org/wiki/Ocean_acoustic_tomography)

13. Seismic tomography; a technique for imaging Earth's sub-surface characteristics aiming to understand the deep geologic structures; (wikipedia.org/wiki/Seismic_tomography)

14. Muon tomography; a technique that uses cosmic ray muons to generate 3D-images of volumes using information contained in the Coulomb scattering of the muons (Figure 1.4); (wikipedia.org/wiki/Muon_tomography)

15. Deconvolution; the problem here is to reverse the effects of convolution on recorded data, i.e., to solve the linear equation $g * x = y$, where y is the recorded signal, x is the signal that we

wish to recover (but has been convolved with some other signal g before we recorded it), and g might represent the transfer function. Deconvolution techniques are widely used in the areas of signal processing and image processing; (wikipedia.org/wiki/Deconvolution)

16. Parameter identification in parabolic PDE's, e.g., determining the volatility[1] in mathematical models for financial markets; (wikipedia.org/wiki/Volatility_(finance))

17. Parameter identification in elliptic PDE's, e.g., determining the diffusion coefficient from measurements of the Dirichlet to Neumann map; (wikipedia.org/wiki/Poincaré-Steklov_operator)

18. Can you hear the shape of a drum? (see Figure (1.1)) To hear the shape of a drum is to determine the shape of the drumhead from the sound it makes, i.e., from the list of overtones; (wikipedia.org/wiki/Hearing_the_shape_of_a_drum)

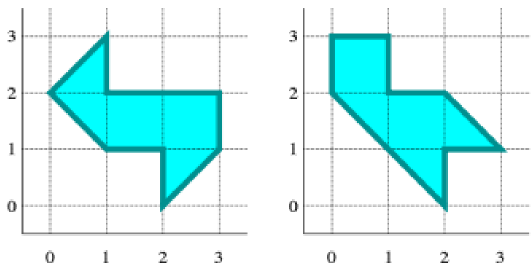

Figure 1.1: These two drums, with membranes of different shapes, would sound the same because the eigenfrequencies are all equal. The frequencies at which a drumhead can vibrate depend on its shape. The Helmholtz equation allows the calculation of the the frequencies if the shape is known. These frequencies are the eigenvalues of the Laplacian in the space. (source: Wikipedia)

[1]Volatility is a measure for variation of price of a financial instrument over time.

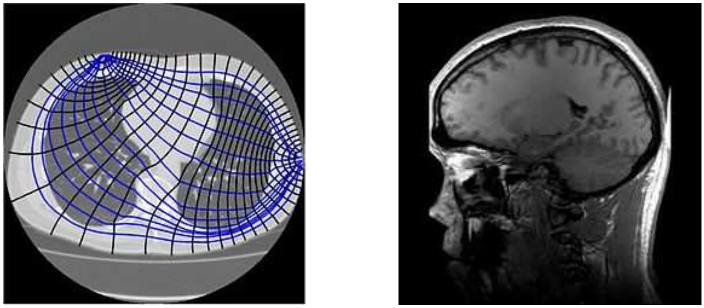

Figure 1.2: Medical imaging techniques (source: Wikipedia).
EIT (left hand side), a cross section of a human thorax from an X-ray CT showing current stream lines and equi-potentials from drive electrodes (lines are bent by the change in conductivity between different organs).
MRI (right hand side), para-sagittal MRI of the head, with aliasing artifacts (nose and forehead appear at the back of the head).

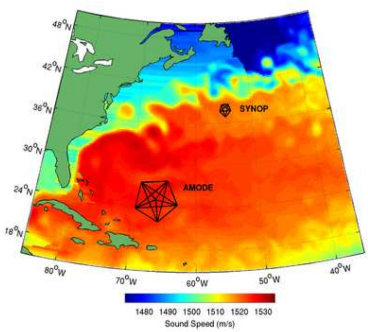

Figure 1.3: Ocean acoustic tomography (source: Wikipedia).
The western North Atlantic showing the locations of two experiments that employed ocean acoustic tomography. AMODE (Acoustic Mid-Ocean Dynamics Experiment, designed to study ocean dynamics in an area away from the Gulf Stream, 1990) and SYNOP (Synoptically Measure Aspects of the Gulf Stream, 1988). The colors show a snapshot of sound speed at 300 m depth derived from a high-resolution numerical ocean model.

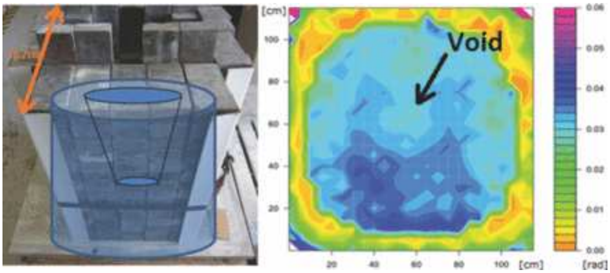

Figure 1.4: Muon scattering tomography (source: Wikipedia).
Imaging of a reactor mockup using the Muon Mini Tracker (MMT) at Los Alamos. The MMT consists of two muon trackers made up of sealed drift tubes. In the demonstration, cosmic-ray muons passing through a physical arrangement of concrete and lead; materials similar to a reactor were measured. The reactor mockup consisted of two layers of concrete shielding blocks, and a lead assembly in between. Lead with a conical void (similar in shape to the melted core of the Three Mile Island reactor) was imaged through the concrete walls. It took 3 weeks to accumulate 8×10^4 muon events. This test object was successfully imaged.

Let us assume that we have a mathematical model of a physical process. Moreover, we also assume that this model gives a precise description of the system behind the process, as well as its operating conditions, and explains the principal quantities of the model (see Figure 1.5), namely:

input, system parameters, output.

In most cases the description of the system is given in terms of a set of equations (e.g., ordinary differential equations (ODE's), partial differential equations (PDE's) integral equations, ...), containing certain parameters. The analysis of a given physical process via the corresponding mathematical model may be divided into three distinct types of problems:

(A) **Direct Problem:** Given the input and the system parameter, find out the output of the model.

(B) **Reconstruction Problem:** Given the system parameters and the output, find out which input has led to this output.

(C) **Identification Problem**. Given the input and the output, determine the system parameters which are in agreement with the relation between input and output.

Problems of type (A) are called direct (or forward) problems, since they are oriented towards a cause-effect sequence. In this sense, problems of type (B) and (C) are called inverse problems, because they consist of finding out unknown causes of known (observed) consequences.

It is worth noticing that the solution of a direct problem is part of the formulation of the corresponding inverse problem, and vice-versa (see Example 1.1.1 below).

Moreover, it follows immediately from definitions (A), (B) and (C) above, that the solution of one of these problems involves some treatment of the other problems as well.

A complete discussion of a model by solving the related inverse problems is among the main goals of the **inverse problem theory**.

In what follows, we present a brief mathematical description of the input, the output and the system parameters in a functional analytical framework:

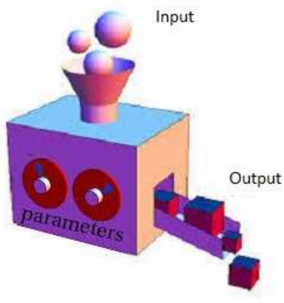

Figure 1.5: Principal quantities of a mathematical model describing an inverse problem: input, system parameters, output.

X : space of input quantities;
Y : space of output quantities;
P : space of system parameters;
$A(p)$: system operator from X into Y, associated to the parameter $p \in P$.

In these simple terms we may state the above defined problems in the following way:

(A) Given $x \in X$ and $p \in P$, find $y := A(p)\,x$.

(B) Given $y \in Y$ and $p \in P$, find $x \in X$ s.t. $A(p)\,x = y$.

(C) Given $y \in Y$ and $x \in X$, find $p \in P$ s.t. $A(p)\,x = y$.

At first glance, direct problems (A) seem to be much easier to solve than inverse problems (B) or (C). However, for the computation of $y := A(p)\,x$, it may be necessary to solve differential or integral equations, tasks which may be of the same order of complexity as the solution of the equations related to the inverse problems.

Example 1.1.1 (Differentiation of data)**.** *Let's consider the problem of finding the integral of a given function. This task can be performed, both analytically and numerically, in a very stable way.*
When this problem is considered as a direct (forward) problem, then to differentiate a given function is the corresponding inverse problem. A mathematical description is given as follows:

Direct Problem: *Given a continuous function* $x : [0, 1] \to \mathbb{R}$, *compute* $y(t) := \int_0^t x(s)\,ds$, $t \in [0, 1]$.

Inverse Problem: *Given a differentiable function* $y : [0, 1] \to \mathbb{R}$, *determine* $x(t) := y'(t)$, $t \in [0, 1]$.

We are interested in the inverse problem. Additionally y *should be considered as the result of measurements (a very reasonable assumption in real life problems). Therefore, the data* y *are noisy and we may not expect that noisy data are continuously differentiable.*

At this point we distinguish: $\widetilde{y}$ *is the noisy data (the actually available data), obtained by measurements and contaminated by noise;* y *is the exact data, i.e., data that would be available if we were able to perform perfect measurements.*

Therefore, the inverse problem has no obvious solution. Moreover, the problem should not be formulated in the space of continuous functions, since perturbations due to noise lead to functions which are not continuous.

The differentiation of (measured) data is involved in many relevant inverse problems, e.g., in a mechanical system one may ask for hidden forces. Since Newton's law relates forces to velocities and accelerations, one has to differentiate observed data. We shall see that, in the problem of X-ray tomography, differentiation is implicitly present as well. □

In certain simple examples, inverse problems can be converted formally into direct problems. For example, if the system operator A has a known inverse, then the reconstruction problem is solved by $x := A^{-1}y$. However, the explicit determination of the inverse does not help if the output y is not in the domain of definition of A^{-1}. This situation is typical in applications, due to the fact that the output may be only partially known and/or distorted by noise.

In the linear case, i.e., if $A(p)$ is a linear map for every $p \in P$, problem (B) has been extensively studied and corresponding theory is well-developed. The state of the art in the nonlinear case is somewhat less satisfactory. Linearization is a very successful tool to find an acceptable solution to a nonlinear problem but, in general, this strategy provides only partial answers.

The identification problem (C), when formulated in a general setting, results in a rather difficult challenge, since it almost always gives raise to a (highly) nonlinear problem with many local solutions. Moreover, the input and/or output functions may be available only incompletely.

1.2 Ill-posed problems

One of the main tasks in the current research in inverse problems is the (stable) computation of approximate solutions of an operator equation, from a given set of observed data. The theory related to this task splits into two distinct parts: The first one deals with the ideal case in which the data are assumed to be exactly and completely known (the so called "exact data case"). The other one treats practical situations that arise when only incomplete and/or imprecise data are available (i.e., the "noisy data case").

It might be thought that the knowledge of the (exact) solution to an inverse problem in the exact data case, would prove itself useful also in the (practical) noisy data case. This is unfortunately not correct. It turns out in inverse problems that solutions obtained by analytic inversion formulas (whenever available) are very sensitive to the way in which the data set is completed, as well as to errors in it.

In order to achieve complete understanding of an inverse problem, the questions of **existence**, **uniqueness**, **stability** and **solution methods** are to be considered.
The questions of existence and uniqueness are of great importance in testing the assumptions behind any mathematical model. If the answer to the uniqueness question is negative, then one knows that even perfect data do not provide enough information to recover the physical quantity to be determined.
What concerns the stability question, one has to determine whether or not the solution depends continuously on the data. Stability is necessary if one wants to make sure that, a variation of the given data in a sufficiently small range leads to an arbitrarily small change in the solution. Obviously, one has to answer the stability question in a satisfactory way, before trying to devise reliable solution methods for solving an inverse problem.

The concept of stability is essential to discuss the main subject of this section, namely the **ill-posed problems**. This concept was introduced in 1902 by the french mathematician Jacques Hadamard[2] in connection with the study of boundary value problems for partial differential equations. He was the one who designated the unstable problems "ill-posed problems". The nature of inverse problems (which include irreversibility, causality, unmodelled structures, ...) leads to ill-posedness as an intrinsic characteristic of these problems.[3]

[2]See wikipedia.org/wiki/Jacques_Hadamard.

[3]Hadamard believed – as a matter of fact, many scientists still do – that ill-posed problems are actually incorrectly posed and, therefore, artificial in that they would not describe physical systems. He was wrong in this regard!

Figure 1.6: Jacques Hadamard.

When solving ill-posed problems numerically, we must certainly expect some difficulties, since any errors can act as a perturbation on the original equation, and so may cause arbitrarily large variations in the solution. Observational errors have the same effect. Since errors cannot be completely avoided, there may be a range of plausible solutions and we have to find out a reasonable solution. These ambiguities in the solution of inverse problems (which are unstable by nature) can be reduced by incorporating some *a-priori* information (whenever available) that limits the class of possible solutions. By "a-priori information" we mean some piece of information, which has been obtained independently of the observed data (e.g., smoothness, boundedness, sparsity, ...). This a-priori information may be given in the form of deterministic or statistical information. Here we shall restrict ourselves to deterministic considerations only.

We conclude this section presenting a tutorial example of an ill-posed problem. Actually, we revisit the problem introduced in Example 1.1.1 and consider a numerical version of the same inverse problem.

Example 1.2.1 (Numerical differentiation of data)**.** *Suppose that we have for the continuous function* $y : [0,1] \to \mathbb{R}$ *a measured function* $y^\delta : [0,1] \to \mathbb{R}$*, which is contaminated by noise in the following sense:*

$$|y^\delta(t) - y(t)| \ \leq \ \delta, \quad \textit{for all } t \in [0,1]\,.$$

In order to to reconstruct the derivative $x := y'$ *of* y *at* $\tau \in (0,1)$*, it seams reasonable to use the central difference approximation (CD) scheme*

$$x^{\delta,h}(\tau) \ := \ CD(y^\delta;\tau,h) \ := \ \frac{y^\delta(\tau+h) - y^\delta(\tau-h)}{2h}\,.$$

Thus, we obtain the error estimate

$$\begin{aligned} \big|x(\tau) - x^{\delta,h}(\tau)\big| \ & \leq \ \big|x(\tau) - CD(y;\tau,h)\big| \\ & \quad + \big|CD(y;\tau,h) - x^{\delta,h}(\tau)\big| \qquad (1.1) \\ & = \ \left|x(\tau) - \frac{y(\tau+h) - y(\tau-h)}{2h}\right| \\ & \quad + \left|\frac{(y-y^\delta)(\tau+h) - (y-y^\delta)(\tau-h)}{2h}\right|. \end{aligned}$$

The fist term on the right hand side of (1.1) *is called* **approximation error** *while the second term is the* **data error**.

Next we introduce a typical a-priori information about the exact solution: If we know a bound

$$|x'(t)| \ \leq \ E\,, \quad \textit{for all } t \in [0,1]\,, \tag{1.2}$$

where $E > 0$*, we are able to derive the estimate*

$$\big|x(\tau) - x^{\delta,h}(\tau)\big| \ \leq \ E\,h \,+\, \delta\,h^{-1}\,. \tag{1.3}$$

At this point, it becomes clear that the best to do is to choose $h > 0$*, s.t. it balances the two terms on the right hand side of* (1.3)*. This leads to the choice*

$$h(\delta) \ := \ E^{\frac{1}{2}}\,\delta^{\frac{1}{2}}$$

(for simplicity, we assume that $\{\tau - h, \tau + h\} \subset [0,1]$). The above choice of h results in the estimate

$$\left|x(\tau) - x^{\delta,h(\delta)}(\tau)\right| \;\leq\; 2E^{\frac{1}{2}}\,\delta^{\frac{1}{2}}\,. \tag{1.4}$$

From this very simple example we learn some important lessons:

- *The error estimate (1.1) consists of two main terms: the first one due to the approximation of the inverse mapping (first term on the rhs of (1.1)); the other one due to measurement errors.*
- *The first term can be estimated by $E\,h$, and converges to zero as $h \to 0$.*
- *The second term can be estimated by $\delta\,h^{-1}$ and, no matter how small the level of noise $\delta > 0$, it becomes unbounded as $h \to 0$.*
- *The balance of these two terms gives the "best possible" reconstruction result (under the a-priori assumption (1.2)).*

The estimate (1.1) for the reconstruction error is depicted in Figure 1.7. This picture describes a typical scenario for approximations in ill-posed problems.

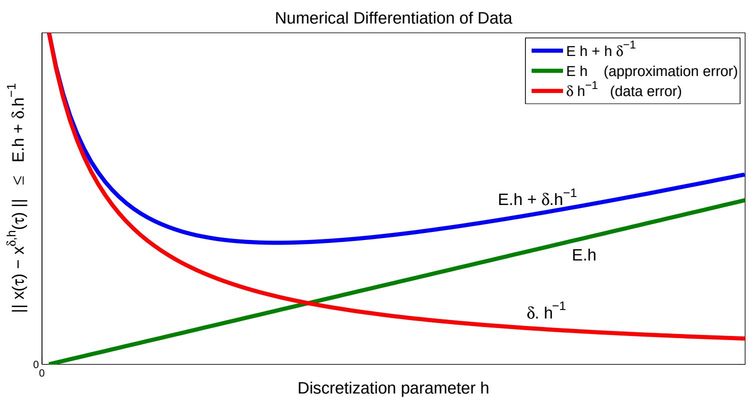

Figure 1.7: Error estimate for the inverse problem of numerical differentiation of data: $E\,h$ estimates the **approximation error** while $\delta\,h^{-1}$ estimates the **data error**.

In contrast to well-posed problems, it is not the best strategy to discretize finer and finer. One may consider ill-posed problems under the motto "When the imprecise is preciser" *(see title of [50]).* □

1.3 Regularization theory

Regularization theory is the area of mathematics dedicated to the analysis of methods (either direct or iterative) for obtaining stable solutions for ill-posed problems. The main results of this section appear in a more general form in [28, Chapter 2].

The exact data case

In order to introduce some basic concepts of regularization theory, we consider in this section a very simple functional analytical framework:

- Let F be a linear compact operator acting from the Hilbert spaces X into the Hilbert space Y.
- Our goal is to find a solution to the operator equation

$$F\,x = y\,, \tag{1.5}$$

  where the data $y \in Y$ is assumed to be exactly known (the noisy data case is considered later in this section).

In order to solve this ill-posed problem, we wish to construct a family of linear bounded operators $\{R_\alpha\}_{\alpha>0}$, such that $R_\alpha : Y \to X$ approximate $F^\dagger$ (the generalized inverse of F) in the sense that

$$\lim_{\alpha\to 0} R_\alpha\, y \;=\; x^\dagger \;:=\; F^\dagger\, y\,,$$

for each $y \in D(F^\dagger)$, where $x^\dagger$ is the least square solution of (1.5).

In what follows we adopt the notation $\widetilde{F} := F^*F : X \to X$ and $\widehat{F} := F\,F^* : Y \to Y$. Consequently, $x^\dagger \in X$ solves the normal equation $\widetilde{F}\,x^\dagger = F^*y$.

If $\widetilde{F}$ were invertible, one could think of computing $x^\dagger = \widetilde{F}^{-1}F^*y$. This corresponds to

$$x^\dagger \;=\; R(\widetilde{F})\,F^*y\,, \quad \text{whith} \quad R(t) := t^{-1}\,.$$

However, this is not a stable procedure since $\widetilde{F}$ is also a compact operator. A possible alternative is the following: even if $\widetilde{F}$ is not invertible, we can try to approximate $x^\dagger$ by elements $x_\alpha \in X$ of the form

$$x_\alpha \ := \ R_\alpha(\widetilde{F})F^*y\,, \ \ \alpha > 0\,,$$

where R_α is a real continuous function defined on the spectrum of $\widetilde{F}$, $\sigma(\widetilde{F}) \subset [0, \|F\|^2]$, which approximates the function $R(t) = 1/t$. It is worth noticing that $R_\alpha(\widetilde{F})F^* = F^*R_\alpha(\widehat{F})$. Moreover, the operators $R_\alpha(\widetilde{F}) : Y \to X$ are continuous for each $\alpha > 0$.

Remark 1.3.1. *Notice that we use the same notation to represent the operators $R_\alpha : Y \to X$, (approximating $F^\dagger$) as well as the real functions $R_\alpha : [0, \|F\|^2] \to \mathbb{R}$ (approximating $R(t) = 1/t$). As a matter of fact, the operators R_α are defined by $R_\alpha(\widetilde{F})F^* : Y \to X$, $\alpha > 0$.* □

Next we make some assumptions on the real functions R_α, which are sufficient to ensure that

$$\lim_{\alpha\to 0} x_\alpha \ = \ \lim_{\alpha\to 0} R_\alpha(\widetilde{F})F^*y \ = \ F^\dagger y \ = \ x^\dagger\,,$$

for each $y \in D(F^\dagger)$.

Assumption A1.

(A1.1) $\lim_{\alpha\to 0} R_\alpha(t) = 1/t$, for each $t > 0$;

(A1.2) $|t\,R_\alpha(t)|$, is uniformly bounded for $t \in [0, \|F\|^2]$ and $\alpha > 0$.

Theorem 1.3.2. *Let $\{R_\alpha\}_{\alpha>0}$ be a family of continuous real valued functions on $[0, \|F\|^2]$ satisfying Assumption (A1). The following assertions hold true:*

a) For each $y \in D(F^\dagger)$, $R_\alpha(\widetilde{F})F^*y \to F^\dagger y$ as $\alpha \to 0$;

b) If $y \notin D(F^\dagger)$ then, for any sequence $\alpha_n \to 0$, $\{R_{\alpha_n}(\widetilde{F})F^*y\}$ is not weakly convergent.

Some useful remarks:

- Assertion (a) in the above theorem brings a very positive message. Assertion (b), however, shows that we should be very cautious when dealing with ill-posed problems. It tell us that if $y \notin D(F^\dagger)$, the sequence $\{R_\alpha(\widetilde{F})F^*y\}_{\alpha>0}$ does not have weakly convergent subsequences.

- Since, in Hilbert spaces, every bounded sequence has a weakly convergent subsequence, it follows from assertion (b) that $y \notin D(F^\dagger)$ implies $\lim\limits_{\alpha\to 0} \|R_\alpha(\widetilde{F})F^*y\| = \infty$.

- Let us denote by $\Pi : Y \to \overline{R(F)}$ the orthogonal projection from Y onto the closure of $R(F)$. Theorem 1.3.2 shows that, in order to obtain convergence of x_α towards $x^\dagger$ it is necessary and sufficient that $\Pi\, y \in R(F)$.

We conclude the discussion of the exact data case by presenting some convergence rates for the approximations x_α, i.e., determining how fast the **approximation error** $e_\alpha := \|x_\alpha - x^\dagger\|$ converges to zero as $\alpha \to 0$.

From the above discussion, we know that condition $\Pi\, y \in R(F)$ (or, equivalently, $y \in D(F^\dagger)$) is not enough to obtain rates of convergence. Notice, however, that for every $x \in X$ we have

$$\Pi_{N(F)^\perp}\, x \;=\; \Pi_{N(\widetilde{F})^\perp}\, x \;=\; \lim_{\nu\to 0^+} \widetilde{F}^\nu\, x \qquad (1.6)$$

(here Π_H denotes the orthogonal projection of X onto the closed subspace $H \subset X$). This fact suggests that the stronger condition $\Pi\, y \in R(F\widetilde{F}^\nu)$, with $\nu > 0$, is a good candidate to prove the desired convergence rates.

Notice that if $\Pi\, y = F\widetilde{F}^\nu\, w$ for some $w \in X$, then $x^\dagger = F^\dagger y = \widetilde{F}^\nu w$. Reciprocally, if $x^\dagger = \widetilde{F}^\nu w$ for $w \in X$, then $\Pi\, y = F\widetilde{F}^\nu\, w$. Thus, the condition $\Pi\, y \in R(F\widetilde{F}^\nu)$, with $\nu > 0$ can be equivalently written in the form of the

Source condition: $x^\dagger = \widetilde{F}^\nu w$ for some $w \in X$, and $\nu > 0$.

Additionally to (A1), we make the assumption

Assumption B1. Assume that $t^\nu \, |1 - t\, R_\alpha(t)| \;\leq\; \omega(\alpha, \nu)$, for $t \in [0, \|F\|^2]$, where $\lim_{\alpha \to 0} \omega(\alpha, \nu) = 0$, for every $\nu > 0$.

The function $\omega(\cdot, \cdot)$ described above is called *convergence rate* function. We are now ready to state our main **convergence rate result**:

Theorem 1.3.3. *Let $\{R_\alpha\}_{\alpha>0}$ be a family of continuous functions on $[0, \|F\|^2]$ satisfying assumptions (A1) and (B1), and $\alpha > 0$. Moreover, suppose that the least square solution $x^\dagger$ satisfies a source condition for some $\nu \geq 1$ and $w \in X$. Then*

$$\|e_\alpha\| \;\leq\; \omega(\alpha, \nu)\, \|w\| \, .$$

For the convergence analysis of iterative regularization methods (see Example 1.3.9 below) another hypothesis proves to be more useful, namely $\Pi\, y \in R(\widehat{F}^\nu)$ for some $\nu \geq 1$.

Remark 1.3.4. *Notice that $\Pi\, y \in R(\widehat{F}^\nu)$ for some $\nu \geq 1$ is equivalent to $\Pi\, y \in R(F\, \widetilde{F}^{\nu-1} F^*)$. Hence, we obtain directly from Theorem 1.3.3 a rate of convergence of the order $\omega(\alpha, \nu - 1)$.* □

Theorem 1.3.5. *Let $\{R_\alpha\}_{\alpha>0}$ be a family of continuous functions on $[0, \|F\|^2]$ satisfying assumptions (A1) and (B1), and $\alpha > 0$. Moreover, suppose that $\Pi\, y = \widehat{F}^\nu\, w$ for some $\nu \geq 1$ and $w \in X$. The following assertions hold true:*

a) $\|e_\alpha\| \;\leq\; \omega(\alpha, \nu)\, \|F\, e_\alpha\|\, \|w\|$;

b) $\|e_\alpha\| \;\leq\; \big(\omega(\alpha, \nu - 1)\, \omega(\alpha, \nu)\big)^{1/2}\, \|w\|$.

The noisy data case

For the remaining of this section we shall consider the case of inexact data. If the data y in (1.5) is only imprecisely known, i.e., if only some noisy version y^δ is available satisfying

$$\|y - y^\delta\| \;\leq\; \delta \, , \tag{1.7}$$

where $\delta > 0$ is an *a-priori* known noise level, we still want to find a stable way of computing solutions for the ill-posed operator equation

$$F\,x = y^\delta\,. \tag{1.8}$$

A natural way is to use the available data to compute the approximations

$$x_\alpha^\delta \;:=\; R_\alpha(\widetilde{F})F^*y^\delta\,, \quad \alpha > 0\,,$$

These approximations are said to regular (or stable) if they converge (in some sense) to the minimal norm solution $x^\dagger$ as $\delta \to 0$. In other words, the approximations are regular, whenever there exists some choice of the regularization parameter α in terms of the noise level δ (i.e., a real function $\alpha : \delta \mapsto \alpha(\delta)$) such that

$$\lim_{\delta\to 0} x_{\alpha(\delta)}^\delta \;=\; \lim_{\delta\to 0} R_{\alpha(\delta)}(\widetilde{F})F^*y^\delta \;=\; F^\dagger y \;=\; x^\dagger\,. \tag{1.9}$$

This means that a **regularization method** consists not only of a choice of regularization functions $\{R_\alpha\}$, but also of a choice of the parameter function $\alpha(\delta)$ to determine the regularization parameter. The pair $(\{R_\alpha\}, \alpha(\cdot))$ determines a regularization method.

The choice of the regularization parameter function $\alpha(\cdot)$ may be either *a-priori* or *a-posteriori* (see, e.g., [21, 4]). Nevertheless, the mating of $\alpha(\cdot)$ with the noise present in the data is the most sensitive task in the regularization theory.

Before stating the first regularity results, we introduce some useful notation. From Assumption (A1) we conclude the existence of a constant $C > 0$ and a function $r(\alpha)$ such that

$$|t\,R_\alpha(t)| \;\le C^2, \quad \forall\; t \in [0, \|F\|^2], \quad \forall\; \alpha > 0\,.$$

$$r(\alpha) \;:=\; \max\{\; |R_\alpha(t)|\,, \quad \forall\; t \in [0, \|F\|^2]\;\}$$

(notice that (A1.1) implies that $\lim\limits_{\alpha\to 0} r(\alpha) = \infty$).

Theorem 1.3.6. *Let $\{R_\alpha\}_{\alpha>0}$ be a family of continuous functions on $[0, \|F\|^2]$ satisfying Assumption (A1), $y^\delta \in Y$ some noisy data satisfying (1.7), and $\alpha > 0$. The following assertions hold true:*

a) $\|F\,(x_\alpha - x_\alpha^\delta)\| \;\le\; C^2\,\delta$;

b) $\|x_\alpha - x_\alpha^\delta\| \;\leq\; C\,\delta\, r(\alpha)^{1/2}$*;*

where the constant $C > 0$ *and the function* $r(\alpha)$ *are defined as above.*

We are now ready to establish a sufficient condition on $\alpha(\delta)$ in order to prove regularity of the approximations x_α^δ in the noisy data case, i.e., in order to obtain (1.9).

Theorem 1.3.7. *Let* $\{R_\alpha\}_{\alpha>0}$ *be a family of continuous functions on* $[0, \|F\|^2]$ *satisfying Assumption (A1). Suppose that* $y \in D(F^\dagger)$*,* $\alpha(\delta) \to 0$ *and* $\delta^2 r(\alpha(\delta)) \to 0$*, as* $\delta \to 0$*. Then* $\lim\limits_{\delta\to 0} x_{\alpha(\delta)}^\delta = x^\dagger$*.*

Proof. Notice that

$$\|x^\dagger - x_{\alpha(\delta)}^\delta\| \;\leq\; \|x^\dagger - x_{\alpha(\delta)}\| + \|x_{\alpha(\delta)} - x_{\alpha(\delta)}^\delta\| \tag{1.10}$$

Since $y \in D(F^\dagger)$, Theorem 1.3.2 (a) and assumption $\lim\limits_{\delta\to 0} \alpha(\delta) = 0$, guarantee that $\|x^\dagger - x_{\alpha(\delta)}\| \to 0$, as $\delta \to 0$. On the other hand, from Theorem 1.3.6 (b) and the assumption $\lim\limits_{\delta\to 0} \delta^2 r(\alpha(\delta)) = 0$ we conlcude that $\|x_{\alpha(\delta)} - x_{\alpha(\delta)}^\delta\| \to 0$ as $\delta \to 0$. □

Some useful remarks:

- Assertion (a) in Theorem 1.3.2 is called **convergence result**. It means that, it we have exact data $y \in D(F^\dagger)$, the family of operators $\{R_\alpha\}$ generate approximate solutions x_α satisfying $\|x^\dagger - x_\alpha\| \to 0$ as $\alpha \to 0$.

- Assertion (b) in Theorem 1.3.6 is called **stability result**. It means that, if only noisy data $y^\delta \in Y$ satisfying (1.7) is available, then the distance between the "ideal approximate solution" x_α and the "computable approximate solution" x_α^δ is bounded by $C\,\delta\, r(\alpha)^{1/2}$ (this bound may explode as $\alpha \to 0$; indeed, as already observed, (A1.1) implies $\lim\limits_{\alpha\to 0} r(\alpha) = \infty$.

- It becomes clear from estimate (1.10) that "convergence" and "stability" are the key ingredients to prove Theorem 1.3.7, which is called **semi-convergence result**. According to this theorem, if one can has better and better measured noisy data y^δ with $\delta \to 0$, then the "computable approximate solutions" $x_{\alpha(\delta)}^\delta$ converge to $x^\dagger$ as $\delta \to 0$.

- The assumptions on the parameter choice function $\alpha(\delta)$ on Theorem 1.3.7 mean that $\alpha(\delta)$ must converge to zero as $\delta \to 0$, but not very fast, since $\delta^2 r(\alpha(\delta))$ must also converge to zero, as $\delta \to 0$.

- Suppose that fixed noisy data $y_\delta \in Y$ is available (with $\delta > 0$). The the regularization parameter α plays the same role as the discretization level $h > 0$ in Example 1.3.8.
  The **approximation error** $\|x^\dagger - x_\alpha\|$ in (1.10) corresponds to the term $|x(\tau) - CD(y;\tau,h)|$ in (1.1), while the **data error** $\|x_\alpha - x_\alpha^\delta\|$ in (1.10) corresponds to $|CD(y;\tau,h) - x^{\delta,h}(\tau)|$ in (1.1).
  If we freely choose $\alpha > 0$ (disregarding the parameter choice function $\alpha(\delta)$), the behaviour of "approximation error" and "data error" is exactly as depicted in Figure 1.7 for the problem of numerical differentiation of data.

Example 1.3.8. *(Tikhonov regularizarion)*
Consider the family of functions defined by

$$R_\alpha(t) \;:=\; (t+\alpha)^{-1}\,, \quad \alpha > 0\,,$$

that is

$$x_\alpha \;:=\; (\widetilde{F} + \alpha I)^{-1} F^* y\,.$$

This is called Tikhonov regularization. *Notice that Assumption* (A1) *is satisfied. Moreover, since*

$$|t\,R_\alpha(t)| \;\le\; 1 \quad \text{and} \quad \max_{t\ge 0} |R_\alpha(t)| \;=\; \alpha^{-1}\,, \tag{1.11}$$

we are allowed to choose $C := 1$ and $r(\alpha) := \alpha^{-1}$ (see Exercise 1.10). Therefore, the conclusion of Theorem 1.3.7 holds for the Tikhonov regularization method.

Moreover, for this method we may choose the convergence rate function $\omega(\alpha,\nu) := \alpha^\nu$, for $\nu \in (0,1]$ (prove!). □

Example 1.3.9. *(Landweber method – iterative regularization)*
The Landweber (or Landweber-Friedman) iterative method for the operator equation (1.5) is defined by

$$x_0 \;:=\; \lambda\,F^* y\,,\;\; x_{k+1} \;:=\; x_k - \lambda\,F^*(F\,x_k - y) \;=\; (I - \lambda\,\widetilde{F})\,x_k + \lambda\,F^* y\,,$$

where the positive constant λ *satisfies* $0 < \lambda < 2\|F\|^{-2}$ *(see Exercise 1.11). This method corresponds to the family of functions*

$$R_k(t) \;:=\; \lambda \sum_{j=0}^{k} (1-\lambda t)^j\,, \quad k \ge 0\,.$$

Notice that the regularization parameter is now the iteration number $k = k(\delta)$*, where* $k(\delta)$ *is the parameter choice function.*[4]

For this method we can choose

$$C \;:=\; 1 \quad \text{and} \quad r(k) \;:=\; \lambda\,(k+1) \tag{1.12}$$

and find that the conclusion of Theorem 1.3.7 also holds for this iterative regularization method (see Exercise 1.12). □

Example 1.3.10. *(Spectral cut-off method)*
The spectral cut-off method (or truncated singular function expansion) is defined by the family of operators

$$R_k(t) \;:=\; \begin{cases} 1/t\,, & t \ge \mu_k^{-2} \\ 0\,, & t < \mu_{k+1}^{-2} \end{cases} \tag{1.13}$$

where $\{u_j, v_j; \mu_j\}$ *is a sungular system for* F*. With this choice, one obtains the approximations*

$$x_k \;:=\; \sum_{j=1}^{k} \mu_j \langle y, u_j\rangle\, v_j \tag{1.14}$$

(see Exercise 1.13). Moreover, one can choose $C := 1$ *and* $r(k) := 1/k$*, such that Theorem 1.3.7 also holds for the spectral cut-off method.* □

1.4 Bibliographical comments

In the 1970's, the monograph of Tikhonov and Arsenin [82] can be condidered as the starting point of a systematic study of inverse problems. Nowadays, there exists a vast amount of literature on several aspects of inverse problems and ill-posedness. Instead of giving a complete list of relevant contributions we mention only some monographs [2, 21, 28, 51, 69, 4] and survey articles [35, 86].

[4] This situation can be easily fitted in the above framework by setting $\alpha(\delta) := k(\delta)^{-1}$, i.e. $\alpha(\delta)$ is a piecewise constant function with $\lim_{\delta\to 0} \alpha(\delta) = 0$.

1.5 Exercises

1.1. Find a polynomial p with coefficients in $\mathbb{C}$ with given zeros $\xi_1, \ldots, \xi_n$. If this problem is considered as an inverse problem, what is the formulation of the corresponding direct problem?

1.2. The problem of computing the eigenvalues of a given matrix is a classical problem in the linear algebra theory. If this problem is considered as a direct problem, what is the formulation of the corresponding inverse problem?

1.3. Show that under the stronger a-priori assumption

$$|x''(t)| \;\le\; E\,, \quad \text{for all} \;\; t \in [0,1]\,,$$

the inequality (1.3) can be improved and an estimate of the type

$$|x^{\delta,h(\delta)}(\tau) - x(\tau)| \;\le\; c\,E^{1/3}\,\delta^{2/3}$$

is possible (here $c > 0$ is some constant independent of δ and E).

1.4. A model for population growth is given by the ODE

$$u'(t) \;=\; q(t)\,u(t)\,, \; t \ge 0\,,$$

where the u represents the size of the population and q describes the growth rate. Derive a method to reconstruct q from the observation $u : [0,1] \to \mathbb{R}$, when q is a time dependent function from $[0,1]$ to $\mathbb{R}$.

1.5. *Can you hear the length of a string?*
Consider the boundary value problem

$$u'' \;=\; f\,, \; u(0) = u(L) = 0\,,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function. Suppose that the solution u and f are known. Find the length $L > 0$ of the interval.

1.6. Consider the boundary value problem

$$u'' + qu \;=\; f\,, \; u(0) = u(L) = 0\,,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function. Find sufficient conditions on f, such that q can be computed from the observation $u(\tau)$ for some point $\tau \in (0, L)$.

1.7. Prove equation (1.6).

1.8. Prove that the hypotesis $\Pi\, y \in R(F\widetilde{F}^{\nu})$, on the data, is equivalent to the hypotesis $x^{\dagger} \in R(\widetilde{F}^{\nu})$, on the solution.

1.9. Prove the assertion in Remark 1.3.4.

1.10. Prove the estimates for $|t\, R_{\alpha}(t)|$ and $\max\limits_{t \geq 0} |R_{\alpha}(t)|$ in (1.11).

1.11. Prove that the choice of λ in Example 1.3.9 is sufficient to guarantee that $\|I - \lambda\, \widetilde{F}\| \leq 1$.

1.12. Prove that the choices of C and $r(k)$ in (1.12) are in agreement with Assumption (A1) above.

1.13. Prove that x_k in (1.14) satisfies $x_k = R_k(\widetilde{F}) F^{*} y$, where R_k is defined as in (1.13).

Chapter 2

Tikhonov regularization

In this chapter we present the Tikhonov type regularization method and we summarise the main convergence results available in the literature. The adjective "type" refers to the extension of the classical Tikhonov method mainly by setting the penalisation term to be a general convex functional (instead of the usual quadratic norm) while the discrepancy term base on least squares is preserved.

This variation allow us not only to reconstruct a solution with special properties, but also to extend theoretical results for both linear and nonlinear operators defined between general topological spaces, e.g., Banach spaces. In the other hand we need to be acquainted with more sophisticated concepts and tools brought from smooth optimisation and functional analysis. For a review we recommend the reader to survey the books [15, 70, 20].

On the following we shall display a collection of results from [12, 74, 75, 42, 6], organised in a schematic way.

2.1 Tikhonov type methods

The general methods of mathematical analysis were best adapted to the solution of well-posed problems and they are no longer meaningful in most applications in the sense of ill-posed problems. One of the earliest works in this field and the most outstanding was done

Figure 2.1: Andrey Tikhonov.

by Andrey N. Tikhonov [1]. He succeeded in giving a precise mathematical definition of *approximated solution* for general classes of such problems and in constructing "optimal" solutions.

Tikhonov was a Soviet and Russian mathematician. He made important contributions in a number of different fields in mathematics, e.g., in topology, functional analysis, mathematical physics, and certain classes of ill-posed problems. Certainly, *Tikhonov regularization*, the most widely used method to solve ill-posed inverse problems, is named in his honour.

Nevertheless, we should make a note that Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations from the work of Tikhonov [81] and David L. Phillips [71]. Some authors use the term Tikhonov-Phillips regularization.

We focus on the quadratic regularization methods for solving ill-

[1] See www.keldysh.ru/ANTikhonov-100/ANTikhonov-essay.html.

posed operator equations of the form

$$F(u) = g \ , \tag{2.1}$$

where $F : \mathscr{D}(F) \subset \mathcal{U} \to \mathcal{H}$ is an operator between infinite dimensional Banach spaces. Both linear and nonlinear problems are considered.

The Tikhonov type regularization consists of minimizing

$$J_\alpha^\delta(u) = \frac{1}{2}\|F(u) - g_\delta\|^2 + \alpha\mathcal{R}(u) \ , \tag{2.2}$$

where $\alpha \in \mathbb{R}_+$ is the regularization parameter and $\mathcal{R}$ is a proper convex functional. Moreover, we assume the noisy data g_δ is available under the deterministic assumption

$$\|g - g_\delta\| \leq \delta \ . \tag{2.3}$$

If the underlying equation has (infinite) many solutions, we select one among all *admissible solutions* which minimizes the functional $\mathcal{R}$; we call it the **$\mathcal{R}$-minimizing solution**.

The functional J_α^δ presented above represents a generalisation of the classical Tikhonov regularization [81, 29]. Consequently, the following questions should be considered on the new approach:

- For $\alpha > 0$, does a solution of (2.2) exist? Does the solution depends continuously on the data g_δ?

- Is the method convergent? (i.e., if the data g is exact and $\alpha \to 0$, do the minimizers of (2.2) converge to a solution of (2.1)?)

- Is the method stable in the following sense: if $\alpha = \alpha(\delta)$ is chosen appropriately, do the minimizers of (2.2) converge to a solution of (2.1) as $\delta \to 0$?

- What is the rate of convergence? How should the parameter $\alpha = \alpha(\delta)$ be chosen in order to get optimal convergence rates?

Existence and stability results can be found in the original articles cited above. In this chapter we focus on the last question and we repeat theorems (combined with a short proof) of error estimates and convergence rates.

To accomplish our task we assume throughout this chapter the following assumptions:

Assumption A2.

(A2.1) Given the Banach spaces $\mathcal{U}$ and $\mathcal{H}$ one associates the topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{H}}$, respectively, which are weaker than the norm topologies;

(A2.2) The topological duals of $\mathcal{U}$ and $\mathcal{H}$ are denoted by $\mathcal{U}^*$ and $\mathcal{H}^*$, respectively;

(A2.3) The norm $\|\cdot\|_{\mathcal{U}}$ is sequentially lower semi-continuous with respect to $\tau_{\mathcal{H}}$, i.e., for $u_k \to u$ with respect to the $\tau_{\mathcal{U}}$ topology, $\mathcal{R}(u) \leq \liminf_k \mathcal{R}(u_k)$;

(A2.4) $\mathscr{D}(F)$ has empty interior with respect to the norm topology and is $\tau_{\mathcal{U}}$-closed. Moreover[2], $\mathscr{D}(F) \cap \mathit{dom}\ \mathcal{R} \neq \emptyset$;

(A2.5) $F : \mathscr{D}(F) \subseteq \mathcal{U} \to \mathcal{H}$ is continuous from $(\mathcal{U}, \tau_{\mathcal{U}})$ to $(\mathcal{H}, \tau_{\mathcal{H}})$;

(A2.6) The functional $\mathcal{R} : \mathcal{U} \to [0, +\infty]$ is proper, convex, bounded from below and $\tau_{\mathcal{U}}$ lower semi-continuous;

(A2.7) For every $M > 0$, $\alpha > 0$, the sets

$$\mathcal{M}_{\alpha}(M) = \left\{ u \in \mathcal{U} \mid J_{\alpha}^{\delta}(u) \leq M \right\}$$

are $\tau_{\mathcal{U}}$ compact, i.e. every sequence (u_k) in $\mathcal{M}_{\alpha}(M)$ has a subsequence, which is convergent in $\mathcal{U}$ with respect to the $\tau_{\mathcal{U}}$ topology.

Convergence rates and error estimates with respect to the generalised *Bregman distances* were derived originally introduced in [11]. Even though this tool does not satisfy neither symmetry nor triangle inequality, it is still the key ingredient whenever we consider convex penalisation.

2.1.1 Well-posedness

In this section we display the main results on well-posedness, stability, existence and convergence of the regularization methods consisting in minimization of (2.2).

[2]on the following *dom* $\mathcal{R}$ denotes the *effective domain*, i.e., the set of elements where the functional $\mathcal{R}$ is bounded.

Theorem 2.1.1 ([42, Thm 3.1]). *Assume that $\alpha > 0$, $g_\delta \in \mathcal{H}$. Let the Assumption A2 be satisfied. Then there exists a minimizer of (2.2).*

Proof. See Exercise 2.1. □

Theorem 2.1.2 ([42, Thm 3.2]). *The minimizers of (2.2) are stable with respect to the data g_δ. That is, if $(u^j)_j$ is a sequence converging to $g_\delta \in \mathcal{H}$ with respect to the norm-topology, then every sequence $(u^j)_j$ satisfying*

$$u^j \in \arg\min\left\{\|F(u) - g_\delta\|^2 + \alpha\mathcal{R}(u) \;\middle|\; u \in \mathcal{U}\right\} \qquad (2.4)$$

has a subsequence, which converges with respect to the $\tau_\mathcal{U}$ topology, and the limit of each $\tau_\mathcal{U}$-convergent subsequence is a minimizer $\overline{u}$ of (2.2). Moreover, for each $\tau_\mathcal{U}$-convergent subsequence

$$(u^{j_m})_m \quad \text{and} \quad (\mathcal{R}(u^{j_m}))_m$$

converges to $\mathcal{R}(\overline{u})$.

Proof. See Exercise 2.2. □

Theorem 2.1.3 ([42, Thm 3.4]). *Let Assumption A2 be satisfied. If there exists a solution of (2.2), then there exists a $\mathcal{R}$-minimizing solution.*

Proof. See Exercise 2.3. □

Theorem 2.1.4 ([42, Thm 3.5]). *Let Assumption A2 be satisfied. Moreover, we assume that there exists a solution of (2.2) (Then, according to Theorem 2.1.3 there exists an $\mathcal{R}$-minimizing solution).*

Assume that the sequence δ_j converges monotonically to 0 and $g_j := g_{\delta_j}$ satisfies $\|g - g_j\| \leq \delta_j$.

Moreover, assume that $\alpha(\delta)$ satisfies

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^p}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0$$

and $\alpha(\cdot)$ is monotonically increasing.

A sequence $(u^j)_j$ satisfying (2.4) has a convergent subsequence with respect to the $\tau_\mathcal{U}$ topology. A limit of each $\tau_\mathcal{U}$-convergent subsequence is an $\mathcal{R}$-minimizing solution. If in addition the $\mathcal{R}$-minimizing solution $\overline{u}$ is unique, then $u^j \to \overline{u}$ with respect to $\tau_\mathcal{U}$.

Proof. See Exercise 2.4. □

2.2 Convergence rate results for linear problems

In this section we consider the linear case. Therefore the Equation (2.1) shall be denoted by $Fu = g$, where the operator F is defined from a Banach space into a Hilbert space. The main results of this section were proposed originally in [12, 74].

2.2.1 Rates of convergence for SC of type I

First of all we have to decide which "solution" we aim to recover for the underlying problem. Therefore in this section we assume that the noise free data g is *attainable*, i.e., $g \in \mathscr{R}(F)$ and so we define u an *admissible solution* if u satisfies

$$Fu = g. \tag{2.5}$$

In particular, among all admissible solutions, we denote $\overline{u}$ the $\mathcal{R}$-minimizing solution of (2.5).

Secondly, error estimates between the regularised solution u_δ^α and $\overline{u}$ can be obtained only under additional smoothness assumption. This assumption, also called source condition, can be stated in the following (slightly) different ways:

1. there exist at least one element ξ in $\partial\mathcal{R}(\overline{u})$ which belongs to the range of the adjoint operator of F;

2. there exists an element $\omega \in \mathcal{H}$ such that

$$F^*\omega =: \xi \in \partial\mathcal{R}(\overline{u}). \tag{2.6}$$

In summary we say the *Source Condition of type I* (SC-I) is satisfied if there is an element $\xi \in \partial\mathcal{R}(\overline{u}) \subseteq \mathcal{U}^*$ in the range of the operator F^*, i.e.,

$$\mathscr{R}(F^*) \cap \partial\mathcal{R}(\overline{u}) \neq \varnothing. \tag{2.7}$$

This assumption enable us to derive the upcoming stability result.

Theorem 2.2.1 ([12, Thm 2]). *Let (2.3) hold and let $\overline{u}$ be a $\mathcal{R}$-minimizing solution of (2.1) such that the source condition (2.7) and (2.5) are satisfied. Then, for each minimizer u_δ^α of (2.2) the estimate*

$$D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big) \leq \frac{1}{2\alpha}\left(\alpha\,\|\omega\| + \delta\right)^2 \tag{2.8}$$

holds for $\alpha > 0$. In particular, if $\alpha \sim \delta$, then $D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big) = \mathcal{O}(\delta)$.

Proof. We note that $\left\|F\overline{u} - g_\delta\right\| \leq \delta^2$, by (2.5) and (2.3). Since u_δ^α is a minimizer of the regularised problem (2.2), we have

$$\frac{1}{2}\left\|Fu_\delta^\alpha - g_\delta\right\| + \alpha\mathcal{R}(u_\delta^\alpha) \ \leq \ \frac{\delta^2}{2} + \alpha\mathcal{R}(\overline{u})\,.$$

Let $D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big)$ the Bregman distance between u_δ^α and $\overline{u}$, so the above inequality becomes

$$\frac{1}{2}\left\|Fu_\delta^\alpha - g_\delta\right\| + \alpha\left(D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big) + \langle F^*\omega, u_\delta^\alpha - \overline{u}\rangle\right) \ \leq \ \frac{\delta^2}{2}\,.$$

Hence, using (2.3) and Cauchy-Schwarz inequality we can derive the estimate

$$\frac{1}{2}\left\|Fu_\delta^\alpha - g_\delta\right\| + \langle \alpha\omega\ ,\ Fu_\delta^\alpha - g_\delta\rangle_{\mathcal{H}} + \alpha D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big) \ \leq \ \frac{\delta^2}{2} + \alpha\,\|\omega\|\,\delta\,.$$

Using the the equality $\|a+b\| = \|a\| + 2\langle a, b\rangle + \|b\|$, it is easy to see that

$$\frac{1}{2}\left\|Fu_\delta^\alpha - g_\delta + \alpha\omega\right\| + \alpha D_{\mathcal{R}}^{F^*\omega}\big(u_\delta^\alpha, \overline{u}\big) \ \leq \ \frac{\alpha^2}{2}\,\|\omega\| + \alpha\delta\,\|\omega\| + \frac{\delta^2}{2}\ ,$$

which yields (2.8) for $\alpha > 0$. □

Theorem 2.2.2 ([12, Thm 1]). *If $\overline{u}$ is a $\mathcal{R}$-minimizing solution of (2.1) such that the source condition (2.7) and (2.5) are satisfied, then for each minimizer u^α of (2.2) with exact data, the estimate*

$$D_{\mathcal{R}}^{F^*\omega}\big(u^\alpha, \overline{u}\big) \leq \frac{\alpha}{2}\,\|\omega\|^2$$

holds true.

Proof. See Exercise 2.5. □

2.2.2 Rates of convergence for SC of type II

In this section we use another type of source condition, which is stronger than the one assumed in previous subsection. We relax the definition of *admissible solution*, where it is understood in the context of least-squares[3], i.e.,

$$F^*Fu = F^*g\,. \tag{2.9}$$

Note that we do not require $g \in \mathscr{R}(F)$. Moreover, we still denote $\overline{u}$ the $\mathcal{R}$-minimizing solution, but instead with respect to (2.9).

Likewise in the previous section, we introduce the *Source Condition of type II* (SC-II)[4] as follows: there exists one element $\xi \in \partial\mathcal{R}\left(\overline{u}\right) \subset \mathcal{U}^*$ in the range of the operator F^*F,

$$\xi \in \mathscr{R}(F^*F) \cap \partial\mathcal{R}\left(\overline{u}\right) \neq \varnothing\,. \tag{2.10}$$

This condition is equivalent to the existence of $\omega \in \mathcal{U}\setminus\{0\}$ such that $\xi = F^*F\omega$, where F^* is the adjoint operator of F and $F^*F : \mathcal{U} \to \mathcal{U}^*$.

Theorem 2.2.3 ([74, Thm 2.2])**.** *Let* (2.3) *hold and let* $\overline{u}$ *be a* $\mathcal{R}$*-minimizing solution of* (2.1) *such that the source condition* (2.10) *as well as* (2.9) *are satisfied. Then the following inequalities hold for any* $\alpha > 0$*:*

$$D_{\mathcal{R}}^{F^*F\omega}\left(u_\delta^\alpha, \overline{u}\right) \leq D_{\mathcal{R}}^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right) + \frac{\delta^2}{\alpha} + \frac{\delta}{\alpha}\sqrt{\delta^2 + 2\alpha D_{\mathcal{R}}^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right)}, \tag{2.11}$$

$$\left\|Fu_\delta^\alpha - F\overline{u}\right\| \leq \alpha\left\|F\omega\right\| + \delta + \sqrt{\delta^2 + 2\alpha D_{\mathcal{R}}^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right)}\,. \tag{2.12}$$

Proof. Since u_δ^α is a minimizer of (2.2), it follows from algebraic ma-

[3] in the literature this definition of generalised solution is also known as *best-approximate solution*.

[4] also called *source condition of second kind*.

nipulation and from the definition of Bregman distance that

$$\begin{aligned} 0 \;&\geq\; \frac{1}{2}\left[\left\|Fu_\delta^\alpha - g_\delta\right\| - \left\|Fu - g_\delta\right\|\right] + \alpha\mathcal{R}(u_\delta^\alpha) - \alpha\mathcal{R}(u) \\ &=\; \frac{1}{2}\left[\left\|Fu_\delta^\alpha\right\| - \left\|Fu\right\|\right] - \left\langle F\left(u_\delta^\alpha - u\right)\ ,\ g_\delta\right\rangle_{\mathcal{H}} - \alpha D_{\mathcal{R}}^{F^*F\omega}\big(u,\overline{u}\big) \\ &\quad + \alpha\left\langle F\omega\ ,\ F\left(u_\delta^\alpha - u\right)\right\rangle_{\mathcal{H}} + \alpha D_{\mathcal{R}}^{F^*F\omega}\big(u_\delta^\alpha,\overline{u}\big)\,. \end{aligned} \tag{2.13}$$

Notice that

$$\begin{aligned} \left\|Fu_\delta^\alpha\right\| - \left\|Fu\right\| \;&=\; \left\|F\left(u_\delta^\alpha - \overline{u} + \alpha\omega\right)\right\| - \left\|F\left(u - \overline{u} + \alpha\omega\right)\right\| \\ &\quad + 2\left\langle Fu_\delta^\alpha - Fu\ ,\ F\overline{u} - \alpha F\omega\right\rangle_{\mathcal{H}}\,. \end{aligned}$$

Moreover, by (2.9), we have

$$\left\langle F\left(u_\delta^\alpha - u\right)\ ,\ g_\delta - F\overline{u}\right\rangle_{\mathcal{H}} = \left\langle F\left(u_\delta^\alpha - u\right)\ ,\ g_\delta - g\right\rangle_{\mathcal{H}}.$$

Therefore, it follows from (2.13) that

$$\begin{aligned} &\frac{1}{2}\left\|F\left(u_\delta^\alpha - \overline{u} + \alpha\omega\right)\right\| + \alpha D_{\mathcal{R}}^{F^*F\omega}\big(u_\delta^\alpha,\overline{u}\big) \\ \leq\;\; &\left\langle F\left(u_\delta^\alpha - u\right)\ ,\ g_\delta - g\right\rangle_{\mathcal{H}} + \alpha D_{\mathcal{R}}^{F^*F\omega}\big(u,\overline{u}\big) + \frac{1}{2}\left\|F\left(u - \overline{u} + \alpha\omega\right)\right\| \end{aligned}$$

for every $u \in \mathcal{U}$, $\alpha \geq 0$ and $\delta \geq 0$.

Replacing u by $\overline{u} - \alpha\omega$ in the last inequality, using (2.3), relations $\langle a, b\rangle \leq |\langle a, b\rangle| \leq \|a\|\,\|b\|$, and defining $\gamma = \left\|F\left(u_\delta^\alpha - \overline{u} + \alpha\omega\right)\right\|$ we obtain

$$\frac{1}{2}\gamma^2 + \alpha D_{\mathcal{R}}^{F^*F\omega}\big(u_\delta^\alpha,\overline{u}\big) \;\leq\; \delta\gamma + \alpha D_{\mathcal{R}}^{F^*F\omega}\big(\overline{u} - \alpha\omega,\overline{u}\big)\,.$$

We estimate separately each term on the left-hand side by right-hand side. One of the estimates is an inequality in the form of a polynomial of the second degree for γ, which gives us the inequality

$$\gamma \leq \delta + \sqrt{\delta^2 + 2\alpha D_{\mathcal{R}}^{F^*F\omega}\big(\overline{u} - \alpha\omega,\overline{u}\big)}\,.$$

This inequality together with the other estimate, gives us (2.11). Now, (2.12) follows from the fact that $\left\|F\left(u_\delta^\alpha - \overline{u}\right)\right\| \leq \gamma + \alpha\left\|F\omega\right\|$. □

Theorem 2.2.4 ([74, Thm 2.1]). *Let $\alpha \geq 0$ be given. If $\overline{u}$ is a $\mathcal{R}$-minimizing solution of (2.1) satisfying the source condition (2.10) as well as (2.9), then the following inequalities hold true:*

$$D_{\mathcal{R}}^{F^*F\omega}\left(u^\alpha, \overline{u}\right) \leq D_{\mathcal{R}}^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right),$$

$$\|Fu^\alpha - F\overline{u}\| \leq \alpha \|F\omega\| + \sqrt{2\alpha D_{\mathcal{R}}^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right)}.$$

Proof. See Exercise 2.6. □

Corollary 2.2.5 ([74]). *Let the assumptions of the Theorem 2.2.3 hold true. Further, assume that $\mathcal{R}$ is twice differentiable in a neighbourhood U of $\overline{u}$ and there exists a number $M > 0$ such that for any $v \in \mathcal{U}$ and $u \in U$ the inequality*

$$\langle \mathcal{R}''(u)v, v\rangle \leq M \|v\|^2 \tag{2.14}$$

hold true. Then, for the parameter choice $\alpha \sim \delta^{\frac{2}{3}}$ we have

$$D_{\mathcal{R}}^{\xi}\left(u_\delta^\alpha, \overline{u}\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}\right).$$

Moreover, for exact data we have $D_{\mathcal{R}}^{\xi}\left(u^\alpha, \overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$.

Proof. Using Taylor's expansion at the element $\overline{u}$ we obtain

$$\mathcal{R}(u) \;=\; \mathcal{R}(\overline{u}) + \langle \mathcal{R}'(\overline{u}), u - \overline{u}\rangle + \frac{1}{2}\langle \mathcal{R}''(\mu)(u - \overline{u}), u - \overline{u}\rangle$$

for some $\mu \in [u, \overline{u}]$. Let $u = \overline{u} - \alpha\omega$ in the above equality. For sufficiently small α, it follows from assumption (2.14) and the definition of the Bregman distance, with $\xi = \mathcal{R}'(\overline{u})$, that

$$\begin{aligned} D_{\mathcal{R}}^{\xi}\left(\overline{u} - \alpha\omega, \overline{u}\right) &= \frac{1}{2}\langle \mathcal{R}''(\mu)(-\alpha\omega), -\alpha\omega\rangle \\ &\leq \alpha^2 \frac{M}{2} \|\omega\|_{\mathcal{U}}^2 . \end{aligned}$$

Note that $D_{\mathcal{R}}^{\xi}\left(\overline{u} - \alpha\omega, \overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$, so the desired rates of convergence follow from Theorems 2.2.3 and 2.2.4. □

2.3 Convergence rate results for non-linear problems

This section displays a collection the convergence analysis for the linear problems. In contrast with other classical conditions, the following analysis covers the case when both $\mathcal{U}$ and $\mathcal{H}$ are Banach spaces.

Back to [22] we learn through two examples of linear problems the interesting effect: ill-posedness of a linear problem need not imply ill-posedness of its linearisation. Also that the converse implication need not be true. A well-posed linear problem may have ill-posed linearisation. Hence we need additional assumptions concerning both operator and its linearisation.

This assumption is known as *linearity condition* and it is based on first-order Taylor expansion of the operator F around $\overline{u}$. The linearity condition assumed in this section is given originally in [75] and stated as follows.

Assumption B2. Assume that a $\mathcal{R}$-minimizing solution $\overline{u}$ of (2.1) exists and that the operator $F : \mathscr{D}(F) \subseteq \mathcal{U} \to \mathcal{H}$ is Gâteaux differentiable. Moreover, we assume that there exists $\rho > 0$ such that, for every $u \in \mathscr{D}(F) \cap \mathcal{B}_\rho\left(\overline{u}\right)$

$$\left\|F\left(u\right) - F\left(\overline{u}\right) - F'\left(\overline{u}\right)\left(u - \overline{u}\right)\right\| \leq c D_{\mathcal{R}}^{\xi}\left(u, \overline{u}\right) \ , \ c > 0 \qquad (2.15)$$

and $\xi \in \partial\mathcal{R}\left(\overline{u}\right)$.

2.3.1 Rates of convergence for SC of type I

In comparison with the source condition (2.7) introduced on previous section, the extension of the *Source Condition of type I* to linear problems are done with respect to the linearisation of the operator and its adjoint. Namely, we assume that

$$\xi \in \mathscr{R}(F'\left(\overline{u}\right)^*) \cap \partial\mathcal{R}\left(\overline{u}\right) \neq \varnothing \qquad (2.16)$$

where $\overline{u}$ is a $\mathcal{R}$-minimizing solution of (2.1).

Note that the derivative of operator F is defined between the Banach space $\mathcal{U}$ and $\mathscr{L}\left(\mathcal{U}, \mathcal{H}\right)$, the space of the linear transformations from $\mathcal{U}$ into $\mathcal{H}$. When we apply the derivative at $\overline{u} \in \mathcal{U}$ we have a linear operator $F'(\overline{u}) : \mathcal{U} \to \mathcal{H}$ and so we define its adjoint

$$F'(\overline{u})^* : \mathcal{H}^* \to \mathcal{U}^*.$$

The source condition (2.16) is stated equivalently as follows: there exists an element $\omega \in \mathcal{H}^*$ such that

$$\xi = F'(\overline{u})^* \omega \in \partial\mathcal{R}(\overline{u}) \,. \tag{2.17}$$

Theorem 2.3.1 ([75, Thm 3.2])**.** *Let the Assumptions A2, B2 and relation (2.3) hold true. Moreover, assume that there exists $\omega \in \mathcal{H}^*$ such that (2.17) is satisfied and $c\|\omega\|_{\mathcal{H}^*} < 1$. Then, the following estimates hold:*

$$\|F(u_\delta^\alpha) - F(\overline{u})\| \le 2\alpha \|\omega\|_{\mathcal{H}^*} + 2\left(\alpha^2 \|\omega\|_{\mathcal{H}^*}^2 + \delta^2\right)^{\frac{1}{2}} ,$$

$$D_{\mathcal{R}}^{F'(\overline{u})^*\omega}(u_\delta^\alpha, \overline{u}) \le \left(\frac{2}{1 - c\|\omega\|_{\mathcal{H}^*}}\right) \cdot \left[\frac{\delta^2}{2\alpha} + \alpha\|\omega\|_{\mathcal{H}^*}^2 + \|\omega\|_{\mathcal{H}^*}\left(\alpha^2\|\omega\|_{\mathcal{H}^*}^2 + \delta^2\right)^{\frac{1}{2}}\right] .$$

In particular, if $\alpha \sim \delta$, then

$$\|F(u_\delta^\alpha) - F(\overline{u})\| = \mathcal{O}(\delta) \ \text{ and } D_{\mathcal{R}}^{F'(\overline{u})^*\omega}(u_\delta^\alpha, \overline{u}) = \mathcal{O}(\delta) \,.$$

Proof. Since u_δ^α is the minimizer of (2.2), it follows from the definition of the Bregman distance that

$$\frac{1}{2}\left\|F(u_\delta^\alpha) - g_\delta\right\| \le \frac{1}{2}\delta^2 - \alpha\left(D_{\mathcal{R}}^{F'(\overline{u})^*\omega}(u_\delta^\alpha, \overline{u}) + \langle F'(\overline{u})^*\omega, u_\delta^\alpha - \overline{u}\rangle\right).$$

By using (2.3) and (2.1) we obtain

$$\frac{1}{2}\left\|F(u_\delta^\alpha) - F(\overline{u})\right\| \quad \le \quad \left\|F(u_\delta^\alpha) - g_\delta\right\| + \delta^2 \,.$$

Now, using the last two inequalities above, the definition of Bregman distance, the linearity condition and the assumption

$\left(c\left\|\omega\right\|_{\mathcal{H}^*}-1\right)<0$, we obtain

$$
\begin{aligned}
\frac{1}{4}\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)\right\| \leq & \frac{1}{2}\left(\left\|F\left(u_\delta^\alpha\right)-g_\delta\right\|+\delta^2\right) \\
\leq & \delta^2-\alpha D_{\mathcal{R}}^{F'(\overline{u})^*\omega}\left(u_\delta^\alpha,\overline{u}\right)+\alpha\langle\omega,-F'\left(\overline{u}\right)\left(u_\delta^\alpha-\overline{u}\right)\rangle \\
\leq & \delta^2-\alpha D_{\mathcal{R}}^{F'(\overline{u})^*\omega}\left(u_\delta^\alpha,\overline{u}\right) \\
& +\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)\right\| \\
& +\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)-F'\left(\overline{u}\right)\left(u_\delta^\alpha-\overline{u}\right)\right\| \\
= & \delta^2+\alpha\left(c\left\|\omega\right\|_{\mathcal{H}^*}-1\right)D_{\mathcal{R}}^{F'(\overline{u})^*\omega}\left(u_\delta^\alpha,\overline{u}\right) \\
& +\ \alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)\right\| \qquad (2.18) \\
\leq & \delta^2+\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)\right\| \qquad (2.19)
\end{aligned}
$$

From (2.19) we obtain an inequality in the form of a polynomial of second degree for the variable $\gamma=\left\|F\left(u_\delta^\alpha\right)-F\left(\overline{u}\right)\right\|$. This gives us the first estimate stated by the theorem. For the second estimate we use (2.18) and the previous estimate for γ. □

Theorem 2.3.2. *Let the Assumptions A2 and B2 hold true. Moreover, assume the existence of $\omega\in\mathcal{H}^*$ such that (2.17) is satisfied and $c\left\|\omega\right\|_{\mathcal{H}^*}<1$. Then, the following estimates hold:*

$$\left\|F\left(u^\alpha\right)-F\left(\overline{u}\right)\right\|\leq 4\alpha\left\|\omega\right\|_{\mathcal{H}^*}\,,$$

$$D_{\mathcal{R}}^{F'(\overline{u})^*\omega}\left(u^\alpha,\overline{u}\right)\leq\frac{4\alpha\left\|\omega\right\|_{\mathcal{H}^*}^2}{1-c\left\|\omega\right\|_{\mathcal{H}^*}}\,.$$

Proof. See Exercise 2.7. □

2.3.2 Rates of convergence for SC of type II

Similarly as in the previous subsection, the extension of the *Source Condition of type II* (2.10) to linear problems is given as:

$$\xi\in\mathscr{R}(F'\left(\overline{u}\right)^*F'\left(\overline{u}\right))\cap\partial\mathcal{R}\left(\overline{u}\right)\neq\varnothing$$

where $\overline{u}$ is a $\mathcal{R}$-minimizing solution of (2.1).

The assumption above has the following equivalent formulation: there exists an element $\omega\in\mathcal{U}$ such that

$$\xi=F'\left(\overline{u}\right)^*F'\left(\overline{u}\right)\omega\in\partial\mathcal{R}\left(\overline{u}\right)\,. \qquad (2.20)$$

Theorem 2.3.3 ([75, Thm 3.4]). *Let the Assumptions A2, B2 hold as well as estimate (2.3). Moreover, let $\mathcal{H}$ be a Hilbert space and assume the existence of a $\mathcal{R}$-minimizing solution $\overline{u}$ of (2.1) in the interior of $\mathscr{D}(F)$. Assume also the existence of $\omega \in \mathcal{U}$ such that (2.20) is satisfied and $c\,\|F'(\overline{u})\,\omega\| < 1$. Then, for α sufficiently small the following estimates hold:*

$$\|F(u_\delta^\alpha) - F(\overline{u})\| \leq \alpha \|F'(\overline{u})\,\omega\| + h(\alpha,\delta)\,,$$

$$D_{\mathcal{R}}^{\xi}\big(u_\delta^\alpha, \overline{u}\big) \leq \frac{\alpha s + (cs)^2/2 + \delta h(\alpha,\delta) + cs\,(\delta + \alpha\,\|F'(\overline{u})\,\omega\|)}{\alpha\,(1 - c\,\|F'(\overline{u})\,\omega\|)}\,, \tag{2.21}$$

where $h(\alpha,\delta) := \delta + \sqrt{(\delta+cs)^2 + 2\alpha s\,(1 + c\,\|F'(\overline{u})\,\omega\|)}$ *and* $s = D_{\mathcal{R}}^{\xi}\big(\overline{u} - \alpha\omega, \overline{u}\big)$*.*

Proof. Since u_δ^α is the minimizer of (2.2), it follows that

$$\begin{aligned} 0 &\geq \frac{1}{2}\big\|F(u_\delta^\alpha) - g_\delta\big\| - \frac{1}{2}\big\|F(u) - g_\delta\big\| + \alpha\,(\mathcal{R}(u_\delta^\alpha) - \mathcal{R}(u)) \\ &= \frac{1}{2}\big\|F(u_\delta^\alpha)\big\| - \frac{1}{2}\big\|F(u)\big\| + \langle F(u) - F(u_\delta^\alpha)\ , g_\delta\rangle_{\mathcal{H}} \\ &\quad + \alpha\,(\mathcal{R}(u_\delta^\alpha) - \mathcal{R}(u)) \\ &= \varrho\,(u_\delta^\alpha) - \varrho\,(u)\ . \end{aligned} \tag{2.22}$$

where $\varrho\,(u) = \frac{1}{2}\big\|F(u) - q\big\| + \alpha D_{\mathcal{R}}^{\xi}\big(u, \overline{u}\big) - \langle F(u)\ , g_\delta - q\rangle_{\mathcal{H}} + \alpha\langle \xi, u\rangle$, $q = F(\overline{u}) - \alpha F'(\overline{u})\,\omega$ and ξ is given by source condition (2.20).

From (2.22) we have $\varrho\,(u_\delta^\alpha) \leq \varrho\,(u)$. By the definition of $\varrho\,(\cdot)$, taking $u = \overline{u} - \alpha\omega$ and setting $v = F(u_\delta^\alpha) - F(\overline{u}) + \alpha F'(\overline{u})\,\omega$ we obtain

$$\frac{1}{2}\,\|v\| + \alpha D_{\mathcal{R}}^{\xi}\big(u_\delta^\alpha, \overline{u}\big) \quad \leq \quad \alpha s + T_1 + T_2 + T_3\,, \tag{2.23}$$

where s is given in the theorem, and

$$T_1 = \frac{1}{2}\big\|F(\overline{u} - \alpha\omega) - F(\overline{u}) + \alpha F'(\overline{u})\,\omega\big\|\,,$$

$$T_2 = \left|\langle F(u_\delta^\alpha) - F(\overline{u} - \alpha\omega)\ , g_\delta - g\rangle_{\mathcal{H}}\right|\,,$$

$$T_3 = \alpha \left\langle F'\left(\overline{u}\right)\omega \ , \ F\left(u_\delta^\alpha\right) - F\left(\overline{u} - \alpha\omega\right) - F'\left(\overline{u}\right)\left(u_\delta^\alpha - \left(\overline{u} - \alpha\omega\right)\right)\right\rangle_{\mathcal{H}} .$$

The next step is to estimate each one of the constants T_j above, $j = 1, 2$ and 3. We use the linear condition (2.15), Cauchy-Schwarz, and some algebraic manipulation to obtain $T_1 \leq \frac{c^2 s^2}{2}$,

$$\begin{aligned} T_2 \leq & \left|\left\langle v \ , \ g_\delta - g\right\rangle_{\mathcal{H}}\right| \\ & + \left|\left\langle F\left(\overline{u} - \alpha\omega\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega - \ , \ g_\delta - g\right\rangle_{\mathcal{H}}\right| \\ \leq & \left\|v\right\| \left\|g_\delta - g\right\| + c D_{\mathcal{R}}^{\xi}\left(\overline{u} - \alpha\omega, \overline{u}\right) \left\|g_\delta - g\right\| \\ \leq & \delta \left\|v\right\| + \delta c s \, , \end{aligned}$$

and

$$\begin{aligned} T_3 &= \alpha \left\langle F'\left(\overline{u}\right)\omega \ , \ F\left(u_\delta^\alpha\right) - F\left(\overline{u}\right) - F'\left(\overline{u}\right)\left(u_\delta^\alpha - \overline{u}\right)\right\rangle_{\mathcal{H}} \\ & \quad + \alpha \left\langle F'\left(\overline{u}\right)\omega \ , \ -\left(F\left(\overline{u} - \alpha\omega\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega\right)\right\rangle_{\mathcal{H}} \\ &\leq \alpha \left\|F'\left(\overline{u}\right)\omega\right\| \left\|F\left(u_\delta^\alpha\right) - F\left(\overline{u}\right) - F'\left(\overline{u}\right)\left(u_\delta^\alpha - \overline{u}\right)\right\| \\ & \quad + \alpha \left\|F'\left(\overline{u}\right)\omega\right\| \left\|F\left(\overline{u} - \alpha\omega\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega\right\| \\ &\leq \alpha \left\|F'\left(\overline{u}\right)\omega\right\| c D_{\mathcal{R}}^{\xi}\left(u_\delta^\alpha, \overline{u}\right) + \alpha \left\|F'\left(\overline{u}\right)\omega\right\| c D_{\mathcal{R}}^{\xi}\left(\overline{u} - \alpha\omega, \overline{u}\right) \\ &= \alpha c \left\|F'\left(\overline{u}\right)\omega\right\| D_{\mathcal{R}}^{\xi}\left(u_\delta^\alpha, \overline{u}\right) + \alpha c s \left\|F'\left(\overline{u}\right)\omega\right\| . \end{aligned}$$

Using these estimates in (2.23), we obtain

$$\begin{aligned} \left\|v\right\| + 2\alpha D_{\mathcal{R}}^{\xi}\left(u_\delta^\alpha, \overline{u}\right)\left[1 - c\left\|F'\left(\overline{u}\right)\omega\right\|\right] \leq & \ 2\delta\left\|v\right\| + 2\alpha s + (cs)^2 \\ & + 2\delta c s + 2\alpha c s \left\|F'\left(\overline{u}\right)\omega\right\| . \end{aligned}$$

Analogously as in the proof of Theorem 2.2.3, each term on the left-hand side of the last inequality is estimated separately by the right-hand side. This allows the derivation of an inequality described by a polynomial of second degree. From this inequality, the theorem follows. □

Theorem 2.3.4. *Let Assumptions A2, B2 hold and assume $\mathcal{H}$ to be a Hilbert space. Moreover, assume the existence of a $\mathcal{R}$-minimizing solution $\overline{u}$ of (2.1) in the interior of $\mathscr{D}(F)$, also the existence of $\omega \in \mathcal{U}$ such that (2.20) is satisfied and $c\left\|F'\left(\overline{u}\right)\omega\right\| < 1$. Then, for α sufficiently small the following estimates hold:*

$$\left\|F\left(u^\alpha\right) - F\left(\overline{u}\right)\right\| \leq \alpha\left\|F'\left(\overline{u}\right)\omega\right\| + \sqrt{\left(cs\right)^2 + 2\alpha s\left(1 + c\left\|F'\left(\overline{u}\right)\omega\right\|\right)} \, ,$$

$$D_{\mathcal{R}}^{\xi}\left(u^{\alpha},\overline{u}\right) \leq \frac{\alpha s + (cs)^2/2 + \alpha c s \left\|F'\left(\overline{u}\right)\omega\right\|_{\mathcal{H}}}{\alpha\left(1 - c\left\|F'\left(\overline{u}\right)\omega\right\|_{\mathcal{H}}\right)}\,, \tag{2.24}$$

where $s = D_{\mathcal{R}}^{\xi}\left(\overline{u} - \alpha\omega, \overline{u}\right)$.

Proof. See Exercise 2.8. □

Corollary 2.3.5 ([75, Prop 3.5])**.** *Let assumptions of the Theorem 2.3.3 hold true. Moreover, assume that* $\mathcal{R}$ *is twice differentiable in a neighbourhood* U *of* $\overline{u}$*, and that there exists a number* $M > 0$ *such that for all* $u \in U$ *and for all* $v \in \mathcal{U}$*, the inequality* $\langle \mathcal{R}''(u)v, v\rangle \leq M\left\|v\right\|$ *holds. Then, for the choice of parameter* $\alpha \sim \delta^{\frac{2}{3}}$ *we have* $D_{\mathcal{R}}^{\xi}\left(u_{\delta}^{\alpha},\overline{u}\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}\right)$*, while for exact data we obtain* $D_{\mathcal{R}}^{\xi}\left(u_{\delta}^{\alpha},\overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$ *.*

Proof. See Exercise 2.9. □

2.4 Bibliographical comments

We briefly comment on two new trends for deriving convergences rates, namely, *variational inequalities* and *approximated source condition*.

Since the first convergence rates results for linear problems given in [22] until the results [12, 74, 75] presented previously, the results of Engl and co-workers seems to be fully generalised. Nevertheless another paper concerning convergence rates came out [42] bringing new insights. The authors observed the following:

> In all these papers relatively strong regularity assumptions are made. However, it has been observed numerically that violations of the smoothness assumptions of the operator do not necessarily affect the convergence rate negatively. We take this observation and weaken the smoothness assumptions on the operator and prove a novel convergence rate result. The most significant difference in this result from the previous ones is that the source condition is formulated as a *variational inequality* and not as an equation as previously.

We display the **variational inequality** (VI) proposed in [42, Assumption 4.1], regardless auxiliary assumptions found in the paper.

Assumption C2. There exist numbers $c_1, c_2 \in [0, \infty)$, where $c_1 < 1$, and $\xi \in \partial\mathcal{R}(\overline{u})$ such that

$$\langle \xi, u - \overline{u} \rangle \leq c_1 D_{\mathcal{R}}^{\xi}(u, \overline{u}) + c_2 \|F(u) - F(\overline{u})\|$$

for all $u \in \mathcal{M}_{\alpha_{\max}}(\rho)$ where $\rho > \alpha_{\max}\left(\mathcal{R}(\overline{u}) + \frac{\delta^2}{\alpha}\right)$.

Additionally, it was proved that standard linearity conditions imply the new VI. Under this assumption one can derive the same rate of convergence obtained in Section 2.3. For more details see [42, 23, 44].

In [41] an alternative concept for proving convergence rates for linear problems in Hilbert spaces is presented, when the source condition

$$\overline{u} = F^*\omega, \quad \omega \in \mathcal{H}^* \tag{2.25}$$

is injured.

Instead we have an **approximated source condition** like

$$\overline{u} = F^*\omega + r,$$

where $r \in \mathcal{U}$. The theory is based on the decay rate of so-called *distance functions* which measures the degree of violation of the solution with respect to a prescribed benchmark source condition, e.g. (2.25). For the linear case the distance function is defined intuitively as

$$d(\rho) = \inf\left\{\|\overline{u} - F^*\omega\| \mid \omega \in \mathcal{H}^*, \|\omega\| \leq \rho\right\}.$$

The article [37] points out that this approach can be generalised to Banach spaces, as well as to linear operators. Afterwards, with the aid of this distance functions, the authors of [38] presented error bounds and convergence rates for regularised solutions of linear problems for Tikhonov type functionals when the reference source condition is not satisfied.

2.5 Exercises

2.1. Prove Theorem 2.1.1.

2.2. Prove Theorem 2.1.2.

2.3. Prove Theorem 2.1.3.

2.4. Prove Theorem 2.1.4.

2.5. Prove Theorem 2.2.2.

2.6. Prove Theorem 2.2.4.

2.7. Prove Theorem 2.3.2.

2.8. Prove Theorem 2.3.4.

2.9. Prove Corollary 2.3.5.

Chapter 3

Iterative regularization: Landweber type methods

The *Landweber*[1] *method* [55] is a classical iterative regularization method for solving ill-posed problems [55, 21, 48]. In this chapter we focus on a novel variant of this method, namely the *Landweber-Kaczmarz* iteration [52], which is designed to efficiently solve large systems of ill-posed equations in a stable way, and has been object of extensive study over the last decade [33, 31, 30, 56, 45].

3.1 Landweber-Kaczmarz method: Hilbert space approach

In this section we analyze novel iterative regularization techniques for the solution of systems of nonlinear ill–posed operator equations in Hilbert spaces. The basic idea consists in considering separately each equation of this system and incorporating a loping strategy. The first technique is a Kaczmarz type iteration, equipped with a

[1] See www.iihr.uiowa.edu/about/iihr-archives/landweber-archives.

novel stopping criteria. The second method is obtained using an embedding strategy, and again a Kaczmarz type iteration. We prove well-posedness, stability and convergence of both methods.

Figure 3.1: Louis Landweber.

3.1.1 Mathematical problem and iterative methods

We consider the problem of determining some physical quantity x from data $(y^i)_{i=0}^{N-1}$, which is functionally related by

$$\mathrm{F}_i(x) = y^i\,, \quad i = 0, \dots, N-1\,. \tag{3.1}$$

Here $\mathrm{F}_i : D_i \subseteq X \to Y$ are operators between separable Hilbert spaces X and Y. We are specially interested in the situation where the data is not exactly known, i.e., we have only an approximation $y^{\delta,i}$ of the exact data, satisfying

$$\|y^{\delta,i} - y^i\| < \delta^i\,. \tag{3.2}$$

Standard methods for the solution of such systems are based on rewriting (3.1) as a single equation

$$\mathrm{F}(x) = \mathbf{y}\,, \quad i = 0, \ldots, N-1\,, \tag{3.3}$$

where $\mathrm{F} := 1/\sqrt{N} \cdot (\mathrm{F}_0, \ldots, F_{N-1})$ and $\mathbf{y} = 1/\sqrt{N} \cdot (y^0, \ldots, y^{N-1})$. There are at least two basic concepts for solving ill posed equations of the form (3.3): *Iterative* regularization methods (cf., e.g., [55, 34, 21, 1, 48]) and *Tikhonov type* regularization methods [65, 82, 78, 66, 21]. However these methods become inefficient if N is large or the evaluations of $\mathrm{F}_i(x)$ and $\mathrm{F}_i'(x)^*$ are expensive. In such a situation Kaczmarz type methods [47, 69] which cyclically consider each equation in (3.1) separately, are much faster [68] and are often the method of choice in practice. On the other hand, only few theoretical results about regularizing properties of Kaczmarz methods are available, so far.

The *Landweber–Kaczmarz approach* for the solution of (3.1), (3.2) analyzed here consists in incorporating a bang-bang relaxation parameter in the classical Landweber–Kaczmarz method [52], combined with a new stopping rule. Namely,

$$x_{n+1} = x_n - \omega_n \mathrm{F}_{[n]}'(x_n)^* (\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]})\,, \tag{3.4}$$

with

$$\omega_n := \omega_n(\delta, y^\delta) = \begin{cases} 1 & \|\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]}\| > \tau\delta^{[n]} \\ 0 & \text{otherwise} \end{cases}\,, \tag{3.5}$$

where $\tau > 2$ is an appropriate chosen positive constant and $[n] := n \mod N \in \{0, \ldots, N-1\}$. The iteration terminates if all ω_n become zero within a cycle, that is if $\|\mathrm{F}_i(x_n) - y^{\delta,i}\| \le \tau\delta^i$ for all $i \in \{0, \ldots, N-1\}$. We shall refer to this method as *loping Landweber–Kaczmarz method* (lLK). Its worth mentioning that, for noise free data, $\omega_n = 1$ for all n and therefore, in this special situation, our iteration is identical to the classical Landweber–Kaczmarz method

$$x_{n+1} = x_n - \mathrm{F}_{[n]}'(x_n)^* (\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]})\,, \tag{3.6}$$

which is a special case of [68, Eq. (5.1)].

However, for noisy data, the lLK method is fundamentally different to (3.6): The parameter ω_n effects that the iterates defined in (3.4) become stationary and *all components* of the residual vector $\|F_i(x_n) - y^{\delta,i}\|$ fall below some threshold, making (3.4) a convergent regularization method. The convergence of the residuals in the maximum norm better exploits the error estimates (3.2) than standard methods, where only *squared average* $1/N \cdot \sum_{i=0}^{N-1} \|F_i(x_n) - y^{\delta,i}\|^2$ of the residuals falls below a certain threshold. Moreover, especially after a large number of iterations, ω_n will vanish for some n. Therefore, the computational expensive evaluation of $\mathrm{F}_{[n]}[x_n]^*$ might be loped, making the Landweber–Kaczmarz method in (3.4) a fast alternative to conventional regularization techniques for system of equations.

The second regularization strategy considered in this section is an embedding approach, which consists in rewriting (3.1) into an system of equations on the space X^N

$$\mathrm{F}_i(x^i) = y^i\,, \qquad i = 0, \dots, N-1\,, \tag{3.7}$$

with the additional constraint

$$\sum_{i=0}^{N-1} \|x^{i+1} - x^i\|^2 = 0\,, \tag{3.8}$$

where we set $x^N := x^0$. Notice that if x is a solution of (3.1), then the *constant vector* $(x^i = x)_{i=0}^{N-1}$ is a solution of system (3.7), (3.8), and vice versa. This system of equations is solved using a block Kaczmarz strategy of the form

$$\mathbf{x}_{n+1/2} = \mathbf{x}_n - \omega_n \mathbf{F}'(\mathbf{x}_n)^*(\mathbf{F}(\mathbf{x}_n) - \mathbf{y}^\delta) \tag{3.9}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_{n+1/2} - \omega_{n+1/2}\mathbf{G}(\mathbf{x}_{n+1/2})\,, \tag{3.10}$$

where $\mathbf{x} := (x^i)_i \in X^N$, $\mathbf{y}^\delta := (y^{\delta,i})_i \in Y^N$, $\mathbf{F}(\mathbf{x}) := (\mathrm{F}_i(x^i))_i \in Y^N$,

$$\begin{aligned} \omega_n &= \begin{cases} 1 & \|\mathbf{F}(\mathbf{x}_n) - \mathbf{y}^\delta\| > \tau\delta \\ 0 & \text{otherwise} \end{cases}\,, \\ \omega_{n+1/2} &= \begin{cases} 1 & \|\mathbf{G}(\mathbf{x}_{n+1/2})\| > \tau\epsilon(\delta) \\ 0 & \text{otherwise} \end{cases}\,, \end{aligned} \tag{3.11}$$

with $\delta := \max\{\delta^i\}$. The strictly increasing function $\epsilon : [0,\infty) \to [0,\infty)$ satisfies $\epsilon(\delta) \to 0$, as $\delta \to 0$, and guaranties the existence of a finite stopping index. A natural choice is $\epsilon(\delta) = \delta$. Moreover, up to a positive multiplicative constant, $\mathbf{G}$ corresponds to the steepest descent direction of the functional

$$\mathcal{G}(\mathbf{x}) := \sum_{i=0}^{N-1} \|x^{i+1} - x^i\|^2 \tag{3.12}$$

on X^N. Notice that (3.10) can also be interpreted as a Landweber–Kaczmarz step with respect to the equation

$$\lambda D(\mathbf{x}) = 0\,, \tag{3.13}$$

where $D(\mathbf{x}) = (x^{i+1} - x^i)_i \in X^N$ and λ is a small positive parameter such that $\|\lambda D\| \leq 1$. Since equation (3.1) is embedded into a system of equations on a higher dimensional function space we call the resulting regularization technique *embedded Landweber–Kaczmarz* (eLK) method. As shown in Section 3.1.3, (3.9), (3.10) generalizes the Landweber method for solving (3.3).

3.1.2 Analysis of the lLK method

In this section we present the convergence analysis of the *loping Landweber–Kaczmarz* (lLK) method. The novelty of this approach consists in omitting an update in the Landweber Kaczmarz iteration, within one cycle, if the corresponding i–th residual is below some threshold, see (3.5). Consequently, the lLK method is not stopped until all residuals are below the specified threshold. Therefore, it is the natural counterpart of the Landweber–Kaczmarz iteration [47, 69] for ill–posed problems.

The following assumptions are standard in the convergence analysis of iterative regularization methods [21, 34, 48]. We assume that F_i is *Fréchet differentiable* and that there exists $\rho > 0$ with

$$\|\mathrm{F}_i'(x)\|_Y \leq 1\,, \qquad x \in B_\rho(x_0) \subset \bigcap_{i=0}^{N-1} D_i\,. \tag{3.14}$$

Here $B_\rho(x_0)$ denotes the closed ball of radius ρ around the starting value x_0, D_i is the domain of F_i, and $\mathrm{F}_i'(x)$ is the Fréchet derivative of F_i at x.

Moreover, we assume that the *local tangential cone condition*

$$\|\mathrm{F}_i(x) - \mathrm{F}_i(\bar{x}) - \mathrm{F}_i'(x)(x - \bar{x})\|_Y \le \eta \|\mathrm{F}_i(x) - \mathrm{F}_i(\bar{x})\|_Y\,, \qquad x, \bar{x} \in B_\rho(x_0) \subset D_i \tag{3.15}$$

holds for some $\eta < 1/2$. This is a central assumption in the analysis of iterative methods for the solution of nonlinear ill–posed problems [21, 48].

In the analysis of the lLK method we assume that τ (used in the definition (3.5) of ω_n) satisfies

$$\tau > 2\frac{1+\eta}{1-2\eta} > 2\,. \tag{3.16}$$

Note that, for noise free data, the lLK method is equivalent to the classical Landweber–Kaczmarz method, since $\omega_n = 1$ for all $n \in \mathbb{N}$.

In the case of noisy data, iterative regularization methods require early termination, which is enforced by an appropriate stopping criteria. In order to motivate the stopping criteria, we derive in the following lemma an estimate related to the monotonicity of the sequence x_n defined in (3.4).

Lemma 3.1.1. *Let x be a solution of (3.1) where F_i are Fréchet differentiable in $B_\rho(x_0)$, satisfying (3.14), (3.15). Moreover, let x_n be the sequence defined in (3.4), (3.5). Then*

$$\|x_{n+1} - x\|^2 - \|x_n - x\|^2 \;\le\; \omega_n \|\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]}\| \cdot \cdot \Big(2(1+\eta)\delta^{[n]} - (1-2\eta)\|\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]}\| \Big), \tag{3.17}$$

where $[n] = \mod(n, N)$.

Proof. The proof follows the lines of [34, Proposition 2.2]. Notice that if ω_n is different from zero, inequality (3.17) follows analogously as in [34]. In the case $\omega_n = 0$, (3.17) follows from $x_n = x_{n+1}$. □

Motivated, by Lemma 3.1.1 we define the termination index $n_*^\delta = n_*^\delta(y^\delta)$ as the smallest integer multiple of N such that

$$x_{n_*^\delta} = x_{n_*^\delta + 1} = \cdots = x_{n_*^\delta + N}\,. \tag{3.18}$$

Now we have the following monotonicity result:

Lemma 3.1.2. *Let x, F_i and x_n be defined as in Lemma 3.1.1 and n_*^δ be defined by (3.18). Then we have*

$$\|x_{n+1} - x\| \leq \|x_n - x\| \,, \quad n = 0, \dots, n_*^\delta \,. \tag{3.19}$$

Moreover, the stoping rule (3.18) implies $\omega_{n_*^\delta + i} = 0$ for all $i \in \{0, \dots, N-1\}$, i.e.,

$$\|\mathrm{F}_i(x_{n_*^\delta}) - y^{\delta,i}\| \leq \tau\delta^i \,, \quad i = 0, \dots, N-1 \,. \tag{3.20}$$

Proof. If $\omega_n = 0$, then (3.19) holds since the iteration stagnates. Otherwise, from the definitions of ω_n in (3.5) and τ in (3.16), it follows that

$$2(1+\eta)\delta^i - (1-2\eta)\|\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]}\| < 0 \,, \tag{3.21}$$

and the right hand side in (3.17) becomes nonpositive.

To prove the second assertion we use (3.17) for $n = n_*^\delta + i$, for $i \in \{0, \dots, N-1\}$. By noting that $x_{n_*^\delta + i} = x_{n_*^\delta}$ and $[n_*^\delta + i] = i$, we obtain

$$0 \leq \omega_{n_*^\delta + i} \cdot \|\mathrm{F}_i(x_{n_*^\delta}) - y^{\delta,i}\| \Big(2(1+\eta)\delta^i - (1-2\eta)\|y^{\delta,i} - \mathrm{F}_i(x_{n_*^\delta})\| \Big),$$

for $i \in \{0, \dots, N-1\}$. Suppose $\omega_{n_*^\delta + i} \neq 0$, then

$$2(1+\eta)\delta^i - (1-2\eta)\|y^{\delta,i} - \mathrm{F}_i(x_{n_*^\delta})\| \geq 0,$$

which contradicts the definition of $\omega_{n_*^\delta + i}$. □

Note that for $n > n_*^\delta$, $\omega_n \equiv 0$ and therefore $x_n = x_{n_*^\delta}$. This shows that the Landweber–Kaczmarz method becomes stationary after n_*^δ.

Remark 3.1.3. *Similar to the nonlinear Landweber iteration one obtains the estimate*

$$\begin{aligned} \frac{n_*^\delta \cdot \big(\tau \min_i(\delta^i)\big)^2}{N} &\leq \sum_{n=0}^{n_*^\delta - 1} \omega_n \|y^{\delta,[n]} - \mathrm{F}_{[n]}(x_n)\|^2 \\ &\leq \frac{\tau \|x - x_{n_*^\delta}\|^2}{(1-2\eta)\tau - 2(1+\eta)} \,. \end{aligned} \tag{3.22}$$

Here we use the notation of Lemma 3.1.1.

From Remark 3.1.3 it follows that, in the case of noisy data, $n_*^\delta < \infty$ and the iteration terminates after a finite number of steps. Next, we state the main result of this section, namely that the Landweber–Kaczmarz method is a convergent regularization method.

Theorem 3.1.4. *Assume that* F_i *are Fréchet-differentiable in* $B_\rho(x_0)$*, satisfy* (3.14)*,* (3.15) *and the system* (3.1) *has a solution in* $B_{\rho/2}(x_0)$*. Then*

1. *For exact data* $y^{\delta,i} = y^i$*, the sequence* x_n *in* (3.4) *converges to a solution of* (3.1)*. Moreover, if* $x^\dagger$ *denotes the unique solution of* (3.1) *with minimal distance to* x_0 *and*

$$\mathcal{N}(\mathrm{F}_i'(x^\dagger)) \subseteq \mathcal{N}(\mathrm{F}_i'(x))\,, \quad x \in B_\rho(x_0)\,, \tag{3.23}$$

for $i \in \{0, \ldots, N-1\}$*, then* $x_n \to x^\dagger$*.*

2. *For noisy data the loping Landweber–Kaczmarz iterates* $x_{n_*^\delta}$ *converge to a solution of* (3.1) *as* $\delta \to 0$*. If in addition* (3.23) *holds, then* $x_{n_*^\delta}$ *converges to* $x^\dagger$ *as* $\delta \to 0$*.*

Proof. The proof of the first item is analogous to the proof in [52, Proposition 4.3] (see also [48]). We emphasize that, for exact data, the iteration (3.4) reduces to the classical Landweber–Kaczmarz method, which allows to apply the corresponding result of [52].

The proof of the second item is analogous to the proof of the corresponding result for the Landweber iteration as in [34, Theorem 2.9]. For the first case within this proof, (3.20) is required. For the second case we need the monotony result from Lemma 3.1.2. □

In the case of noisy data (i.e. the second item of Theorem 3.1.4), it has been shown in [52] that the Landweber–Kaczmarz iteration

$$x_{n+1} = x_n - \mathrm{F}_{[n]}'(x_n)^*(\mathrm{F}_{[n]}(x_n) - y^{\delta,[n]})\,, \tag{3.24}$$

is convergent if it is terminated after the $\tilde{n}^\delta$–th step, where $\tilde{n}^\delta$ is the smallest iteration index that satisfies

$$\|\mathrm{F}_{[\tilde{n}^\delta]}(x_{\tilde{n}^\delta}) - y^{\delta,[\tilde{n}^\delta]}\| \le \tau\delta^{[\tilde{n}^\delta]}\,. \tag{3.25}$$

Therefore, in general, only one of the components of the residual vector $\left(\|\mathrm{F}_i(x_{\tilde{n}^\delta}) - y^{\delta,i}\|\right)_i$ is smaller than $\tau\delta^i$, namely the *active component* $\|\mathrm{F}_{[\tilde{n}^\delta]}(x_{\tilde{n}^\delta}) - y^{\delta,[\tilde{n}^\delta]}\|$. However, the argumentation in [52] is incomplete, in the sense that the case when $\tilde{n}^\delta$ stagnates as $\delta \to 0$, has not been considered. Hence, [52, Theorem 4.4] requires the additional assumption that $\tilde{n}^\delta \to \infty$, as $\delta \to 0$, which is usually the case in practice.

3.1.3 Analysis of the eLK method

In the *embedded Landweber–Kaczmarz* (eLK) method for the solution of (3.1), $x \in X$ is substituted by a vector $\mathbf{x} = (x^i)_{i=0}^{N-1}$. In (3.9) each component of $\mathbf{x}$ is updated independently according to one of the system equations. In the balancing step (3.10), the difference between the components of $\mathbf{x}$ is minimized.

In order to determine $\mathbf{x}_{n+1/2}$, each of its components $x^i_{n+1/2}$ can be evaluated independently:

$$x^i_{n+1/2} = x^i_n - \omega_n \mathrm{F}'_i(x^i_n)^* \left(\mathrm{F}_i(x^i_n) - y^{\delta,i}\right), \; i = 0, \dots, N-1 \,.$$

In the balancing step (3.10), $\mathbf{x}_{n+1}$ is determined from $\mathbf{x}_{n+1/2}$ by a matrix multiplication with the sparse matrix $I_{X^N} - \omega_{n+1/2}\mathbf{G}$, where

$$\mathbf{G} = \lambda^2 \begin{pmatrix} 2I & -I & 0 & & -I \\ -I & 2I & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & 2I & -I \\ -I & & 0 & -I & 2I \end{pmatrix} \in \mathcal{L}(X^N, X^N) \,.$$

Here λ is a small positive parameter such that $\|\lambda D\| \le 1$, and the operator $\mathbf{G}$ is a discrete variant of $-\lambda^2$ times the second derivative operator and therefore penalizes for varying components. As already mentioned in the introduction, the the balancing step (3.10) is a Landweber–Kaczmarz step with respect to the equation (3.13). The operator D is linear and bounded, which guaranties the existence of a positive constant λ such that λD satisfies (3.14), which will be needed in the analysis of the embedded Landweber method. The iteration defined in (3.9), (3.10) is terminated when for the first time

$$\mathbf{x}_{n_\star^\delta+1} = \mathbf{x}_{n_\star^\delta+1/2} = \mathbf{x}_{n_\star^\delta} \,. \tag{3.26}$$

The artificial noise level $\epsilon : [0,\infty) \to [0,\infty)$ satisfies $\epsilon(\delta) \to 0$, as $\delta \to 0$ and guaranties the existence of a finite stopping index in the eLK method.

In the sequel we shall apply the results of the Section 3.1.2 to prove convergence of the eLK method. As initial guess we use a constant vector $\mathbf{x}_0 := (x_0)_i$ whose components are identical to x_0. Moreover, our convergence analysis will again require the scaling assumption (3.14) and the tangential cone condition (3.15) to be satisfied near x_0.

Remark 3.1.5 (Comparison with the classical Landweber iteration)**.** *Let* $\mathrm{F} := 1/\sqrt{N} \cdot (\mathrm{F}_0, \ldots, F_{N-1})$ *and* $\mathbf{y}^\delta := 1/\sqrt{N} \cdot (y^{\delta,0}, \ldots, y^{\delta,N-1})$. *The Landweber iteration for the solution of* $\mathrm{F}(x) = \mathbf{y}^\delta$, *see* (3.3), *is [34, 21]*

$$\begin{aligned} x_{n+1} &= x_n - \mathrm{F}'[x_n]^*(\mathrm{F}(x_n) - \mathbf{y}^\delta) \\ &= x_n - \tfrac{1}{N} \cdot \sum_{i=0}^{N-1} \mathrm{F}_i'(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i}) \\ &= \tfrac{1}{N} \cdot \sum_{i=0}^{N-1} \Big(x_n - \mathrm{F}_i'(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i}) \Big) \,. \end{aligned}$$

If we set $x_{n+1/2}^i := x_n - \mathrm{F}_i'(x_n)^*(\mathrm{F}_i(x_n) - y^{\delta,i})$ *then the Landweber method can be rewritten in form similar to* (3.9), (3.10), *namely*

$$\begin{aligned} x_{n+1/2}^i &= x_n - F_i'(x_n)^*(F_i(x_n) - y^{\delta,i}) \,, \\ x_{n+1} &= \tfrac{1}{N} \cdot \sum_{i=0}^{N-1} x_{n+1/2}^i \,. \end{aligned} \tag{3.27}$$

Hence, the distinction between the Landweber and the eLK method is that (3.27) *in the Landweber method makes all components equal, whereas the balancing step* (3.10) *in the embedded Landweber–Kaczmarz method leaves them distinct.*

In order to illustrate the idea behind the eLK method, we exemplarily consider the case $N = 3$. *In this case the mechanism of*

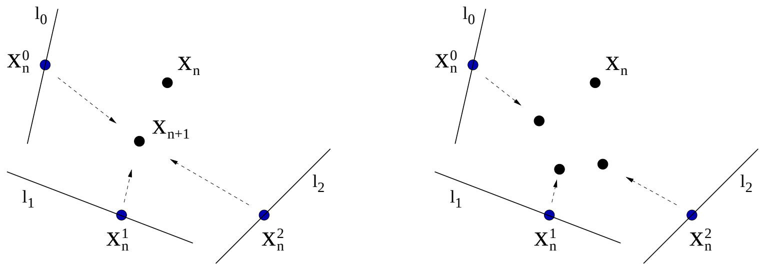

Figure 3.2: Landweber versus eLK method for solving $a_i \cdot x = y^i$, $i \in \{0,1,2\}$ with $\|a_1\| = 1$. In this case $x^i_{n+1/2} := x_n - a_i^*(a_i \cdot x_n - y^i)$ is the orthogonal projection of x_n on $l_i := \{x \in \mathbb{R}^2 : a_i \cdot x = y^i\}$. Each step of the Landweber iteration generates a single element in X (left picture), namely the average $1/3 \sum_{i=0}^{2} x_n^i$. In contrast, a cycle in the embedded Landweber–Kaczmarz method generates a vector in X^N (right picture), where each component of $\mathbf{x}_{n+1}$ is a linear combination of the x_n^i.

the embedded iteration is explained in Figure 3.2 in contrast to the Landweber method.

In the next theorem we prove that the termination index is well defined, as well as convergence and stability of the eLK method.

Theorem 3.1.6. *Assume that the operators* F_i *are Fréchet-differentiable in* $B_\rho(x_0)$ *and satisfy* (3.14), (3.15)*. Moreover, we assume that* (3.1) *has a solution in* $B_{\rho/2}(x_0)$*. Then we have:*

1. *For exact data* $y^{\delta,i} = y^i$*, the sequence* $\mathbf{x}^n$ *in* (3.9), (3.10) *converges to a constant vector* $(x)_i$*, where* x *is a solution* (3.1) *in* $B_{\rho/2}(x_0)$*. Additionally, if the operators* F_i *satisfy* (3.23)*, then the sequence* $\mathbf{x}^n$ *converges to the constant vector* $\mathbf{x}^\dagger = (x^\dagger)_i$*, where* $x^\dagger$ *is the unique solution of minimal distance to* x_0*.*

2. *For noisy data* $\delta > 0$*,* (3.26) *defines a finite termination index* $n_\star^\delta$*. Moreover, the embedded Landweber–Kaczmarz iteration* $\mathbf{x}^{n_\star^\delta}$ *converges to a constant vector* $\mathbf{x} = (x)_i$*, where* x *is a solution* (3.1) *in* $B_{\rho/2}(x_0)$*, as* $\delta \to 0$*. If in addition* (3.23) *holds, then each component of* $\mathbf{x}^{n_\star^\delta}$ *converges to* $x^\dagger$*, as* $\delta \to 0$*.*

Proof. In order to prove the first item we apply Theorem 3.1.4, item 1 to the system (3.7), (3.13). From (3.14) it follows that $\|\mathbf{F}[\mathbf{x}]\| \le 1$ for $\mathbf{x} \in B_\rho(x_0)^N$. Moreover, since D is bounded linear, $\|\lambda D\| \le 1$ for sufficiently small λ. The tangential cone condition (3.15) for F_i implies

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}}) - \mathbf{F}'(\mathbf{x})(\mathbf{x} - \bar{\mathbf{x}})\| \le \eta \|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}})\| \, , \qquad \mathbf{x}, \bar{\mathbf{x}} \in B_\rho(x_0)^N \, .$$

Moreover, since λD is a linear operator, the tangential cone condition is obviously satisfied for λD with the same η. Therefore, by applying Theorem 3.1.4, item 1 we conclude that $\mathbf{x}_n$ converges to a solution $\tilde{\mathbf{x}}$ of (3.1), (3.13). From (3.13) it follows that $\tilde{\mathbf{x}} = (\tilde{x})_i$ is a constant vector. Therefore, $\mathrm{F}_i(\tilde{x}) = y^i$, proving the assertion.

Additionally, let $\mathbf{x}^\dagger$ denote the solution of (3.7), (3.13) with minimal distance to $(x_0)_i$. As an auxiliary result we show that $\mathbf{x}^\dagger = (x^\dagger)_i$, where $x^\dagger$ is the unique solution of (3.1) with minimal distance to x_0. Due to (3.13) we have $\mathbf{x}^\dagger = (\tilde{x})_i$, for some $\tilde{x} \in X$. Moreover, the vector $(x^\dagger)_i$ is a solution of (3.1), (3.13) and

$$\|\tilde{x} - x_0\|^2 = \tfrac{1}{N} \sum_{i=0}^{N-1} \|\tilde{x} - x_0\|^2 \le \tfrac{1}{N} \sum_{i=0}^{N-1} \|x^\dagger - x_0\|^2 = \|x^\dagger - x_0\|^2 .$$

Therefore $\mathbf{x}^\dagger = (x^\dagger)_i$. Now, if (3.23) is satisfied, then

$$\mathcal{N}(\mathbf{F}'(\mathbf{x}^\dagger)) \ \subseteq \ \mathcal{N}(\mathbf{F}'(\mathbf{x})) \, , \quad \mathbf{x} \in B_\rho(x_0)^N$$

and by applying Theorem 3.1.4 we conclude that $\mathbf{x}^n \to \mathbf{x}^\dagger$.

The proof of the second item follows from Theorem 3.1.4, item 2 in an analogous way as above. □

As consequence of Theorem 3.1.6, if $n_\star^\delta$ is defined by (3.26) and $\mathbf{x}^{n_\star^\delta} = (x_i^{n_\star^\delta})_i$, then

$$x^{n_\star^\delta} \ := \ \sum_{i=0}^{N-1} x_i^{n_\star^\delta} \longrightarrow x^\dagger \, , \tag{3.28}$$

as $\delta \to 0$. However, Theorem 3.1.6 guaranties even more: All components $x_i^{n_\star^\delta}$ converge to $x^\dagger$ as the noise level tend to zero. Moreover, due to the averaging process in (3.28) the noise level in the actual regularized solution $x^{n_\star^\delta}$ becomes noticeable reduced.

3.2 Landweber-Kaczmarz method: Banach space approach

In this section we investigate the *Landweber-Kaczmarz method in Banach spaces* (LKB) for obtaining regularized approximate solutions for systems of nonlinear operator equations modelled by ill-posed operators acting between Banach spaces.

3.2.1 Systems of nonlinear ill-posed equations

The *inverse problem* we are interested in consists of determining an unknown physical quantity $x \in X$ from the set of data $(y_1, \dots, y_m) \in Y^m$, where X, Y are Banach spaces, X uniformly convex and smooth [14], and $m \geq 1$.

In practical situations, we do not know the data exactly. Instead, we have only approximate measured data $y_i^\delta \in Y$ satisfying

$$\|y_i^\delta - y_i\| \leq \delta_i \,, \quad i = 1, \dots, m \,, \tag{3.29}$$

with $\delta_i > 0$ (noise level). The finite set of data above is obtained by indirect measurements of the parameter, this process being described by the model

$$F_i(x) = y_i \,, \quad i = 1, \dots, m \,, \tag{3.30}$$

where $F_i : D_i \subset X \to Y$, and D_i are the corresponding domains of definition.

Example 3.2.1. *A tutorial example of an inverse problem of the form* (3.30) *is the identification of the space-dependent coefficient* $a(x)$ *(bounded away from zero) in the elliptic model*

$$-\nabla(a\nabla u) \; = \; f \,, \text{ in } \Omega \qquad\qquad u \; = \; 0 \,, \text{ at } \partial\Omega \,,$$

where $\Omega \subset \mathbb{R}^2$ *is an open bounded domain with regular (smooth) boundary* $\partial\Omega$. *Available data for the identification problem are* $u|_{\Omega_i}$, *i.e., the restrictions of the solution* u *to given open sets* $\Omega_i \subset \Omega$, $i = 1, \dots, m$.

In the standard Hilbert space setting [21, 16] we have $F_i : H^2(\Omega) = X \supset D_i \ni a \mapsto (\Delta_a^{-1} f)|_{\Omega_i} \in Y_i = L^2(\Omega_i)$*, where* $\Delta_a : H^2(\Omega) \cap H_0^1(\Omega) \ni u \mapsto -\nabla(a\nabla u) \in L^2(\Omega)$ *and* $D_i = D := \{a \in X;\, a(x) \geq \underline{a} > 0,\ a.e.\ in\ \Omega\}$, $i = 1, \cdots, m$.

A possible Banach space setting for this problem is analyzed in [49] (for the case $m = 1$ *and* $\Omega_1 = \Omega$*), where the choice* $X = W^{1,q}(\Omega)$, $Y_i = L^r(\Omega_i)$ *with* $q > 2$ *and* $r \in (1, \infty)$ *is considered. In particular, it follows from [49, Corollary 3] that the convergence analysis results derived here can be applied to this parameter identification problem (see Assumption A3).*

3.2.2 Regularization in Banach spaces

The starting point of our approach is the Landweber method [76, 49] for solving ill-posed problems in Banach spaces.[2] In the case of a single operator equation, i.e., $m = 1$ in (3.30), this method is defined by

$$\begin{aligned} x_n^* &= J_p(x_n) - \mu_n F'(x_n)^* J_r\big(F(x_n) - y^\delta\big)\,, \\ x_{n+1} &= J_q(x_n^*)\,, \end{aligned} \tag{3.31}$$

where $F'(x)$ is the Fréchet derivative of F at point x, and J_p, J_r, J_q are duality mappings from X, Y, X^* to their duals respectively. Moreover, $x_0 \in D$ and $p, q, r \in (1, \infty)$ satisfy $p + q = pq$.

The step-size μ_n depends on the constant of the tangential cone condition, the constant of the discrepancy principle, the residual at x_n, and a constant describing geometrical properties of the Banach spaces (see [76, Section 3]).

Convergence analysis for the linear case $F \in \mathcal{L}(X, Y)$ can be found in [76], while convergence for nonlinear operator equations is derived in [49], where X is assumed to be uniformly smooth and uniformly convex (actually, X is assumed to be p-convex, which is equivalent to the dual being q-smooth, i.e., there exists a constant $C_q > 0$ such that for all x^*, $y^* \in X^*$ it follows $\|x^* - y^*\|^q \leq \|x^*\|^q - q\langle J_q(x^*), y^*\rangle + C_q\|y^*\|^q$; see [49, Section 2.2]). For a detailed definition of smoothness, uniform smoothness and uniform convexity in Banach spaces, we refer the reader to [14, 76].

[2] See also [1, 21, 48] for the analysis of the Landweber method in Hilbert spaces.

3.2.3 The LKB method

The *Landweber-Kaczmarz method in Banach spaces* LKB consists in incorporating the (cyclic) Kaczmarz strategy to the Landweber method depicted in in (3.31) for solving the system of operator equations in (3.30).

This strategy is analog to the one proposed in [33, 31] regarding the Landweber-Kaczmarz (LK) iteration in Hilbert spaces. See also [17] for the Steepest-Descent-Kaczmarz (SDK) iteration, [32] for the Expectation-Maximization-Kaczmarz (EMK) iteration, [3] for the Levenberg-Marquardt-Kaczmarz (LMK) iteration, and [16] for the iterated-Tikhonov-Kaczmarz (ITK) iteration.

Motivated by the ideas in the above mentioned papers (in particular by the approach in [32], where $X = L^1(\Omega)$ and convergence is measured with respect to the Kullback-Leibler distance), we propose next the LBK method, which is sketched as follows:

$$\begin{aligned} x_n^* &= J_p(x_n) - \mu_n F'_{i_n}(x_n)^* J_r\big(F_{i_n}(x_n) - y^\delta_{i_n}\big)\,, \\ x_{n+1} &= J_q(x_n^*)\,, \end{aligned} \tag{3.32}$$

for $n = 0, 1, \ldots$ Moreover, $i_n := (n \bmod m) + 1 \in \{1, ..., m\}$, and $x_0 \in X \backslash \{0\}$ is an initial guess, possibly incorporating *a priori* knowledge about the exact solution (which may not be unique).

Here $\mu_n \geq 0$ is chosen analogously as in (3.31) if $\|F_{i_n}(x_n) - y^\delta_{i_n}\| \geq \tau\delta_{i_n}$ (see Section 3.2.5 for the precise definition of μ_n and the discrepancy parameter $\tau > 0$). Otherwise, we set $\mu_n = 0$. Consequently, $x_{n+1} = J_q(x_n^*) = J_q(J_p(x_n)) = x_n$ every time the residual of the iterate x_n w.r.t. the i_n-th equation of system (3.30) drops below the discrepancy level given by $\tau\delta_{i_n}$.

Due to the bang-bang strategy used in to define the sequence of parameters (μ_n), the iteration in (3.32) is alternatively called loping Landweber-Kaczmarz method in Banach spaces.

As usual in Kaczmarz type algorithms, a group of m subsequent steps (beginning at some integer multiple of m) is called a *cycle*. The iteration should be terminated when, for the first time, all of the residuals $\|F_{i_n}(x_{n+1}) - y^\delta_{i_n}\|$ drop below a specified threshold within a cycle. That is, we stop the iteration at the step

$$\hat{n} \ := \ \min\{\ell m + (m-1) : \ \ell \in \mathbb{N}, \ \ \|F_i(x_{\ell m+i-1}) - y_i^\delta\| \le \tau\delta_i \,, \quad \text{for } 1 \le i \le m\} \,. \quad (3.33)$$

In other words, writing $\hat{n} := \hat{\ell}m+(m-1)$, (3.33) can be interpreted as $\|F_i(x_{\hat{\ell}m+i-1}) - y_i^\delta\| \le \tau\delta_i$, $i = 1, \ldots, m$. In the case of noise free data ($\delta_i = 0$ in (3.29)) the stop criteria in (3.33) may never be reached, i.e., $\hat{n} = \infty$ for $\delta_i = 0$.

3.2.4 Mathematical background

Overview on convex analysis

Let X be a (nontrivial) real Banach space with topological dual X^*. By $\|\cdot\|$ we denote the norm on X and X^*. The duality product on $X \times X^*$ is a bilinear symmetric mapping, denoted by $\langle\cdot,\cdot\rangle$, and defined as $\langle x, x^*\rangle = x^*(x)$, for all $(x, x^*) \in X \times X^*$.

Let $f : X \to (-\infty, \infty]$ be convex, proper and lower semicontinuous. Recall that f is convex lower semicontinuous when its epigraph $\text{epi}(f) := \{(x, \lambda) \in X \times \mathbb{R} \,:\, f(x) \le \lambda\}$ is a closed convex subset of $X \times \mathbb{R}$. Moreover, f is proper when its domain $\text{dom}(f) := \{x \in X : f(x) < \infty\}$ is nonempty. The *subdifferential* of f is the (point-to-set) operator $\partial f : X \to 2^{X^*}$ defined at $x \in X$ by

$$\partial f(x) = \{x^* \in X^* \,:\, f(y) \ge f(x) + \langle x^*, y - x\rangle, \ \forall y \in X\}. \quad (3.34)$$

Notice that $\partial f(x) = \emptyset$ whenever $x \notin \text{dom}(f)$. The domain of ∂f is the set $\text{dom}(\partial f) = \{x \in X \,:\, \partial f(x) \ne \emptyset\}$. Next we present a very useful characterization of ∂f using the concept of *Fenchel Conjugation*. The Fenchel-conjugate of f is the lower semicontinuous convex function $f^* : X^* \to (-\infty, \infty]$ defined at $x^* \in X^*$ by

$$f^*(x^*) = \sup_{x \in X} \langle x, x^*\rangle - f(x). \quad (3.35)$$

It is well known that f^* is also proper whenever f is proper. It follows directly from (3.35) the *Fenchel-Young* inequality

$$f(x) + f^*(x^*) \ge \langle x, x^*\rangle, \ \ \forall (x, x^*) \in X \times X^*. \quad (3.36)$$

Proposition 3.2.2. Let $f : X \to (-\infty, \infty]$ be proper convex lower semicontinuous and $(x, x^*) \in X \times X^*$. Then $x^* \in \partial f(x) \iff f(x) + f^*(x^*) = \langle x, x^* \rangle$.

Proof. See Exercise 3.1. □

An important example considered here is given by $f(x) = p^{-1}\|x\|^p$, where $p \in (1, \infty)$. In this particular case, the following result can be found in [14].

Proposition 3.2.3. Let $p \in (1, \infty)$ and $f : X \ni x \mapsto p^{-1}\|x\|^p \in \mathbb{R}$. Then

$$f^* : X^* \to \mathbb{R},\ x^* \mapsto q^{-1}\|x^*\|^q\,, \quad \text{where} \ \ p + q = pq\,.$$

Proof. See Exercise 3.2. □

For $p \in (1, \infty)$, the duality mapping $J_p : X \to 2^{X^*}$ is defined by

$$J_p := \partial p^{-1}\| \cdot \|^p\,.$$

From the proposition above, we conclude that

$$x^* \in J_p(x) \iff p^{-1}\|x\|^p + q^{-1}\|x^*\|^q = \langle x, x^* \rangle\,,\ p + q = pq.$$

It follows from the above identity that $J_p(0) = \{0\}$. On the other hand, when $x \neq 0$, $J_p(x)$ may not be singleton.

Proposition 3.2.4. Let X and the duality mapping J_p be defined as above. The following identities hold:

$$\begin{aligned} J_p(x) &= \{x^* \in X^* \,:\, \|x^*\| = \|x\|^{p-1} \ \textit{and} \ \langle x, x^* \rangle = \|x\|\,\|x^*\|\} \\ &= \{x^* \in X^* \,:\, \|x^*\| = \|x\|^{p-1} \ \textit{and} \ \langle x, x^* \rangle = \|x\|^p\} \\ &= \{x^* \in X^* \,:\, \|x^*\| = \|x\|^{p-1} \ \textit{and} \ \langle x, x^* \rangle = \|x^*\|^q\}\,. \end{aligned}$$

Moreover, $J_p(x) \neq \emptyset$ for all $x \in X$.

Proof. Take $x^* \in J_p(x)$. Then $p^{-1}\|x\|^p + q^{-1}\|x^*\|^q = \langle x, x^* \rangle$. Now, using the Young inequality for the real numbers $\|x\|$ and $\|x^*\|$, we obtain

$$\|x\|\,\|x^*\| \ \leq \ p^{-1}\|x\|^p + q^{-1}\|x^*\|^q \ = \ \langle x, x^* \rangle \ \leq \ \|x\|\,\|x^*\|\,,$$

from what follows $\langle x, x^* \rangle = \|x\| \, \|x^*\|$ and $\|x\| \, \|x^*\| = p^{-1}\|x\|^p + q^{-1}\|x^*\|^q$. Consider the real function $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(t) = p^{-1}t^p$. The above identity tells us that $\|x^*\| \in \partial\psi(\|x\|) = \{\|x\|^{p-1}\}$. Thus, $\|x^*\| = \|x\|^{p-1}$. The other identities as well as the reverse inclusion follow from analog reasoning.

Now take an arbitrary $x \in X$. From the Hahn-Banach theorem, it follows the existence of $z^* \in X^*$ such that $\|z^*\| = 1$ and $\|x\| = \langle x, z^* \rangle$. Then, defining $x^* := \|x\|^{p-1} z^*$, we obtain $\|x^*\| = \|x\|^{p-1}$ and $\langle x, x^* \rangle = \|x\|^{p-1}\langle x, z^* \rangle = \|x\| \, \|x^*\|$, i.e, $x^* \in J_p(x)$. □

Since $f(x) = p^{-1}\|x\|^p$ is a continuous convex functions, $J_p(x)$ is a singleton at $x \in X$ iff f is Gâteaux differentiable at x [10, Corollary 4.2.5]. This motivates us to consider X a *smooth* Banach space, i.e., a Banach space having a Gâteaux differentiable norm $\| \cdot \|_X$ on $X \backslash \{0\}$. As already observed, $J_p(0) = \{0\}$ in any Banach space. In particular in a smooth Banach space $f(x) = p^{-1}\|x\|^p$ is Gâteaux differentiable everywhere.

The following proposition gives a trivial characterization of smooth Banach spaces.

Proposition 3.2.5. X is smooth iff for each $x \neq 0$ there exists a unique $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.

Proof. See Exercise 3.3. □

The next theorem describes a coercivity result related to geometrical properties of uniformly smooth Banach spaces. For details on the proof (as well as the precise definition of the constant G_q) we refer the reader to [76, Section 2.1] or [84].

Theorem 3.2.6. *Let X be uniformly convex, $q \in (1, \infty)$ and $\rho_{X^*}(\cdot)$ the smoothness modulus of X^* [14]. There exists a positive constant G_q such that the function*

$$\tilde{\sigma}(x^*, y^*) \;:=\; q\, G_q \int_0^1 (\|x^* - ty^*\| \vee \|x^*\|)^q \, t^{-1} \rho_{X^*}\Big(t\|y^*\| \,/\, 2(\|x^* - ty^*\| \vee \|x^*\|)\Big)\, dt$$

satisfies [1]

$$\|x^*\|^q - q\,\langle J_q(x^*), y^*\rangle + \tilde\sigma_q(x^*, y^*) \ \geq \ \|x^* - y^*\|^q\,,\ \forall\ x^*, y^* \in X^*\,.$$

Reciprocally, in uniformly convex Banach spaces we have [76].

Theorem 3.2.7. *Let X be uniformly convex and $p \in (1, \infty)$. There exists a constant K_p such that the function*

$$\sigma(x, y) := pK_p \int_0^1 (\|x - ty\| \vee \|x\|)^p\, t^{-1} \delta_X\Big(t\|y\| \,/\, 2(\|x - ty\| \vee \|x\|)\Big)\, dt$$

satisfies

$$\|x - y\|^p \ \geq \ \|x\|^p - p\langle J_p(x), y\rangle + \sigma_p(x, y)\,,\ \forall\ x, y \in X\,.$$

Proof. See Exercise 3.4. □

Overview on Bregman distances

Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function which is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}(f))$. Moreover, denote by f' the Gâteaux derivative of f. The *Bregman distance* induced by f is defined as $D_f : \mathrm{dom}(f) \times \mathrm{int}(\mathrm{dom}(f)) \to \mathbb{R}$

$$D_f(y, x) = f(y) - (f(x) + \langle f'(x), y - x\rangle)\,.$$

The following proposition is a useful characterization of Bregman distances using Fenchel conjugate functions.

Proposition 3.2.8. Let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous convex function which happens to be Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}(f))$. Then $D_f(y, x) = f(y) + f^*(f'(x)) - \langle f'(x), y\rangle$, for all $(y, x) \in \mathrm{dom}(f) \times \mathrm{int}(\mathrm{dom}(f))$.

Proof. Let $x \in \mathrm{int}(\mathrm{dom}(f))$. Since $f'(x) \in \partial f(x)$ we have $f(x) + f^*(f'(x)) = \langle x, f'(x)\rangle$. Thus,

$$\begin{aligned} D_f(y, x) &= f(y) + (\langle x, f'(x)\rangle - f(x)) - \langle f'(x), y\rangle \\ &= f(y) + f^*(f'(x)) - \langle f'(x), y\rangle\,, \end{aligned}$$

completing the proof. □

[1] We adopt the notation $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, for a, $b \in \mathbb{R}$.

Based on this proposition, we derive the following two corollaries, which are used in forthcoming convergence analysis. The proofs of these corollaries follow as a particular case $f(x) = p^{-1}\|x\|^p$ ($p \in (1,\infty)$). We use the notation D_p instead of D_f.

Corollary 3.2.9. *Let X be a smooth Banach space. Then $J_p : X \to X^*$ is a single-valued mapping for which $D_p : X \times X \to \mathbb{R}$ satisfies*

$$\begin{aligned} D_p(y,x) &= p^{-1}\|y\|^p + q^{-1}\|J_p(x)\|^q - \langle y, J_p(x)\rangle \\ &= p^{-1}\|y\|^p + q^{-1}\|x\|^p - \langle y, J_p(x)\rangle. \end{aligned}$$

Proof. See Exercise 3.5. □

Corollary 3.2.10. *Let X be a smooth Banach space. Then $J_p : X \to X^*$ is a single-valued mapping for which $D_p : X \times X \to \mathbb{R}$ satisfies*

$$D_p(y,x) = q^{-1}\left(\|x\|^p - \|y\|^p\right) + \langle J_p(y) - J_p(x), y\rangle.$$

Proof. See Exercise 3.6. □

3.2.5 Algorithmic implementation of LKB

In this section we introduce an algorithm for solving system (3.30) with data satisfying (3.29), namely a numerical implementation of the LKB method. From now on we assume the Banach space X to be uniformly convex and smooth, e.g., L_p spaces for $p \in (1,\infty)$.[2] These assumptions are crucial for the analysis derived in this section as well as in the forthcoming one.

We denote by

$$\begin{aligned} \mathcal{B}^1_p(x,r) &= \{y \in X \,:\, D_p(x,y) \le r\}, \\ \mathcal{B}^2_p(x,r) &= \{y \in X \,:\, D_p(y,x) \le r\}. \end{aligned}$$

the balls of radius $r > 0$ with respect to the Bregman distance $D_p(\cdot,\cdot)$.

A solution of (3.30) is any $\bar{x} \in D$ satisfying simultaneously the operator equations in (3.30), while a *minimum-norm solution* of (3.30) in S ($S \subset X$) is any solution $x^\dagger \in S$ satisfying

$$\|x^\dagger\| = \min\{\|x\| \,:\, x \in S \text{ is a solution of } (3.30)\}.$$

[2]Notice that L_1 and L_∞ are not uniformly convex [76, Example 2.2].

Assumption A3. Let p, q, $r \in (1, \infty)$ be given with $p+q = pq$. The following assumptions will be required in the forthcoming analysis:

(A3.1) Each operator F_i is of class C^1 in D. Moreover, the system of operator equations (3.30) has a solution $\bar{x} \in X$ satisfying $x_0 \in \mathcal{B}_p^1(\bar{x}, \bar{\rho}) \subset D$, for some $\bar{\rho} > 0$. Further, we require $D_p(\bar{x}, x_0) \le p^{-1}\|\bar{x}\|^p$. The element x_0 will be used as initial guess of the Landweber-Kaczmarz algorithm.

(A3.2) The family $\{F_i\}_{1 \le i \le m}$ satisfies the tangential cone condition in $\mathcal{B}_p^1(\bar{x}, \bar{\rho})$, i.e., there exists $\eta \in (0, 1)$ such that

$$\|F_i(y) - F_i(x) - F_i'(x)(y - x)\| \;\le\; \eta \,\|F_i(y) - F_i(x)\| \,,$$

for all x, $y \in \mathcal{B}_p^1(\bar{x}, \bar{\rho})$, $i = 1, \cdots, m$.

(A3.3) The family $\{F_i\}_{1 \le i \le m}$ satisfies the tangential cone condition in $\mathcal{B}_p^2(x_0, \rho_0) \subset D$ for some $\rho_0 > 0$, i.e., there exists $\eta \in (0, 1)$ such that

$$\|F_i(y) - F_i(x) - F_i'(x)(y - x)\| \;\le\; \eta \,\|F_i(y) - F_i(x)\| \,,$$

for all x, $y \in \mathcal{B}_p^2(x_0, \rho_0)$, $i = 1, \cdots, m$.

(A3.4) For every $x \in \mathcal{B}_p^1(\bar{x}, \bar{\rho})$ we have $\|F_i'(x)\| \le 1$, $i = 1, 2, \cdots, m$.

In the sequel we formulate our *Landweber-Kaczmarz* algorithm for approximating a solution of (3.30), with data given as in (3.29):

Algorithm 3.2.11. Under assumptions (A3.1), (A3.2), choose $c \in (0, 1)$, and $\tau \in (0, \infty)$ such that $\beta := \eta + \tau^{-1}(1 + \eta) < 1$.

Step 0: Set $n = 0$ and take $x_0 \ne 0$ satisfying (A3.1);

Step 1: Set $i_n := n (\text{mod } m) + 1$ and evaluate the residual $R_n = F_{i_n}(x_n) - y^\delta_{i_n}$;

Step 2: IF $(\|R_n\| \le \tau \delta_{i_n})$ THEN
$\mu_n := 0$;
ELSE
Find $\tau_n \in (0, 1]$ solving the equation

$$\rho_{X^*}(\tau_n)\, \tau_n^{-1} \;=\; \Big(c(1 - \beta)\, \|R_n\| \cdot \cdot \big[2^q\, G_q (1 \vee \|F_{i_n}'(x_n)\|)\, \|x_n\| \big]^{-1} \Big) \wedge \rho_{X^*}(1)\,; \quad (3.37)$$

$\mu_n := \tau_n \|x_n\|^{p-1} \,/\, \big[(1 \vee \|F'_{i_n}(x_n)\|)\, \|R_n\|^{r-1}\big];$
ENDIF
$x_n^* := J_p(x_n) - \mu_n F'_{i_n}(x_n)^* J_r(F_{i_n}(x_n) - y_{i_n}^\delta);$

$$x_{n+1} \;=\; J_q(x_n^*)\,; \tag{3.38}$$

Step 3: IF $(i_n = m)$ AND $(x_{n+1} = x_n = \cdots = x_{n-(m-1)})$ THEN STOP;

Step 4: SET $n = n + 1$; GO TO Step 1.

The next remark guarantees that the above algorithm is well defined.

Remark 3.2.12. *It is worth noticing that a solution $\tau_n \in (0,1]$ of equation (3.37) can always be found. Indeed, since X^* is uniformly smooth, the function $(0,\infty) \ni \tau \mapsto \rho_{X^*}(\tau)/\tau \in (0,1]$ is continuous and satisfies $\lim_{\tau\to 0} \rho_{X^*}(\tau)/\tau = 0$ (see, e.g., [76, Definition 2.1] or [14]). For each $n \in \mathbb{N}$, define*

$$\lambda_n := \Big(c(1-\beta)\, \|R_n\| \, \big[2^q\, G_q (1 \vee \|F'_{i_n}(x_n)\|)\, \|x_n\|\big]^{-1} \Big) \wedge \rho_{X^*}(1)\,. \tag{3.39}$$

It follows from [76, Section 2.1] that $\rho_{X^*}(1) \le 1$. Therefore, $\lambda_n \in (0,1]$, $n \in \mathbb{N}$ and we can can find $\sigma_n \in (0,1]$ satisfying $\rho_{X^*}(\sigma_n)/\sigma_n < \lambda_n \le \rho_{X^*}(1)$. Finally, the mean value theorem guarantees the existence of corresponding $\tau_n \in (0,1]$, such that $\lambda_n = \rho_{x^*}(\tau_n)/\tau_n$, $n \in \mathbb{N}$.

Algorithm 3.2.11 should be stopped at the smallest iteration index $\hat{n} \in \mathbb{N}$ of the form $\hat{n} = \hat{\ell} m + (m-1)$, $\hat{\ell} \in \mathbb{N}$, which satisfies

$$\|F_{i_n}(x_n) - y_{i_n}^\delta\| \;\le\; \tau \delta_{i_n}\,, \qquad n = \hat{\ell} m, \ldots, \hat{\ell} m + (m-1) \tag{3.40}$$

(notice that $i_{\hat{n}} = m$). In this case, $x_{\hat{n}} = x_{\hat{n}-1} = \cdots = x_{\hat{n}-(m-1)}$ within the $\hat{\ell}^{th}$ cycle. The next result guarantees monotonicity of the iteration error (w.r.t. the Bregman distance D_p) until the discrepancy principle in (3.40) is reached.

Lemma 3.2.13 (Monotonicity)**.** *Let assumptions (A3.1), (A3.2) be satisfied and (x_n) be a sequence generated by Algorithm 3.2.11. Then*

$$D_p(\bar{x}, x_{n+1}) \;\le\; D_p(\bar{x}, x_n)\,, \quad n = 0, 1, \cdots, \hat{n}\,,$$

where $\hat{n} = \hat{\ell} m + (m-1)$ is defined by (3.40). From the above inequality, it follows that $x_n \in \mathcal{B}_p^1(\bar{x}, \bar{\rho}) \subset D$, $n = 0, 1, \cdots, \hat{n}$.

Proof. Let $0 \le n \le \hat{n}$ and assume that x_n is a nonzero vector satisfying $x_n \in \mathcal{B}^1_p(\bar{x}, \bar{\rho})$. From assumption (A3.1) follows $x_n \in D$.

If $\|R_n\| \le \tau\delta_{i_n}$, then $x_{n+1} = x_n$ and the lemma follows trivially. Otherwise, it follows from Corollary 3.2.9 that

$$D_p(\bar{x}, x_{n+1}) \;=\; p^{-1}\|\bar{x}\|^p + q^{-1}\|J_p(x_{n+1})\|^q - \langle \bar{x}, J_p(x_{n+1})\rangle\,. \quad (3.41)$$

Since $R_n = F_{i_n}(x_n) - y^\delta_{i_n}$, we conclude from (3.38) and $J_q = (J_p)^{-1}$ [14] that

$$J_p(x_{n+1}) \;=\; J_p(x_n) \;-\; \mu_n F'_{i_n}(x_n)^* J_r(R_n)\,.$$

Thus, it follows from Theorem 3.2.6 that

$$\begin{aligned}
\|J_p(x_{n+1})\|^q \;&=\; \|J_p(x_n) - \mu_n F'_{i_n}(x_n)^* J_r(R_n)\|^q \\
&\le\; \|J_p(x_n)\|^q - q\mu_n\langle J_q(J_p(x_n)), F'_{i_n}(x_n)^* J_r(R_n)\rangle \\
&\quad + \tilde{\sigma}_q(J_p(x_n), \mu_n F'_{i_n}(x_n)^* J_r(R_n)) \\
&=\; \|J_p(x_n)\|^q - q\mu_n\langle x_n, F'_{i_n}(x_n)^* J_r(R_n)\rangle \\
&\quad + \tilde{\sigma}_q(J_p(x_n), \mu_n F'_{i_n}(x_n)^* J_r(R_n))\,. \qquad (3.42)
\end{aligned}$$

In order to estimate the last term on the right hand side of (3.42), notice that for all $t \in [0,1]$ the inequality

$$\begin{aligned}
\|J_p(x_n) \;-\; t\mu_n F'_{i_n}(x_n)^* J_r(R_n)\| \vee \|J_p(x_n)\| \\
\le\;\; \|x_n\|^{p-1} + \mu_n(1 \vee \|F'_{i_n}(x_n)\|)\|R_n\|^{r-1} \\
\le\;\; (1+\tau_n)\|x_n\|^{p-1} \;\le\; 2\|x_n\|^{p-1}
\end{aligned}$$

holds true (to obtain the first inequality we used Proposition 3.2.4). Moreover,

$$\|J_p(x_n) - t\mu_n F'_{i_n}(x_n)^* J_r(R_n)\| \vee \|J_p(x_n)\| \ge \|J_p(x_n)\| = \|x_n\|^{p-1}.$$

From the last two inequalities together with the monotonicity of $\rho_{X^*}(t)/t$, it follows that (see Theorem 3.2.6)

$$\begin{aligned}
&\tilde{\sigma}_q(J_p(x_n), \mu_n F'_{i_n}(x_n)^* J_r(R_n)) \;\le \\
&qG_q \int_0^1 \frac{(2\|x_n\|^{p-1})^q}{t}\rho_{X^*}\left(\frac{t\mu_n(1 \vee \|F'_{i_n}(x_n)\|)\|R_n\|^{r-1}}{\|x_n\|^{p-1}}\right) dt. \qquad (3.43)
\end{aligned}$$

Consequently,

$$\begin{aligned}\tilde\sigma_q(J_p(x_n)\,,\,\mu_n F'_{i_n}(x_n)^* J_r(R_n)) & \\ \leq\ & 2^q\, q\, G_q\, \|x_n\|^p \int_0^1 \rho_{X^*}(t\tau_n)/t dt \\ =\ & 2^q\, q\, G_q\, \|x_n\|^p \int_0^{\tau_n} \rho_{X^*}(t)/t dt \\ \leq\ & 2^q\, q\, G_q\, \rho_{X^*}(\tau_n)/\tau_n \|x_n\|^p \int_0^{\tau_n} dt \\ =\ & 2^q\, q\, G_q\, \rho_{X^*}(\tau_n) \|x_n\|^p\,. \end{aligned} \tag{3.44}$$

Now, substituting (3.44) in (3.42) we get the estimate

$$\begin{aligned}\|J_p(x_{n+1})\|^q \ \leq\ & \|J_p(x_n)\|^q - q\mu_n \langle x_n, F'_{i_n}(x_n)^* J_r(R_n)\rangle \\ & + \ q\, 2^q\, G_q\, \rho_{X^*}(\tau_n)\|x_n\|^p\,.\end{aligned}$$

From this last inequality, Corollary 3.2.9 and (3.41) we obtain

$$\begin{aligned}D_p(\bar x, x_{n+1}) \leq D_p(\bar x, x_n) - \mu_n\, \langle x_n - \bar x, F'_{i_n}(x_n)^* J_r(R_n)\rangle & \\ + \ 2^q\, G_q\, \rho_{X^*}(\tau_n)\|x_n\|^p. & \end{aligned} \tag{3.45}$$

Next we estimate the term $\langle x_n - \bar x, F'_{i_n}(x_n)^* J_r(R_n)\rangle$ in (3.45). Since $\bar x$, $x_n \in \mathcal{B}^1_p(\bar x, \bar\rho)$, it follows from (A3.2) and simple algebraic manipulations (see Exercise 3.7).

$$\begin{aligned}\langle \bar x - x_n, F'_{i_n}(x_n)^* J_r(R_n)\rangle & \\ =\ & \langle y_{i_n} - F_{i_n}(x_n) - F'_{i_n}(x_n)(\bar x - x_n), -J_r(R_n)\rangle \\ & - \langle \tilde R_n, J_r(R_n)\rangle \\ \leq\ & \eta \|\tilde R_n\| \|J_r(R_n)\| - \langle R_n, J_r(R_n)\rangle + \langle y_{i_n} - y^\delta_{i_n}, J_r(R_n)\rangle \\ \leq\ & \eta\, (\|R_n\| + \delta_{i_n})\, \|R_n\|^{r-1} - \|R_n\|^r + \delta_{i_n} \|R_n\|^{r-1} \\ =\ & (\eta(\|R_n\| + \delta_{i_n}) + \delta_{i_n}) \|R_n\|^{r-1} - \|R_n\|^r \\ \leq\ & [(\eta + \tau^{-1}(1+\eta)]\, \|R_n\|)\, \|R_n\|^{r-1} - \|R_n\|^r \\ =\ & -(1-\beta)\|R_n\|^r\,, \end{aligned} \tag{3.46}$$

where $\tilde R_n := F_{i_n}(x_n) - y_{i_n}$ and $\beta > 0$ is defined as in Algorithm 3.2.11.

Substituting this last inequality in (3.45) yields

$$D_p(\bar{x}, x_{n+1}) \ \leq \\ D_p(\bar{x}, x_n) \ - \ (1-\beta)\mu_n \|R_n\|^r \ + \ 2^q\, G_q\, \rho_{X^*}(\tau_n)\, \|x_n\|^p . \quad (3.47)$$

Moreover, from the formulas for μ_n and τ_n (see Algorithm 3.2.11) we can estimate the last two terms on the right hand side of (3.47) by

$$\begin{aligned} -\,(1-\beta)\,\mu_n\,\|R_n\|^r \ + \ 2^q\, G_q\, \rho_{X^*}(\tau_n)\|x_n\|^p & \\ = \ & -(1-\beta)\frac{\tau_n \|x_n\|^{p-1}\|R_n\|}{1 \vee \|F'_{i_n}(x_n)\|} + 2^q\, G_q\, \rho_{X^*}(\tau_n)\|x_n\|^p \\ = \ & -(1-\beta)\frac{\tau_n \|x_n\|^{p-1}\|R_n\|}{1 \vee \|F'_{i_n}(x_n)\|} \cdot \\ & \cdot \left(1 - \frac{2^q\, G_q (1 \vee \|F'_{i_n}(x_n)\|)\|x_n\|}{(1-\beta)\|R_n\|} \frac{\rho_{X^*}(\tau_n)}{\tau_n} \right) \\ \leq \ & -(1-\beta)(1-c)\frac{\tau_n \|x_n\|^{p-1}\|R_n\|}{1 \vee \|F'_{i_n}(x_n)\|} \, . \end{aligned} \quad (3.48)$$

Finally, substituting (3.48) in (3.47) we obtain

$$D_p(\bar{x}, x_{n+1}) \ \leq \ D_p(\bar{x}, x_n) \\ -\,(1-\beta)(1-c)\tau_n \|x_n\|^{p-1}\|R_n\| \left[1 \vee \|F'_{i_n}(x_n)\| \right]^{-1} , \quad (3.49)$$

concluding the proof. □

Remark 3.2.14. *In the proof of Lemma 3.2.13 we used the fact that the elements $x_n \in X$ generated by Algorithm 3.2.11 are a nonzero vectors. This can be verified by an inductive argument. Indeed, $x_0 \neq 0$ is chosen in Algorithm 3.2.11. Assume $x_k \neq 0$, $k = 0, \dots, n$. If $\|R_n\| \leq \tau\delta_{i_n}$, then $x_{n+1} = x_n$ is also a nonzero vector. Otherwise, $\|R_n\| > \tau\delta_{i_n} > 0$ and it follows from (3.49) that $D_p(\bar{x}, x_{n+1}) < D_p(\bar{x}, x_n) \leq \cdots \leq D_p(\bar{x}, x_0) \leq p^{-1}\|\bar{x}\|^p$ (the last inequality follows from the choice of x_0 in (A3.1)). If x_{n+1} were the null vector, we would have $p^{-1}\|\bar{x}\|^p = D_p(\bar{x}, 0) < D_p(\bar{x}, x_n) \leq p^{-1}\|\bar{x}\|^p$ (the identity follows from Corollary 3.2.9), which is clearly a contradiction. Therefore, x_n is a nonzero vector, for $n = 0, 1, \dots, \hat{n}$.*

In the case of exact data ($\delta_i = 0$), we have $x_n \neq 0$, $n \in \mathbb{N}$.

The following lemma guarantees that, in the noisy data case, Algorithm 3.2.11 is stopped after a finite number of cycles, i.e., $\hat{n} < \infty$ in (3.40).

Lemma 3.2.15. *Let assumptions* (A3.1), (A3.2), (A3.4) *be satisfied and* (x_n) *be a sequence generated by Algorithm* 3.2.11*. Then*

$$\sum_{n\in\hat{\Sigma}} \tau_n \|x_n\|^{p-1} \|R_n\| \ \le\ (1-\beta)^{-1}(1-c)^{-1} D_p(\bar{x}, x_0)\,, \qquad (3.50)$$

where $\hat{\Sigma} := \{n \in \{0, 1, \cdots, \hat{n}-1\} :\ \|R_n\| > \tau\delta_{i_n}\}$*. Additionally,*
i) In the noisy data case, $\min\{\delta_i\}_{1\le i\le m} > 0$*, Algorithm* 3.2.11 *is stopped after finitely many steps;*
ii) In the noise free case we have $\lim_{n\to\infty} \|R_n\| = 0$*.*

Proof. Given $n \in \hat{\Sigma}$, it follows from (3.49) and (A3.4) that

$$(1-\beta)(1-c)\tau_n \|x_n\|^{p-1} \|R_n\| \le D_p(\bar{x}, x_n) - D_p(\bar{x}, x_{n+1})\,. \quad (3.51)$$

Moreover, if $n \notin \hat{\Sigma}$ and $n < \hat{n}$, we have $0 \le D_p(\bar{x}, x_n) - D_p(\bar{x}, x_{n+1})$. Inequality (3.50) follows now from a telescopic sum argument using the above inequalities.

Add i): Assume by contradiction that Algorithm 3.2.11 is never stopped by the discrepancy principle. Therefore, $\hat{n}$ defined in (3.40) is not finite. Consequently, $\hat{\Sigma}$ is an infinite set (at least one step is performed in each iteration cycle).

Since $(D_p(\bar{x}, x_n))_{n\in\hat{\Sigma}}$ is bounded, it follows that $(\|x_n\|)_{n\in\hat{\Sigma}}$ is bounded [76, Theorem 2.12(b)]. Therefore, the sequence $(\lambda_n)_{n\in\hat{\Sigma}}$ in (3.39), is bounded away from zero (see (3.37) and Remark 3.2.12), from what follows that $(\tau_n)_{n\in\hat{\Sigma}}$ is bounded away from zero as well. From this fact and (3.50) we obtain

$$\sum_{n\in\hat{\Sigma}} \|x_n\|^{p-1} < \infty\,.$$

Consequently, $(x_n)_{n\in\hat{\Sigma}}$ converges to zero in X and, arguing with the continuity of $D_p(\bar{x}, \cdot)$ [76, Theorem 2.12(c)] or [14]), we conclude

$$p^{-1}\|\bar{x}\|^p \ =\ D_p(\bar{x}, 0) \ =\ \lim_{n\in\hat{\Sigma}} D_p(\bar{x}, x_n) \ \le$$
$$\le\ D_p(\bar{x}, x_{n'+1}) \ <\ D_p(\bar{x}, x_{n'}) \ \le\ p^{-1}\|\bar{x}\|^p\,,$$

where $n' \in \mathbb{N}$ is an arbitrary element of $\hat{\Sigma}$ (notice that (3.49) holds with strict inequality for all $n' \in \hat{\Sigma}$). This is clearly a contradiction. Thus, $\hat{n}$ must be finite.

Add ii): Notice that in the noise free case we have $\delta_i = 0$, $i = 1, 2, \cdots, m$. In this particular case, (3.51) holds for all $n \in \mathbb{N}$. Consequently,

$$\sum_{n\in\mathbb{N}} \tau_n \|x_n\|^{p-1} \|R_n\| \ \le\ (1-\beta)^{-1}(1-c)^{-1} D_p(\bar{x}, x_0)\,.$$

Assume the existence of $\varepsilon > 0$ such that the inequality $\|R_{n_k}\| > \varepsilon$ holds true for some subsequence, and define $\hat{\Sigma} := \{n_k;\, k \in \mathbb{N}\}$. Using the same reasoning as in the proof of the second assertion we arrive at a contradiction, concluding the proof. □

3.2.6 Convergence analysis

In this section the main results of the manuscript are presented. A convergence analysis of the proposed method is given, and stability results are derived. We start the presentation discussing a result related to the existence of minimum-norm solutions.

Lemma 3.2.16. *Assume the continuous Fréchet differentiability of the operators F_i in D. Moreover, assume that* (A3.3) *is satisfied and also that problem* (3.30) *is solvable in $\mathcal{B}^2_p(x_0, \rho_0)$, where $x_0 \in X$ and $\rho_0 > 0$ is chosen as in* (A3.3).
i) There exists a unique minimum-norm solution $x^\dagger$ of (3.30) *in the ball $\mathcal{B}^2_p(x_0, \rho_0)$.*
ii) If $x^\dagger \in \mathrm{int}\big(\mathcal{B}^2_p(x_0, \rho_0)\big)$, it can be characterized as the solution of (3.30) *in $\mathcal{B}^2_p(x_0, \rho_0)$ satisfying the condition*

$$J_p(x^\dagger) \ \in\ \mathcal{N}(F_i'(x^\dagger))^\perp, \qquad i = 1, 2, \cdots, m \tag{3.52}$$

(here $A^\perp \subset X^*$ denotes the annihilator of $A \subset X$, while $\mathcal{N}(\cdot)$ represents the null-space of a linear operator).

Proof. As an immediate consequence of (A3.3) we obtain [34, Proposition 2.1]

$$F_i(z) \ =\ F_i(x) \iff z - x \in \mathcal{N}(F_i'(x))\,, \quad i = 1, 2, \cdots m\,, \tag{3.53}$$

for $x,\, z \in \mathcal{B}_p^2(x_0, \rho_0)$. Next we define for each $x \in \mathcal{B}_p^2(x_0, \rho_0)$ the set $M_x := \{z \in \mathcal{B}_p^2(x_0, \rho_0) : F_i(z) = F_i(x),\, i = 1, 2, \cdots, m\}$. Notice that $M_x \neq \emptyset$, for all $x \in \mathcal{B}_p^2(x_0, \rho_0)$. Moreover, it follows from (3.53) that

$$M_x = \bigcap_{i=1}^{m} \Big(x + \mathcal{N}(F_i^{'}(x))\Big) \cap \mathcal{B}_p^2(x_0, \rho_0). \tag{3.54}$$

Since $D_p(\cdot, x_0)$ is continuous (see Corollary 3.2.9) and $\mathcal{B}_p^2(x_0, \rho_0)$ is convex (by definition), it follows from (3.54) that M_x is nonempty closed and convex, for all $x \in \mathcal{B}_p^2(x_0, \rho_0)$. Therefore, there exists a unique $x^\dagger \in X$ corresponding to the projection of 0 on $M_{\bar{x}}$, where $\bar{x}$ is a solution of (3.30) in $\mathcal{B}_p^2(x_0, \rho_0)$ [14]. This proves the first assertion.
Add ii): From the definition of $x^\dagger$ and $M_{\bar{x}} = M_{x^\dagger}$, we conclude that [76, Theorem 2.5 (h)]

$$\langle J_p(x^\dagger), x^\dagger \rangle \;\leq\; \langle J_p(x^\dagger), y \rangle\,, \quad \forall y \in M_{x^\dagger}\,. \tag{3.55}$$

From the assumption $x^\dagger \in \mathrm{int}\big(\mathcal{B}_p^2(x_0, \rho_0)\big)$, it follows that given $h \in \cap_{i=1}^m \mathcal{N}(F_i'(x^\dagger))$, there exists a $\varepsilon_0 > 0$ such that

$$x^\dagger + \varepsilon h\,, \quad x^\dagger - \varepsilon h \;\in\; M_{x^\dagger}\,, \qquad \forall \varepsilon \in [0, \varepsilon_0)\,. \tag{3.56}$$

Thus, (3.52) follows from (3.55), (3.56) in an straightforward way. In order to prove uniqueness, let $\tilde{x}$ be any solution of (3.30) in $\mathcal{B}_p^2(x_0, \rho_0)$ satisfying

$$J_p(\tilde{x}) \;\in\; \mathcal{N}(F_i^{'}(\tilde{x}))^\perp\,, \qquad i = 1, 2, \cdots, m\,. \tag{3.57}$$

Let $i \in \{1, 2, \cdots, m\}$. We claim that

$$\mathcal{N}(F_i^{'}(x^\dagger)) \subset \mathcal{N}(F_i^{'}(\tilde{x})). \tag{3.58}$$

Indeed, let $h \in \mathcal{N}(F_i^{'}(x^\dagger))$ and set $x_\theta = (1-\theta)x^\dagger + \theta\tilde{x}$, with $\theta \in \mathbb{R}$. Since $x^\dagger \in \mathrm{int}\big(\mathcal{B}_p^2(x_0, \rho_0)\big)$, we obtain a $\theta_0 > 0$ such that $x_\theta \in \mathrm{int}\big(\mathcal{B}_p^2(x_0, \rho_0)\big)$, for all $\theta \in [0, \theta_0)$. Take $\theta \in (0, \theta_0)$ and define $x_{\theta,\mu} = x_\theta + \mu h$, for $\mu \in \mathbb{R}$. Using the same reasoning we obtain $\mu_0 > 0$ such that $x_{\theta,\mu} \in B_p^2(x_0, \rho_0)$, $\forall \mu \in [0, \mu_0)$.

For a fixed $\mu \in (0, \mu_0)$, note that $x_{\theta,\mu} - x^\dagger = \theta(\tilde{x} - x^\dagger) + \mu h$. Using (3.53) we get $\tilde{x} - x^\dagger \in \mathcal{N}(F_i^{'}(x^\dagger))$ and consequently $x_{\theta,\mu} -$

$x^\dagger \in \mathcal{N}(F_i^{'}(x^\dagger))$. From (3.53) it follows that $F(x_{\theta,\mu}) = F(x^\dagger)$ and consequently $F(x_{\theta,\mu}) = F(\tilde{x})$. Applying the same reasoning as above (based on (3.53)) we conclude that $x_{\theta,\mu} - \tilde{x} \in \mathcal{N}(F_i^{'}(\tilde{x}))$.
Since $x_{\theta,\mu} - \tilde{x} = (1-\theta)(x^\dagger - \tilde{x}) + \mu h$ and $x^\dagger - \tilde{x} \in \mathcal{N}(F_i^{'}(\tilde{x}))$ it follows $h \in \mathcal{N}(F_i^{'}(\tilde{x}))$, completing the proof of our claim.

Combining (3.57) and (3.58) we obtain $J_p(\tilde{x}) \in \mathcal{N}(F_i^{'}(x^\dagger))^\perp$. Consequently, $J_p(x^\dagger) - J_p(\tilde{x}) \in \mathcal{N}(F_i^{'}(x^\dagger))^\perp$. Since $x^\dagger - \tilde{x} \in \mathcal{N}(F_i^{'}(x^\dagger))$ we conclude that $\langle J_p(x^\dagger) - J_p(\tilde{x}), x^\dagger - \tilde{x}\rangle = 0$. Moreover, since J_p is strictly monotone [76, Theorem 2.5(e)], we obtain $x^\dagger = \tilde{x}$. □

Theorem 3.2.17 (Convergence for exact data)**.** *Assume that $\delta_i = 0$, $i = 1, 2, \cdots, m$. Let the Assumption A3 be satisfied (for simplicity we assume $\bar{\rho} = \rho_0$). Then any iteration (x_n) generated by Algorithm 3.2.11 converges strongly to a solution of (3.30).*
Additionally, if $x^\dagger \in \operatorname{int}\big(\mathcal{B}_p^2(x_0, \rho_0)\big)$, $J_p(x_0) \in \mathcal{N}(F_i'(x^\dagger))^\perp$ and $\mathcal{N}(F_i'(x^\dagger)) \subset \mathcal{N}(F_i'(x))$, $x \in B_p^1(\bar{x}, \bar{\rho})$, $i = 1, 2, \cdots, m$, then (x_n) converges strongly to $x^\dagger$.

Proof. From Lemma 3.2.13 it follows that $D_p(\bar{x}, x_n)$ is bounded and so $(\|x_n\|)$ is bounded. In particular, $(J_p(x_n))$ is also bounded. Define $\varepsilon_n = q^{-1}\|x_n\|^p - \langle \bar{x}, J_p(x_n)\rangle$, $n \in \mathbb{N}$. From Lemma 3.2.13 and Corollary 3.2.9 it follows that (ε_n) is bounded and monotone nonincreasing. Thus, there exists $\varepsilon \in \mathbb{R}$ such that $\varepsilon_n \to \varepsilon$, as $n \to \infty$.

Let $m, n \in \mathbb{N}$ such that $m > n$. It follows from Corollary 3.2.10 that

$$
\begin{aligned}
D_p(x_n, x_m) &= q^{-1}\left(\|x_m\|^p - \|x_n\|^p\right) + \langle J_p(x_n) - J_p(x_m), x_n\rangle \\
&= (\varepsilon_m - \varepsilon_n) + \langle J_p(x_n) - J_p(x_m), x_n - \bar{x}\rangle. \quad (3.59)
\end{aligned}
$$

The first term of (3.59) converges to zero, as $m, n \to \infty$. Notice that

$$
\begin{aligned}
|\langle J_p(x_n) - J_p(x_m), x_n - \bar{x}\rangle| &= \Big|\Big\langle \sum_{k=n}^{m-1} (J_p(x_{k+1}) - J_p(x_k)), x_k - \bar{x}\Big\rangle\Big| \\
&\overset{(3.38)}{=} \Big|\Big\langle \sum_{k=n}^{m-1} \mu_k F_{i_k}'(x_k)^* J_r(R_k), x_k - \bar{x}\Big\rangle\Big| \\
&\le \sum_{k=n}^{m-1} \mu_k \|J_r(R_k)\| \|F_{i_k}'(x_k)(x_k - \bar{x})\|.
\end{aligned}
$$

Moreover, from (A3.2) we have

$$\begin{aligned}
\|F'_{i_k}(x_k)(x_k-\bar{x})\| &\leq \|F_{i_k}(x_k)-F_{i_k}(\bar{x})-F'_{i_k}(x_k)(x_k-\bar{x})\| \\
&\quad + \|F_{i_k}(x_k)-F_{i_k}(\bar{x})\| \\
&\leq \|F_{i_n}(x_{n_k})-F_{i_n}(x_n)-F'_{i_n}(x_n)(x_{n_k}-x_n)\| \\
&\quad + \|F_{i_n}(\bar{x})-F_{i_n}(x_n)-F'_{i_n}(x_n)(\bar{x}-x_n)\| \\
&\quad + \|F_{i_n}(x_{n_k})-F_{i_n}(x_n)\| \\
&\quad + \|F_{i_n}(\bar{x})-F_{i_n}(x_n)\| \\
&\leq (1+\bar{\eta})\big(\|F_{i_n}(x_{n_k})-F_{i_n}(x_n)\| \\
&\quad + \|F_{i_n}(\bar{x})-F_{i_n}(x_n)\|\big) \\
&\leq (1+\bar{\eta})\big(\|R_{n_k}\|+2\|R_n\|\big) \\
&\leq (1+\eta)\,\|R_k\|\,,
\end{aligned}$$

where the last inequality follows from the fact that $n < n_k$. Therefore, using (A3.4) and the definition of μ_k in Algorithm 3.2.11, we can estimate

$$\begin{aligned}
|\langle J_p(x_n)-J_p(x_m), x_n-\bar{x}\rangle| &\leq (1+\eta)\sum_{k=n}^{m-1} \mu_k\|R_k\|^{r-1}\|R_k\| \\
&= (1+\eta)\sum_{k=n}^{m-1} \tfrac{\tau_k\|x_k\|^{p-1}\|R_k\|^r}{(1\vee\|F'_{i_k}(x_k)\|)\|R_k\|^{r-1}} \\
&\leq (1+\eta)\sum_{k=n}^{m-1} \tau_k\|x_k\|^{p-1}\|R_k\|
\end{aligned}$$

(notice that the last two sums are carried out only for the terms with $\mu_k \neq 0$). Consequently, $\langle J_p(x_n)-J_p(x_m), x_n-\bar{x}\rangle$ converges to zero, from what follows $D_p(x_n,x_m)\to 0$, as $m,\ n\to\infty$. Therefore, we conclude that (x_n) is a Cauchy sequence, converging to some element $\tilde{x}\in X$ [76, Theorem 2.12(b)]. Since $x_n\in B_p^1(\bar{x},\bar{\rho})\subset D$, for $n\in\mathbb{N}$, it follows that $\tilde{x}\in D$. Moreover, from the continuity of $D_p(\cdot,\tilde{x})$, we have $D_p(x_n,\tilde{x})\to D_p(\tilde{x},\tilde{x})=0$, proving that $\|x_n-\tilde{x}\|\to 0$.

Let $i\in\{1,2,\cdots,m\}$ and $\varepsilon>0$. Since F_i is continuous, we have $F_i(x_n)\to F_i(\tilde{x})$, $n\to\infty$. This fact together with $R_n\to 0$, allow us to find an $n_0\in\mathbb{N}$ such that

$$\|F_i(x_n)-F_i(\tilde{x})\| < \varepsilon/2\,, \qquad \|F_{i_n}(x_n)-y_{i_n}\| < \varepsilon/2\,, \quad \forall n\geq n_0\,.$$

Let $\tilde{n} \geq n_0$ be such that $i_{\tilde{n}} = i$. Then $\|F_i(\tilde{x}) - y_i\| \leq \|F_i(x_{\tilde{n}}) - F_i(\tilde{x})\| + \|F_{i_{\tilde{n}}}(x_{\tilde{n}}) - y_{i_{\tilde{n}}}\| < \varepsilon$. Thus, $F_i(\tilde{x}) = y_i$, proving that $\tilde{x}$ is a solution of (3.30).

For each $n \in \mathbb{N}$ it follows from (3.38) and the theorem assumption that

$$J_p(x_n) - J_p(x_0) \;\in\; \bigcap_{k=0}^{n-1} \mathcal{N}(F'_{i_k}(x_k))^\perp \;\subset\; \bigcap_{k=0}^{n-1} \mathcal{N}(F'_{i_k}(x^\dagger))^\perp .$$

Moreover, due to $J_p(x_0) \in \mathcal{N}(F'_i(x^\dagger))^\perp$, $i = 1, 2, \cdots, m$, we have $J_p(x_n) \in \bigcap_{j=1}^m \mathcal{N}(F'_j(x^\dagger))^\perp$, $n \geq m$. Then $J_p(x_n) \in \mathcal{N}(F'_i(x^\dagger))^\perp$, for $n \geq m$. Since J_p is continuous and $x_n \to \tilde{x}$, we conclude that $J_p(\tilde{x}) \in \mathcal{N}(F'_i(x^\dagger))^\perp$. However, due to $\mathcal{N}(F_i^{'}(\tilde{x})) = \mathcal{N}(F_i^{'}(x^\dagger))$ (which follows from $F_i(\tilde{x}) = F_i(x^\dagger)$) we conclude that $J_p(\tilde{x}) \in \mathcal{N}(F'_i(\tilde{x}))^\perp$, proving that $\tilde{x} = x^\dagger$. □

In the sequel we prove a convergence result in the noisy data case. For simplicity of the presentation, we assume for the rest of this section that $\delta_1 = \delta_2 = \cdots = \delta_m = \delta > 0$. Moreover, we denote by (x_n), (x_n^δ) the iterations generated by Algorithm 3.2.11 with exact data and noisy data respectively.

Theorem 3.2.18 (Semi-convergence)**.** *Let Y be an uniformly smooth Banach space and Assumption A3 be satisfied (for simplicity we assume $\bar{\rho} = \rho_0$). Moreover, let $(\delta_k > 0)_{k \in \mathbb{N}}$ be a sequence satisfying $\delta_k \to 0$ and $y_i^k \in Y$ be corresponding noisy data satisfying $\|y_i^k - y_i\| \leq \delta_k$, $i = 1, \ldots, m$, and $k \in \mathbb{N}$.*
If (for each $k \in \mathbb{N}$) the iterations $(x_n^{\delta_k})$ are stopped according to the discrepancy principle (3.40) at $\hat{n}_k = \hat{n}(\delta_k)$, then $(x_{\hat{n}_k}^{\delta_k})$ converges (strongly) to a solution $\tilde{x} \in B_p^1(\bar{x}, \bar{\rho})$ of (3.30) as $k \to \infty$.
Additionally, if $x^\dagger \in \operatorname{int}\big(\mathcal{B}_p^2(x_0, \rho_0)\big)$, $J_p(x_0) \in \mathcal{N}(F'_i(x^\dagger))^\perp$ and $\mathcal{N}(F'_i(x^\dagger)) \subset \mathcal{N}(F'_i(x))$, $x \in B_p^1(\bar{x}, \bar{\rho})$, $i = 1, 2, \cdots, m$, then $(x_{\hat{n}_k}^{\delta_k})$ converges (strongly) to $x^\dagger$ as $k \to \infty$.

Proof. For each $k \in \mathbb{N}$ we can write $\hat{n}_k$ in (3.40) in the form $\hat{\ell}_k m + (m-1)$. Thus, $x_{\hat{n}_k}^{\delta_k} = x_{\hat{n}_k - 1}^{\delta_k} = \cdots = x_{\hat{n}_k - (m-1)}^{\delta_k}$ and

$$\big\|F_{i_n}\big(x_n^{\delta_k}\big) - y_{i_n}^k\big\| \;\leq\; \tau\,\delta_k\,, \qquad n = \hat{\ell}_k m, \cdots, \hat{\ell}_k m + (m-1)\,.$$

Since $i_n = 1, 2, \cdots, m$ as $n = \hat{\ell}_k m, \cdots, \hat{\ell}_k m + (m-1)$, it follows that

$$\big\| F_i\big(x_{\hat{n}_k}^{\delta_k}\big) - y_i^k \big\| \;\leq\; \tau\, \delta_k\,, \qquad i = 1, 2, \cdots, m\,. \tag{3.60}$$

At this point we must consider two cases separately:

Case 1: The sequence $(\hat{n}_k) \in \mathbb{N}$ is bounded.
If this is the case, we can assume the existence of $\hat{n} \in \mathbb{N}$ such that $\hat{n}_k = \hat{n}$, for all $k \in \mathbb{N}$. Notice that, for each $k \in \mathbb{N}$, the sequence element $x_{\hat{n}}^{\delta_k}$ depends continuously on the corresponding data $\big(y_i^k\big)_{i=1}^m$ (this is the point where the uniform smoothness of Y is required). Therefore, it follows that

$$x_{\hat{n}}^{\delta_k} \to x_{\hat{n}}\,, \qquad F_i\big(x_{\hat{n}}^{\delta_k}\big) \to F_i(x_{\hat{n}})\,,\; k \to \infty\,, \tag{3.61}$$

for each $i = 1, 2, \cdots, m$. Since each operator F_i is continuous, taking limit as $k \to \infty$ in (3.60) gives $F_i(x_{\hat{n}}) = y_i$, $i = 1, 2, \cdots, m$, which proves that $\tilde{x} := x_{\hat{n}}$ is a solution of (3.30).

Case 2: The sequence $(\hat{n}_k) \in \mathbb{N}$ is unbounded.
We can assume that $\hat{n}_k \to \infty$, monotonically. Due to Theorem 3.2.17, $(x_{\hat{n}_k})$ converges to some solution $\tilde{x} \in B_p^1(\bar{x}, \bar{\rho})$ of (3.30). Therefore, $D_p(\tilde{x}, x_{\hat{n}_k}) \to 0$. Thus, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$D_p(\tilde{x}, x_{\hat{n}_k}) \;<\; \varepsilon/2\,, \qquad \forall \hat{n}_k \geq N\,.$$

Since $x_N^{\delta_k} \to x_N$ as $k \to \infty$, and $D_p(\tilde{x}, \cdot)$ is continuous, there exists $\tilde{k} \in \mathbb{N}$ such that

$$\big| D_p\big(\tilde{x}, x_N^{\delta_k}\big) - D_p(\tilde{x}, x_N) \big| \;<\; \varepsilon/2\,, \qquad \forall k \geq \tilde{k}\,.$$

Consequently,

$$D_p(\tilde{x}, x_N^{\delta_k}) \;=\; D_p(\tilde{x}, x_N) + D_p\big(\tilde{x}, x_N^{\delta_k}\big) - D_p(\tilde{x}, x_N) \;<\; \varepsilon\,, \qquad \forall k \geq \tilde{k}\,.$$

Since $D_p(\tilde{x}, x_{\hat{n}_k}^{\delta_k}) \leq D_p(\tilde{x}, x_N)$, for all $\hat{n}_k > N$, it follows that $D_p(\tilde{x}, x_{\hat{n}_k}^{\delta_k}) < \varepsilon$ for k large enough. Therefore, due to [76, Theorem 2.12(d)] or [14], we conclude that $(x_{\hat{n}_k}^{\delta_k})$ converges to $\tilde{x}$.

To prove the last assertion, it is enough to observe that, due to the extra assumption, $\tilde{x} = x^\dagger$ must hold. □

3.3 Bibliographical comments

Standard methods for the solution of system (3.30) are based in the use of *Iterative type* regularization methods [1, 21, 48, 76, 49] or *Tikhonov type* regularization methods [21, 66, 82, 74] after rewriting (3.30) as a single equation $F(x) = y$, where

$$F := (F_1, \ldots, F_m) : \bigcap_{i=1}^{m} D_i =: D \to Y^m \tag{3.62}$$

and $y := (y_1, \ldots, y_m)$. However these methods become inefficient if m is large or the evaluations of $F_i(x)$ and $F_i'(x)^*$ are expensive. In such a situation, Kaczmarz type methods [47, 63, 68] which cyclically consider each equation in (3.30) separately are much faster [68] and are often the method of choice in practice.

Ill-posed operator equations in Banach spaces is a fast growing area of research. Over the last seven years several theoretical results have been derived in this field, e.g,

- The classical paper on regularization of ill-posed problems in Banach spaces by Resmerita [74];

- Tikhonov regularization in Banach spaces is also investigated in [9], where two distinct iterative methods for finding the minimizer of norm-based Tikhonov functionals are proposed and analyzed (convergence is proven). Moreover, convergence rates results for Tikhonov regularization in Banach spaces are considered in [43].

- In [76] a nonlinear extension of the Landweber method to linear operator equations in Banach spaces is investigated using duality mappings. The same authors considered in [77] the solution of convex split feasibility problems in Banach spaces by cyclic projections. See also [40, 39] for convergence analysis of modified Landweber iterations in Banach spaces;

- In [49] the nonlinear Landweber method and the IRGN method are considered for a single (nonlinear) operator equation in Banach spaces, and convergence results are derived. Moreover, the applicability of the proposed methods to parameter identification problems for elliptic PDEs is investigated;

- The Gauss-Newton method in Banach spaces is considered in [1] for a single operator equation in the special case $X = Y$. A convergence result is obtained and convergence rates (under strong source conditions) are provided.

3.4 Exercises

3.1. Prove Proposition 3.2.2.

3.2. Prove Proposition 3.2.3.

3.3. Prove Proposition 3.2.5.

3.4. Prove Theorem 3.2.7.

3.5. Prove Corolary 3.2.9.

3.6. Prove Corolary 3.2.10.

3.7. Prove the inequality (3.46). (Hint: use Proposition 3.2.4)

3.8. Prove (3.53).

3.9. Derive equality (3.59).

Chapter 4

Double regularization

In the classical Least Squares approach the system matrix is assumed to be free from error and all the errors are confined to the observation vector. However in many applications this assumption is often unrealistic. Therefore a new technique was proposed: *Total Least Squares*, or shortly, TLS[1]. This concept has been independently develop in various literatures, namely, *error-in-variables*, *rigorous least squares*, or (in a special case) *orthogonal regression*, listing only few in statistical literature. It also leads to a procedure investigated in this chapter named *regularized total least squares*.

In this chapter we shall introduce the TLS fitting technique and the two variations: the *regularized* TLS and the *dual regularized* TLS. Additionally to this survey we present a new technique introduced recently in [7], extending results to Hilbert spaces and deriving rates of convergence using Bregman distances and convex optimisation techniques; similar as presented in the previous chapters.

4.1 Total least squares

Gene Howard Golub[2] was an American mathematician with remarkable work in the field of numerical linear algebra; listing only a few

[1] We hope to not mislead to *truncate least squares* notation.

[2] See wikipedia.org/wiki/Gene_H._Golub.

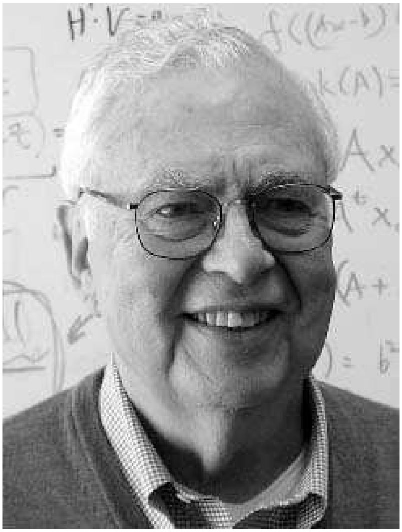

Figure 4.1: Gene H. Golub.

topics: least-squares problems, singular value decomposition, domain decomposition, differentiation of pseudo-inverses, inverse eigenvalue problem, conjugate gradient method, Gauss quadrature.

In 1980 Golub and Van Loan [26] investigated a fitting technique based on the least squares problem for solving a matrix equation with incorrect matrix and data vector, named *total least squares* (TLS) method. On the following we presented the TLS method and we compare it briefly with another classical approach; more details can be found in [83, 62] and references therein.

Let A_0 be a matrix in $\mathbb{R}^{m\times n}$ and y_0 a vector in $\mathbb{R}^{m\times 1}$, obtained after the discretisation of the linear operator equation $Fu = g$, where $F : \mathcal{U} \to \mathcal{H}$ is a mapping between two Hilbert spaces. We then consider[3] solving the equation

$$A_0 x = y_0 \tag{4.1}$$

[3] We introduce the problem only for the one dimensional case $Ax = y$, i.e., when x and y are vectors. In the book [83, Chap 3] it is also considered the multidimensional case $AX = Y$, where all elements are matrices.

where both A_0 and y_0 information out of inaccurate measurement or inherent some roundoff errors. More precisely, it is available only the following pair

$$\|y_0 - y_\delta\|_2 \leq \delta \tag{4.2}$$

and

$$\|A_0 - A_\epsilon\|_F \leq \epsilon. \tag{4.3}$$

In particular the classical *least squares* (LS) approach, proposed by Carl Friedrich Gauss (1777-1855), the measurements A_0 are assumed to be free of error; hence, all the errors are confined to the observation vector y_δ. The LS solution is given by solving the following minimization problem

$$\begin{array}{ll} \text{minimize}_y & \|y - y_\delta\|_2 \\ \text{subject to} & y \in \mathscr{R}(A_0) \end{array}$$

or equivalently

$$\underset{x}{\text{minimize}} \|A_0 x - y_\delta\|_2. \tag{4.4}$$

Solutions of the ordinary LS problem are characterised by the following theorem.

Theorem 4.1.1 ([83, Cor 2.1]). *If* $\text{rank}(A_0) = n$ *then* (4.4) *has a unique LS solution, given by*

$$x^{LS} = (A_0^T A_0)^{-1} A_0^T y_\delta. \tag{4.5}$$

the corresponding LS correction is given by the residual

$$r = y_\delta - A_0 x^{LS} = y_\delta - y^{LS}, \qquad y^{LS} = P_{A_0} y_\delta$$

where $P_{A_0} = A_0(A_0^T A_0)^{-1} A_0^T$ *is the orthogonal projector onto* $\mathscr{R}(A_0)$.

Proof. See Exercise 4.1. □

This approach is frequently unrealistic: sampling errors, human errors and instrument errors may imply inaccuracies of the data matrix A_0 as well (e.g., due discretisation, approximation of differential or integral models).

Therefore the need of an approach which amounts to fitting a "best" subspace to the measurement data (A_ϵ, y_δ) leads to the TLS approach. In comparison to LS method the new minimization problem is with respect to the pair (A, y). The element paring $\tilde{A}x^{TLS} = \tilde{y}$ is then called the *total least squares solution*, where $\tilde{A}$ and $\tilde{y}$ are the arguments which minimizes the following constrained problem[4]

$$\begin{array}{cc} \text{minimize}_{(A,y)} & \big\| [A, y] - [A_\epsilon, y_\delta] \big\|_F \\ \text{subject to} & y \in \mathscr{R}(A) \end{array} . \tag{4.6}$$

The basic principle of TLS is that the noisy data $[A_\epsilon, y_\delta]$, while not satisfying a linear relation, are modified with minimal effort, as measured by the Frobenius norm, in a "nearby" matrix $[\tilde{A}, \tilde{y}]$ that is rank-deficient so that the set $\tilde{A}x = \tilde{y}$ is compatible. This matrix $[\tilde{A}, \tilde{y}]$ is a rank one modification of the data matrix $[A_\epsilon, y_\delta]$.

The foundation is the *singular value decomposition* (SVD), an important role in a number of matrix approximation problems [27]; see its definition in the upcoming theorem.

Theorem 4.1.2 ([27, Thm 2.5.1]). *If $A \in \mathbb{R}^{m\times n}$ then there exist orthonormal matrices $U = [u_1, \ldots, u_m] \in \mathbb{R}^{m\times m}$ and $V = [v_1, \ldots, v_n] \in \mathbb{R}^{n\times n}$ such that*

$$U^T A V = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_p), \quad \sigma_1 \geq \cdots \geq \sigma_p \geq 0$$

where $p = \min\{m, n\}$.

Proof. See Exercise 4.2. □

The triplet (u_i, σ_i, v_i) reveals a great deal about the structure of A. For instance, defining r as the number of nonzeros singular values, i.e., $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots \sigma_p = 0$ it is known that

$$\operatorname{rank}(A) = r \quad \text{and} \quad A = U_r \Sigma_r {V_r}^T = \sum_{i=1}^{r} \sigma_i u_i {v_i}^T \tag{4.7}$$

where U_r (equivalently Σ_r and V_r) denotes the first r columns of the matrix U (equivalently Σ and V). The Equation (4.7) displays the decomposition of the matrix A of rank r in a sum of r matrices of rank one.

[4]we use the same Matlab's notation to add the vector y as a new column to the matrix A and so create an extended matrix $[A, y] \in \mathbb{R}^{m\times(n+1)}$

Through SVD we can define the Frobenious norm of a matrix A as

$$\|A\|_F^2 := \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 = \sigma_1^2 + \cdots + \sigma_p^2, \quad p = \min\{m,n\}$$

while the 2-norm

$$\|A\|_2 := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1.$$

The core of matrix approximation problem is stated by Eckart-Young ([19] with Frobenious norm) and Mirsky ([64] with 2-norm) and summarised on the next result, known as *Eckart-Young-Mirsky (matrix approximation) theorem.*

Theorem 4.1.3 ([83, Thm 2.3])**.** *Let the SVD of $A \in \mathbb{R}^{m\times n}$ be given by $A = \sum_{i=1}^{r} \sigma_i u_i {v_i}^T$ with $r = \operatorname{rank}(A)$. If $k < r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i {v_i}^T$, then*

$$\min_{\operatorname{rank}(D)=k} \|A - D\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

and

$$\min_{\operatorname{rank}(D)=k} \|A - D\|_F = \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^{p} \sigma_i^2}, \quad p = \min\{m,n\}$$

Proof. See Exercise 4.3. □

On the following we give a close form characterising the TLS solution, similar as (4.5) for the LS solution.

Theorem 4.1.4 ([83, Thm 2.7])**.** *Let $A_\epsilon = U'\Sigma'V'^T$ (respectively, $[A_\epsilon, y_\delta] = U\Sigma V^T$) be the SVD decomposition of A_ϵ (respectively, $[A_\epsilon, y_\delta]$). If $\sigma_n' > \sigma_{n+1}$, then*

$$x^{TLS} = (A_\epsilon^T A_\epsilon - \sigma_{n+1}^2 I)^{-1} A_\epsilon^T y_\delta \tag{4.8}$$

and

$$\sigma_{n+1}^2 \left[1 + \sum_{i=1}^{n} \frac{({u_i'}^T y_\delta)^2}{{\sigma_i'}^2 - \sigma_{n+1}^2}\right] = \min \big\|A_\epsilon x - y_\delta\big\|_2^2 \tag{4.9}$$

Proof. See Exercise 4.4. □

In order to illustrate the effect of the use of TLS as opposed to LS, we consider here the simplest example of *parameter estimation in 1D*.

Example 4.1.5. *Find the slope m of the linear equation*

$$xm = y$$

for given a set of eight pairs measurements (x_i, y_i), where $x_i = y_i$ for $1 \leq i \leq 8$. It is easy to find out that the slop (solution) is $m = 1$. Although this example is straightforward and well-posed, we can learn the following geometric interpretation: the LS solution displayed on Figure 4.2 with measurements on the left-hand side fits the curve on the horizontal direction, since the axis y is free of noise; meanwhile, the LS solution displayed on Figure 4.3 with measurements on the right-hand side fits the curve on the vertical direction, since the axis x is fixed (free of noise).

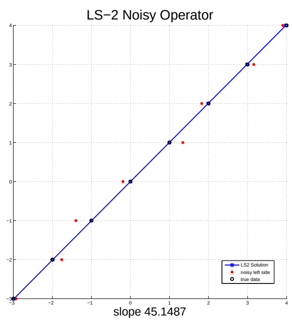

Figure 4.2: Solution for the data (x_ϵ, y_0), i.e., only noise on the left-hand side.

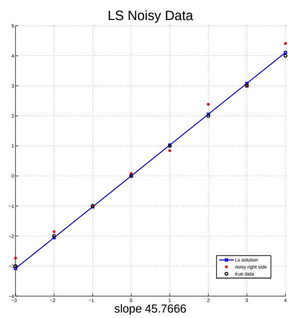

Figure 4.3: Solution for the data (x_0, y_δ), i.e., only noise on the right-hand side.

The TLS solution on Figure 4.4 illustrates the estimation with noise on both directions and now the deviations are orthogonal to the fitted line, i.e., it minimizes the sum of squares of their lengths.

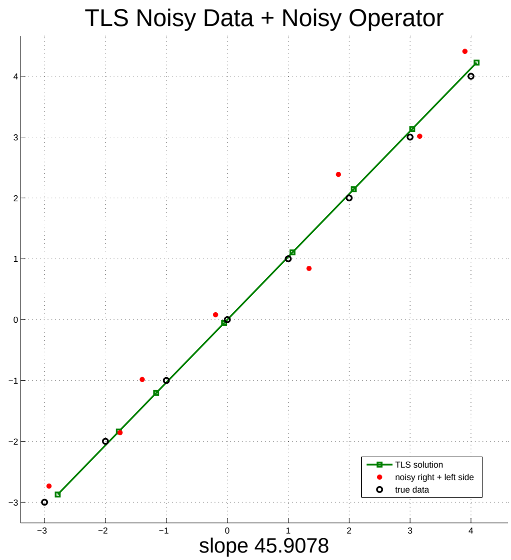

Figure 4.4: Solution for the data (x_ϵ, y_δ), i.e., noise on the both sides.

Therefore, this estimation procedure is sometimes called as orthogonal regression.

Van Loan commented on her book [83] that in typical applications, gains of 10–15 percent in accuracy can be obtained using TLS over the standard LS method, almost at no extra computational cost. Moreover, it becomes more effective when more measurements can be made.

Another formulation for the TLS, investigate e.g. in [60], of the set $A_\epsilon x \approx y_\delta$ is given through the following constrained problem

$$\begin{array}{ll} \text{minimize} & \left\|A - A_\epsilon\right\|_F^2 + \left\|y - y_\delta\right\|_2^2 \\ \text{subject to} & Ax = y \end{array} . \tag{4.10}$$

This formulation emphasises the perpendicular distance by minimizing the sum of squared misfit in each direction. One can also recast this constrained minimization as an unconstrained problem, by replacing $y = Ax$ in the second term of Equation (4.10).

In the upcoming section we extend this approach to the regularized version, that is, adding a stabilisation term.

4.1.1 Regularized total least squares

Since our focus is on very ill-posed problems the approach introduced in the previous section is no longer efficient. We can observe from the discretisation of ill-posed problems, such as integral equations of the first kind, that the singular values of the discrete operator decay gradually to zero. The need of a stabilisation term leads us to regularization methods, e.g., the Tikhonov method already defined in the Chapter 2. We introduce now one equivalent formulation (see commentaries on [25]) called *regularized least squares* problem, as the following constrained optimisation problem

$$\begin{array}{ll} \text{minimize} & \left\| A_0 x - y_\delta \right\|_2^2 \\ \text{subject to} & \left\| Lx \right\|^2 \leq M \end{array} . \tag{4.11}$$

This idea can be carried over when both sides of the underlying Equation (4.1) are contaminated with some noise, i.e., using TLS instead of the LS misfit term.

So was Tikhonov regularization recast as a TLS formulation and the resulting was coined *regularized total least squares* method (R-TLS), see [26, 36, 25]. Intuitively it is added some constrained to the TLS problem (4.10). Consequently, in a finite dimensional setting[5], the R-TLS method can be formulated as

$$\begin{array}{ll} \text{minimize} & \left\| A - A_\epsilon \right\|_F^2 + \left\| y - y_\delta \right\|_2^2 \\ \text{subject to} & \begin{cases} Ax = y \\ \left\| Lx \right\|_2^2 \leq M \, . \end{cases} \end{array} \tag{4.12}$$

The optimal pair $(\hat{A}, \hat{y})$ minimizes the residual in the operator and in the data, measured by Frobenius and Euclidian norm, respectively. Moreover, the solution pair is connected via the equation $\hat{A}x = \hat{y}$,

[5] we keep the same notation as in the infinite dimensional setup.

where the element x belongs to a ball in $\mathcal{V}$ of radius M. The "size" of the ball is measured by a linear and invertible operator L (often the identity). Any element x^R satisfying these constraineds defines a *R-TLS solution.*

The *Karush-Kuhn-Tucker* (KKT[6]) condition for the optimisation problem introduced in (4.12) are summarised in the upcoming result.

Theorem 4.1.6 ([25, Thm 2.1])**.** *If the inequality constrained is active, then*

$$\left(A_\epsilon^T A_\epsilon + \alpha L^T L + \beta I\right) x^R = A_\epsilon^T y_\delta \text{ and } \left\|Lx^R\right\| = M$$

with $\alpha = \mu(1 + \left\|x^R\right\|^2)$, $\beta = -\dfrac{\left\|A_\epsilon x^R - y_\delta\right\|^2}{1 + \left\|x^R\right\|^2}$ *and* $\mu > 0$ *is the Lagrange multiplier. The two parameters are related by*

$$\alpha M^2 = y_\delta^T (y_\delta - A_\epsilon x^R) + \beta.$$

Moreover, the TLS residual satisfies

$$\left\|[A_\epsilon, y_\delta] - [\hat{A}, \hat{y}]\right\|_F^2 = -\beta \tag{4.13}$$

Proof. See Exercise 4.5. □

The main drawback on this approach is the following: the method requires a reliable bound M for the norm $\left\|Lx^{true}\right\|^2$, where such estimation for the true solution is not known. In [60] there is an example showing the dependence and instability of the method for different values of M.

Observe that the R-TLS residual given in (4.13) is a *weighted LS* misfit term. In other words, it is minimized the LS error with weight

$$w(x) = \frac{1}{1 + \|x\|^2} \,. \tag{4.14}$$

Moreover, the solution of both problems are the same, as stated in the next theorem.

[6]KKT are first order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied; see more details in [70].

Theorem 4.1.7 ([60, Thm 2.3])**.** *The R-TLS problem solution of the problem* (4.12) *is the solution of the constrained minimization problem*

$$\underset{x}{\text{minimize}} \frac{\left\|A_\epsilon x - y_\delta\right\|^2}{1+\left\|x\right\|^2} \quad \text{subject to} \quad \left\|Lx\right\| \le M \tag{4.15}$$

Proof. See Exercise 4.6. □

In the next section we comment on another approach to deal with this class of problems. Moreover, this approach leads to error bounds.

4.1.2 Dual regularized total least squares

The accuracy of the R-TLS depends heavily on the right choice of M, which is usually difficult to obtain, as commented previously.

An alternative is the *dual regularized total least square* (D-RTLS) method proposed few years ago [60, 61, 59]. When some reliable bounds for the noise levels δ and ϵ are known it makes sense to look for approximations $(\hat{A}, \hat{x}, \hat{y})$ which satisfy the side conditions

$$Ax = y, \quad \left\|y - y_\delta\right\| \le \delta \quad \text{and} \quad \left\|A - A_\epsilon\right\| \le \epsilon.$$

The solution set characterised by these three side conditions is nonempty, according to [60]. This is the major advantage of the dual method over the R-TLS, because we can avoid the dependence of the bound M.

Selecting from the solution set the element which minimizes $\|Lx\|$ leads us to a problem in which some estimate $(\hat{A}, \hat{x}, \hat{y})$ for the true (A_0, x^{true}, y_0) is determined by solving the constrained minimization problem

$$\begin{array}{ll} \text{minimize} & \|Lx\|_2^2 \\ \text{subject to} & \begin{cases} Ax = y \\ \|y - y_\delta\|_2^2 \le \delta \\ \|A - A_\epsilon\|_F^2 \le \epsilon\,, \end{cases} \end{array} \tag{4.16}$$

where $\|\cdot\|_F$ still denotes again the Frobenius norm. Please note that most of the available results on this method do again require a finite dimensional setup, see, e.g., [60, 61, 80].

Theorem 4.1.8 ([60, Thm 3.2]). *If the two inequalities constraineds are active, then the dual R-TLS solution x^D of the problem (4.16) is a solution of the equation*

$$\left(A_\epsilon^T A_\epsilon + \alpha L^T L + \beta I\right)x^D = A_\epsilon^T y_\delta$$

with $\alpha = \dfrac{\nu + \mu\left\|x^D\right\|^2}{\nu\mu}$, $\beta = -\dfrac{\mu\left\|A_\epsilon x^D - y_\delta\right\|^2}{\nu + \mu\left\|x^D\right\|^2}$ *and* $\nu, \mu > 0$ *are Lagrange multipliers. Moreover,*

$$\left\|A_\epsilon x^D - y_\delta\right\| = \delta + \epsilon\left\|x^D\right\| \qquad \text{and} \qquad \beta = -\frac{\epsilon\left(\delta + \epsilon\left\|x^D\right\|\right)}{\left\|x^D\right\|} \tag{4.17}$$

Proof. See Exercise 4.7. □

As result of the above theorem (see [60, Remark 3.4]), if the two constraineds of the dual problem are active, then we can also characterise either by the *constrained* minimization problem

$$\begin{array}{cc} \text{minimize} & \|Lx\| \\ \text{subject to} & \|A_\epsilon x - y_\delta\| = \delta + \epsilon\,\|x\| \end{array}$$

or by the *unconstrained* minimization problem

$$\underset{x}{\text{minimize}}\, \|A_\epsilon x - y_\delta\|^2 + \alpha\,\|Lx\|^2 - (\delta + \epsilon\big\|x\big\|)^2$$

wiht α chosen by the nonlinear equation $\|A_\epsilon x - y_\delta\| = \delta + \epsilon\,\|x\|$.

The relation of constrained and unconstrained minimization problems is essential for understanding the new regularization method proposed in the upcoming section.

Additionally to this short revision we list two important theorems concerning error bounds for both R-TLS and D-RTLS method, for the standard case $L = I$ (identity operator). As indicated in the article [60] these are the first results to prove order optimal error bounds so far given in the literature and they depend on the following classical source condition

$$x^\dagger = A_0^*\omega \quad \omega \in \mathcal{U}. \tag{4.18}$$

This SC-I is the same type assumed on the previous chapter, see (2.6) and (2.17), respectively, for the linear and nonlinear case.

Theorem 4.1.9 ([60, Thm 6.2]). *Assume that the exact solution $x^\dagger$ of the problem (4.1) satisfies the SC (4.18) and let x^D be the D-RTLS solution of the problem (4.16). Then*

$$\left\|x^D - x^\dagger\right\| \leq 2\,\|\omega\|^{1/2}\sqrt{\delta + \epsilon\,\|x^\dagger\|}.$$

Proof. See Exercise 4.8. □

In contrast we present also convergence rate for the R-TLS solution, that is to say, both of order $\mathcal{O}\left(\sqrt{\delta+\epsilon}\right)$.

Theorem 4.1.10 ([60, Thm 6.1]). *Assume that the exact solution $x^\dagger$ of the problem (4.1) satisfies the SC (4.18) and the side condition $\left\|x^\dagger\right\| = M$. Let in addition x^R be the R-TLS solution of the problem (4.12), then*

$$\left\|x^R - x^\dagger\right\| \leq (2+2\sqrt{2})^{1/2}\,\|\omega\| \max\left\{1, M^{1/2}\right\}\sqrt{\delta+\epsilon}.$$

Proof. See Exercise 4.9. □

4.2 Total least squares with double regularization

In our approach, we would like to restrict our attention to linear operators that can be mainly characterised by a function, as it is, e.g., the case for linear integral operators, where the kernel function determines the behaviour of the operator. Moreover, we will assume that the noise in the operator is due to an incorrect characterising function. This approach will allow us to treat the problem of finding a solution of an operator equation from incorrect data and operator in the framework of Tikhonov regularization rather than as a constrained minimization problem.

In this chapter we introduce the proposed method as well as its mathematical setting. We focus on analysing its regularization properties: existence, stability and convergence. Additionally we study source condition and derive convergence rates with respect to Bregman distance.

4.2.1 Problem formulation

We aim at the inversion of linear operator equation

$$A_0 f = g_0$$

from noisy data g_δ and incorrect operator A_ϵ. Additionally we assume that the operators $A_0, A_\epsilon : \mathcal{V} \to \mathcal{H}$, $\mathcal{V}, \mathcal{H}$ Hilbert spaces, can be characterised by functions $k_0, k_\epsilon \in \mathcal{U}$. To be more specific, we consider operators

$$\begin{aligned} A_k : \quad \mathcal{V} &\longrightarrow \quad \mathcal{H} \\ v &\longmapsto \quad B(k, v)\ , \end{aligned}$$

i.e., $A_k v := B(k, v)$, where B is a *bilinear* operator

$$B : \mathcal{U} \times \mathcal{V} \to \mathcal{H}$$

fulfilling, for some $C > 0$,

$$\big\|B(k, f)\big\|_{\mathcal{H}} \leq C\big\|k\big\|_{\mathcal{U}}\big\|f\big\|_{\mathcal{V}}. \tag{4.19}$$

From (4.19) follows immediately

$$\big\|B(k, \cdot)\big\|_{\mathcal{V}\to\mathcal{H}} \leq C\big\|k\big\|_{\mathcal{U}}. \tag{4.20}$$

Associated with the bilinear operator B, we also define the linear operator

$$\begin{aligned} F\,(:) \quad \mathcal{U} &\longrightarrow \quad \mathcal{H} \\ u &\longmapsto \quad B(u, f)\ , \end{aligned}$$

i.e., $F\,(u) := B(u, f)$.

From now on, let us identify A_0 with A_{k_0} and A_ϵ with A_{k_ϵ}. From (4.20) we deduce immediately

$$\|A_0 - A_\epsilon\| \leq C\|k_0 - k_\epsilon\|\ , \tag{4.21}$$

i.e., the operator error norm is controlled by the error norm of the characterising functions. Now we can formulate our problem as follows:

$$\text{Solve} \qquad A_0 f = g_0 \tag{4.22a}$$

$$\text{from noisy data } g_\delta \text{ with} \qquad \|g_0 - g_\delta\| \leq \delta \tag{4.22b}$$

$$\text{and noisy function } k_\epsilon \text{ with} \qquad \|k_0 - k_\epsilon\| \leq \epsilon\ . \tag{4.22c}$$

Please note that the problem with explicitly known k_0 (or the operator A_0) is often ill-posed and needs regularization for a stable inversion. Therefore we will also propose a regularizing scheme for the problem (4.22a)-(4.22c). Now let us give some examples.

Example 4.2.1. *Consider a linear integral operator A_0 defined by*

$$(A_0 f)(s) := \int_\Omega k_0(s,t) f(t) dt = B(k_0, f)$$

with $\mathcal{V} = \mathcal{H} = L_2(\Omega)$ and let k_0 be a function in $\mathcal{U} = L_2(\Omega^2)$. Then the bilinear operator B yields

$$\|B(k_0, f)\| \leq \|k_0\|_{\mathcal{U}} \|f\|_{\mathcal{V}}.$$

The considered class of operators also contains deconvolution problems, which are important in imaging, as well as blind deconvolution problems [53, 13, 46], where it is assumed that also the exact convolution kernel is unknown.

Example 4.2.2. *In medical imaging, the data of Single Photon Emission Computed Tomography (SPECT) is described by the attenuated Radon transform [67, 18, 72]:*

$$Af(s,\omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) \cdot e^{-\int_t^\infty \mu(s\omega^\perp + \tau\omega)\, d\tau} dt \ .$$

The function μ is the density distribution of the body. In general, the density distribution is also unknown. Modern scanner, however, perform a CT scan in parallel. Due to measurement errors, the reconstructed density distribution is also incorrect. Setting

$$k_\epsilon(s,t,\omega) = e^{-\int_t^\infty \mu_\epsilon(s\omega^\perp + \tau\omega)\, d\tau} ,$$

we have

$$A_\epsilon f = B(k_\epsilon, f) ,$$

and similar estimates as in (4.19) can be obtained.

4.2.2 Double regularized total least squares

Due to our assumptions on the structure of the operator A_0, the inverse problem of identifying the function f^{true} from noisy measurements g_δ and inexact operator A_ϵ can now be rewritten as the task of solving the inverse problem

$$B(k_0, f) = g_0 \tag{4.23}$$

from noisy measurements (k_ϵ, g_δ) fulfilling

$$\left\|g_0 - g_\delta\right\|_{\mathcal{H}} \leq \delta, \tag{4.24a}$$

and

$$\left\|k_0 - k_\epsilon\right\|_{\mathcal{U}} \leq \epsilon. \tag{4.24b}$$

In most applications, the “inversion” of B will be ill-posed (e.g., if B is defined via a Fredholm integral operator), and a regularization strategy is needed for a stable solution of the problem (4.23).

The structure of our problem allows to reformulate (4.22a)-(4.22c) as an unconstrained Tikhonov type problem:

$$\underset{(k,f)}{\text{minimize}}\ J^{\delta,\varepsilon}_{\alpha,\beta}\big(k,f\big) := \frac{1}{2}T^{\delta,\varepsilon}\left(k,f\right) + R_{\alpha,\beta}\left(k,f\right), \tag{4.25a}$$

where

$$T^{\delta,\varepsilon}\left(k,f\right) = \left\|B(k,f) - g_\delta\right\|^2 + \gamma\left\|k - k_\epsilon\right\|^2 \tag{4.25b}$$

and

$$R_{\alpha,\beta}\left(k,f\right) = \frac{\alpha}{2}\left\|Lf\right\|^2 + \beta\mathcal{R}(k). \tag{4.25c}$$

Here, α and β are the regularization parameters which have to be chosen properly, γ is a scaling parameter, L is a bounded linear and continuously invertible operator and $\mathcal{R} : X \subset \mathcal{U} \to [0, +\infty]$ is proper, convex and weakly lower semi-continuous functional . We wish to note that most of the available papers assume that L is a densely defined, unbounded self-adjoint and strictly positive operator, see, e.g. [60, 58]. For our analysis, however, boundedness is needed and it is an open question whether the analysis could be extended to cover unbounded operators, too.

We call this scheme the *double regularized total least squares method* (dbl-RTLS). Please note that the method is closely related to the total least squares method, as the term $\|k - k_\epsilon\|^2$ controls the error in the operator. The functional $J^{\delta,\varepsilon}_{\alpha,\beta}$ is composed as the sum of two terms: one which measures the discrepancy of data and operator, and one which promotes stability. The functional $T^{\delta,\varepsilon}$ is a *data-fidelity* term based on the TLS technique, whereas the functional $R_{\alpha,\beta}$ acts as a *penalty* term which stabilizes the inversion with respect to the pair (k, f). As a consequence, we have two regularization parameters, which also occurs in *double regularization*, see, e.g., [85].

The domain of the functional $J^{\delta,\varepsilon}_{\alpha,\beta} : (\mathcal{U} \cap X) \times \mathcal{V} \longrightarrow \mathbb{R}$ can be extended over $\mathcal{U} \times \mathcal{V}$ by setting $\mathcal{R}(k) = +\infty$ whenever $k \in \mathcal{U} \setminus X$. Then $\mathcal{R}$ is proper, convex and weak lower semi-continuous functional in $\mathcal{U}$.

4.2.3 Regularization properties

In this section we shall analyse some analytical properties of the proposed dbl-RTLS method. In particular, we prove its well-posedness as a regularization method, i.e., the minimizers of the regularization functional $J^{\delta,\varepsilon}_{\alpha,\beta}$ exist for every $\alpha, \beta > 0$, depend continuously on both g_δ and k_ϵ, and converge to a solution of $B(k_0, f) = g_0$ as both noise level approaches zero, provided the regularization parameters α and β are chosen appropriately.

For the pair $(k, f) \in \mathcal{U} \times \mathcal{V}$ we use the canonical inner product

$$\langle (k_1, f_1), (k_2, f_2) \rangle_{\mathcal{U}\times\mathcal{V}} := \langle k_1, k_2 \rangle_{\mathcal{U}} + \langle f_1, f_2 \rangle_{\mathcal{V}} \ ,$$

i.e., convergence is defined componentwise. For the upcoming results, we need the following assumption on the operator B:

Assumption A4. Let the operator B be strongly continuous, i.e., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ then $B(k^n, f^n) \to B(\bar{k}, \bar{f})$.

Proposition 4.2.3. Let $J^{\delta,\varepsilon}_{\alpha,\beta}$ be the functional defined in (4.25). Assume that L is a bounded linear and continuously invertible operator and B fulfills Assumption A4. Then $J^{\delta,\varepsilon}_{\alpha,\beta}$ is a positive, weakly lower semi-continuous and coercive functional.

Proof. By the definition of $T^{\delta,\varepsilon}$, $\mathcal{R}$ and Assumption A4, $J^{\delta,\varepsilon}_{\alpha,\beta}$ is positive and w-lsc. As the operator L is continuously invertible, there exists a constant $c > 0$ such that

$$c\|f\| \leq \|Lf\|$$

for all $f \in \mathscr{D}(L)$. We get

$$J^{\delta,\varepsilon}_{\alpha,\beta}(k,f) \geq \gamma\|k - k_\epsilon\|^2 + \frac{\alpha c}{2}\|f\|^2 \to \infty$$

as $\|(k,f)\|^2 := \|k\|^2 + \|f\|^2 \to \infty$ and therefore $J^{\delta,\varepsilon}_{\alpha,\beta}$ is coercive. □

We point out here that the problem (4.23) may not even have a solution for any given noisy measurements (k_ϵ, g_δ) whereas the regularized problem (4.25) does, as stated below:

Theorem 4.2.4 (Existence)**.** *Let the assumptions of Proposition 4.2.3 hold. Then the functional $J^{\delta,\varepsilon}_{\alpha,\beta}(k,f)$ has a global minimizer.*

Proof. By Proposition 4.2.3, $J^{\delta,\varepsilon}_{\alpha,\beta}(k,f)$ is positive, proper and coercive, i.e., there exists $(k,f) \in \mathscr{D}(J^{\delta,\varepsilon}_{\alpha,\beta})$ such that $J^{\delta,\varepsilon}_{\alpha,\beta}(k,f) < \infty$.

Let $\nu = \inf\{\, J^{\delta,\varepsilon}_{\alpha,\beta}(k,f) \mid (k,f) \in \operatorname{dom} J^{\delta,\varepsilon}_{\alpha,\beta}\}$. Then, there exists $M > 0$ and a sequence $(k^j, f^j) \in \operatorname{dom} J^{\delta,\varepsilon}_{\alpha,\beta}$ such that $J(k^j, f^j) \to \nu$ and

$$J^{\delta,\varepsilon}_{\alpha,\beta}(k^j, f^j) \leq M \quad \forall j.$$

In particular we have

$$\frac{1}{2}\alpha\|Lf^j\|^2 \leq M \quad \text{and} \quad \frac{1}{2}\gamma\|k^j - k_\epsilon\|^2 \leq M.$$

Using

$$\|k^j\| - \|k_\epsilon\| \leq \|k^j - k_\epsilon\| \leq \left(\frac{2M}{\gamma}\right)^{1/2}$$

it follows

$$\|k^j\| \leq \left(\frac{2M}{\gamma}\right)^{1/2} + \|k_\epsilon\| \quad \text{and} \quad \|f^j\| \leq \left(\frac{2M}{\alpha c^2}\right)^{1/2},$$

i.e., the sequences (k^j) and (f^j) are bounded. Thus there exist subsequences of (k^j), (f^j) (for simplicity, again denoted by (k^j) and (f^j)) s.t.

$$k^j \rightharpoonup \bar{k} \quad \text{and} \quad f^j \rightharpoonup \bar{f},$$

and thus

$$(k^j, f^j) \rightharpoonup (\bar{k}, \bar{f}) \in (\mathcal{U} \cap X) \times \mathcal{V}.$$

By the w-lsc of the functional $J^{\delta,\varepsilon}_{\alpha,\beta}$ we obtain

$$\nu \leq J^{\delta,\varepsilon}_{\alpha,\beta}\big(\bar{k}, \bar{f}\big) \leq \liminf J^{\delta,\varepsilon}_{\alpha,\beta}\big(k^j, f^j\big) = \lim J^{\delta,\varepsilon}_{\alpha,\beta}\big(k^j, f^j\big) = \nu$$

Hence $\nu = J^{\delta,\varepsilon}_{\alpha,\beta}\big(\bar{k}, \bar{f}\big)$ is the minimum of the functional and $(\bar{k}, \bar{f})$ is a global minimizer,

$$(\bar{k}, \bar{f}) = \arg\min\{\ J^{\delta,\varepsilon}_{\alpha,\beta}\big(k, f\big)\ \mid\ (k, f) \in \mathscr{D}(J^{\delta,\varepsilon}_{\alpha,\beta})\}.$$

□

The stability property of the standard Tikhonov regularization strategy for problems with noisy right hand side is well known. We next investigate this property for the Tikhonov type regularization scheme (4.25) for perturbations on both (k_ϵ, g_δ).

Theorem 4.2.5 (Stability)**.** *Let $\alpha, \beta > 0$ be fixed the regularization parameters, L a bounded and continuously invertible operator and $(g_{\delta_j})_j$, $(k_{\epsilon_j})_j$ sequences with $g_{\delta_j} \to g_\delta$ and $k_{\epsilon_j} \to k_\epsilon$. If (k^j, f^j) denote minimizers of $J^{\delta_j,\varepsilon_j}_{\alpha,\beta}$ with data g_{δ_j} and characterising function k_{ϵ_j}, then there exists a convergent subsequence of $(k^j, f^j)_j$. The limit of every convergent subsequence is a minimizer of the functional $J^{\delta,\varepsilon}_{\alpha,\beta}$.*

Proof. By the definition of (k^j, f^j) as minimizers of $J^{\delta_j,\varepsilon_j}_{\alpha,\beta}$ we have

$$J^{\delta_j,\varepsilon_j}_{\alpha,\beta}\big(k^j, f^j\big) \leq J^{\delta_j,\varepsilon_j}_{\alpha,\beta}\big(k, f\big) \quad \forall (k, f) \in \mathscr{D}(J^{\delta,\varepsilon}_{\alpha,\beta}), \tag{4.26}$$

With $(\tilde{k}, \tilde{f}) := (k^{\delta,\epsilon}_{\alpha,\beta}, f^{\delta,\epsilon}_{\alpha,\beta})$ we get $J^{\delta_j,\varepsilon_j}_{\alpha,\beta}\big(\tilde{k}, \tilde{f}\big) \to J^{\delta,\varepsilon}_{\alpha,\beta}\big(\tilde{k}, \tilde{f}\big)$. Hence, there exists a $\tilde{c} > 0$ so that $J^{\delta_j,\varepsilon_j}_{\alpha,\beta}\big(\tilde{k}, \tilde{f}\big) \leq \tilde{c}$ for j sufficiently large. In particular, we observe with (4.26) that $\big(\|k^j - k_{\epsilon_j}\|\big)_j$ as well as $\big(\|Lf^j\|\big)_j$ are uniformly bounded.

Analogous to the proof of Theorem 4.2.4 we conclude that the sequence $(k^j, f^j)_j$ is uniformly bounded. Hence there exists a subsequence (for simplicity also denoted by$(k^j, f^j)_j$) such that

$$k^j \rightharpoonup \bar{k} \quad \text{and} \quad f^j \rightharpoonup \bar{f}.$$

By the weak lower semicontinuity (w-lsc) of the norm and continuity of B we have

$$\left\|B(\bar{k}, \bar{f}) - g_\delta\right\| \leq \liminf_j \left\|B(k^j, f^j) - g_{\delta_j}\right\|$$

and

$$\left\|\bar{k} - k_\epsilon\right\| \leq \liminf_j \left\|k^j - k_{\epsilon_j}\right\|.$$

Moreover, (4.26) implies

$$\begin{aligned} J_{\alpha,\beta}^{\delta,\varepsilon}\big(\bar{k}, \bar{f}\big) &\leq \liminf_j J_{\alpha,\beta}^{\delta_j,\varepsilon_j}\big(k^j, f^j\big) \\ &\leq \limsup_j J_{\alpha,\beta}^{\delta_j,\varepsilon_j}\big(k, f\big) \\ &= \lim_j J_{\alpha,\beta}^{\delta_j,\varepsilon_j}\big(k, f\big) \\ &= J_{\alpha,\beta}^{\delta,\varepsilon}\big(k, f\big) \end{aligned}$$

for all $(k, f) \in \mathscr{D}(J_{\alpha,\beta}^{\delta,\varepsilon})$. In particular, $J_{\alpha,\beta}^{\delta,\varepsilon}\big(\bar{k}, \bar{f}\big) \leq J_{\alpha,\beta}^{\delta,\varepsilon}\big(\tilde{k}, \tilde{f}\big)$. Since $\big(\tilde{k}, \tilde{f}\big)$ is by definition a minimizer of $J_{\alpha,\beta}^{\delta,\varepsilon}$, we conclude $J_{\alpha,\beta}^{\delta,\varepsilon}\big(\bar{k}, \bar{f}\big) = J_{\alpha,\beta}^{\delta,\varepsilon}\big(\tilde{k}, \tilde{f}\big)$ and thus

$$\lim_{j\to\infty} J_{\alpha,\beta}^{\delta_j,\varepsilon_j}\big(k^j, f^j\big) = J_{\alpha,\beta}^{\delta,\varepsilon}\big(\bar{k}, \bar{f}\big). \tag{4.27}$$

It remains to show

$$k^j \to \bar{k} \quad \text{and} \quad f^j \to \bar{f}.$$

As the sequences are weakly convergent, convergence of the sequences holds if

$$\left\|k^j\right\| \to \left\|\bar{k}\right\| \quad \text{and} \quad \left\|f^j\right\| \to \left\|\bar{f}\right\|.$$

The norms on $\mathcal{U}$ and $\mathcal{V}$ are w-lsc, thus it is sufficient to show

$$\|\bar{k}\| \geq \limsup \|k^j\| \quad \text{and} \quad \|\bar{f}\| \geq \limsup \|f^j\|.$$

The operator L is bounded and continuously invertible, therefore $f^j \to \bar{f}$ if and only if $Lf^j \to L\bar{f}$. Therefore, we accomplish the prove for the sequence $(Lf^j)_j$. Now suppose there exists τ_1 as

$$\tau_1 := \limsup \|Lf^j\| > \|L\bar{f}\|$$

and there exists a subsequence $(f^n)_n$ of $(f^j)_j$ such that $Lf^n \rightharpoonup L\bar{f}$ and $\|Lf^n\| \to \tau_1$.
From the first part of this proof (4.27), it holds

$$\lim_{j\to\infty} J_{\alpha,\beta}^{\delta_j,\varepsilon_j}\left(k^j, f^j\right) = J_{\alpha,\beta}^{\delta,\varepsilon}\left(\bar{k}, \bar{f}\right).$$

Using (4.25) we observe

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{1}{2}\left\|B(k^n, f^n) - g_{\delta_n}\right\|^2 + \frac{\gamma}{2}\left\|k^n - k_{\epsilon_n}\right\|^2 + \beta\mathcal{R}(k^n) \right) \\
=&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \frac{\gamma}{2}\left\|\bar{k} - k_\epsilon\right\|^2 + \beta\mathcal{R}(\bar{k}) \qquad (4.28)\\
&+ \frac{\alpha}{2}\left(\left\|L\bar{f}\right\|^2 - \lim_{n\to\infty}\left\|Lf^n\right\|^2\right) \\
=&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \frac{\gamma}{2}\left\|\bar{k} - k_\epsilon\right\|^2 + \beta\mathcal{R}(\bar{k}) + \frac{\alpha}{2}\left(\left\|L\bar{f}\right\|^2 - {\tau_1}^2\right) \\
<&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \frac{\gamma}{2}\left\|\bar{k} - k_\epsilon\right\|^2 + \beta\mathcal{R}(\bar{k}) \ ,
\end{aligned}$$

which is a contradiction to the w-lsc property of the involved norms and the functional $\mathcal{R}$. Thus $Lf^j \to L\bar{f}$ and

$$f^j \to \bar{f}.$$

The same idea can be used in order to prove convergence of the characterising functions. Suppose there exists τ_2 s.t.

$$\tau_2 := \limsup \|k^j - k_\epsilon\| > \|\bar{k} - k_\epsilon\|$$

and there exists a subsequence $(k^n)_n$ of $(k^j)_j$ such that $(k^n - k_\epsilon) \rightharpoonup (\bar{k} - k_\epsilon)$ and $\|k^n - k_\epsilon\| \to \tau_2$.

By the triangle inequality we get

$$\left\|k^n - k_\epsilon\right\| - \left\|k_{\epsilon_n} - k_\epsilon\right\| \leq \left\|k^n - k_{\epsilon_n}\right\| \leq \left\|k^n - k_\epsilon\right\| + \left\|k_{\epsilon_n} - k_\epsilon\right\|,$$

and thus

$$\lim_{n\to\infty}\left\|k^n - k_{\epsilon_n}\right\| = \lim_{n\to\infty}\left\|k^n - k_\epsilon\right\|.$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{2}\left\|B(k^n, f^n) - g_{\delta_n}\right\|^2 + \beta\mathcal{R}(k^n)\right)\\
=&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \frac{\gamma}{2}\left(\left\|\bar{k} - k_\epsilon\right\|^2 - \lim_{n\to\infty}\left\|k^n - k_{\epsilon_n}\right\|^2\right) + \beta\mathcal{R}(\bar{k})\\
=&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \frac{\gamma}{2}\left(\left\|\bar{k} - k_\epsilon\right\|^2 - {\tau_2}^2\right) + \beta\mathcal{R}(\bar{k})\\
<&\frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_\delta\right\|^2 + \beta\mathcal{R}(\bar{k}) \ ,
\end{aligned}$$

which is again a contradiction to the w-lsc of the involved norms and functionals. □

In the following, we investigate the regularization property of our approach, i.e., we show, under an appropriate parameter choice rule, that the minimizers $(k^{\delta,\epsilon}_{\alpha,\beta}, f^{\delta,\epsilon}_{\alpha,\beta})$ of the functional (4.25) converge to an exact solution as the noise level (δ, ϵ) goes to zero.

Let us first clarify our notion of a solution. In principle, the equation

$$B(k, f) = g$$

might have different pairs (k, f) as solution. However, as $k_\epsilon \to k_0$ as $\epsilon \to 0$, we get k_0 for free in the limit, that is, we are interested in reconstructing solutions of the equation

$$B(k_0, f) = g.$$

In particular, we want to reconstruct a solution with minimal value of $\|Lf\|$, and therefore define:

Definition 4.2.6. *We call $f^\dagger$ a minimum-norm solution if*

$$f^\dagger = \arg\min_f \left\{\|Lf\| \ \mid \ B(k_0, f) = g_0\right\}.$$

The definition above is the standard *minimum-norm solution* for the classical Tikhonov regularization (see for instance [21]).

Furthermore, we have to introduce a regularization parameter choice which depends on both noise level, defined through (4.29) in the upcoming theorem.

Theorem 4.2.7 (convergence)**.** *Let the sequences of data g_{δ_j} and k_{ϵ_j} with $\|g_{\delta_j} - g_0\| \le \delta_j$ and $\|k_{\epsilon_j} - k_0\| \le \epsilon_j$ be given with $\epsilon_j \to 0$ and $\delta_j \to 0$. Assume that the regularization parameters $\alpha_j = \alpha(\epsilon_j, \delta_j)$ and $\beta_j = \beta(\epsilon_j, \delta_j)$ fulfill $\alpha_j \to 0$, $\beta_j \to 0$, as well as*

$$\lim_{j\to\infty} \frac{\delta_j^2 + \gamma\epsilon_j^2}{\alpha_j} = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = \eta \tag{4.29}$$

for some $0 < \eta < \infty$.

Let the sequence

$$(k^j, f^j)_j := \big(k^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}, f^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}\big)_j$$

be the minimizer of (4.25), obtained from the noisy data g_{δ_j} and k_{ϵ_j}, regularization parameters α_j and β_j and scaling parameter γ.

Then there exists a convergent subsequence of $(k^j, f^j)_j$ with $k^j \to k_0$ and the limit of every convergent subsequence of $(f^j)_j$ is a minimum-norm solution of (4.23).

Proof. The minimizing property of (k^j, f^j) guarantees

$$J^{\delta_j,\varepsilon_j}_{\alpha_j,\beta_j}(k^j, f^j) \le J^{\delta_j,\varepsilon_j}_{\alpha_j,\beta_j}(k, f), \quad \forall (k, f) \in \mathscr{D}(J^{\delta,\varepsilon}_{\alpha,\beta}).$$

In particular,

$$\begin{aligned} 0 \le J^{\delta_j,\varepsilon_j}_{\alpha_j,\beta_j}(k^j, f^j) \le& J^{\delta_j,\varepsilon_j}_{\alpha_j,\beta_j}(k_0, f^\dagger) \\ \le& \frac{\delta_j^2 + \gamma\epsilon_j^2}{2} + \frac{\alpha_j}{2}\|Lf^\dagger\|^2 + \beta_j \mathcal{R}(k_0), \end{aligned} \tag{4.30}$$

where $f^\dagger$ denotes a minimum-norm solution of the main equation $B(k_0, f) = g_0$, see Definition 4.2.6.

Combining this estimate with the assumptions on the regularization parameters, we conclude that the sequences

$$\|B(k^j, f^j) - g_{\delta_j}\|^2, \|k^j - k_{\epsilon_j}\|^2, \|Lf^j\|^2, \mathcal{R}(k^j)$$

are uniformly bounded and by the invertibility of L, the sequence $(k^j, f^j)_j$ is uniformly bounded.

Therefore there exists a weakly convergent subsequence of $(k^j, f^j)_j$ denoted by $(k^m, f^m)_m := (k^{j_m}, f^{j_m})_{j_m}$ with

$$(k^m, f^m) \rightharpoonup (\bar{k}, \bar{f}) .$$

In the following we will prove that for the weak limit $(\bar{k}, \bar{f})$ holds $\bar{k} = k_0$ and $\bar{f}$ is a minimum-norm solution.

By the weak lower semi-continuity of the norm we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left\|B(\bar{k}, \bar{f}) - g_0\right\|^2 + \frac{\gamma}{2}\left\|\bar{k} - k_0\right\|^2 \\ &\leq \liminf_{m\to\infty} \left\{ \frac{1}{2}\left\|B(k^m, f^m) - g_{\delta_m}\right\|^2 + \frac{\gamma}{2}\left\|k^m - k_{\epsilon_m}\right\|^2 \right\} \\ &\overset{(4.30)}{\leq} \liminf_{m\to\infty} \left\{ \frac{\delta_m^2 + \gamma\epsilon_m^2}{2} + \frac{\alpha_m}{2}\left\|Lf^\dagger\right\|^2 + \beta_m \mathcal{R}(k_0) \right\} \\ &= 0, \end{aligned}$$

where the last equality follows from the parameter choice rule.

In particular, we have

$$\bar{k} = k_0 \quad \text{and} \quad B(\bar{k}, \bar{f}) = g_0 .$$

From (4.30) follows

$$\frac{1}{2}\left\|Lf^m\right\|^2 + \frac{\beta_m}{\alpha_m}\mathcal{R}(k^m) \leq \frac{\delta_m^2 + \gamma\epsilon_m^2}{2\alpha_m} + \frac{1}{2}\left\|Lf^\dagger\right\|^2 + \frac{\beta_m}{\alpha_m}\mathcal{R}(k_0) \ .$$

Again, weak lower semi-continuity of the norm and the functional $\mathcal{R}$ result in

$$\begin{aligned} \frac{1}{2}\left\|L\bar{f}\right\|^2 + \eta\mathcal{R}(\bar{k}) &\leq \liminf_{m\to\infty} \left\{ \frac{1}{2}\left\|Lf^m\right\|^2 + \eta\mathcal{R}(k^m) \right\} \\ &= \liminf_{m\to\infty} \left\{ \frac{1}{2}\left\|Lf^m\right\|^2 + \frac{\beta_m}{\alpha_m}\mathcal{R}(k^m) \right\} \\ &\leq \liminf_{m\to\infty} \left\{ \frac{\delta_m^2 + \gamma\epsilon_m^2}{2\alpha_m} + \frac{1}{2}\left\|Lf^\dagger\right\|^2 + \frac{\beta_m}{\alpha_m}\mathcal{R}(k_0) \right\} \\ &\overset{(4.29)}{=} \frac{1}{2}\left\|Lf^\dagger\right\|^2 + \eta\mathcal{R}(k_0) \ . \end{aligned}$$

As $\bar{k} = k_0$ we conclude that $\bar{f}$ is a minimum-norm solution and

$$\begin{aligned}\frac{1}{2}\left\|L\bar{f}\right\|^2 + \eta\mathcal{R}(\bar{k}) &= \lim_{m\to\infty}\left\{\frac{1}{2}\left\|Lf^m\right\|^2 + \frac{\beta_m}{\alpha_m}\mathcal{R}(k^m)\right\} \quad (4.31)\\ &= \frac{1}{2}\left\|Lf^\dagger\right\|^2 + \eta\mathcal{R}(k_0).\end{aligned}$$

So far we showed the existence of a subsequence $(k^m, f^m)_m$ which converges weakly to $(k_0, \bar{f})$, where $\bar{f}$ is a minimizing solution. It remains to show that the sequence also converges in the strong topology of $\mathcal{U} \times \mathcal{V}$.

In order to show $f^m \to \bar{f}$ in $\mathcal{V}$, we prove $Lf^m \to L\bar{f}$. Since is $Lf^m \rightharpoonup L\bar{f}$ it is sufficient to show

$$\left\|Lf^m\right\| \to \left\|L\bar{f}\right\|,$$

or, as the norm is w.-l.s.c.,

$$\limsup_{m\to\infty}\left\|Lf^m\right\| \leq \left\|L\bar{f}\right\|.$$

Assume that the above inequality does not hold. Then there exists a constant τ_1 such that

$$\tau_1 := \limsup_{m\to\infty}\left\|Lf^m\right\|^2 > \left\|L\bar{f}\right\|^2$$

and there exists a subsequence of $(Lf^m)_m$ denoted by the sequence $(Lf^n)_n := (Lf^{m_n})_{m_n}$ such that

$$Lf^n \rightharpoonup L\bar{f} \quad \text{and} \quad \left\|Lf^n\right\|^2 \to \tau_1.$$

From (4.31) and the hypothesis stated above

$$\begin{aligned}\limsup_{n\to\infty}\frac{\beta_n}{\alpha_n}\mathcal{R}(k^n) &= \eta\mathcal{R}(k_0) + \frac{1}{2}\left(\left\|L\bar{f}\right\|^2 - \limsup_{n\to\infty}\left\|Lf^n\right\|^2\right)\\ &< \eta\mathcal{R}(k_0),\end{aligned}$$

which is a contradiction to the w.-l.s.c. of $\mathcal{R}$. Thus

$$\limsup_{m\to\infty}\left\|Lf^m\right\| \leq \left\|L\bar{f}\right\|,$$

i.e., $f^m \to \bar{f}$ in $\mathcal{V}$.

The convergence of the sequence $(k^m)_m$ in the topology of $\mathcal{U}$ follows straightforward by

$$\begin{aligned}\|k^m - k_0\| &\leq \|k^m - k_{\epsilon_m}\| + \|k_{\epsilon_m} - k_0\| \\ &\leq \|k^m - k_{\epsilon_m}\| + \epsilon_m \overset{(4.30)}{\longrightarrow} 0 \text{ as } m \to \infty.\end{aligned}$$

Moreover, if $f^\dagger$ is unique, the assertion about the convergence of the whole sequence $(k^j, f^j)_j$ follows from the fact that then every subsequence of the sequence converges towards the same limit $(k_0, f^\dagger)$. □

Remark 4.2.8. *Note that the easiest parameter choice rule fulfilling condition* (4.29) *is given by*

$$\beta = \eta\alpha, \qquad \eta > 0.$$

For this specific choice, we only have one regularization parameter left, and the problem (4.25) *reduces to*

$$\underset{(k,f)}{\text{minimize}} \;\; J_\alpha(k,f) := \frac{1}{2}T^{\delta,\varepsilon}(k,f) + \alpha\Phi(k,f)\,, \tag{4.32}$$

where $T^{\delta,\varepsilon}$ is defined in (4.25b) *and*

$$\Phi(k,f) := \frac{1}{2}\|Lf\|^2 + \eta\mathcal{R}(k). \tag{4.33}$$

It is well known that, under the general assumptions, the rate of convergence of $(k^j, f^j)_j \to (k_0, f^\dagger)$ for $(\delta_j, \epsilon_j) \to 0$ can be in general arbitrarily slow. For linear and nonlinear inverse problems convergence rates were obtained if *source conditions* are satisfied (see [22, 21, 12, 74] and Chapter 2).

For our analysis, we will use the following source condition:

$$\mathscr{R}(B'(k_0, f^\dagger)^*) \cap \partial\Phi(k_0, f^\dagger) \neq \emptyset,$$

where $\partial\Phi$ denotes the subdifferential of the functional Φ defined in (4.33). This condition says there exists a subgradient $(\xi_{k_0}, \xi_{f^\dagger})$ of Φ s.t.

$$(\xi_{k_0}, \xi_{f^\dagger}) = B'(k_0, f^\dagger)^*\omega, \quad \omega \in \mathcal{H}.$$

Convergence rates are often given with respect to the *Bregman distance* generated by the regularization functional Φ. In our setting, the distance is defined by

$$D_\Phi^{(\xi_{\bar u},\xi_{\bar v})}\big((u,v),(\bar u,\bar v)\big) = \Phi(u,v) - \Phi(\bar u,\bar v) - \langle(\xi_{\bar u},\xi_{\bar v}),(u,v)-(\bar u,\bar v)\rangle \quad (4.34)$$

for $(\xi_{\bar u},\xi_{\bar v}) \in \partial\Phi\big(\bar u,\bar v\big)$.

Lemma 4.2.9. *Let Φ be the functional defined in (4.33) with $L = I$. Then the Bregman distance is given by*

$$D_\Phi^{(\xi_{\bar u},\xi_{\bar v})}\big((u,v),(\bar u,\bar v)\big) = \frac{1}{2}\big\|v-\bar v\big\|^2 + \eta D_{\mathcal R}^{\zeta}\big(u,\bar u\big), \quad (4.35)$$

with $\zeta \in \partial\mathcal R\big(\bar u\big)$.

Proof. By definition of Bregman distance we have

$$\begin{aligned}
D_\Phi^{(\xi_{\bar u},\xi_{\bar v})}\big((u,v),(\bar u,\bar v)\big) &= \Big(\tfrac12\|v\|^2 + \eta\mathcal R(u)\Big) - \Big(\tfrac12\|\bar v\|^2 + \eta\mathcal R(\bar u)\Big) \\
&\quad - \langle(\xi_{\bar u},\xi_{\bar v}),(u-\bar u, v-\bar v)\rangle \\
&= \tfrac12\|v\|^2 - \tfrac12\|\bar v\|^2 - \langle\xi_{\bar v}, v-\bar v\rangle \\
&\quad + \eta\mathcal R(u) - \eta\mathcal R(\bar u) - \langle\xi_{\bar u}, u-\bar u\rangle \\
&= \tfrac12\|v-\bar v\|^2 + \eta D_{\mathcal R}^{\zeta}\big(u,\bar u\big)
\end{aligned}$$

with $\zeta = \frac{1}{\eta}\xi_{\bar u}$. Note that the functional Φ is composed as a sum of a differentiable and a convex functional. Therefore, the subgradient of the first functional is an unitary set and it holds (see, e.g.,[15])

$$\begin{aligned}
\partial\Phi\big(\bar u,\bar v\big) &= \partial\Big(\|\bar v\|^2 + \eta\mathcal R(\bar u)\Big) \\
&= \Big\{(\xi_{\bar u},\xi_{\bar v}) \in \mathcal U^* \times \mathcal V^* \;\big|\; \xi_{\bar v} \in \partial\|\bar v\|^2 \text{ and } \xi_{\bar u} \in \eta\partial\mathcal R\big(\bar u\big)\Big\}
\end{aligned}$$

□

For the convergence rate analysis, we need the following result:

Lemma 4.2.10. *Let $B : \mathcal{U} \times \mathcal{V} \to \mathcal{H}$ be a bilinear operator with $\|B(k,f)\| \le C\,\|k\|\,\|f\|$. Then its Fréchet derivative at point (k,f) is given by*

$$B'(k,f)(u,v) = B(u,f) + B(k,v),$$

$(u,v) \in \mathcal{U} \times \mathcal{V}$. Moreover, the remainder of the Taylor expansion can be estimated by

$$\|B(k+u, f+v) - B(k,f) - B'(k,f)(u,v)\| \le \frac{C}{2}\big\|(u,v)\big\|^2 . \quad (4.36)$$

Proof. See Exercise 4.10. □

The following theorem gives an error estimate within an infinite dimensional setting, similar to the results found in [60, 80]. Please note that we have not only an error estimate for the solution f, but also for the characterising function k, i.e., we are able to derive convergence rate for the operator via (4.21).

Theorem 4.2.11 (Convergence rates)**.** *Let $g_\delta \in \mathcal{H}$ with $\big\|g_0 - g_\delta\big\| \le \delta$, $k_\epsilon \in \mathcal{U}$ with $\big\|k_0 - k_\epsilon\big\| \le \epsilon$ and let $f^\dagger$ be a minimum norm solution. For the regularization parameter $0 < \alpha < \infty$, let (k^α, f^α) denote the minimizer of (4.32) with $L = I$. Moreover, assume that the following conditions hold:*

(i) There exists $\omega \in \mathcal{H}$ satisfying

$$(\xi_{k_0}, \xi_{f^\dagger}) = B'(k_0, f^\dagger)^* \omega,$$

with $(\xi_{k_0}, \xi_{f^\dagger}) \in \partial\Phi\big(k_0, f^\dagger\big)$.

(ii) $C\big\|\omega\big\|_{\mathcal{H}} < \min\big\{1, \frac{\gamma}{2\alpha}\big\}$, where C is the constant in (4.36).

Then, for the parameter choice $\alpha \sim (\delta + \epsilon)$ holds

$$\big\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\big\|_{\mathcal{H}} = \mathcal{O}\left(\delta + \epsilon\right)$$

and

$$D_\Phi^\xi\big((k^\alpha, f^\alpha), (k_0, f^\dagger)\big) = \mathcal{O}\left(\delta + \epsilon\right).$$

Proof. Since (k^α, f^α) is a minimizer of J_α, defined in (4.32), it follows

$$J_\alpha\big(k^\alpha, f^\alpha\big) \le J_\alpha\big(k, f\big) \quad \forall (k,f) \in \mathcal{U} \times \mathcal{V}.$$

In particular,

$$\begin{aligned} J_\alpha\big(k^\alpha, f^\alpha\big) &\le J_\alpha\big(k_0, f^\dagger\big) \\ &\le \frac{\delta^2}{2} + \frac{\gamma\epsilon^2}{2} + \alpha\Phi(k_0, f^\dagger). \end{aligned} \tag{4.37}$$

Using the definition of the Bregman distance (at the subgradient $(\xi_{k_0}, \xi_{f^\dagger}) \in \partial\Phi\big(k_0, f^\dagger\big)$), we rewrite (4.37) as

$$\begin{aligned} \frac{1}{2}\big\|B(k^\alpha, f^\alpha) - g_\delta\big\|^2 &+ \frac{\gamma}{2}\big\|k^\alpha - k_\epsilon\big\|^2 \\ &\le \frac{\delta^2 + \gamma\epsilon^2}{2} + \alpha\left(\Phi(k_0, f^\dagger) - \Phi(k^\alpha, f^\alpha)\right) \\ &= \frac{\delta^2 + \gamma\epsilon^2}{2} - \alpha\Big[D_\Phi^{\xi^\dagger}\left((k^\alpha, f^\alpha), (k_0, f^\dagger)\right) \\ &\quad + \langle(\xi_{k_0}, \xi_{f^\dagger}), (k^\alpha, f^\alpha) - (k_0, f^\dagger)\rangle\Big]. \end{aligned} \tag{4.38}$$

Using

$$\begin{aligned} \frac{1}{2}\big\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\big\|^2 &\le \big\|B(k^\alpha, f^\alpha) - g_\delta\big\|^2 + \big\|g_\delta - g_0\big\|^2 \\ &\le \big\|B(k^\alpha, f^\alpha) - g_\delta\big\|^2 + \delta^2 \end{aligned}$$

and

$$\begin{aligned} \frac{\gamma}{2}\big\|k^\alpha - k_0\big\|^2 &\le \gamma\big\|k^\alpha - k_\epsilon\big\|^2 + \gamma\big\|k_\epsilon - k_0\big\|^2 \\ &\le \gamma\big\|k^\alpha - k_\epsilon\big\|^2 + \gamma\epsilon^2, \end{aligned}$$

we get

$$\begin{aligned} &\frac{1}{4}\big\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\big\|^2 + \frac{\gamma}{4}\big\|k^\alpha - k_0\big\|^2 \\ &\quad\le \frac{1}{2}\big\|B(k^\alpha, f^\alpha) - g_\delta\big\|^2 + \frac{\gamma}{2}\big\|k^\alpha - k_\epsilon\big\|^2 + \left(\frac{\delta^2 + \gamma\epsilon^2}{2}\right) \\ &\quad\overset{(4.38)}{\le} \left(\delta^2 + \gamma\epsilon^2\right) - \alpha\Big[D_\Phi^{\xi^\dagger}\left((k^\alpha, f^\alpha), (k_0, f^\dagger)\right) \\ &\qquad + \langle(\xi_{k_0}, \xi_{f^\dagger}), (k^\alpha, f^\alpha) - (k_0, f^\dagger)\rangle\Big]. \end{aligned}$$

Denoting

$$r := B(k^\alpha, f^\alpha) - B(k_0, f^\dagger) - B'(k_0, f^\dagger)((k^\alpha, f^\alpha) - (k_0, f^\dagger))$$

and using the source condition *(i)*, the last term in the above inequality can be estimated as

$$\begin{aligned}
&-\langle(\xi_{k_0}, \xi_{f^\dagger}), (k^\alpha, f^\alpha) - (k_0, f^\dagger)\rangle \\
&\quad = \quad -\langle B'(k_0, f^\dagger)^*\omega, (k^\alpha, f^\alpha) - (k_0, f^\dagger)\rangle \\
&\quad = \quad \langle \omega, -B'(k_0, f^\dagger)\big((k^\alpha, f^\alpha) - (k_0, f^\dagger)\big)\rangle \\
&\quad = \quad \langle \omega, B(k_0, f^\dagger) - B(k^\alpha, f^\alpha) + r\rangle \\
&\quad \le \quad \|\omega\| \left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| + \|\omega\| \|r\| \\
&\quad \overset{(4.36)}{\le} \quad \|\omega\| \left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| \\
&\qquad\qquad + \frac{C}{2} \|\omega\| \left\|(k^\alpha, f^\alpha) - (k_0, f^\dagger)\right\|^2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\frac{1}{4}\left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\|^2 + \frac{\gamma}{4}\left\|k^\alpha - k_0\right\|^2 \\
&\qquad + \alpha D_\Phi^{\xi^\dagger}\big((k^\alpha, f^\alpha), (k_0, f^\dagger)\big) \qquad (4.39)\\
&\quad \le \big(\delta^2 + \gamma\epsilon^2\big) + \alpha \|\omega\| \left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| \\
&\qquad + \alpha\frac{C}{2} \|\omega\| \left\|(k^\alpha, f^\alpha) - (k_0, f^\dagger)\right\|^2.
\end{aligned}$$

Using (4.35) and the definition of the norm on $\mathcal{U} \times \mathcal{V}$, (4.39) can be rewritten as

$$\begin{aligned}
&\frac{1}{4}\left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\|^2 + \frac{\alpha}{2}\left(1 - C\|\omega\|\right)\left\|f^\alpha - f^\dagger\right\|^2 \\
&\qquad + \alpha\ \eta D_{\mathcal{R}}^{\zeta}\big(k^\alpha, k_0\big) \\
&\quad \le \big(\delta^2 + \gamma\epsilon^2\big) + \alpha \|\omega\| \left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| \\
&\qquad + \frac{1}{2}\left(\alpha C\|\omega\| - \frac{\gamma}{2}\right)\left\|k^\alpha - k_0\right\|^2 \\
&\quad \le \big(\delta^2 + \gamma\epsilon^2\big) + \alpha \|\omega\| \left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\|, \qquad (4.40)
\end{aligned}$$

as $\left(C\|\omega\| - \frac{\gamma}{2\alpha}\right) \le 0$ according to *(ii)*. As $(1 - C\|\omega\|)$ as well as the Bregman distance are nonnegative, we derive

$$\frac{1}{4}\left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\|^2 - \alpha\|\omega\|\left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| \\ - \left(\delta^2 + \gamma\epsilon^2\right) \le 0,$$

which only holds for

$$\left\|B(k^\alpha, f^\alpha) - B(k_0, f^\dagger)\right\| \le 2\alpha\|\omega\| + 2\sqrt{\alpha^2\|\omega\|^2 + (\delta^2 + \gamma\epsilon^2)}.$$

Using the above inequality to estimate the right-hand side of (4.40) yields

$$\left\|f^\alpha - f^\dagger\right\|^2 \le \left(\frac{2}{1 - C\|\omega\|}\right) \cdot \\ \left(\frac{\delta^2 + \gamma\epsilon^2}{\alpha} + 2\alpha\|\omega\|^2 + 2\|\omega\|\sqrt{\alpha^2\|\omega\|^2 + (\delta^2 + \gamma\epsilon^2)}\right)$$

and

$$D_{\mathcal{R}}^{\zeta}\left(k^\alpha, k_0\right) \le \frac{\delta^2 + \gamma\epsilon^2}{\eta\alpha} + \frac{2\|\omega\|}{\eta}\left(\alpha\|\omega\| + \sqrt{\alpha^2\|\omega\|^2 + (\delta^2 + \gamma\epsilon^2)}\right),$$

and for the parameter choice $\alpha \sim (\delta + \epsilon)$ follows the convergence rate $\mathcal{O}(\delta + \epsilon)$. □

Remark 4.2.12. *The assumptions of Theorem 4.2.11 include the condition*

$$C\|\omega\|_{\mathcal{H}} < \min\left\{1, \frac{\gamma}{2\alpha}\right\} .$$

Note that $\frac{\gamma}{(2\alpha)} < 1$ *for* α *small enough (i.e., for small noise level* δ *and* ϵ*), and thus (ii) reduces to the standard smallness assumption common for convergence rates for nonlinear ill-posed problems, see [21].*

4.2.4 Numerical example

In order to illustrate our analytical results we present first reconstructions from a convolution operator. That is, the kernel function is defined by $k_0(s,t) := k_0(s-t)$ over $\Omega = [0,1]$, see also Example 4.2.1 in Section 4.2.1, and we want to solve the integral equation

$$\int_\Omega k_0(s-t)f(t)dt = g_0(s)$$

from given measurements k_ϵ and g_δ satisfying (4.24). For our test, we defined k_0 and f as

$$k_0 = \begin{cases} 1 & x \in [0.1, 0.4] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f = \begin{cases} 1-5|t-0.3| & t \in [0.1, 0.5] \\ 0 & \text{otherwise} \end{cases},$$

respectively, the characteristic and the hat function. An example of

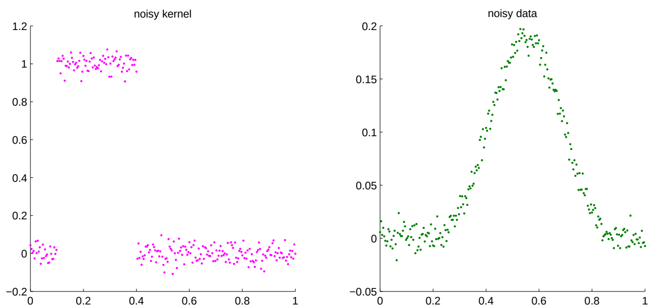

Figure 4.5: Simulated measurements for k_0 (left) and g_0 (right), both with 10% relative error.

noisy measurements k_ϵ and g_δ is displayed in Figure 4.5.

The functions k and f were expanded in a wavelet basis, as for example,

$$k = \sum_{l \in \mathbb{Z}} \langle k, \phi_{0,l} \rangle \phi_{0,l} + \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \langle k, \psi_{j,l} \rangle \psi_{j,l} \ ,$$

where $\{\phi_\lambda\}_\lambda$ and $\{\psi_\lambda\}_\lambda$ are the pair of scaling and wavelet function associated to Haar wavelet basis. The convolution operator was implemented in terms of the wavelet basis as well. For our numerical tests, we used the Haar wavelet. The integration interval $\Omega = [0,1]$ was discretized into $N = 2^8$ points, the maximum level considered by the Haar wavelet is $J = 6$. The functional $\mathcal{R}$ was defined as

$$\mathcal{R}(k) := \|k\|_{\ell_1} = \sum_{\lambda \in \Lambda} |\langle k, \psi_\lambda \rangle|,$$

where $\Lambda = \big\{\{l\} \cup (j,l) \mid j \in \mathbb{N}_0, l \leq 2^j - 1\big\}$.

In order to find the optimal set of coefficients minimizing (4.25) we used Matlab's internal function `fminsearch`.

Figure 4.6 displays the numerical solutions for three different (relative) error levels: 10%, 5% and 1%. The scaling parameter was set to $\gamma = 1$ and the regularization parameters are chosen according to the noise level, i.e., $\alpha = 0.01(\delta + \varepsilon)$ and $\beta = 0.2(\delta + \varepsilon)$, ($\eta = 20$) was chosen. Our numerical results confirm our analysis. In particular it is observed that the reconstruction quality increases with decreasing noise level, see also Table 4.1.

	$\left\|k^{\text{rec}} - k_0\right\|_1$	$\left\|f^{\text{rec}} - f^{\text{true}}\right\|_1$	$\left\|k^{\text{rec}} - k_0\right\|_2$	$\left\|f^{\text{rec}} - f^{\text{true}}\right\|_2$
10%	6.7543e-02	1.8733e-01	8.1216e-03	1.7436e-02
5%	4.0605e-02	1.7173e-01	6.9089e-03	1.5719e-02
1%	2.0139e-02	1.1345e-01	6.5219e-03	8.0168e-03

Table 4.1: Relative error measured by the L_1- and L_2-norm.

Please note that the optimisation with the `fminsearch` routine is by no means efficient. In the upcoming chapter we shall propose a fast iterative optimisation routine for the minimization of (4.25).

4.3 Bibliographical comments

Heretofore we listed only few approaches to treat ill-posed problems with error in both operator and data, namely, the first regularized version of TLS (R-TLS) method proposed in 1999 and the D-RTLS, which was the first approach given with rates of convergence.

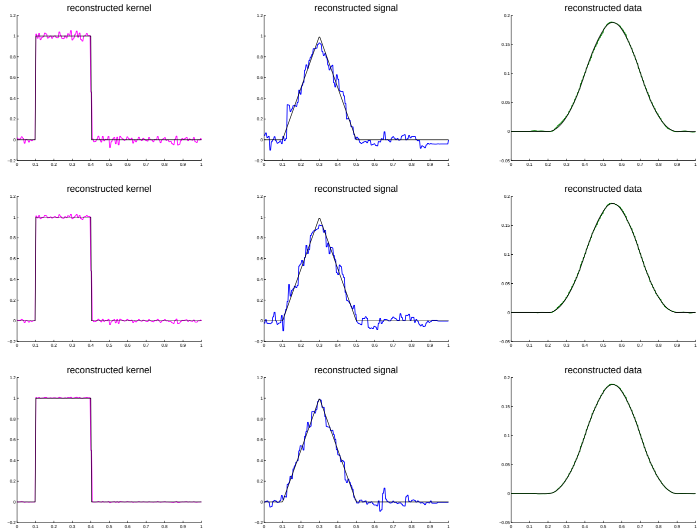

Figure 4.6: Reconstruction of the characterising function k_0, the signal f (solution) and the data g_0. From top to bottom: reconstruction with 10%, 5% and 1% relative error (both for g_δ and k_ϵ). The reconstructions are coloured from [7].

One efficient algorithm for solving the R-TLS problem was developed in [73], based on the minimization of the *Rayleigh quotient*

$$\phi(x) := \frac{\left\| A_\epsilon x - y_\delta \right\|^2}{1 + \left\| x \right\|^2}.$$

To be more precise, it solves the equivalent problem (4.15), also known as *weighted LS* or *normalised residual* problem, instead of minimizing the constrained functional (4.12). Usually one refers to this formulation as *regularized Rayleigh quotient form for total least squares* (RQ-RTLS).

Adding a quadratic constrained to the TLS minimization problem can be solved via a *quadratic eigenvalue problem* [24]. It results in an iterative procedure for solving the R-TLS proposed in [79], named as *regularized total least squares solved by quadratic eigenvalue problem*

(RTLSQEP). The authors of [54] also analysed the same problem, focusing in the efficiency of solving the R-TLS in mainly two different approaches: via a sequence of quadratic eigenvalue problems and via a sequence of linear eigenvalue problems.

A typical algorithm for solving the D-RTLS is based on *model function*, see e.g., [61, 59]. The D-RTLS solution x^D has a close form given in the Theorem 4.1.8, but it depends on two variables, i.e., $x^D = x(\alpha, \delta)$. The parameters α and β are found to be the solution of the (highly) nonlinear system (4.17). The main idea is to approximate the unconstrained minimization problem by a simple function which relates the derivatives of this functional with respect to each parameter. The "simple" function is called *model function* and it is denoted by $m(\alpha, \beta)$, to emphasise its parametrisation, and it should solve a differential equation. We skip further comments and formulas, recommending to the reader the article [61] and references therein for more details.

Finally we cite a very new approach towards nonlinear operators [57]. In this article is considered a standard Tikhonov type functional for solving a nonlinear operator equation of the form $F_0(u) = g_0$, where additionally to the noisy data g_δ it is assumed the noisy operator F_ϵ holds

$$\sup_u \big\| F_0(u) - F_\epsilon(u) \big\| \leq \epsilon$$

with a known constant ϵ referring to the operator noise level. A regularized solution is obtained, as well as convergence rates.

For numerical implementation on dbl-RTLS we recommend the upcoming article [8], which shall introduce an alternating minimization algorithm for functionals over two variables.

4.4 Exercises

4.1. Prove Theorem 4.1.1.

4.2. Prove Theorem 4.1.2.

4.3. Prove Theorem 4.1.3.

4.4. Prove Theorem 4.1.4.

4.5. Prove Theorem 4.1.6.

4.6. Prove Theorem 4.1.7.

4.7. Prove Theorem 4.1.8.

4.8. Prove Theorem 4.1.9.

4.9. Prove Theorem 4.1.10.

4.10. Prove Lemma 4.2.10.

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