

University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

---

Mathematical and Statistical Sciences Faculty  
Publications and Presentations

College of Sciences

---

9-28-2020

## Multiple positive solutions to the fractional Kirchhoff problem with critical indefinite nonlinearities

Jie Yang

Haibo Chen

Zhaosheng Feng

*The University of Texas Rio Grande Valley*

Follow this and additional works at: [https://scholarworks.utrgv.edu/mss\\_fac](https://scholarworks.utrgv.edu/mss_fac)



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Yang, J., Chen, H., & Feng, Z. (2020). Multiple positive solutions to the fractional Kirchhoff problem with critical indefinite nonlinearities. *Electronic Journal of Differential Equations*, 2020(101), 1–21.

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in Mathematical and Statistical Sciences Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact [justin.white@utrgv.edu](mailto:justin.white@utrgv.edu), [william.flores01@utrgv.edu](mailto:william.flores01@utrgv.edu).

**MULTIPLE POSITIVE SOLUTIONS TO THE FRACTIONAL  
 KIRCHHOFF PROBLEM WITH CRITICAL INDEFINITE  
 NONLINEARITIES**

JIE YANG, HAIBO CHEN, ZHAOSHENG FENG

ABSTRACT. This article concerns the existence and multiplicity of positive solutions to the fractional Kirchhoff equation with critical indefinite nonlinearities by applying the Nehari manifold approach and fibering maps.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we study the existence and multiplicity of positive solutions to the fractional Kirchhoff type problem

$$M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy\right) (-\Delta)^s u = f_\lambda(x)|u|^{q-2}u + g(x)|u|^{2_s^*-2}u, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
(1.1)

where  $\Omega \subset \mathbb{R}^N$  is an open bounded domain with the Lipschitz boundary  $\partial\Omega$ , dimension  $N > 2s$  with  $s \in (0, 1)$ ,  $2_s^* = \frac{2N}{N-2s}$  is the fractional critical Sobolev exponent and  $0 < s < 1 < q < \min\{2, \frac{N}{N-2s}\} < \infty$ . Here,  $M(t) = a + bt^{m-1}$  with  $m > 1$ ,  $a, b > 0$ ,  $f_\lambda \in L^{q^*}(\Omega)$ ,  $q^* = \frac{2_s^*}{2_s^* - q}$ ,  $f_\lambda = \lambda f_+ - f_-$  with  $\lambda > 0$ , and  $f_\pm = \max\{\pm f, 0\}$  and  $g \in L^\infty(\Omega)$ . Furthermore,  $g$  satisfies the condition

$$(A1) \quad g(x) = \max_{x \in \bar{\Omega}} g(x) \equiv 1 \text{ in } B_\rho(0) \text{ for some } \rho > 0.$$

We denote by  $(-\Delta)^s$  the usual fractional Laplacian operator which is defined (up to normalization factors) as follows (see for instance [18] and the references therein for further details on the fractional Laplacian) by

$$(-\Delta)^s u(x) = 2\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$
(1.2)

where P.V. stands for the principle value.

When  $M(t) \equiv 1$ ,  $\lambda = 1$  and  $s = 1$ , equation (1.1) can be reduced to the semilinear elliptic problem

$$-\Delta u = f(x)|u|^{q-2} + g(x)|u|^{2_s^*-2}u, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega,$$
(1.3)

---

2010 *Mathematics Subject Classification*. 35A15, 35B33, 35R11.

*Key words and phrases*. Fractional Kirchhoff equation; Nehari manifold; fibering maps.

©2020 Texas State University.

Submitted June 12, 2020. Published September 28, 2020.

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 3$ ),  $1 < q < 2$ , and the weight functions  $f, g$  are continuous and sign-changing. By using the Nehari manifold, fibering maps and Ljusternik-Schnirelmann category, Wu [25] proved that there existed at least three positive solutions of (1.3). Xie-Chen [26] presented a multiplicity result on the Kirchhoff-type problems in the bounded domain by using a similar strategy. A number of works dealt with the fractional differential equations [3, 6, 7, 11, 21] and some recent results on problem (1.3) can be seen in [4, 5, 10, 12, 13, 14, 15, 16, 22, 23, 27] and the references therein.

As we know, the variational problems involving fractional and nonlocal operators are much more complicated and challenging. In the last decade, considerable attention focused on the fractional Laplacian operator and nonlocal operator. We refer to [19] for the Brezis-Nirenberg type results for the following elliptic equation involving the fractional Laplacian  $(-\Delta)^s$  ( $0 < s < 1$ ) in a bounded domain,

$$\begin{aligned} (-\Delta)^s u &= \lambda u + |u|^{2_s^*-2} u, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{aligned}$$

where  $\lambda > 0$ ,  $s \in (0, 1)$  is fixed,  $2_s^* = \frac{2N}{N-2s}$ ,  $\Omega \subset \mathbb{R}^N$  ( $N > 2s$ ) is open, bounded and with the Lipschitz boundary, and  $(-\Delta)^s$  is the fractional Laplace operator. The classical Brezis-Nirenberg result was generalized to the case of nonlocal fractional operators through variational techniques. The existence of multiple solutions to the fractional Laplacian equations of Kirchhoff type was considered in [17] and two positive solutions for proper selection of positive parameter  $\lambda$  was obtained.

The main purpose of this article is to establish the existence and multiplicity of positive solutions to problem (1.1) with the critical growth and sign-changing weight functions. Our results encompass and improve the corresponding results presented in [26] for the fractional Kirchhoff type equations involving the critical growth.

The energy functional associated with problem (1.1) is

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{b}{2m} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^m \\ &\quad - \frac{1}{q} \int_{\Omega} f_\lambda(x) |u|^q dx - \frac{1}{2_s^*} \int_{\Omega} g(x) |u|^{2_s^*} dx \end{aligned}$$

for  $u \in H_0^s(\Omega)$ . We can prove that  $I_\lambda \in C^1(H_0^s(\Omega), \mathbb{R})$  and a critical point of  $I_\lambda$  in  $H_0^s(\Omega)$  corresponds to a weak solution of problem (1.1). We summarize our main results as follows.

**Theorem 1.1.** *Assume that  $m < \frac{N}{N-2s}$ ,  $f_\pm \not\equiv 0$  and condition (A1) holds. Then there exist  $0 < \Lambda_* \leq \Lambda_0$  and  $\bar{b} > 0$  such that*

- (i) *for any  $\lambda \in (0, \Lambda_0)$ , problem (1.1) admits at least one positive solution  $u_1$  with  $I_\lambda(u_1) < 0$ , and  $u_1$  is a ground state solution;*
- (ii) *for any  $\lambda \in (0, \Lambda_*)$  and  $b \in (0, \bar{b})$ , problem (1.1) admits at least two positive solutions  $u_1$  and  $u_2$  satisfying  $I_\lambda(u_1) < 0 < I_\lambda(u_2)$ , and  $u_1$  is a ground state solution.*

**Theorem 1.2.** *Assume that  $m = \frac{N}{N-2s}$ ,  $f_\pm \not\equiv 0$  and condition (A1) holds. Then the following two statements hold:*

- (i) *For  $b \geq 1/S^m$  and any  $\lambda > 0$ , problem (1.1) admits at least one positive solution.*

- (ii) For  $b < 1/S^m$ , there exist  $0 < \tilde{\Lambda}_* \leq \Lambda_0$  and  $\tilde{b} > 0$  such that
  - (1) for any  $\lambda \in (0, \Lambda_0)$ , problem (1.1) admits at least one positive solution;
  - (2) for any  $\lambda \in (0, \tilde{\Lambda}_*)$  and  $b \in (0, \tilde{b})$ , problem (1.1) admits at least two positive solutions  $u_1$  and  $u_2$  satisfying  $I_\lambda(u_1) < 0 < I_\lambda(u_2)$ , and  $u_1$  is a ground state solution.

**Theorem 1.3.** Assume that  $m > \frac{N}{N-2s}$ ,  $f_- \equiv 0$ , and condition (A1) holds. Then there exist  $b^*, \Lambda^* > 0$  such that for any  $b \in (0, b^*)$  and  $\lambda \in (0, \Lambda^*)$ , problem (1.1) admits at least three positive solutions  $u_b, u_\lambda, u_{\lambda,b}$  with

$$I_\lambda(u_\lambda) < I_\lambda(u_b) < 0 < I_\lambda(u_{\lambda,b}),$$

and  $u_\lambda$  is a ground state solution.

Note that the corresponding results in [26] are generalized to the nonlocal fractional Kirchhoff problem and the existence results are extended in the sense that the restriction on the Kirchhoff coefficient  $M$  is eliminated.

When  $g(x) \equiv 1$ , by Theorems 1.1 and 1.2, we obtain the existence and multiplicity of positive solutions to the problem

$$M \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right) (-\Delta)^s u = f_\lambda(x) |u|^{q-2} u + |u|^{2_s^*-2} u, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where  $M(t) = a + bt^{m-1}$  with  $a, b > 0$  for  $t \geq 0$  and  $m \in [1, 2_s^*/2]$ , which generalizes [17, Theorem 1.1].

In view of [2, 5], problem (1.1) appears more complicated because of the lack of compactness and the nonlocal nature of the fractional Laplacian. Theorems 1.1–1.3 can be regarded as generalizations of [26] for fractional Laplacian operators.

The rest of this paper is organized as follows. In Section 2, we present mathematical notation and technical lemmas. We prove Theorems 1.1 and 1.2 in Section 3, and prove Theorem 1.3 in Section 4.

## 2. PRELIMINARY RESULTS

In this section, we introduce some notation, definitions and useful lemmas which will be used in the proofs of main results. We define the Hilbert space  $H^s(\mathbb{R}^N)$  by

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}, \quad (2.1)$$

where the term

$$[u]_s = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}$$

is the so-called Gagliardo semi-norm of  $u$ . In view of (1.2) and [18, Proposition 3.6], we have

$$\|(-\Delta)^{s/2} u\|_2^2 = \frac{1}{C_s} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where  $C_s$  is a positive constant depending on  $s$ . We define  $\mathcal{D}^{s,2}(\mathbb{R}^N)$  as the closure of  $\mathcal{C}_0^\infty(\mathbb{R}^N)$  with the norm

$$\|u\|_{\mathcal{D}^{s,2}} = \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \right)^{1/2}.$$

Then  $\mathcal{D}^{s,2}(\mathbb{R}^N)$  is continuously embedded into  $L^{2_s^*}(\mathbb{R}^N)$ . As in [7, Theorem 1.1], let  $S$  be the best constant of the fractional Sobolev embedding  $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$  defined by

$$S = \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy}{\left( \int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{2/2_s^*}}, \quad (2.2)$$

which is well-defined and strictly positive.

We define

$$E_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

with the norm

$$\|u\|_{E_0} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2},$$

which is equivalent to (2.1) [19, 20]. The embedding  $E_0 \hookrightarrow L^r(\Omega)$  is continuous for any  $r \in [1, 2_s^*]$  and compact whenever  $r \in [1, 2_s^*)$ . We recall that  $(E_0, \|\cdot\|_{E_0})$  is a Hilbert space with the inner product defined by

$$\langle u, v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx dy.$$

For simplicity, we will just denote  $\|\cdot\|_{E_0}$  and  $\|\cdot\|_{L^p(\Omega)}$  by  $\|\cdot\|$  and  $|\cdot|_p$ , respectively. Throughout this paper, the letters  $C, C_i, i = 1, 2, \dots$  denote positive constants which may vary from line to line but independent of the associated terms and parameters.

As we see,  $I_\lambda$  is of class  $C^1$  in  $E_0$  and for any  $v \in E_0$  it holds

$$\begin{aligned} \langle I'_\lambda(u), v \rangle &= M(\|u\|^2) \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx dy \\ &\quad - \int_{\Omega} f_\lambda(x) |u|^{q-2} u v dx - \int_{\Omega} g(x) |u|^{2_s^*-2} u v dx. \end{aligned}$$

Define the Nehari manifold associated with  $I_\lambda$  by

$$N_\lambda = \{u \in E_0 \setminus \{0\} : \langle I'_\lambda(u), u \rangle = 0\}.$$

It is well-known that the Nehari manifold is closely related to the behavior of the fibering map  $\phi_u : t \in \mathbb{R}^+ \rightarrow I_\lambda(tu)$  [2, 8]. Thus, we have

$$\begin{aligned} \phi'_u(t) &= at\|u\|^2 + bt^{2m-1}\|u\|^{2m} - t^{q-1} \int_{\Omega} f_\lambda(x) |u|^q dx - t^{2_s^*-1} \int_{\Omega} g(x) |u|^{2_s^*} dx, \\ \phi''_u(t) &= a\|u\|^2 + (2m-1)bt^{2m-2}\|u\|^{2m} \\ &\quad - (q-1)t^{q-2} \int_{\Omega} f_\lambda(x) |u|^q dx - (2_s^*-1)t^{2_s^*-2} \int_{\Omega} g(x) |u|^{2_s^*} dx. \end{aligned}$$

Then  $u \in N_\lambda$  if and only if  $\phi'_u(1) = 0$ . Moreover, for  $u \in N_\lambda$  we have

$$\phi''_u(1) = a(2-q)\|u\|^2 + b(2m-q)\|u\|^{2m} - (2_s^*-q) \int_{\Omega} g(x) |u|^{2_s^*} dx, \quad (2.3)$$

or

$$\phi''_u(1) = a(2 - 2_s^*)\|u\|^2 + b(2m - 2_s^*)\|u\|^{2m} - (q - 2_s^*) \int_{\Omega} f_{\lambda}(x)|u|^q dx. \quad (2.4)$$

We split  $N_{\lambda}$  into three parts:

$$\begin{aligned} N_{\lambda}^+ &= \{u \in N_{\lambda} | \phi''_u(1) > 0\}, \\ N_{\lambda}^- &= \{u \in N_{\lambda} | \phi''_u(1) < 0\}, \\ N_{\lambda}^0 &= \{u \in N_{\lambda} | \phi''_u(1) = 0\}, \end{aligned}$$

and define

$$\begin{aligned} H^+ &= \{u \in E_0 | \int_{\Omega} f_{\lambda}(x)|u|^q dx > 0\}, & H^- &= \{u \in E_0 | \int_{\Omega} f_{\lambda}(x)|u|^q dx \leq 0\}, \\ G^+ &= \{u \in E_0 | \int_{\Omega} g(x)|u|^{2_s^*} dx > 0\}, & G^- &= \{u \in E_0 | \int_{\Omega} g(x)|u|^{2_s^*} dx \leq 0\}. \end{aligned}$$

In view of  $m \leq \frac{N}{N-2_s}$  and following [17, Lemma 3.2], we can derive the following lemma immediately.

**Lemma 2.1.** *If  $u$  is a minimizer of  $I_{\lambda}$  on  $N_{\lambda}$  such that  $u \notin N_{\lambda}^0$ , then  $I'_{\lambda}(u) = 0$  in  $E_0^{-1}$ .*

**Lemma 2.2.** *For any  $\lambda > 0$ , the functional  $I_{\lambda}$  is coercive and bounded below on  $N_{\lambda}$ .*

*Proof.* For  $u \in N_{\lambda}$ , from (2.2) and Hölder’s inequality, we have

$$\begin{aligned} I_{\lambda}(u) &= I_{\lambda}(u) - \frac{1}{2_s^*} \langle I'_{\lambda}(u), u \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)a\|u\|^2 + \left(\frac{1}{2m} - \frac{1}{2_s^*}\right)b\|u\|^{2m} - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) \int_{\Omega} f_{\lambda}(x)|u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right)a\|u\|^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right)\lambda|f_{+}|_{q^*} S^{-q/2} \|u\|^q. \end{aligned}$$

Recalling that  $1 < q < 2$ , we obtain that  $I_{\lambda}$  is coercive and bounded below on  $N_{\lambda}$ . □

Let

$$\lambda_1 = \left[ \frac{a(2 - q)}{2_s^* - q} \right]^{\frac{2-q}{2_s^*-2}} \frac{a(2_s^* - 2)S^{\frac{2_s^*-q}{2_s^*-2}}}{(2_s^* - q)|f_{+}|_{q^*}}. \quad (2.5)$$

**Lemma 2.3.** *There exists  $\lambda_1 > 0$  such that  $N_{\lambda}^0 = \emptyset$  for  $\lambda \in (0, \lambda_1)$ .*

*Proof.* By contradiction assume that for some  $\lambda \in (0, \lambda_1)$ , there is a function  $u \in N_{\lambda}^0$ . Then from (2.3) and (2.4), we have

$$a(2 - q)\|u\|^2 + b(2m - q)\|u\|^{2m} - (2_s^* - q) \int_{\Omega} g(x)|u|^{2_s^*} dx = 0, \quad (2.6)$$

$$a(2 - 2_s^*)\|u\|^2 + b(2m - 2_s^*)\|u\|^{2m} - (q - 2_s^*) \int_{\Omega} f_{\lambda}(x)|u|^q dx = 0. \quad (2.7)$$

It follows from (A1), (2.6) and (2.2) that

$$\|u\|^2 \leq \frac{2_s^* - q}{a(2 - q)} |u|_{2_s^*}^{2_s^*} \leq \frac{2_s^* - q}{a(2 - q)} S^{-\frac{2_s^*}{2}} \|u\|^{2_s^*}. \quad (2.8)$$

Similarly, from (2.2), (2.7) and Hölder's inequality, we can deduce that

$$\|u\|^2 \leq \frac{2_s^* - q}{a(2_s^* - 2)} \lambda \int_{\Omega} f_+ |u|^q dx \leq \frac{2_s^* - q}{a(2_s^* - 2)} \lambda |f_+|_{q^*} S^{-q/2} \|u\|^q. \quad (2.9)$$

Combining (2.8) and (2.9) yields

$$\left[ \frac{a(2 - q)}{2_s^* - q} S^{\frac{2_s^*}{2}} \right]^{\frac{1}{2_s^* - 2}} \leq \|u\| \leq \left[ \frac{(2_s^* - q) \lambda |f_+|_{q^*} S^{-q/2}}{a(2_s^* - 2)} \right]^{\frac{1}{2 - q}}.$$

Therefore,

$$\lambda \geq \left[ \frac{a(2 - q)}{2_s^* - q} \right]^{\frac{2 - q}{2_s^* - 2}} \frac{a(2_s^* - 2) S^{\frac{2_s^* - q}{2_s^* - 2}}}{(2_s^* - q) |f_+|_{q^*}} = \lambda_1.$$

This is a contradiction.  $\square$

We define

$$\lambda_2 = \begin{cases} \lambda_1, & m < \frac{N}{N - 2s}, \\ \left( \frac{1}{1 - bS^m} \right)^{\frac{2 - q}{2_s^* - 2}} \lambda_1, & m = \frac{N}{N - 2s}, b < 1/S^m. \end{cases} \quad (2.10)$$

The lemma below shows that the component sets  $N_{\lambda}^+$  and  $N_{\lambda}^-$  are nonempty.

**Lemma 2.4.** *Assume  $m < \frac{N}{N - 2s}$ . Then the following two statements are true.*

- (i) *For any  $u \in G^+ \cap H^+$  and  $\lambda \in (0, \lambda_2)$ , there exist  $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$  such that  $t^+u \in N_{\lambda}^+$ ,  $t^-u \in N_{\lambda}^-$  and*

$$I_{\lambda}(t^+u) = \inf_{0 \leq t \leq t^-} I_{\lambda}(tu), \quad I_{\lambda}(t^-u) = \sup_{t \geq t_{\max}} I_{\lambda}(tu).$$

- (ii) *For any  $u \in G^+ \cap H^-$  and  $\lambda > 0$ , there exists a unique  $t^- = t^-(u) > t_{\max}$  such that  $t^-u \in N_{\lambda}^-$  and*

$$I_{\lambda}(t^-u) = \sup_{t \geq 0} I_{\lambda}(tu).$$

*Proof.* Fix  $u \in E_0 \setminus \{0\}$  and define  $\psi_u(t) : \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\psi_u(t) = at^{2-q} \|u\|^2 + bt^{2m-q} \|u\|^{2m} - t^{2_s^*-q} \int_{\Omega} g(x) |u|^{2_s^*} dx. \quad (2.11)$$

We remark that  $tu \in N_{\lambda}$  if and only if  $\psi_u(t) = \int_{\Omega} f_{\lambda} |u|^q dx$ .

- (i) Let  $u \in G^+ \cap H^+$ . From (2.11), it is easy to check that

$$\psi_u(0) = 0, \quad \lim_{t \rightarrow \infty} \psi_u(t) = -\infty, \quad \lim_{t \rightarrow 0^+} \psi'_u(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \psi'_u(t) < 0.$$

Define  $\psi'_u(t) = t^{1-q} h_u(t)$ , where

$$h_u(t) = a(2 - q) \|u\|^2 + (2m - q) bt^{2m-2} \|u\|^{2m} - (2_s^* - q) t^{2_s^*-2} \int_{\Omega} g(x) |u|^{2_s^*} dx.$$

Then, there exists a unique  $t_0 > 0$  such that  $h'_u(t_0) = 0$ , where

$$t_0 = \left( \frac{(2m - q)(2m - 2)b \|u\|^{2m}}{(2_s^* - q)(2_s^* - 2) \int_{\Omega} g(x) |u|^{2_s^*} dx} \right)^{\frac{1}{2_s^* - 2m}}.$$

From  $m < \frac{N}{N - 2s}$  it follows that  $\lim_{t \rightarrow 0^+} h_u(t) > 0$  and  $\lim_{t \rightarrow \infty} h_u(t) = -\infty$ . This implies that there is a unique  $t_{\max} > t_0$  such that  $h_u(t_{\max}) = 0$ . Hence,  $\psi'_u(t) >$

0 for  $t \in (0, t_{\max})$ ,  $\psi'_u(t) < 0$  for  $t \in (t_{\max}, \infty)$  and  $\psi'_u(t_{\max}) = 0$ . Moreover,  $\psi_u(t_{\max}) = \max_{t>0} \psi_u(t) \geq \max_{t>0} \bar{\psi}_u(t)$ , where

$$\bar{\psi}_u(t) = at^{2-q}\|u\|^2 - t^{2^*-q} \int_{\Omega} g(x)|u|^{2^*} dx.$$

From (2.2) it follows that

$$\begin{aligned} \max_{t>0} \bar{\psi}_u(t) &= \|u\|^q \frac{a(2_s^* - 2)}{2_s^* - q} \left( \frac{(2 - q)a\|u\|^{2_s^*}}{(2_s^* - q) \int_{\Omega} g(x)|u|^{2_s^*} dx} \right)^{\frac{2-q}{2_s^*-2}} \\ &\geq \|u\|^q \frac{a(2_s^* - 2)}{2_s^* - q} \left( \frac{(2 - q)aS^{\frac{2_s^*}{2}}}{2_s^* - q} \right)^{\frac{2-q}{2_s^*-2}}. \end{aligned}$$

For  $u \in H^+$ , it holds

$$\psi_u(0) = 0 < \int_{\Omega} f_{\lambda}(x)|u|^q dx \leq \lambda \int_{\Omega} f_+|u|^q dx \leq \lambda|f_+|_{q^*} S^{-q/2}\|u\|^q.$$

So, if

$$\lambda < \lambda_1 = \frac{a(2_s^* - 2)S^{\frac{2_s^*-q}{2}}}{(2_s^* - q)|f_+|_{q^*}} \left( \frac{(2 - q)a}{2_s^* - q} \right)^{\frac{2-q}{2_s^*-2}},$$

there exist unique  $t^+ = t^+(u) < t_{\max}$  and  $t^- = t^-(u) > t_{\max}$  such that

$$\psi_u(t^+) = \int_{\Omega} f_{\lambda}(x)|u|^q dx = \psi_u(t^-), \quad \psi'_u(t^+) > 0, \quad \psi'_u(t^-) < 0,$$

which implies  $t^+u, t^-u \in N_{\lambda}$ . According to  $\phi''_u(1) = t^{q+1}\psi'_u(t)$ , we can deduce that  $t^+u \in N_{\lambda}^+$  and  $t^-u \in N_{\lambda}^-$ . Since  $\phi'_u(t) = t^{q-1}(\psi_u(t) - \int_{\Omega} f_{\lambda}(x)|u|^q dx)$ , it is clear that  $\phi'_u(t) < 0$  for  $t \in [0, t^+)$  and  $\phi'_u(t) > 0$  for  $t \in (t^+, t^-)$ . This indicates that  $I_{\lambda}(t^+u) = \inf_{0 \leq t \leq t^-} I_{\lambda}(tu)$ .

Similarly, from  $\phi'_u(t) > 0$  for  $t \in (t^+, t^-)$  and  $\phi'_u(t) < 0$  for  $t \in (t^-, \infty)$ , we can obtain  $I_{\lambda}(t^-u) = \sup_{t \geq t_{\max}} I_{\lambda}(tu)$ .

(ii) The proof is essentially the same as that in Part (i), so we omit it. □

As in Lemma 2.4, we can deduce the following two lemmas.

**Lemma 2.5.** *Assume that  $m = \frac{N}{N-2s}$  and  $b \geq 1/S^m$ . Then for any  $u \in H^+$ , there exists a unique  $0 < t^+ < t_{\max}$  such that  $t^+u \in N_{\lambda}$  and  $I_{\lambda}(t^+u) = \inf_{t \geq 0} I_{\lambda}(tu)$ .*

**Lemma 2.6.** *Assume that  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ . Then the following two statements are true.*

(i) *For any  $u \in H^+$  and  $\lambda \in (0, \lambda_2)$ , there exist  $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$  such that  $t^+u \in N_{\lambda}^+$ ,  $t^-u \in N_{\lambda}^-$  and*

$$I_{\lambda}(t^+u) = \inf_{0 \leq t \leq t^-} I_{\lambda}(tu), \quad I_{\lambda}(t^-u) = \sup_{t \geq t_{\max}} I_{\lambda}(tu).$$

(ii) *For any  $u \in H^-$  and  $\lambda > 0$ , there exists a unique  $t^- = t^-(u) > t_{\max}$  such that  $t^-u \in N_{\lambda}^-$  and*

$$I_{\lambda}(t^-u) = \sup_{t \geq 0} I_{\lambda}(tu).$$

**Lemma 2.7.** *Assume  $\lambda \in (0, \lambda_1)$ . Then for any  $u \in N_{\lambda}^+$  and  $v \in N_{\lambda}^-$ , there exist  $B_0 > B_{\lambda} > 0$  such that  $\|v\| > B_0 > B_{\lambda} > \|u\|$ .*



*Proof.* Let  $u \in N_\lambda^+ \subset N_\lambda$ . In view of (2.2) and (2.4), it follows from Hölder's inequality that

$$a(2_s^* - 2)\|u\|^2 < (2_s^* - q) \int_\Omega f_\lambda(x)|u|^q dx \leq (2_s^* - q)\lambda S^{-q/2}|f_+|_{q^*}\|u\|^q.$$

Then

$$\|u\| < \left( \frac{(2_s^* - q)\lambda S^{-q/2}|f_+|_{q^*}}{a(2_s^* - 2)} \right)^{\frac{1}{2-q}} = B_\lambda.$$

Similarly, if  $v \in N_\lambda^- \subset N_\lambda$ , from (2.3) and (A1) we have

$$a(2 - q)\|v\|^2 < (2_s^* - q) \int_\Omega g(x)|v|^{2_s^*} dx \leq (2_s^* - q)S^{-2_s^*/2}\|v\|^{2_s^*}.$$

Hence, we have

$$\|v\| > \left( \frac{a(2 - q)S^{\frac{2_s^*}{2}}}{2_s^* - q} \right)^{\frac{1}{2_s^*-2}} = B_0.$$

By a direct calculation, we can verify that  $B_0 > B_\lambda$  for  $\lambda \in (0, \lambda_1)$ , where  $\lambda_1$  is given in (2.5).  $\square$

**Corollary 2.8** ([11]). *For any  $\lambda \in (0, \lambda_1)$ ,  $N_\lambda^-$  is a closed set in  $E_0$  topology.*

### 3. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we discuss the existence and multiplicity of solutions to problem (1.1) when  $m \leq \frac{N}{N-2s}$ . From Lemmas 2.3, 2.4 and 2.6, if  $m < \frac{N}{N-2s}$  or  $m = \frac{N}{N-2s}$ , and  $b < 1/S^m$  holds for any  $\lambda \in (0, \lambda_1)$ , then  $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ . Now, we study the infimum of  $I_\lambda$  on the  $N_\lambda^\pm$  by defining  $c_\lambda^\pm = \inf_{N_\lambda^\pm} I_\lambda(u)$  and  $\lambda_3 = \frac{q}{2}\lambda_1$ .

**Lemma 3.1.** *Assume that  $m < \frac{N}{N-2s}$  or  $m = \frac{N}{N-2s}$ , and  $b < 1/S^m$ . Then*

- (i) *for any  $\lambda \in (0, \lambda_1)$ , we have  $c_\lambda^+ = \inf_{u \in N_\lambda^+} I_\lambda(u) < 0$ ;*
- (ii) *for any  $\lambda \in (0, \lambda_3)$ , we have  $c_\lambda^- \geq \alpha > 0$ . In particular, if  $\lambda \in (0, \lambda_1)$ , then*

$$c_\lambda^+ = \inf_{u \in N_\lambda} I_\lambda(u).$$

*Proof.* (i) For  $u \in N_\lambda^+$ , it follows from (2.4) that

$$\int_\Omega f_\lambda(x)|u|^q dx \geq \left( \frac{2_s^* - 2}{2_s^* - q} \right) a\|u\|^2 + \left( \frac{2_s^* - 2m}{2_s^* - q} \right) b\|u\|^{2m}. \quad (3.1)$$

By (3.1), we obtain

$$\begin{aligned} c_\lambda^+ &\leq I_\lambda(u) - \frac{1}{2_s^*} \langle I'_\lambda(u), u \rangle \\ &= \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a\|u\|^2 + \left( \frac{1}{2m} - \frac{1}{2_s^*} \right) b\|u\|^{2m} - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \int_\Omega f_\lambda(x)|u|^q dx \\ &\leq -\left( \frac{1}{q} - \frac{1}{2} \right) \left( 1 - \frac{2}{2_s^*} \right) a\|u\|^2 - \left( \frac{1}{q} - \frac{1}{2m} \right) \left( 1 - \frac{2m}{2_s^*} \right) b\|u\|^{2m} \\ &< 0. \end{aligned}$$

(ii) For  $u \in N_\lambda^-$ , applying Lemma 2.7 and  $\lambda \in (0, \lambda_3)$ , we deduce

$$I_\lambda(u) = \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a\|u\|^2 + \left( \frac{1}{2m} - \frac{1}{2_s^*} \right) b\|u\|^{2m} - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \int_\Omega f_\lambda(x)|u|^q dx$$

$$\begin{aligned} &\geq \|u\|^q \left[ \frac{as}{N} \left( \frac{a(2-q)}{2_s^* - q} S^{\frac{2_s^*}{2}} \right)^{\frac{2-q}{2_s^* - 2}} - \lambda \left( \frac{1}{q} - \frac{1}{2_s^*} \right) |f_+|_{q^*} S^{-q/2} \right] \\ &\geq \frac{(2_s^* - q) |f_+|_{q^*} \|u\|^q}{2_s^* q S^{\frac{q}{2}}} (\lambda_3 - \lambda) \\ &\geq \alpha > 0. \end{aligned}$$

□

**Lemma 3.2.** *For each  $u \in N_\lambda^\pm$  and  $\lambda \in (0, \lambda_1)$ , there is a number  $\epsilon$  and a differentiable function  $\zeta : B(0, \epsilon) \subseteq E \rightarrow \mathbb{R}$  such that  $\zeta(0) = 1$ , the function  $\zeta(v)(u - v) \in N_\lambda^\pm$ , and*

$$\begin{aligned} &\langle \zeta'(0), v \rangle \\ &= \frac{2a\langle u, v \rangle + 2mb\|u\|^{2(m-1)}\langle u, v \rangle - q \int_\Omega f_\lambda |u|^{q-2} u v dx - 2_s^* \int_\Omega g |u|^{2_s^*-2} u v dx}{(2-q)a\|u\|^2 + (2m-q)b\|u\|^{2m} - (2_s^* - q) \int_\Omega g |u|^{2_s^*} dx}, \end{aligned}$$

where

$$\langle u, v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy$$

for  $v \in B_\epsilon(0) = \{v \in E_0 : \|v\| \leq \epsilon\}$ .

The proof of the above lemma is similar to that of [11, Lemma 3.4], we omit it here.

**Lemma 3.3.** *Assume that  $\lambda \in (0, \lambda_1)$ . Then there exists a minimizing sequence  $\{u_k\} \subset N_\lambda$  such that*

$$I_\lambda(u_k) \rightarrow c_\lambda \quad \text{and} \quad \|I'_\lambda(u_k)\|_{E_0^{-1}} \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{3.2}$$

with  $c_\lambda = \inf_{u \in N_\lambda} I_\lambda(u)$ .

*Proof.* It follows from Lemma 2.2 and the Ekeland's variational principle [9] that there exists a minimizing sequence  $\{u_k\} \subset N_\lambda$  such that

$$c_\lambda < I_\lambda(u_k) < c_\lambda + \frac{1}{k}, \tag{3.3}$$

$$I_\lambda(u_k) < I_\lambda(u) + \frac{1}{k} \|u - u_k\|, \quad u \in N_\lambda. \tag{3.4}$$

From (3.3) and Lemma 2.2, we have  $\sup_k \|u_k\| < \infty$ . Now, we claim that  $\|I'_\lambda(u_k)\|_{E_0^{-1}} \rightarrow 0$  as  $k \rightarrow \infty$ . From Lemma 3.2, we know the differentiable functions  $\zeta_k : B_{\epsilon_k}(0) \rightarrow \mathbb{R}$  for some  $\epsilon_k > 0$  such that  $\zeta_k(v)(u_k - v) \in N_\lambda$  for  $v \in B_{\epsilon_k}(0)$ . For a fixed  $k$ , we take  $0 < \varrho < \epsilon_k$  and define  $v_\varrho = \varrho u / \|u\|$  with  $u \in E_0, u \neq 0$  and  $\omega_\varrho = \zeta_k(v_\varrho)(u_k - v_\varrho)$ . Then it is easy to see that  $\omega_\varrho \in N_\lambda$ . By (3.4), we can deduce that

$$I_\lambda(\omega_\varrho) - I_\lambda(u_k) \geq -\frac{1}{k} \|\omega_\varrho - u_k\|,$$

which implies

$$\langle I'_\lambda(u_k), \omega_\varrho - u_k \rangle + o_k(\|\omega_\varrho - u_k\|) \geq -\frac{1}{k} \|\omega_\varrho - u_k\|.$$

Therefore,

$$-\langle I'_\lambda(u_k), v_\varrho \rangle + (\zeta_k(v_\varrho) - 1) \langle I'_\lambda(u_k), u_k - v_\varrho \rangle \geq -\frac{1}{k} \|\omega_\varrho - u_k\| + o_k(\|\omega_\varrho - u_k\|).$$

Then  $\langle I'_\lambda(\omega_\varrho), u_k - v_\varrho \rangle = 0$  yields

$$\begin{aligned} & -\varrho \langle I'_\lambda(u_k), \frac{u}{\|u\|} \rangle + (\zeta_k(v_\varrho) - 1) \langle I'_\lambda(u_k) - I'_\lambda(\omega_\varrho), u_k - v_\varrho \rangle \\ & \geq -\frac{1}{k} \|\omega_\varrho - u_k\| + o_k(\|\omega_\varrho - u_k\|). \end{aligned}$$

That is,

$$\begin{aligned} \langle I'_\lambda(u_k), \frac{u}{\|u\|} \rangle & \leq \frac{1}{k\varrho} \|\omega_\varrho - u_k\| + \frac{o_k(\|\omega_\varrho - u_k\|)}{\varrho} \\ & \quad + \frac{(\zeta_k(v_\varrho) - 1)}{\varrho} \langle I'_\lambda(u_k) - I'_\lambda(\omega_\varrho), u_k - v_\varrho \rangle. \end{aligned} \tag{3.5}$$

Since  $\|\omega_\varrho - u_k\| \leq \rho|\zeta_k(v_\varrho)| + |\zeta_k(v_\varrho) - 1|\|u_k\|$  and  $\lim_{\varrho \rightarrow 0} \frac{|\zeta_k(v_\varrho) - 1|}{\varrho} \leq \|\zeta'_k(0)\|$ , taking the limit  $\varrho \rightarrow 0^+$  in (3.5), we obtain

$$\langle I'_\lambda(u_k), \frac{u}{\|u\|} \rangle \leq \frac{C}{k} (1 + \|\zeta'_k(0)\|)$$

for some  $C > 0$  independent of  $u$ .

It suffices to show that  $\|\zeta'_k(0)\|$  is bounded. Assume by contradiction that  $\langle \zeta'(0), v \rangle = \infty$ . It follows from Lemma 3.2 and Hölder's inequality that

$$\langle \zeta'_k(0), v \rangle = \frac{C\|v\|}{(2 - q)a\|u_k\|^p + (2m - q)b\|u_k\|^{2m} - (2_s^* - q) \int_\Omega g(x)|u_k|^{2_s^*} dx}$$

for some  $C > 0$ , which implies that there exists a subsequence  $\{u_k\}$  such that

$$(2 - q)a\|u_k\|^2 + (2m - q)b\|u_k\|^{2m} - (2_s^* - q) \int_\Omega g(x)|u_k|^{2_s^*} dx = o_k(1). \tag{3.6}$$

Analogously, we can obtain

$$a(2 - 2_s^*)\|u_k\|^2 + b(2m - 2_s^*)\|u_k\|^{2m} - (q - 2_s^*) \int_\Omega f_\lambda(x)|u_k|^q dx = o_k(1). \tag{3.7}$$

From (3.6) and (3.7), as in the proof of Lemma 2.3, we can see that  $\lambda \geq \lambda_1$ , which is impossible.  $\square$

We define

$$c_\lambda^* := \frac{s}{N} (aS)^{\frac{N}{2s}} - D\lambda^{\frac{2}{2-q}}, \tag{3.8}$$

where

$$D = \frac{(2 - q)(2_s^* - q)|f_+|^{\frac{2}{2-q}}}{2q2_s^*} \left( \frac{2_s^* - q}{(2_s^* - 2)S} \right)^{\frac{q}{2-q}}.$$

**Lemma 3.4.** *Assume that  $m \leq \frac{N}{N-2s}$ . Then  $I_\lambda$  satisfies the (PS) condition at the level  $c_\lambda < c_\lambda^*$ , where  $c_\lambda^*$  is given in (3.8).*

*Proof.* Let  $\{u_n\}$  be a  $(PS)_{c_\lambda}$  sequence satisfying (3.2). It follows from Lemma 2.2 that  $\{u_n\}$  is bounded in  $E_0$ . Hence, we may assume that, up to a subsequence, there exists  $u \in E_0$  such that

$$\begin{aligned} u_n & \rightarrow u, \quad \text{a. e. in } \Omega, \\ u_n & \rightharpoonup u, \quad \text{weakly in } E_0, \\ u_n & \rightarrow u, \quad \text{strongly in } L^r(\Omega), \quad 1 \leq r < 2_s^*. \end{aligned} \tag{3.9}$$

Meanwhile, there exists  $\bar{h} \in L^2(\Omega)$  such that  $|u_n(x)| \leq \bar{h}(x)$  a.e. in  $\Omega$ . Note that  $\lim_{n \rightarrow \infty} \|u_n\| = \beta$  and  $M$  is continuous. We derive that  $M(\|u_n\|^2) \rightarrow M(\beta^2)$  as  $n \rightarrow \infty$ . Set  $v_n = u_n - u$ . We can assume that  $\lim_{n \rightarrow \infty} \|v_n\| = d_1 > 0$ . Otherwise, the conclusion follows. From [1, Lemma 2.7], (3.9) and condition (A1), we have

$$\begin{aligned} \|u_n\|^2 &= \|u_n - u\|^2 + \|u\|^2 + o_n(1), \\ \int_{\Omega} g(x)|u_n|^{2_s^*} dx &= \int_{\Omega} g(x)|u_n - u|^{2_s^*} dx + \int_{\Omega} g(x)|u|^{2_s^*} dx + o_n(1), \end{aligned} \tag{3.10}$$

as  $n \rightarrow \infty$ . By (3.9)-(3.10), we obtain

$$\begin{aligned} o_n(1) &= \langle I'_\lambda(u_n), u_n \rangle \\ &= M(\|u_n\|^2)\|u_n\|^2 - \int_{\Omega} f_\lambda(x)|u|^q dx - \int_{\Omega} g(x)|u|^{2_s^*} dx \\ &\quad - \int_{\Omega} g(x)|v_n|^{2_s^*} dx, \end{aligned} \tag{3.11}$$

and

$$o_n(1) = \langle I'_\lambda(u_n), u \rangle = M(\|u_n\|^2)\|u\|^2 - \int_{\Omega} f_\lambda(x)|u|^q dx - \int_{\Omega} g(x)|u|^{2_s^*} dx. \tag{3.12}$$

As a consequence of (3.11) and (3.12), we obtain

$$M(\|u_n\|^2)\|v_n\|^2 - \int_{\Omega} g(x)|v_n|^{2_s^*} dx = o_n(1).$$

Let  $\lim_{n \rightarrow \infty} \int_{\Omega} g(x)|v_n|^{2_s^*} dx = d_2$ . We derive

$$(a + b\beta^{2(m-1)})d_1^2 = d_2, \tag{3.13}$$

which implies  $d_2 > 0$ . Moreover, from the definition of  $S$  in (2.2), we have

$$d_1^2 \geq Sd_2^{2/2_s^*}. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$d_1^2 \geq a^{\frac{N-2s}{2s}} S^{\frac{N}{2s}}. \tag{3.15}$$

It follows from Hölder's inequality that

$$\begin{aligned} c_\lambda &= \lim_{n \rightarrow \infty} \left( I_\lambda(u_n) - \frac{1}{2_s^*} \langle I'_\lambda(u_n), u_n \rangle \right) \\ &= \lim_{n \rightarrow \infty} \left\{ \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a \|u_n\|^2 + \left( \frac{1}{2m} - \frac{1}{2_s^*} \right) b \|u_n\|^{2m} - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \int_{\Omega} f_\lambda |u_n|^q dx \right\} \\ &\geq \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a d_1^2 + \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a \|u\|^2 - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \lambda |f_+|_{q^*} S^{-q/2} \|u\|^q. \end{aligned}$$

Setting

$$F_\lambda(t) = \left( \frac{1}{2} - \frac{1}{2_s^*} \right) a t^2 - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \lambda |f_+|_{q^*} S^{-q/2} t^q,$$

we deduce that  $F_\lambda(t)$  attains its minimum as

$$\min_{t \geq 0} F_\lambda(t) = -\frac{(2-q)(2_s^*-q)(\lambda|f_+|_{q^*})^{\frac{2}{2-q}}}{22_s^*q} \left( \frac{2_s^*-q}{(2_s^*-2)S} \right)^{\frac{q}{2-q}} = -D\lambda^{\frac{2}{2-q}},$$

where

$$D = \frac{(2-q)(2_s^* - q)|f_+|_{\frac{2^*}{2-q}}}{2q2_s^*} \left( \frac{2_s^* - q}{(2_s^* - 2)S} \right)^{\frac{q}{2-q}}.$$

By applying (3.15), we obtain

$$c_\lambda \geq \frac{s}{N} (aS)^{\frac{N}{2s}} - D\lambda^{\frac{2}{2-q}} = c_\lambda^*,$$

which yields a contradiction with the hypothesis  $c_\lambda < c_\lambda^*$ . □

We define

$$\lambda_4 := \left( \frac{s}{N} (aS)^{\frac{N}{2s}} / D \right)^{\frac{2-q}{2}}$$

and  $\Lambda_0 = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ , where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are given in (2.5), (2.10) and Lemma 3.1, respectively.

**Proposition 3.5.** *Assume that  $m < \frac{N}{N-2s}$  or  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ . Then for  $\lambda \in (0, \Lambda_0)$ ,  $I_\lambda$  has a minimizer  $u_1$  in  $N_\lambda$ , which is a positive solution to problem (1.1) with  $I_\lambda(u_1) = c_\lambda^+$  and  $\|u_1\| \rightarrow 0$  as  $\lambda \rightarrow 0$ .*

*Proof.* For  $\lambda \in (0, \Lambda_0)$ , combining the definition of  $c_\lambda^*$  and Lemma 3.1 gives

$$c_\lambda^+ < 0 < c_\lambda^*.$$

In view of the Ekeland’s variational principle [9], there exists a  $(PS)_{c_\lambda^+}$  sequence  $\{u_n\} \subset N_\lambda^+$  satisfying (3.2). It follows from Lemma 3.4 that there exists  $u_1 \in N_\lambda$  such that

$$I_\lambda'(u_1) = 0, \quad I_\lambda(u_1) = c_\lambda^+ < 0,$$

We now show that  $u_1 \in N_\lambda^+$ . Consider the case  $m < \frac{N}{N-2s}$ , while the case  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$  follows similarly. Suppose by contradiction that  $u_1 \in N_\lambda^-$ . Combining this with (2.3), we have  $u_1 \in G^+$ . On the other hand, from  $u_1 \in N_\lambda$  and  $I_\lambda(u_1) = c_\lambda^+ < 0$ , we can see that  $u_1 \in H^+$ . Hence, from Lemma 2.4, we can infer that there exist  $t^-(u_1) > t^+(u_1) > 0$  such that  $t^-u_1 \in N_\lambda^-$  and  $t^+u_1 \in N_\lambda^+$ . This implies  $t^- = 1$  and  $t^+ < 1$ . Therefore, there exists  $\tilde{t} \in (t^+, t^-)$  such that

$$I_\lambda(t^+u_1) = \min_{0 \leq t \leq t^-} I_\lambda(tu_1) < I_\lambda(\tilde{t}u_1) < I_\lambda(t^-u_1) = I_\lambda(u_1) = c_\lambda^+,$$

which yields a contradiction. This implies  $u_1 \in N_\lambda^+$ .

Furthermore, we show that  $u_1$  is positive. Note that  $I_\lambda(u) \neq I_\lambda(|u|)$  and  $\|u\| \neq \||u|\|$  in  $E_0$ . We consider the positive part of problem (1.1) by defining

$$I_\lambda^+(u) = \frac{a}{2} \|u\|^2 + \frac{b}{2m} \|u\|^{2m} - \frac{1}{q} \int_\Omega f_\lambda(x) |u^+|^q dx - \frac{1}{2_s^*} \int_\Omega g(x) |u^+|^{2_s^*} dx.$$

Then there exists a critical point  $u_1 \in N_\lambda^+$  for  $I_\lambda^+$ . That is, for any  $v \in E_0$  it holds

$$\begin{aligned} M(\|u_1\|^2) & \int_{\mathbb{R}^{2N}} \frac{(u_1(x) - u_1(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ & = \int_\Omega f_\lambda(x) |u_1^+|^{q-1} v dx - \int_\Omega g(x) |u_1^+|^{2_s^*-1} v dx. \end{aligned} \tag{3.16}$$

Taking  $v = u_1^- = \min\{u_1, 0\}$  as a test function in (3.16) and applying the inequality

$$\begin{aligned} (u_1(x) - u_1(y))(u_1^-(x) - u_1^-(y)) & = -u_1^+(x)u_1^-(y) - u_1^-(x)u_1^+(y) - [u_1^-(x) - u_1^-(y)]^2 \\ & \leq -[u_1^-(x) - u_1^-(y)]^2, \end{aligned}$$

we obtain

$$(a + b\|u_1^-\|^{2(m-1)}) \int_{\mathbb{R}^{2N}} \frac{|u_1^-(x) - u_1^-(y)|^2}{|x - y|^{N+2s}} dx dy = o(1),$$

which implies  $\|u_1^-\| = 0$ , i.e.  $u_1 \geq 0$  in  $\mathbb{R}^N$ . Moreover, by the strong maximum principle [3], we know that  $u_1$  is positive.

Then, we prove that  $u_1$  is a local minimizer of  $I_\lambda$  in  $E_0$ . From Lemmas 2.4 and 2.6, we have  $t^+(u_1) = 1 < t_{\max}(u_1)$ . From continuity of  $u \mapsto t_{\max}(u)$ , for the fixed  $\epsilon > 0$ , there exists  $\delta_1 = \delta_1(\epsilon) > 0$  such that  $t_{\max}(u_1 - u) > 1 + \epsilon$  for all  $\|u\| < \delta_1$ . Meanwhile, by Lemma 3.2, we can see that for a given  $\delta_2 > 0$ , there exists a  $C^1$  map  $\zeta : B_{\delta_2}(0) \rightarrow \mathbb{R}^+$  such that  $\zeta(u)(u_1 - u) \in N_\lambda^+$  and  $\zeta(0) = 1$ . Hence, taking into account  $0 < \delta = \min\{\delta_1, \delta_2\}$  and the uniqueness of zeros of fibering map, we have  $t^+(u_1 - u) = \zeta(u) < 1 + \epsilon < t_{\max}(u_1 - u)$  for all  $\|u\| < \delta$ . By  $t_{\max}(u_1 - u) > 1$ , we obtain  $I_\lambda(u_1) \leq I_\lambda(t^+(u_1 - u)(u_1 - u)) \leq I_\lambda(u_1 - u)$ , which implies that  $u_1$  is a local minimizer of  $I_\lambda$  in  $E_0$ .

By Lemma 2.1, we obtain that  $u_1$  is a positive solution to problem (1.1). By Lemma 2.7, we arrive at the desired result.  $\square$

In [19], it is shown that the infimum in (2.2) is attained by

$$u_\epsilon(x) = \frac{\epsilon^{(N-2s)/2}}{(\epsilon^2 + |x|^2)^{(N-2s)/2}}, \quad \epsilon > 0, \tag{3.17}$$

which satisfies

$$\int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x - y|^{N+2s}} dx dy = S|u_\epsilon|_{2_s^*}^{2_s^*}.$$

We define

$$u_{\epsilon,\eta}(x) = \eta(x)u_\epsilon(x), \tag{3.18}$$

where  $\eta(x) \in C_0^\infty(B_\rho(0))$  satisfies  $0 \leq \eta \leq 1$  in  $B_\rho(0)$ ,  $\eta \equiv 1$  in  $B_{\rho/2}(0)$  and  $\eta \equiv 0$  in  $\mathbb{R}^N \setminus B_\rho(0)$ , for some  $\rho > 0$  sufficiently small as given in condition (A1). From [19], we have

$$\|u_{\epsilon,\eta}\|^2 \leq S^{N/(2s)} + O(\epsilon^{N-2s}) \quad \text{and} \quad |u_{\epsilon,\eta}|_{2_s^*}^{2_s^*} = S^{N/(2s)} + O(\epsilon^N). \tag{3.19}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned} \int_{B_\rho(0)} |u_{\epsilon,\eta}|^q dx &\leq C \left( \int_{B_\epsilon(0)} \frac{1}{\epsilon^{q(N-2s)/2}} dx + \int_{B_\rho(0) \setminus B_\epsilon(0)} \frac{\epsilon^{q(N-2s)/2}}{|x|^{q(N-2s)}} dx \right) \\ &= C\omega_N \left( \epsilon^{N - \frac{(N-2s)q}{2}} + \epsilon^{\frac{q(N-2s)}{2}} \int_\epsilon^\rho r^{N-1-q(N-2s)} dr \right) \\ &= O(\epsilon^{N - \frac{(N-2s)q}{2}}) + O(\epsilon^{\frac{q(N-2s)}{2}}) \\ &= O(\epsilon^{\frac{q(N-2s)}{2}}), \end{aligned}$$

where  $1 < q < N/(N - 2s)$ , and  $\omega_N$  denotes the unit sphere in  $\mathbb{R}^N$ .

In view of condition (A1) and the definition of  $\eta$ , we have the following lemma.

**Lemma 3.6** ([11, 19]). *For small  $\epsilon > 0$ , the following statements are true.*

- (i)  $\int_{B_\rho(0)} |u_{\epsilon,\eta}|^q dx = O(\epsilon^{q(N-2s)/2})$ ;
- (ii)  $\int_{B_\rho(0)} |u_{\epsilon,\eta}|^{2_s^* - 1} dx \geq C\epsilon^{\frac{N-2s}{2}}$ ;
- (iii)  $\int_{B_\rho(0)} g(x)|u_{\epsilon,\eta}|^{2_s^*} dx = S^{\frac{N}{2s}} + O(\epsilon^N)$ .

We consider the following two sets

$$A_1 = \{u \in E \setminus \{0\} : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) > 1\} \cup \{0\},$$

$$A_2 = \{u \in E \setminus \{0\} : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) < 1\},$$

where  $t^-$  is given in Lemma 2.4. It follows from [25] that  $t^-(u)$  is continuous for  $u \in E_0 \setminus \{0\}$  and  $N_\lambda^- = \{u \in E_0 \setminus \{0\} : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) = 1\}$  splits  $E_0$  into two connected parts  $A_1$  and  $A_2$ . It follows from Lemmas 2.4 and 2.6 that for any  $u \in N_\lambda^+$  and  $\lambda \in (0, \lambda_2)$ , we have  $1 < t_{\max}(u) < t^-(u)$ . Then  $N_\lambda^+ \subset A_1$ . Particularly,  $u_\lambda^+ \in A_1$ .

**Lemma 3.7.** *Assume that  $m < \frac{N}{N-2s}$  or  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ . Then for any  $\epsilon > 0$ , there exists  $t_1 > 0$  such that  $u_1 + t_1 u_{\epsilon,\eta} \in A_2$ .*

*Proof.* We just prove the case of  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ , since the case of  $m < \frac{N}{N-2s}$  can be processed in a similar manner.

We claim that there exists a constant  $\tilde{c} > 0$  such that  $0 < t^-\left(\frac{u_1 + t u_{\epsilon,\eta}}{\|u_1 + t u_{\epsilon,\eta}\|}\right) < \tilde{c}$  for  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ . Otherwise, there is a sequence  $\{t_n\} \subset \mathbb{R}^+$  such that  $t_n \rightarrow \infty$  and  $t^-\left(\frac{u_1 + t_n u_{\epsilon,\eta}}{\|u_1 + t_n u_{\epsilon,\eta}\|}\right) \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $v_n = \frac{u_1 + t_n u_{\epsilon,\eta}}{\|u_1 + t_n u_{\epsilon,\eta}\|}$ . From (3.19), we deduce that

$$\begin{aligned} \int_{B_\rho(0)} g|v_n|^{2^*_s} dx - b\|v_n\|^{2^*_s} &= \frac{\int_{B_\rho(0)} g|u_1 + t_n u_{\epsilon,\eta}|^{2^*_s} dx}{\|u_1 + t_n u_{\epsilon,\eta}\|^{2^*_s}} - b \\ &\rightarrow \frac{\int_{B_\rho(0)} g|u_{\epsilon,\eta}|^{2^*_s} dx}{\|u_{\epsilon,\eta}\|^{2^*_s}} - b \\ &\geq 1/S^m - b + O(\epsilon^N) > 0 \end{aligned}$$

for  $0 < \epsilon < \epsilon_1$  with some  $\epsilon_1 > 0$ , as  $n \rightarrow \infty$ . Thus,  $I_\lambda(t^-(v_n)v_n) \rightarrow -\infty$  as  $n \rightarrow \infty$  for  $m = \frac{N}{N-2s}$  and  $\epsilon \in (0, \epsilon_1)$ , which contradicts Lemma 2.2. According to [25, Lemma 3.6], we obtain  $u_1 + t_1 u_{\epsilon,\eta} \in A_2$  immediately.  $\square$

**Lemma 3.8.** *Assume that  $m < \frac{N}{N-2s}$  or  $m = \frac{N}{N-2s}$  and  $b < 1/S^m$ . Then there exist  $\Lambda_* \in (0, \Lambda_0]$  and  $\bar{b} > 0$  such that for any  $\lambda \in (0, \Lambda_*)$  and  $b \in (0, \bar{b})$  it holds*

$$\sup_{t \geq 0} I_\lambda(u_1 + t u_{\epsilon,\eta}) < c_\lambda^*,$$

where  $c_\lambda^*$  is given in (3.8).

*Proof.* For any  $\alpha, \beta \geq 0$  and  $m \geq 1$ , we recall the inequality

$$(\alpha + \beta)^m \leq \alpha^m + C_m(\alpha^m + \beta^m) + m\alpha^{m-1}\beta,$$

where  $C_m > 0$  is a constant depending on  $m$ . It follows from Young's inequality that

$$\begin{aligned} &\frac{b}{2m} \|u_1 + t u_{\epsilon,\eta}\|^{2m} \\ &\leq \frac{b}{2m} \|u_1\|^{2m} + b \|u_1\|^{2(m-1)} t \int_{\mathbb{R}^{2N}} \frac{(u_1(x) - u_1(y))(u_{\epsilon,\eta}(x) - u_{\epsilon,\eta}(y))}{|x - y|^{N+2s}} \quad (3.20) \\ &\quad + b C_m \|u_1\|^{2m} + b D_m t^{2m} \|u_{\epsilon,\eta}\|^{2m}, \end{aligned}$$

where  $D_m > 0$ . Since  $u_1$  is a critical point of  $I_\lambda$ , we obtain

$$\langle I'_\lambda(u_1), tu_{\epsilon,\eta} \rangle = 0. \tag{3.21}$$

In view of the inequality

$$(\alpha + \beta)^p - \alpha^p - \beta^p - p\alpha^{p-1}\beta \geq C\alpha\beta^{p-1}, \quad \alpha, \beta \geq 0, \quad p > 2,$$

it follows from the definition of  $\eta$ , (3.20), (3.21) and condition (A1) that

$$\begin{aligned} & I_\lambda(u_1 + tu_{\epsilon,\eta}) \\ &= \frac{a}{2} \|u_1 + tu_{\epsilon,\eta}\|^2 + \frac{b}{2m} \|u_1 + tu_{\epsilon,\eta}\|^{2m} \\ &\quad - \frac{1}{q} \int_{B_\rho(0)} f_\lambda |u_1 + tu_{\epsilon,\eta}|^q dx - \frac{1}{2_s^*} \int_{B_\rho(0)} g |u_1 + tu_{\epsilon,\eta}|^{2_s^*} dx \\ &\leq I_\lambda(u_1) + \frac{a}{2} \|tu_{\epsilon,\eta}\|^2 + bC_m \|u_1\|^{2m} + bD_m \|tu_{\epsilon,\eta}\|^{2m} \\ &\quad - \frac{1}{q} \int_{B_\rho(0)} f_\lambda \left( \int_0^{tu_{\epsilon,\eta}} [|u_1 + s|^{q-1} - |u_1|^{q-1}] ds \right) dx \\ &\quad - \frac{1}{2_s^*} \int_{B_\rho(0)} g [|u_1 + tu_{\epsilon,\eta}|^{2_s^*} - |u_1|^{2_s^*} - 2_s^* tu_{\epsilon,\eta} |u_1|^{2_s^*-1}] dx \\ &\leq c_\lambda^+ + \frac{a}{2} \|tu_{\epsilon,\eta}\|^2 + bC_m \|u_1\|^{2m} + bD_m \|tu_{\epsilon,\eta}\|^{2m} + C|f_-|_\infty |tu_{\epsilon,\eta}|_q^q \\ &\quad - \frac{1}{2_s^*} \int_{B_\rho(0)} g |tu_{\epsilon,\eta}|^{2_s^*} dx - C \int_{B_\rho(0)} |u_1| |tu_{\epsilon,\eta}|^{2_s^*-1} dx. \end{aligned} \tag{3.22}$$

We now consider

$$\begin{aligned} J_\lambda(tu_{\epsilon,\eta}) &= \frac{a}{2} \|tu_{\epsilon,\eta}\|^2 + bC_m \|u_1\|^{2m} + bD_m \|tu_{\epsilon,\eta}\|^{2m} + C|f_-|_\infty |tu_{\epsilon,\eta}|_q^q \\ &\quad - \frac{1}{2_s^*} \int_{B_\rho(0)} g |tu_{\epsilon,\eta}|^{2_s^*} dx - C \int_{B_\rho(0)} |u_1| |tu_{\epsilon,\eta}|^{2_s^*-1} dx. \end{aligned}$$

**Claim 1.** There exist  $t_\epsilon$  and  $t_2 > 0$  independent of  $\epsilon$  and  $\lambda$  such that

$$t_2 \leq t_\epsilon \leq t_1, \quad J_\lambda(t_\epsilon u_{\epsilon,\eta}) = \sup_{t \geq 0} J_\lambda(tu_{\epsilon,\eta}), \quad \frac{d}{dt} J_\lambda(tu_{\epsilon,\eta})|_{t=t_\epsilon} = 0, \tag{3.23}$$

where  $t_1$  is given in Lemma 3.7. Since  $\lambda \in (0, \lambda_3)$ , from Lemma 3.1 we have

$$0 < \alpha < \alpha - c_\lambda^+ \leq c_\lambda^- - c_\lambda^+ \leq \sup_{t \geq 0} I_\lambda(u_1 + tu_{\epsilon,\eta}) - c_\lambda^+ \leq J_\lambda(t_\epsilon u_{\epsilon,\eta}),$$

which implies  $t_2 \leq t_\epsilon$  for some  $t_2 > 0$ .

To find the estimate of  $\sup_{t \geq 0} J_\lambda(tu_{\epsilon,\eta})$ , we define

$$h(t) = \frac{at^2}{2} \|u_{\epsilon,\eta}\|^2 - \frac{t^{2_s^*}}{2_s^*} \int_{B_\rho(0)} g |u_{\epsilon,\eta}|^{2_s^*} dx.$$

By (3.19) and Lemma 3.6, we obtain

$$\sup_{t \geq 0} h(t) \leq \frac{s}{N} (aS)^{\frac{N}{2s}} + O(\epsilon^{N-2s}). \tag{3.24}$$



From (3.23), (3.24) and Lemmas 3.6 and 2.7, we deduce that

$$\begin{aligned}
 J_\lambda(tu_{\epsilon,z}) &\leq \frac{s}{N} (aS)^{\frac{N}{2s}} + bC_m \|u_1\|^{2m} + bD_m t_1^{2m} \|u_{\epsilon,\eta}\|^{2m} + O(\epsilon^{N-2s}) \\
 &\quad + Ct_1^q |f_-|_\infty |u_{\epsilon,\eta}|_q^q - Ct_2^{2^*-1} \int_{B_\rho(0)} |u_1| |u_{\epsilon,\eta}|^{2^*-1} dx \\
 &\leq \frac{s}{N} (aS)^{\frac{N}{2s}} + O(\epsilon^{N-2s}) + Cb\lambda^{\frac{2m}{2-q}} + Cb + C\epsilon^{\frac{q(N-2s)}{2}} - C\epsilon^{\frac{N-2s}{2}} \\
 &\leq \frac{s}{N} (aS)^{\frac{N}{2s}} + Cb\lambda^{\frac{2m}{2-q}} + Cb - C\epsilon_0^{\frac{N-2s}{2}}
 \end{aligned} \tag{3.25}$$

for some  $\epsilon_0 > 0$ . Thus, there exist two positive numbers  $\Lambda_* \in (0, \Lambda_0]$  and  $\bar{b} > 0$  such that for any  $\lambda \in (0, \Lambda_*)$  and  $b \in (0, \bar{b})$  it holds

$$Cb\lambda^{\frac{2m}{2-q}} + Cb + D\lambda^{\frac{2}{2-q}} < C\epsilon_0^{\frac{N-2s}{2}}.$$

Combining this and (3.22)-(3.25) and by Lemma 3.8, we arrive at the desired result.  $\square$

*Proof of Theorem 1.1.* Clearly, (i) follows from Proposition 3.5. (ii) Let  $\lambda \in (0, \Lambda_*)$ . From Proposition 3.5 and Lemma 3.7, we obtain  $u_1 \in A_1$  and  $u_1 + t_1 u_{\epsilon,\eta} \in A_2$ . We define a path  $\gamma(s) = u_1 + st_1 u_{\epsilon,\eta}$  for  $s \in [0, 1]$ . Since  $\gamma(0) \in A_1$  and  $\gamma(1) \in A_2$ , there exists  $s \in (0, 1)$  such that  $u_1 + st_1 u_{\epsilon,\eta} \in N_\lambda^-$ , which implies  $c_\lambda^- \leq \sup_{t \geq 0} I_\lambda(u_1 + tu_{\epsilon,\eta})$ . According to Lemma 3.8, we obtain  $c_\lambda^- < c_\lambda^*$  for any  $\lambda \in (0, \Lambda_*)$  and  $b \in (0, \bar{b})$ . In view of Corollary 2.8,  $N_\lambda^-$  is a closed set. By Proposition 3.5, there exists  $u_2 \in N_\lambda^-$  such that  $I'_\lambda(u_2) = 0$  and  $I_\lambda(u_2) = c_\lambda^-$ . This indicates that  $u_2$  is also a positive solution to problem (1.1).  $\square$

*Proof of Theorem 1.2.* (i) From Lemma 2.5, we obtain  $N_\lambda^+ = N_\lambda$  and define  $c_\lambda = \inf_{u \in N_\lambda} I_\lambda(u)$ . It is clear that  $c_\lambda < 0$ . By Proposition 3.5, there exists a  $(PS)_{c_\lambda}$  sequence  $\{u_n\} \subset N_\lambda^+$  for  $I_\lambda$ . It follows from Lemma 2.2 that  $\{u_n\}$  is bounded in  $E_0$ . Hence, up to a subsequence, there is  $u \in E_0$  satisfying (3.9). Denote  $v_n = u_n - u$ , for  $b \geq 1/S^m$ . Then

$$\int_\Omega g(x) |u|^{2^*} dx - b \|u\|^{2m} < 0.$$

Combining this (3.11) and (3.12) yields

$$a \|v_n\|^2 \leq a \|v_n\|^2 + b \|u\|^{2(m-1)} \|v_n\|^2 + b \|v_n\|^{2m} - \int_\Omega g(x) |u|^{2^*} dx = o_n(1),$$

which implies  $u_n \rightarrow u$  in  $E_0$ . According to Proposition 3.5, we see that  $u$  is a positive solution to problem (1.1).

The proof of (ii) is similar to that of Theorem 1.1, so we omit it.  $\square$

#### 4. PROOF OF THEOREM 1.3

In this section, we assume that  $m > \frac{N}{N-2s}$ ,  $f_- \equiv 0$  and condition (A1) holds. Let us start with a compactness result.

**Lemma 4.1.**  $I_\lambda$  satisfies the (PS) condition if

$$c < c_{\lambda,b}^* := \frac{s}{N} (aS)^{\frac{N}{2s}} - D_1 b - D_2 \lambda^{\frac{2}{2-q}}.$$

*Proof.* Let  $\{u_n\}$  be a  $(PS)_c$  sequence satisfying (3.2). We claim that  $\{u_n\}$  is bounded in  $E_0$ . By way of contradiction, we assume that there is a subsequence of the original sequence such that  $\|u_n\| \rightarrow \infty$ , as  $n \rightarrow \infty$ . We define  $w_n = u_n/\|u_n\|$ . By the Sobolev and Hölder’s inequalities, we obtain

$$\int_{\Omega} g(x)|w_n|^{2_s^*} dx \leq S^{-2_s^*/2} \quad \text{and} \quad \int_{\Omega} \lambda f(x)|w_n|^q dx \leq \lambda S^{-q/2}|f|_{q^*}.$$

Therefore,

$$\begin{aligned} \frac{c + o_n(1)}{\|u_n\|^{2_s^*}} &= \frac{a}{2} \frac{\|u_n\|^2}{\|u_n\|^{2_s^*}} + \frac{b}{2m} \frac{\|u_n\|^{2m}}{\|u_n\|^{2_s^*}} - \frac{1}{q} \frac{\int_{\Omega} \lambda f(x)|u|^q dx}{\|u_n\|^{2_s^*}} - \frac{1}{2_s^*} \int_{\Omega} g(x)|w_n|^{2_s^*} dx \\ &\geq \frac{b}{2m} \|u_n\|^{2m-2_s^*} - \frac{S^{-2_s^*/2}}{2_s^*} + o_n(1) \rightarrow \infty. \end{aligned}$$

In view of  $m > N/(N - 2s)$ , this yields a contradiction.

Hence, up to a subsequence, there exists  $u \in E_0$  satisfying (3.9). Similar to the proof of Lemma 3.4, setting  $v_n = u_n - u$ , we can suppose that  $\lim_{n \rightarrow \infty} \|v_n\| = d_1 > 0$ . Using Hölder’s inequality, (3.15) and  $m > \frac{N}{N-2s}$ , we deduce that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \left( I_{\lambda}(u_n) - \frac{1}{2_s^*} \langle I'_{\lambda}(u_n), u_n \rangle \right) \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{as}{N} \|u_n\|^2 - \left( \frac{1}{2_s^*} - \frac{1}{2m} \right) b \|u_n\|^{2m} - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \lambda \int_{\Omega} f|u_n|^q dx \right\} \\ &\geq \frac{as}{N} d_1^2 + \frac{as}{N} \|u\|^2 - D_1 b - \left( \frac{1}{q} - \frac{1}{2_s^*} \right) \lambda |f|_{q^*} \|u\|^q \\ &\geq \frac{s}{N} (aS)^{\frac{N}{2s}} - D_1 b - D_2 \lambda^{\frac{2}{2-q}}, \end{aligned}$$

where  $D_1 = D_1(N, m, s)$  and  $D_2 = D_2(N, q, S, a, |f|_{q^*})$ . This contradicts the hypothesis of  $c < c_{\lambda,b}^*$ . □

**Lemma 4.2.** *There exist  $\lambda_5 > 0$  and  $r > 0$  such that for any  $\lambda \in (0, \lambda_5)$  it holds*

$$\inf_{u \in E_0, \|u\|=r} I_{\lambda}(u) = \tilde{\alpha} > 0.$$

*In particular, when  $\lambda = 0$ , there exists an  $r_0 > r$  such that  $I_0(u) > 0$  for all  $u \in B_{r_0} \setminus \{0\}$ .*

*Proof.* For  $u \in E_0$ , we have

$$\begin{aligned} I_{\lambda}(u) &= \frac{a}{2} \|u\|^2 + \frac{b}{2m} \|u\|^{2m} - \frac{1}{q} \int_{\Omega} \lambda f(x)|u|^q dx - \frac{1}{2_s^*} \int_{\Omega} g(x)|u|^{2_s^*} dx \\ &\geq \frac{a}{2} \|u\|^2 - \frac{\lambda |f|_{q^*}}{q S^{q/2}} \|u\|^q - \frac{1}{2_s^* S^{2_s^*/2}} \|u\|^{2_s^*} \\ &\geq \left( \frac{a}{2} \|u\|^{2-q} - \frac{\lambda |f|_{q^*}}{q S^{q/2}} - \frac{1}{2_s^* S^{2_s^*/2}} \|u\|^{2_s^*-q} \right) \|u\|^q. \end{aligned} \tag{4.1}$$

We define

$$l(t) = \frac{a}{2} t^{2-q} - \frac{1}{2_s^*} S^{-2_s^*/2} t^{2_s^*-q}$$

for  $t \geq 0$ . In view of  $2 < 2_s^*$ , for each  $u \in E_0$  with

$$\|u\| = r := \left[ \frac{a 2_s^* S^{2_s^*/2} (2-q)}{2(2_s^* - q)} \right]^{1/(2_s^*-2)},$$

we obtain  $\max_{t \geq 0} l(t) = l(r) > 0$ . Thus, by taking

$$\lambda < \lambda_5 = \frac{l(r)q}{S^{-q/2}|f|_{q^*}},$$

we obtain

$$I_\lambda(u) \geq \left( l(r) - \lambda \frac{|f|_{q^*}}{qS^{2/q}} \right) r^q =: \tilde{\alpha} > 0.$$

When  $\lambda = 0$ , from (4.1), there exists an  $r_0 = [(2_s^* - q)/(2 - q)]^{1/(2_s^* - 2)} r > r$  such that  $I_0(u) > 0$  for  $u \in B_{r_0} \setminus \{0\}$ .  $\square$

**Lemma 4.3.** *Let  $\lambda_5$  be given in Lemma 4.2. Then there exist two positive constants  $b_1$  and  $0 < \lambda_6 \leq \lambda_5$  such that for each  $b \in (0, b_1)$  and  $\lambda \in (0, \lambda_6)$ , problem (1.1) admits a positive solution  $u_b$  with  $I_\lambda(u_b) < 0$ .*

*Proof.* It follows from Lemma 4.2 that there is an  $r > 0$  such that  $I_\lambda(u) \geq 0$  for  $\|u\| = r$ . For  $u \in E_0 \setminus \{0\}$  and  $t > 0$  sufficiently small, we have

$$I_\lambda(tu) = \frac{at^2}{2} \|u\|^2 + \frac{bt^{2m}}{2m} \|u\|^{2m} - \frac{t^q}{q} \int_\Omega \lambda f(x) |u|^q dx - \frac{t^{2_s^*}}{2_s^*} \int_\Omega g(x) |u|^{2_s^*} dx < 0.$$

Thus, we obtain

$$m_\lambda := \inf\{I_\lambda(u) : u \in \bar{B}_r\} < 0. \quad (4.2)$$

By Ekeland's variational principle [9], there exists a minimizing sequence  $\{u_n\} \subset \bar{B}_r$  such as

$$I_\lambda(u_n) \rightarrow m_\lambda, \quad \|I'_\lambda(u_n)\|_{E_0^{-1}} \rightarrow 0,$$

as  $n \rightarrow \infty$ . On the other hand, it is easy to see that there exist  $b_1 > 0$  and  $\lambda_6 \leq \lambda_5$  such that

$$c_{\lambda,b}^* > 0, \quad b \in (0, b_1), \quad \lambda \in (0, \lambda_6),$$

where  $c_{\lambda,b}^*$  is given in Lemma 4.1. It follows from (4.2) and Lemma 4.1 that there exists  $u_b \in E_0$  such that  $u_n \rightarrow u_b$ , i.e.  $u_b$  is a nontrivial solution of problem (1.1). By Proposition 3.5, we see that  $u_b$  is a positive solution of (1.1).  $\square$

**Lemma 4.4.** *Let  $r$  and  $b_1$  be given in Lemmas 4.2 and 4.3, respectively. Then there exist  $0 < b_2 \leq b_1$  and  $e_1 \in E_0$  with  $\|e_1\| > r$  such that  $I_\lambda(e_1) < 0$  for  $b \in (0, b_2)$ .*

*Proof.* If  $b = 0$ , we consider the functional  $I_\lambda$  denoted by

$$I_{\lambda,0}(tu_{\epsilon,\eta}) = \frac{at^2}{2} \|u_{\epsilon,\eta}\|^2 - \frac{t^q}{q} \int_{B_\rho(0)} \lambda f(x) |u_{\epsilon,\eta}|^q dx - \frac{t^{2_s^*}}{2_s^*} \int_{B_\rho(0)} g(x) |u_{\epsilon,\eta}|^{2_s^*} dx.$$

Recalling Fatou's Lemma and Lemma 3.6 (iii), we can see that for small  $\epsilon > 0$  it holds

$$\lim_{t \rightarrow \infty} \frac{I_{\lambda,0}(tu_{\epsilon,\eta})}{t^{2_s^*}} \leq -\frac{1}{2_s^*} S^{\frac{N}{2_s^*}} + C\epsilon^N < 0.$$

Namely, there exists a large  $T > 0$  satisfying  $\|Tu_{\epsilon,\eta}\| > r$  and  $I_{\lambda,0}(Tu_{\epsilon,\eta}) < 0$ . Since  $I_\lambda(Tu_{\epsilon,\eta}) \rightarrow I_{\lambda,0}(Tu_{\epsilon,\eta})$  as  $b \rightarrow 0^+$ , we deduce that there exists  $0 < b_2 \leq b_1$  such that  $I_\lambda(Tu_{\epsilon,\eta}) < 0$  for any  $b \in (0, b_2)$ .  $\square$

**Lemma 4.5.** *Let  $\lambda_6$  and  $b_2$  be given in Lemmas 4.3 and 4.4, respectively. Then there exists  $\alpha_b < 0$  such that for any  $b \in (0, b_2)$  it holds*

$$\alpha_b \leq \bar{m}_\lambda := \inf\{I_\lambda(u) : u \in E_0\} < 0.$$

Furthermore, for any  $b \in (0, b_2)$  and  $\lambda \in (0, \lambda_6)$ , problem (1.1) admits a positive solution  $u_\lambda$  with  $I_\lambda(u_\lambda) = \bar{m}_\lambda$ .

*Proof.* Note that

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2m}\|u\|^{2m} - \frac{1}{q} \int_\Omega \lambda f(x)|u|^q dx - \frac{1}{2_s^*} \int_\Omega g(x)|u|^{2_s^*} dx \\ &\geq \frac{b}{2m}\|u\|^{2m} - \frac{1}{q}\lambda|f|_{q^*} S^{-q/2}\|u\|^q - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \end{aligned}$$

Let

$$\bar{A} = \frac{2m\lambda|f|_{q^*}}{bqS^{q/2}}, \quad \bar{B} = \frac{2m}{2_s^* b S^{2_s^*/2}}, \quad \Phi_{\bar{A}, \bar{B}}(t) = t^{2m} - \bar{A}t^q - \bar{B}t^{2_s^*}.$$

From [22, Lemm 2.3], for any  $b > 0$  there exist  $t_3, t_4 > 0$  such that

$$\alpha_b = \min_{t \geq 0} \Phi_{\bar{A}, \bar{B}}(t) = \Phi_{\bar{A}, \bar{B}}(t_3) < 0$$

and  $\Phi_{\bar{A}, \bar{B}}(t) \geq 0$  for  $t \geq t_4$ . While, by Lemma 4.4, it is easy to see that for any  $b \in (0, b_2)$  it holds

$$\bar{m}_\lambda := \inf\{I_\lambda(u) : u \in E_0\} < 0. \tag{4.3}$$

Then using a similar strategy of Lemma 4.3, we obtain that  $u_\lambda$  is a positive solution of problem (1.1).  $\square$

To obtain two distinct solutions to (1.1), we need to show that the infimum  $\bar{m}_\lambda < m_\lambda$ .

**Lemma 4.6.** *There exists  $0 < \lambda_7 \leq \lambda_6$  such that  $\bar{m}_\lambda < m_\lambda$  for  $b \in (0, b_2)$  and  $\lambda \in (0, \lambda_7)$ .*

*Proof.* For  $\lambda = 0$ , let  $m_0$  be given as in (4.3). So, for any  $b \in (0, b_2)$ , problem (1.1) admits a positive solution  $u_{0,b}$  satisfying  $I_0(u_{0,b}) = m_0 := \inf\{I_0(u) : u \in E_0\} < 0$ . Taking into account  $f_- = 0$ , we deduce

$$\bar{m}_\lambda \leq I_\lambda(u_{0,b}) = I_0(u_{0,b}) - \lambda \int_\Omega f|u_{0,b}|^q dx = m_0 - \lambda \int_\Omega f|u_{0,b}|^q dx \leq m_0. \tag{4.4}$$

In view of Lemma 4.2 and (4.2), we have  $m_\lambda \rightarrow 0$  as  $\lambda \rightarrow 0$ . Then, there exists  $0 < \lambda_7 \leq \lambda_6$  such that  $m_0 < m_\lambda$  for any  $b \in (0, b_2)$  and  $\lambda \in (0, \lambda_7)$ . Combining this and (4.4), we arrive at the desired result.  $\square$

It follows from Lemmas 4.2 and 4.4 that  $I_\lambda$  has the mountain pass geometry. Using the mountain pass theorem [24], there exists a  $(PS)_{c_{\lambda,b}}$  sequence  $\{u_n\} \subset E_0$ , that is

$$I_\lambda(u_n) \rightarrow c_{\lambda,b}, \quad \|I'_\lambda(u_n)\|_{E_0^{-1}} \rightarrow 0.$$

We note that  $c_{\lambda,b}$  has the characteristic property

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0, 1], E_0) : \gamma(0) = u_b, \gamma(1) = u_b + Tu_{\epsilon,\eta}\}.$$

Similar to Lemma 3.8, we can obtain the following lemma.

**Lemma 4.7.** *There exist  $0 < \Lambda^* \leq \lambda_7$  and  $0 < b^* \leq b_2$  such that for any  $\lambda \in (0, \Lambda^*)$  and  $b \in (0, b^*)$  it holds*

$$c_{\lambda,b} \leq \sup_{t \geq 0} I_\lambda(u_b + tu_{\epsilon,\eta}) < c_{\lambda,b}^*.$$

*Proof of Theorem 1.3.* By Lemmas 4.2 and 4.4, it is easy to see that  $I_\lambda$  has the mountain pass geometry. i.e. there exists a bounded  $(PS)_{c_{\lambda,b}}$  sequence  $\{u_n\}$ . It follows from Lemmas 4.1 and 4.7 that, up to a subsequence, there exists  $u_{\lambda,b} \in E_0$  such that  $I_\lambda(u_{\lambda,b}) = c_{\lambda,b} > 0$ . By Lemmas 4.3 and 4.5 and 4.6, problem (1.1) admits three positive solutions  $u_b$ ,  $u_\lambda$  and  $u_{\lambda,b}$ .  $\square$

**Acknowledgments.** H. B. Chen was supported by the National Natural Science Foundation of China (No. 11671403). J. Yang was Supported by the Scientific Research Fund of Hunan Provincial Education Department (No. 19B450), and by the Natural Science Foundation of Hunan Province (No. 2019JJ50473).

#### REFERENCES

- [1] C. O. Alves, V. Ambrosio, T. Isernia; Existence, multiplicity and concentration for a class of fractional  $p$ & $q$  Laplacian problems in  $\mathbb{R}^N$ , *Commun. Pure Appl. Anal.*, **18** (2019), 2009-2045.
- [2] K. J. Brown, Y. Zhang; The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, *J. Differential Equations*, **193** (2003), 481-499.
- [3] X. Cabre, Y. Sire; Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31** (2014), 23-53.
- [4] G. F. Che, H. B. Chen; Existence and asymptotic behavior of positive ground state solutions for coupled nonlinear fractional Kirchhoff-type systems, *Comput. Math. Appl.*, **77** (2019), 173-188.
- [5] C. Y. Chen, Y. C. Kuo, T. F. Wu; The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, *J. Differential Equations*, **250** (2011), 1876-1908.
- [6] M. Chhetri, P. Girg, E. Hollifield; Existence of positive solutions for fractional Laplacian equations: theory and numerical experiments, *Electron. J. Differential Equations*, **2020** (2020), no. 81, 1-31.
- [7] A. Cotsoolis, N. Tavoularis; Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.*, **295** (2004), 225-236.
- [8] R. Drabek, S. I. Pohozaev; Positive solutions for the  $p$ -Laplacian: Application of the fibering method, *Proc. Roy. Soc. Edinburgh*, **127** (1997), 703-726.
- [9] I. Ekeland; On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324-353.
- [10] G. M. Figueiredo; Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.*, **401** (2013), 706-713.
- [11] A. Fiscella, P. K. Mishra; The Nehari manifold for fractional Kirchhoff problems involving singular and critical terms, *Nonlinear Anal.*, **186** (2019), 6-32.
- [12] A. Fiscella, E. Valdinoci; A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156-170.
- [13] X. M. He, W. M. Zou; Ground states for nonlinear Kirchhoff equations with critical growth, *Ann. Mat. Pura Appl.*, **193** (2014), 473-500.
- [14] Y. He, G. B. Li, S. J. Peng; Concentrating bound states for Kirchhoff type problems in  $\mathbb{R}^3$  involving critical Sobolev exponents, *Adv. Nonlinear Stud.*, **14** (2014), 483-510.
- [15] S. H. Liang, S. Y. Shi; Soliton solutions to Kirchhoff type problems involving the critical growth in  $\mathbb{R}^N$ , *Nonlinear Anal.*, **81** (2013), 31-41.
- [16] J. Liu, J. F. Liao, C. L. Tang; Positive solutions for Kirchhoff-type equations with critical exponent in  $\mathbb{R}^N$ , *J. Math. Anal. Appl.*, **429** (2015), 1153-1172.
- [17] J. Marcos do Ó, X. M. He, P. K. Mishra; Fractional Kirchhoff problem with critical indefinite nonlinearity, *Math. Nachr.*, **292** (2019), 615-632.
- [18] E. Di Nezza, G. Palatucci, E. Valdinoci; Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521-573.
- [19] R. Servadei, E. Valdinoci; The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, **367** (2015), 67-102.
- [20] R. Servadei, E. Valdinoci; Mountain Pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887-898.

- [21] Y. Su, H. B. Chen, S. L. Liu, X. W. Fang; Fractional Schrodinger-Poisson systems with weighted Hardy potential and critical exponent, *Electron. J. Differential Equations*, **2020** (2020), no. 01, 1-17.
- [22] J. T. Sun, T. F. Wu; Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **146** (2016), 435-448.
- [23] X. Tang, S. Chen; Ground state solutions of Nehari-Pohozaev type for Kirchhoff type problems with general potentials, *Calc. Var.*, **56** (2017), 1-25.
- [24] M. Williem; *Minimax Theorems*, Birkhäuser, Boston, 1996.
- [25] T.F. Wu; Three positive solutions for Dirichlet problems involving critical Sobolev exponent and sign-changing weight, *J. Differential Equations*, **249** (2010), 1459-1578.
- [26] W. H. Xie, H. B. Chen; Multiple positive solutions for the critical Kirchhoff type problems involving sign-changing weight functions, *J. Math. Anal. Appl.*, **479** (2019), 135-161.
- [27] L. Xu, H. B. Chen; Nontrivial solutions for Kirchhoff-type problems with a parameter, *J. Math. Anal. Appl.*, **433** (2016), 455-472.

JIE YANG

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA, HUNAN 410083, CHINA.

DEPARTMENT OF MATHEMATICS, HUIHUA UNIVERSITY, HUIHUA, HUNAN 418008, CHINA

*Email address:* [dafeyang@163.com](mailto:dafeyang@163.com)

HAIBO CHEN

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA, HUNAN 410083, CHINA

*Email address:* [math.chb@163.com](mailto:math.chb@163.com)

ZHAOSHENG FENG

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF TEXAS RIO GRANDE VALLEY, EDINBURG, TX 78539, USA

*Email address:* [zhaosheng.feng@utrgv.edu](mailto:zhaosheng.feng@utrgv.edu)