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On the noninterpolation of polyhedral maps

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Abstract

In this paper we show that if attention is restricted to polyhedral embeddings of graphs, no theorem analogous to the Duke interpolation theorem for 2-cell embeddings is true. We also give two interesting classes of graphs: (i) a class in which the members have polyhedral embeddings in the torus and also in orientable manifolds of arbitrarily high genus, (ii) and another in which the members have polyhedral embeddings in the projective plane and also in orientable and nonorientable manifolds of arbitrarily low Euler characteristic.

1. Introduction

A well-known theorem of Duke states that if a graph has 2-cell embeddings in the orientable manifolds of genera m and n, then G has a 2-cell embedding in the orientable manifold of genus i for each i between m and n [3]. We consider *polyhedral* embeddings of graphs in orientable manifolds, and show that in contrast to Duke's theorem, a graph can have polyhedral embeddings in manifolds of genera m and n without having polyhedral embeddings in all of the manifolds of genus between m and n. We also consider an interesting class of graphs which have polyhedral embeddings in the projective plane, and also in manifolds, both orientable and nonorientable, of arbitrarily low Euler characteristic.

2. Definitions and notation

By a *manifold* we shall mean any 2-dimensional compact manifold without boundary. We assume throughout that our graphs have no 2-valent vertices, as this does not

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affect the embeddings of the graph. If a graph G is embedded in a manifold M then the closures of the connected components of $M-G$ will be called the *faces* of G. When there is no possibility of ambiguity we shall also use the term *face* for its bounding circuit.

An embedding of a graph G in a manifold is a *closed 2-cell embedding* provided that each face is a closed 2-cell. If in addition, in G the intersection of two faces is either empty, a vertex, or an edge, then we say that the embedding is *polyhedral*. Note that this is equivalent to saying that the intersection of each two faces is connected. In this case, we say that the faces of G meet *properly*. A polyhedral embedding of a graph in a manifold M will be called a *polyhedral map* (in M).

A *circuit* in G is a sequence of vertices $v_1, \ldots, v_n$ such that v_i and v_{i+1} are joined by an edge for $1 \leqslant i \leqslant n-1$ and v_n and v_1 are joined by an edge. We call a circuit in G *simple* provided that each v_i appears only once in the sequence. A circuit C in graph G is a *separating circuit* if and only if it has a diagonal (i.e. an edge not on C whose endpoints lie on C) or $G-C$ is disconnected; otherwise, C is *nonseparating*.

A simple circuit in a graph embedded in a manifold M is called *planar* if and only if it bounds a cell that is a subset of M. Simple circuits that do not bound such cells are called *nonplanar*. A *path* is a sequence of vertices $v_1, \ldots, v_n$ such that v_i and v_{i+1} are joined by an edge for $1 \leqslant i \leqslant n-1$. If each v_i appears only once in the sequence then P is a *simple path*.

3. Examples

Theorem 3.1. *There exist polyhedral embeddings of graphs in orientable manifolds of arbitrarily high genus that also have polyhedral embeddings in the torus.*

Proof. We begin with a family of maps on the torus, for which all faces are 4-sided and all vertices are 4-valent, called the $n \times m$ picture frames (see Fig. 1). The graph of the $n \times m$ picture frame is the cartesian product of the simple circuit of length n with the simple circuit of length m. We refer to the n circuits of length m as $r_1, \ldots, r_n$ and the m circuits of length n as $s_1, \ldots, s_m$. Collectively, these circuits are known as the *meridians* of the graph.

Let T be a $2n \times 2m$ picture frame. We construct a manifold M by taking the faces of M to be alternating faces of T (checkerboard fashion, as shown by the shaded faces in Fig. 1) and cells that span the circuits $r_1, \ldots, r_{2n}$ and $s_1, \ldots, s_{2m}$. Each edge of T lies on exactly two of these faces of M. Each vertex of T lies in exactly four of these faces and the union of these four faces is a cell. Thus, M is a manifold, and the graph of T is polyhedrally embedded on it.

The Euler characteristic of M is

$$V-E+F=4nm-8nm+(2nm+2n+2m)=2n+2m-2nm.$$

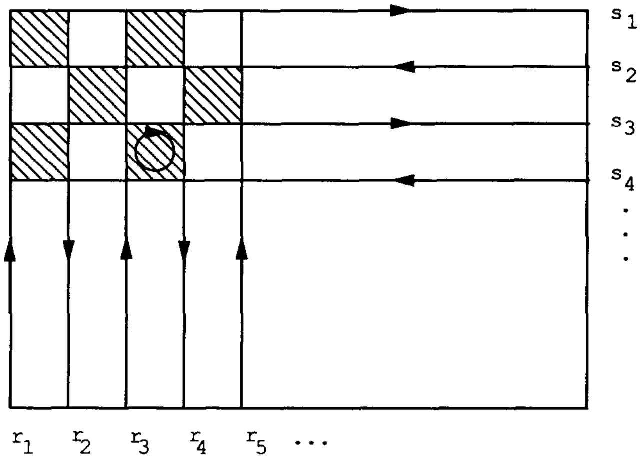

Fig. 1.

For large values of m and n, we get large negative Euler characteristics, and thus arbitrarily large genera. To see that M is orientable, we give an orientation to the r_i's and s_i's (alternating directions as in Fig. 1). This gives an orientation for the faces of M that are not faces of the picture frame, and induces a compatible orientation on the picture frame faces in M. $\square$

Theorem 3.2. *There exist graphs with polyhedral embeddings in manifolds (both orientable and nonorientable) of arbitrarily low Euler characteristic which also have polyhedral embeddings in the projective plane.*

Proof. Our proof is similar to the proof of Theorem 3.1. We consider a family of graphs $\{G_i\}$ embedded in the projective plane. Fig. 2 shows graphs G_4 and G_6. Each G_i is the union of i nonplanar circuits C_i, each two intersecting on exactly one vertex. In each G_i there is one face F that has an edge on each circuit.

We construct a manifold M by taking as faces, a set S of alternate faces of G_{2i} as indicated by shading in Fig. 2, together with cells spanning each of the C_i's. As in Theorem 3.1, the surface M is a manifold. Clearly, all faces meet properly for $i \geqslant 3$, so the embedding in M is polyhedral.

To see that M is orientable, we take an arbitrary orientation of one of the C_i's. Now, alternating C_i's in the disc representation of G_{2i} are given orientations in the opposite direction (see figure). This orientation of the C_i's admits a compatible orientation of the faces of G_i in M. Note that this process yields a polyhedral embedding of G_6 on the torus.

To get a nonorientable manifold M', we use the set of faces of G_{2i} not in S and the circuits C_i spanned by cells. To see that M' is not orientable, take an orientation of F. This will force orientations on each C_i going in the same direction. Now, however, if

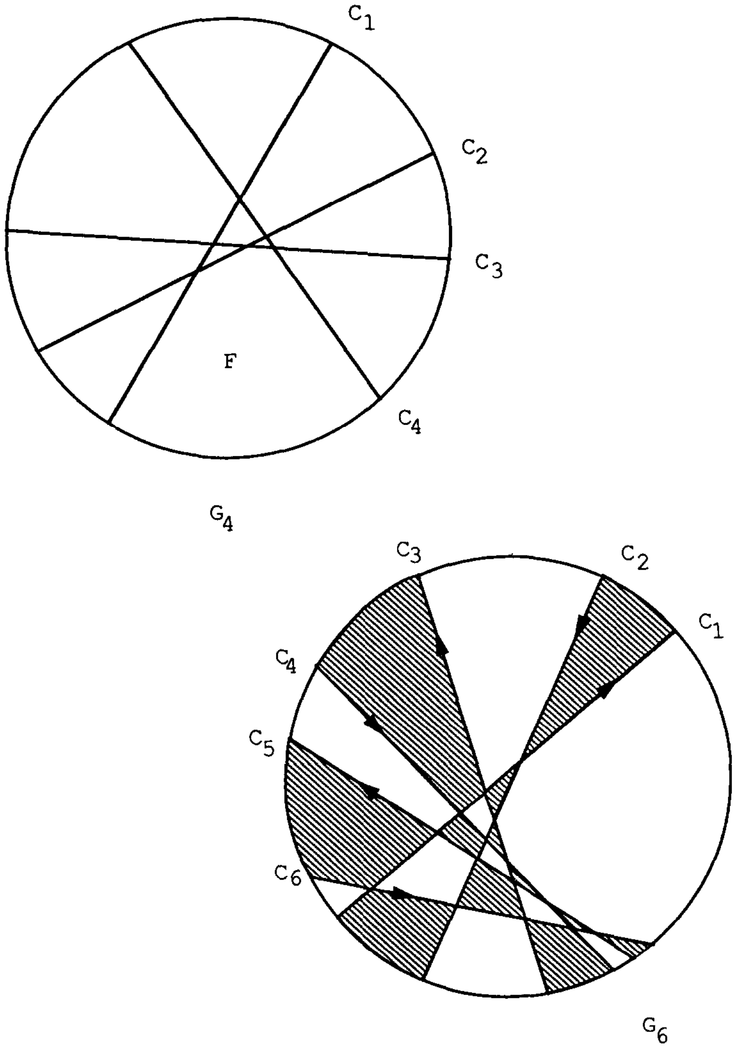

Fig. 2.

we take any 4-sided face of G_{2i} in M', say one meeting F on a vertex, the direction of the orientations along its edges will not be in one direction around the boundary of F, and thus M' is not orientable. Computation of the Euler characteristic will show that M and M' can have arbitrarily high genus. □

4. The noninterpolation of polyhedral maps

In this section, we prove the following theorem.

Theorem 4.1. *If the $2n \times 2n$ picture frame $(n \geqslant 3)$ is polyhedrally embeddable in an orientable surface of genus $\gamma > 1$, then $\gamma > \frac{5}{4}n - \frac{3}{2}$.*

In particular, this implies that a Duke-interpolation-type theorem does not hold for polyhedral embeddings, since the 6×6 picture frame is polyhedrally embeddable in the torus, and by Theorem 3.1, the orientable surface of genus 4, but not in the double torus. Throughout this section, P_n will refer to the graph of the $2n \times 2n$ picture frame $(n \geqslant 3)$, the *standard embedding* of P_n will refer to the embedding given in Fig. 1, and M will refer to some polyhedral embedding of P_n other than the standard one. Note that a 2-cell embedding of P_3 on the double torus can be constructed from the standard toroidal embedding by inserting a handle across two faces which meet at a vertex and using it to switch the way the edges are connected (see Fig. 3). The embedding thus produced can be directly seen to be nonpolyhedral since the two 8-sided faces meet each other improperly. Of course, Theorem 4.1 guarantees that no polyhedral embedding can be found.

We shall need several theorems of Barnette about polyhedral maps in manifolds. The first is that the dual of such a map is again a polyhedral map ([2], see also [6]). The second is that the vertices of a face of a polyhedral map do not disconnect the graph. This follows from duality and a theorem of Barnette's [1] that implies that the set of faces of a polyhedral map missing a vertex form a strongly connected complex.

These two theorems imply the following results.

Lemma 4.2. *Given any two vertices x and y of a polyhedral map M and any face F of M, there is a path from x to y in M missing F except possibly at x and y.*

Corollary 4.3. *A face F of a polyhedral map is a nonseparating circuit.*

Proof. By the previous lemma, the complement of the face is a connected graph, and since faces meet properly, no edge not on F has both its endpoints on F. □

Now, by the *length* of a face F, we mean the length of the bounding circuit of F, which we will denote $\pounds(F)$. Note that because each face of M must be a nonseparating circuit, no such face can meet a square face of the standard embedding in more than two edges without being the square face itself. Finally, if a nonseparating circuit C of P_n meets a face S of the standard embedding in two edges, we call S a *corner* of C.

Lemma 4.4. *If every face of M is either of length 4 or of length $2n$, then M is either the standard embedding of P_n or is of genus $(n-1)^2$.*

Proof. Note that the only circuits of length 4 in P_n $(n \geqslant 3)$ are the square faces of the standard embedding, and that the only nonseparating circuits of length $2n$ are its meridians. If all faces of M are square, then since the only circuits of length 4 in P_n $(n \geqslant 3)$ are the $4n^2$ ones which bound faces in the standard embedding, M has exactly

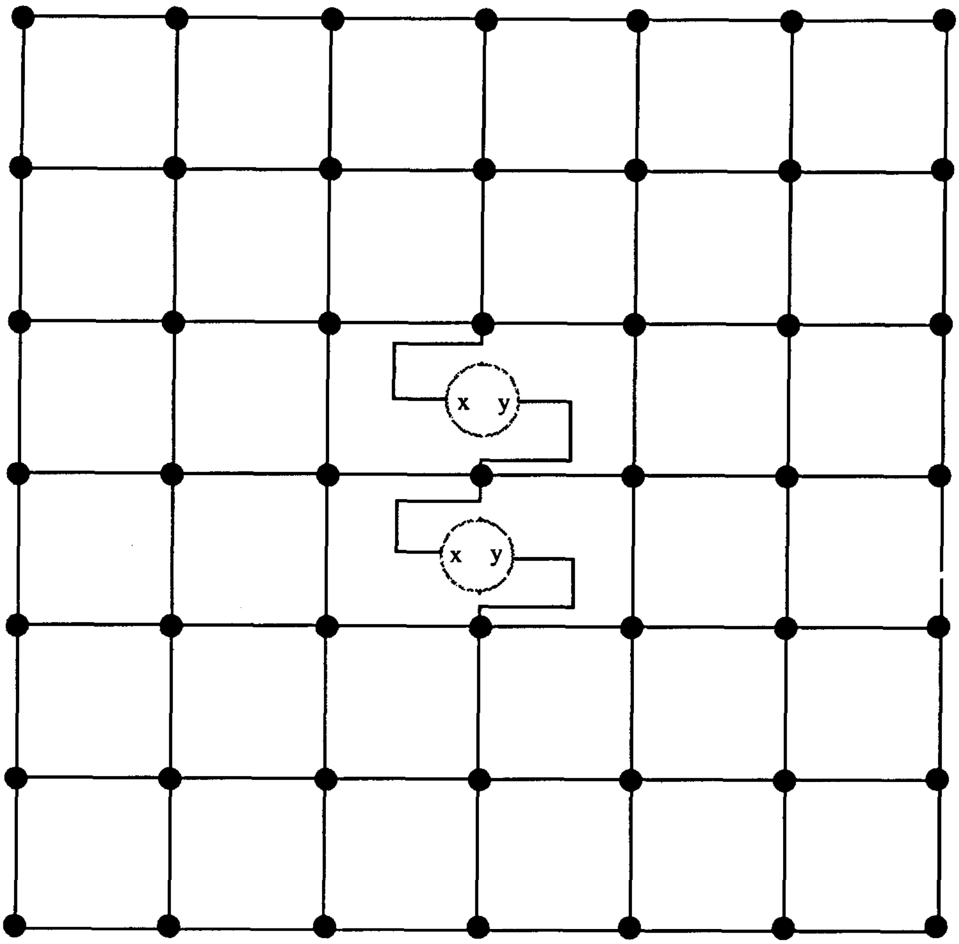

Fig. 3.

the same circuits bounding faces as the standard embedding, and therefore M is identical to the standard embedding. Thus, we may assume that there is a face F of length $2n$. Let e be an edge in F. There is no nonseparating circuit of length $2n$ other than F which contains e, and thus the second face containing e must be a square. If $f \in F$ is an edge which shares a vertex with e, the square faces G and H containing e and f cannot share an edge, for if they did, there could not be two faces containing the fourth edge meeting vertex $F \cap G \cap H$. Thus, the $2n$ square faces of M meeting F must lie on alternating sides of F. Similar arguments will show that there must be $2n$ faces of length $2n$ meeting F in a vertex alone, and in fact that all the $4n$ meridians of P_n must bound faces in M. Since the meridians are the only nonseparating circuits of length $2n$, all other faces of M must be square. Since P_n has $8n^2$ edges and $16n^2 = 2E = \sum_{R \in M} \pounds(R) = (4n)(2n) + 4s$, where s is the number of square faces of M, then

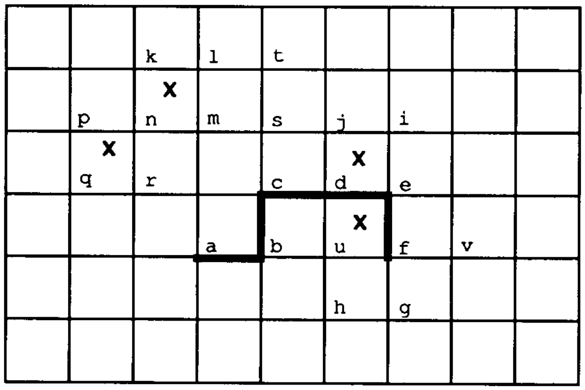

Fig. 4.

$16n^2=8n^2+4s$, i.e. so $s=2n^2$; thus, M has $4n+2n^2$ faces. Hence, by Euler's formula, $2-2\gamma=V-E+F=4n^2-8n^2+4n+2n^2=-2n^2+4n$, so $1-\gamma=-n^2+2n$, i.e. $\gamma=n^2-2n+1=(n-1)^2$. □

Lemma 4.5. *Let F be a face of M with $\pounds(F)>2n$. Then F has two disjoint corners not joined by an edge.*

Proof. Since $\pounds(F)>2n$ and F is a simple circuit, clearly F must contain at least two corners. If F has exactly two, they must be disjoint and not joined by an edge, or F will fail to be a simple circuit. Thus, we may assume that F has at least three corners H, J, and K.

Let $F\cap H=\{ab, bc\}$ (see Fig. 4), and suppose that when travelling along F in the indicated direction, K is the second corner encountered, after H. If K is disjoint from H and not connected to it by an edge, we are done. Otherwise K must be one of the four squares marked by an "x" in Fig. 4. If K is the face containing vertices d, e, and f, then F must contain the path $abcdef$. Vertex g must be next on F, because if v is the next vertex then the face containing e, f, and v will not be joined to H by an edge, while vertex u is ruled out as the next vertex, because otherwise F would have a diagonal. Since F is nonseparating, $h\notin F$. Thus, F must either have a corner disjoint from H and not joined to it by an edge, or else contain the entire meridian determined by edge ef. The latter is not possible, and so the former must be true.

Now, if K is face $jdei$, then F must either contain path $abcdei$ or path $abcsjie$. If K is face $klmn$ then F must contain either path $abcsmnk$ or path $abcstlkn$. If K is face $pqrn$, then F must contain path $abcsmnpq$. In each of these cases, F can be shown to contain a corner having the requisite properties by arguments similar to those used above. □

Proof of Theorem 4.1. Any planar circuit in the standard embedding, other than the faces of that embedding, is separating. Clearly, the meridians are the shortest non-planar circuits; thus, if P_n has an embedding M other than the standard one, it must have a face of length $\geqslant 2n$. If all such faces are of length $2n$, then by Lemma 4.4, $\gamma = (n-1)^2 > \frac{5}{4}n - \frac{3}{2}$. Thus, we may assume there is a face F of length $> 2n$.

Let H and K be the two faces of the standard embedding guaranteed by Lemma 4.4. Let H consist of vertices $bcde$, and suppose that $H \cap F = \{bc, cd\}$. Now, H cannot be a face in M, since it meets F improperly. Thus, one of the two faces of M containing edge ed must have length $\geqslant 2n$, and likewise for one of the two faces of M containing edge eb. Neither of these two faces can be identical with F, for if F contained a third edge of H, F would be H itself. The two faces cannot be identical with each other, for if they were, that face would meet F improperly.

Analogously, there are two faces of length $\geqslant 2n$, different from F and from each other, derived from K. Neither of the faces associated with K can be identical with either of the faces associated with H because, due to the fact that H and K are disjoint and not joined by an edge, such a face would meet F improperly. Thus, M has at least four faces of length $\geqslant 2n$ and one of length $> 2n$.

Now, P_n has $4n^2$ vertices and $8n^2$ edges. Let f denote the number of faces of M. Then a standard counting argument reveals that $16n^2 = \sum_{R \in M} £(R) > 10n + (f-5)4$, and thus that $4n^2 - \frac{5}{2}n + 5 > f$ and so by Euler's equation,

$$2 - 2\gamma = 4n^2 - 8n^2 + f < -4n^2 + 4n^2 - \tfrac{5}{2}n + 5,$$

and the desired inequality follows immediately. $\square$

Note that the assumption of polyhedrality is essential to the proof of Theorem 4.1, and thus the question of whether an interpolation theorem holds for closed 2-cell embeddings is still open. We also conjecture that the lower bound on γ given in Theorem 4.1 is not the best possible, and that in fact the only two polyhedral embeddings of any of the $2m \times 2n$ picture frames are the standard ones and the one given in Theorem 3.1. By mimicking the proof of Theorem 4.1 using the existence (in M) of a single face of length $\geqslant 2n$, it is possible to get a different lower bound which, although worse, is still sufficient to show noninterpolation. However, prefer to get the better bound, since we find the question of the polyhedral embedding range of the picture frames interesting in its own right.

Siran [5] has a result similar to ours: he shows that signed graph embeddings do not necessarily interpolate. Finally, other examples of graphs which have polyhedral embeddings in surfaces of different genera can be produced by applying results of C. Schulz. Satz 6 of [4] states that the boundary complex of a 4-dimensional prism over a 3-polytope with no faces of odd length contains exactly three subsurfaces which contain all vertices and edges of the prism (Schulz refers to such subsurfaces as *Hamilton–Flächen*). This theorem may be applied, e.g. to the 2-fold prism over the $2n$-gon to produce many examples which will in general have different genera.

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