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Abstract

We give various conditions on pinched-torus polyhedral maps which are necessary for their graphs to be embeddable in the projective plane. Our other main result is that even if the graph of a polyhedral map in the pinched torus is embeddable in a projective plane, the map induced by the embedding cannot be polyhedral, but must have all faces bounded by cycles. Finally, we give a class of examples of graphs which have polyhedral embeddings on the pinched torus and also on orientable surfaces of arbitrary high genus.

1. Introduction

In [7] it is shown that no polyhedral map on a 2-manifold is planar, and in [3] necessary conditions are given for such maps to be projective planar. In [6] is given a more restrictive condition for polyhedral maps on the torus to be projective planar. In this paper, we give similar conditions necessary for a polyhedral map on the pinched torus to be projective planar. We also show that if a polyhedral map on the pinched torus is embeddable in the projective plane, then the embedding cannot produce a polyhedral map, but must produce a closed 2-cell map. We give examples to show that in general, this situation is unique to projective plane embeddings by exhibiting a family of polyhedral maps on the pinched torus which have polyhedral embeddings on orientable surfaces of arbitrarily high genus. This family of examples is the pinched-toroidal analog of a family of examples given for the torus in [1], and is also similar to a class of examples given for the torus by Thomassen [7].

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2. Definitions

A *surface* is a compact 2-manifold, and we denote the orientable surface of genus k by S_k. A *pseudosurface* P is a topological space obtained from a surface S by choosing finitely many sets $V_1, \ldots, V_k$ of finitely many different points of S, and identifying all the points in V_i, for $i = 1, \ldots, k$. We denote such a P by $S(|V_1|, \ldots, |V_k|)$. The point in P obtained from V_i is denoted v_i, and such points of P are called *pinch points*. The *index* of a pinch point v_i is $|V_i|$. By convention, we label pinch points so that $|V_i| \leqslant |V_{i+1}|$. We follow the convention that in any embedding of a graph in a pseudosurface there is a vertex at each pinch point. Such vertices are called *pinch vertices*.

In this paper, all graphs are without loops, multiple edges, or 2-valent vertices. A *map* is an embedding of a graph in a surface or pseudosurface. A map is called *closed 2-cell* provided that the closure of each region is a 2-cell. If no ambiguity is likely to arise, we often do not distinguish a region from its bounding circuit, nor a map from its underlying graph. Two intersecting regions of a map are said to *meet properly* if their union is not multiply connected. A *polyhedral* map is one in which every two regions meet properly if at all. Note that we use the term *face* interchangeably with the term *region*.

The *pinched torus* is the pseudosurface $S_0(2)$. Note that in the literature the pinched torus is also referred to as the *pinched sphere* and the *spindle surface*. A polyhedral map on the pinched torus is called a *pinched polyhedral map*. The link of the pinch vertex of a pinched polyhedral map M consists of two disjoint cycles, called the *center circuits* of M. The closed annulus bounded by the center circuits is called the *center strip* of M.

Removing an edge from a map means just that, along with coalescing any 2-valent vertices which may be created into the edges in which they lie. *Shrinking an edge* of a map means contracting it to a point, along with coalescing any created multiple edges which bound a face. The inverses (in the sense of 'inverse relation' and not 'inverse function') of these two operations are called *face splitting* and *vertex splitting*, respectively. A polyhedral map is said to be *R-minimal* (resp. *S-minimal*) if removing (resp. shrinking) any edge yields a nonpolyhedral map. If a map is minimal with respect to both, it is called *diminimal*. The one diminimal pinched polyhedral map was determined by the author [5], and is shown in Fig. 1. Note that the pinched torus is represented as a 'lens', where the upper and lower arcs are identified to form an elongated sphere, and then the two pointed ends are identified to form the pinch-point. Often vertices and/or edges of the map will lie on the boundaries of the lens. If x, y, and z are vertices on a cycle C of a graph G, then $C[x, y, +z]$ denotes the path on C from x to y which includes z, whereas $C[x, y, -1]$ denotes the path on C from x to y which misses z.

A *diagonal* of a cycle C in a graph G is an edge $e \notin C$, but with its endpoints in C. A *nonseparating circuit* of a graph G is a cycle C in G with no diagonals such that $G - C$ is connected. Note that diagonals are also known as *chords*, and nonseparating circuits are also known as *peripheral* circuits. Finally, a *nonplanar* circuit of a map

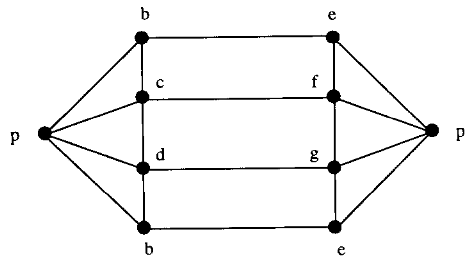

Fig. 1.

M in a surface or pseudosurface S is one which does not bound a cellular region of S. Note that in contrast to the case of surfaces, a nonplanar circuit of a map on a pseudosurface may in fact be contractible to a point. It is known that for surfaces, a map being polyhedral is equivalent to the condition that it is 3-connected and every nonplanar curve in the surface meets the map in at least three points. This is quite clearly true in the case of pseudosurfaces as well. Note that any terms concerning graphs which are not defined here can be found in [4], whereas any terms concerning maps which are not defined here can be found in [1–3, 5].

3. Preliminary lemmas

In [2], Barnette proves the following lemma.

Lemma 3.1. *A face of a polyhedral map on a surface is a nonseparating circuit.*

However, his proof uses map duality and is thus invalid for maps on pseudosurfaces. Thus we prove the following lemma.

Lemma 3.2. *A face of a polyhedral map on a pseudosurface is a nonseparating circuit.*

Proof. Let $v_1, \ldots, v_k$ be the pinch vertices of M. Let n_i be the index of v_i. Let N be a polyhedral map on an appropriate surface, containing vertices $v(1,1), \ldots, v(1,n_1)$, $v(2,1), \ldots, v(2,n_2), \ldots, v(n,1), \ldots, v(n,n_k)$ such that $V_i = \{v(i,1), \ldots, v(i,n_i)\}$. That is, N is a polyhedral map on a surface obtained by 'detaching' the pinch vertices of M.

Clearly the regions of N are in one-to-one correspondence with the faces of M. Let x and y be two vertices of M, and let F be a region of M, with $x, y \notin F$. Since no face of a polyhedral map has a diagonal, we need only show that $M - F$ is connected.

If $v_i \notin F$ for any i, then by Lemma 3.1 there is a path from x_0 to y_0 in N missing F, which, upon identification of each V_i to recover M, becomes a trail from x_0 to y_0, which contains a path in M from x_0 to y_0.

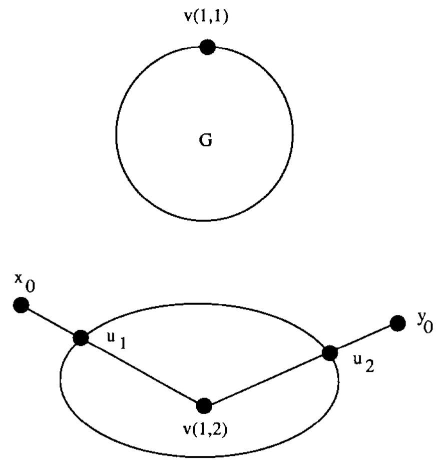

Fig. 2.

On the other hand, if $v_i \in F$ for some values of i, then without loss of generality (w.l. o.g.), we may assume that $v_1, \ldots, v_j \in F, j \leqslant k$. Let G be the region of N corresponding to F. Without loss of generality, we may assume that $v(1,1), v(2,1), \ldots, v(j,1) \in G$. If in N there is an $x_0 y_0$ path missing G which does not touch any of $v(m,n)$, $1 \leqslant m \leqslant j$, $2 \leqslant n \leqslant n_m$, then that path, upon identification of each V_i, will yield an xy trial in M. Thus we may assume that every $x_0 y_0$ path missing G meets some $v(m,n)$ with $1 \leqslant m \leqslant j$, $2 \leqslant n \leqslant n_m$. Let P be such a path with a minimum number of such meetings. Without loss of generality, we may assume that $v(1,2) \in P$. Let u_1 be the vertex in $P \cap \mathrm{link}_N v(1,2)$ which is closest to x_0. Note that $\mathrm{link}_N v(1,2)$ cannot contain any of $v(m,n)$, $1 \leqslant m \leqslant j$, $2 \leqslant n \leqslant n_m$ since, if it did, F would meet itself or some other face in $\mathrm{star}_M v_1$ improperly in M. Thus replacing the segment $u_1 - v(1,2) - u_2$ in P with either path from u_1 to u_2 on $\mathrm{link}_N v(1,2)$ will produce an $x_0 y_0$ path in N with fewer such intersections than P. This contradicts facts hypothesized about P, and so the lemma is proved (see Fig. 2). □

We represent the projective plane Π as a disk with antipodal points identified. The following result should be clear.

Lemma 3.3. *A nonseparating circuit in a graph embedded in a surface or pseudosurface must be either nonplanar or be a face of the map.*

In our representations of maps embedded on Π, it is usually convenient to have a nonplanar cycle of the map on the boundary of the disk. The topology of Π is such that any such cycle can be chosen to lie on the boundary of the disk. Due to the identification, the cycle appears *twice around the boundary* of the disk. We will also need the following lemma.

Lemma 3.4. *Let C be a nonseparating circuit in a graph G. Let H be obtained from G by shrinking C to a vertex. If H is a nonplanar graph, then C must be a region in any Π-embedding of G.*

Proof. If C is not a face in a Π-embedding of G, then by Lemma 3.3, C is a nonplanar circuit, and so can be drawn twice around the boundary of the disk. Contracting C to a vertex yields an embedding of H on the sphere, contrary to assumption. □

4. Π-Embeddings of pinched polyhedral maps

Riskin proved [5] that there are 70 R-minimal pinched polyhedral maps, and that they are all obtainable from the one diminimal pinched polyhedral map (Fig. 1) by splitting zero or more of the 4-valent vertices into pairs of 3-valent vertices.

Lemma 3.5. *Let M be a pinched polyhedral map. Then M contains a submap N which is homeomorphic to one of the* 70 *R-minimal pinched polyhedral maps, and which has the same center circuits as M.*

Proof. Removing all removable edges from M without coalescing 2-valent vertices will yield a submap homeomorphic to one of the 70 R-minimal pinched polyhedral maps. Note that removing a removable edge changes which edges are removable, and thus there are generally many ways in which this can be done. Suppose the lemma is false. Among all submaps of M homeomorphic to an R-minimal pinched polyhedral map, let N be one minimal with respect to the number of edges in the symmetric difference of its center circuits with the center circuits of M. Let C and D be the center circuits of M and let P and Q be the center circuits of N. We may assume w.l.o.g. that there is an edge $e \in P$ with $e \notin C \cup D$, and that P lies in a region bounded by C and Q. Since N is homeomorphic to one of the R-minimal pinched polyhedral maps, there are paths S and T in N from the pinch vertex p to P which cross C. Let b, c, d, and f be vertices contained in $S \cap C$, $T \cap C$, $P \cap S$, and $P \cap T$, respectively (see Fig. 3). Now, let F be that bc-path on C which forms a planar circuit with $S[b,d] \cup P[d,f,+e] \cup T[c,f]$. Then replacing $P[d,f,+e]$ in N by $F \cup S[b,d] \cup T[c,f]$ yields a map homeomorphic to an R-minimal pinched polyhedral map, but whose center circuits meet the center circuits of M in more edges. □

We restate this result as follows.

Corollary 4.1. *Given a pinched polyhedral map M, it is possible to remove all removable edges of M in such a way that the center circuits of the R-minimal pinched polyhedral map thus obtained are composed solely of edges or unions of edges from the center circuits of M.*

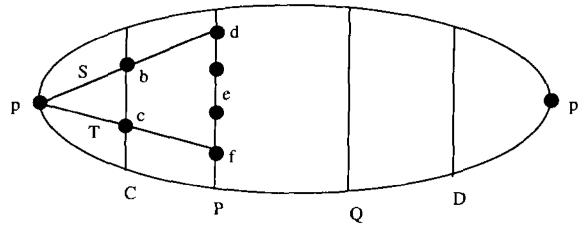

Fig. 3.

Lemma 4.2. *Let C be a nonplanar, nonseparating circuit in the center strip of a pinched polyhedral map M. If C is disjoint from at least one of the center circuits of M, then C must be a face in any Π-embedding of M.*

Proof. Removing all removable edges from M in accordance with Corollary 4.1 yields one of the 70 R-minimal pinched polyhedral maps N. N has three nonplanar cycles P, Q, and R that meet each of the center circuits of N in at least a vertex and at most an edge, and having $P \cap Q \cap R = \{p\}$, where p is the pinch vertex. Since the center circuits of M and N are coincident, adding the edges back to N of which C is comprised will produce a nonplanar, nonseparating circuit in N which is still disjoint from one of the centre circuits. Then contracting C to a point yields a graph of which K_5 is a subcontraction. Thus by Lemma 3.4, C is a face in any Π-embedding of M. □

The next two theorems are analogous to a theorem of Riskin [6] which states that a polyhedral map on the torus with four disjoint homotopic nonplanar circuits is not Π-embeddable.

Theorem 4.3. *If a pinched polyhedral map M has three disjoint circuits homotopic to the pinch vertex, then M is not Π-embeddable.*

Proof. Let A, B, and C be the circuits mentioned above. We may assume w.l.o.g. that A and B bound a region which contains C, and thus that C lies in the interior of the center strip of M. Removing all removable edges of M pursuant to Corollary 4.1, and replacing the edges of which C is comprised yield a map of which the map N given in Fig. 4 is a subcontraction. Clearly if M is Π-embeddable, then so is N. By Lemma 4.2, cycle *efge* must be a face in any Π-embedding of N. By Lemma 3.4, both cycle *befcb* and cycle *ehife* must also be faces in a Π-embedding of N. That makes three faces containing edge *ef*, so no Π-embedding of N is possible. □

Theorem 4.4. *If a pinched polyhedral map M has 4 nonplanar circuits not contractible to the pinch vertex and containing the pinch vertex but otherwise disjoint, then M is not Π-embeddable.*

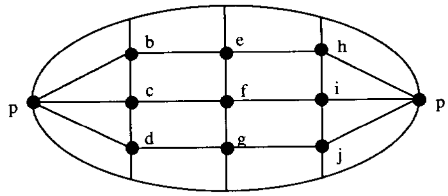

Fig. 4.

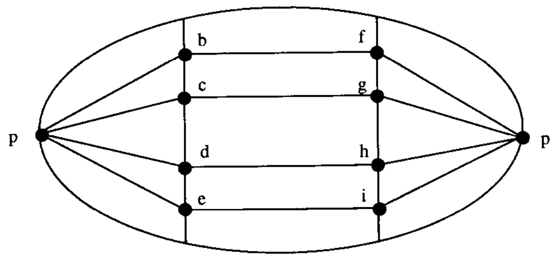

Fig. 5.

Proof. By arguments completely analogous to those used in the proof of Theorem 4.3, M has the map N, shown in Fig. 5, as a subcontraction. By Lemma 4.2, cycles *bcdeb* and *fghif* must be faces in a Π-embedding of N, and by Lemma 3.4, each of the four square faces in the center strip must be as well. But now the faces containing vertex c 'close out' a rotation at c which does not contain edge pc. □

Using methods very similar to those used in the proof of Theorem 4.4, the following lemma can be easily proved.

Lemma 4.5. *At least one of the* 6 *triangles containing the pinch vertex must be a face in any* Π*-embedding of the diminimal pinched polyhedral map.*

We will also need the following lemma.

Lemma 4.6. *Up to map-isomorphism, there are exactly two* Π*-embeddings of the diminimal pinched polyhedral map* D.

Proof. By Lemma 4.5, we may assume that cycle *pbcp* is a face in a Π-embedding of D (all vertices are labelled as in Fig. 1). By Lemma 3.4, *bcdb* must also be a face in a Π-embedding, and thus cycle *befcb* must fail to be a face, and so the disk of the

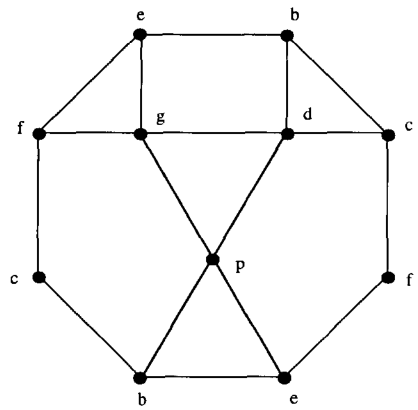

Fig. 6.

Π-embedding can be drawn with that cycle twice around the boundary. Furthermore, Lemma 3.4 forces *efge* to be a face, and there is exactly one way for the three faces *pbcp*, *bcdb*, and *efge* to lie in relation to *befcb*. Vertex p must lie in the octagonal region *cdgfcbef*, and so the places of edges *pd*, *pe*, *pb*, and *pg* are determined (see Fig. 6). Up to map-isomorphism, there are exactly two ways in which the remaining edges *pc* and *pf* can be added, yielding the two Π-embeddings of D (see Fig. 7). □

We are now in a position to prove the following theorem.

Theorem 4.7. *Every Π-embedding of a pinched polyhedral map is a closed 2-cell embedding.*

Proof. Let M be a Π-embeddable pinched polyhedral map, with Π-embedding P. Removing all edges from P which are removable in M yields a Π-embedding of one of the 70 R-minimal pinched polyhedral maps N. Since the graph of N is obtainable from the graph of the diminimal pinched polyhedral map D via vertex splittings, shrinking selected edges of N will yield a Π-embedding of D, which must be one of the three given in Lemma 4.6, and which is therefore a closed 2-cell embedding. P can be recovered from the Π-embedding of D by splitting selected vertices in the surface of Π, and then splitting selected faces. Since neither of these operations can change the closed 2-cell status of an embedding, P itself must be closed 2-cell. □

We need the following theorem due to Barnette [3] and, independently, to Vitray [8]

Theorem 4.8. *A graph has a polyhedral Π-embedding iff G is 3-connected, Π-embeddable, and $G-x$ is nonplanar for each vertex x.*

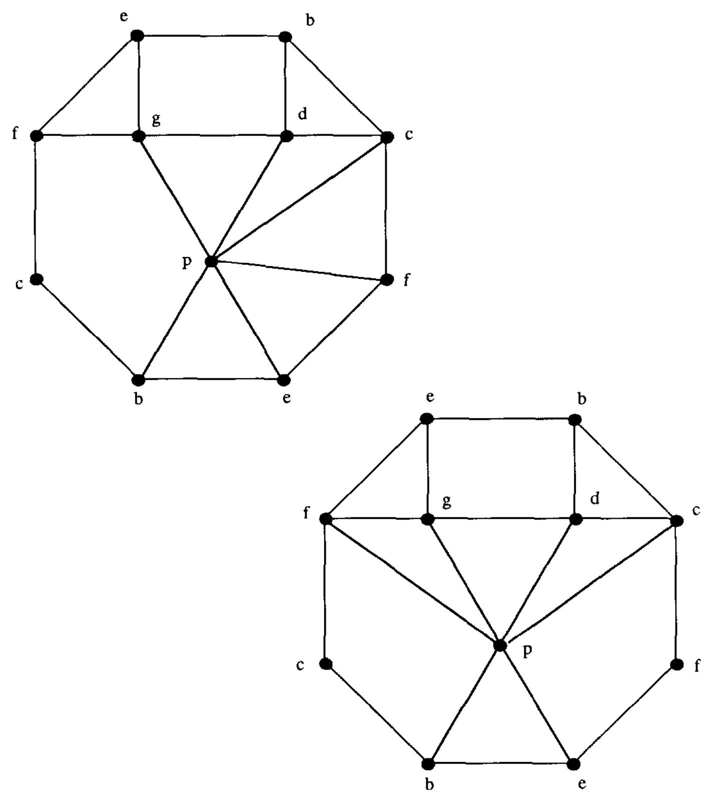

Fig. 7.

We have the following corollary to this theorem.

Corollary 4.9. *No pinched polyhedral map has a polyhedral Π-embedding.*

Theorem 4.7 and Corollary 4.9 combine to yield Theorem 4.10.

Theorem 4.10. *A Π-embedding of a pinched polyhedral map must be closed 2-cell, but cannot be polyhedral.*

This is in strong contrast to embeddings of pinched polyhedral maps into other surfaces. For example, we define $P_{m,n}$ to be the pinched polyhedral map consisting of n triangular faces containing the pinch vertex on either side, and $m-2$ annular bands of n rectangular faces composing the center strip, as shown in Fig. 8(a). The map $P_{2j,2k}$, $j,k \geqslant 2$, can be given a polyhedral embedding on an orientable surface by the following process: First remove half of the faces in a 'checkerboard' fashion (see Fig. 8(b)). Then let each of the $2k-1$ nonplanar circuits homotopic to the pinch

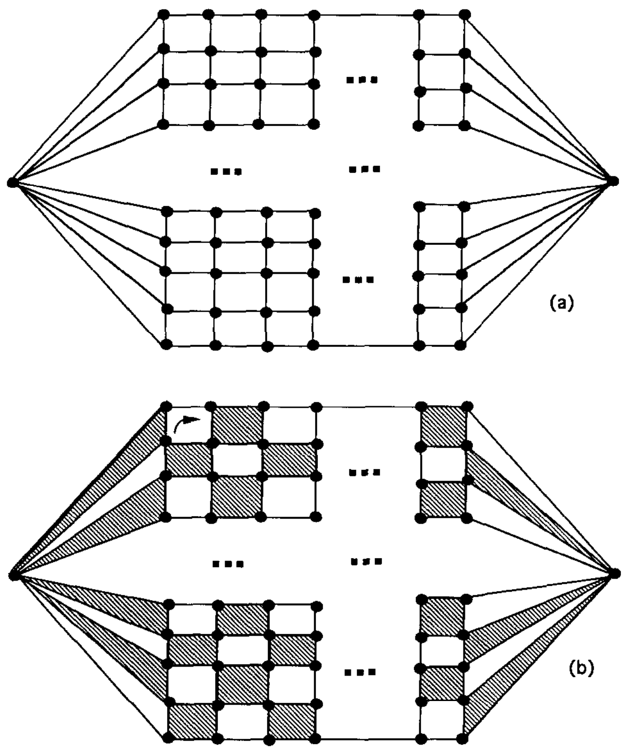

Fig. 8.

vertex, and each of the $2j$ nonplanar circuits through the pinch vertex bound cells. Each edge is thus in two regions, none of which meet improperly, so that the graph is polyhedrally embedded in a topological space. The fact that the graph is embedded in a surface can be seen from the fact that the link of each vertex is a cycle. The fact that the surface is orientable can be seen from the fact that an orientation on one of the square faces, as shown in Fig. 8(b), induces an orientation on the whole map. Finally, Euler's formula can be employed to show that, using this method, $P_{2j,2k}$ is embedded on the orientable surface of genus $(j-1)(k-1)$. Note that either Theorem 4.3 or Theorem 4.4 imply that none of $P_{2j,2k}$, $j, k \geqslant 2$, are projective planar.

Acknowledgment

I would like to thank the referee for a suggestion which greatly shortened my proof of Theorem 4.4, and for pointing out an error in my original statement of Lemma 4.6.

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