Filtering for A Class of Nonlinear Discrete-Time Stochastic Systems with State Delays

Zidong Wang, James Lam and Xiaohui Liu

Abstract

In this paper, the filtering problem is investigated for a class of nonlinear discrete-time stochastic systems with state delays. We aim at designing a full-order filter such that the dynamics of the estimation error is guaranteed to be stochastically, exponentially, ultimately bounded in the mean square, for all admissible nonlinearities and time-delays. First, an algebraic matrix inequality approach is developed to deal with the filter analysis problem, and sufficient conditions are derived for the existence of the desired filters. Then, based on the generalized inverse theory, the filter design problem is tackled and a set of the desired filters is explicitly characterized. A simulation example is provided to demonstrate the usefulness of the proposed design method.

Keywords

Filtering; Nonlinear systems; Stochastic systems; Time-delay; Algebraic matrix inequalities.

I. Introduction

One of the fundamental problems in control systems and signal processing is the estimation of the state variables of a dynamic system through available noisy measurements. For linear systems, there are two approaches available, namely, the Luenberger observer design in the deterministic framework and the Kalman filter design in the stochastic one.

Nonlinear filtering has been an active area of research over the past three decades. With respect to some recent representative work on this general topic in the deterministic case, we refer the reader to [5], [10], [11] and the references therein. For the stochastic case, the nonlinear filtering problem has received considerable attention, and a number of traditional approaches have been proposed in the literature, such as Gram-charlier expansion, Edgeworth expansion, extended Kalman filters, weighted sum of gaussian densities, generalized least-squares approximation and statistically linearized filters, see [7] for a survey. Among others, some later developments include the bound-optimal filters, exponentially bounded filters, exact finite dimensional filters, approximations by Markov chains, minimum variance filters, approximation of the Kushner equation, wavelet transform, etc. It is remarkable that, Tarn and Rasis [13] have tackled the nonlinear filtering problem through the concepts of observer for stochastic nonlinear systems, and have proposed an important stochastic stability approach to designing the observers with guaranteed convergence. In [4], the radial basis function neural networks have been exploited to approximate and estimate the nonlinear stochastic dynamics, and systematic procedures have been provided. Unlike the linear case, in most literature mentioned above, the solution to the nonlinear filtering problem has been given as a nonexplicit representation.

This work was supported in part by the EPSRC under Grant GR/S27658/01 and Grant GR/R35018/01, the Nuffield Foundation under Grant NAL/00630/G, the William M. W. Mong Engineering Research Fund of the University of Hong Kong, and the Alexander von Humboldt Foundation of Germany.

Z. Wang and X. Liu are with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom. Email: Zidong.Wang@brunel.ac.uk.

J. Lam is with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong.

On the other hand, the dynamic behavior of many industrial processes contains inherent time delays. Time delays may result from the distributed nature of the system, material transport, or from the time required to measure some of the variables. In the past few years, there has been rapidly growing interest in robust and/or H_∞ filtering for linear systems with certain types of time-delays, see [3] for a survey. In the stochastic framework, for example, the Kalman filter design problem has been tackled in [6], [20] for *linear* continuous- and discrete-time cases, respectively. In [19], the asymptotic stability problem for a general class of nonlinear stochastic time-delay systems has been thoroughly investigated. In [14], [15], [16], the filtering problems have been studied for some *continuous-time* nonlinear stochastic time-delay systems. It is well known that discrete-time systems play a very important role in digital signal analysis and processing. However, despite its importance, up to now, the filtering problem for general nonlinear *discrete* time-delay systems has not been fully investigated and remains open.

In this paper, we are concerned with the filtering problem for a class of nonlinear discrete time-delay stochastic systems. The system under study involves stochastic disturbances, time-delay and inherent nonlinearities. The nonlinearities are assumed to have the similar form as in [4], [15], [16]. We aim at designing a full-order filter such that the dynamics of the estimation error is constrained to be stochastically, exponentially, ultimately bounded in the mean square, for all admissible nonlinearities and time-delays. First, an algebraic matrix inequality approach is developed to deal with the filter analysis problem, and sufficient conditions are derived for the existence of the desired filters. Then, based on the generalized inverse theory, the filter design problem is tackled and a set of the desired filters is explicitly characterized. A simulation example is provided to demonstrate the usefulness of the proposed design method.

Notation. The notations in this paper are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices, and $\mathbb{Z}$ is the set of positive integers. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. $|\cdot|$ means the Euclidean norm in $\mathbb{R}^n$. If A is a real matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^T A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\mathcal{E}\{\cdot\}$ stand for the mathematical expectation operator with respect to the given probability measure P. The expected value of a random variable x is denoted by $\mathcal{E}\{x\}$ and the expected value of x conditional on y is represented by $\mathcal{E}\{x|y\}$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem formulation and assumptions

Let us consider the nonlinear discrete-time state delayed stochastic system described by

$$x(k+1) = f(x(k), u(k)) + g(x(k-d)) + E_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + E_2 w(k) \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the deterministic input, $y(k) \in \mathbb{R}^p$ is the measurement output, and $f(\cdot,\cdot) \in \mathbb{R}^n$ and $g(\cdot) \in \mathbb{R}^n$ are nonlinear vector functions. $d \in \mathbb{Z}$ is a positive integer time delay of the system state. We assume $x(k-d) = 0$ when $k - d < 0$, $k \in \mathbb{Z}$. Here, $w(k) \in \mathbb{R}^q$ is a zero mean Gaussian white noise sequence with $\mathcal{E}\{|w(k)|^2\} \leq \theta$ for some positive constant θ. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(k)$. E_1, E_2 are known constant matrices with appropriate dimensions.

Assumption 1: The nonlinear functions $f(\cdot,\cdot)$ and $g(\cdot)$ are assumed to satisfy $f(0,0)=0$, $g(0)=0$, and

$$\left| f(x(k)+\xi, u(k)+\delta) - f(x(k),u(k)) - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \xi \\ \delta \end{bmatrix} \right| \le a_1 \left| \begin{bmatrix} \xi \\ \delta \end{bmatrix} \right|, \tag{3}$$

$$|g(x(k-d)+\xi) - g(x(k-d) - A_d\xi| \le a_2|\xi|, \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $A_d \in \mathbb{R}^{n\times n}$ are known constant matrices, $\xi \in \mathbb{R}^n$, $\delta \in \mathbb{R}^m$ are vectors, a_1 and a_2 are known positive constants.

Remark 1: The nonlinear descriptions (3)-(4), which have been adopted in [13], [15], [16], quantify the maximum possible derivations from a linear model with (A, B, A_d).

The full-order filter considered in this paper is of the form

$$\hat{x}(k+1) = f(\hat{x}(k), u(k)) + g(\hat{x}(k-d)) + K[y(k) - C\hat{x}(k)] \tag{5}$$

where $\hat{x}$ is the state estimate and the constant matrix K is the filter gain to be designed.

Let the error state be $e(k) = x(k) - \hat{x}(k)$, then it follows from (1)-(2) and (5) that

$$\begin{aligned} e(k+1) \;=\; & f(x(k),u(k)) - f(\hat{x}(k),u(k)) + g(x(k-d)) - g(\hat{x}(k-d)) \\ & - KCe(k) + (E_1 - KE_2)w(k). \end{aligned} \tag{6}$$

For notational convenience, we define

$$l(k) \;=\; f(x(k),u(k)) - f(\hat{x}(k),u(k)) - Ae(k), \tag{7}$$

$$m(k-d) \;=\; g(x(k-d)) - g(\hat{x}(k-d)) - A_d e(k-d), \tag{8}$$

and then obtain from (6) that

$$e(k+1) = (A-KC)e(k) + A_d e(k-d) + l(k) + m(k-d) + (E_1 - KE_2)w(k). \tag{9}$$

Now, take the initial estimate of the state $x(0)$ to be equal to the known mean of the initial state $\bar{x}(0)$. Let $e(k)$ denote the state trajectory from the initial data $e(0)$. To this end, we introduce the following concept of exponential ultimate boundedness.

Definition 1: The dynamics of the estimation error $e(k)$ (i.e., the solution of the system (9)) is exponentially ultimately bounded in the mean square if there exist constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that

$$\mathcal{E}\left\{ |e(k)|^2 \middle| e(0) \right\} \le \alpha^k \beta + \gamma. \tag{10}$$

where $\alpha \in [0,1)$, $\beta > 0$ and $\gamma > 0$. In this case, the filter (5) is said to be exponential.

Remark 2: The exponential ultimate boundedness of the error dynamics means that, the estimation error will initially decrease exponentially in the mean square, and remain within a region in the steady state, again in the mean square sense. Such a region is defined in terms of the norm $(\mathcal{E}\{|e(k)|^2\})^{1/2}$ of the Hilbert space of random vectors, and is specified by the coefficient γ. In other words, the steady-state estimation error variance will be bounded.

The objective of this paper is to design an exponential filter for the nonlinear time-delay system (1)-(2). More specifically, we are interested in designing the filter parameter K such that the dynamics of the estimation error (i.e., the solution of the system (9)) is guaranteed to be stochastically exponentially ultimately bounded in the mean square.

III. Main results and proofs

In this section, we will consider both the filter analysis and filter design issues. For the filter analysis problem, given the filter structure, we will establish the conditions under which the estimation error is stochastically exponentially ultimately bounded in the mean square. For the filter design problem, we will try to derive the *explicit* expression of the expected filter parameter in terms of the positive definite solution to Riccati-like matrix inequalities.

A. Filter analysis

The following simple lemma will be used several times in the proof of our main results.

Lemma 1: [16] Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have $2x^Ty \leq \varepsilon x^Tx + \varepsilon^{-1}y^Ty$.

The following theorem provides sufficient conditions for the error dynamics of system (9) to be stochastically exponentially ultimately bounded in the mean square.

Theorem 1: Let the filter parameter K be given. If there exist positive scalars ε_1, ε_2, ε_3 and a positive definite matrix R such that the following matrix equation

$$\begin{aligned}(1+\varepsilon_1+\varepsilon_2)(A-KC)^TP(A-KC)-P+(1+\varepsilon_1^{-1}+\varepsilon_3)A_d^TPA_d \\ +2(a_1^2+a_2^2)(1+\varepsilon_2^{-1}+\varepsilon_3^{-1})\lambda_{\max}(P)I+R=0\end{aligned} \tag{11}$$

has a positive definite solution P, then the system (9) is exponentially ultimately bounded in the mean square.

Proof: Let

$$\Theta_k := \left[\begin{array}{cccc} e(k-d) & e(k-d-1) & \cdots & e(k) \end{array}\right].$$

Define a Lyapunov functional candidate for system (9) as

$$V_k(\Theta_k) = e^T(k)Pe(k) + \sum_{i=k-d}^{k-1} e^T(i)Qe(i), \tag{12}$$

where $P > 0$ is the solution to (11), and

$$Q = (1+\varepsilon_1^{-1}+\varepsilon_3)A_d^TPA_d + 2a_2^2(1+\varepsilon_2^{-1}+\varepsilon_3^{-1})\lambda_{\max}(P)I > 0. \tag{13}$$

Then, one has from (9) that

$$\begin{aligned}\Delta V_k &:= \mathcal{E}\{V_{k+1}(\Theta_{k+1})|\Theta_k\} - V_k(\Theta_k) \\ &= \mathcal{E}\{e^T(k+1)Pe(k+1)\} + e^T(k)(Q-P)e(k) - e^T(k-d)Qe(k-d) \\ &= e^T(k)[(A-KC)^TP(A-KC)-P+Q]e(k) + 2e^T(k)(A-KC)^TPA_de(k-d) \\ &\quad +2e^T(k)(A-KC)^TP[l(k)+m(k-d)] + e^T(k-d)A_d^TPA_de(k-d) \\ &\quad +2e^T(k-d)A_d^TP[l(k)+m(k-d)] + [l(k)+m(k-d)]^TP[l(k)+m(k-d)] \\ &\quad +\mathcal{E}\{w^T(k)(E_1-KE_2)^T(E_1-KE_2)w(k)\} - e^T(k-d)Qe(k-d).\end{aligned} \tag{14}$$

Next, it follows from Lemma 1 that

$$2e^T(k)(A-KC)^T PA_d e(k-d) \\ \leq \varepsilon_1 e^T(k)(A-KC)^T P(A-KC)e(k) + \varepsilon_1^{-1} e^T(k-d)A_d^T PA_d e(k-d), \tag{15}$$

$$2e^T(k)(A-KC)^T P[l(k)+m(k-d)] \\ \leq \varepsilon_2 e^T(k)(A-KC)^T P(A-KC)e(k) + \varepsilon_2^{-1}[l(k)+m(k-d)]^T P[l(k)+m(k-d)], \tag{16}$$

$$2e^T(k-d)A_d^T P[l(k)+m(k-d)] \\ \leq \varepsilon_3 e^T(k-d)A_d^T PA_d e(k-d) + \varepsilon_3^{-1}[l(k)+m(k-d)]^T P[l(k)+m(k-d)]. \tag{17}$$

Furthermore, noticing the Assumption 1 and the definitions (7)-(8), we have

$$l^T(k)l(k) = |f(x(k),u(k)) - f(\hat{x}(k),u(k)) - Ae(k)|^2 \leq a_1^2|e(k)|^2 = a_1^2 e^T(k)e(k), \tag{18}$$

$$\begin{aligned} m^T(k-d)m(k-d) &= |g(x(k-d)) - g(\hat{x}(k-d)) - A_d e(k-d)|^2 \\ &\leq a_2^2|e(k-d))|^2 = a_2^2 e^T(k-d)e(k-d), \end{aligned} \tag{19}$$

and hence it follows again from Lemma 1 that

$$\begin{aligned} &[l(k)+m(k-d)]^T P[l(k)+m(k-d)] \\ &\quad\leq \lambda_{\max}(P)[l(k)+m(k-d)]^T[l(k)+m(k-d)] \\ &\quad\leq \lambda_{\max}(P)[2l^T(k)l(k) + 2m^T(k-d)m(k-d)] \\ &\quad\leq \lambda_{\max}(P)[2a_1^2 e^T(k)e(k) + 2a_2^2 e^T(k-d)e(k-d)]. \end{aligned} \tag{20}$$

For simplicity, we denote

$$\begin{aligned} \Pi := (1+\varepsilon_1+\varepsilon_2)(A-KC)^T P(A-KC) - P + (1+\varepsilon_1^{-1}+\varepsilon_3)A_d^T PA_d \\ +2(a_1^2+a_2^2)(1+\varepsilon_2^{-1}+\varepsilon_3^{-1})\lambda_{\max}(P)I, \end{aligned} \tag{21}$$

and then (11) indicates that $\Pi = -R < 0$.

Since $\mathcal{E}\{|w(k)|^2\} \leq \theta$, it can be easily seen that

$$\mathcal{E}\{w^T(k)(E_1-KE_2)^T(E_1-KE_2)w(k)\} \leq \theta\lambda_{\max}\{(E_1-KE_2)^T(E_1-KE_2)\} := \phi \tag{22}$$

Considering the definition of Q in (13), the relationships (18)-(22), after tedious algebraic manipulation, we obtain from (14) that

$$\Delta V(k) \leq e^T(k)\Pi e(k) + \phi \leq -\lambda_{\min}(R)|e(k)|^2 + \phi, \tag{23}$$

where the matrix $\Pi < 0$ and the scalar $\phi \geq 0$ are defined in (21) and (22), respectively.

Based on (23), the exponentially ultimate boundedness behavior of the estimation error can be proven by following the same line as in [18]. In order to make the presentation concise, we give the details in Appendix. This completes the proof of this theorem. ∎

B. Filter design

The purpose of this subsection is to give an explicit expression of the set of desired filters. By means of Theorem 1, we shall deal with the following two problems: a) find the existence conditions for the positive definite matrix P under which there exists a filter gain K satisfying (11), and b) derive the characterization of expected filter gains. A lemma given below will be needed in the development of the design procedure.

Lemma 2: [9] Let $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{m\times p}$ $(m \leq p)$. There exists a matrix V that satisfies simultaneously $Y = XV$, $VV^T = I$ if and only if $XX^T = YY^T$. In this case, a general solution for V can be expressed as

$$V = V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T, \quad U \in \mathbb{R}^{(n-r_X)\times(p-r_X)}, \quad UU^T = I \tag{24}$$

where V_X and V_Y come from the singular value decomposition of X and Y, respectively,

$$X = U_X \begin{bmatrix} Z_X & 0 \\ 0 & 0 \end{bmatrix} V_X^T = \begin{bmatrix} U_{X1} & U_{X2} \end{bmatrix} \begin{bmatrix} Z_X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{X1}^T \\ V_{X2}^T \end{bmatrix} \tag{25}$$

$$Y = U_Y \begin{bmatrix} Z_Y & 0 \\ 0 & 0 \end{bmatrix} V_Y^T = \begin{bmatrix} U_{Y1} & U_{Y2} \end{bmatrix} \begin{bmatrix} Z_Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{Y1}^T \\ V_{Y2}^T \end{bmatrix} \tag{26}$$

and $r_X = \text{rank}(X)$, $U_X = U_Y$, $Z_X = Z_Y$.

For presentation convenience, we define

$$\Omega := (1+\varepsilon_1+\varepsilon_2)^{-1}\big[P - (1+\varepsilon_1^{-1}+\varepsilon_3)A_d^T P A_d - 2(a_1^2+a_2^2)(1+\varepsilon_2^{-1}+\varepsilon_3^{-1})\lambda_{\max}(P)I - R\big]. \tag{27}$$

Suppose the conditions of Theorem 1 are satisfied. Hence, we have

$$(A-KC)^T P(A-KC) = \Omega \tag{28}$$

Since the left-hand side of (28) is non-negative definite, Ω is required to satisfy

$$\Omega \geq 0 \tag{29}$$

Now, assume that (29) is true and let $\Omega^{1/2}$ be the square root of Ω. Then equation (11) can be rewritten as

$$[(A-KC)^T P^{1/2}][(A-KC)^T P^{1/2}]^T = (\Omega^{1/2})(\Omega^{1/2})^T. \tag{30}$$

It follows from Lemma 2 that, (30) holds if and only if there exists an orthogonal matrix V $(V \in \mathbb{R}^{n\times n})$ satisfying $(A-KC)^T P^{1/2} = \Omega^{1/2}V$, or

$$C^T K^T = A^T - \Omega^{1/2} V P^{-1/2}. \tag{31}$$

It is easily seen from [1] that, there exists an orthogonal matrix V such that (31) has a solution for K, if and only if there exists an orthogonal matrix V such that

$$[I - C^T(C^T)^+](A^T - \Omega^{1/2} V P^{-1/2}) = 0, \tag{32}$$

where $(C^T)^+$ denotes the Moore-Penrose inverse of C^T.

By denoting

$$X := [I - C^T(C^T)^+]\Omega^{1/2}, \tag{33}$$

$$Y := [I - C^T(C^T)^+]A^T P^{1/2}, \tag{34}$$

we can rearrange (32) as $XV = Y$, and from Lemma 2, $XV = Y$ holds if and only if $XX^T = YY^T$, which can be expressed as

$$[I - C^T(C^T)^+](\Omega - A^T P A)[I - C^T(C^T)^+] = 0. \tag{35}$$

So far, it should be clear that, there exists a filter gain matrix K such that (11) holds if and only if there exist positive scalars ε_1, ε_2, ε_3 and a positive definite matrix R such that (29) and (35) have a positive definite solution $P > 0$ ($P > R$). In other words, (29) and (35) serve as the existence conditions for a filter gain matrix to satisfy (11).

Assume that (29) and (35) hold. The rest we need to do now is to derive the explicit expression of the desired filter gains. It follows from [1] that a general solution to (31) is given by

$$K = \{(C^T)^+[A^T - \Omega^{1/2}VP^{-1/2}] + [I - (C^T)^+C^T]Z\}^T, \tag{36}$$

where $Z \in \mathbb{R}^{p\times n}$ is arbitrary, V is any orthogonal matrix satisfying $XV = Y$ and can be expressed, again by Lemma 2, as

$$V = V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T, \quad U \in \mathbb{R}^{(n-r_X)\times(p-r_X)}, \tag{37}$$

where X and Y are defined in (33) and (34), respectively, the matrix U is arbitrary orthogonal, and $r_X = \text{rank}(X)$.

Finally, by means of Theorem 1 and the above derivation, the characterization of the desired filter gains is given as follows.

Theorem 2: If there exist positive scalars ε_1, ε_2, ε_3 and a positive definite matrix R such that (29) and (35) have a positive definite solution $P > 0$ ($P > R$), then with the filter gain given by

$$K = \{(C^T)^+[A^T - \Omega^{1/2}V_X \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_Y^T P^{-1/2}] + [I - (C^T)^+C^T]Z\}^T, \tag{38}$$

where X, Y, U and r_X are defined previously, the system (9) is exponentially ultimately bounded in the mean square.

Remark 3: In practical applications, it is very desirable to solve directly the matrix inequality (29) subject to the constraint (35), and then obtain the expected filter parameters readily from (38). First, the positive scalars ε_1, ε_2, ε_3 can be determined by using the optimization approach proposed in [17] and the references therein, in order to reduce the possible conservatism that may result from the inequalities (15)-(17). Then, when we deal with the matrix inequality (29) subject to the constraints (35), the local numerical searching algorithms suggested by [2] and [8] are very effective for a relatively low-order model. A related discussion of the solving algorithms for matrix inequalities can be found in [12]. We also mention that there is a considerable freedom in the filter design, such the choices of the matrices Z and U in (38), which could be further used to improve other filtering properties.

Remark 4: We point out that the main results can be easily extended to the multiple state delayed case. Also, it is not difficult to obtain parallel results for the case where there are bounded nonlinearities and uncertain disturbances. That is, $f(\cdot,\cdot)$ and $g(\cdot)$ satisfy $f(0,0) = 0$, $g(0) = 0$, and

$$\left| f(x(k)+\sigma, u(k)+\delta) - f(x(k), u(k)) - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \sigma \\ \delta \end{bmatrix} \right| \le a_1 \left| \begin{bmatrix} \sigma \\ \delta \end{bmatrix} \right| + b_1, \tag{39}$$

$$|g(x(k-\tau)+\sigma) - g(x(k-\tau)) - A_d\sigma| \le a_2|\sigma| + b_2, \tag{40}$$

where the new parameters $b_1 > 0$ and $b_2 > 0$ account for the possible uncertain disturbances. The reason why we discuss the relatively simple system (1)-(2) associated with (3)-(4) is just to make our theory more understandable and to avoid unnecessarily complicated notations.

IV. Numerical simulation

In this section, a simple simulation example is presented to illustrate the usefulness of the proposed filter design method.

Let the nonlinear discrete-time stochastic state delayed system be given by

$$\begin{aligned} x_1(k+1) &= 0.2x_1(k) - 0.01x_2(k) + 0.1\sin x_1(k) \\ &\quad + 0.1x_1(k-1) + 0.2\cos(x_2(k-1)) + 0.2w(t), \\ x_2(k+1) &= 0.01x_1(k) + 0.2x_2(k) + 0.1\sin x_2(k) \\ &\quad + 0.1x_2(k-1) - 0.2\cos(x_1(k-1) - x_2(k-1)) + 0.2w(t), \\ y_1(t) &= x_1(t) + 0.1w(t). \end{aligned}$$

Considering the system (1)-(2) with the constraints (3)-(4), we can obtain that

$$A = \begin{bmatrix} 0.2 & -0.01 \\ 0.01 & 0.2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B = 0, \quad E_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad d = 1, \quad a_1 = 0.12, \quad a_2 = 0.25.$$

We aim at designing an exponential filter for the nonlinear time-delay system (1)-(2), such that the dynamics of the estimation error is stochastically exponentially ultimately bounded in the mean square.

Firstly, by using the method discussed in the previous section, we may choose the appropriate parameters ε_1, ε_2, ε_3, and obtain P as follows:

$$\varepsilon_1 = 4.8, \quad \varepsilon_2 = 8.2, \quad \varepsilon_3 = 0.7, \quad P = \begin{bmatrix} 3.0650 & 0 \\ 0 & 3.0650 \end{bmatrix}.$$

Let the positive definite matrix R be of the form $R = [r_{ij}]_{2\times 2}$ $(i, j = 1, 2)$. The condition (35) implies that $r_{22} = 0.0835$. Then, based on the constraint (29), we can select other elements of R as $r_{11} = 0.1$, $r_{12} = r_{21} = 0$, and hence obtain the matrices Ω, V_X, V_Y as the following:

$$\Omega = \begin{bmatrix} 0.1229 & 0 \\ 0 & 0.1229 \end{bmatrix}, \quad V_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_Y = \begin{bmatrix} 0.0499 & -0.9988 \\ -0.9988 & -0.0499 \end{bmatrix}.$$

Note that in the expression (38), the choice of the arbitrary matrix Z does not affect the solution since $I - (C^T)^+ C^T = 0$. Therefore, letting $U = 1$ (38) leads to the following desired filter gain:

$$K_1 = \begin{bmatrix} 0.4000 \\ 0.0200 \end{bmatrix}.$$

The responses of error dynamics to initial conditions are shown in Fig. 1 and Fig. 2, which demonstrate that the estimation error is exponentially ultimately bounded in the mean square.

V. Conclusions

In this paper we have considered the filter design problem for a class of nonlinear stochastic discrete time-delay systems. We have investigated both the filter analysis and design issues. The existence conditions as well as the analytical parameterization of desired filters are derived. The method relies not on the optimization

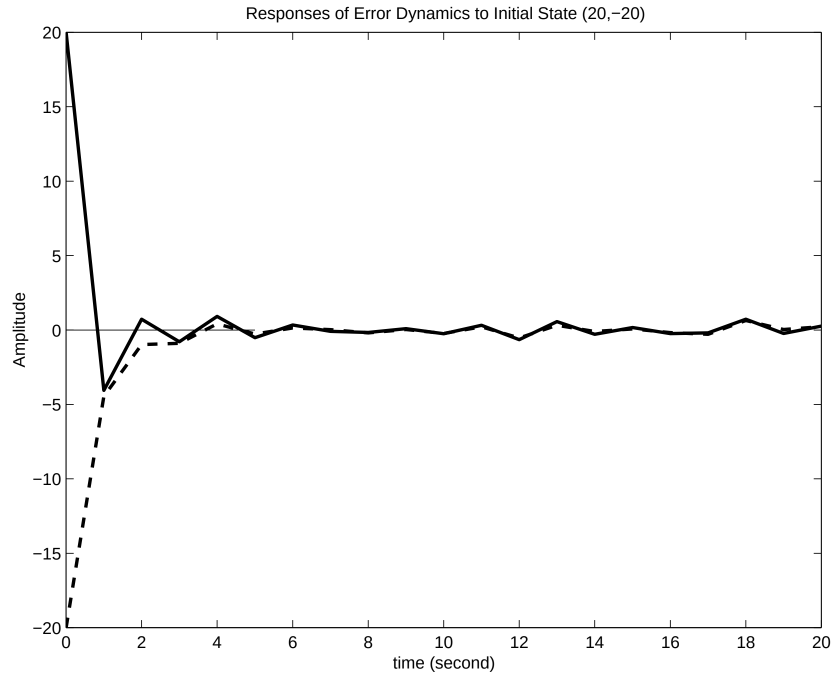

Fig. 1. x_1 (solid), x_2 (dashed).

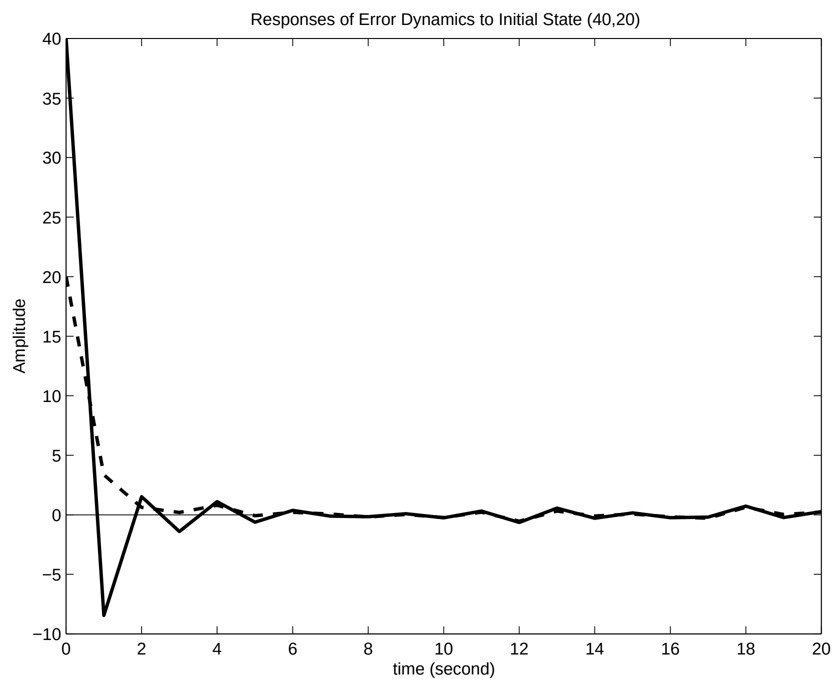

Fig. 2. x_1 (solid), x_2 (dashed).

theory but on Lyapunov type stochastic stability results that can guarantee a mean square exponential rate of convergence for the estimation error. It has been pointed out that, the desired exponential filters for this class of nonlinear discrete time-delay systems, when they exist, are usually a large set, and the remaining freedom can be used to meet other expected performance requirements. The results of this paper have been demonstrated by a numerical simulation example. Finally, we may generalize our results to more complex systems such as sampled-data systems and stochastic parameter systems, which gives us future research topics.

VI. Appendix

Proof of the exponentially ultimate boundedness in Theorem 1.

In this appendix, based on the relation (23), we proceed to prove the exponentially ultimate boundedness behavior of the estimation error dynamics for the system (9).

If $e(k)=0$ for some finite k, then it is straightforward to show that the system (9) is stochastically bounded in the mean square. We now assume that $e(k)\neq 0$. Taking the expectation of both sides of (23) and using the definition of ΔV_k, we have

$$\mathcal{E}\{V_{k+1}(\Theta_{k+1})\}-\mathcal{E}\{V_k(\Theta_k)\}\leq -\lambda_{\min}(R)\mathcal{E}\{|e(k)|^2\}+\phi. \tag{41}$$

It follows readily from (12) that

$$V_k(\Theta_k)\leq \lambda_{\max}(P)|e(k)|^2+\lambda_{\max}(Q)\sum_{i=k-d}^{k-1}|e(i)|^2, \tag{42}$$

which, together with (41), shows that for any scalar $\mu>1$,

$$\begin{aligned}
&\mathcal{E}\{\mu^{k+1}V_{k+1}(\Theta_{k+1})\}-\mathcal{E}\{\mu^k V_k(\Theta_k)\}\\
=\;& \mu^{k+1}\Big[\mathcal{E}\{V_{k+1}(\Theta_{k+1})\}-\mathcal{E}\{V_k(\Theta_k)\}\Big]+\mu^k(\mu-1)\mathcal{E}\{V_k(\Theta_k)\}\\
\leq\;& \mu^k\Big[-\mu\lambda_{\min}(R)+(\mu-1)\lambda_{\max}(P)\Big]\mathcal{E}\{|e(k)|^2\}\\
&+\mu^k(\mu-1)\lambda_{\max}(Q)\sum_{i=k-d}^{k-1}\mathcal{E}\{|e(i)|^2\}+\mu^{k+1}\phi.
\end{aligned} \tag{43}$$

For any integer $T\geq d+1$, summing up both sides of (43) from 0 to T with respect to k, we have

$$\begin{aligned}
&\mathcal{E}\{\mu^T V_T(\Theta_T)\}-\mathcal{E}\{V_0(\Theta_0)\}\\
\leq\;& a(\mu)\sum_{k=0}^{T-1}\mu^k\mathcal{E}\{|e(k)|^2\}+b(\mu)\sum_{k=0}^{T-1}\sum_{i=k-d}^{k-1}\mu^k\mathcal{E}\{|e(i)|^2\}+\frac{\mu(\mu^{T+1}-1)}{\mu-1}\phi,
\end{aligned} \tag{44}$$

where

$$a(\mu)=-\mu\lambda_{\min}(R)+(\mu-1)\lambda_{\max}(P),\;\; b(\mu)=(\mu-1)\lambda_{\max}(Q). \tag{45}$$

Note that for $d\geq 1$,

$$\begin{aligned}
&\sum_{k=0}^{T-1}\sum_{i=k-d}^{k-1}\mu^k\mathcal{E}\{|e(i)|^2\}\\
\leq\;& \Big(\sum_{i=-d}^{-1}\sum_{k=0}^{i+d}+\sum_{i=0}^{T-d-1}\sum_{k=i+1}^{i+d}+\sum_{i=T-d}^{T-1}\sum_{k=i+1}^{T-1}\Big)\mu^k\mathcal{E}\{|e(i)|^2\}\\
\leq\;& \frac{\mu^d-1}{\mu-1}\sum_{i=-d}^{-1}\mathcal{E}\{|e(i)|^2\}+\frac{\mu(\mu^d-1)}{\mu-1}\sum_{i=0}^{T-1}\mu^i\mathcal{E}\{|e(i)|^2\}+\frac{\mu(\mu^{d-1}-1)}{\mu-1}\sum_{i=0}^{T-1}\mu^i\mathcal{E}\{|e(i)|^2\}.
\end{aligned} \tag{46}$$

Then, it follows from (44) and (46) that

$$\begin{aligned}
&\mathcal{E}\{\mu^T V_T(\Theta_T)\}-\mathcal{E}\{V_0(\Theta_0)\}\\
\leq\;& \frac{b(\mu)(\mu^d-1)d}{\mu-1}\sup_{-d\leq i\leq 0}\mathcal{E}\{|e(i)|^2\}+\zeta(\mu)\sum_{k=0}^{T-1}\mu^k\mathcal{E}\{|e(k)|^2\}+\frac{\mu(\mu^{T+1}-1)}{\mu-1}\phi,
\end{aligned} \tag{47}$$

where

$$\zeta(\mu) = a(\mu) + \frac{2\mu b(\mu)(\mu^d - 1)}{\mu - 1}.$$

Since $\zeta(1) = -\lambda_{\min}(R) < 0$ and $\lim_{\mu\to+\infty}\zeta(\mu) = +\infty$, there exists a scalar $\mu_0 > 1$ such that $\zeta(\mu_0) = 0$. Therefore, we can obtain from (47) that, for any integer $T \geq d+1$,

$$\mathcal{E}\{\mu_0^T V_T(\Theta_T)\} - \mathcal{E}\{V_0(\Theta_0)\} \leq d\lambda_{\max}(Q)(\mu_0^d - 1) \sup_{-d\leq i\leq 0} \mathcal{E}\{|e(i)|^2\} + \frac{\mu_0(\mu_0^{T+1} - 1)}{\mu_0 - 1}\phi, \tag{48}$$

and subsequently,

$$\mathcal{E}\{\mu_0^T V_T(\Theta_T)\} \leq \Big[d\lambda_{\max}(Q)(\mu_0^d - 1) + d\max(\lambda_{\max}(P), \lambda_{\max}(Q))\Big] \sup_{-d\leq i\leq 0} \mathcal{E}\{|e(i)|^2\} + \frac{\mu_0(\mu_0^{T+1} - 1)}{\mu_0 - 1}\phi. \tag{49}$$

Finally, it follows easily from Definition 1 that the error dynamical system (9) is exponentially ultimately bounded in the mean square. The proof is now complete.

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized inverses: theory and applications*, New York: Wiley, 1974.

[2] E. Beran and K. Grigoriadis, A combined alternating projections and semidefinite programming algorithm for low-order control design, in: *Prep. 13th IFAC World Congress*, San Francisco, CA, USA, vol. C, pp. 85-90, 1996.

[3] E. K. Boukas and Z. -K. Liu, *Deterministic and stochastic time-delay systems*, Birkhauser, Boston, 2002.

[4] V. T. S. Elanayar and Y. C. Shin, Radial basis function neural network for approximation and estimation of nonlinear stochastic dynamic systems, *IEEE Trans. Neural Networks*, vol. 5, pp. 594-603, Jul. 1994.

[5] W. H. Fleming and W. M. McEneaney, A max-plus-based algorithm for a Hamilton–Jacobi–Bellman equation of nonlinear filtering, *SIAM J. Control Optimization*, vol. 38, pp. 683-710, 2000.

[6] E. Fridman, U. Shaked and L. Xie, Robust H_2 filtering of linear systems with time delays, *Int. J. Robust & Nonlinear Control*, vol. 13, pp. 983-1010, 2003.

[7] A. Gelb, *Applied optimal estimation*, (Cambridge: Cambridge University Press, 1974)

[8] J. C. Geromel, P. L. D. Peres and S. R. Souza, Output feedback stabilization of uncertain systems through a min/max problem, in: *Prep. 12th IFAC World Congress*, Sydney, Australia, vol. 6, pp. 35-38, 1993.

[9] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Int. J. Control*, vol. 39, pp. 1115-1193, 1984.

[10] S. K. Nguang and P. Shi, Nonlinear H_∞ filtering of sampled-data systems, *Automatica*, vol. 36, pp. 303-310, 2000.

[11] S. K. Nguang and P. Shi, H_∞ filtering design for uncertain nonlinear systems under sampled measurements, *Int. J. Systems Sciences*, vol. 32, pp. 889-898, 2001.

[12] A. Saberi, P. Sannuti and B. M. Chen, *H_2 optimal control*, (Prentice Hall International, London, 1995)

[13] T. -J. Tarn and Y. Rasis, Observers for nonlinear stochastic systems, *IEEE Trans. Automat. Control*, vol. 21, pp. 441-448, Aug. 1976.

[14] Z. Wang and K. J. Burnham, Robust filtering for a class of stochastic uncertain nonlinear time-delay systems via exponential state estimation, *IEEE Trans. Signal Processing*, vol. 49, pp. 794-804, Apr. 2001.

[15] Z. Wang and D. W. C. Ho, Filtering on nonlinear time-delay stochastic systems, *Automatica*, vol. 39, pp. 101-109, 2003.

[16] Z. Wang, J. Lam and X. Liu, Nonlinear filtering for state delayed systems with Markovian switching, *IEEE Trans. Signal Processing*, vol. 51, pp. 2321-2328, Sept. 2003.

[17] L. Xie, Y. C. Soh and C. E. de Souza, Robust Kalman filtering for uncertain discrete-time systems, *IEEE Trans. Automat. Control*, vol. 39, pp. 1310-1314, Jun. 1994.

[18] S. Xu and J. Lam, Robust output feedback stabilization of uncertain discrete time-delay stochastic systems with multiplicative noise, *Dynamics of Continuous, Discrete and Impulsive Systems*, accepted for publication, 2004.

[19] C. Yuan and X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching, *Automatica*, vol. 40, no. 3, pp. 343-354, 2004.

[20] X. Zhu, Y. C. Soh and L. Xie, Robust Kalman filters design for discrete time-delay systems, *Circuits, Systems, and Signal Processing*, vol. 21, no. 3, pp. 319-335, 2002.