Robust H_∞ Control for A Class of Nonlinear Stochastic Systems with Mixed Time-Delay

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Abstract

This paper is concerned with the problem of robust H_∞ control for a class of uncertain nonlinear Itô-type stochastic systems with mixed time delays. The parameter uncertainties are assumed to be norm-bounded, the mixed time delays comprise both the discrete and distributed delays, and the the sector nonlinearities appear in both the system states and delayed states. The problem addressed is the design of a linear state feedback controller such that, in the simultaneous presence of parameter uncertainties, system nonlinearities and mixed time-delays, the resulting closed-loop system is asymptotically stable in the mean square and also achieves a prescribed H_∞ disturbance rejection attenuation level. By using the Lyapunov stability theory and the Itô differential role, some new techniques are developed to derive the sufficient conditions guaranteeing the existence of the desired feedback controllers. A unified linear matrix inequality (LMI) is proposed to deal with the problem under consideration and a numerical example is exploited to show the usefulness of the results obtained.

Keywords

Itô stochastic system; H_∞ control; Mixed time delays; Lyapunov-Krasovskii functional; Linear matrix inequality.

I. Introduction

Nonlinear systems and stochastic systems are arguably two of the most important kinds of complex systems that have had successful applications in control and communication problems, such as attitude control of satellites and missile control, macroeconomic system control, chemical process control, etc. In the past years, control of nonlinear stochastic systems has been a topic of recurring interest in the past years, and a great number of results on this subject have been reported in the literature; see, for example, [16] for a survey.

It is noticed that, recently, a variety of nonlinear stochastic systems have received renewed research interests. For example, in [3], a minimax dynamic game approach has been developed for the controller design problem of the nonlinear stochastic systems that employ risk-sensitive performance criteria. The stabilization problem has been investigated in [4, 5] for nonlinear stochastic systems, and a stochastic counterpart of the input-to-state stabilization results has been provided. In [15], under an infinite-horizon risk-sensitive cost criterion, the problem of output feedback control design has been studied for a class of strict feedback stochastic nonlinear systems. In [25], the decentralized global stabilization problem has been dealt with by using a Lyapunov-based recursive design method. Most recently, in [2], an H_∞-type theory has been developed for a large class of discrete-time nonlinear stochastic systems. It is worth mentioning that, among different descriptions of the nonlinearities, the so-called *sector nonlinearity* [10] has gained much attention for *deterministic* systems, and both the control analysis and model reduction problems have been investigated, see [9, 13, 14].

On the other hand, since time delays are encountered in various physical and engineering systems and often result in instability and performance degradation, increasing attention has recently been focused on robust and/or H_∞ control problems for linear systems with certain types of time-delays, see [1] for a survey. Within

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the stochastic framework, the stability analysis problem for linear time-delay systems has been studied by many authors. For example, in [23], the stability analysis problem for uncertain stochastic fuzzy systems with time-delays has been considered. In [26], an LMI approach has been developed to cope with the robust H_∞ control problem for linear uncertain stochastic systems with state delay. In [17], the robust integral sliding mode control problem has been studied for uncertain stochastic systems with time-varying delays, and the H_∞ performance has been analyzed in [7] for continuous-time stochastic systems with polytopic uncertainties. In [8], the robust $L_2 - L_\infty$ filtering problem has been thoroughly studied for uncertain stochastic time-delay systems, and the filter design has been elegantly cast into a convex optimization problem with little conservatism. As for nonlinear stochastic time-delay systems, the related results have been scattered, and most of the results have been concerned with the stability analysis issue with only discrete time-delays, see e.g. [6, 21, 22]. So far, the robust H_∞ control problem for nonlinear stochastic systems with mixed time-delays has not been fully investigated and remains important.

In this paper, we will consider the robust H_∞ control problem for a class of uncertain continuous-time Itô-type stochastic systems involving sector nonlinearities and mixed time delays. The parameter uncertainties are assumed to be norm-bounded, the mixed time delays comprise both the discrete and distributed delays, and the sector nonlinearities appear in the system states and all delayed states. An effective linear matrix inequality (LMI) approach is proposed to design the state feedback controllers such that, for all admissible nonlinearities and time-delays, the overall uncertain closed-loop system is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We first investigate the sufficient conditions for the uncertain nonlinear stochastic time-delay systems to be stable in the mean square, and then derive the explicit expression of the desired controller gains. A numerical example is provided to show the usefulness and effectiveness of the proposed design method.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^T A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. The asterisk $\star$ in a matrix is used to denote term that is induced by symmetry. Matrices, if not explicitly specified, are assumed to have compatible dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation

Consider, on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, the following uncertain nonlinear Itô stochastic system with time delays of the form:

$$\begin{aligned}(\Sigma) : dx(t) &= [\mathcal{F}(x(t), x(t-\tau(t)), t) + B_1(t)u(t) + D_1(t)v(t)]dt \\ &\quad + [\mathcal{G}(x(t), x(t-\tau(t)), t) + B_2(t)u(t) + D_2(t)v(t)]dw(t), \qquad (1)\\ y(t) &= Cx(t) + Bu(t), \qquad (2)\\ x(t) &= \phi(t), \quad t \in [-\bar{\tau}, 0], \qquad (3)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input; $y(t) \in \mathbb{R}^q$ is the controlled output; C and B are known real constant matrices; and $w(t)$ is a zero-mean scalar Wiener process (Brownian motion) on

$(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\mathbb{E}[w(t)] = 0, \quad \mathbb{E}[w^2(t)] = t.$$

Also, for the exogenous disturbance signal $v(t) \in \mathbb{R}^p$, it is assumed that $v(\cdot) \in \mathcal{L}_{\mathcal{E}2}([0,\infty);\mathbb{R}^p)$, where $\mathcal{L}_{\mathcal{E}2}([0,\infty);\mathbb{R}^p)$ is the space of non-anticipatory square integrable stochastic process $f(\cdot) = (f(t))_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ with the following norm:

$$\|f\|_{\mathcal{E}2} = \left\{\mathbb{E}\int_0^{+\infty} |f(t)|^2 dt\right\}^{1/2} = \left\{\int_0^{+\infty} \mathbb{E}|f(t)|^2 dt\right\}^{1/2}.$$

Furthermore, suppose that $\mathcal{F}(\cdot,\cdot,\cdot)$ and $\mathcal{G}(\cdot,\cdot,\cdot)$ are nonlinear vector functions that can be decomposed as follows:

$$\begin{aligned}\mathcal{F}(x(t), x(t-\tau(t)), t) = {} & A(t)x(t) + f(x(t)) + A_{d_1}(t)x(t-\tau(t)) + f_{d_1}(x(t-\tau(t))) \\ & + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds, \end{aligned} \tag{4}$$

$$\mathcal{G}(x(t), x(t-\tau(t)), t) = G(t)x(t) + G_{d_1}(t)x(t-\tau(t)) + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds \tag{5}$$

with

$$A(t) = A + \Delta A(t), \quad A_{d_1}(t) = A_{d_1} + \Delta A_{d_1}(t), \quad A_{d_2}(t) = A_{d_2} + \Delta A_{d_2}(t), \tag{6}$$

$$G(t) = G + \Delta G(t), \quad G_{d_1}(t) = G_{d_1} + \Delta G_{d_1}(t), \quad G_{d_2}(t) = G_{d_2} + \Delta G_{d_2}(t). \tag{7}$$

Also, the matrices $B_1(\cdot), B_2(\cdot), D_1(\cdot)$ and $D_2(\cdot)$ satisfy

$$B_1(t) = B_1 + \Delta B_1(t),\ B_2(t) = B_2 + \Delta B_2(t),\ D_1(t) = D_1 + \Delta D_1(t),\ D_2(t) = D_2 + \Delta D_2(t). \tag{8}$$

Here, the scalar $\tau(t) \geq 0$ represents the time-varying discrete time delays satisfying $\dot{\tau} \leq h < 1$ (h is a fixed constant), while τ_0 describes the size of the distributed time delays. $A, A_{d_1}, A_{d_2}, G, G_{d_1}, G_{d_2}, B_1, B_2, D_1$ and D_2 are known real constant matrices, and $\Delta A(t), \Delta A_{d_1}(t), \Delta A_{d_2}(t), \Delta G(t), \Delta G_{d_1}(t)$, $\Delta G_{d_2}(t), \Delta B_1(t)$, $\Delta B_2(t), \Delta D_1(t)$ and $\Delta D_2(t)$ are unknown matrices representing time-varying uncertainties, which are assumed to satisfy the following condition:

$$\begin{bmatrix} \Delta A(t) & \Delta B_1(t) & \Delta A_{d_1}(t) & \Delta A_{d_2}(t) & \Delta D_1(t) \\ \Delta G(t) & \Delta B_2(t) & \Delta G_{d_1}(t) & \Delta G_{d_2}(t) & \Delta D_2(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(t) \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix} \tag{9}$$

where M_i $(i = 1,2)$ and N_i $(i = 1,2,3,4,5)$ are known real constant matrices and $F(t)$ is the unknown Lebesque-measurable matrix-valued function subject to the following condition:

$$F^T(t)F(t) \leq I,\ \forall t. \tag{10}$$

Remark 1: The conditions (9)-(10) are referred to as the *admissible conditions*. These conditions have been frequently used to describe parameter uncertainties in many papers dealing with filtering and control problems for uncertain systems, see e.g. [1, 17–19, 23, 26].

The vector-valued nonlinear functions f, f_{d_1}, f_{d_2} are assumed to satisfy the following *sector-bounded* conditions:

$$[f(x) - L_1x]^T[f(x) - L_2x] \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{11}$$

$$[f_{d_1}(x) - U_1x]^T[f_{d_1}(x) - U_2x] \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{12}$$

$$[f_{d_2}(x) - W_1x]^T[f_{d_2}(x) - W_2x] \leq 0, \quad \forall x \in \mathbb{R}^n. \tag{13}$$

where $L_1, L_2, U_1, U_2, W_1, W_2 \in \mathbb{R}^{n\times n}$ are known real constant matrices, and $L = L_1 - L_2$, $U = U_1 - U_2$ and $W = W_1 - W_2$ are symmetric positive definite matrices.

Remark 2: It is customary that the nonlinear functions f, f_{d_1}, f_{d_2} are said to belong to sectors $[L_1,\ L_2]$, $[U_1,\ U_2]$ and $[W_1,\ W_2]$, respectively [10]. The nonlinear descriptions in (11)-(13) are quite general that include the usual Lipschitz conditions as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see e.g. [9, 13, 14].

With the above assumptions, the system (1)-(3) can be rewritten as

$$\begin{aligned}
(\Sigma') : dx(t) &= [A(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds \\
&\quad + B_1(t)u(t) + D_1(t)v(t)]dt + [G(t)x(t) + G_{d_1}(t)x(t-\tau(t)) \\
&\quad + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds + B_2(t)u(t) + D_2(t)v(t)]dw(t), &(14)\\
y(t) &= Cx(t) + Bu(t), &(15)\\
x(t) &= \phi(t),\ t \in [-\bar{\tau}, 0], &(16)
\end{aligned}$$

where $\bar{\tau}$ is a positive constant satisfying $\tau(t) \leq \bar{\tau}$ $(\forall t \geq 0)$ and $\tau_0 \leq \bar{\tau}$.

Substituting the state feedback law

$$u(t) = Kx(t)$$

into the system (Σ') gives the following closed-loop system:

$$\begin{aligned}
(\Sigma_c) : dx(t) &= [A_K(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds \\
&\quad + D_1(t)v(t)]dt + [G_K(t)x(t) + G_{d_1}(t)x(t-\tau(t)) \\
&\quad + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds + D_2(t)v(t)]dw(t), &(17)\\
y(t) &= C_K x(t), &(18)\\
x(t) &= \phi(t),\ t \in [-\bar{\tau}, 0], &(19)
\end{aligned}$$

where

$$A_K(t) = A(t) + B_1(t)K,\ G_K(t) = G(t) + B_2(t)K,\ C_K = C + BK.$$

In this paper, we aim at developing the techniques of robust stochastic stabilization and robust H_∞ control for uncertain nonlinear Itô stochastic systems (17)-(19) with mixed time delays. More specifically, a state feedback controller of the form $u(t) = Kx(t)$ is to be designed such that

(1) The closed-loop system (Σ_c) with $v(t) = 0$ is mean-square asymptotically stable for all admissible uncertainties.

(2) Under zero initial condition, the closed-loop system satisfies $\|y\|_{\mathcal{E}2} \leq \gamma \|v\|_{\mathcal{E}2}$ for any nonzero $v(\cdot) \in \mathcal{L}_{\mathcal{E}2}([0, +\infty); \mathbb{R}^{n\times m})$.

Remark 3: For the precise definition of the stability, we refer the readers to [11, 12].

III. Main Results

The following lemmas are essential in establishing our main results.

Lemma 1: Let $\mathcal{D}, \mathcal{S}$ and F be real matrices of appropriate dimensions with F satisfying $F^T F \leq I$. Then, for any scalar $\varepsilon > 0$,

$$\mathcal{D}F\mathcal{S} + (\mathcal{D}F\mathcal{S})^T \leq \varepsilon^{-1}\mathcal{D}\mathcal{D}^T + \varepsilon\mathcal{S}^T\mathcal{S}.$$

Lemma 2: (Schur Complement) Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^T$ and $\Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Let us now first deal with the controller analysis problem for the closed-loop system (Σ_c), and derive sufficient condition in the form of LMIs under which the robust mean-square asymptotic stability can be guaranteed for the closed-loop system (Σ_c) with $v(t) = 0$.

Theorem 1: Let the controller gain K be given and suppose that the admissible conditions hold. Then, the closed-loop system (Σ_c) with $v(t) = 0$ is robustly asymptotically stable in the mean square if there exist three positive definite matrices X, Q, R and four positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 such that the following LMI holds:

$$\Psi < 0. \tag{20}$$

Here

$$\Psi = \begin{bmatrix} \Omega & A_{d_1}X & I - XL_2 & I & -XW_2 & A_{d_2} & \Xi^T & X & 0 & XN_1^T & Y^TN_2^T & 0 & 0 \\ \star & -(1-h)Q & 0 & -XU_2 & 0 & 0 & XG_{d_1}^T & 0 & X & 0 & 0 & XN_3^T & 0 \\ \star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \tau_0^2 R - I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -R & G_{d_2}^T & 0 & 0 & 0 & 0 & 0 & N_4^T \\ \star & \star & \star & \star & \star & \star & \Upsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Lambda^{-1} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \breve{U}_1^{-1} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_3 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_4 I \end{bmatrix},$$

where

$$\begin{aligned} \breve{L}_1 &= (L_1^TL_2 + L_2^TL_1)/2;\ \breve{L}_2 = -(L_1^T + L_2^T)/2; & (21) \\ \breve{U}_1 &= (U_1^TU_2 + U_2^TU_1)/2; \breve{U}_2 = -(U_1^T + U_2^T)/2; & (22) \\ \breve{W}_1 &= (W_1^TW_2 + W_2^TW_1)/2;\ \breve{W}_2 = -(W_1^T + W_2^T)/2; & (23) \\ Y &= KX;\ \Lambda = \breve{L}_1 + \breve{W}_1; & (24) \\ \Omega &= AX + XA^T + B_1Y + Y^TB_1^T + Q + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)M_1M_1^T, & (25) \\ \Xi &= GX + B_2Y + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)M_2M_1^T, & (26) \\ \Upsilon &= -X + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)M_2M_2^T. & (27) \end{aligned}$$

Proof: Consider the closed-loop system (Σ_c) with $v(t) = 0$. In order to analyze the stochastic stability, we introduce the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V_0(t) &= x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)\hat{Q}x(s)ds + \int_{t-\tau_0}^{t}\left(\int_s^t f_{d_2}^T(x(\theta))d\theta\right)R\left(\int_s^t f_{d_2}(x(\theta))d\theta\right)ds \\ &\quad + \int_0^{\tau_0}\int_{t-s}^{t}(\theta - t + s)f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta ds, \end{aligned}$$

where $P = X^{-1}$ and $\hat{Q} = X^{-1}QX^{-1}$.

By Itô differential formula [12, 20], the stochastic differential of $V_0(t)$ along the trajectory of system (Σ_c) with $v(t) = 0$ is given by

$$dV_0(t) = \mathcal{L}V_0(t)dt + 2x^T(t)P[G_K(t)x(t) + G_{d_1}(t)x(t-\tau(t)) + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds]dw(t),$$

where

$$\begin{aligned}
\mathcal{L}V_0(t) &= 2x^T(t)P\Big[A_K(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds\Big] \\
&\quad + x^T(t)\hat{Q}x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))\hat{Q}x(t-\tau(t)) \\
&\quad - \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right]^T R \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right] + 2\int_{t-\tau_0}^{t} f_{d_2}^T(x(t))R\left[\int_{s}^{t} f_{d_2}(x(\theta))d\theta\right]ds \\
&\quad + \int_0^{\tau_0} s f_{d_2}^T(x(t))Rf_{d_2}(x(t))ds - \int_0^{\tau_0}\int_{t-s}^{t} f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta ds + \bar{G}_0^T(t)P\bar{G}_0(t) \qquad (28)
\end{aligned}$$

where

$$\bar{G}_0(t) = G_K(t)x(t) + G_{d_1}(t)x(t-\tau(t)) + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds.$$

Some of the terms in (28) can be calculated as follows:

$$\begin{aligned}
& -(1-\dot{\tau}(t))x^T(t-\tau(t))\hat{Q}x(t-\tau(t)) \\
\leq\; & -(1-h)x^T(t-\tau(t))\hat{Q}x(t-\tau(t)); \qquad (29) \\
& 2\int_{t-\tau_0}^{t} f_{d_2}^T(x(t))R\left[\int_{s}^{t} f_{d_2}(x(\theta))d\theta\right]ds \\
=\; & 2\int_{t-\tau_0}^{t} d\theta \int_{t-\tau_0}^{\theta} f_{d_2}^T(x(t))Rf_{d_2}(x(\theta))ds \\
=\; & 2\int_{t-\tau_0}^{t} (\theta - t + \tau_0) f_{d_2}^T(x(t))Rf_{d_2}(x(\theta))d\theta \quad \text{(by Lemma 1)} \\
\leq\; & \int_{t-\tau_0}^{t} (\theta - t + \tau_0)\left[f_{d_2}^T(x(t))Rf_{d_2}(x(t)) + f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))\right]d\theta \\
=\; & \frac{\tau_0^2}{2} f_{d_2}^T(x(t))Rf_{d_2}(x(t)) + \int_{t-\tau_0}^{t} (\theta - t + \tau_0) f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta; \qquad (30) \\
& \int_0^{\tau_0} s f_{d_2}^T(x(t))Rf_{d_2}(x(t))ds \\
=\; & \frac{\tau_0^2}{2} f_{d_2}^T(x(t))Rf_{d_2}(x(t)); \qquad (31) \\
& -\int_0^{\tau_0}\int_{t-s}^{t} f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta ds \\
=\; & -\int_{t-\tau_0}^{t}\int_{t-\theta}^{\tau_0} f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))ds \\
=\; & -\int_{t-\tau_0}^{t} (\theta - t + \tau_0) f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta. \qquad (32)
\end{aligned}$$

Substituting (29)-(32) into (28) leads to

$$\begin{aligned}
\mathcal{L}V_0(t) &\leq 2x^T(t)P\Big[A_K(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds\Big] \\
&\quad + x^T(t)\hat{Q}x(t) - (1-h)x^T(t-\tau(t))\hat{Q}x(t-\tau(t)) \\
&\quad - \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right]^T R \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right] + \tau_0^2 f_{d_2}^T(x(t))Rf_{d_2}(x(t)) + \bar{G}_0^T(t)P\bar{G}_0(t) \\
&= \xi_0^T(t)\Psi_1(t)\xi_0(t) + \xi_0^T(t)\hat{G}_0^T(t)P\hat{G}_0(t)\xi_0(t), \qquad (33)
\end{aligned}$$

where

$$\begin{aligned}
\hat{G}_0(t) &= \begin{bmatrix} G_K(t) & G_{d_1}(t) & 0 & 0 & 0 & G_{d_2}(t) \end{bmatrix} \\
\xi_0(t) &= \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & f^T(x(t)) & f_{d_1}^T(x(t-\tau(t))) & f_{d_2}^T(x(t)) & \int_{t-\tau_0}^{t} f_{d_2}^T(x(s))ds \end{bmatrix}^T \\
\Psi_1(t) &= \left[\begin{array}{cccccc} PA_K(t)+A_K^T(t)P+\hat{Q} & PA_{d_1}(t) & P & P & 0 & PA_{d_2}(t) \\ A_{d_1}^T(t)P & -(1-h)\hat{Q} & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_0^2 R & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R \end{array}\right].
\end{aligned}$$

Notice that (11) is equivalent to

$$\left[\begin{array}{c} x(t) \\ f(x(t)) \end{array}\right]^T \left[\begin{array}{cc} \breve{L}_1 & \breve{L}_2 \\ \breve{L}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t) \\ f(x(t)) \end{array}\right] \leq 0, \tag{34}$$

where $\breve{L}_1, \breve{L}_2$ are defined in (21)

Similarly, it follows from (12)-(13) that

$$\left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_1}(x(t-\tau(t))) \end{array}\right]^T \left[\begin{array}{cc} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_1}(x(t-\tau(t))) \end{array}\right] \leq 0, \tag{35}$$

$$\left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_2}(x(t)) \end{array}\right]^T \left[\begin{array}{cc} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_2}(x(t)) \end{array}\right] \leq 0, \tag{36}$$

where $\breve{U}_1, \breve{U}_2, \breve{W}_1, \breve{W}_2$ are defined in (22)-(23).

Now, from (34)-(36), we can have

$$\begin{aligned}
\mathcal{L}V_0(t) \leq\ & \mathcal{L}V_0(t) - \Bigg\{ \left[\begin{array}{c} x(t) \\ f(x(t)) \end{array}\right]^T \left[\begin{array}{cc} \breve{L}_1 & \breve{L}_2 \\ \breve{L}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t) \\ f(x(t)) \end{array}\right] \\
& + \left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_1}(x(t-\tau(t))) \end{array}\right]^T \left[\begin{array}{cc} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t-\tau(t)) \\ f_{d_1}(x(t-\tau(t))) \end{array}\right] \\
& + \left[\begin{array}{c} x(t) \\ f_{d_2}(x(t)) \end{array}\right]^T \left[\begin{array}{cc} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{array}\right] \left[\begin{array}{c} x(t) \\ f_{d_2}(x(t)) \end{array}\right] \Bigg\} \\
=\ & \xi_0^T(t) \Big[\Psi_2(t) + \hat{G}_0^T(t) P \hat{G}_0(s) \Big] \xi_0(t),
\end{aligned} \tag{37}$$

where

$$\Psi_2(t) = \left[\begin{array}{cccccc} \Pi_1(t) & PA_{d_1}(t) & P-L_2 & P & -W_2 & PA_{d_2}(t) \\ A_{d_1}^T(t)P & \Theta_1 & 0 & -U_2 & 0 & 0 \\ P-L_2^T & 0 & -I & 0 & 0 & 0 \\ P & -U_2^T & 0 & -I & 0 & 0 \\ -W_2^T & 0 & 0 & 0 & \tau_0^2 R - I & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R \end{array}\right], \tag{38}$$

and $\Pi_1(t) = PA_K(t) + A_K^T(t)P + \hat{Q} - \breve{L}_1 - \breve{W}_1, \Theta_1 = -(1-h)\hat{Q} - \breve{U}_1$.

Recall that our goal is to show

$$\Psi_2(t) + \hat{G}_0^T(t)P\hat{G}_0(t) < 0. \tag{39}$$

Notice that, by Lemma 2 (Schur Complement), (39) is equivalent to

$$\Psi_3(t) < 0, \tag{40}$$

where

$$\Psi_3(t) = \begin{bmatrix} \Pi_1(t) & PA_{d_1}(t) & P - L_2 & P & -W_2 & PA_{d_2}(t) & G_K^T(t) \\ A_{d_1}^T(t)P & \Theta_1 & 0 & -U_2 & 0 & 0 & G_{d_1}^T(t) \\ P - L_2^T & 0 & -I & 0 & 0 & 0 & 0 \\ P & -U_2^T & 0 & -I & 0 & 0 & 0 \\ -W_2^T & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R & G_{d_2}^T(t) \\ G_K(t) & G_{d_1}(t) & 0 & 0 & 0 & G_{d_2}(t) & -P^{-1} \end{bmatrix}, \tag{41}$$

and therefore it remains to show $\Psi_3(t) < 0, \ \forall t > 0$.

Denote $\hat{X}_0 = \mathrm{diag}(X, X, I, I, I, I, I)$ and let

$$\begin{aligned} \Psi_4(t) &= \hat{X}_0 \Psi_3(t) \hat{X}_0 \\ &= \begin{bmatrix} \Pi_2(t) & A_{d_1}(t)X & I - XL_2 & I & -XW_2 & A_{d_2}(t) & XG^T(t) + Y^T B_2^T(t) \\ A_{d_1}^T(t) & \Theta_2 & 0 & -XU_2 & 0 & 0 & XG_d^T(t) \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & G_{d_2}^T(t) \\ G(t)X + B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & -X \end{bmatrix}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} \Theta_2 &= -(1-h)Q - X\breve{U}_1 X, & (43) \\ \Pi_2(t) &= A_K(t)X + XA_K^T(t) + Q + X[-\breve{L}_1 - \breve{W}_1]X \\ &= A(t)X + B_1(t)Y + XA^T(t) + Y^T B_1^T(t) + Q - X\Lambda X, & (44) \end{aligned}$$

and Λ is defined in (24).

It then follows that

$$\Psi_4(t) = \Psi_5(t) + \hat{X}^T \begin{bmatrix} -\Lambda & 0 \\ 0 & -\breve{U}_1 \end{bmatrix} \hat{X}, \tag{45}$$

where

$$\begin{aligned} \hat{X} &= \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Psi_5(t) &= \begin{bmatrix} \Pi_3(t) & A_{d_1}(t)X & I - XL_2 & I & -XW_2 & A_{d_2}(t) & XG^T(t) + Y^T B_2^T(t) \\ XA_{d_1}^T(t) & -(1-h)Q & 0 & -XU_2 & 0 & 0 & XG_d^T(t) \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & G_{d_2}^T(t) \\ G(t)X + B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & -X \end{bmatrix}, \\ \Pi_3(t) &= A(t)X + B_1(t)Y + XA^T(t) + Y^T B_1^T(t) + Q. \end{aligned}$$

It is obvious that $\Psi_3(t) < 0$ is equivalent to $\Psi_4(t) < 0$. Again, by Lemma 2 (Schur Complement), $\Psi_4(t) < 0$ is equivalent to

$$\Psi_5(t) < 0, \tag{46}$$

where

$$\Psi_5(t) = \begin{bmatrix} \Pi_3(t) & A_{d_1}(t)X & I - XL_2 & I & -XW_2 & A_{d_2}(t) & XG^T(t) + Y^T B_2^T(t) & X & 0 \\ XA_{d_1}^T(t) & -(1-h)Q & 0 & -XU_2 & 0 & 0 & XG_{d_1}^T(t) & 0 & X \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & G_{d_2}^T(t) & 0 & 0 \\ G(t)X + B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & -X & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}. \tag{47}$$

It can be rewritten that

$$\Psi_5(t) = \Psi_5 + \Delta\Psi_5(t), \tag{48}$$

with

$$\Psi_5 = \begin{bmatrix} \Pi_3 & A_{d_1}X & I - XL_2 & I & -XW_2 & A_{d_2} & XG^T + Y^T B_2^T & X & 0 \\ XA_{d_1}^T & -(1-h)Q & 0 & -XU_2 & 0 & 0 & XG_{d_1}^T & 0 & X \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 & 0 \\ A_{d_2}^T & 0 & 0 & 0 & 0 & -R & G_{d_2}^T & 0 & 0 \\ GX + B_2 Y & G_{d_1}X & 0 & 0 & 0 & G_{d_2} & -X & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}, \tag{49}$$

and

$$\Delta\Psi_5(t) = \begin{bmatrix} \Delta\Pi_3(t) & \Delta A_{d_1}(t)X & 0 & 0 & 0 & \Delta A_{d_2}(t) & X\Delta G^T(t) + Y^T \Delta B_2^T(t) & 0 & 0 \\ X\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & X\Delta G_{d_1}^T(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0 \\ \Delta G(t)X + \Delta B_2(t)Y & \Delta G_{d_1}(t)X & 0 & 0 & 0 & \Delta G_{d_2}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{50}$$

where

$$\Pi_3 = AX + XA^T + B_1 Y + Y^T B_1^T + Q, \tag{51}$$

$$\Delta\Pi_3(t) = \Delta A(t)X + \Delta B_1(t)Y + X\Delta A^T(t) + Y^T \Delta B_1^T(t). \tag{52}$$

It is now clear that

$$\begin{aligned}\Delta\Psi_5(t) &= \begin{bmatrix}\Delta A^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G^T(t) & 0 & 0\end{bmatrix}^T \hat{X}_1 \\ &\quad + \hat{X}_1^T\begin{bmatrix}\Delta A^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G^T(t) & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}\Delta B_1^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta {B_2}^T(t) & 0 & 0\end{bmatrix}^T \hat{Y}_1 \\ &\quad + \hat{Y}_1^T\begin{bmatrix}\Delta B_1^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta B_2^T(t) & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_1}^T(t) & 0 & 0\end{bmatrix}^T \hat{X}_2 \\ &\quad + \hat{X}_2^T\begin{bmatrix}\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_1}^T(t) & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}\Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0\end{bmatrix}^T \hat{I}_1 \\ &\quad + \hat{I}_1^T\begin{bmatrix}\Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0\end{bmatrix},\end{aligned} \tag{53}$$

where

$$\begin{aligned}\hat{X}_1 &= \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \hat{X}_2 &= \begin{bmatrix} 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \hat{Y}_1 &= \begin{bmatrix} Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \hat{I}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0\end{bmatrix}.\end{aligned}$$

Letting

$$\begin{aligned}\hat{M} &= \begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix}^T, \\ \hat{N}_4 &= \begin{bmatrix}0 & 0 & 0 & 0 & N_4 & 0 & 0 & 0 & 0\end{bmatrix},\end{aligned}$$

we have from (9) and Lemma 1 that

$$\begin{aligned}\Delta\Psi_5(t) &= \begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix}^T F(t)N_1\hat{X}_1 \\ &\quad + \hat{X}_1^T N_1^T F^T(t)\begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix}^T F(t)N_2\hat{Y}_1 \\ &\quad + \hat{Y}_1^T N_2^T F^T(t)\begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix}^T F(t)N_3\hat{X}_2 \\ &\quad + \hat{X}_2^T N_3^T F^T(t)\begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix} \\ &\quad + \begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix}^T F(t)\hat{N}_4 \\ &\quad + \hat{N}_4^T F^T(t)\begin{bmatrix}M_1^T & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0\end{bmatrix} \\ &\leq \varepsilon_1^{-1}\hat{X}_1^T N_1^T N_1\hat{X}_1 + \varepsilon_2^{-1}\hat{Y}_1^T N_2^T N_2\hat{Y}_1 + \varepsilon_3^{-1}\hat{X}_2^T N_3^T N_3\hat{X}_2 + \varepsilon_4^{-1}\hat{N}_4^T\hat{N}_4 \\ &\quad + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)\hat{M}\hat{M}^T.\end{aligned} \tag{54}$$

Hence, from (48), (49) and (54), it follows that:

$$\Psi_5(t) \leq \Psi_6 + \varepsilon_1^{-1}\hat{X}_1^T N_1^T N_1\hat{X}_1 + \varepsilon_2^{-1}\hat{Y}_1^T N_2^T N_2\hat{Y}_1 + \varepsilon_3^{-1}\hat{X}_2^T N_3^T N_3\hat{X}_2 + \varepsilon_4^{-1}\hat{N}_4^T\hat{N}_4, \tag{55}$$

where

$$\Psi_6 = \begin{bmatrix} \Omega & A_{d_1}X & I - XL_2 & I & -XW_2 & A_{d_2} & \Xi^T & X & 0 \\ XA_{d_1}^T & -(1-h)Q & 0 & -XU_2 & 0 & 0 & XG_{d_1}^T & 0 & X \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 & 0 \\ A_{d_2}^T & 0 & 0 & 0 & 0 & -R & G_{d_2}^T & 0 & 0 \\ \Xi & G_{d_1}X & 0 & 0 & 0 & G_{d_2} & \Upsilon & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}, \tag{56}$$

and Ω, Ξ and Υ are defined in (25)-(27), respectively.

From Lemma 2 (Schur Complement), one can infer that $\Psi < 0$ is equivalent to that the right hand side of (55) is negative definite. We now arrive at

$$\Psi_3(t) < 0, \tag{57}$$

and it then follows from [11, 12] that the proof of this theorem is complete. ■

Next, let us further consider the H_∞ performance of the closed-loop system (Σ_c).

Theorem 2: Let the controller gain K be given and $\gamma > 0$ be a given positive constant. Then, under the admissible conditions, the closed-loop system (Σ_c) is robustly mean-square asymptotically stable for $v(t) = 0$ and satisfies $\|y\|_{\mathcal{E}2} \leq \gamma \|v\|_{\mathcal{E}2}$ for any nonzero $v(\cdot) \in \mathcal{L}_{\mathcal{E}2}([0, +\infty); \mathbb{R}^{n \times m})$ under the zero initial condition, if there exist three positive definite matrices X, Q, R and five positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5 such that the following LMI holds:

$$\Phi_0 < 0, \tag{58}$$

where

$$
\Phi_0 = \left[\begin{array}{cccccccccccccccc}
\Omega_0 & A_{d_1}X & I-XL_2 & I & -XW_2 & A_{d_2} & D_1 & \Xi_0^T & X & 0 & XC^T+Y^TB^T & XN_1^T & Y^TN_2^T & 0 & 0 & 0 \\
\star & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_{d_1}^T & 0 & X & 0 & 0 & 0 & XN_3^T & 0 & 0 \\
\star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \tau_0^2R-I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -R & 0 & G_{d_2}^T & 0 & 0 & 0 & 0 & 0 & 0 & N_4^T & 0 \\
\star & \star & \star & \star & \star & \star & -\gamma^2 I & D_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_5^T \\
\star & \star & \star & \star & \star & \star & \star & \Upsilon_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Lambda^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \breve{U}_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_3 I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_4 I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_5 I
\end{array}\right] \tag{59}
$$

and $\breve{L}_1, \breve{L}_2, \breve{U}_1, \breve{U}_2, \breve{W}_1, \breve{W}_2$ and Y are defined as in Theorem 1, and

$$
\begin{aligned}
\Omega_0 &= AX + XA^T + B_1Y + Y^TB_1^T + Q + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)M_1M_1^T, \\
\Xi_0 &= GX + B_2Y + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)M_2M_1^T, \\
\Upsilon_0 &= -X + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)M_2M_2^T
\end{aligned}
$$

Proof: First it is easy to see that $\Phi_0 < 0$ implies that $\Psi < 0$ where Ψ is defined in (20). Therefore, by Theorem 1, the closed-loop system (Σ_c) with $v(t) = 0$ is robustly asymptotically stable in the mean square.

Next, we need to establish H_∞ performance of the closed-loop system (17)-(19) under zero initial condition, i.e., show that the closed-loop system with zero condition satisfies $\|y(t)\|_{\mathcal{E}2} \le \gamma \|v\|_{\mathcal{E}2}$ for nonzero $v(t)$.

Define the following Lyapunov candidate for system (Σ_c):

$$
\begin{aligned}
V(t) &= x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)\hat{Q}x(s)ds + \int_{t-\tau_0}^{t}\left(\int_s^t f_{d_2}^T(s(\theta))d\theta\right) R \left(\int_s^t f_{d_2}(x(\theta))d\theta\right) ds \\
&\quad + \int_0^{\tau_0}\int_{t-s}^{t} (\theta - t + s) f_{d_2}^T(x(\theta)) R f_{d_2}(x(\theta)) d\theta ds,
\end{aligned} \tag{60}
$$

where $P = X^{-1}$ and $\hat{Q} = X^{-1}QX^{-1}$.

By Itô differential formula, the stochastic differential of $V(t)$ along the trajectory of system (Σ_e) is given by

$$dV(t) = \mathcal{L}V(t)dt + 2x^T(t)P[G_K(t)x(t) + G_{d_1}(t)x(t-\tau(t)) + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds + D_2(t)v(t)]dw(t), \quad (61)$$

where

$$\begin{aligned}
\mathcal{L}V(t) &= 2x^T(t)P\Big[A_K(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds \\
&\quad + B_1(t)u(t) + D_1(t)v(t)\Big] + x^T(t)\hat{Q}x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))\hat{Q}x(t-\tau(t)) \\
&\quad - \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right]^T R\left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right] + 2\int_{t-\tau_0}^{t} f_{d_2}^T(x(t))R\left[\int_{s}^{t} f_{d_2}(x(\theta))d\theta\right]ds \\
&\quad + \int_0^{\tau_0} s f_{d_2}^T(x(t))Rf_{d_2}(x(t))ds - \int_0^{\tau_0}\int_{t-s}^{t} f_{d_2}^T(x(\theta))Rf_{d_2}(x(\theta))d\theta ds + \bar{G}^T(t)P\bar{G}(t) \qquad (62)
\end{aligned}$$

and

$$\bar{G}(t) = G_K(t)x(t) + G_{d_1}(t)x(t-\tau(t)) + G_{d_2}(t)\int_{t-\tau_0}^{t} f_{d_2}(x(s))ds + D_2(t)v(t). \quad (63)$$

Along the same line as in the proof of the Theorem 1, one can obtain

$$\begin{aligned}
\mathcal{L}V(t) &\leq 2x^T(t)P\Big[A_K(t)x(t) + A_{d_1}(t)x(t-\tau(t)) + f(x(t)) + f_{d_1}(x(t-\tau(t))) + A_{d_2}(t)\int_{t-t_0}^{t} f_{d_2}(x(s))ds \\
&\quad + B_1(t)u(t) + D_1(t)v(t)\Big] + x^T(t)\hat{Q}x(t) - (1-h)x^T(t-\tau(t))\hat{Q}x(t-\tau(t)) \\
&\quad - \left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right]^T R\left[\int_{t-\tau_0}^{t} f_{d_2}(x(\theta))d\theta\right] + \tau_0^2 f_{d_2}^T(x(t))Rf_{d_2}(x(t)) + \bar{G}^T(t)P\bar{G}(t) \\
&= \xi^T(t)\Phi_1(t)\xi(t) + \xi^T(t)\hat{G}^T(t)P\hat{G}(t)\xi(t), \qquad (64)
\end{aligned}$$

where

$$\begin{aligned}
\hat{G}(t) &= \begin{bmatrix} G_K(t) & G_{d_1}(t) & 0 & 0 & 0 & G_{d_2}(t) & D_2(t) \end{bmatrix} \\
\xi(t) &= \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & f^T(x(t)) & f_{d_1}^T(x(t-\tau(t))) & f_{d_2}^T(x(t)) & \int_{t-\tau_0}^{t} f_{d_2}^T(x(s))ds & v^T(t) \end{bmatrix}^T \\
\Phi_1(t) &= \begin{bmatrix}
PA_K(t) + A_K^T(t)P + \hat{Q} & PA_{d_1}(t) & P & P & 0 & PA_{d_2}(t) & PD_1(t) \\
A_{d_1}^T(t)P & -(1-h)\hat{Q} & 0 & 0 & 0 & 0 & 0 \\
P & 0 & 0 & 0 & 0 & 0 & 0 \\
P & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau_0^2 R & 0 & 0 \\
A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R & 0 \\
D_1^T(t)P & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\end{aligned}$$

In order to establish the H_∞ performance of the closed-loop system under the zero initial condition, we introduce

$$J(t) = \mathbb{E}\int_0^t [z^T(s)z(s) - \gamma^2 v^T(s)v(s)]ds \quad (65)$$

where $t > 0$.

Our goal is to show $J(t) < 0$. Based on the zero initial condition and $\mathbb{E}V(t) \geq 0$, it can be shown that for any nonzero $v(t) \in \mathcal{L}_{\mathcal{E}2}([0,+\infty);\mathbb{R}^p)$ and $t > 0$, the following holds

$$\begin{aligned} J(t) &= \mathbb{E}\int_0^t \left[z^T(s)z(s) - \gamma^2 v^T(s)v(s) + \mathcal{L}V(s)\right]ds - \mathbb{E}V(t) \\ &\leq \mathbb{E}\int_0^t \left[x^T(s)C_K^T C_K x(s) - \gamma^2 v^T(s)v(s) + \mathcal{L}V(s)\right]ds. \\ &= \mathbb{E}\int_0^t \xi^T(s)\Phi_2(t)\xi(s) + \xi^T(s)\hat{G}^T(s)P\hat{G}(s)\xi(s)\big]ds, \end{aligned} \tag{66}$$

where

$$\Phi_2(t) = \begin{bmatrix} PA_K(t) + A_K^T(t)P + \hat{Q} + C_K^T C_K & PA_{d_1}(t) & P & P & 0 & PA_{d_2}(t) & PD_1(t) \\ A_{d_1}^T(t)P & -(1-h)\hat{Q} & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_0^2 R & 0 & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R & 0 \\ D_1^T(t)P & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}. \tag{67}$$

It follows from (34)-(36) that

$$\begin{aligned} J(t) &\leq J(t) - \mathbb{E}\int_0^t \left\{ \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} \breve{L}_1 & \breve{L}_2 \\ \breve{L}_2^T & I \end{bmatrix} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} \right. \\ &\quad + \begin{bmatrix} x(s-\tau(s)) \\ f_{d_1}(x(s-\tau(s))) \end{bmatrix}^T \begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(s-\tau(s)) \\ f_{d_1}(x(s-\tau(s))) \end{bmatrix} \\ &\quad \left. + \begin{bmatrix} x(s) \\ f_{d_2}(x(s)) \end{bmatrix}^T \begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix} \begin{bmatrix} x(s) \\ f_{d_2}(x(s)) \end{bmatrix} \right\} ds \\ &= \mathbb{E}\int_0^t \xi^T(s)\Big[\Phi_3(s) + \hat{G}^T(s)P\hat{G}(s)\Big]\xi(s)ds, \end{aligned} \tag{68}$$

where

$$\Phi_3(t) = \begin{bmatrix} \Omega_1(t) & PA_{d_1}(t) & P - L_2 & P & -W_2 & PA_{d_2}(t) & PD_1(t) \\ A_{d_1}^T(t)P & \Gamma_1 & 0 & -U_2 & 0 & 0 & 0 \\ P - L_2^T & 0 & -I & 0 & 0 & 0 & 0 \\ P & -U_2^T & 0 & -I & 0 & 0 & 0 \\ -W_2^T & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R & 0 \\ D_1^T(t)P & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{69}$$

and $\Omega_1(t) = PA_K(t) + A_K^T(t)P + \hat{Q} + C_K^T C_K - \breve{L}_1 - \breve{W}_1, \Gamma_1 = -(1-h)\hat{Q} - \breve{U}_1$.

In order to guarantee $J(t) < 0$, it suffices to show

$$\Phi_3(s) + \hat{G}^T(s)P\hat{G}(s) < 0. \tag{70}$$

It is noticed from Lemma 2 (Schur Complement) that (70) is equivalent to

$$\Phi_4(t) < 0, \tag{71}$$

where

$$\Phi_4(t) = \begin{bmatrix} \Omega_1(t) & PA_{d_1}(t) & P-L_2 & P & -W_2 & PA_{d_2}(t) & PD_1(t) & G_K^T(t) \\ A_{d_1}^T(t)P & \Gamma_1 & 0 & -U_2 & 0 & 0 & 0 & G_{d_1}^T(t) \\ P-L_2^T & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ P & -U_2^T & 0 & -I & 0 & 0 & 0 & 0 \\ -W_2^T & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 \\ A_{d_2}^T(t)P & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T(t) \\ D_1^T(t)P & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & D_2^T(t) \\ G_K(t) & G_{d_1}(t) & 0 & 0 & 0 & G_{d_2}(t) & D_2(t) & -P^{-1} \end{bmatrix}. \tag{72}$$

Therefore, we just need to show that $\Phi_4(t) < 0,\ \forall t > 0$.

Denote $\bar{X} = \mathrm{diag}(X, X, I, I, I, I, I, I)$, and let

$$\begin{aligned} \Phi_5(t) &= \bar{X}\Phi_4(t)\bar{X} \\ &= \begin{bmatrix} \Omega_2(t) & A_{d_1}(t)X & I-XL_2 & I & -XW_2 & A_{d_2}(t) & D_1(t) & XG^T(t)+Y^TB_2^T(t) \\ A_{d_1}^T(t) & \Gamma_2 & 0 & -XU_2 & 0 & 0 & 0 & XG_d^T(t) \\ I-L_2^TX & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^TX & 0 & -I & 0 & 0 & 0 & 0 \\ -W_2^TX & 0 & 0 & 0 & \tau_0^2R-I & 0 & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T(t) \\ D_1^T(t) & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & D_2^T(t) \\ G(t)X+B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & D_2(t) & -X \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Gamma_2 &= -(1-h)Q - X\breve{U}_1X, \\ \Omega_2(t) &= A_K(t)X + XA_K^T(t) + Q + X[C_K^TC_K - \breve{L}_1 - \breve{W}_1]X \\ &= A(t)X + B_1(t)Y + XA^T(t) + Y^TB_1^T(t) + Q - X\Lambda X + XC_K^TC_KX. \end{aligned}$$

Obviously, $\Phi_4 < 0$ is equivalent to $\Phi_5 < 0$, and we can write

$$\Phi_5(t) = \Phi_6(t) + \tilde{X}^T \begin{bmatrix} -\Lambda & 0 \\ 0 & -\breve{U}_1 \end{bmatrix} \tilde{X} + \bar{X}_1^T C_K^T C_K \bar{X}_1,$$

where

$$\bar{X}_0 = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_6(t) = \begin{bmatrix} \Omega_3(t) & A_{d_1}(t)X & I-XL_2 & I & -XW_2 & A_{d_2}(t) & D_1(t) & XG^T(t)+Y^TB_2^T(t) \\ XA_{d_1}^T(t) & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_d^T(t) \\ I-L_2^TX & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^TX & 0 & -I & 0 & 0 & 0 & 0 \\ -W_2^TX & 0 & 0 & 0 & \tau_0^2R-I & 0 & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T(t) \\ D_1^T(t) & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & E^T(t) \\ G(t)X+B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & D_2(t) & -X \end{bmatrix},$$

$$\Omega_3(t) = A(t)X + B_1(t)Y + XA^T(t) + Y^TB_1^T(t) + Q.$$

By Lemma 2 (Schur Complement), $\Phi_5(t) < 0$ is equivalent to

$$\Phi_7(t) < 0,$$

where

$$\Phi_7(t) = \begin{bmatrix} \Omega_3(t) & A_{d_1}(t)X & I - XL_2 & I & -XW_2 & A_{d_2}(t) & D_1(t) & XG^T(t) + Y^T B_2^T(t) & X & 0 & XC^T + Y^T B^T \\ XA_{d_1}^T(t) & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_{d_1}^T(t) & 0 & X & 0 \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{d_2}^T(t) & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T(t) & 0 & 0 & 0 \\ D_1^T(t) & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & D_2^T(t) & 0 & 0 & 0 \\ G(t)X + B_2(t)Y & G_{d_1}(t)X & 0 & 0 & 0 & G_{d_2}(t) & D_2(t) & -X & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} & 0 \\ CX + BY & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

Now, we can decompose $\Phi_7(t)$ as

$$\Phi_7(t) = \Phi_7 + \Delta\Phi_7(t). \tag{73}$$

Here

$$\Phi_7 = \begin{bmatrix} \Omega_3 & A_{d_1}X & I - XL_2 & I & -XW_2 & A_{d_2} & D_1 & XG^T + Y^T B_2^T & X & 0 & XC^T + Y^T B^T \\ XA_{d_1}^T & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_{d_1}^T & 0 & X & 0 \\ I - L_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^T X & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -W_2^T X & 0 & 0 & 0 & \tau_0^2 R - I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{d_2}^T & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T & 0 & 0 & 0 \\ D_1^T & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & D_2^T & 0 & 0 & 0 \\ GX + B_2Y & G_{d_1}X & 0 & 0 & 0 & G_{d_2} & D_2 & -X & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} & 0 \\ CX + BY & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

and

$$\Delta\Phi_7(t) = \begin{bmatrix} \Delta\Omega_3(t) & \Delta A_{d_1}(t)X & 0 & 0 & 0 & \Delta A_{d_2}(t) & \Delta D_1(t) & X\Delta G^T(t) + Y^T \Delta B_2^T(t) & 0 & 0 & 0 \\ X\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & X\Delta G_{d_1}^T(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0 & 0 \\ \Delta D_1^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta D_2^T(t) & 0 & 0 & 0 \\ \Delta G(t)X + \Delta B_2(t)Y & \Delta G_{d_1}(t)X & 0 & 0 & 0 & \Delta G_{d_2}(t) & \Delta D_2(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \Omega_3 &= AX + XA^T + B_1Y + Y^T B_1^T + Q, \\ \Delta\Omega_3(t) &= \Delta A(t)X + \Delta B_1(t)Y + X\Delta A^T(t) + Y^T \Delta B_1^T(t). \end{aligned}$$

It is easy to see that

$$\begin{aligned}\Delta\Phi_7(t) &= \begin{bmatrix}\Delta A^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G^T(t) & 0 & 0 & 0\end{bmatrix}^T \bar{X}_1 \\ &+ \bar{X}_1^T \begin{bmatrix}\Delta A^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G^T(t) & 0 & 0 & 0\end{bmatrix} \\ &+ \begin{bmatrix}\Delta B_1^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta {B_2}^T(t) & 0 & 0 & 0\end{bmatrix}^T \bar{Y}_1 \\ &+ \bar{Y}_1^T \begin{bmatrix}\Delta B_1^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta B_2^T(t) & 0 & 0 & 0\end{bmatrix} \\ &+ \begin{bmatrix}\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_1}^T(t) & 0 & 0 & 0\end{bmatrix}^T \bar{X}_2 \\ &+ \bar{X}_2^T \begin{bmatrix}\Delta A_{d_1}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_1}^T(t) & 0 & 0 & 0\end{bmatrix} \\ &+ \begin{bmatrix}\Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0 & 0\end{bmatrix}^T \bar{I}_1 \\ &+ \bar{I}_1^T \begin{bmatrix}\Delta A_{d_2}^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta G_{d_2}^T(t) & 0 & 0 & 0\end{bmatrix} \\ &+ \begin{bmatrix}\Delta D_1^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta D_2^T(t) & 0 & 0 & 0\end{bmatrix}^T \bar{I}_2 \\ &+ \bar{I}_2^T \begin{bmatrix}\Delta D_1^T(t) & 0 & 0 & 0 & 0 & 0 & 0 & \Delta D_2^T(t) & 0 & 0 & 0\end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}\bar{X}_1 &= \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{X}_2 &= \begin{bmatrix} 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{Y}_1 &= \begin{bmatrix} Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{I}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{I}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

Similar to the derivation in the proof of Theorem 1, it can be inferred that

$$\begin{aligned}\Delta\Phi_7(t) \;\leq\; & \varepsilon_1^{-1}\bar{X}_1^T N_1^T N_1 \bar{X}_1 + {\varepsilon_2}^{-1}\bar{Y}_1^T N_2^T N_2 \bar{Y}_1 + \varepsilon_3^{-1}\bar{X}_2^T N_3^T N_3 \bar{X}_2 + \varepsilon_4^{-1}\bar{N}_4^T \bar{N}_4 \\ & + \varepsilon_5^{-1}\bar{N}_4^T \bar{N}_5 + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)\bar{M}\bar{M}^T,\end{aligned}$$

where

$$\begin{aligned}\bar{M} &= \begin{bmatrix} M_1^T & 0 & 0 & 0 & 0 & 0 & 0 & M_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\ \bar{N}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & N_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \bar{N}_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & N_5 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

Hence, from (73)-(74), it follows that:

$$\Phi_7(t) \leq \Phi_8 + \varepsilon_1^{-1}\bar{X}_1^T N_1^T N_1 \bar{X}_1 + {\varepsilon_2}^{-1}\bar{Y}_1^T N_2^T N_2 \bar{Y}_1 + \varepsilon_3^{-1}\bar{X}_2^T N_3^T N_3 \bar{X}_2 + \varepsilon_4^{-1}\bar{N}_4^T \bar{N}_4 + \varepsilon_5^{-1}\bar{N}_5^T \bar{N}_5, \tag{74}$$

where

$$\Phi_8 = \begin{bmatrix} \Omega & A_{d_1}X & I-XL_2 & I & -XW_2 & A_{d_2} & D_1 & \Xi^T & X & 0 & XC^T+Y^TB^T \\ XA_{d_1}^T & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_{d_1}^T & 0 & X & 0 \\ I-L_2^TX & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & -U_2^TX & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -W_2^TX & 0 & 0 & 0 & \tau_0^2R-I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{d_2}^T & 0 & 0 & 0 & 0 & -R & 0 & G_{d_2}^T & 0 & 0 & 0 \\ D_1^T & 0 & 0 & 0 & 0 & 0 & -\gamma^2I & D_2^T & 0 & 0 & 0 \\ \Xi & G_{d_1}X & 0 & 0 & 0 & G_{d_2} & D_2 & \Upsilon & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-1} & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} & 0 \\ CX+BY & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}.$$

From Lemma 2 (Schur Complement), we know that $\Phi_0 < 0$ is equivalent to that the right hand side of (74) is negative definite. To this end, we can conclude that

$$\Phi_4(t) < 0,$$

which implies

$$J(t) \leq 0, \quad \forall t > 0.$$

Letting $t \to +\infty$, we obtain

$$\|y\|_{\mathcal{E}2} \leq \gamma \|v\|_{\mathcal{E}2}.$$

This completes the proof of this theorem. ■

By now, we have established the conditions under which the closed-loop system is robustly asymptotically stable in the mean square and also the H_∞ performance requirement is satisfied. We are now ready to deal with the design problem of the H_∞ controller for the system (Σ'.) The following result can be easily accessible from Theorem 2, hence the proof is omitted.

Theorem 3: Let γ be a given positive constant. Suppose that the admissible conditions hold. Then , for the nonlinear Itô stochastic system (Σ'), a state feedback controller can be designed such that the closed-loop system (Σ_c) is robustly mean-square asymptotically stable with disturbance attenuation γ if there exist four definite matrices $X > 0, Q > 0, R > 0$ and Y, and five positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5 such that the following LMI holds:

$$\Phi < 0,$$

where

$$\Phi = \left[\begin{array}{cccccccccccccccc}
\Omega_0 & A_{d_1}X & I - XL_2 & I & -XW_2 & A_{d_2} & D_1 & \Xi_0^T & X & 0 & XC^T + Y^TB^T & XN_1^T & Y^TN_2^T & 0 & 0 & 0 \\
\star & -(1-h)Q & 0 & -XU_2 & 0 & 0 & 0 & XG_{d_1}^T & 0 & X & 0 & 0 & 0 & XN_3^T & 0 & 0 \\
\star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \tau_0^2 R - I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -R & 0 & G_{d_2}^T & 0 & 0 & 0 & 0 & 0 & 0 & N_4^T & 0 \\
\star & \star & \star & \star & \star & \star & -\gamma^2 I & D_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_5^T \\
\star & \star & \star & \star & \star & \star & \star & \Upsilon_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Lambda^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \breve{U}_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_3 I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_4 I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_5 I
\end{array}\right] \qquad (75)$$

and $\breve{L}_1, \breve{L}_2, \breve{U}_1, \breve{U}_2, \breve{W}_1, \breve{W}_2, \Lambda_0, \Omega_0, \Xi_0$ and Υ are defined as in Theorem 2. Moreover, the state feedback gain matrix can be designed as

$$K = YX^{-1}.$$

Remark 4: The robust H_∞ controller design problem is solved in Theorem 3 for the addressed uncertain nonlinear stochastic time-delay systems with time-varying norm-bounded parameter uncertainties appearing in both the state and input matrices. Note that the time delays considered include both the discrete- and distributed-type delays. In [11], the first order approximation methodologies have been used to deal with functional differential delayed equations. Different from the method used in [11], in this paper, a computationally efficient LMI approach is developed to derive the sufficient condition for the existence of state feedback controllers that ensure the mean-square asymptotic stability of the resulting closed-loop system and reduce the effect of the disturbance input on the controlled output to a prescribed level for all admissible uncertainties.

Remark 5: It should be pointed out that the main results are dependent on the distributed delay as well as the upper bound of the derivative of the time-varying delay. The feasibility of the controller design problem can be readily checked by the solvability of an LMI, which can be done by resorting to the Matlab LMI toolbox. In next section, an illustrative example will be provided to show the potential of the proposed techniques.

IV. Numerical Example

In this section, a numerical example is presented to demonstrate the effectiveness of the developed method on the design of robust H_∞ control for the uncertain nonlinear Itô stochastic systems with mixed time delays.

Consider the system (Σ') with the following parameters:

$$\begin{aligned}
A &= \begin{bmatrix} -4.5 & 1.1 & 0.7 \\ -0.3 & -4.5 & 0.5 \\ 0 & 0.6 & 0 \end{bmatrix}, \; A_{d_1} = \begin{bmatrix} 1.2 & -0.5 & 0.7 \\ 0.3 & -1.2 & 0.4 \\ -0.5 & -1.2 & -0.3 \end{bmatrix}, \; A_{d_2} = \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ -0.4 & -0.5 & 0.4 \\ -0.5 & 0.8 & -0.6 \end{bmatrix}, \\
G &= \begin{bmatrix} -0.4 & 0.5 & 0.6 \\ -0.5 & 0.7 & 0.6 \\ -0.5 & -0.6 & 0.8 \end{bmatrix}, \; G_{d_1} = \begin{bmatrix} -0.9 & 0.4 & 0.6 \\ -0.5 & 0.6 & 0.4 \\ 0.7 & -0.6 & 0.7 \end{bmatrix}, \; G_{d_2} = \begin{bmatrix} 0.7 & 0.2 & -0.4 \\ 0.6 & -0.5 & 0.4 \\ -0.6 & -0.7 & -0.6 \end{bmatrix}, \\
C &= \begin{bmatrix} -0.2 & -0.3 & 0.2 \\ -0.2 & -0.1 & 0.2 \\ -0.3 & 0.2 & -0.1 \end{bmatrix}, \; B = \begin{bmatrix} 0.4 & 0.1 \\ -0.2 & 0.3 \\ -0.2 & -0.1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 0.1 \\ 0.3 & -6 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.4 \\ 0.3 & -0.2 \end{bmatrix}, \\
D_1 &= \begin{bmatrix} -0.7 & 0.6 \\ -0.4 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.2 & 0.4 \\ 0.6 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, \; M_1 = \begin{bmatrix} -0.2 \\ 0.3 \\ 0.2 \end{bmatrix}, \; M_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}, \; N_1 = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.1 \end{bmatrix}^T, \\
N_2 &= \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \; N_3 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \end{bmatrix}, \; N_4 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, \; N_5 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \; \tau_0 = 0.2, \\
f(x) &= f_{d_1}(x) = f_{d_2}(x) = \begin{pmatrix} 0.5x_1 - 0.2x_2 + 0.1x_3 + 0.2x_1 \sin x_2 \\ -0.2x_1 + 0.5x_2 + 0.1x_3 + 0.2x_3 \sin x_2 \\ 0.1x_1 + 0.1x_2 0.4x_3 + 0.2x_2 \sin x_3 \end{pmatrix}.
\end{aligned}$$

It is easy to check that the open-loop system matrix A is unstable, and it can also be verified that

$$L_1 = U_1 = W_1 = \begin{bmatrix} 0.6 & -0.2 & 0.1 \\ -0.1 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}, \; L_2 = U_2 = W_2 = \begin{bmatrix} -0.4 & 0.2 & -0.1 \\ 0.3 & -0.5 & -0.1 \\ -0.1 & -0.1 & -0.4 \end{bmatrix}.$$

Furthermore, the H_∞ performance level is taken as $\gamma = 0.9$. With the above parameters and by using the Matlab LMI Toolbox, we solve the LMI (75) and obtain

$$\begin{aligned}
X &= \begin{bmatrix} 2.6182 & 1.5805 & 0.2069 \\ 1.5805 & 4.9573 & -1.2649 \\ 0.2069 & -1.2649 & 4.9807 \end{bmatrix}, \; Q = \begin{bmatrix} 7.6300 & 1.5068 & 3.6396 \\ 1.5068 & 12.7325 & -13.1956 \\ 3.6396 & -13.1956 & 77.2616 \end{bmatrix}, \\
R &= \begin{bmatrix} 12.0836 & -3.5690 & 0.0917 \\ -3.5690 & 12.0813 & 0.0950 \\ 0.0917 & 0.0950 & 15.4713 \end{bmatrix}, \; Y = \begin{bmatrix} -2.6066 & 0.6721 & -11.9963 \\ 0.4532 & 1.4437 & 26.0484 \end{bmatrix}, \\
\varepsilon_1 &= 6.8742, \; \varepsilon_2 = 6.7686, \; \varepsilon_3 = 7.3258, \; \varepsilon_4 = 4.9824, \; \varepsilon_5 = 9.4372.
\end{aligned}$$

Therefore, the state feedback gain matrix can be designed as

$$K = YX^{-1} = \begin{bmatrix} -0.6252 & -0.2920 & -2.4568 \\ -1.7045 & 2.3388 & 5.8947 \end{bmatrix}.$$

V. Conclusions

In this paper, we have studied the robust H_∞ control problem for a class of uncertain continuous-time Itô-type stochastic systems involving sector nonlinearities and mixed time delays. An effective linear matrix inequality (LMI) approach has been proposed to design the state feedback controllers such that, for all admissible nonlinearities and time-delays, the overall uncertain closed-loop system is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We have first investigated the sufficient conditions for the uncertain nonlinear stochastic time-delay systems to be stable in the mean square, and then derived the explicit expression of the desired controller gains. A numerical example has been provided to show the usefulness and effectiveness of the proposed design method.

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