H_∞ Filtering for Uncertain Stochastic Time-Delay Systems with Sector-Bounded Nonlinearities

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Abstract

In this paper, we deal with the robust H_∞ filtering problem for a class of uncertain nonlinear time-delay stochastic systems. The system under consideration contains parameter uncertainties, Itô-type stochastic disturbances, time-varying delays, as well as sector-bounded nonlinearities. We aim at designing a full-order filter such that, for all admissible uncertainties, nonlinearities and time-delays, the dynamics of the filtering error is guaranteed to be robustly asymptotically stable in the mean square, while achieving the prescribed H_∞ disturbance rejection attenuation level. By using the Lyapunov stability theory and Itô's differential rule, sufficient conditions are first established to ensure the existence of the desired filters, which are expressed in the form of a linear matrix inequality (LMI). Then, the explicit expression of the desired filter gains is also characterized. Finally, a numerical example is exploited to show the usefulness of the results derived.

Keywords

Itô stochastic system; H_∞ filtering; Robust filtering; Nonlinear filtering; Time delays; Lyapunov-Krasovskii functional; Linear matrix inequality.

I. Introduction

It is well known that Kalman filtering approach is one of the most effective ways to deal with the state estimation problems [1]. One drawback with Kalman filters, which has been well recognized, is that the system model under consideration is required to be exactly known and the disturbances are restricted to be stationary Gaussian noises with known statistics. However, these assumptions are not always satisfied in practical applications [17]. Therefore, in the past decade, much research effort has been paid to the robust filtering problems with respect to various filtering performance criteria, such as the H_∞ specification, the minimum variance requirement and the so-called admissible variance constraint, see [6, 8, 10, 16, 17, 23, 25–28, 30, 31] and the references therein.

On the other hand, time-delays are frequently encountered in many practical engineering systems, such as communication, electronics, hydraulic and chemical systems. Their existence may induce instability, oscillation and poor performance of systems. Therefore, in designing filters, the possible time delays should be taken into account in order to make sure that the filtering error dynamics converges in the expected way. In the past few years, many results have been reported in the literature on robust and/or H_∞ filtering for time-delay systems, see [2] for a survey. As for stochastic systems, for example, the Kalman filter design problem has been investigated in [6, 19, 20] for linear continuous- and discrete-time time-delay systems.

Filtering for nonlinear dynamical system is an important research area that has attracted considerable interest. A large number of suboptimal approaches have been developed to solve the nonlinear filtering problem, which include Gram-charlier expansion, Edgeworth expansion, extended Kalman filters, weighted sum

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of Gaussian densities, generalized least-squares approximation and statistically linearized filters, see [3] for a survey. Among others, some later developments (e.g. [22,29]) include the bound-optimal filters, exponentially bounded filters, exact finite dimensional filters, approximations by Markov chains, minimum variance filters, approximation of the Kushner equation, wavelet transform, particle filters, etc. However, most existing literature has dealt with the nonlinear systems with *white noises*. Another important type of noises/disturbances described by Brownian motions (or Wiener processes) has seldom been addressed for the filtering problems [31]. Note that stochastic systems with Brownian motions, governed by the Itô differential equations, have attracted much research attention over the past few decades due to the extensive application of stochastic modelling in mechanical systems, economics, and other areas [24]. Unfortunately, to the best of the authors' knowledge, up to now, the robust H_∞ filtering problem for *uncertain nonlinear Itô-type stochastic time-delay* systems has not been fully investigated and remains open.

In this paper, we are concerned with the robust H_∞ filtering problem for a class of uncertain nonlinear time-delay Itô stochastic systems. The system under study involves parameter uncertainties, Itô-type stochastic disturbances, time-varying delays and inherent sector-like nonlinearities. Note that, among different descriptions of the nonlinearities, the so-called *sector nonlinearity* [12] has gained much attention for *deterministic* systems, and both the control analysis and model reduction problems have been investigated, see [9,13,14]. We first investigate the sufficient conditions for the filtering error system to be stable in the mean square, and then derive the explicit expression of the desired controller gains. A numerical example is provided to show the usefulness and effectiveness of the proposed design method.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h,0];\mathbb{R}^n)$ denote the family of continuous functions φ from $[-h,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h,0];\mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0}\mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

II. Problem Formulation

Consider the following uncertain nonlinear time-delay Itô stochastic system defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$(\Sigma): \quad dx(t) = [\mathcal{F}(x(t), x(t-\tau(t)), t) + D_1(t)v(t)]dt + [\mathcal{G}(x(t), x(t-\tau(t)), t) + E(t)v(t)]dw(t), \tag{1}$$

$$y(t) = \varphi(x(t), x(t-\tau(t)), t) + D_2(t)v(t), \tag{2}$$

$$z(t) = Lx(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^r$ is the output or measurement; $z(t) \in \mathbb{R}^q$ is the signal to be estimated; $w(t)$ is a zero-mean scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with $\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[w^2(t)] = t$. The exogenous disturbance signal $v(t) \in \mathbb{R}^p$ is assumed to obey $v(\cdot) \in \mathcal{L}_{\mathcal{E}2}([0,\infty);\mathbb{R}^p)$, where $\mathcal{L}_{\mathcal{E}2}([0,\infty);\mathbb{R}^p)$ is the space of non-anticipatory square integrable stochastic process $f(\cdot) = (f(t))_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ with the following norm:

$$\|f\|_{\mathcal{E}2} = \left\{\mathbb{E}\int_0^{+\infty}|f(t)|^2dt\right\}^{1/2} = \left\{\int_0^{+\infty}\mathbb{E}|f(t)|^2dt\right\}^{1/2}.$$

Furthermore, L is a real constant matrix, the scalar $\tau(t) \geq 0$ represents the time-varying delays satisfying $\dot{\tau} \leq h < 1$, and $\mathcal{F}(\cdot,\cdot,\cdot)$, $\mathcal{G}(\cdot,\cdot,\cdot)$ and $\varphi(\cdot)$ are nonlinear vector functions which are decomposed as follows:

$$\begin{aligned}\mathcal{F}(x(t),x(t-\tau(t)),t) &= A(t)x(t) + f(x(t)) + A_d(t)x(t-\tau(t)) + f_d(x(t-\tau(t))),\\ \mathcal{G}(x(t),x(t-\tau(t)),t) &= B(t)x(t) + B_d(t)x(t-\tau(t)),\\ \varphi(x(t),x(t-\tau(t)),t) &= C(t)x(t) + \phi(x(t)) + C_d(t)x(t-\tau(t)) + g(x(t-\tau(t)))\end{aligned}$$

with $A(t) = A + \Delta A(t)$, $A_d(t) = A_d + \Delta A_d(t)$, $B(t) = B + \Delta B(t)$, $B_d(t) = B_d + \Delta B_d(t)$, $C(t) = C + \Delta C(t)$, $C_d(t) = C_d + \Delta C_d(t)$. Also, $D_1(\cdot)$, $D_2(\cdot)$ and $E(\cdot)$ satisfy $D_1(t) = D_1 + \Delta D_1(t)$, $D_2(t) = D_2 + \Delta D_2(t)$, and $E(t) = E + \Delta E(t)$, respectively. Here, $A, A_d, B, B_d, C, D_1, D_2$ and E are known real constant matrices, while $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D_1(t)$, $\Delta D_2(t)$ and $\Delta E(t)$ are unknown matrices representing time-varying uncertainties, which are assumed to satisfy the following conditions:

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta D_1(t) \\ \Delta B(t) & \Delta B_d(t) & \Delta E(t) \\ \Delta C(t) & \Delta C_d(t) & \Delta D_2(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} F(t) \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}, \tag{4}$$

where $M_i (i = 1, 2, 3)$ and $N_i (i = 1, 2, 3)$ are known real constant matrices and $F(t)$ is the unknown Lebesque-measurable matrix-valued function subject to the following condition:

$$F^T(t)F(t) \leq I, \ \forall t. \tag{5}$$

The conditions (4)-(5) are referred to as the *admissible* conditions. For vector-valued functions f, f_d, g and ϕ, we assume:

$$[f(x) - f(y) - R_1(x-y)]^T[f(x) - f(y) - R_2(x-y)] \leq 0, \quad \forall x, y \in \mathbb{R}^n, \tag{6}$$
$$[f_d(x) - f_d(y) - U_1(x-y)]^T[f_d(x) - f_d(y) - U_2(x-y)] \leq 0, \quad \forall x, y \in \mathbb{R}^n, \tag{7}$$
$$[g(x) - g(y) - S_1(x-y)]^T[g(x) - g(y) - S_2(x-y)] \leq 0, \quad \forall x, y \in \mathbb{R}^n, \tag{8}$$
$$[\phi(x) - \phi(y) - W_1(x-y)]^T[\phi(x) - \phi(y) - W_2(x-y)] \leq 0, \quad \forall x, y \in \mathbb{R}^n, \tag{9}$$

where $R_1, R_2, U_1, U_2 \in \mathbb{R}^{n\times n}$ and $S_1, S_2, W_1, W_2 \in R^{r\times n}$ are known real constant matrices.

Remark 1: As in [12], the nonlinear functions f, f_d, ϕ, g are said to belong to sectors [12]. In other words, the nonlinearities are bounded by sectors. The nonlinear descriptions in (6)-(9) are quite general that include the usual Lipschitz conditions as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied, see e.g. [9, 13, 14].

In what follows, for presentation simplicity and without loss of generality, we always assume that:

$$f(0) = 0, \ f_d(0) = 0, \ g(0) = 0, \ \phi(0) = 0. \tag{10}$$

With the above assumptions, the system (1)-(3) can be rewritten as

$$(\Sigma'): \quad \begin{aligned} dx(t) &= [A(t)x(t) + f(x(t)) + A_d(t)x(t-\tau(t)) + f_d(x(t-\tau(t))) + D_1(t)v(t)]dt \\ &\quad + [B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)]dw(t), \end{aligned} \tag{11}$$
$$y(t) = C(t)x(t) + \phi(x(t)) + C_d(t)x(t-\tau(t)) + g(x(t-\tau(t))) + D_2(t)v(t), \tag{12}$$
$$z(t) = Lx(t). \tag{13}$$

In this paper, we aim at obtaining the estimation $\hat{z}(t)$ of the output $z(t)$ in (Σ'). To be more specific, we are interested in constructing the following full-order filter:

$$(\Sigma_f): \quad d\hat{x}(t) = A_f\hat{x}(t)dt + B_f y(t)dt, \tag{14}$$
$$\hat{z}(t) = L\hat{x}(t), \tag{15}$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{z} \in \mathbb{R}^q$, and the constant matrices A_f and B_f are filter parameters to be determined.

Let $\tilde{x} = x(t) - \hat{x}(t)$ and $\tilde{z} = z(t) - \hat{z}(t)$. Then, from the systems (Σ') and (Σ_f), the filtering error dynamics can be described by:

$$\begin{aligned}(\Sigma_e):\ dx(t) &= [A(t)x(t) + A_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t))) + D_1(t)v(t)]dt \\ &\quad + [B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)]dw(t), \qquad (16)\\ d\tilde{x}(t) &= \Big[\tilde{C}(t)x(t) + A_f\tilde{x}(t) + \tilde{C}_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t))) \\ &\quad - B_f g(x(t-\tau(t))) - B_f\phi(x(t)) + \tilde{D}(t)v(t)\Big]dt \\ &\quad + [B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)]dw(t), \qquad (17)\\ \tilde{z}(t) &= L\tilde{x}(t), \qquad (18)\end{aligned}$$

where $\tilde{C}(t) = A(t) - A_f - B_f C(t)$, $\tilde{C}_d(t) = A_d(t) - B_f C_d(t)$, and $\tilde{D}(t) = D_1(t) - B_f D_2(t)$.

Assumption 1: The system (Σ') in (11)-(13) is asymptotically mean-square stable.

Remark 2: Assumption 1 is a prerequisite for the filtering error system (Σ_e) to be asymptotically mean-square stable. Since the filter (Σ_f) does not affect the state of the original system and $x(t)$ is part of the states of (Σ_e), the exponential mean-square stability of $x(t)$ is a necessary condition of the exponential mean-square stability of (Σ_e).

We are now in a position to formulate the robust H_∞ filter design problem to be addressed in this paper as follows: given a disturbance attenuation level $\gamma > 0$, design the parameters A_f and B_f for the filter (14)-(15) such that the filtering error system (Σ_e) is robustly asymptotically stable in the mean square for $v(t) = 0$ and satisfies $\|\tilde{z}\|_{\mathcal{E}2} \leq \gamma\|v\|_{\mathcal{E}2}$ under the zero-initial condition for any nonzero $v(t) \in \mathcal{L}_{\mathcal{E}2}([0,\infty);\mathbb{R}^p)$.

III. Main Results

First, we deal with the stability analysis problem for the filtering error system (Σ_e) with $v(t) = 0$ and derive an LMI condition that can guarantee the mean-square asymptotic stability of (Σ_e) with $v(t) = 0$.

Theorem 1: Let the filter parameters A_f and B_f be given. Then the filtering error system (Σ_e) with $v(t) \equiv 0$ is robustly asymptotically stable in the mean square if there exist six positive scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varepsilon_1, \varepsilon_2$ and three positive definite matrices P_1, P_2, P_3 such that the following LMI

$$\Psi = \begin{bmatrix} \Pi & \Sigma_{\tilde{C}}^T & \Omega & P_1 - \lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & B^T P_{12} & P_1 M_1 & 0 \\ * & \Sigma_1 + \Sigma_1^T & \Sigma_{\tilde{C}_d} & P_2 & P_2 & -\Sigma_2 & -\Sigma_2 & 0 & P_2 M_1 & \Sigma_2 M_3 \\ * & * & \Theta & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & B_d^T P_{12} & 0 & 0 \\ * & * & * & -\lambda_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\lambda_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\lambda_4 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -P_{12} & P_{12}M_2 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_1 I_1 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \qquad (19)$$

holds, where

$$\begin{aligned}
\Sigma_1 &= P_2A_f;\ \Sigma_2 = P_2B_f; &(20)\\
\breve{R}_1 &= (R_1^TR_2 + R_2^TR_1)/2;\ \breve{R}_2 = -(R_1^T + R_2^T)/2; &(21)\\
\breve{U}_1 &= (U_1^TU_2 + U_2^TU_1)/2;\ \breve{U}_2 = -(U_1^T + U_2^T)/2; &(22)\\
\breve{S}_1 &= (S_1^TS_2 + S_2^TS_1)/2;\ \breve{S}_2 = -(S_1^T + S_2^T)/2; &(23)\\
\breve{W}_1 &= (W_1^TW_2 + W_2^TW_1)/2;\ \breve{W}_2 = -(W_1^T + W_2^T)/2; &(24)\\
\Pi &= P_1A + A^TP_1 + P_3 - \lambda_1\breve{R}_1 - \lambda_4\breve{W}_1 + (\varepsilon_1 + \varepsilon_2)N_1^TN_1; &(25)\\
\Theta &= -(1-h)P_3 - \lambda_2\breve{U}_1 - \lambda_3\breve{S}_1 + (\varepsilon_1 + \varepsilon_2)N_2^TN_2; &(26)\\
\Omega &= P_1A_d + (\varepsilon_1 + \varepsilon_2)N_1^TN_2; &(27)\\
\Sigma_{\tilde{C}} &= P_2A - \Sigma_1 - \Sigma_2C; &(28)\\
\Sigma_{\tilde{C}_d} &= P_2A_d - \Sigma_2C_d; &(29)\\
P_{12} &= P_1 + P_2. &(30)
\end{aligned}$$

Proof: Construct the Lyapunov-Krasovskii functional as follows:

$$V_0(t) = x^T(t)P_1x(t) + \tilde{x}^T(t)P_2\tilde{x}(t) + \int_{t-\tau(t)}^{t} x^T(s)P_3x(s)ds. \quad (31)$$

By Itô differential formula [15, 21] and noticing that $v(t) \equiv 0$, the stochastic differential of $V_0(t)$ along the trajectory of system (Σ_e) with $v(t) = 0$ is given by

$$dV_0(t) = \mathcal{L}V_0(t)dt + 2[x^T(t)P_1 + \tilde{x}^T(t)P_2][B(t)x(t) + B_d(t)x(t-\tau(t))]dw(t), \quad (32)$$

where

$$\begin{aligned}
\mathcal{L}V_0(t) &= 2x^T(t)P_1\Big[A(t)x(t) + A_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t)))\Big]\\
&\quad + 2\tilde{x}(t)P_2\Big[\tilde{C}(t)x(t) + A_f\tilde{x}(t) + \tilde{C}_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t)))\\
&\quad - B_fg(x(t-\tau(t))) - B_f\phi(x(t))\Big] + x^T(t)P_3x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))P_3x(t-\tau(t))\\
&\quad + \Big[B(t)x(t) + B_d(t)x(t-\tau(t))\Big]^TP_{12}\Big[B(t)x(t) + B_d(t)x(t-\tau(t))\Big]. \quad (33)
\end{aligned}$$

Considering the fact that $\dot{\tau}(t) \le h < 1$, it is easy to see that

$$\mathcal{L}V_0(t) \le \xi_0^T(t)\Psi_1(t)\xi(t) + \vartheta_0^T(t)P_{12}\vartheta_0(t), \quad (34)$$

with

$$\begin{aligned}
\xi_0(t) &= [x^T(t)\ \tilde{x}^T(t)\ x^T(t-\tau(t))\ f^T(x(s)))\ f_d^T(x(s-\tau(s)))\ g^T(x(s-\tau(s)))\ \phi^T(x(s))]^T,\\
\vartheta_0(t) &= B(t)x(t) + B_d(t)x(t-\tau(t)),\\
\Psi_1(t) &= \begin{bmatrix}
P_1A(t) + A^T(t)P_1 + P_3 & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1 & P_1 & 0 & 0\\
P_2\tilde{C}(t) & P_2A_f + A_f^TP_2 & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f\\
A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2 & -(1-h)P_3 & 0 & 0 & 0 & 0\\
P_1 & P_2 & 0 & 0 & 0 & 0 & 0\\
P_1 & P_2 & 0 & 0 & 0 & 0 & 0\\
0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0\\
0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\end{aligned}$$

From (6) and (10), one has $[f(x)-R_1x]^T[f(x)-R_2x]\le 0$, which implies $[f(x(t))-R_1x(t)]^T[f(x(t))-R_2x(t)]\le 0$, or equivalently,

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} \breve{R}_1 & \breve{R}_2 \\ \breve{R}_2^T & I \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \le 0, \tag{35}$$

where $\breve{R}_1, \breve{R}_2$ are defined in (21).

Similarly, it follows from (7)-(10) that

$$\begin{bmatrix} x(t-\tau(t)) \\ f_d(x(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ f_d(x(t-\tau(t))) \end{bmatrix} \le 0, \tag{36}$$

$$\begin{bmatrix} x(t-\tau(t)) \\ g(x(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \\ \breve{S}_2^T & I \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ g(x(t-\tau(t))) \end{bmatrix} \le 0, \tag{37}$$

$$\begin{bmatrix} x(t) \\ \phi(x(t)) \end{bmatrix}^T \begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix} \begin{bmatrix} x(t) \\ \phi(x(t)) \end{bmatrix} \le 0, \tag{38}$$

where $\breve{U}_1, \breve{U}_2, \breve{S}_1, \breve{S}_2, \breve{W}_1$ and $\breve{W}_2$ are defined in (22)-(24).

It implies from (35)-(38) that

$$\begin{aligned} \mathcal{L}V_0(t) \le\ & \mathcal{L}V_0(t) - \lambda_1 \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} \breve{R}_1 & \breve{R}_2 \\ \breve{R}_2^T & I \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \\ & - \lambda_2 \begin{bmatrix} x(t-\tau(t)) \\ f_d(x(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} \breve{U}_1 & \breve{U}_2 \\ \breve{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ f_d(x(t-\tau(t))) \end{bmatrix} \\ & - \lambda_3 \begin{bmatrix} x(t-\tau(t)) \\ g(x(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \\ \breve{S}_2^T & I \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ g(x(t-\tau(t))) \end{bmatrix} \\ & - \lambda_4 \begin{bmatrix} x(t) \\ \phi(x(t)) \end{bmatrix}^T \begin{bmatrix} \breve{W}_1 & \breve{W}_2 \\ \breve{W}_2^T & I \end{bmatrix} \begin{bmatrix} x(t) \\ \phi(x(t)) \end{bmatrix} \\ \le\ & \xi_0^T(t)\Psi_2\xi_0(t) + \vartheta_0^T(t)P_{12}\vartheta_0(t) = \xi_0^T(t)[\Psi_2 + \bar{\vartheta}_0^T(t)P_{12}\bar{\vartheta}_0(t)]\xi_0(t), \end{aligned} \tag{39}$$

where

$$\begin{aligned} \bar{\vartheta}_0(t) &= \begin{bmatrix} B(t) & 0 & B_d(t) & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Psi_2(t) &= \begin{bmatrix} \Pi_1(t) & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 \\ P_2\tilde{C}(t) & P_2A_f + A_f^TP_2 & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f \\ A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2 & \Theta_1 & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1 I & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2 I & 0 & 0 \\ 0 & -B_f^TP_2 & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3 I & 0 \\ -\lambda_4\breve{W}_2^T & -B_f^TP_2 & 0 & 0 & 0 & 0 & -\lambda_4 I \end{bmatrix}, \end{aligned}$$

and $\Pi_1(t) = P_1A(t) + A^T(t)P_1 + P_3 - \lambda_1\breve{R}_1 - \lambda_4\breve{W}_1, \Theta_1 = -(1-h)P_3 - \lambda_2\breve{U}_1 - \lambda_3\breve{S}_1$.

Since $\mathbb{E}dV_0(t) = \mathbb{E}\mathcal{L}V(t)dt$, in order to show that the filtering error system is robustly asymptotically stable in the mean square with $v(t)=0$, we just need to prove that $\Psi_2 + \bar{\vartheta}_0^T(t)P_{12}\bar{\vartheta}_0(t) < 0$ which, by Schur Complement, is equivalent to

$$\Psi_3(t) < 0, \tag{40}$$

where

$$\Psi_3(t) = \begin{bmatrix} \Psi_2(t) & \bar{\vartheta}^T(t)P_{12} \\ P_{12}\bar{\vartheta}(t) & -P_{12} \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_1(t) & \Sigma_{\tilde{C}}^T & P_1A_d(t) & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & B^T(t)P_{12} \\ \Sigma_{\tilde{C}} & \Sigma_1+\Sigma_1^T & P_2\tilde{C}_d(t) & P_2 & P_2 & -\Sigma_2 & -\Sigma_2 & 0 \\ A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2^T & \Theta_1 & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & B_d^T(t)P_{12} \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2 I & 0 & 0 & 0 \\ 0 & -\Sigma_2^T & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3 I & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -\Sigma_2^T & 0 & 0 & 0 & 0 & -\lambda_4 I & 0 \\ P_{12}B(t) & 0 & P_{12}B_d(t) & 0 & 0 & 0 & 0 & -P_{12} \end{bmatrix}.$$

Notice that we can rewrite $\Psi_3(t)$ as follows:

$$\Psi_3(t) = \Psi_3 + \Delta\Psi_3(t), \tag{41}$$

where

$$\Psi_3 = \begin{bmatrix} \Pi_1 & \Sigma_{\tilde{C}}^T & P_1A_d & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & B^TP_{12} \\ \Sigma_{\tilde{C}} & \Sigma_1+\Sigma_1^T & \Sigma_{\tilde{C}_d} & P_2 & P_2 & -\Sigma_2 & -\Sigma_2 & 0 \\ A_d^TP_1 & \Sigma_{\tilde{C}_d}^T & \Theta_1 & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & B_d^TP_{12} \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2 I & 0 & 0 & 0 \\ 0 & -\Sigma_2^T & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3 I & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -\Sigma_2^T & 0 & 0 & 0 & 0 & -\lambda_4 I & 0 \\ P_{12}B & 0 & P_{12}B_d & 0 & 0 & 0 & 0 & -P_{12} \end{bmatrix}, \tag{42}$$

with $\Pi_1 = P_1A + A^TP_1 + P_3 - \lambda_1\breve{R}_1 - \lambda_4\breve{W}_1$ and

$$\Delta\Psi_3(t) = \begin{bmatrix} P_1\Delta A(t)+\Delta A^T(t)P_1 & \Delta A^T(t)P_2-\Delta C^T(t)\Sigma_2 & P_1\Delta A_d(t) & 0 & 0 & 0 & 0 & \Delta B^T(t)P_{12} \\ P_2\Delta A(t)-\Sigma_2\Delta C(t) & 0 & P_2A_d(t)-\Sigma_2\Delta C_d(t) & 0 & 0 & 0 & 0 & 0 \\ \Delta A_d^T(t)P_1 & A_d^T(t)P_2-\Delta C_d^T(t)\Sigma_2^T & 0 & 0 & 0 & 0 & 0 & \Delta B_d^T(t)P_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{12}\Delta B(t) & 0 & P_{12}\Delta B_d(t) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (4), it follows readily that

$$\Delta\Psi_3(t) = \hat{M}F(t)\hat{N} + \hat{N}^TF^T(t)\hat{M}^T - \hat{\Sigma}F(t)\hat{N} - \hat{N}^TF^T(t)\hat{\Sigma}^T,$$

where $\hat{M} = [M_1^TP_1 \quad M_1^TP_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad M_2^TP_{12}]^T$, $\hat{\Sigma} = [0 \quad M_3^T\Sigma_2^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$, and $\hat{N} = [N_1 \quad 0 \quad N_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$. Then, it is not difficult to see that

$$\Delta\Psi_3(t) \leq \varepsilon_1^{-1}\hat{M}\hat{M}^T + \varepsilon_2^{-1}\hat{\Sigma}\hat{\Sigma}^T + (\varepsilon_1+\varepsilon_2)\hat{N}^T\hat{N}. \tag{43}$$

Hence, from (41)-(43), it follows that:

$$\Psi_3(t) \leq \Psi_4 + \varepsilon_1^{-1}\hat{M}\hat{M}^T + \varepsilon_2^{-1}\hat{\Sigma}\hat{\Sigma}^T, \tag{44}$$

where

$$\Psi_4 = \begin{bmatrix} \Pi & \Sigma_{\tilde{C}}^T & \Omega & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & B^TP_{12} \\ \Sigma_{\tilde{C}} & \Sigma_1+\Sigma_1^T & \Sigma_{\tilde{C}_d} & P_2 & P_2 & -\Sigma_2 & -\Sigma_2 & 0 \\ \Omega^T & \Sigma_{\tilde{C}_d}^T & \Theta & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & B_d^TP_{12} \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2 I & 0 & 0 & 0 \\ 0 & -\Sigma_2^T & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3 I & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -\Sigma_2^T & 0 & 0 & 0 & 0 & -\lambda_4 I & 0 \\ P_{12}B & 0 & P_{12}B_d & 0 & 0 & 0 & 0 & -P_{12} \end{bmatrix}. \tag{45}$$

Observing (19) and using Schur Complement, it can be inferred that the right hand side of (44) is negative definite, and therefore $\Psi_3(t) < 0$. To this end, we can conclude from the Lyapunov stability theory that the filtering error system with $v(t) = 0$ is robustly asymptotically stable in the mean square. ■

Now, based on Theorem 1, we are able to focus on the analysis of the H_∞ performance of the filtering process in the following theorem.

Theorem 2: Given the filter parameters A_f and B_f and let γ be a known positive constant. Then the filtering error system (Σ_e) is robustly asymptotically stable in the mean square for $v(t) = 0$, and filtering error satisfies $\|\tilde{z}\|_{\mathcal{E}2} \leq \|v\|_{\mathcal{E}2}$ under zero initial condition if there exist three matrices $P_1 > 0, P_2 > 0, P_3 > 0$ and eight positive constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 such that the following LMI holds:

$$\Phi < 0, \tag{46}$$

where

$$\Phi = \left[\begin{array}{ccccccccccccc} \Pi & \Sigma_{\tilde{C}}^T & \Omega & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & P_1D_1 & B^TP_{12} & P_1M_1 & 0 & P_1M_1 & 0 \\ \Sigma_{\tilde{C}} & \Sigma_1+\Sigma_1^T+L^TL & \Sigma_{\tilde{C}_d} & P_2 & P_2 & -\Sigma_2 & -\Sigma_2 & \Sigma_{\tilde{D}} & 0 & P_2M_1 & \Sigma_2M_3 & P_2M_1 & \Sigma_2M_3 \\ \Omega^T & C_d^T\Sigma_2^T & \Theta & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & 0 & B_d^TP_{12} & 0 & 0 & 0 & 0 \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\Sigma_2^T & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -\Sigma_2^T & 0 & 0 & 0 & 0 & -\lambda_4 I & 0 & 0 & 0 & 0 & 0 & 0 \\ D_1^TP_1 & \Sigma_{\tilde{D}} & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I+\Upsilon & E^TP_{12} & 0 & 0 & 0 & 0 \\ P_{12}B & 0 & P_{12}B_d & 0 & 0 & 0 & 0 & P_{12}E & -P_{12} & P_{12}M_2 & 0 & P_{12}M_2 & 0 \\ M_1^TP_1 & M_1^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 & M_2^TP_{12} & -\varepsilon_1 I & 0 & 0 & 0 \\ 0 & M_3^T\Sigma_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I & 0 & 0 \\ M_1^TP_1 & M_1^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 & M_2^TP_{12} & 0 & 0 & -\varepsilon_3 I & 0 \\ 0 & M_3^T\Sigma_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_4 I \end{array} \right], \tag{47}$$

with $\Upsilon = (\varepsilon_3 + \varepsilon_4)N_3^TN_3$.

Proof: First, it is not difficult to verify that $\Psi < 0$ under the condition $\Phi < 0$. Therefore, according to Theorem 1, the filtering error system (Σ_e) with $v(t) = 0$ is robustly asymptotically stable in the mean square. It remains to deal with the H_∞ performance, i.e., show that under the given conditions the filtering error $\tilde{z}$ satisfies $\|\tilde{z}\|_{\mathcal{E}2} \leq \gamma\|v\|_{\mathcal{E}2}$.

Define the following Lyapunov candidate for system (Σ_e):

$$V(t) = x^T(t)P_1x(t) + \tilde{x}^T(t)P_2\tilde{x}(t) + \int_{t-\tau(t)}^{t} x^T(s)P_3x(s)ds. \tag{48}$$

Similar to the proof of Theorem 1 (but we do not impose the condition $v(t) \equiv 0$ now), from Itô differential formula, the stochastic differential of $V(t)$ along the trajectory of system (Σ_e) is given by

$$dV(t) = \mathcal{L}V(t)dt + 2[x^T(t)P_1 + \tilde{x}^T(t)P_2][B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)]dw(t), \tag{49}$$

where

$$\begin{aligned}
\mathcal{L}V(t) &= 2x^T(t)P_1\Big[A(t)x(t) + A_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t))) + D_1(t)v(t)\Big] \\
&\quad + 2\tilde{x}(t)P_2\Big[\tilde{C}(t)x(t) + A_f\tilde{x}(t) + \tilde{C}_d(t)x(t-\tau(t)) + f(x(t)) + f_d(x(t-\tau(t))) \\
&\quad - B_f g(x(t-\tau(t))) - B_f\phi(x(t)) + \tilde{D}(t)v(t)\Big] \\
&\quad + x^T(t)P_3x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))P_3x(t-\tau(t)) \\
&\quad + \Big[B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)\Big]^T(P_1+P_2)\Big[B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t)\Big] \\
&\leq \xi^T(t)\Phi_1(t)\xi(t) + \vartheta^T(t)P_{12}\vartheta(t)
\end{aligned} \tag{50}$$

with

$$\xi(t) = [x^T(t)\ \tilde{x}^T(t)\ x^T(t-\tau(t))\ f^T(x(t)))\ f_d^T(x(t-\tau(t)))\ g^T(x(t-\tau(t)))\ \phi^T(x(t))\ v^T(t)]^T, \tag{51}$$

$$\vartheta(t) = B(t)x(t) + B_d(t)x(t-\tau(t)) + E(t)v(t), \tag{52}$$

$$\Phi_1(t) = \begin{bmatrix}
P_1A(t) + A^T(t)P_1 + P_3 & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1 & P_1 & 0 & 0 & P_1D_1(t) \\
P_2\tilde{C}(t) & P_2A_f + A_f^TP_2 & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f & P_2\tilde{D}(t) \\
A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2 & -(1-h)P_3 & 0 & 0 & 0 & 0 & 0 \\
P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
D_1^T(t)P_1 & \tilde{D}^T(t)P_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{53}$$

To establish the H_∞ performance under the zero initial condition, we introduce

$$J(t) = \mathbb{E}\int_0^t [\tilde{z}^T(s)\tilde{z}(s) - \gamma^2v^T(s)v(s)]ds \tag{54}$$

where $t > 0$. Our goal is to prove that $J(t) < 0$. With the zero initial condition and $\mathbb{E}V(t) \geq 0$, it can be seen that for any nonzero $v(t) \in \mathcal{L}_{\mathcal{E}2}([0, +\infty); \mathbb{R}^p)$ and $t > 0$, we have

$$\begin{aligned}
J(t) &= \mathbb{E}\int_0^t \big[\tilde{z}^T(s)\tilde{z}(s) - \gamma^2v^T(s)v(s) + \mathcal{L}V(s)\big]ds - \mathbb{E}V(t) \\
&\leq \mathbb{E}\int_0^t \big[\tilde{x}(s)^TL^TL\tilde{x}(s) - \gamma^2v^T(s)v(s) + \mathcal{L}V(s)\big]ds. \\
&= \mathbb{E}\int_0^t \big[\xi^T(s)\Phi_2\xi(s) + \vartheta^T(s)P_{12}\vartheta(s)\big]ds,
\end{aligned}$$

where

$$\Phi_2(t)=\begin{bmatrix} P_1A(t)+A^T(t)P_1+P_3 & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1 & P_1 & 0 & 0 & P_1D_1(t) \\ P_2\tilde{C}(t) & P_2A_f+A_f^TP_2+L^TL & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f & P_2\tilde{D}(t) \\ A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2 & -(1-h)P_3 & 0 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_f^TP_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ D_1^T(t)P_1 & \tilde{D}^T(t)P_2 & 0 & 0 & 0 & 0 & 0 & -\gamma^2I \end{bmatrix}.$$

From the definition (52) of $\vartheta(s)$, it is easy to see that

$$\vartheta(t)=\begin{bmatrix}B(t) & 0 & B_d(t) & 0 & 0 & 0 & 0 & E(t)\end{bmatrix}\xi(t)=\bar{\vartheta}(t)\xi(t), \tag{55}$$

where $\bar{\vartheta}(t)=\begin{bmatrix}B(t) & 0 & B_d(t) & 0 & 0 & 0 & 0 & E(t)\end{bmatrix}$. Then, it follows from (35)-(38) that

$$\begin{aligned} J(t) \;\leq\;& J(t)-\mathbb{E}\int_0^t\Bigg\{\lambda_1\begin{bmatrix}x(s)\\ f(x(s))\end{bmatrix}^T\begin{bmatrix}\breve{R}_1 & \breve{R}_2\\ \breve{R}_2^T & I\end{bmatrix}\begin{bmatrix}x(s)\\ f(x(s))\end{bmatrix} \\ &+\lambda_2\begin{bmatrix}x(s-\tau(s))\\ f_d(x(s-\tau(s)))\end{bmatrix}^T\begin{bmatrix}\breve{U}_1 & \breve{U}_2\\ \breve{U}_2^T & I\end{bmatrix}\begin{bmatrix}x(s-\tau(s))\\ f_d(x(s-\tau(s)))\end{bmatrix} \\ &+\lambda_3\begin{bmatrix}x(s-\tau(s))\\ g(x(s-\tau(s)))\end{bmatrix}^T\begin{bmatrix}\breve{S}_1 & \breve{S}_2\\ \breve{S}_2^T & I\end{bmatrix}\begin{bmatrix}x(s-\tau(s))\\ g(x(s-\tau(s)))\end{bmatrix} \\ &+\lambda_4\begin{bmatrix}x(s)\\ \phi(x(s))\end{bmatrix}^T\begin{bmatrix}\breve{W}_1 & \breve{W}_2\\ \breve{W}_2^T & I\end{bmatrix}\begin{bmatrix}x(s)\\ \phi(x(s))\end{bmatrix}\Bigg\}ds \\ =\;& \mathbb{E}\int_0^t\xi^T(s)\Phi_3(s)\xi(s)+\vartheta^T(s)P_{12}\vartheta(s)ds=\mathbb{E}\int_0^t\xi^T(s)\Big[\Phi_3(s)+\bar{\vartheta}^T(s)P_{12}\bar{\vartheta}(s)\Big]\xi(s)ds, \end{aligned} \tag{56}$$

where

$$\Phi_3(t)=\begin{bmatrix} \Pi_1(t) & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & P_1D_1(t) \\ P_2\tilde{C}(t) & P_2A_f+A_f^TP_2+L^TL & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f & P_2\tilde{D}(t) \\ A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2 & \Theta_1 & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & 0 \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1I & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2I & 0 & 0 & 0 \\ 0 & -B_f^TP_2 & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3I & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -B_f^TP_2 & 0 & 0 & 0 & 0 & -\lambda_4I & 0 \\ D_1^T(t)P_1 & \tilde{D}^T(t)P_2 & 0 & 0 & 0 & 0 & 0 & -\gamma^2I \end{bmatrix}.$$

Then, from Schur Complement, we can have $\Phi_3(t)+\bar{\vartheta}^T(t)P_{12}\bar{\vartheta}(t)<0$, which is equivalent to $\Phi_4(t)<0$, where

$$\begin{aligned}\Phi_4(t)&=\begin{bmatrix}\Phi_3(t) & \bar{\vartheta}^T(t)P_{12}\\ P_{12}\bar{\vartheta}(t) & -P_{12}\end{bmatrix}\\ &=\begin{bmatrix} \Pi_1(t) & \tilde{C}^T(t)P_2 & P_1A_d(t) & P_1-\lambda_1\breve{R}_2 & P_1 & 0 & -\lambda_4\breve{W}_2 & P_1D_1(t) & B^T(t)P_{12} \\ P_2\tilde{C}(t) & P_2A_f+A_f^TP_2+L^TL & P_2\tilde{C}_d(t) & P_2 & P_2 & -P_2B_f & -P_2B_f & P_2\tilde{D}(t) & 0 \\ A_d^T(t)P_1 & \tilde{C}_d^T(t)P_2^T & \Theta_1 & 0 & -\lambda_2\breve{U}_2 & -\lambda_3\breve{S}_2 & 0 & 0 & B_d^T(t)P_{12} \\ P_1-\lambda_1\breve{R}_2^T & P_2 & 0 & -\lambda_1I & 0 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -\lambda_2\breve{U}_2^T & 0 & -\lambda_2I & 0 & 0 & 0 & 0 \\ 0 & -B_f^TP_2 & -\lambda_3\breve{S}_2^T & 0 & 0 & -\lambda_3I & 0 & 0 & 0 \\ -\lambda_4\breve{W}_2^T & -B_f^TP_2 & 0 & 0 & 0 & 0 & -\lambda_4I & 0 & 0 \\ D_1^T(t)P_1 & \tilde{D}^T(t)P_2^T & 0 & 0 & 0 & 0 & 0 & -\gamma^2I & E^T(t)P_{12} \\ P_{12}B(t) & 0 & P_{12}B_d(t) & 0 & 0 & 0 & 0 & P_{12}E(t) & -P_{12} \end{bmatrix}.\end{aligned}$$

In order to show $J(t) < 0$, it suffices to prove that $\Phi_4(t) < 0,\ \forall t > 0$. The rest of the proof is similar to that in Theorem 1, and is thus omitted. ■

Finally, we are ready to deal with the design problem for the robust H_∞ filters. The following result can be readily derived from Theorem 2, hence its proof is not given here.

Theorem 3: For the uncertain stochastic system (Σ) or (Σ'). For a given disturbance attenuation level $\gamma > 0$, the robust H_∞ filtering problem is solvable by a filter (Σ_f) if there exist five matrices $\Sigma_1, \Sigma_2, P_1 > 0, P_2 > 0, P_3 > 0$ and eight positive constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 such that the LMI (46) holds. In this case, the filtering parameters can be designed as

$$A_f = P_2^{-1}\Sigma_1, \quad B_f = P_2^{-1}\Sigma_2. \tag{57}$$

Remark 3: Theorem 3 shows that the feasibility of the filter design problem can be readily checked by the solvability of an LMI, which can be determined by using the Matlab LMI toolbox in a straightforward way. In the next section, an illustrative example will be provided to show the usefulness of the proposed techniques.

IV. Numerical Example

Consider the system (Σ'), where the nominal system matrix A and the measurement output matrix C are taken from the linearized model of an F-404 aircraft engine system in [5]:

$$A = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ 0.1643 & -0.4000 & -0.3788 \\ 0.3107 & 0 & -2.2300 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Virtually all aircraft engine systems are in some way disturbed by uncontrolled external forces. The disturbances may assume a myriad of forms such as wind gusts, gravity gradients, structural vibrations, or sensor and actuator noise, and may enter the systems in many different ways. These perturbations generally degrade the performance of the system and, in some cases, may even jeopardize the outcome of the engineering task. For example, random vibration of aircraft engine system, even in light aircraft, is important because random vibration analysis is needed to conduct accurate fatigue analysis and affect the design of engine control systems [11], so that the accurate fatigue life may be computed, and the engine design may be changed early and inexpensively if needed. As in [4], let the motion of the F-404 aircraft engine be determined by the system of stochastic differential equations derived from the basic aerodynamics, and the stochastic part of the motion is due to the changing wind. On the other hand, the time delay in the filtering process of an aircraft is mainly due to the computational load on the navigation computer, and there also exists a small amount of time delay in sensor signal processing.

Suppose that, when modeling the aircraft engine system, there exist modeling errors (parameter uncertainties), linearization errors (nonlinear disturbances), time delays and Itô-type stochastic perturbations. Accordingly, in addition to the main system parameters A and C, we set other parameters as follows:

$$A_d = \begin{bmatrix} 0.006 & -0.006 & 0.008 \\ 0.004 & -0.015 & 0.006 \\ -0.007 & -0.011 & -0.004 \end{bmatrix}, \; D_1 = \begin{bmatrix} -0.07 & 0.08 \\ -0.05 & 0.11 \\ 0.09 & -0.06 \end{bmatrix}, \; B = \begin{bmatrix} -0.05 & 0.08 & 0.06 \\ -0.05 & 0.11 & 0.07 \\ 0.06 & -0.08 & 0.12 \end{bmatrix},$$

$$B_d = \begin{bmatrix} -0.05 & 0.08 & 0.06 \\ -0.05 & 0.11 & 0.07 \\ 0.06 & -0.08 & 0.12 \end{bmatrix}, \; E = \begin{bmatrix} -0.1 & 0.08 \\ -0.06 & 0.22 \\ 0.04 & -0.08 \end{bmatrix}, \; C_d = \begin{bmatrix} 0.03 & 0.02 & 0.02 \\ -0.01 & 0.06 & 0.05 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -0.06 & 0.05 \\ -0.04 & 0.07 \end{bmatrix}, \ L = \begin{bmatrix} 0.42 & 0.35 & 0.28 \\ 0.28 & 0.49 & 0.14 \end{bmatrix}, \ R_1 = U_1 = \begin{bmatrix} 0.02 & 0.01 & 0.03 \\ 0.02 & 0.04 & 0.01 \\ 0.03 & 0.04 & 0.03 \end{bmatrix},$$

$$R_2 = U_2 = \begin{bmatrix} -0.04 & -0.01 & -0.02 \\ -0.02 & -0.02 & -0.01 \\ -0.01 & -0.04 & -0.02 \end{bmatrix}, \ S_1 = W_1 = \begin{bmatrix} 0.03 & 0.01 & 0.02 \\ 0.02 & 0.04 & 0.01 \end{bmatrix},$$

$$S_2 = W_2 = \begin{bmatrix} -0.04 & -0.02 & -0.01 \\ -0.01 & -0.03 & -0.03 \end{bmatrix}, \ M_1 = M_2 = \begin{bmatrix} 0.02 & 0.03 & 0.02 \end{bmatrix}^T,$$

$$M_3 = \begin{bmatrix} 0.02 & 0.03 \end{bmatrix}^T, \ N_1 = N_2 = \begin{bmatrix} 0.03 & 0.02 & 0.02 \end{bmatrix}, \ N_3 = \begin{bmatrix} 0.02 & 0.03 \end{bmatrix}, \ h = 0.2.$$

The H_∞ performance level is taken as $\gamma = 0.9$. With the above parameters and by using the Matlab LMI toolbox, we solve the LMI (46) and obtain

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.4171 & 0.0841 & 0.3792 \\ 0.0841 & 1.3486 & -0.0277 \\ 0.3792 & -0.0277 & 3.0670 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1.2684 & -0.0008 & -0.0870 \\ -0.0008 & 1.6909 & 0.1680 \\ -0.0870 & 0.1680 & 2.2160 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 1.9754 & -0.1121 & -1.8369 \\ -0.1121 & 0.3629 & 0.1450 \\ -1.8369 & 0.1450 & 4.9151 \end{bmatrix}, \ \Sigma_1 = \begin{bmatrix} -2.3828 & -0.3098 & 2.2705 \\ -0.0127 & -2.3171 & -0.7701 \\ -0.8407 & 0.1939 & -3.4088 \end{bmatrix}, \\ \Sigma_2 &= \begin{bmatrix} 1.0717 & 0.2774 \\ 0.2141 & 2.0586 \\ 1.5035 & -0.2626 \end{bmatrix}, \ \lambda_1 = 7.8068, \ \lambda_2 = 4.4338, \ \lambda_3 = 7.7825, \ \lambda_4 = 5.6870, \\ \varepsilon_1 &= 5.2056, \ \varepsilon_2 = 5.1990, \ \varepsilon_3 = 5.2576, \ \varepsilon_4 = 5.2510. \end{aligned}$$

Therefore, the filter parameters can be designed as

$$A_f = P_2^{-1}\Sigma_1 = \begin{bmatrix} -1.9099 & -0.2325 & 1.6905 \\ 0.0371 & -1.3887 & -0.3107 \\ -0.4572 & 0.1837 & -1.4483 \end{bmatrix}, \ B_f = P_2^{-1}\Sigma_2 = \begin{bmatrix} 0.8936 & 0.2054 \\ 0.0565 & 1.2378 \\ 0.7093 & -0.2043 \end{bmatrix}.$$

V. Conclusions

In this paper, we have studied the robust H_∞ filtering problem for a class of uncertain nonlinear time-delay stochastic systems. The system under study involves parameter uncertainties, stochastic disturbances, time-varying delays and inherent sector-like nonlinearities. An effective linear matrix inequality (LMI) approach has been proposed to design the filters such that, for all admissible nonlinearities and time-delays, the overall uncertain filtering error dynamics is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We have first investigated the sufficient conditions for the filtering error dynamics to be stable in the mean square, and then derived the explicit expression of the desired controller gains. A numerical example has been provided to show the usefulness and effectiveness of the proposed design method. It is possible to extend the main results to the discrete-time systems by using delay-dependent techniques [6, 7], which is one of the future research topics.

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