Asymptotic Stability for Neural Networks with Mixed Time-Delays: The Discrete-Time Case

Yurong Liu, Zidong Wang* and Xiaohui Liu

Abstract

This paper is concerned with the stability analysis problem for a new class of discrete-time recurrent neural networks with mixed time-delays. The mixed time-delays that consist of both the discrete and distributed time-delays are addressed, for the first time, when analyzing the asymptotic stability for discrete-time neural networks. The activation functions are not required to be differentiable or strictly monotonic. The existence of the equilibrium point is first proved under mild conditions. By constructing a new Lyapnuov-Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the discrete-time neural networks to be globally asymptotically stable. As an extension, we further consider the stability analysis problem for the same class of neural networks but with state-dependent stochastic disturbances. All the conditions obtained are expressed in terms of LMIs whose feasibility can be easily checked by using the numerically efficient Matlab LMI Toolbox. A simulation example is presented to show the usefulness of the derived LMI-based stability condition.

Keywords

Discrete-time neural networks; stochastic neural networks; asymptotic stability; discrete time-delays; distributed time-delays; Lyapunov-Krasovskii functional; linear matrix inequality.

I. Introduction

In the past few decades, recurrent neural networks (RNNs) have received intensive interest due to their wide applications in a variety of areas including such as pattern recognition, associative memory and combinational optimization. Dynamical behaviors (e.g. stability, instability, periodic oscillatory and chaos) of the neural networks are known to be crucial in applications. For instance, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium. Therefore, stability analysis of neural networks has received much attention and various stability conditions have been obtained.

Time delay is an inherent feature of signal transmission between neurons, and becomes one of the main sources for causing instability and poor performances of neural networks (see e.g. [1,5,6]). According to the way it occurs, time-delay can be classified as two types: discrete and distributed. Discrete time-delay is relatively easier to be identified in practice and, therefore, stability analysis for RNNs with discrete delays has been an attractive subject of research in the past few years. Various sufficient conditions, either delay-dependent

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or delay-independent, have been proposed to guarantee the global asymptotic or exponential stability for the RNNs, see e.g. [3, 19, 23, 24] for some recent publications. On the other hand, due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, a neural network usually has a spatial nature. Therefore, it is necessary to introducing continuously *distributed delays* over a certain duration of time such that the distant past has less influence compared to the recent behavior of the state [18, 21]. Recently, the global stability analysis problem for general RNNs with *both discrete and distributed delays* (or called *mixed time-delays*) has received increasing research attention and many relevant results have been reported in the literature, see e.g. [12, 23, 24] and the references therein.

It should be pointed out that, to date, almost all results concerning dynamics analysis problems for RNNs with mixed time-delays have been on continuous-time models. In implementing and applications of neural networks, however, *discrete-time* neural networks play a more important role than their continuous-time counterparts in today's digital world. If one wants to simulate or compute the continuous-time neural network, it is essential to formulate the discrete-time analogue so as to investigate the dynamical characteristics [15, 16, 20]. In the past few years, various stability criteria have been proposed for discrete-time neural networks (DNNs) in the literature, see e.g. [7, 22, 26–29] for DNNs without time delays and [4, 8, 9, 25] for DNNs with discrete time-delays. Note that pioneering work has been carried out in [17] for preserving exponential stability in discrete-time analogues of artificial neural networks with distributed delays.

It has now been well recognized that, in implementations of neural networks, stochastic disturbances are nearly inevitable owing to thermal noise in electronic devices. It has also been shown that certain stochastic inputs could destabilize a neural network. Therefore, the stability analysis problem for discrete-time stochastic neural networks with time-delays becomes more significant from the practical point of view, and initial results related to this problem has recently been published in [13] and the references therein. Unfortunately, so far, the stability analysis problem for *discrete-time stochastic* neural networks with *mixed time-delays* has not been fully investigated yet and remains challenging. The major difficulty stems from the question that how to represent the distributed time-delays in the discrete-time domain and then establish a unified framework to handle both the discrete and distributed time-delays. The main purpose of the present research is to make the first attempt to shorten such a gap.

In this paper, we study the asymptotic stability problem for a new class of *discrete-time* stochastic neural networks with both discrete and distributed time-delays. We first deal with the deterministic neural network. The existence of the equilibrium point is proved under mild conditions on the activation functions, where neither differentiability nor monotonicity is needed . By constructing a new Lyapnuov-Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the discrete-time neural networks to be globally asymptotically stable. As an extension, we then consider the stability analysis problem for the same class of neural networks but with state-dependent stochastic disturbances. All the conditions obtained are expressed in terms of LMIs whose feasibility can be easily checked by using the numerically efficient Matlab LMI Toolbox. Note that LMIs can be easily solved by using the Matlab LMI toolbox, and no tuning of parameters is required [2]. A simulation example is presented to show the usefulness of the derived LMI-based stability condition.

Notations: The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry. For vector or matrix z, $z \succeq 0$ means that each entry of z is nonnegative. I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. Matrices, if not explicitly specified, are assumed to have compatible dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation

Consider the following $n-$neuron discrete-time neural network with discrete and distributed delays of the form:

$$\begin{aligned} u_i(k+1) &= a_i u(k) + \sum_{j=1}^{n} b_{ij}\hat{f}_j(u_j(k)) + \sum_{j=1}^{n} c_{ij}\hat{g}_j(u_j(k-\tau(k))) \\ &\quad + \sum_{j=1}^{n} d_{ij} \sum_{m=1}^{+\infty} \mu_m \hat{h}_j(u_j(k-m)) + J_j, \qquad i = 1,2,...,n, \end{aligned} \tag{1}$$

or, in an equivalent vector form

$$u(k+1) = Au(k) + B\hat{F}(u(k)) + C\hat{G}(u(k-\tau(k))) + D\sum_{m=1}^{+\infty}\mu_m \hat{H}(u(k-m)) + J \tag{2}$$

where $u(k) = (u_1(k), u_2(k), ..., u_n(k))^T$ is the neural state vector, $A = \mathrm{diag}\{a_1, a_2, ..., a_n\}$ with $|a_i| < 1$ is the state feedback coefficient matrix; the $n \times n$ matrices $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$ and $D = [d_{ij}]_{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix and distributively delayed connection weight matrix. The positive integer $\tau(k)$ denotes the time-varying delay satisfying

$$\tau_m \le \tau(k) \le \tau_M, \quad k \in \mathbb{N}, \tag{3}$$

where τ_m and $\tau_M(k)$ are known positive integers. In (2), $\hat{F}(u(k)) = [\hat{f}_1(u_1(k)), \hat{f}_2(u_2(k)), ..., \hat{f}_n(u_n(k))]^T$, $\hat{G}(u(k)) = [\hat{g}_1(u_1(k)), \hat{g}_2(u_2(k)), ..., \hat{g}_n(u_n(k))]^T$ and $\hat{H}(u(k)) = [\hat{h}_1(u_1(k)), \hat{h}_2(u_2(k)), ..., \hat{h}_n(u_n(k))]^T$ denote the neuron activation functions. The constant vector $J = [J_1, J_2, ..., J_n]^T$ is the exogenous input and μ_m ($m = 1, 2, ...$) are scalar constants.

Remark 1: The model (1) or (2) is quite general and can be seen as the discrete analog of the following well-studied continuous-time RNN with mixed time delay:

$$\frac{du}{dt} = Au + BF(u(t)) + CG(u(t-\tau(t)))) + D\int_{-\infty}^{t} k(t-s)H(u(s))ds + J.$$

The activation functions are usually assumed to continuous, differentiable, monotonically increasing and bounded, such as the sigmoid-type of function. However, in many electronic circuits, the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network. In this paper, we make following assumptions for the neuron activation functions.

Assumption 1: For $i \in \{1, 2, ..., n\}$, the neuron activation functions $\hat{f}_i(\cdot)$, $\hat{g}_i(\cdot)$ and $\hat{h}_i(\cdot)$ in (1) or (2) are continuous and bounded.

Assumption 2: For $i \in \{1, 2, ..., n\}$, the neuron activation functions in (1) or (2) satisfies

$$l_i^- \leq \frac{\hat{f}_i(s_1) - \hat{f}_i(s_2)}{s_1 - s_2} \leq l_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R}, \tag{4}$$

$$v_i^- \leq \frac{\hat{g}_i(s_1) - \hat{g}_i(s_2)}{s_1 - s_2} \leq v_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R}, \tag{5}$$

$$\sigma_i^- \leq \frac{\hat{h}_i(s_1) - \hat{h}_i(s_2)}{s_1 - s_2} \leq \sigma_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R}, \tag{6}$$

where $l_i^-,\ l_i^+,\ v_i^-,\ v_i^+,\ \sigma_i^-,\ \sigma_i^+$ are some constants.

Remark 2: Assumption 2 was first introduced in [12, 13]. The constants $l_i^-,\ l_i^+,\ v_i^-,\ v_i^+,\ \sigma_i^-,\ \sigma_i^+$ in Assumption 2 are allowed to be positive, negative or zero. Hence, the resulting activation functions may be non-monotonic, and more general than the usual sigmoid functions and Lipschitz-type conditions. Such a description is very precise/tight in quantifying the lower and upper bounds of the activation functions, hence very helpful for using LMI-based approach to reduce the possible conservatism.

Assumption 3: The constant $\mu_m \geq 0$ satisfies the following convergent condition:

$$\sum_{m=1}^{+\infty} \mu_m < +\infty \ \text{ and } \ \sum_{m=1}^{+\infty} m\mu_m < +\infty. \tag{7}$$

Remark 3: Assumptions 1 and 3 make sense as they guarantee that the term $D \sum_{m=1}^{+\infty} \mu_m \hat{H}(x(k-m))$ in (2) is convergent, which is necessary for the subsequent analysis.

Proposition 1: Under Assumption 1 and Assumption 3, there exists an equilibrium point for system (2).

Proof: It is easy to verify that, under the given conditions, the function $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(u) = Au + B\hat{F}(u) + C\hat{G}(u) + D \sum_{m=1}^{+\infty} \mu_m \hat{H}(u)$ is continuous. Then what remains is just a standard exercise to prove the existence of the equilibrium point by using Brower' Fixed Point theorem. The proof is therefore omitted here for simplicity. ■

Definition 1: Let $u^* = [u_1^*, u_2^*, ..., u_n^*]$. The discrete-time neural network (2) is said to be globally asymptotically stable if each solution $u(k)$ of the neural network (2) satisfies

$$\lim_{k \to +\infty} |u(k) - u^*| = 0.$$

In the rest of this paper, we will focus on the problem of stability analysis for the discrete time neural network (2). By utilizing a new Lyapunov-Krasoviskii functional, we will develop an LMI approach to derive sufficient conditions under which the neural network (2) is globally asymptotically stable. To facilitate the readers, we will start with deterministic neural networks and then extend the main results to stochastic neural networks without major difficulties.

III. Asymptotic Stability of Deterministic Discrete-time Neural Network

The following lemmas will be needed in our main derivation.

Lemma 1: Let $X,\ Y$ be any n-dimensional real vectors and P be a $n \times n$ positive semi-definite matrix. Then, the following matrix inequality holds:

$$2X^T PY \leq X^T PX + Y^T PY.$$

Lemma 2: [14] Let $M \in \mathbb{R}^n$ be a positive semi-definite matrix, $\mathbf{x}_i \in \mathbb{R}^n$ and $a_i \geq 0$ $(i = 1, 2, ...)$. If the series concerned are convergent, the following inequality holds:

$$\left(\sum_{i=1}^{+\infty} a_i \mathbf{x}_i\right)^T M \left(\sum_{i=1}^{+\infty} a_i \mathbf{x}_i\right) \le \left(\sum_{i=1}^{+\infty} a_i\right) \sum_{i=1}^{+\infty} a_i \mathbf{x}_i^T M \mathbf{x}_i \tag{8}$$

We are now ready to state our main results of this paper in the following theorem.

Theorem 1: Under Assumptions 1–3, the discrete time neural network (2) is globally asymptotically stable if there exist three diagonal matrices $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\} > 0$, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, ..., \gamma_n\} > 0$, and $\Delta = \mathrm{diag}\{\delta_1, \delta_2, ..., \delta_n\} > 0$, and three positive definite matrices P, Q and R such that the following LMI holds:

$$\Omega = \begin{bmatrix} \Xi_1 & A^TPB + \Lambda L_2 & \Gamma\Upsilon_2 & A^TPC & \Delta\Sigma_2 & A^TPD \\ * & B^TPB - \Lambda & 0 & B^TPC & 0 & B^TPD \\ * & * & \Xi_2 & 0 & 0 & 0 \\ * & * & * & C^TPC - Q & 0 & 0 \\ * & * & * & * & \bar{\mu}R - \Delta & 0 \\ * & * & * & * & * & D^TPD - \bar{\mu}^{-1}R \end{bmatrix} < 0, \tag{9}$$

where

$$L_1 = \mathrm{diag}\{l_1^+ l_1^-, l_2^+ l_2^-, ..., l_n^+ l_n^-\}, \quad L_2 = \mathrm{diag}\{\frac{l_1^+ + l_1^-}{2}, \frac{l_2^+ + l_2^-}{2}, ..., \frac{l_n^+ + l_n^-}{2}\}, \tag{10}$$

$$\Upsilon_1 = \mathrm{diag}\{v_1^+ v_1^-, v_2^+ v_2^-, ..., v_n^+ v_n^-\}, \quad \Upsilon_2 = \mathrm{diag}\{\frac{v_1^+ + v_1^-}{2}, \frac{v_2^+ + v_2^-}{2}, ..., \frac{v_n^+ + v_n^-}{2}\}, \tag{11}$$

$$\Sigma_1 = \mathrm{diag}\{\sigma_1^+ \sigma_1^-, \sigma_2^+ \sigma_2^-, ..., \sigma_n^+ \sigma_n^-\}, \quad \Sigma_2 = \mathrm{diag}\{\frac{\sigma_1^+ + \sigma_1^-}{2}, \frac{\sigma_2^+ + \sigma_2^-}{2}, ..., \frac{\sigma_n^+ + \sigma_n^-}{2}\}. \tag{12}$$

$$\Xi_1 = A^TPA - P - \Lambda L_1 - \Gamma\Upsilon_1 - \Delta\Sigma_1, \quad \Xi_2 = (d_M - d_m + 1)Q - \Gamma, \tag{13}$$

$$\bar{\mu} = \sum_{m=1}^{+\infty} \mu_k \tag{14}$$

Proof: First, by Proposition 1, the discrete time neural network (2) has an equilibrium point u^*. For convenience, we shift the equilibrium u^* to origin by letting $x(k) = u(k) - u^*$, and then the system (2) can be transformed into

$$x(k+1) = Ax(k) + BF(x(k)) + CG(x(k - \tau(k))) + D\sum_{m=1}^{+\infty} \mu_m H(x(k-m)), \tag{15}$$

where $x(k) = [x_1(k), x_2(k), ..., x_n(k)]^T \in \mathbb{R}^n$ is the state vector of the transformed system, and the transformed neuron activation functions are

$$\begin{aligned} F(x(k)) : &= (f_1(x_1(k)), f_2(x_2(k)), ..., f_n(x_n(k)))^T = \hat{F}(u(k)) - \hat{F}(u^*), \\ G(x(k)) : &= (g_1(x_1(k)), g_2(x_2(k)), ..., g_n(x_n(k)))^T = \hat{G}(u(k)) - \hat{G}(u^*), \\ H(x(k)) : &= (h_1(x_1(k)), h_2(x_2(k)), ..., h_n(x_n(k)))^T = \hat{H}(u(k)) - \hat{H}(u^*). \end{aligned}$$

By Assumption 2, it can be verified readily that the transformed neuron activation functions satisfy

$$l_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le l_i^+, \tag{16}$$

$$v_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le v_i^+, \tag{17}$$

$$\sigma_i^- \le \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \tag{18}$$

In order to show the stability of the neural network (2), we just need to deal with the stability of the system (15). To this end, we introduce the following Lyapunov-Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{19}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k)Px(k), &(20)\\
V_2(k) &= \sum_{i=k-\tau(k)}^{k-1} G^T(x(i))QG(x(i)), &(21)\\
V_3(k) &= \sum_{j=k-\tau_M+1}^{k-\tau_m} \sum_{i=j}^{k-1} G^T(x(i))QG(x(i)), &(22)\\
V_4(k) &= \sum_{i=1}^{+\infty} \mu_i \sum_{j=k-i}^{k-1} H^T(x(j))RH(x(j)) &(23)
\end{aligned}$$

Notice that from the conditions in (7), $V_4(k)$ is convergent. Now, calculating the difference of $V(k)$ along the system (15), we have

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k), \tag{24}$$

where

$$\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k)\\
&= \Big(Ax(k) + BF(x(k)) + CG(x(k-\tau(k))) + D\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\Big)^T P\Big(Ax(k) + BF(x(k))\\
&\quad + CG(x(k-\tau(k))) + D\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\Big) - x^T(k)Px(k)\\
&= x^T(k)A^T PAx(k) + F^T(x(k))B^T PBF(x(k)) + G^T(x(k-\tau(k)))C^T PCG(x(k-\tau(k)))\\
&\quad + \Big(D\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\Big)^T PD\sum_{m=1}^{+\infty} \mu_m H(x(k-m)) + 2x^T(k)A^T PBF(x(k))\\
&\quad + 2x^T(k)A^T PCG(x(k-\tau(k))) + 2x^T(k)A^T PD\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\\
&\quad + 2F^T(x(k))B^T PCG(x(k-\tau(k))) + 2F^T(x(k))B^T PD\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\\
&\quad + 2\Big(CG(x(k-\tau(k)))\Big)^T PD\sum_{m=1}^{+\infty} \mu_m H(x(k-m)) - x^T(k)Px(k), \qquad (25)
\end{aligned}$$

$$\begin{aligned}\Delta V_2(k) &= V_2(k+1) - V_2(k) \\ &= \sum_{i=k+1-\tau(k+1)}^{k} G^T(x(i))QG(x(i)) - \sum_{i=k-\tau(k)}^{k-1} G^T(x(i))QG(x(i)) \\ &= G^T(x(k))QG(x(k)) - G^T(x(k-\tau(k)))QG(x(k-\tau(k))) \\ &\quad + \sum_{i=k-\tau(k+1)+1}^{k-1} G^T(x(i))QG(x(i)) - \sum_{i=k-\tau(k)+1}^{k-1} G^T(x(i))QG(x(i)) \\ &= G^T(x(k))QG(x(k)) - G^T(x(k-\tau(k)))QG(x(k-\tau(k))) + \sum_{i=k-\tau_m+1}^{k-1} G^T(x(i))QG(x(i)) \\ &\quad + \sum_{i=k-\tau(k+1)+1}^{k-\tau_m} G^T(x(i))QG(x(i)) - \sum_{i=k-\tau(k)+1}^{k-1} G^T(x(i))QG(x(i)) \\ &\leq G^T(x(k))QG(x(k)) - G^T(x(k-\tau(k)))QG(x(k-\tau(k))) + \sum_{i=k-\tau_M+1}^{k-\tau_m} G^T(x(i))QG(x(i)),\end{aligned} \tag{26}$$

$$\begin{aligned}\Delta V_3(k) &= V_3(k+1) - V_3(k) \\ &= \sum_{j=k-\tau_M+2}^{k-\tau_m+1} \sum_{i=j}^{k} G^T(x(i))QG(x(i)) - \sum_{j=k-\tau_M+1}^{k-\tau_m} \sum_{i=j}^{k-1} G^T(x(i))QG(x(i)) \\ &= \sum_{j=k-\tau_M+1}^{k-\tau_m} \sum_{i=j+1}^{k} G^T(x(i))QG(x(i)) - \sum_{j=k-\tau_M+1}^{k-\tau_m} \sum_{i=j}^{k-1} G^T(x(i))QG(x(i)) \\ &= \sum_{j=k-\tau_M+1}^{k-\tau_m} \big(G^T(x(k))QG(x(k)) - G^T(x(j))QG(x(j))\big) \\ &= (\tau_M - \tau_m)G^T(x(k))QG(x(k)) - \sum_{i=k-\tau_M+1}^{k-\tau_m} G^T(x(i))QG(x(i)),\end{aligned} \tag{27}$$

and

$$\begin{aligned}\Delta V_4(k) &= V_4(k+1) - V_4(k) \\ &= \sum_{i=1}^{+\infty} \mu_i \sum_{j=k+1-i}^{k} H^T(x(j))RH(x(j)) - \sum_{i=1}^{+\infty} \mu_i \sum_{j=k-i}^{k-1} H^T(x(j))RH(x(j)) \\ &= \sum_{i=1}^{+\infty} \mu_i \Big(H^T(x(k))RH(x(k)) - H^T(x(k-i))RH(x(k-i))\Big) \\ &= \sum_{i=1}^{+\infty} \mu_i H^T(x(k))RH(x(k)) - \sum_{i=1}^{+\infty} \mu_i H^T(x(k-i))RH(x(k-i)) \\ &= \bar{\mu} H^T(x(k))RH(x(k)) - \sum_{i=1}^{+\infty} \mu_i H^T(x(k-i))RH(x(k-i)) \\ &\qquad \text{(By Lemma 2)} \\ &\leq \bar{\mu} H^T(x(k))RH(x(k)) - \frac{1}{\bar{\mu}} \left(\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\right)^T R \left(\sum_{m=1}^{+\infty} \mu_m H(x(k-m))\right).\end{aligned} \tag{28}$$

Substituting (25)-(28) into (24) leads to

$$
\begin{aligned}
\Delta V(k) \leq{} & x^T(k)A^TPAx(k) + F^T(x(k))B^TPBF(x(k)) + G^T(x(k-\tau(k)))C^TPCG(x(k-\tau(k))) \\
& + \Big(D\sum_{m=1}^{+\infty}\mu_m H(x(k-m))\Big)^T PD\sum_{m=1}^{+\infty}\mu_m H(x(k-m)) + 2x^T(k)A^TPBF(x(k)) \\
& + 2x^T(k)A^TPCG(x(k-\tau(k))) + 2x^T(k)A^TPD\sum_{m=1}^{+\infty}\mu_m H(x(k-m)) \\
& + 2F^T(x(k))B^TPCG(x(k-\tau(k))) + 2F^T(x(k))B^TPD\sum_{m=1}^{+\infty}\mu_m H(x(k-m)) \\
& + 2\Big(CG(x(k-\tau(k)))\Big)^T PD\sum_{m=1}^{+\infty}\mu_m H(k-m) - x^T(k)Px(k), \\
& + (1+\tau_M-\tau_m)G^T(x(k))QG(x(k)) - G^T(x(k-\tau(k)))QG(x(k-\tau(k))) \\
& + \bar{\mu}H^T(x(k))RH(x(k)) - \frac{1}{\bar{\mu}}\left(\sum_{m=1}^{+\infty}\mu_m H(x(k-m))\right)^T R\left(\sum_{m=1}^{+\infty}\mu_m H(x(k-m))\right) \\
={} & \xi^T(k)\Omega_1\xi(k),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
\xi(k) &= \left[x^T(k) \;\; F^T(x(k)) \;\; G^T(x(k)) \;\; G^T(x(k-\tau(k))) \;\; H^T(x(k)) \;\; \sum_{m=1}^{+\infty}\mu_m H^T(x(k-m))\right]^T, \\
\Omega_1 &= \begin{bmatrix}
A^TPA-P & A^TPB & 0 & A^TPC & 0 & A^TPD \\
* & B^TPB & 0 & B^TPC & 0 & B^TPD \\
* & * & (d_M-d_m+1)Q & 0 & 0 & 0 \\
* & * & * & C^TPC-Q & 0 & 0 \\
* & * & * & * & \bar{\mu}R & 0 \\
* & * & * & * & * & D^TPD-\frac{1}{\bar{\mu}}R
\end{bmatrix}.
\end{aligned}
$$

Similar to [13], from (16), we have

$$
(f_i(x_i(k)) - l_i^+x_i(k))(f_i(x_i(k)) - l_i^-x_i(x)) \leq 0, \quad i = 1, 2, ..., n,
$$

which is equivalent to

$$
\begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix}^T \begin{bmatrix} l_i^+l_i^-e_ie_i^T & -\frac{l_i^++l_i^-}{2}e_ie_i^T \\ -\frac{l_i^++l_i^-}{2}e_ie_i^T & e_ie_i^T \end{bmatrix} \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix} \leq 0, \quad k = 1, 2, ..., n,
$$

where e_k denotes the unit column vector having "1" element on its kth row and zeros elsewhere.

Consequently,

$$
\sum_{i=1}^{n}\lambda_i \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix}^T \begin{bmatrix} l_i^+l_i^-e_ie_i^T & -\frac{l_i^++l_i^-}{2}e_ie_i^T \\ -\frac{l_i^++l_i^-}{2}e_ie_i^T & e_ie_i^T \end{bmatrix} \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix} \leq 0,
$$

namely

$$
\begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix}^T \begin{bmatrix} \Lambda L_1 & -\Lambda L_2 \\ -\Lambda L_2 & \Lambda \end{bmatrix} \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix} \leq 0. \tag{30}
$$

Similarly , from (17)-(18), we have

$$\begin{bmatrix} x(k) \\ G(x(k)) \end{bmatrix}^T \begin{bmatrix} \Gamma\Upsilon_1 & -\Gamma\Upsilon_2 \\ -\Gamma\Upsilon_2 & \Gamma \end{bmatrix} \begin{bmatrix} x(k) \\ G(x(k)) \end{bmatrix} \leq 0, \tag{31}$$

$$\begin{bmatrix} x(k) \\ H(x(k)) \end{bmatrix}^T \begin{bmatrix} \Delta\Sigma_1 & -\Delta\Sigma_2 \\ -\Delta\Sigma_2 & \Delta \end{bmatrix} \begin{bmatrix} x(k) \\ H(x(k)) \end{bmatrix} \leq 0. \tag{32}$$

Therefore, from (29) and (30)-(32), we obtain

$$\begin{aligned} \Delta V(k) \leq \xi^T(k)\Omega_1\xi(k) &- \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix}^T \begin{bmatrix} \Lambda L_1 & -\Lambda L_2 \\ -\Lambda L_2 & \Lambda \end{bmatrix} \begin{bmatrix} x(k) \\ F(x(k)) \end{bmatrix} \\ &- \begin{bmatrix} x(k) \\ G(x(k)) \end{bmatrix}^T \begin{bmatrix} \Gamma\Upsilon_1 & -\Gamma\Upsilon_2 \\ -\Gamma\Upsilon_2 & \Gamma \end{bmatrix} \begin{bmatrix} x(k) \\ G(x(k)) \end{bmatrix} \\ &- \begin{bmatrix} x(k) \\ H(x(k)) \end{bmatrix}^T \begin{bmatrix} \Delta\Sigma_1 & -\Delta\Sigma_2 \\ -\Delta\Sigma_2 & \Delta \end{bmatrix} \begin{bmatrix} x(k) \\ H(x(k)) \end{bmatrix} \\ = \xi^T(k)\Omega\xi(k). \end{aligned} \tag{33}$$

Since $\lambda_{\max}(\Omega) < 0$, from (33), it is obvious that

$$\Delta V(k) \leq \lambda_{\max}(\Omega)|x(k)|^2. \tag{34}$$

Letting N be a positive integer, the summation of both sides of (34) from 1 to N with respect to k yields

$$V(N) - V(0) \leq \lambda_{\max}(\Omega) \sum_{k=1}^{N} |x(k)|^2,$$

which implies that

$$-\lambda_{\max}(\Omega) \sum_{k=1}^{N} |x(k)|^2 \leq V(0).$$

By letting $N \to +\infty$, it can be seen that the series $\sum_{k=1}^{+\infty} |x(k)|^2$ is convergent, and therefore $|x(k)|^2 \to 0$. ■

If the distributed delay term disappears, i.e., $D = 0$, the neural network (2) reduces to

$$u(k+1) = Au(k) + B\hat{F}(u(k)) + C\hat{G}(u(k-\tau(k))) + J. \tag{35}$$

For the neural network (35), we have the following stability result.

Corollary 1: Under Assumptions 1 and 2, the DRNN (35) is globally asymptotically stable if there exist two diagonal matrices $\Lambda = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\} > 0$ and $\Gamma = \text{diag}\{\gamma_1, \gamma_2, ..., \gamma_n\} > 0$, and two positive definite matrices P and Q such that the following LMI holds:

$$\begin{bmatrix} W_1 & A^TPB + \Lambda L_2 & \Gamma\Upsilon_2 & A^TPC \\ * & B^TPB - \Lambda & 0 & B^TPC \\ * & * & \Xi_2 & 0 \\ * & * & * & C^TPC - Q \end{bmatrix} < 0$$

where $W_1 = A^TPA - P - \Lambda L_1 - \Gamma\Upsilon_1$ and Ξ_2 is defined in (13).

IV. Asymptotic Stability of Stochastic Discrete-time Neural Network

As discussed in the introduction, in the real world, stochastic disturbances is probably one of the main resources of the performance degradations of the implemented neural networks. In this section, based on the system (15), we further consider the following n-neuron discrete-time stochastic delayed neural network on a probability space $(\mathbf{\Omega}, \mathscr{F}, \mathcal{P})$:

$$x(k+1) = Ax(k) + BF(x(k)) + CG(x(k-\tau(k))) + D\sum_{m=1}^{+\infty}\mu_m H(x(k-m)) + \psi(x(k), x(k-\tau(k)), k)w(k), \quad (36)$$

where $w(k)$ is a scalar Wiener process (Brownian Motion) on $(\mathbf{\Omega}, \mathscr{F}, \mathcal{P})$ with

$$\mathbb{E}[w(k)] = 0, \ \mathbb{E}[w^2(k)] = 1, \text{ and } \ \mathbb{E}[w(i)w(j)] = 0 (i \neq j), \quad (37)$$

with $\mathbb{E}[\cdot]$ being the mathematical expectation operator. In (36), $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear vector function representing the disturbance intensity.

Assumption 4: The activation functions $F(x(k)) = (f_1(x_1(k)), f_2(x_2(k)), ..., f_n(x_n(k)))^T$, $G(x(k)) = (g_1(x_1(k)), g_2(x_2(k)), ..., g_n(x_n(k)))^T$ and $H(x(k)) = (h_1(x_1(k)), h_2(x_2(k)), ..., h_n(x_n(k)))^T$ satisfy $F(0) = G(0) = H(0)$ and (16)–(18).

Assumption 5: There exist a constant matrix K such that

$$\psi^T(x, y, k)\psi(x, y, k) \leq |Kx|, \ \forall x, \ y \in \mathbb{R}^n,$$

Definition 2: The stochastic neural network (36) is said to be globally asymptotically stable in the mean square if, for each solution $x(k)$ of (36), the following holds

$$\lim_{k\to+\infty} \mathbb{E}[|x(k)|^2] = 0.$$

For the stochastic neural network (36), we have the following stability results.

Theorem 2: Under Assumptions 1–5, the discrete time neural network (36) is globally asymptotically stable in the mean square if there exist a constant $\lambda_0 > 0$, three diagonal matrices $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\} > 0$, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, ..., \gamma_n\} > 0$, and $\Delta = \mathrm{diag}\{\delta_1, \delta_2, ..., \delta_n\} > 0$, and three positive definite matrices P, Q and R such that the following LMIs hold:

$$P < \lambda_0 I, \quad \begin{bmatrix} \Pi_1 & A^TPB + \Lambda L_2 & \Gamma\Upsilon_2 & A^TPC & \Delta\Sigma_2 & A^TPD \\ * & B^TPB - \Lambda & 0 & B^TPC & 0 & B^TPD \\ * & * & \Xi_2 & 0 & 0 & 0 \\ * & * & * & C^TPC - Q & 0 & 0 \\ * & * & * & * & \bar{\mu}R - \Delta & 0 \\ * & * & * & * & * & D^TPD - \frac{1}{\bar{\mu}}R \end{bmatrix} < 0, \quad (38)$$

where $\Pi_1 = A^TPA - P - \Lambda L_1 - \Gamma\Upsilon_1 - \Delta\Sigma_1 - \lambda_0 K^TK$ and Ξ_2 is defined as in Theorem 1.

Proof: The proof is a fairly straightforward combination of that of Theorem 1 and that of the main results in [13] concerning stochastic analysis, and is therefore omitted here to avoid duplication. ■

Remark 4: In Theorem 1 and Theorem 2, the criteria are established that ensures that the discrete neural networks with mixed delays are globally stable and such criteria are expressed in terms of the solution to certain LMIs. Note that LMIs can be effectively solved and checked by the algorithms such as the interior-point method from Matlab toolbox.

V. Numerical Example

In this section, a numerical example is presented to demonstrate the usefulness of the developed method on the asymptotic stability of the stochastic delayed neural network (36) with mixed time delays.

Consider the system (36) with the following parameters:

$$\begin{aligned} A &= \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \; B = \begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0 \\ 0 & -0.1 & -0.4 \end{bmatrix}, \; C = \begin{bmatrix} 0.4 & 0.2 & -0.1 \\ 0 & 0.2 & 0.3 \\ -0.1 & 0 & 0.2 \end{bmatrix}, \\ D &= \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.2 & 0.3 & 0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}, \; F(x(k)) = G(x(k)) = H(x(k)) = \begin{bmatrix} \tanh(0.8x_1(k)) \\ \tanh(0.6x_1(k)) \\ \tanh(-0.6x_1(k)) \end{bmatrix}, \\ \tau(k) &= 3 + [1 + (-1)^k]/2, \; \mu_k = e^{-4k} \end{aligned}$$

and

$$\psi(x(k), x(k-\tau(k))) = \operatorname{diag}\{0.2\cos(x_2(k-\tau(k)))|\sin x_1(k)|, \; 0.3x_2(k), \; 0.3\sin x_3(k)\}.$$

Given the above parameters, it can be verified that $\tau_m = 3, \tau_M = 4, \bar{\tau} = e^{-2}$ and

$$L_1 = \Upsilon_1 = \Delta_1 = \operatorname{diag}\{0, \; 0, \; 0\}, \quad L_2 = \Upsilon_2 = \Sigma_2 = \operatorname{diag}\{0.4, \; 0.3, \; -0.3\}, \quad K = \operatorname{diag}\{0.2, 0.3, 0.3\}.$$

By using the Matlab LMI Toolbox, we solve LMI (38) and obtain the feasible solutions as follows:

$$P = \begin{bmatrix} 2.7271 & 0.4618 & 0.2441 \\ 0.4618 & 3.2760 & -0.1929 \\ 0.2441 & -0.1929 & 3.6660 \end{bmatrix}, \; Q = \begin{bmatrix} 1.0238 & 0.2599 & -0.2992 \\ 0.2599 & 1.4755 & 0.1735 \\ -0.2992 & 0.1735 & 1.0787 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.5425 & -0.0825 & 0.0011 \\ -0.0825 & 1.0591 & -0.0809 \\ 0.0011 & -0.0809 & 0.6914 \end{bmatrix}, \; \Lambda = \begin{bmatrix} 1.4937 & 0 & 0 \\ 0 & 4.4576 & 0 \\ 0 & 0 & 5.9550 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 4.2627 & 0 & 0 \\ 0 & 7.0306 & 0 \\ 0 & 0 & 3.9344 \end{bmatrix}, \; \Delta = \begin{bmatrix} 0.3788 & 0 & 0 \\ 0 & 1.8656 & 0 \\ 0 & 0 & 1.2923 \end{bmatrix}, \; \lambda_0 = 4.2548.$$

Then, it follows from Theorem 2 that the system (36) with given parameters is globally asymptotically stable in the mean square, which is further verified by the simulation result given in Fig. 1.

VI. Conclusions

In this paper, we have studied the stability analysis problem for a new class of discrete-time recurrent neural networks with mixed time-delays that consist of both the discrete and distributed time-delays. The activation functions are not required to be differentiable or strictly monotonic. The existence of the equilibrium point has first been proved under mild conditions. By constructing a new Lyapnuov-Krasovskii functional, a linear matrix inequality (LMI) approach has been developed to establish sufficient conditions for the discrete-time neural networks to be globally asymptotically stable. As an extension, we have further considered the stability analysis problem for the same class of neural networks but with state-dependent stochastic disturbances. All the conditions obtained have been expressed in terms of LMIs whose feasibility can be easily checked by using the numerically efficient Matlab LMI Toolbox. A simulation example has been presented to show the

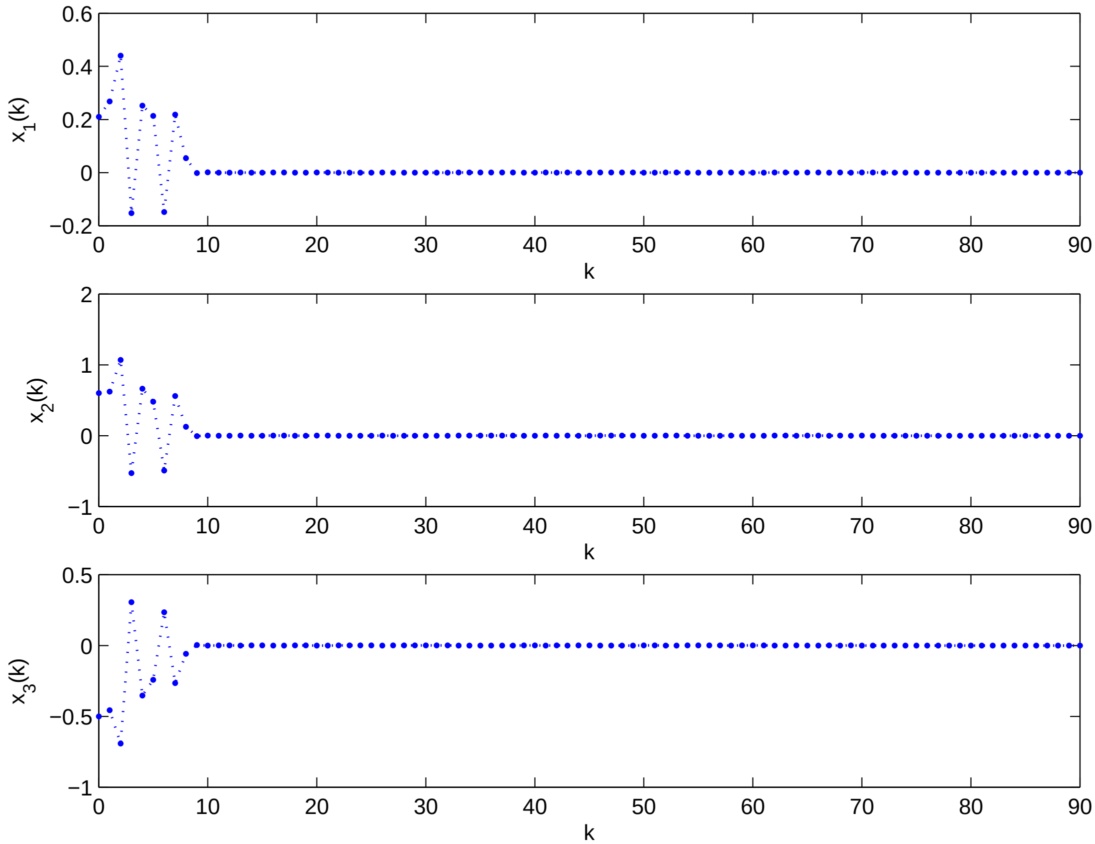

Fig. 1. State trajectories of the discrete-time stochastic neural network in the example.

usefulness of the derived LMI-based stability condition. One of the future research topics is to deal with the discrete-time complex networks with mixed time-delays [10, 11, 14].

References

[1] Arik, S. (2000) Stability analysis of delayed neural networks. *IEEE Transactions on Circuits Systems -I*, 47(7), 1089-1092.

[2] Boyd, S., EI Ghaoui, L., Feron, E. and Balakrishnan, V. (1994) *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM.

[3] Cao, J. and Song, Q. (2006) Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays. *Nonlinearity*, 19(7), 1601-1617.

[4] Chen, W.-H., Lu, X. and Liang D.-Y. (2006) Global exponential stability for discrete-time neural networks with variable delays, *Physics Letters A*, 358(3), 186-198.

[5] Gao, H., Lam, J. and Wang, C. (2006) Robust energy-to-peak filter design for stochastic time-delay systems, *Systems & Control Letters*, 55(2), 101-111.

[6] Gao, H., Lam, J. and Chen, G. (2006) New Criteria for synchronization stability of general complex dynamical networks with coupling delays, *Physics Letters A*, 360(2), 263-273.

[7] Hu, S. and Wang, J. (2006) Global robust stability of a class of discrete-time interval neural networks, *IEEE Trans. Circuits and Systems I - Regular Papers*, 53(1), 129-138.

[8] Liang, J., Cao, J. and Lam, J. (2005) Convergence of discrete-time recurrent neural networks with variable delay, *Int. J. Bifurcation and Chaos*, 15(2), 581-595.

[9] Liang, J., Cao, J. and Ho, D.W.C. (2005) Discrete-time bidirectional associative memory neural networks with variable delays, *Physics Letters A*, 335(2-3), 226-234.

[10] Liang, J., Wang, Z., Liu, Y. and Liu, X. (2008) Global synchronization control of general delayed discrete-time networks with stochastic coupling and disturbances, *IEEE Trans. Systems, Man, and Cybernetics - Part B*, 38(4), 1073-1083.

[11] Liang, J., Wang, Z. and Liu, X. (2008) Exponential synchronization of stochastic delayed discrete-time complex networks, *Nonlinear Dynamics*, 53(1-2), 153-165.

[12] Liu, Y., Wang, Z. and Liu, X. (2006) Global exponential stability of generalized recurrent neural networks with discrete and distributed delays, *Neural Networks*, 19(5), 667-675.

[13] Liu, Y., Wang, Z. and Liu, X. (2007) Robust stability of discrete-time stochastic neural networks with time-varying delays, *Neurocomputing*, 71(4-6), 823-833.

[14] Liu, Y., Wang, Z., Liang, J. and Liu, X. (2008) Synchronization and state estimation for discrete-time complex networks with distributed delays, *IEEE Trans. Systems, Man, and Cybernetics - Part B*, 38(5), 1314-1325.

[15] Mohamad, S. and Naim, A. (2002) Discrete-time analogues of integrodifferential equations modelling bidirectional neural networks, *J. Comput. Appl. Math.* 138, 1-20.

[16] Mohamad, S. and Gopalsamy, K. (2003) Exponential stability of continuous-time and discrete-time cellular neural networks with delays, *Applied Mathematics and Computation*, 135(1), 17-38.

[17] Mohamad, S. (2008) Exponential stability preservation in discrete-time analogues of artificial neural networks with distributed delays, *Journal of Computational and Applied Mathematics*, 215(1), 270-287.

[18] Principle, J. C., Kuo, J.-M. and Celebi, S. (1994) An analysis of the gamma memory in dynamic neural networks. *IEEE Trans. Neural Networks*, 5(2), 337-361.

[19] Song, Q. and Cao, J. (2006) Periodic solutions and its exponential stability of reaction-diffusion recurrent neural networks with continuously distributed delays, *Nonlinear Analysis - Real World Applications*, 7(1), 65-80.

[20] Stuart, A. and Humphries, A. (1996) *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, 1996.

[21] Tank, D. W. and Hopfield, J. J. (1987) Neural computation by concentrating information in time. *Proc. Nat. Acad. Sci.*, 84, 1896-1991.

[22] Wang, L. and Xu, Z. (2006) Sufficient and necessary conditions for global exponential stability of discrete-time recurrent neural networks, *IEEE Trans. Circuits and Systems I - Regular Papers*, 53(6), 1373-1380.

[23] Wang, Z., Liu, Y. and Liu, X. (2005) On global asymptotic stability of neural networks with discrete and distributed delays, *Physics Letters A*, 345(4-6), 299-308.

[24] Wang, Z., Liu, Y., Li, M. and Liu, X. (2006) Stability analysis for stochastic Cohen-Grossberg neural networks with mixed time delays, *IEEE Trans. Neural Networks*, 17(3), 814-820.

[25] Xiang, H., Yan, K. and Wang, B. (2005) Existence and global stability of periodic solution for delayed discrete high-order Hopfield-type neural networks, *Discrete Dynamics in Nature and Society*, 3, 281-297.

[26] Xiong, W. and Cao, J. (2005) Global exponential stability of discrete-time Cohen-Grossberg neural networks, *Neurocomputing*, 64, 433-446.

[27] Yuan, Z., Hu, D. and Huang, L. (2005) Stability and bifurcation analysis on a discrete-time neural network, *J. Computational and Applied Mathematics*, 177(1), 89-100.

[28] Zhao, H. and Wang, L. (2006) Stability and bifurcation for discrete-time Cohen-Grossberg neural network, *Applied Mathematics and Computation*, 179(2), 787-798.

[29] Zou, L. and Zhou, Z. (2006) Periodic solutions for nonautonomous discrete-time neural networks, *Applied Mathematics Letters*, 19(2), 174-185.