Robust H_∞ control of time-varying systems with stochastic non-linearities: the finite-horizon case

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Abstract: This paper is concerned with the robust H_∞ control problem for the class of uncertain non-linear discrete time-varying stochastic systems with a covariance constraint. All the system parameters are time-varying and the uncertainties enter into the state matrix. The non-linearities under consideration are described by statistical means and they cover several classes of well-studied non-linearities. The purpose of the addressed problem is to design a dynamic output-feedback controller such that, the H_∞ disturbance rejection attenuation level is achieved in the finite-horizon case while the state covariance is not more than an individual upper bound at each time point. An algorithm is developed to deal with the addressed problem by means of recursive linear matrix inequalities (RLMIs). It is shown that the robust H_∞ control problem is solvable if the series of RLMIs is feasible. An illustrative simulation example is given to show the applicability and effectiveness of the proposed algorithm.

Keywords: stochastic systems, non-linear systems, time-varying systems, H_∞ control, robust control

1 INTRODUCTION

In the past few decades, stochastic control and filtering problems have received a considerable amount of attention and an extensive literature now exists in the areas of engineering, social sciences, and ecological systems [**1–6**]. In particular covariance control (CC) theory has attracted significant attention due mainly to the fact that the performance objectives of many engineering systems can be naturally expressed in terms of the upper bounds of steady-state variances. The CC theory proposed in [**7**, **8**] aims to solve the covariance-constrained control or filtering problems while satisfying multiple performance indices such as L_1 and H_∞ norm constraints. The basic CC theory has been extended to the class of uncertain non-linear stochastic systems with or without missing measurements, see [**4**, **6**, **9**, **10**]. It should be pointed out that, so far, almost all the literature concerning CC theory has been on time-invariant systems, and the Riccati matrix equation or linear matrix inequality approaches have been widely applied because of the numerical efficiency of the MATLAB toolbox.

In the real world, there are virtually no strictly time-invariant systems since the working circumstances, operating points, or equipment deterioration levels are inherently time-varying in nature. Therefore, time-varying stochastic systems have started to receive attention in recent years. For example, in [**2**, **11**], H_∞ controllers as well as filters were designed for a special type of non-linear system called time-varying stochastic systems with multiplicative noises. Recently, in [**10**], a finite-horizon filter was studied for stochastic systems with an error variance constraint. Unfortunately, despite the importance of the time-varying nature in system modelling, the CC problem for time-varying non-linear systems has been largely overlooked, not to mention the simultaneous consideration of the H_∞ constraints. Recognizing the great importance of the

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time-varying nature of real-time models, the purpose of this paper is to investigate the finite-horizon H_∞ control problem for a class of uncertain time-varying systems with stochastic non-linearities.

The aim of this paper is to propose an output-feedback controller design technique for a class of uncertain non-linear discrete time-varying stochastic systems with a covariance constraint, where the considered non-linearities are described by statistical means and contain several widely studied non-linearities as special cases. The controlled system is expected to achieve the prescribed H_∞ disturbance rejection attenuation level in the finite horizon, and also to make sure that the state covariance is not more than an individual upper bound at each time point. Using the method proposed in [**11**], a recursive linear matrix inequality (RLMI) approach is developed to solve the addressed problem. An illustrative simulation example is given to show the applicability and effectiveness of the proposed algorithm.

There are two major contributions made by this paper.

1. The robust H_∞ control problem is considered, for the first time, for a class of time-varying systems with stochastic non-linearities.
2. A novel RLMI approach is developed to handle the addressed problem which is then demonstrated via a numerical example.

The rest of this paper is set out as follows. Section 2 formulates the robust H_∞ dynamic output-feedback controller design problem for uncertain discrete time-varying non-linear stochastic systems with a state covariance constraint. In section 3, the H_∞ noise attenuation level and state covariance performances of the closed-loop system are analysed separately, and a sufficient condition is then presented for the addressed controller design problem via the RLMI method. The controller design technique is given in section 4 by means of a series of RLMIs with properly chosen initial conditions. In section 5, an illustrative numerical example is provided to show the effectiveness and usefulness of the proposed approach. Conclusions are drawn in section 6.

Notation

The following notation will be used in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ matrices, and $\mathbb{I}^+$ denotes the set of non-negative integers. The notation $\mathbf{X}\geqslant\mathbf{Y}$ (respectively $\mathbf{X}>\mathbf{Y}$), where $\mathbf{X}$ and $\mathbf{Y}$ are real symmetric matrices, means that $\mathbf{X}-\mathbf{Y}$ is positive semi-definite (respectively positive-definite). $\|\boldsymbol{x}\|$ represents the 2-norm of the variable $\boldsymbol{x}$. $\mathrm{E}\{\boldsymbol{x}\}$ stands for the expectation of stochastic variable $\boldsymbol{x}$ and $\mathrm{E}\{\boldsymbol{x}|y\}$ for the expectation of $\boldsymbol{x}$ conditional on y. The superscript 'T' denotes the transpose. $\mathrm{tr}(\mathbf{A})$ represents the trace of a matrix $\mathbf{A}$. $\mathrm{diag}\{\mathbf{F}_1,\mathbf{F}_2,\ldots,\mathbf{F}_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $\mathbf{F}_1$, $\mathbf{F}_2$, …, $\mathbf{F}_n$. The symbol '*' in a matrix means that the corresponding term of the matrix can be obtained by symmetric property.

2 PROBLEM FORMULATION

Consider the following uncertain discrete time-varying non-linear stochastic system defined on $k\in[0,N]$

$$\begin{cases}\boldsymbol{x}(k+1)=(\mathbf{A}(k)+\Delta\mathbf{A}(k))\boldsymbol{x}(k)+\mathbf{B}(k)\boldsymbol{u}(k)\\ \qquad\qquad +\boldsymbol{f}(\boldsymbol{x}(k),k)+\mathbf{D}_1(k)\omega(k)\\ \boldsymbol{y}(k)=\mathbf{C}(k)\boldsymbol{x}(k)+\boldsymbol{g}(\boldsymbol{x}(k),k)+\mathbf{D}_2(k)\omega(k)\end{cases}\tag{1}$$

where $\boldsymbol{x}(k)\in\mathbb{R}^n$ is the state, $\boldsymbol{y}(k)\in\mathbb{R}^r$ is the output, $\boldsymbol{u}(k)\in\mathbb{R}^m$ is the control input, $\omega(k)\in\mathbb{R}^p$ is a zero mean Gaussian white noise sequence with covariance $\mathbf{W}(k)>0$, and $\mathbf{A}(k)$, $\mathbf{B}(k)$, $\mathbf{C}(k)$, $\mathbf{D}_1(k)$, and $\mathbf{D}_2(k)$ are known real time-varying matrices with appropriate dimensions. $\Delta\mathbf{A}(k)$ is a real-valued time-varying matrix that represents parametric uncertainties and has the following form

$$\Delta\mathbf{A}(k)=\mathbf{H}(k)\mathbf{F}(k)\mathbf{E}(k),\quad \mathbf{F}(k)\mathbf{F}(k)^{\mathrm{T}}\leqslant\mathbf{I}\tag{2}$$

where $\mathbf{H}(k)$ and $\mathbf{E}(k)$ are known time-varying matrices with appropriate dimensions. $\mathbf{F}(k)$ represents the time-varying uncertainty. The uncertainty in $\Delta\mathbf{A}(k)$ is said to be admissible if equation (2) holds.

The non-linear stochastic functions $\boldsymbol{f}(\boldsymbol{x}(k),k)$ and $\boldsymbol{g}(\boldsymbol{x}(k),k)$ are assumed to have the following first moments for all $\boldsymbol{x}(k)$ and k

$$\mathrm{E}\left\{\begin{bmatrix}\boldsymbol{f}(\boldsymbol{x}(k),k)\\ \boldsymbol{g}(\boldsymbol{x}(k),k)\end{bmatrix}\middle|\boldsymbol{x}(k)\right\}=0\tag{3}$$

with the covariance given by

$$\mathrm{E}\left\{\begin{bmatrix}\boldsymbol{f}(\boldsymbol{x}(k),k)\\ \boldsymbol{g}(\boldsymbol{x}(k),k)\end{bmatrix}\begin{bmatrix}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(j),j) & \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x}(j),j)\end{bmatrix}\middle|\boldsymbol{x}(k)\right\}=0,\quad k\neq j\tag{4}$$

and

$$\mathrm{E}\left\{\begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}(k),k) \\ \boldsymbol{g}(\boldsymbol{x}(k),k)\end{bmatrix}\begin{bmatrix}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(k),k) & \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x}(k),k)\end{bmatrix}\middle|\boldsymbol{x}(k)\right\} = \sum_{i=1}^{q}\begin{bmatrix}\boldsymbol{\Omega}_{11}^{i} & \boldsymbol{\Omega}_{12}^{i} \\ (\boldsymbol{\Omega}_{12}^{i})^{\mathrm{T}} & \boldsymbol{\Omega}_{22}^{i}\end{bmatrix}\mathrm{E}\{\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{\Gamma}_i\boldsymbol{x}(k)\} \tag{5}$$

where $\boldsymbol{\Omega}_{jl}^{i}$ and $\boldsymbol{\Gamma}_i$ ($j,l=1,2;\ i=1,2,\cdots,q$) are known matrices.

Remark 1

The non-linearity description in equations (3) to (5) covers several classes of well-studied non-linear systems, for example, a system with state-dependent multiplicative noises and a system whose state power depends on the sector-bounded (or sign) of the non-linear state function of the state, see [**5**, **6**].

Applying the following full-order dynamic output feedback control law

$$\begin{cases}\boldsymbol{x}_{\mathrm{g}}(k+1)=\mathbf{A}_{\mathrm{g}}(k)\boldsymbol{x}_{\mathrm{g}}(k)+\mathbf{B}_{\mathrm{g}}(k)\boldsymbol{y}(k)\\ \boldsymbol{u}(k)=\mathbf{C}_{\mathrm{g}}(k)\boldsymbol{x}_{\mathrm{g}}(k)\end{cases} \tag{6}$$

to the system (1), the following closed-loop system can be obtained

$$\begin{cases}\boldsymbol{z}(k+1)=\hat{\mathbf{A}}(k)\boldsymbol{z}(k)+\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)+\hat{\mathbf{D}}(k)\boldsymbol{\omega}(k)\\ \boldsymbol{y}(k)=\hat{\mathbf{C}}(k)\boldsymbol{z}(k)+\boldsymbol{g}(\boldsymbol{x}(k),k)+\mathbf{D}_2(k)\boldsymbol{\omega}(k)\end{cases} \tag{7}$$

where

$$\boldsymbol{z}(k)=\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}_{\mathrm{g}}(k)\end{bmatrix},\quad \boldsymbol{h}(\boldsymbol{x}(k),k)=\begin{bmatrix}\boldsymbol{f}(\boldsymbol{x}(k),k)\\ \boldsymbol{g}(\boldsymbol{x}(k),k)\end{bmatrix}$$

$$\hat{\mathbf{A}}(k)=\begin{bmatrix}\mathbf{A}(k)+\Delta\mathbf{A}(k) & \mathbf{B}(k)\mathbf{C}_{\mathrm{g}}(k)\\ \mathbf{B}_{\mathrm{g}}(k)\mathbf{C}(k) & \mathbf{A}_{\mathrm{g}}(k)\end{bmatrix}$$

$$\hat{\mathbf{G}}(k)=\begin{bmatrix}\mathbf{I} & 0\\ 0 & \mathbf{B}_{\mathrm{g}}(k)\end{bmatrix},\quad \hat{\mathbf{D}}(k)=\begin{bmatrix}\mathbf{D}_1(k)\\ \mathbf{B}_{\mathrm{g}}(k)\mathbf{D}_2(k)\end{bmatrix}$$

$$\hat{\mathbf{C}}(k)=[\,\mathbf{C}(k)\quad 0\,]$$

The objective of this paper is to design a finite-horizon dynamic output-feedback controller of form (6) such that the following two requirements are satisfied simultaneously.

Requirement 1

For given scalar $\gamma>0$, matrix $\mathbf{S}>0$ and the initial state $\boldsymbol{x}(0)$, the H_∞ performance index

$$\mathbf{J}:=\mathrm{E}\left(\|\boldsymbol{y}(k)\|_{[0,N-1]}^{2}-\gamma^2\|\boldsymbol{\omega}(k)\|_{[0,N-1]}^{2}\right)-\gamma^2\boldsymbol{x}^{\mathrm{T}}(0)\mathbf{S}\boldsymbol{x}(0)<0 \tag{8}$$

is achieved for all admissible parameter uncertainties and all stochastic non-linearities.

Requirement 2

For a sequence of specified definite matrices $\{\boldsymbol{\Theta}(k)\}_{0<k\leqslant N}$, at each sampling instant k, the system state covariance satisfies

$$\mathbf{X}(k):=\mathrm{E}\{\boldsymbol{x}(k)\boldsymbol{x}^{\mathrm{T}}(k)\}\leqslant\boldsymbol{\Theta}(k),\quad \forall k \tag{9}$$

This control problem is referred to as the robust H_∞ output-feedback control problem for uncertain non-linear discrete time-varying stochastic systems with a covariance constraint.

3 H_∞ AND COVARIANCE PERFORMANCES ANALYSIS

In this section, in terms of two matrix inequalities, the H_∞ and covariance performances will first be analysed separately for the closed-loop system (7). Then, a theorem which combines the two performance indices in a unified framework is presented via the RLMI algorithm.

3.1 H_∞ Performance

In this subsection, a sufficient condition is given for the closed-loop system (7) to satisfy the prescribed H_∞ noise attenuation level. The following assignments are made for notational convenience

$$\hat{\boldsymbol{\Gamma}}_i=\begin{bmatrix}\boldsymbol{\Gamma}_i & 0\\ 0 & 0\end{bmatrix},\quad \hat{\boldsymbol{\Omega}}_i=\begin{bmatrix}\boldsymbol{\Omega}_{11}^{i} & \boldsymbol{\Omega}_{12}^{i}\\ (\boldsymbol{\Omega}_{12}^{i})^{\mathrm{T}} & \boldsymbol{\Omega}_{22}^{i}\end{bmatrix},\quad \hat{\mathbf{S}}=\begin{bmatrix}\mathbf{S} & 0\\ 0 & 0\end{bmatrix}$$

Theorem 1

Consider the system (7). Let the controller feedback gain matrices $\mathbf{A}_{\mathrm{g}}(k)$, $\mathbf{B}_{\mathrm{g}}(k)$, and $\mathbf{C}_{\mathrm{g}}(k)$ be given. For a positive scalar $\gamma>0$ and a positive definite matrix $\mathbf{S}>0$, the H_∞ performance index requirement defined in equation (8) is achieved for all non-zero $\boldsymbol{\omega}(k)$ if, with the initial condition $\mathbf{Q}(0)\leqslant\gamma^2\hat{\mathbf{S}}$, there exists a sequence of positive-definite matrices $\{\mathbf{Q}(k)\}_{1\leqslant k\leqslant N}$ satisfying the following recursive matrix inequalities

$$\boldsymbol{\Upsilon}(k):=\begin{bmatrix}\boldsymbol{\Upsilon}_{11}(k) & \boldsymbol{\Upsilon}_{12}(k)\\ \boldsymbol{\Upsilon}_{12}^{\mathrm{T}}(k) & \boldsymbol{\Upsilon}_{22}(k)\end{bmatrix}<0 \tag{10}$$

where

$$\Upsilon_{11}(k)=\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k)-\mathbf{Q}(k)+\sum_{i=1}^{q}\hat{\Gamma}_i \times \mathrm{tr}\left[\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\right] + \hat{\mathbf{C}}^{\mathrm{T}}(k)\hat{\mathbf{C}}(k)+\sum_{i=1}^{q}\hat{\Gamma}_i\times \mathrm{tr}\left[\boldsymbol{\Omega}_{22}^{i}\right]$$

$$\Upsilon_{12}(k)=\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)+\hat{\mathbf{C}}^{\mathrm{T}}(k)\mathbf{D}_2(k)$$

$$\Upsilon_{22}(k)=-\gamma^2\mathbf{I}+\hat{\mathbf{D}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)+\mathbf{D}_2^{\mathrm{T}}(k)\mathbf{D}_2(k)$$

Proof

Define

$$\mathbf{J}(k) := \boldsymbol{z}^{\mathrm{T}}(k+1)\mathbf{Q}(k+1)\boldsymbol{z}(k+1)-\boldsymbol{z}^{\mathrm{T}}(k)\mathbf{Q}(k)\boldsymbol{z}(k) \tag{11}$$

Substituting equation (7) into $\mathbf{J}(k)$ leads to

$$\begin{aligned}\mathrm{E}\{\mathbf{J}(k)\}=&\mathrm{E}\Big\{\left[\hat{\mathbf{A}}(k)\boldsymbol{z}(k)+\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)+\hat{\mathbf{D}}(k)\omega(k)\right]^{\mathrm{T}} \\ &\times\mathbf{Q}(k+1)\left[\hat{\mathbf{A}}(k)\boldsymbol{z}(k)+\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k) +\hat{\mathbf{D}}(k)\omega(k)\right]-\boldsymbol{z}^{\mathrm{T}}(k)\mathbf{Q}(k)\boldsymbol{z}(k)\Big\}\\ =&\mathrm{E}\Big\{\boldsymbol{z}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k)\boldsymbol{z}(k)\\ &+\omega^{\mathrm{T}}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)\omega(k)\\ &+\boldsymbol{z}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)\omega(k)\\ &+\omega^{\mathrm{T}}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k)\boldsymbol{z}(k)\\ &-\boldsymbol{z}^{\mathrm{T}}(k)\mathbf{Q}(k)\boldsymbol{z}(k)+\left(\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)\right)^{\mathrm{T}}\\ &\times\mathbf{Q}(k+1)\left(\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)\right)\Big\}\end{aligned} \tag{12}$$

Taking equation (5) into consideration it becomes possible to write that

$$\begin{aligned}&\mathrm{E}\left\{\left(\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)\right)^{\mathrm{T}}\mathbf{Q}(k+1)\left(\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k),k)\right)\right\}\\ &=\mathrm{E}\left\{\boldsymbol{x}^{\mathrm{T}}(k)\sum_{i=1}^{q}\boldsymbol{\Gamma}_i\times\mathrm{tr}\left[\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\right]\boldsymbol{x}(k)\right\}\\ &=\mathrm{E}\left\{\boldsymbol{z}^{\mathrm{T}}(k)\sum_{i=1}^{q}\hat{\boldsymbol{\Gamma}}_i\times\mathrm{tr}\left[\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\right]\boldsymbol{z}(k)\right\}\end{aligned} \tag{13}$$

and therefore

$$\begin{aligned}\mathrm{E}\{\mathbf{J}(k)\}=&\mathrm{E}\Big\{\boldsymbol{z}^{\mathrm{T}}(k)\Big(\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k)-\mathbf{Q}(k)\\ &+\sum_{i=1}^{q}\hat{\boldsymbol{\Gamma}}_i\times\mathrm{tr}\left[\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\right]\Big)\boldsymbol{z}(k)\\ &+\omega^{\mathrm{T}}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)\omega(k)\\ &+\boldsymbol{z}^{\mathrm{T}}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{D}}(k)\omega(k)\\ &+\omega^{\mathrm{T}}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k)\boldsymbol{z}(k)\Big\}\end{aligned} \tag{14}$$

Adding the zero term $\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)-\gamma^2\omega^{\mathrm{T}}(k)\omega(k)-\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)+\gamma^2\omega^{\mathrm{T}}(k)\omega(k)$ to $\mathrm{E}\{\mathbf{J}(k)\}$ results in

$$\mathrm{E}\{\mathbf{J}(k)\}=\mathrm{E}\left\{\begin{bmatrix}\boldsymbol{z}^{\mathrm{T}}(k) & \omega^{\mathrm{T}}(k)\end{bmatrix}\Upsilon(k)\begin{bmatrix}\boldsymbol{z}(k)\\ \omega(k)\end{bmatrix} -\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)+\gamma^2\omega^{\mathrm{T}}(k)\omega(k)\right\} \tag{15}$$

Summing up equation (15) on both sides from zero to $N-1$ with respect to k results in

$$\begin{aligned}\sum_{k=0}^{N-1}\mathrm{E}\{\mathbf{J}(k)\}&=\mathrm{E}\{\boldsymbol{z}^{\mathrm{T}}(N)\mathbf{Q}(N)\boldsymbol{z}(N)\}-\boldsymbol{z}^{\mathrm{T}}(0)\mathbf{Q}(0)\boldsymbol{z}(0)\\ &=\mathrm{E}\left\{\sum_{k=0}^{N-1}\begin{bmatrix}\boldsymbol{z}^{\mathrm{T}}(k) & \omega^{\mathrm{T}}(k)\end{bmatrix}\Upsilon(k)\begin{bmatrix}\boldsymbol{z}(k)\\ \omega(k)\end{bmatrix}\right\}\\ &\quad-\mathrm{E}\left\{\sum_{k=0}^{N-1}\left(\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)-\gamma^2\omega^{\mathrm{T}}(k)\omega(k)\right)\right\}\end{aligned} \tag{16}$$

Hence, the H_∞ performance index defined in equation (8) is given by

$$\begin{aligned}\mathbf{J}=&\mathrm{E}\left\{\sum_{k=0}^{N-1}\begin{bmatrix}\boldsymbol{z}^{\mathrm{T}}(k) & \omega^{\mathrm{T}}(k)\end{bmatrix}\Upsilon(k)\begin{bmatrix}\boldsymbol{z}(k)\\ \omega(k)\end{bmatrix}\right\}\\ &-\mathrm{E}\{\boldsymbol{z}^{\mathrm{T}}(N)\mathbf{Q}(N)\boldsymbol{z}(N)\}+\boldsymbol{z}^{\mathrm{T}}(0)\left(-\gamma^2\hat{\mathbf{S}}+\mathbf{Q}(0)\right)\boldsymbol{z}(0)\end{aligned} \tag{17}$$

Noting that $\Upsilon(k)<0$, $\mathbf{Q}(N)>0$ and the initial condition $\mathbf{Q}(0)\leqslant\gamma^2\hat{\mathbf{S}}$, then $\mathbf{J}<0$ which completes the proof. ∎

3.2 Variance Analysis

This subsection discusses how to obtain the state covariance of the closed-loop system (7). First, define the state covariance matrix of system (7) by

$$\mathbf{Z}(k) := \mathrm{E}\left\{\boldsymbol{z}(k)\boldsymbol{z}(k)^{\mathrm{T}}\right\} \tag{18}$$

It is easy to see that

$$\mathbf{X}(k) = [\mathbf{I} \quad 0]\mathbf{Z}(k)\begin{bmatrix}\mathbf{I}\\0\end{bmatrix} \tag{19}$$

Defining a function as

$$\mathscr{F}(\mathbf{Y}(k)) := \hat{\mathbf{A}}(k)\mathbf{Y}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k) + \sum_{i=1}^{q}\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\hat{\mathbf{G}}^{\mathrm{T}}(k) \times \mathrm{tr}\left[\hat{\boldsymbol{\Gamma}}_i\mathbf{Y}(k)\right] + \hat{\mathbf{D}}(k)\mathbf{W}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k) \tag{20}$$

then the upper bound of the covariance $\mathbf{Z}(k)$ can be obtained as given in Theorem 2.

Theorem 2

Consider the system (7). Given the controller feedback gain $\mathbf{A}_{\mathrm{g}}(k)$, $\mathbf{B}_{\mathrm{g}}(k)$, and $\mathbf{C}_{\mathrm{g}}(k)$. If there exists a sequence of positive definite matrices $\{\mathbf{P}(k)\}_{1 \leqslant k \leqslant N+1}$ satisfying the following matrix inequality

$$\mathbf{P}(k+1) \geqslant \mathscr{F}(\mathbf{P}(k)) \tag{21}$$

with the initial condition $\mathbf{P}(0) = \mathbf{Z}(0)$, then $\mathbf{P}(k) \geqslant \mathbf{Z}(k)$, $\forall k \in \{1, 2, \cdots, N+1\}$.

Proof

The Lyapunov-type equation that governs the evolution of state covariance $\mathbf{Z}_k$ of closed-loop systems (7) is given by

$$\begin{aligned}\mathbf{Z}(k+1) &= \hat{\mathbf{A}}(k)\mathbf{Z}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k) \\ &\quad + \mathrm{E}\left\{\hat{\mathbf{G}}(k)\boldsymbol{h}(\boldsymbol{x}(k), k)\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{x}(k), k)\hat{\mathbf{G}}^{\mathrm{T}}(k)\right\} \\ &\quad + \hat{\mathbf{D}}(k)\mathbf{W}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k) \\ &= \hat{\mathbf{A}}(k)\mathbf{Z}(k)\hat{\mathbf{A}}^{\mathrm{T}}(k) + \sum_{i=1}^{q}\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\hat{\mathbf{G}}^{\mathrm{T}}(k) \\ &\quad \times \mathrm{tr}\left[\hat{\boldsymbol{\Gamma}}_i\mathbf{Z}(k)\right] + \hat{\mathbf{D}}(k)\mathbf{W}(k)\hat{\mathbf{D}}^{\mathrm{T}}(k) \\ &= \mathscr{F}(\mathbf{Z}(k))\end{aligned} \tag{22}$$

The following proof is done by induction. Obviously, $\mathbf{P}(0) \geqslant \mathbf{Z}(0)$. Suppose $\mathbf{P}(k) \geqslant \mathbf{Z}(k)$, then

$$\mathrm{tr}\left[\widehat{\boldsymbol{\Gamma}}_i\mathbf{P}(k)\right] \geqslant \mathrm{tr}\left[\widehat{\boldsymbol{\Gamma}}_i\mathbf{Z}(k)\right] \tag{23}$$

Therefore

$$\mathbf{P}(k+1) \geqslant \mathscr{F}(\mathbf{P}(k)) \geqslant \mathscr{F}(\mathbf{Z}(k)) = \mathbf{Z}(k+1) \tag{24}$$

and then proof is complete. ■

In the following stage, to conclude the above analysis, a theorem is presented which tends to take both H_∞ performance index and covariance constraint into consideration in a unified framework via the RLMIs method. Before the main results are given two lemmas that are needed in later derivations are presented.

Lemma 1: (Schur complement)

Given constant matrices $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ where $\mathbf{S}_1 = \mathbf{S}_1^{\mathrm{T}}$ and $0 < \mathbf{S}_2 = \mathbf{S}_2^{\mathrm{T}}$, then $\mathbf{S}_1 + \mathbf{S}_3^{\mathrm{T}}\mathbf{S}_2^{-1}\mathbf{S}_3 < 0$ if and only if

$$\begin{bmatrix}\mathbf{S}_1 & \mathbf{S}_3^{\mathrm{T}}\\ \mathbf{S}_3 & -\mathbf{S}_2\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}-\mathbf{S}_2 & \mathbf{S}_3\\ \mathbf{S}_3^{\mathrm{T}} & \mathbf{S}_1\end{bmatrix} < 0 \tag{25}$$

Lemma 2: (S-procedure)

Let $\mathbf{L} = \mathbf{L}^{\mathrm{T}}$, $\mathbf{H}$, and $\mathbf{E}$ be real matrices of appropriate dimensions and $\mathbf{F}$ satisfy equation (2). Then $\mathbf{L} + \mathbf{HFE} + \mathbf{E}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} < 0$ if and only if there exists a positive scalar ε such that $\mathbf{L} + \varepsilon\mathbf{HH}^{\mathrm{T}} + \varepsilon^{-1}\mathbf{E}^{\mathrm{T}}\mathbf{E} < 0$ or, equivalently

$$\begin{bmatrix}\mathbf{L} & \varepsilon\mathbf{H} & \mathbf{E}^{\mathrm{T}}\\ \varepsilon\mathbf{H}^{\mathrm{T}} & -\varepsilon\mathbf{I} & 0\\ \mathbf{E} & 0 & -\varepsilon\mathbf{I}\end{bmatrix} < 0 \tag{26}$$

Without loss of any generality, denoting

$$\hat{\boldsymbol{\Omega}}_i = \boldsymbol{\pi}_i\boldsymbol{\pi}_i^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{\pi}_{1i}\\ \boldsymbol{\pi}_{2i}\end{bmatrix}\begin{bmatrix}\boldsymbol{\pi}_{1i}\\ \boldsymbol{\pi}_{2i}\end{bmatrix}^{\mathrm{T}}$$

where $\boldsymbol{\pi}_i = \left[\boldsymbol{\pi}_{1i}^{\mathrm{T}} \quad \boldsymbol{\pi}_{2i}^{\mathrm{T}}\right]^{\mathrm{T}}$ $(i = 1, 2, \cdots, q)$ are column vectors of appropriate dimensions, then the following theorem can be given.

Theorem 3

Consider the system (7). Given the controller feedback gain $\mathbf{A}_{\mathrm{g}}(k)$, $\mathbf{B}_{\mathrm{g}}(k)$, and $\mathbf{C}_{\mathrm{g}}(k)$, a positive scalar $\gamma > 0$ and a positive-definite matrix $\mathbf{S} > 0$. If there exist families of positive definite matrices $\{\mathbf{Q}(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{P}(k)\}_{1 \leqslant k \leqslant N}$, and $\{\boldsymbol{\eta}_i(k)\}_{0 \leqslant k \leqslant N}$ $(i = 1, 2, \cdots, q)$ satisfying the following recursive matrix inequalities

$$\begin{bmatrix} -\boldsymbol{\eta}_i(k) & \boldsymbol{\pi}_i^{\mathrm{T}}\hat{\mathbf{G}}^{\mathrm{T}}(k) \\ * & -\mathbf{Q}^{-1}(k+1) \end{bmatrix} < 0 \tag{27}$$

$$\begin{bmatrix} \hat{\boldsymbol{\Lambda}}(k) & \hat{\mathbf{C}}^{\mathrm{T}}(k)\mathbf{D}_2(k) & \hat{\mathbf{A}}^{\mathrm{T}}(k) \\ * & -\gamma^2\mathbf{I}+\mathbf{D}_2^{\mathrm{T}}(k)\mathbf{D}_2(k) & \hat{\mathbf{D}}^{\mathrm{T}}(k) \\ * & * & -\mathbf{Q}^{-1}(k+1) \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} -\mathbf{P}(k+1) & \hat{\mathbf{A}}(k)\mathbf{P}(k) & \hat{\boldsymbol{\Phi}}_{13}(k) & \hat{\mathbf{D}}(k) \\ * & -\mathbf{P}(k) & 0 & 0 \\ * & * & \hat{\boldsymbol{\Phi}}_{33}(k) & 0 \\ * & * & * & -\mathbf{W}^{-1}(k) \end{bmatrix} < 0 \tag{29}$$

with the initial condition

$$\begin{cases} \mathbf{Q}(0) \leqslant \gamma^2\hat{\mathbf{S}} \\ \mathbf{P}(0) = \mathbf{Z}(0) \end{cases} \tag{30}$$

where

$$\hat{\boldsymbol{\Lambda}}(k) = -\mathbf{Q}(k) + \hat{\mathbf{C}}^{\mathrm{T}}(k)\hat{\mathbf{C}}(k) + \sum_{i=1}^{q}\hat{\boldsymbol{\Gamma}}_i \times \left(\boldsymbol{\eta}_i(k) + \mathrm{tr}\left[\boldsymbol{\Omega}_{22}^i\right]\right)$$

$$\hat{\boldsymbol{\Phi}}_{13}(k) = \left[\hat{\mathbf{G}}(k)\boldsymbol{\pi}_1 \quad \hat{\mathbf{G}}(k)\boldsymbol{\pi}_2 \quad \cdots \quad \hat{\mathbf{G}}(k)\boldsymbol{\pi}_q\right]$$

$$\hat{\boldsymbol{\Phi}}_{33}(k) = \mathrm{diag}\left\{-\rho_1\mathbf{I}, -\rho_2\mathbf{I}, \cdots, -\rho_q\mathbf{I}\right\}$$

$$\rho_i = \left(\mathrm{tr}\left[\hat{\boldsymbol{\Gamma}}_i\mathbf{P}(k)\right]\right)^{-1}, \quad i = 1, 2, \cdots, q$$

then, for the closed-loop system (7), $\mathbf{J}(k) < 0$ and $\mathbf{Z}(k) \leqslant \mathbf{P}(k)$, $\forall k \in \{0, 1, \cdots, N\}$.

Proof

Based on the analysis on H_∞ performance and system state covariance in sections 3.1 and 3.2 it only needs to be shown that, under initial conditions (30), the inequalities (27) and (28) imply equation (10), and the inequality (29) is equivalent to equation (21).

By the Schur Complement, equation (27) is equivalent to

$$\boldsymbol{\pi}_i^{\mathrm{T}}\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\boldsymbol{\pi}_i < \boldsymbol{\eta}_i(k) \tag{31}$$

which, by the property of matrix trace, can be rewritten as

$$\mathrm{tr}\left[\hat{\mathbf{G}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{G}}(k)\hat{\boldsymbol{\Omega}}_i\right] < \boldsymbol{\eta}_i(k) \tag{32}$$

Using the Schur Complement again, equation (28) is equivalent to

$$\hat{\boldsymbol{\Upsilon}}(k) := \begin{bmatrix} \hat{\boldsymbol{\Upsilon}}_{11}(k) & \boldsymbol{\Upsilon}_{12}(k) \\ \boldsymbol{\Upsilon}_{12}^{\mathrm{T}}(k) & \boldsymbol{\Upsilon}_{22}(k) \end{bmatrix} < 0 \tag{33}$$

where $\boldsymbol{\Upsilon}_{12}(k)$ and $\boldsymbol{\Upsilon}_{22}(k)$ are defined in Theorem 1, and

$$\hat{\boldsymbol{\Upsilon}}_{11}(k) = \hat{\mathbf{A}}^{\mathrm{T}}(k)\mathbf{Q}(k+1)\hat{\mathbf{A}}(k) - \mathbf{Q}(k) + \hat{\mathbf{C}}^{\mathrm{T}}(k)\hat{\mathbf{C}}(k) + \sum_{i=1}^{q}\hat{\boldsymbol{\Gamma}}_i \times \left(\boldsymbol{\eta}_i(k) + \mathrm{tr}\left[\boldsymbol{\Omega}_{22}^i\right]\right)$$

Hence, it is easy to see that equation (10) can be implied by equations (27) and (28) under the same initial condition.

Similarly, employing the Schur complement lemma, it can be easily verified that equation (29) is equivalent to equation (21). Thus, according to Theorem 1 and Theorem 2, the H_∞ index defined in equation (8) satisfies $\mathbf{J}(k) < 0$ and, at the same time, the state covariance of closed-loop system (7) achieves $\mathbf{Z}(k) \leqslant \mathbf{P}(k), \forall k \in \{0, 1, \cdots, N\}$. The proof is complete. ■

4 ROBUST FINITE-HORIZON CONTROLLER DESIGN

Based on Theorem 3, an algorithm is proposed in this section to solve the addressed dynamic output-feedback control problem for the uncertain discrete time-varying non-linear stochastic system (1). It will be shown that the controller gains can be obtained by solving a certain set of RLMIs. In other words, at each sampling instant k ($k > 0$), a set of LMIs will be solved to obtain the desired controller gains and, at the same time, certain key parameters are obtained which are needed in solving the LMIs for the $(k+1)$th instant.

The following theorem provides the controller design procedure for system (1).

Theorem 4

For a given disturbance attenuation level $\gamma > 0$, a positive-definite matrix $\mathbf{S} > 0$ and a sequence of pre-specified variance upper bounds $\{\mathbf{\Theta}(k)\}_{0 \leqslant k \leqslant N}$, if there exist families of positive-definite matrices $\{\mathbf{M}(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{N}(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{V}(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{P}_1(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{P}_2(k)\}_{1 \leqslant k \leqslant N}$, $\{\mathbf{P}_3(k)\}_{1 \leqslant k \leqslant N}$, $\{\varepsilon_1(k)\}_{0 \leqslant k \leqslant N}$, $\{\varepsilon_2(k)\}_{0 \leqslant k \leqslant N}$, $\{\mathbf{\eta}_i(k)\}_{0 \leqslant k \leqslant N}$ $(i = 1, 2, \cdots, q)$ and families of real-valued matrices $\{\mathbf{A}_g(k)\}_{0 \leqslant k \leqslant N}$, $\{\mathbf{B}_g(k)\}_{0 \leqslant k \leqslant N}$, and $\{\mathbf{C}_g(k)\}_{0 \leqslant k \leqslant N}$, under initial conditions

$$\begin{cases} \mathbf{Q}_1(0) \leqslant \gamma^2 \mathbf{S} \\ \mathbf{Q}_2(0) = \mathbf{Q}_3(0) = 0 \\ \mathbf{X}(0) = \mathbf{P}_1(0) \leqslant \mathbf{\Theta}(0) \\ \mathbf{P}_2(0) = \mathbf{P}_3(0) = 0 \end{cases} \tag{34}$$

such that the following recursive LMIs

$$\begin{bmatrix} -\mathbf{\eta}_i(k) & \boldsymbol{\pi}_{1i}^{\mathrm{T}} & \boldsymbol{\pi}_{2i}^{\mathrm{T}} \mathbf{B}_g^{\mathrm{T}}(k) \\ * & -\mathbf{M}(k+1) & -\mathbf{V}(k+1) \\ * & * & -\mathbf{N}(k+1) \end{bmatrix} < 0 \tag{35}$$

$$\mathbf{\Lambda}(k) := \begin{bmatrix} \mathbf{\Lambda}_{11}(k) & \mathbf{\Lambda}_{21}^{\mathrm{T}}(k) \\ * & \mathbf{\Lambda}_{22}(k) \end{bmatrix} < 0 \tag{36}$$

$$\mathbf{\Phi}(k) := \begin{bmatrix} \mathbf{\Phi}_{11}(k) & \mathbf{\Phi}_{12}(k) \\ * & \mathbf{\Phi}_{22}(k) \end{bmatrix} < 0 \tag{37}$$

$$\mathbf{P}_1(k+1) - \mathbf{\Theta}(k+1) \leqslant 0 \tag{38}$$

are satisfied with the parameter updated by

$$\begin{cases} \mathbf{Q}_1(k+1) \\ \quad = \left(\mathbf{M}(k+1) - \mathbf{V}(k+1)\mathbf{N}^{-1}(k+1)\mathbf{V}^{\mathrm{T}}(k+1)\right)^{-1} \\ \mathbf{Q}_2(k+1) \\ \quad = \left(\mathbf{N}(k+1) - \mathbf{V}^{\mathrm{T}}(k+1)\mathbf{M}^{-1}(k+1)\mathbf{V}(k+1)\right)^{-1} \\ \mathbf{Q}_3(k+1) = -\mathbf{M}^{-1}(k+1)\mathbf{V}(k+1)(\mathbf{N}(k+1) \\ \qquad\qquad - \mathbf{V}^{\mathrm{T}}(k+1)\mathbf{M}^{-1}(k+1)\mathbf{V}(k+1))^{-1} \end{cases} \tag{39}$$

where

$$\mathbf{\Lambda}_{11}(k) = \begin{bmatrix} \begin{pmatrix} -\mathbf{Q}_1(k) + \mathbf{C}^{\mathrm{T}}(k)\mathbf{C}(k) + \varepsilon_1(k)\mathbf{E}^{\mathrm{T}}(k)\mathbf{E}(k) \\ + \sum_{i=1}^{q} \mathbf{\Gamma}_i\left(\mathbf{\eta}_i(k) + \mathrm{tr}\left[\mathbf{\Omega}_{22}^i\right]\right) \end{pmatrix} & -\mathbf{Q}_3(k) & \mathbf{C}^{\mathrm{T}}(k)\mathbf{D}_2(k) \\ * & -\mathbf{Q}_2(k) & 0 \\ * & * & -\gamma^2\mathbf{I} + \mathbf{D}_2^{\mathrm{T}}(k)\mathbf{D}_2(k) \end{bmatrix}$$

$$\mathbf{\Lambda}_{21}(k) = \begin{bmatrix} \mathbf{A}(k) & \mathbf{B}(k)\mathbf{C}_g(k) & \mathbf{D}_1(k) \\ \mathbf{B}_g(k)\mathbf{C}(k) & \mathbf{A}_g(k) & \mathbf{B}_g(k)\mathbf{D}_2(k) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Lambda}_{22}(k) = \begin{bmatrix} -\mathbf{M}(k+1) & -\mathbf{V}(k+1) & \mathbf{H}(k) \\ * & -\mathbf{N}(k+1) & 0 \\ * & * & -\varepsilon_1(k)\mathbf{I} \end{bmatrix}$$

$$\mathbf{\Phi}_{11}(k) = \begin{bmatrix} -\mathbf{P}_1(k+1) + \varepsilon_2(k)\mathbf{H}(k)\mathbf{H}^{\mathrm{T}}(k) & -\mathbf{P}_3(k+1) & \bar{\mathbf{A}}(k) & \bar{\mathbf{B}}(k) \\ * & -\mathbf{P}_2(k+1) & \bar{\mathbf{C}}(k) & \bar{\mathbf{D}}(k) \\ * & * & -\mathbf{P}_1(k) & -\mathbf{P}_3(k) \\ * & * & * & -\mathbf{P}_2(k) \end{bmatrix}$$

$$\bar{\mathbf{A}}(k) = \mathbf{A}(k)\mathbf{P}_1(k) + \mathbf{B}(k)\mathbf{C}_g(k)\mathbf{P}_3^{\mathrm{T}}(k)$$

$$\bar{\mathbf{B}}(k) = \mathbf{A}(k)\mathbf{P}_3(k) + \mathbf{B}(k)\mathbf{C}_g(k)\mathbf{P}_2(k)$$

$$\bar{\mathbf{C}}(k) = \mathbf{B}_g(k)\mathbf{C}(k)\mathbf{P}_1(k) + \mathbf{A}_g(k)\mathbf{P}_3^{\mathrm{T}}(k)$$

$$\bar{\mathbf{D}}(k) = \mathbf{B}_g(k)\mathbf{C}(k)\mathbf{P}_3(k) + \mathbf{A}_g(k)\mathbf{P}_2(k)$$

$$\boldsymbol{\Phi}_{12}(k)=\begin{bmatrix} \boldsymbol{\pi}_{11} & \boldsymbol{\pi}_{12} & \cdots & \boldsymbol{\pi}_{1q} & \mathbf{D}_1(k) & 0 \\ \mathbf{B}_g(k)\boldsymbol{\pi}_{21} & \mathbf{B}_g(k)\boldsymbol{\pi}_{22} & \cdots & \mathbf{B}_g(k)\boldsymbol{\pi}_{2q} & \mathbf{B}_g(k)\mathbf{D}_2(k) & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathbf{P}_1(k)\mathbf{E}^{\mathrm{T}}(k) \\ 0 & 0 & \cdots & 0 & 0 & \mathbf{P}_3(k)\mathbf{E}^{\mathrm{T}}(k) \end{bmatrix}$$

$$\boldsymbol{\Phi}_{22}(k)=\mathrm{diag}\left\{-\rho_1\mathbf{I},\ -\rho_2\mathbf{I},\ \cdots,\ -\rho_q\mathbf{I},\ -\mathbf{W}^{-1}(k),\ -\varepsilon_2(k)\mathbf{I}\right\}$$

$$\rho_i=(\mathrm{tr}[\boldsymbol{\Gamma}_i\mathbf{P}_1(k)])^{-1},\quad i=1,2,\cdots,q$$

then the addressed robust H_∞ finite-horizon controller design problem is solved for the stochastic non-linear system (1). Moreover, the controller gains $\mathbf{A}_g(k)$, $\mathbf{B}_g(k)$, and $\mathbf{C}_g(k)$ at the sampling instant k $(0 \leqslant k \leqslant N)$ can be obtained by solving the corresponding set of LMIs at time k.

Proof

The proof is based on Theorem 3. First, supposing the variables $\mathbf{Q}(k)$ and $\mathbf{P}(k)$ can be decomposed as follows

$$\mathbf{Q}(k)=\begin{bmatrix} \mathbf{Q}_1(k) & \mathbf{Q}_3(k) \\ \mathbf{Q}_3^{\mathrm{T}}(k) & \mathbf{Q}_2(k) \end{bmatrix},\quad \mathbf{Q}^{-1}(k)=\begin{bmatrix} \mathbf{M}(k) & \mathbf{V}(k) \\ \mathbf{V}^{\mathrm{T}}(k) & \mathbf{N}(k) \end{bmatrix}$$

$$\mathbf{P}(k)=\begin{bmatrix} \mathbf{P}_1(k) & \mathbf{P}_3(k) \\ \mathbf{P}_3^{\mathrm{T}}(k) & \mathbf{P}_2(k) \end{bmatrix} \tag{40}$$

it is easy to see that equation (39) holds and equations (27) and (35) are equivalent to each other.

In order to eliminate the parameter uncertainty $\Delta\mathbf{A}(k)$ in equation (28), it is rewritten in the following form

$$\begin{bmatrix} \hat{\boldsymbol{\Lambda}}(k) & \hat{\mathbf{C}}^{\mathrm{T}}(k)\mathbf{D}_2(k) & \tilde{\mathbf{A}}^{\mathrm{T}}(k) \\ * & -\gamma^2\mathbf{I}+\mathbf{D}_2^{\mathrm{T}}(k)\mathbf{D}_2(k) & \hat{\mathbf{D}}^{\mathrm{T}}(k) \\ * & * & -\mathbf{Q}^{-1}(k+1) \end{bmatrix} + \hat{\mathbf{H}}(k)\mathbf{F}(k)\hat{\mathbf{E}}(k)+\hat{\mathbf{E}}^{\mathrm{T}}(k)\mathbf{F}^{\mathrm{T}}(k)\hat{\mathbf{H}}^{\mathrm{T}}(k)<0 \tag{41}$$

where

$$\tilde{\mathbf{A}}(k)=\begin{bmatrix} \mathbf{A}(k) & \mathbf{B}(k)\mathbf{C}_g(k) \\ \mathbf{B}_g(k)\mathbf{C}(k) & \mathbf{A}_g(k) \end{bmatrix}$$

$$\hat{\mathbf{H}}(k)=\begin{bmatrix} 0 & 0 & 0 & \mathbf{H}^{\mathrm{T}}(k) & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\hat{\mathbf{E}}(k)=[\mathbf{E}(k) \quad 0 \quad 0 \quad 0 \quad 0]$$

Then, by Lemma 2, it can be obtained that equation (28) is equivalent to equation (36). Similarly, using Lemma 2 again, it can be seen that equation (29) is also equivalent to equation (37). Therefore, according to Theorem 3, $\mathbf{J}(k)<0$ and $\mathbf{Z}(k) \leqslant \mathbf{P}(k)$. From equation (38), it is obvious that $\mathbf{X}(k) \leqslant \mathbf{P}_1(k) < \boldsymbol{\Theta}(k)$, $\forall k \in \{0, 1, \cdots, N\}$. It can now be concluded that the Requirements (1 and 2) are simultaneously satisfied. The proof is complete. ■

Based on Theorem 4, the robust controller design (RCD) algorithm can be summarized as follows.

Algorithm RCD

Step 1: Given the H_∞ performance index γ, the positive-definite matrix $\mathbf{S}$ and the state initial condition $\boldsymbol{x}(0)$. Select the initial values for matrices $\{\mathbf{Q}_1(0), \mathbf{Q}_2(0), \mathbf{Q}_3(0), \mathbf{P}_1(0), \mathbf{P}_2(0), \mathbf{P}_3(0)\}$ which satisfy the condition (34) and set $k=0$.

Step 2: Obtain the values of matrices $\{\mathbf{M}(k+1), \mathbf{N}(k+1), \mathbf{V}(k+1), \mathbf{P}_1(k+1), \mathbf{P}_2(k+1), \mathbf{P}_3(k+1)\}$ and the desired controller parameters $\{\mathbf{A}_g(k), \mathbf{B}_g(k), \mathbf{C}_g(k)\}$ for the sampling instant k by solving the LMIs (35) to (38).

Step 3: Set $k=k+1$ and obtain $\{\mathbf{Q}_1(k+1), \mathbf{Q}_2(k+1), \mathbf{Q}_3(k+1)\}$ by the parameter update formula (39).

Step 4: If $k<N$, then go to step 2, else go to step 5.

Step 5: Stop.

Remark 2

It is easy to see that the finite filtering problem for certain non-linear stochastic time-varying systems can be treated as a special case of the dynamic output-feedback control problem studied in this paper, which can be readily solved by means of the proposed RCD algorithm with appropriate modifications. On the other hand, it would be interesting to deal with the corresponding robust steady-state control or filtering problem when the system parameters become time-invariant. It should also be noted that based on the results of this paper as well as the proposed method, it would not be very difficult to extend the technique to study the multiobjective finite-horizon control problem for stochastic networked control systems. This will be attempted in future studies.

5 AN ILLUSTRATIVE EXAMPLE

This section presents an illustrative example to demonstrate the effectiveness of the proposed algorithms. Consider the following discrete time-varying system with stochastic non-linearities

$$\begin{cases} \boldsymbol{x}(k+1) = \left(\begin{bmatrix} 0 & -0.58 \\ 0.2+0.2\sin(2k) & 0.3 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \sin(0.6k) [0.1 \quad 0] \right) \boldsymbol{x}(k) \\ \qquad + \begin{bmatrix} -1.8 \\ -0.61+0.05\cos(k) \end{bmatrix} \boldsymbol{u}(k) \\ \qquad + \boldsymbol{f}(\boldsymbol{x}(k),k) + \begin{bmatrix} 0.1\sin(0.2k) \\ -0.2 \end{bmatrix} \omega(k) \\ \boldsymbol{y}(k) = [-0.375+0.3\sin(1.5k) \quad -0.25] \boldsymbol{x}(k) \\ \qquad + \boldsymbol{g}(\boldsymbol{x}(k),k) + 0.1\omega(k) \end{cases}$$

with the state initial value $\boldsymbol{x}(0) = [0.8 \ -0.7]$ and $\mathbf{S} = \text{diag}\{0.5, 1\}$. Suppose $\omega(k)$ has an identity covariance.

The non-linear functions $\boldsymbol{f}(\boldsymbol{x}(k), k)$ and $\boldsymbol{g}(\boldsymbol{x}(k), k)$ are taken as follows

$$\boldsymbol{f}(\boldsymbol{x}(k),k) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \times (0.3x_1(k)\xi_1(k) + 0.4x_2(k)\xi_2(k))$$

$$\boldsymbol{h}(\boldsymbol{x}(k),k) = 0.05 \times (0.3x_1(k)\xi_1(k) + 0.4x_2(k)\xi_2(k))$$

where $x_i(k)$ $(i=1,2)$ is the ith element of $\boldsymbol{x}(k)$ and $\xi_i(k)$ $(i=1,2)$ are zero mean, uncorrelated Gaussian white noise processes with unity covariances. It is also assumed that $\xi_i(k)$ is uncorrelated with $\omega(k)$. It can be easily checked that the above class of stochastic non-linearities satisfies

$$\mathrm{E}\left\{ \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}(k),k) \\ \boldsymbol{h}(\boldsymbol{x}(k),k) \end{bmatrix} \middle| \boldsymbol{x}(k) \right\} = 0$$

$$\mathrm{E}\left\{ \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}(k),k) \\ \boldsymbol{g}(\boldsymbol{x}(k),k) \end{bmatrix} [\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(k),k) \quad \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{x}(k),k)] \middle| \boldsymbol{x}(k) \right\}$$
$$= \begin{bmatrix} 0.2 \\ 0.3 \\ 0.05 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.05 \end{bmatrix}^{\mathrm{T}} \mathrm{E}\left\{ \boldsymbol{x}^{\mathrm{T}}(k) \begin{bmatrix} 0.09 & 0 \\ 0 & 0.16 \end{bmatrix} \boldsymbol{x}(k) \right\}$$

Set the prespecified performance indices by $\gamma = 1.2$ and $\{\boldsymbol{\Theta}(k)\}_{1 \leqslant k \leqslant N} = \text{diag}\{1.2, 0.9\}$, and choose the parameters' initial values so as to satisfy equation (34). The solvability of the addressed problem with the given initial conditions and prespecified performance indices can be checked using the MATLAB LMI toolbox. The simulation results are shown in Figs 1 to 4, which confirm that the desired finite-horizon performance is well achieved and the proposed RCD algorithm is indeed effective.

6 CONCLUSION

A multiobjective controller design problem via output feedback for a class of discrete time-varying non-

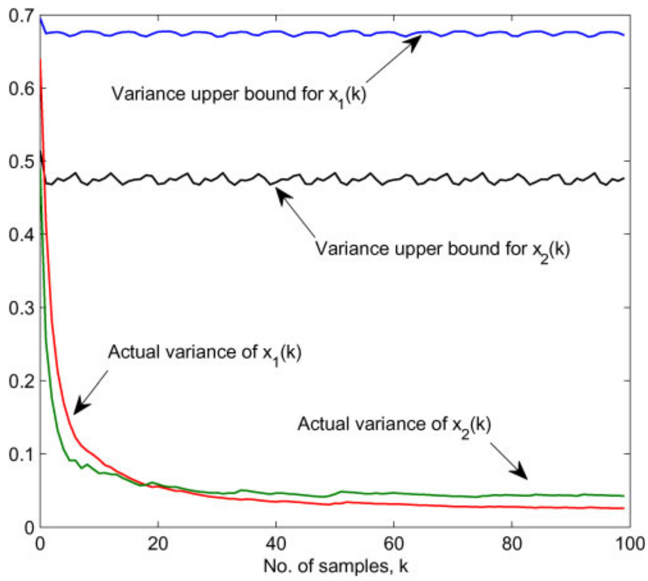

Fig. 1 The variance upper bound and actual variance

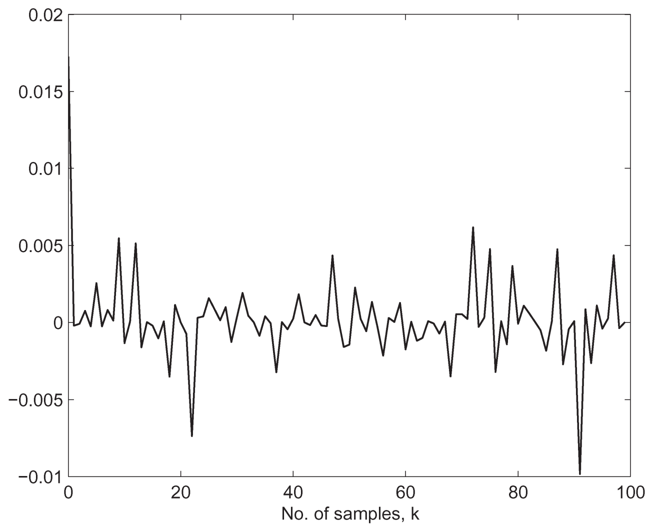

Fig. 2 The input control signal $u(k)$

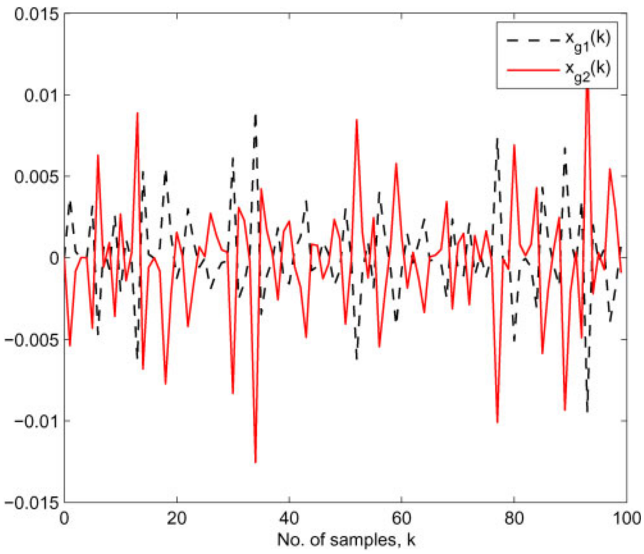

Fig. 3 The states responses of the output-feedback controller

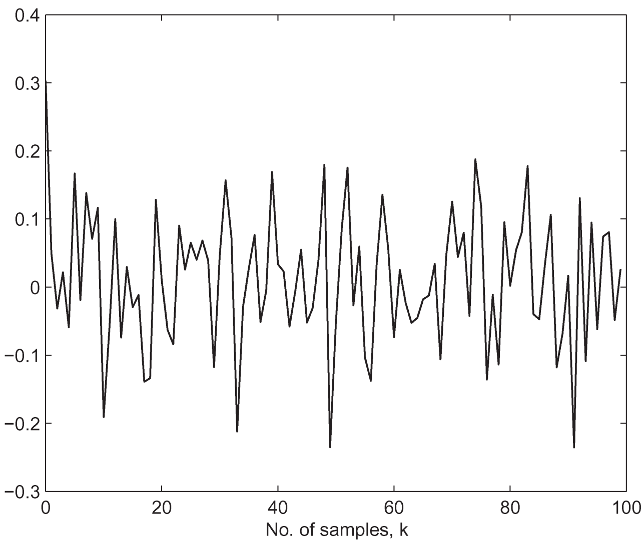

Fig. 4 The system output $y(k)$

linear stochastic systems has been discussed in this paper. The system model considered here with stochastic non-linearities is widely seen in engineering applications. The control task is to achieve the prescribed H_∞ noise attenuation level and system state covariance constraint simultaneously. The H_∞ and covariance performances have been analysed separately, and then a sufficient condition for the solvability of the addressed controller design problem has been given in terms of the feasibility of a series of RLMIs. Finally, an illustrative example has been provided to show the applicability and effectiveness of the proposed algorithm.

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