Robust $\mathcal{H}_\infty$ Finite-Horizon Control for a Class of Stochastic Nonlinear Time-Varying Systems Subject to Sensor and Actuator Saturations

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***Abstract*—This technical note addresses the robust $\mathcal{H}_\infty$ finite-horizon output feedback control problem for a class of uncertain discrete stochastic nonlinear time-varying systems with both sensor and actuator saturations. In the system under investigation, all the system parameters are allowed to be time-varying, the parameter uncertainties are assumed to be of the polytopic type, and the stochastic nonlinearities are described by statistical means which can cover several classes of well-studied nonlinearities. The purpose of the problem addressed is to design an output feedback controller, over a given finite-horizon, such that the $\mathcal{H}_\infty$ disturbance attenuation level is guaranteed for the nonlinear stochastic polytopic system in the presence of saturated sensor and actuator outputs. Sufficient conditions are first established for the robust $\mathcal{H}_\infty$ performance through intensive stochastic analysis, and then a recursive linear matrix inequality (RLMI) approach is employed to design the desired output feedback controller achieving the prescribed $\mathcal{H}_\infty$ disturbance rejection level. Simulation results demonstrate the effectiveness of the developed controller design scheme.**

***Index Terms*—Actuator saturation, discrete time-varying systems, finite-horizon, robust $\mathcal{H}_\infty$ control, sensor saturation, stochastic nonlinear systems.**

I. Introduction

For several decades, stochastic control and nonlinear control are arguably two of the most active research areas in systems and control, and many different kinds of nonlinear stochastic systems have been investigated in the literature, see [3], [11], [17], [18], [20], [27] and the references therein. Among various descriptions of nonlinearities, the so-called stochastic nonlinearities characterized by statistical moments has gained particular attention since they encompass several well-studied nonlinearities in stochastic systems [16]. On the other hand, in practical control systems, sensors and actuators cannot provide unlimited amplitude signal due primarily to the physical, safety or technological constraints. Because of their theoretical significance and practical importance, the problems of filtering and control with actuator saturation have been extensively studied, see, e.g., [5], [9], [14], [21]. Comparing to the vast literature with respect to actuator saturation, the associated results for sensor saturation have been relatively few probably because of the technical difficulty, see [4], [22], [28]. It should be mentioned that very few results have dealt with the systems with simultaneous presence of actuator and sensor saturations [6] although such a presence is quite typical in engineering practice. It is noted that most existing results for nonlinear stochastic control problems with or without saturation have been concerned with time-invariant system over the infinite horizon.

Virtually all model for real-time systems should be time-varying especially those after digital discretization. In recent years, time-varying stochastic systems have stirred considerable research attention due mainly to the insights of their engineering applications, see e.g., [1], [10], [18], [19], [24], [27]. In most existing literature concerning time-varying stochastic systems, however, it has been implicitly assumed that the actuators and sensors can always provide unlimited amplitude signals and therefore ignored the possible effect of amplitude saturation. Very recently, the set-membership filtering problem has been addressed for a class of discrete linear time-varying systems with sensor saturation in [23]. Unfortunately, to the best of the authors' knowledge, the finite-horizon $\mathcal{H}_\infty$ control problem for discrete time-varying nonlinear stochastic systems with polytopic uncertainties has not been adequately investigated, not to mention the case where the actuator and/or sensor saturations are also involved. It is, therefore, the purpose of this technical note to shorten such a gap by employing the recursive linear matrix inequality (RLMI) approach. Note that the RLMI approach has been proposed in [7], [8] which can be used to deal with finite-horizon control and filtering problems for time-varying systems.

In this technical note, we aim to investigate the robust $\mathcal{H}_\infty$ dynamic output-feedback controller design problem for a class of uncertain discrete stochastic nonlinear time-varying systems with both sensors and actuators subject to saturation, where all the system parameters are time-varying, the parameter uncertainty is of polytopic type, and the stochastic nonlinearities are described by statistical means. Note that the system model addressed is quite comprehensive to cover time-varying parameters, stochastic nonlinearities, actuator and sensor saturation as well as parameter uncertainties, hence reflecting the reality closely. The problem addressed represents the first of few attempts to deal with the finite-horizon control problem for stochastic systems with both actuator and sensor saturation. The algorithm developed is computationally appealing in terms of the RLMIs which are suitable for online applications.

Notation: The notation used in the technical note is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$, and I and 0 represent the identity matrix and zero matrix, respectively. The notation $P > 0$ means that P is real symmetric and positive definite; the notation $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\mathrm{tr}(A^T A)}$ and $\|\cdot\|_2$ stands for the usual l_2 norm. In symmetric block matrices or complex matrix expressions, we use an asterisk $*$ to represent a term that is induced by symmetry, and $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix. In addition, $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. $\mathrm{tr}(A)$ represents the trace of a matrix A. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

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II. Problem Formulation

Consider the following uncertain discrete stochastic nonlinear time-varying system with both the sensor and actuator saturations:

$$\begin{cases} x(k+1) = A^{(\varepsilon)}(k)x(k) + B^{(\varepsilon)}(k)\sigma_u(u(k)) \\ \qquad\qquad +f(x(k),k) + D_1^{(\varepsilon)}(k)w(k) \\ y_s(k) = \sigma_y(y(k)) + g(x(k),k) + D_2^{(\varepsilon)}(k)w(k) \\ y(k) = C(k)x(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $y_s(k) \in \mathbb{R}^r$ is the output; $u(k) \in \mathbb{R}^m$ is the control input, and $w(k) \in \mathbb{R}^p$ is the disturbance input which belongs to $l_2[0,\infty)$. All the system matrices in (1) are appropriately dimensioned, of which $C(k)$ is a known time-varying matrix, and $A^{(\varepsilon)}(k)$, $B^{(\varepsilon)}(k)$, $D_1^{(\varepsilon)}(k)$, $D_2^{(\varepsilon)}(k)$ are unknown time-varying matrices which contain polytopic uncertainties (see e.g., [15]) given as follows:

$$\Xi^{(\varepsilon)} := \left(A^{(\varepsilon)}(k), B^{(\varepsilon)}(k), D_1^{(\varepsilon)}(k), D_2^{(\varepsilon)}(k)\right) \in \Re \tag{2}$$

where $\Re$ is a given convex bounded polyhedral domain described by ν vertices

$$\Re := \left\{ \Xi^{(\varepsilon)} \middle| \Xi^{(\varepsilon)} = \sum_{i=1}^{\nu} \varepsilon_i \Xi^{(i)},\ \sum_{i=1}^{\nu} \varepsilon_i = 1,\ \varepsilon_i \geq 0,\ i = 1,2,\ldots,\nu \right\} \tag{3}$$

and $\Xi^{(i)} := \left(A^{(i)}(k), B^{(i)}(k), D_1^{(i)}(k), D_2^{(i)}(k)\right)$ are known matrices for $i = 1,2,\ldots,\nu$.

The nonlinear stochastic functions $f(x(k),k)$ and $g(x(k),k)$ are described by their statistical characteristics as follows [16]:

$$\mathbb{E}\left\{ \begin{bmatrix} f(x(k),k) \\ g(x(k),k) \end{bmatrix} \middle| x(k) \right\} = 0 \tag{4}$$

$$\mathbb{E}\left\{ \begin{bmatrix} f(x(k),k) \\ g(x(k),k) \end{bmatrix} \begin{bmatrix} f^T(x(j),j) & g^T(x(j),j) \end{bmatrix} \middle| x(k) \right\} = 0,\ k \neq j \tag{5}$$

and

$$\begin{aligned} &\mathbb{E}\left\{ \begin{bmatrix} f(x(k),k) \\ g(x(k),k) \end{bmatrix} \begin{bmatrix} f^T(x(k),k) & g^T(x(k),k) \end{bmatrix} \middle| x(k) \right\} \\ &= \sum_{i=1}^{q} \pi_i \pi_i^T \mathbb{E}\left\{ x^T(k)\Gamma_i x(k) \right\} \\ &:= \sum_{i=1}^{q} \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix} \begin{bmatrix} \pi_{1i} \\ \pi_{2i} \end{bmatrix}^T \mathbb{E}\left\{ x^T(k)\Gamma_i x(k) \right\} \\ &:= \sum_{i=1}^{q} \hat{\Theta}_i \mathbb{E}\left\{ x^T(k)\Gamma_i x(k) \right\} \\ &:= \sum_{i=1}^{q} \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ (\Theta_{12}^i)^T & \Theta_{22}^i \end{bmatrix} \mathbb{E}\left\{ x^T(k)\Gamma_i x(k) \right\} \end{aligned} \tag{6}$$

where π_{1i}, π_{2i}, Θ_{jl}^i and Γ_i $(j,l = 1,2;\ i = 1,2,\ldots,q)$ are known matrices.

The saturation function $\sigma(\cdot) : \mathbb{R}^r \mapsto \mathbb{R}^r$ is defined as

$$\sigma(v) = \begin{bmatrix} \sigma_1^T(v_1) & \sigma_2^T(v_2) & \cdots & \sigma_r^T(v_r) \end{bmatrix}^T \tag{7}$$

with $\sigma_i(v_i) = \operatorname{sign}(v_i)\min\{v_{i,\max}, |v_i|\}$, where $v_{i,\max}$ is the ith element of the vector $v_{\max}$, the saturation level.

Definition 1 [12]: A nonlinearity $\Psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if

$$(\Psi(v) - H_1 v)^T (\Psi(v) - H_2 v) \leq 0,\ \forall v \in \mathbb{R}^r \tag{8}$$

for some real matrices $H_1, H_2 \in \mathbb{R}^{r\times r}$, where $H = H_2 - H_1$ is a positive-definite symmetric matrix. In this case, we say that Ψ belongs to the sector $[H_1\ H_2]$.

As in [22], [23], [28], assuming that there exist diagonal matrices K_1, K_2 and R_1, R_2 such that $0 \leq K_1 < I \leq K_2$ and $0 \leq R_1 < I \leq R_2$, then the saturation functions $\sigma_y(y(k))$ and $\sigma_u(u(k))$ in (1) can be decomposed into a linear and a nonlinear part as

$$\sigma_y(y(k)) = K_1 C(k)x(k) + \Psi_y(y(k)) \tag{9}$$

$$\sigma_u(u(k)) = R_1 u(k) + \Psi_u(u(k)) \tag{10}$$

where $\Psi_y(y(k))$ and $\Psi_u(u(k))$ are two nonlinear vector-valued functions satisfying two sector conditions, respectively, with $H_1 = 0$, $H_2 = K$ and $H_1 = 0$, $H_2 = R$, which can be described as follows:

$$\Psi_y^T(y(k))\left(\Psi_y(y(k)) - KC(k)x(k)\right) \leq 0 \tag{11}$$

$$\Psi_u^T(u(k))\left(\Psi_u(u(k)) - Ru(k)\right) \leq 0 \tag{12}$$

where $K = K_2 - K_1$, $R = R_2 - R_1$.

In this technical note, we consider the following time-varying full-order dynamic output feedback controller for the system (1):

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + B_c(k)y_s(k) \\ u(k) = C_c(k)x_c(k) \end{cases} \tag{13}$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state, $A_c(k)$, $B_c(k)$ and $C_c(k)$ are controller parameters to be designed. In this case, the closed-loop system becomes

$$\begin{cases} \bar{x}(k+1) = \bar{A}^{(\varepsilon)}(k)\bar{x}(k) + G^{(\varepsilon)}(k)\bar{\Psi}(k) \\ \qquad\qquad + H(k)h(x(k),k) + \bar{D}^{(\varepsilon)}(k)w(k) \\ y_s(k) = \bar{C}(k)\bar{x}(k) + \bar{H}\bar{\Psi}(k) + \bar{H}h(x(k),k) + D_2^{(\varepsilon)}(k)w(k) \end{cases} \tag{14}$$

where

$$\begin{aligned} &\bar{x}(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \quad \bar{\Psi}(k) = \begin{bmatrix} \Psi_u(u(k)) \\ \Psi_y(y(k)) \end{bmatrix}, \\ &\bar{C}(k) = \begin{bmatrix} K_1 C(k) & 0 \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} 0 & I \end{bmatrix}, \\ &\bar{A}^{(\varepsilon)}(k) = \begin{bmatrix} A^{(\varepsilon)}(k) & B^{(\varepsilon)}(k)R_1 C_c(k) \\ B_c(k)K_1 C(k) & A_c(k) \end{bmatrix}, \\ &G^{(\varepsilon)}(k) = \begin{bmatrix} B^{(\varepsilon)}(k) & 0 \\ 0 & B_c(k) \end{bmatrix}, \quad h(x(k),k) = \begin{bmatrix} f(x(k),k) \\ g(x(k),k) \end{bmatrix}, \\ &\bar{D}^{(\varepsilon)}(k) = \begin{bmatrix} D_1^{(\varepsilon)}(k) \\ B_c(k)D_2^{(\varepsilon)}(k) \end{bmatrix}, \quad H(k) = \begin{bmatrix} I & 0 \\ 0 & B_c(k) \end{bmatrix}. \end{aligned} \tag{15}$$

Our aim in this technical note is to design a finite-horizon dynamic output feedback controller of the form (13) such that, for the given disturbance attenuation level $\gamma > 0$, the positive definite matrix S and the initial state $x(0)$, the saturated output $y_s(k)$ satisfies the following $\mathcal{H}_\infty$ performance constraint:

$$\begin{aligned} J :=& \mathbb{E}\left\{ \|y_s(k)\|_{[0,N-1]}^2 - \gamma^2 \|w(k)\|_{[0,N-1]}^2 \right\} \\ &- \gamma^2 x^T(0)Sx(0) < 0. \end{aligned} \tag{16}$$

The finite-horizon control problem in the presence of actuator and sensor saturations addressed above is referred to as the robust finite-horizon $\mathcal{H}_\infty$ control problem for the uncertain nonlinear discrete time-varying stochastic system (1).

III. Main Results

Lemma 1 (S-Procedure): Let $Y_0(\eta), Y_1(\eta), \cdots, Y_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$, $Y_i(\eta) = \eta^T T_i \eta$, $i = 0, 1, \ldots, p$, with $T_i = T_i^T$. Then, the implication $Y_1(\eta) \le 0, \ldots, Y_p(\eta) \le 0 \Rightarrow Y_0(\eta) \le 0$ holds if there exist $\tau_1, \ldots, \tau_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \le 0. \tag{17}$$

Theorem 1: Let the disturbance attenuation level $\gamma > 0$, families of scalars $\{\tau_1(k)\}_{0 \le k \le N} > 0$, $\{\tau_2(k)\}_{0 \le k \le N} > 0$, a positive definite matrix $S > 0$ and the controller feedback gain matrices $\{A_c(k)\}_{0 \le k \le N}$, $\{B_c(k)\}_{0 \le k \le N}$, $\{C_c(k)\}_{0 \le k \le N}$ be given. For the system (1) subject to the sensor and actuator saturation (9) and (10), the $\mathcal{H}_\infty$ performance index requirement defined in (16) is achieved for all nonzero $w(k)$ if, with the initial condition $P(0) \le \gamma^2 \hat{S}$, there exist a family of positive definite matrices $\{P(k)\}_{0 \le k \le N+1}$ satisfying the following recursive matrix inequalities:

$$\Omega^{(\varepsilon)} = \begin{bmatrix} \Omega_{11}^{(\varepsilon)}(k) & * & * \\ \Omega_{21}^{(\varepsilon)}(k) & \Omega_{22}^{(\varepsilon)}(k) & * \\ \Omega_{31}^{(\varepsilon)}(k) & \Omega_{32}^{(\varepsilon)}(k) & \Omega_{33}^{(\varepsilon)}(k) \end{bmatrix} \le 0 \tag{18}$$

for all $0 \le k \le N$, where

$$\begin{aligned}
\Omega_{11}^{(\varepsilon)}(k) &= \bar{A}^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k) - P(k) + \bar{C}^T(k)\bar{C}(k) \\
&\quad + \sum_{i=1}^{q} \hat{\Gamma}_i \left(\mathrm{tr}\left[H^T(k)P(k+1)H(k)\hat{\Theta}_i \right] + \mathrm{tr}\left[\Theta_{22}^i\right] \right), \\
\Omega_{21}^{(\varepsilon)}(k) &= G^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k) + \bar{H}^T\bar{C}(k) \\
&\quad - \frac{1}{2}\tau_1(k)\hat{C}(k) - \frac{1}{2}\tau_2(k)\hat{C}_c(k), \\
\Omega_{22}^{(\varepsilon)}(k) &= G^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k) + \bar{H}^T\bar{H} \\
&\quad - \frac{1}{2}\tau_1(k)\hat{H} - \frac{1}{2}\tau_2(k)\check{H}, \\
\Omega_{31}^{(\varepsilon)}(k) &= \bar{D}^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k) + D_2^{(\varepsilon)T}(k)\bar{C}(k), \\
\Omega_{32}^{(\varepsilon)}(k) &= \bar{D}^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k) + D_2^{(\varepsilon)T}(k)\bar{H}, \\
\Omega_{33}^{(\varepsilon)}(k) &= \bar{D}^{(\varepsilon)T}(k)P(k+1)\bar{D}^{(\varepsilon)}(k) + D_2^{(\varepsilon)T}(k)D_2^{(\varepsilon)}(k) - \gamma^2 I, \\
\hat{C}(k) &= \begin{bmatrix} 0 \\ -\tilde{C}(k) \end{bmatrix}, \quad \tilde{C}(k) = [\, KC(k) \quad 0 \,], \\
\hat{C}_c(k) &= \begin{bmatrix} -\bar{C}_c(k) \\ 0 \end{bmatrix}, \quad \hat{\Gamma}_i = \begin{bmatrix} \Gamma_i & 0 \\ 0 & 0 \end{bmatrix}, \\
\bar{C}_c(k) &= [\, 0 \quad RC_c(k) \,], \quad \check{H} = \begin{bmatrix} 2I & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{H} &= \begin{bmatrix} 0 & 0 \\ 0 & 2I \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{19}$$

Proof: Define

$$J(k) = \bar{x}^T(k+1)P(k+1)\bar{x}(k+1) - \bar{x}^T(k)P(k)\bar{x}(k). \tag{20}$$

Taking (5) into consideration, we have

$$\begin{aligned}
&\mathbb{E}\left\{ h^T(x(k),k)H^T(k)P(k+1)H(k)h(x(k),k) \right\} \\
&= \mathbb{E}\left\{ \bar{x}^T(k) \sum_{i=1}^{q} \hat{\Gamma}_i \cdot \mathrm{tr}\left[H^T(k)P(k+1)H(k)\hat{\Theta}_i \right] \bar{x}(k) \right\}
\end{aligned} \tag{21}$$

and then obtain from (14) that

$$\begin{aligned}
\mathbb{E}\{J(k)\} = \mathbb{E}\Bigg\{ &\bar{x}^T(k) \Bigg(\bar{A}^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k) + \sum_{i=1}^{q} \hat{\Gamma}_i \\
&\times \mathrm{tr}\left[H^T(k)P(k+1)H(k)\hat{\Theta}_i \right] - P(k) \Bigg) \bar{x}(k) \\
&+ 2\bar{x}^T(k)\bar{A}^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k)\bar{\Psi}(k) \\
&+ \bar{\Psi}^T(k)G^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k)\bar{\Psi}(k) \\
&+ 2w^T(k)\bar{D}^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k)\bar{x}(k) \\
&+ 2w^T(k)\bar{D}^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k)\bar{\Psi}(k) \\
&+ w^T(k)\bar{D}^{(\varepsilon)T}(k)P(k+1)\bar{D}^{(\varepsilon)}(k)w(k) \Bigg\}.
\end{aligned} \tag{22}$$

Adding the zero term $y_s^T(k)y_s(k) - \gamma^2\omega^T(k)\omega(k) - y_s^T(k)y_s(k) + \gamma^2\omega^T(k)\omega(k)$ to $\mathbb{E}\{J(k)\}$ results in

$$\begin{aligned}
\mathbb{E}\left\{ J(k) \right\} &= \mathbb{E}\left\{ [\, \bar{x}^T(k) \quad \bar{\Psi}^T(k) \quad \omega^T(k) \,] \Lambda_k \begin{bmatrix} \bar{x}(k) \\ \bar{\Psi}(k) \\ \omega(k) \end{bmatrix} \right. \\
&\quad \left. - y_s^T(k)y_s(k) + \gamma^2\omega^T(k)\omega(k) \right\} \\
&= \mathbb{E}\left\{ \eta^T(k)\Lambda_k\eta(k) - y_s^T(k)y_s(k) \right. \\
&\quad \left. + \gamma^2\omega^T(k)\omega(k) \right\}
\end{aligned} \tag{23}$$

where Λ_k and $\eta(k)$ are defined in (24), shown at the bottom of the page.

Summing up (23) on both sides from 0 to $N-1$ with respect to k, we obtain

$$\begin{aligned}
\sum_{k=0}^{N-1} \mathbb{E}\{J(k)\} &= \mathbb{E}\left\{ \bar{x}^T(N)P(N)\bar{x}(N) \right\} - \bar{x}^T(0)P(0)\bar{x}(0) \\
&= \mathbb{E}\left\{ \sum_{k=0}^{N-1} \eta^T(k)\Lambda_k\eta(k) \right\} - \mathbb{E}\left\{ \sum_{k=0}^{N-1} (y_s^T(k) \right. \\
&\quad \left. \times y_s(k) - \gamma^2\omega^T(k)\omega(k)) \right\}.
\end{aligned} \tag{25}$$

Hence, the $\mathcal{H}_\infty$ performance index defined in (16) is given by

$$\begin{aligned}
J = &\mathbb{E}\left\{ \sum_{k=0}^{N-1} \eta^T(k)\Lambda_k\eta(k) \right\} - \mathbb{E}\left\{ \bar{x}^T(N)P(N)\bar{x}(N) \right\} \\
&+ \bar{x}^T(0)(P(0) - \gamma^2\hat{S})\bar{x}(0).
\end{aligned} \tag{26}$$

$$\begin{aligned}
\Lambda_k &= \begin{bmatrix} \Omega_{11}^{(\varepsilon)}(k) & * & * \\ G^{(\varepsilon)T}(k)P(k+1)\bar{A}^{(\varepsilon)}(k) + \bar{H}^T\bar{C}(k) & G^{(\varepsilon)T}(k)P(k+1)G^{(\varepsilon)}(k) + \bar{H}^T\bar{H} & * \\ \Omega_{31}^{(\varepsilon)}(k) & \Omega_{32}^{(\varepsilon)}(k) & \Omega_{33}^{(\varepsilon)}(k) \end{bmatrix}, \\
\eta(k) &= [\, \bar{x}^T(k) \quad \bar{\Psi}^T(k) \quad \omega^T(k) \,]^T.
\end{aligned} \tag{24}$$

Noting that $P(N) > 0$ and the initial condition $P(0) \leq \gamma^2 \hat{S}$, we have $J < 0$ when the following inequality holds:

$$\eta^T(k)\Lambda_k \eta(k) \leq 0. \tag{27}$$

Noticing the sensor saturation constraint in (11), we have

$$\Psi_y^T(y(k))\left(\Psi_y(y(k)) - \tilde{C}(k)\bar{x}(k)\right) \leq 0 \tag{28}$$

which can be written by means of $\eta(k)$ as follows:

$$\eta^T(k)\Phi_{yk}\eta(k) \leq 0 \tag{29}$$

where

$$\Phi_{yk} = \frac{1}{2}\begin{bmatrix} 0 & * & * \\ \hat{C}(k) & \hat{H} & * \\ 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

In the same way, we have from the actuator saturation constraint in (12) that

$$\Psi_u^T(u(k))\left(\Psi_u(u(k)) - \bar{C}_c(k)\bar{x}(k)\right) \leq 0 \tag{31}$$

or

$$\eta^T(k)\Phi_{uk}\eta(k) \leq 0 \tag{32}$$

where

$$\Phi_{uk} = \frac{1}{2}\begin{bmatrix} 0 & * & * \\ \hat{C}_c(k) & \check{H} & * \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

Therefore, what we need to do is to find a condition under which (27) holds subject to the constraints (29) and (32). By using the S-procedure described in Lemma 1, such a sufficient condition under which (29) and (32) imply (27) is that there exist positive scalars τ_1 and τ_2 such that

$$\Lambda_k - \tau_1(k)\Phi_{yk} - \tau_2(k)\Phi_{uk} \leq 0 \tag{34}$$

which is equivalent to (18). The proof is now complete. ∎

Up to now, the analysis problem has been dealt with for the $\mathcal{H}_\infty$ output feedback control problem for a class of stochastic nonlinear discrete time-varying systems with sensor and actuator saturation constraints. In the following, we proceed to solve the controller design problem by developing a RLMI approach.

Theorem 2: Let a disturbance attenuation level $\gamma > 0$ and a positive definite matrix $S > 0$ be given. The robust $\mathcal{H}_\infty$ controller (13) can be designed for system (1) with sensor and actuator saturation constraints if there exist families of positive definite matrices $\{M(k)\}_{0\leq k\leq N+1}, \{N(k)\}_{0\leq k\leq N+1}$, families of positive scalars $\{\lambda_i(k)\}_{0\leq k\leq N} > 0$, $(i = 1, 2, \ldots, q)$, $\{\tau_1(k)\}_{0\leq k\leq N} > 0$, $\{\tau_2(k)\}_{0\leq k\leq N} > 0$ and families of real-valued matrices $\{A_c(k)\}_{0\leq k\leq N}$, $\{B_c(k)\}_{0\leq k\leq N}$ and $\{C_c(k)\}_{0\leq k\leq N}$ satisfying the initial condition

$$\begin{bmatrix} P_1(0) - \gamma^2 S & 0 \\ 0 & P_2(0) \end{bmatrix} \leq 0 \tag{35}$$

and the recursive LMIs

$$\begin{bmatrix} -\lambda_i(k) & * & * \\ \pi_{1i} & -M_{k+1} & * \\ B_c(k)\pi_{2i} & 0 & -N_{k+1} \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} \Upsilon_{11}(k) & * & * \\ \Upsilon_{21}^{(i)}(k) & \Upsilon_{22}^{(i)}(k) & * \\ \Upsilon_{31}(k) & \Upsilon_{32}^{(i)}(k) & \Upsilon_{33}(k) \end{bmatrix} \leq 0,\ i = 1, 2, \ldots, \nu \tag{37}$$

are satisfied with the parameter updated by

$$\begin{aligned} P_1(k+1) &= M^{-1}(k+1) \\ P_2(k+1) &= N^{-1}(k+1) \end{aligned} \tag{38}$$

where

$$\begin{aligned}
\Upsilon_{11}(k) &= \begin{bmatrix} -P_1(k) + \sum_{i=1}^{q}\Gamma_i\left(\lambda_i(k) + \mathrm{tr}\left[\Theta_{22}^i\right]\right) & * & * \\ 0 & -P_2(k) & * \\ 0 & \frac{1}{2}RC_c(k) & -\bar{\tau}_2(k)I \end{bmatrix} \\
\Upsilon_{21}^{(i)}(k) &= \begin{bmatrix} \frac{1}{2}\tau_1(k)KC(k) & 0 & 0 \\ D_2^{(i)T}(k)K_1C(k) & 0 & 0 \\ A^{(i)}(k) & B^{(i)}(k)R_1C_c(k) & B^{(i)}(k) \end{bmatrix}, \\
\Upsilon_{22}(k) &= \begin{bmatrix} -\tau_1(k)I & * & * \\ D_2^{(i)T}(k) & D_2^{(i)T}(k)D_2^{(i)}(k) - \gamma^2 I & * \\ 0 & D_1^{(i)}(k) & -M(k+1) \end{bmatrix}, \\
\Upsilon_{31}(k) &= \begin{bmatrix} B_c(k)K_1C(k) & A_c(k) & 0 \\ K_1C(k) & 0 & 0 \end{bmatrix}, \\
\Upsilon_{32}^{(i)}(k) &= \begin{bmatrix} B_c(k) & B_c(k)D_2^{(i)}(k) & 0 \\ I & 0 & 0 \end{bmatrix}, \\
\Upsilon_{33}(k) &= \begin{bmatrix} -N(k+1) & * \\ 0 & -I \end{bmatrix},\ \tau_2 = \bar{\tau}_2^{-1}.
\end{aligned} \tag{39}$$

Here, $A^{(i)}(k)$, $B^{(i)}(k)$, $D_1^{(i)}(k)$, $D_2^{(i)}(k)$ are the matrices at the $i\mathrm{th}$ vertex of the polytope.

Proof: Since the set of system matrices $\Xi^{(\varepsilon)} := \left(A^{(\varepsilon)}(k), B^{(\varepsilon)}(k), D_1^{(\varepsilon)}(k), D_2^{(\varepsilon)}(k)\right)$ belongs to the convex polyhedral set $\Re$, there always exist scalars $\varepsilon_i \geq 0$ $(i = 1, 2, \ldots, \nu)$ such that $\Xi^{(\varepsilon)} = \sum_{i=1}^{\nu}\varepsilon_i \Xi^{(i)}$ with $\sum_{i=1}^{\nu}\varepsilon_i = 1$, where $\Xi^{(i)} = \left(A^{(i)}(k), B^{(i)}(k), D_1^{(i)}(k), D_2^{(i)}(k)\right)$ $(i = 1, 2, \ldots, \nu)$ are ν vertexes of the polytope. By considering (15) together with (19), one can easily see that (18) holds if and only if

$$\begin{bmatrix} \Omega_{11}^{(i)}(k) & * & * \\ \Omega_{21}^{(i)}(k) & \Omega_{22}^{(i)}(k) & * \\ \Omega_{31}^{(i)}(k) & \Omega_{32}^{(i)}(k) & \Omega_{33}^{(i)}(k) \end{bmatrix} \leq 0. \tag{40}$$

Subsequently, we choose the variables $P(k)$ and $P^{-1}(k)$ that can be decomposed as follows:

$$\begin{aligned} P(k) &= \mathrm{diag}\left\{P_1(k), P_2(k)\right\}, \\ P^{-1}(k) &= \mathrm{diag}\left\{M(k), N(k)\right\}. \end{aligned} \tag{41}$$

By Schur Complement [2], (36) is equivalent to

$$\pi_i^T H^T(k)P(k+1)H(k)\pi_i < \lambda_i(k) \quad (i = 1, 2, \ldots, q) \tag{42}$$

which, by the property of matrix trace, can be rewritten as

$$\mathrm{tr}\left[H^T(k)P(k+1)H(k)\hat{\Theta}_i\right] < \lambda_i(k). \tag{43}$$

Noticing (43) and using Schur Complement [2], (40) can be rewritten as (44), shown at the bottom of the next page. It follows that (44) is guaranteed by (37) after some straightforward algebraic manipulations, and the proof of this theorem is then complete. ∎

By means of Theorem 2, the algorithm for designing the robust controller can be outlined as follows.

The controller design algorithm:

Step 1. Give the $\mathcal{H}_\infty$ performance index γ, the positive definite matrix S and the state initial condition $\bar{x}(0)$. Select the initial values for matrices $\{P_1(0), P_2(0)\}$ which satisfy the condition (35) and set $k = 0$.

Step 2. For the sampling instant k, solving the RLMIs (36) and (37) to obtain the values of matrices $\{M(k+1), N(k+1)\}$ as well as the desired controller parameters $\{A_c(k), B_c(k), C_c(k)\}$.

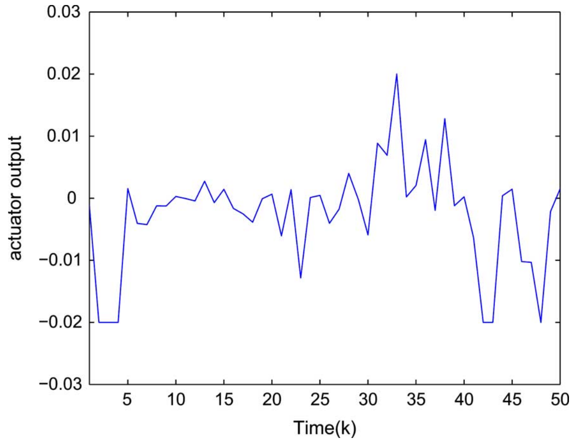

Fig. 1. Actuator output.

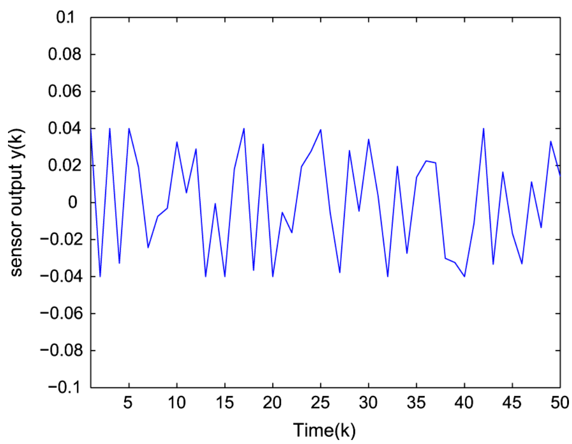

Fig. 2. Sensor output.

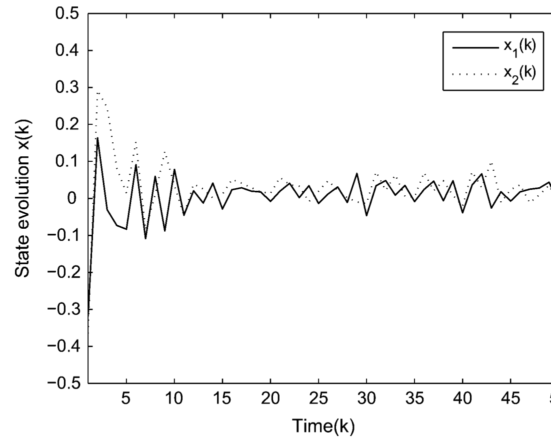

Fig. 3. State evolution $\boldsymbol{x}(\boldsymbol{k})$.

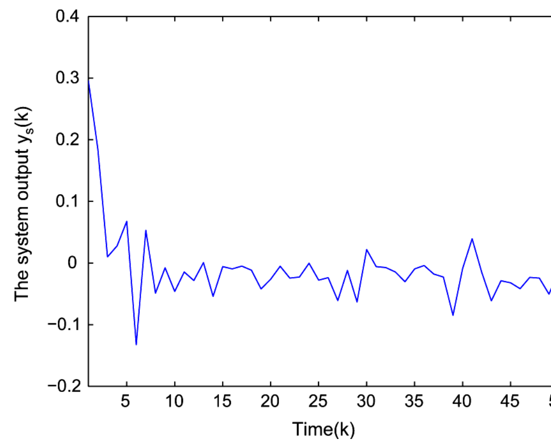

Fig. 4. System output $\boldsymbol{y}_s(\boldsymbol{k})$.

Step 3. Set $k = k + 1$ and obtain $\{P_1(k+1), P_2(k+1)\}$ by the parameter update formula (38).

Step 4. If $k < N$, then go to Step 2, else go to Step 5.

Step 5. Stop.

Remark 1: In Theorem 2, the robust $\mathcal{H}_\infty$ finite-horizon controller is designed by solving a series of recursive linear matrix inequalities (RLMIs) where both the current system measurement and previous system states are employed to control the current system state. Such a recursive control process is particularly useful for real-time implementation such as online process control. On the other hand, we point out that our main results can be extended to deal with the model predictive control problems for Markovian jumping systems [13], [25], [26] over a finite-horizon. Other research topics would be to research into more general nonlinear systems and investigate the corresponding filtering problem over an infinite horizon when the system parameters become time-invariant and the steady-state behavior is of interest. The results will appear in the near future.

IV. An Illustrative Example

Consider the following discrete time-varying stochastic nonlinear systems with sensor and actuator saturations:

$$\begin{cases} x(k+1) = \begin{bmatrix} -0.6 & 0.2+\varepsilon \\ 1.1\sin(5k) & 0.5 \end{bmatrix} x(k) \\ \qquad + \begin{bmatrix} -2 \\ 3\sin(5k) \end{bmatrix} \sigma_u(u(k)) \\ \qquad + f(x(k),k) + \begin{bmatrix} 0.1\sin(3k) \\ -0.3 \end{bmatrix} w(k) \\ y_s(k) = \sigma_y(y(k)) + g(x(k),k) + 0.2w(k) \\ y(k) = \begin{bmatrix} -0.4 & 0.5\sin(5k) \end{bmatrix} x(k) \end{cases} \tag{45}$$

$$\begin{bmatrix} -P(k) + \sum_{i=1}^{q} \hat{\Gamma}_i(\lambda_i(k) + \mathrm{tr}\,[\Theta_{22}^i]) & * & * & * & * \\ -\frac{1}{2}\tau_1(k)\hat{C}(k) - \frac{1}{2}\tau_2(k)\hat{C}_c(k) & -\frac{1}{2}\tau_1(k)\hat{H} - \frac{1}{2}\tau_2(k)\check{H} & * & * & * \\ D_2^{(i)T}(k)\bar{C}(k) & D_2^{(i)T}(k)\bar{H} & D_2^{(i)T}(k)D_2^{(i)}(k) - \gamma^2 I & * & * \\ \bar{A}^{(i)}(k) & G^{(i)}(k) & \bar{D}^{(i)}(k) & -P^{-1}(k+1) & * \\ \bar{C}(k) & \bar{H} & 0 & 0 & -I \end{bmatrix}. \tag{44}$$

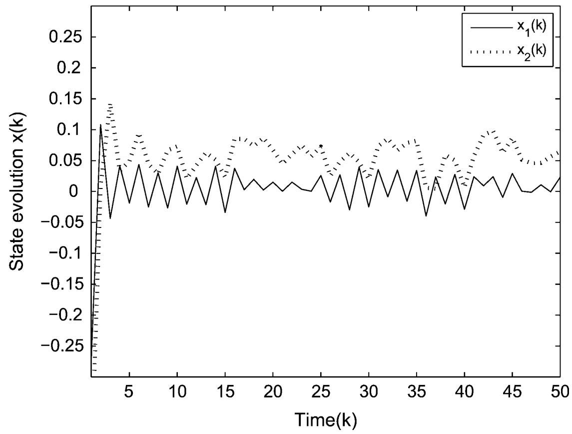

Fig. 5. State evolution $\boldsymbol{x(k)}$ when the saturation gets severe.

where $\sigma_u(u(k))$ and $\sigma_y(y(k))$ are saturation functions described as follows:

$$\begin{cases} \sigma_u(u(k)) = u(k), & \text{if } -V_{ui,\max} \le u(k) \le V_{ui,\max}; \\ \sigma_u(u(k)) = V_{ui,\max}, & \text{if } u(k) > V_{ui,\max}; \\ \sigma_u(u(k)) = -V_{ui,\max}, & \text{if } u(k) < -V_{ui,\max}; \end{cases} \tag{46}$$

$$\begin{cases} \sigma_y(y(k)) = y(k), & \text{if } -V_{yj,\max} \le y(k) \le V_{yj,\max}; \\ \sigma_y(y(k)) = V_{yj,\max}, & \text{if } y(k) > V_{yj,\max}; \\ \sigma_y(y(k)) = -V_{yj,\max}, & \text{if } y(k) < -V_{yj,\max}. \end{cases} \tag{47}$$

In this example, we have $i = j = 1$. Take the saturation values as $V_{u1,\max} = 0.02$ and $V_{y1,\max} = 0.04$ with the state initial value $\bar{x}(0) = [\,0.26 \quad -0.2 \quad 0 \quad 0\,]^T$ and $S = \text{diag}\{1, 1\}$. The exogenous disturbance input is selected as $w(k) = 0.5\sin(4k)$, and other parameters are chosen as $K = 0.3$, $K_1 = 0.7$, $R = 0.4$ and $R_1 = 0.6$.

The nonlinear functions $f(x(k), k)$ and $g(x(k), k)$ are given as follows:

$$\begin{aligned} f(x(k), k) &= \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} \times (0.5x_1(k)\xi_1(k) \\ &\quad + 0.4x_2(k)\xi_2(k)) \\ g(x(k), k) &= 0.8 \times (0.5x_1(k)\xi_1(k) + 0.4x_2(k)\xi_2(k)) \end{aligned}$$

where $x_i(k)$ $(i = 1, 2)$ is the ith element of $x(k)$, and $\xi_i(k)$ $(i = 1, 2)$ are zero mean, uncorrelated Gaussian white noise processes with unity variances that is also uncorrelated with $\omega(k)$. It can be easily checked that the above class of stochastic nonlinearities satisfies (4) and (5) with

$$\begin{aligned} &\mathbb{E}\left\{ \begin{bmatrix} f(x(k), k) \\ g(x(k), k) \end{bmatrix} [\, f^T(x(k), k) \quad g^T(x(k), k)\,] \,\middle|\, x(k) \right\} \\ &= \begin{bmatrix} 0.3 \\ 0.4 \\ 0.8 \end{bmatrix} [\,0.3 \quad 0.4 \quad 0.8\,]\, \mathbb{E}\left\{ x^T(k) \begin{bmatrix} 0.25 & 0 \\ 0 & 0.16 \end{bmatrix} x(k) \right\} \end{aligned}$$

Let $\gamma = 0.5$ and choose the parameters' initial values satisfying (35). The uncertain parameter ε is unknown but assumed to belong to the known range $[\,-0.2 \quad 0.2\,]$. According to controller design algorithm, the RLMIs in Theorem 2 can be solved recursively subject to given initial conditions and prespecified performance indices.

The simulation results are shown in Figs. 1–5, where Fig. 1 plots the actuator output and Fig. 2 depicts the sensor output. Note that both the actuator and sensor outputs are saturated. Fig. 3 shows the state simulation results of the closed-loop system (14), and the system output $y_s(k)$ is depicted in Fig. 4. When the saturation level are changed to $V_{u1,\max} = 0.01$ and $V_{y1,\max} = 0.03$, the state simulation for system (14) is given in Fig. 5, from which we can observe that 1) the saturations do influence the control performances; and 2) the saturation range would have a serious impact on the feasibility of the RLMIs.

V. Conclusion

In this technical note, the problem of robust $\mathcal{H}_\infty$ output feedback control has been discussed for a class of polytopic uncertain stochastic nonlinear discrete time-varying systems with both sensors and actuators subject to saturation. Sufficient conditions have been derived for the closed-loop system under consideration to satisfy the $\mathcal{H}_\infty$ performance constraint. A robust $\mathcal{H}_\infty$ output feedback controller has then been designed by solving a set of recursive LMIs. A numerical simulation example has been used to demonstrate the effectiveness of the control technology presented in this technical note.

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Comments on "A Multichannel IOS Small Gain Theorem for Systems With Multiple Time-Varying Communication Delays"

Björn S. Rüffer, *Member, IEEE*, Rudolf Sailer, and Fabian R. Wirth

***Abstract*—The small-gain condition presented by Polushin *et al.* may be replaced by a strictly weaker one to obtain essentially the same result. The necessary minor modifications of the proof are given. Using essentially the same arguments, a global version of the result is also presented.**

***Index Terms*—Generalized small-gain condition, input-to-output stability, Lyapunov stability, networked control systems, time-varying communication delays.**

I. Introduction

In [1] Polushin *et al.* have presented a small-gain type condition that ensures input-output stability for networked systems in the presence of time delays. In this note we show that the small-gain condition by Polushin *et al.* can be replaced by a less restrictive one. As an extension we obtain a global version of the result, with a global small-gain condition resembling the one of Dashkovskiy *et al.* [2]. By means of an example we show that the modified small-gain conditions are indeed less restrictive than the original one. For brevity we adopt the problem formulation and notations from [1].

It should be noted that in a recent paper [3] a small-gain theorem for large-scale systems under the presence of time-delays has been presented. In contrast to [1] it does not explicitly take into account multiple communication channels. However, it is based on the weaker "cycle-gains are contractions" small-gain condition as in Corollary 2.4.

II. The Generalized Small-Gain Theorem

Based on the setup and notation in [1] we formulate our generalized small-gain condition in a very compact form. We use [1.X] to reference equation/assumption/or result X in [1].

A. Modified Notation

We need a few notations before we can state our main theorem. We write

$$\left.\begin{aligned} \Gamma_U = \begin{pmatrix} \Gamma_{1u} & 0 \\ 0 & \Gamma_{2u} \end{pmatrix}, \quad \Gamma_W = \begin{pmatrix} \Gamma_{1w} & 0 \\ 0 & \Gamma_{2w} \end{pmatrix}, \\ B\left(x_d^+(t)\right) = \begin{pmatrix} \beta_1\left(|x_{1d}(t)|\right) \\ \beta_2\left(|x_{2d}(t)|\right) \end{pmatrix}, \quad \hat{y}^+ = \begin{pmatrix} \hat{y}_1^+ \\ \hat{y}_2^+ \end{pmatrix}, \\ \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & \Psi_2 \\ \Psi_1 & 0 \end{pmatrix}, \\ u^+(t) = \begin{pmatrix} u_1^+(t) \\ u_2^+(t) \end{pmatrix}, \quad \Delta_u = \begin{pmatrix} \Delta_{u1} \\ \Delta_{u2} \end{pmatrix}. \end{aligned}\right\} \tag{1}$$

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