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Robust sliding mode design for uncertain stochastic systems based on H_∞ control method

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SUMMARY

In this paper, the design of sliding mode control is addressed for uncertain stochastic systems modeled by Itô differential equations. There exist the parameter uncertainties in both the state and input matrices, and the *unmatched* external disturbance. The key feature of this work is the integration of sliding mode control method with H_∞ technique such that the robustly stochastic stability with a prescribed disturbance attenuation level γ can be obtained. A sufficient condition for the existence of the desired sliding mode controller is obtained via linear matrix inequalities (LMIs). The reachability of the specified sliding surface is proven. Finally, a numerical simulation is presented to illustrate the proposed method. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Stochastic systems, parameter uncertainties, external disturbance, sliding mode control, H_∞ control

1. INTRODUCTION

The stability and stabilization of stochastic systems, governed by the Itô differential equations, have attracted much attention over the past few decades due to the extensive applications of stochastic modelling in mechanical systems, economics, and other areas [1]. A great number of results on this topic have been reported in the literature, see, e.g., [2, 3, 4], and the references therein. Very recently, the designs of sliding mode control (SMC) for uncertain stochastic systems were also developed in, e.g., [5, 6]. Both the theoretical proof and numerical simulation in the aforementioned works show that SMC is an effective and promising approach for the stabilization of Itô stochastic systems.

It is well known that the key feature of SMC is the insensitiveness of sliding motion on the specified sliding surface to *matched* uncertainties or external disturbances, see [7, 8, 9] and the references therein. However, the sliding motion cannot be detached from the effect of

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unmatched parameter uncertainties, especially, *unmatched* external disturbances. This means that the *unmatched external disturbances* will make the design of SMC become complex and challenging.

On the other hand, the H_∞ control, in the past decades, has been widely employed to deal with the uncertain systems with *external disturbance*. The goal of this problem is to design a controller to stabilize a given system while satisfying a prescribed level of disturbance attenuation. Recently, the H_∞ control for stochastic system has been considered by some researchers. For example, the state feedback controller in [10] was designed for uncertain stochastic system such that the closed-loop systems is robustly asymptotically stable and satisfies a prescribed H_∞ performance. Zhang, *et al.* [11] further presented the stochastic H_2/H_∞ control design for nonlinear stochastic systems with state-dependent noise.

Motivated by the above discussion, it is convinced that the integration of the SMC method with H_∞ technique will have a promising to extend the SMC to the systems with *unmatched* uncertainties and obtain a better dynamic performance. Therefore, in this paper, the design of SMC problem for uncertain Itô stochastic systems with *unmatched* external disturbances will be considered by integrating H_∞ technique. There exist parameter uncertainties in both the state and input matrices, and external disturbances. By utilizing H_∞ technique to attenuate the effect of *unmatched* external disturbance, this paper proposes a novel sliding mode controller that can ensure the robustly stochastic stability with a prescribed disturbance attenuation level γ for the resultant closed-loop system, irrespective of parameter uncertainties and *unmatched* external disturbance. It is shown that the specified sliding surface is attained with probability 1. Moreover, a computational algorithm is given to solve linear matrix inequality (LMI) with equality constraint such that the design of both the sliding surface and the SMC law can be easily obtained by means of convex optimization.

Notations: For a real matrix, $M > 0$ means that M is symmetric and positive definite. I is used to represent an identity matrix of appropriate dimensions. $L_2[0,\infty)$ denotes the space of square-integrable vector functions over $[0,\infty)$. $||\cdot||$ refers the Euclidean vector norm. $||\cdot||_2$ stands for the usual $L_2[0,\infty)$ norm. $(\Omega,\mathcal{F},\mathcal{P})$ is a probability space with Ω the sample space, $\mathcal{F}$ the σ-algebra of subsets of the sample space, and $\mathcal{P}$ the probability measure. $\mathcal{E}\{\cdot\}$ denotes the expectation operator with respect to probability measure $\mathcal{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Formulation of the problem

Consider the uncertain time-delay stochastic systems described by the Itô form:

$$\begin{aligned}(\Sigma): dx(t) &= [(A+\Delta A(t))x(t)+(A_d+\Delta A_d(t))x(t-d) \\ &\quad +(B+\Delta B(t))u(t)+D_1 v(t)]\,dt \\ &\quad +G\left[(E+\Delta E(t))\,x(t)+(E_d+\Delta E_d(t))\right. \\ &\quad \left.\times x(t-d)+D_2 v(t)\right]dw(t), \qquad (1)\\ z(t) &= Cx(t), \qquad (2)\\ x(t) &= \varphi(t), \quad t\in[-d,0] \qquad (3)\end{aligned}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $z(t)\in R^q$ is the controlled output, and $v(t)\in R^p$ is the exogenous disturbance input which belongs to $L_2[0,\infty)\cap L_\infty[0,\infty)$. In this work, it is assumed that the upper bound for $v(t)$ is known. $w(t)$ is a one-dimensional Brownian motion. $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $D_1 \in \mathbb{R}^{n\times p}$, $D_2 \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{m\times n}$, $E_d \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times m}$ are known real constant matrices. Without loss of generality, it is assumed that the input matrix B has full column rank. $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta E(t)$ and $\Delta E_d(t)$ are unknown time-varying matrices representing parameter uncertainties. d is the known constant delay, and $\varphi(t)$ is a continuous vector-valued initial function. In this work, the uncertainty $\Delta B(t)$ is assumed to be matched, i.e., there exists a matrix $\delta(t) \in \mathbb{R}^{m\times m}$ such that $\Delta B(t) = B\delta(t)$ with $||\delta(t)|| \le \rho_B < 1$, where ρ_B is a positive constant. Moreover, the admissible parameter uncertainties $\Delta A(t)$, $\Delta A_d(t)$, $\Delta E(t)$ and $\Delta E_d(t)$ are of the following norm-bounded form:

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) \end{bmatrix} = M_1 F_1(t) \begin{bmatrix} N_a & N_{ad} \end{bmatrix}, \tag{4}$$

$$\begin{bmatrix} \Delta E(t) & \Delta E_d(t) \end{bmatrix} = M_2 F_2(t) \begin{bmatrix} N_e & N_{ed} \end{bmatrix} \tag{5}$$

where M_1, M_2, N_a, N_{ad}, N_e, H_{ed} are known real constant matrices, and $F_1(t)$ and $F_2(t)$ are unknown time-varying matrices with Lebesgue measurable elements satisfying

$$F_1^T(t)F_1(t) \le I, \quad F_2^T(t)F_2(t) \le I, \quad \forall t. \tag{6}$$

It is noted that there exists unmatched external disturbance $v(t)$ in the systems under consideration. In addition, there may exist parameter uncertainties in both the state and control input matrices.

The objective of this work is to design an SMC law such that the desired control performance is obtained for the resultant closed-loop stochastic system despite parameter uncertainties and *unmatched* external disturbance.

Before proceeding, some standard concepts and Lemma will be given as follows, which are useful for the development of our result.

Definition 1. The uncertain stochastic systems in (1) and (3) are said to be robustly stochastically stable if the system associated to (1) and (3) with $u(t) = 0$ and $v(t) = 0$ is mean-square asymptotically stable for all admissible parameter uncertainties.

Definition 2. Given a scalar $\gamma > 0$, the unforced stochastic system in (1)–(3) with $u(t) = 0$ is said to be robustly stochastically stable with disturbance attenuation γ if it is robustly stochastically stable and under zero initial condition, $||z(t)||_{E_2} < \gamma||v(t)||_2$ for all nonzero $v(t) \in L_2[0, \infty)$ and all admissible uncertainties, where

$$||z(t)||_{E_2} = \left(\mathcal{E}\left\{\int_0^\infty |z(t)|^2 dt\right\}\right)^{1/2}.$$

Lemma 1. [12]Let A, E, H, and $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) \le I$. Then, we have

(i) For any scalar $\epsilon > 0$,

$$EF(t)H + H^T F^T(t)E^T \le \epsilon^{-1}EE^T + \epsilon H^T H.$$

(ii) For any real scalar $\epsilon > 0$ and matrix $X > 0$ satisfying $\epsilon I - E^T XE > 0$,

$$(A + EF(t)H)^T X(A + EF(t)H) \le A^T XA + A^T XE \left(\epsilon I - E^T XE\right)^{-1} E^T XA + \epsilon H^T H.$$

3. Sliding mode control with H_∞ performance

In this section, an SMC law will be firstly synthesized such that the resultant closed-loop systems is robustly stochastically stable with disturbance attenuation γ. It is further proven that the reachability of the specified sliding surface $s(t) = 0$ can be ensured by the proposed SMC law. Thus, it is obtained that the synthesized SMC law can guarantee that the state trajectories of uncertain stochastic systems (1)–(3) are driven onto (with probability 1) the sliding surface, and asymptotically tend to zero (in mean-square sense) along the specified sliding surface.

To this end, we choose the sliding surface $s(t)$ as:

$$s(t) = \Gamma B^T P x(t) = 0 \tag{7}$$

where $P \in R^{n\times n}$ is a positive definite matrix to be designed later. $\Gamma \in R^{m\times m}$ is some nonsingular matrix, which is chosen as the identity matrix for simplicity in this work.

Furthermore, we design the sliding mode control law as follows:

$$\begin{aligned} u(t) &= -Kx(t) + u_r(t), && (8)\\ u_r(t) &= \begin{cases} -B^T P\left(Ax(t) + A_d x(t-d)\right) - \rho(x,t)\frac{s(t)}{||s(t)||}, & ||s(t)|| \neq 0 \\ -B^T P\left(Ax(t) + A_d x(t-d)\right), & ||s(t)|| = 0 \end{cases} && (9)\end{aligned}$$

where $K \in R^{m\times n}$ is chosen such that $A - BK$ is Hurwitz, and the positive scalar function $\rho(x,t)$ is given as

$$\begin{aligned}\rho(x,t) \geq \; & \frac{2}{1-\rho_B^2}\left\{\left[||\Phi(A-BK)|| + ||\Phi M_1||||N_a|| + \rho_B||K|| + (1+\rho_B)\,||B^T PA||\right]||x(t)||\right.\\ & + \left[||\Phi A_d|| + ||\Phi M_1||||N_{ad}|| + (1+\rho_B)\,||B^T PA_d||\right]||x(t-d)||\\ & \left. + ||s(t)||||\Phi D_1||||v(t)|| + \mu\right\} \end{aligned}\tag{10}$$

with $\Phi = \left(B^T PB\right)^{-1} B^T P$ and $\mu > 0$ a small known scalar.

Substituting (8) into (1), we obtain the closed-loop system as follows:

$$\begin{aligned}(\Sigma_c) : dx(t) &= \left[(A-BK)\,x(t) + (A_d + \Delta A_d(t))x(t-d) + (B+\Delta B(t))u_r(t)\right.\\ &\quad \left. + \left(\Delta A(t) - \Delta B(t)K\right)x(t) + D_1 v(t)\right]dt\\ &\quad + G\left[(E+\Delta E(t))\,x(t) + (E_d + \Delta E_d(t))\,x(t-d) + D_2 v(t)\right]dw(t), && (11)\\ z(t) &= Cx(t), && (12)\\ x(t) &= \varphi(t), \quad t\in[-d,0]. && (13)\end{aligned}$$

3.1. Analysis of stochastic stability

In the sequel, we shall analyze the dynamic performance of the closed-loop system (Σ_c), and give the sufficient condition for stochastic stability in Theorem 1.

Theorem 1. Given a scalar $\gamma > 0$. Consider the uncertain stochastic system (11)–(13). If there exist matrices $P > 0$, $Q > 0$, and scalars $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$ satisfying LMI:

$$\begin{bmatrix} \Omega_1 & * & * & * & * & * & * & * & * \\ A_d^T P & \Omega_2 & * & * & * & * & * & * & * \\ D_1^T P & 0 & -\gamma^2 I & * & * & * & * & * & * \\ PGE & PGE_d & PGD_2 & -P & * & * & * & * & * \\ M_1^T P & 0 & 0 & 0 & -\varepsilon_1 I & * & * & * & * \\ M_1^T P & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * & * & * \\ B^T P & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3 I & * & * \\ N_e \varepsilon_4 & N_{ed}\varepsilon_4 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_4 I & * \\ M_2^T G^T PGE & M_2^T G^T PGE_d & M_2^T G^T PGD_2 & 0 & 0 & 0 & 0 & 0 & \Omega_3 \end{bmatrix} < 0, \tag{14}$$

with

$$\Omega_1 = P(A - BK) + (A - BK)^T P + Q + \varepsilon_1 N_a^T N_a + \varepsilon_3 \rho_B^2 K^T K + C^T C, \tag{15}$$

$$\Omega_2 = -Q + \varepsilon_2 N_{ad}^T N_{ad}, \ \Omega_3 = M_2^T G^T PGM_2 - \varepsilon_4 I, \tag{16}$$

then the system (Σ_c) is robustly stochastically stable with disturbance attenuation γ.

Proof: Under the condition of theorem 1, we firstly establish the robust stochastic stability of the system (Σ_c). To this end, we consider (11) with $v(t) = 0$, and choose the Lyapunov function candidate as:

$$V(x(t), t) = x(t)^T P x(t) + \int_{t-d}^{t} x(\tau)^T Q x(\tau) d\tau. \tag{17}$$

By Itô 's formula, we obtain the stochastic differential of $V(x(t), t)$ along (11) with $v(t) = 0$ as:

$$dV(x(t), t) = \mathcal{L}V(x(t), t)dt + 2x(t)^T PG \left[(E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-d) \right] dw(t)$$

where

$$\begin{aligned} \mathcal{L}V(x(t), t) = {} & 2x(t)^T P (A - BK)\, x(t) + 2x(t)^T P (A_d + \Delta A_d(t))\, x(t-d) \\ & + 2x(t)^T P (\Delta A(t) - \Delta B(t) K)\, x(t) + 2x(t)^T P (B + \Delta B(t))\, u_r(t) \\ & + \left[(E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-d) \right]^T G^T \\ & \times PG \left[(E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-d) \right] \\ & + x(t)^T Q x(t) - x(t-d)^T Q x(t-d). \end{aligned} \tag{18}$$

Noting the definition of sliding function $s(t)$ in (7) and the expression (9), we have for $||s(t)|| \neq 0$:

$$\begin{aligned} \mathcal{L}V(x(t), t) = {} & x(t)^T \left[P (A - BK) + (A - BK)^T P + Q \right] x(t) + 2x(t)^T P A_d x(t-d) \\ & + 2x(t)^T P \Delta A_d(t) x(t-d) + 2x(t)^T P (\Delta A(t) - \Delta B(t) K)\, x(t) \\ & - 2s(t)^T (I + \delta(t)) B^T P (A x(t) + A_d x(t-d)) \\ & - 2s(t)^T (I + \delta(t)) \rho(x, t) \frac{s(t)}{||s(t)||} - x(t-d)^T Q x(t-d) \\ & + \left[(E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-d) \right]^T G^T \\ & \times PG \left[(E + \Delta E(t))\, x(t) + (E_d + \Delta E_d(t))\, x(t-d) \right]. \end{aligned} \tag{19}$$

By Lemma 1(i), it follows that for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $\varepsilon_3 > 0$:

$$\begin{aligned} 2x(t)^T P\Delta A(t)x(t) &\leq \varepsilon_1^{-1}x(t)^T PM_1M_1^T Px(t) + \varepsilon_1 x(t)^T N_a^T N_a x(t), && (20)\\ 2x(t)^T P\Delta A_d(t)x(t-d) &\leq \varepsilon_2^{-1}x(t)^T PM_1M_1^T Px(t) + \varepsilon_2 x(t-d)^T N_{ad}^T N_{ad} x(t-d), && (21)\\ -2x(t)^T P\Delta B(t)Kx(t) &\leq \varepsilon_3^{-1}x(t)^T PBB^T Px(t) + \varepsilon_3\rho_B^2 x(t)^T K^T Kx(t). && (22)\end{aligned}$$

Further, observing that :

$$\begin{aligned} -2s(t)^T(I+\delta(t))\rho(x,t)\frac{s(t)}{||s(t)||} &\leq -2\rho(x,t)||s(t)|| + \frac{\rho(x,t)}{||s(t)||}\left(s(t)^T\delta(t)\delta(t)^T s(t) + s(t)^T s(t)\right)\\ &\leq \rho(x,t)\left(\rho_B^2 - 1\right)||s(t)||. \qquad (23)\end{aligned}$$

we can obtain from (10) and (23):

$$\begin{aligned} &- 2s(t)^T(I+\delta(t))B^T P\left(Ax(t) + A_d x(t-d)\right) - 2s(t)^T(I+\delta(t))\rho(x,t)\frac{s(t)}{||s(t)||}\\ &\leq 2||s(t)||(1+\rho_B)\left(||B^T PA||||x(t)|| + ||B^T PA_d||||x(t-d)||\right) - \rho(x,t)(1-\rho_B^2)||s(t)||\\ &\leq -2\left[(||\Phi(A-BK)|| + ||\Phi M_1||||N_a|| + \rho_B||K||)\,||x(t)||\right.\\ &\quad \left.+ (||\Phi A_d|| + ||\Phi M_1||||N_{ad}||)\,||x(t-d)|| + \mu\right]||s(t)||\\ &< 0. \qquad (24)\end{aligned}$$

In addition, it is easily shown from (14) that $\varepsilon_4 I - M_2^T G^T PGM_2 > 0$. Thus, we obtain from (5) and Lemma 1 (ii) that:

$$\begin{aligned} &\left[G\bar{E} + GM_2F_2(t)\bar{N}_e\right]^T P\left[G\bar{E} + GM_2F_2(t)\bar{N}_e\right]\\ &\leq \bar{E}^T G^T PG\bar{E} + \bar{E}^T G^T PGM_2\left(\varepsilon_4 I - M_2^T G^T PGM_2\right)^{-1} M_2^T G^T PG\bar{E} + \varepsilon_4\bar{N}_e^T\bar{N}_e \quad (25)\end{aligned}$$

where $\bar{E} = \begin{bmatrix} E & E_d \end{bmatrix}$, $\bar{N}_e = \begin{bmatrix} N_e & N_{ed} \end{bmatrix}$.

Hence, substituting (20)–(25) into (19) yields:

$$\mathcal{L}V(x(t),t) \leq \begin{bmatrix} x(t)^T & x(t-d)^T \end{bmatrix}\Theta\begin{bmatrix} x(t)\\ x(t-d)\end{bmatrix} \qquad (26)$$

where

$$\begin{aligned}\Theta &= \begin{pmatrix} \Pi_1 & PA_d\\ A_d^T P & -Q + \varepsilon_2 N_{ad}^T N_{ad}\end{pmatrix} + \bar{E}^T G^T PG\bar{E} + \varepsilon_4\bar{N}_e^T\bar{N}_e\\ &\quad \bar{E}^T G^T PGM_2\left(\varepsilon_4 I - M_2^T G^T PGM_2\right)^{-1} M_2^T G^T PG\bar{E} \qquad (27)\end{aligned}$$

with

$$\begin{aligned}\Pi_1 &= P(A-BK) + (A-BK)^T P + Q + \varepsilon_1^{-1}PM_1M_1^T P + \varepsilon_1 N_a^T N_a\\ &\quad + \varepsilon_2^{-1}PM_1M_1^T P + \varepsilon_3^{-1}PBB^T P + \varepsilon_3\rho_B^2 K^T K.\end{aligned}$$

On the other hand, it can be seen that LMI (14) implies:

$$\begin{bmatrix} \Pi_2 & * & * & * & * & * & * & *\\ A_d^T P & \Omega_2 & * & * & * & * & * & *\\ PGE & PGE_d & -P & * & * & * & * & *\\ M_1^T P & 0 & 0 & -\varepsilon_1 I & * & * & * & *\\ M_1^T P & 0 & 0 & 0 & -\varepsilon_2 I & * & * & *\\ B^T P & 0 & 0 & 0 & 0 & -\varepsilon_3 I & * & *\\ N_e\varepsilon_4 & N_{ed}\varepsilon_4 & 0 & 0 & 0 & 0 & -\varepsilon_4 I & *\\ M_2^T G^T PGE & M_2^T G^T PGE_d & 0 & 0 & 0 & 0 & 0 & \Omega_3 \end{bmatrix} < 0 \qquad (28)$$

with $\Pi_2 = P(A-BK)+(A-BK)^TP+Q+\varepsilon_1 N_a^T N_a+\varepsilon_3\rho_B^2 K^TK$, Ω_2 and Ω_3 as in (16). By Schur's complement, the matrix inequality (28) implies that $\Theta < 0$. This together with (26) implies that for all $\begin{bmatrix} x(t)^T & x(t-d)^T \end{bmatrix}^T \neq 0$, we have:

$$\mathcal{L}V(x(t),t) < 0. \tag{29}$$

which means that the closed-loop stochastic system (11) with $v(t) = 0$ is robustly stochastically asymptotically stable.

Next, we shall show that the stochastic system (Σ_c) satisfies:

$$||z(t)||_{E_2} < \gamma ||v(t)||_2 \tag{30}$$

for all nonzero $v(t) \in L_2[0,\infty)$. To this end, we assume zero initial condition, that is, $x(t) = 0$ for $t \in [-d, 0]$.

By Itô 's formula, we have:

$$\mathcal{E}\{V(x(t),t\} = \mathcal{E}\{\int_0^t \mathcal{L}V(x(\tau),\tau)d\tau\}$$

where $V(x(t),t)$ is the Lyapunov function candidate as in (17), and

$$\begin{aligned}
\mathcal{L}V(x(t),t) \;\; = \;\; & x(t)^T\left[P\left(A-BK\right)+\left(A-BK\right)^TP+Q\right]x(t)+2x(t)^TPA_dx(t-d) \\
& +2x(t)^TP\Delta A_d(t)x(t-d)+2x(t)^TP\left(\Delta A(t)-\Delta B(t)K\right)x(t) \\
& -2s(t)^T(1+\delta(t))B^TP\left(Ax(t)+A_dx(t-d)\right)+2x(t)^TPD_1v(t) \\
& -2s(t)^T(1+\delta(t))\rho(x,t)\frac{s(t)}{||s(t)||}-x(t-d)^TQx(t-d) \\
& +\left[\left(E+\Delta E(t)\right)x(t)+\left(E_d+\Delta E_d(t)\right)x(t-d)+D_2v(t)\right]^TG^T \\
& \times PG\left[\left(E+\Delta E(t)\right)x(t)+\left(E_d+\Delta E_d(t)\right)x(t-d)+D_2v(t)\right].
\end{aligned} \tag{31}$$

Now, set

$$J(t) = \mathcal{E}\left\{\int_0^t \left[z(\tau)^Tz(\tau)-\gamma^2v(\tau)^Tv(\tau)\right]d\tau\right\} \tag{32}$$

with $t > 0$. And then, it is easy to show that

$$\begin{aligned}
J(t) \;\; = \;\; & \mathcal{E}\left\{\int_0^t \left[z(\tau)^Tz(\tau)-\gamma^2v(\tau)^Tv(\tau)+\mathcal{L}V(x(\tau),\tau)\right]d\tau\right\}-\mathcal{E}\left\{V(x(t),t)\right\} \\
\leq \;\; & \mathcal{E}\left\{\int_0^t \left[z(\tau)^Tz(\tau)-\gamma^2v(\tau)^Tv(\tau)+\mathcal{L}V(x(\tau),\tau)\right]d\tau\right\}
\end{aligned} \tag{33}$$

for all $t > 0$.

By similar lines as in expression (25), we have for $\varepsilon_4 > 0$:

$$\begin{aligned}
& \left[G\hat{E}+GM_2F_2(t)\hat{N}_e\right]^T P\left[G\hat{E}+GM_2F_2(t)\hat{N}_e\right] \\
\leq \;\; & \hat{E}^TG^TPG\hat{E}+\hat{E}^TG^TPGM_2\left(\varepsilon_4 I-M_2^TG^TPGM_2\right)^{-1}M_2^TG^TPG\hat{E}+\varepsilon_4\hat{N}_e^T\hat{N}_e
\end{aligned} \tag{34}$$

where $\hat{E} = \begin{bmatrix} E & E_d & D_2 \end{bmatrix}$, $\hat{N}_e = \begin{bmatrix} N_e & N_{ed} & 0 \end{bmatrix}$.

Thus, considering (20)–(24) and (34), it follows from (33) that:

$$J(t) \leq \mathcal{E}\left\{\int_0^t \left[\begin{array}{ccc} x(\tau)^T & x(\tau-d)^T & v(t)^T \end{array}\right] \Xi \left[\begin{array}{ccc} x(\tau)^T & x(\tau-d)^T & v(t)^T \end{array}\right]^T d\tau\right\} \tag{35}$$

with

$$\begin{aligned} \Xi &= \left(\begin{array}{ccc} \Omega_1 & PA_d & PD_1 \\ A_d^T P & \Omega_2 & 0 \\ D_1^T P & 0 & -\gamma^2 I \end{array}\right) + \hat{E}^T G^T PG\hat{E} \\ &\quad + \hat{E}^T G^T PGM_2 \left(\varepsilon_4 I - M_2^T G^T PGM_2\right)^{-1} M_2^T G^T PG\hat{E} + \varepsilon_4 \hat{N}_e^T \hat{N}_e \end{aligned}$$

where Ω_1 and Ω_2 are given as in (15) and (16). By Schur's complement, it can be shown that $\Xi < 0$ is ensured by LMI (14). This together with (35) implies that $J(t) < 0$ for all $t > 0$. Hence, we obtain (30) from (32). □

Remark 1. It is noted that the condition in Theorem 1 is delay-independent, which might be conservative when the time delay is known and small. Hence, it would be appropriate to extend the current study to delay-dependent issue in future research.

In the next section, we shall show that if the solution P of LMI (14) satisfies a special equality constraint, the reachability of sliding surface in (7) can also be ensured by the SMC law in (8)–(10) .

3.2. *Reachability of sliding surface*

It is known from [13] that the solution $x(t)$ of the system (1) and (3) is given as:

$$\begin{aligned} x(t) &= \varphi(0) + \int_0^t [(A + \Delta A(\tau))x(s) + (A_d + \Delta A_d(\tau))x(\tau - d) \\ &\quad + (B + \Delta B(\tau))u(\tau) + D_1 v(\tau)]\, d\tau \\ &\quad + \int_0^t G\left[(E + \Delta E(\tau))\, x(\tau) + (E_d + \Delta E_d(\tau))\, x(\tau - d) + D_2 v(\tau)\right] dw(\tau). \end{aligned} \tag{36}$$

Here, the last term in (36) is an Itô stochastic integral. Hence, the switching function $s(t)$ in (7) is well defined for the solution $x(t)$ of the stochastic system (1)–(2), and can be expressed as:

$$\begin{aligned} s(t) &= B^T P\varphi(0) + B^T P \int_0^t [(A + \Delta A(\tau))x(\tau) + (A_d + \Delta A_d(\tau))x(\tau - d) \\ &\quad + (B + \Delta B(\tau))u(\tau) + D_1 v(\tau)]\, d\tau \\ &\quad + B^T P \int_0^t G\left[(E + \Delta E(\tau))\, x(\tau) + (E_d + \Delta E_d(\tau))\, x(\tau - d) + D_2 v(\tau)\right] dw(\tau). \end{aligned} \tag{37}$$

It is seen that if the solution P of LMI (14) further satisfies:

$$B^T PG = 0 \tag{38}$$

we have:

$$\begin{aligned} s(t) &= B^T P\varphi(0) + B^T P \int_0^t [(A + \Delta A(\tau))x(\tau) + (A_d + \Delta A_d(\tau))x(\tau - d) \\ &\quad + (B + \Delta B(\tau))u(\tau) + D_1 v(\tau)]\, d\tau. \end{aligned} \tag{39}$$

This means that $s(t)$ varies finitely. That is, it is rational to take the time derivation of $s(t)$ under the condition that $B^TPG = 0$. Hence, we have:

$$\dot{s}(t) = B^T P \left[(A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-d) + (B + \Delta B(t))u(t) + D_1 v(t)\right]. \quad (40)$$

And then, the reachability of the specified sliding surface $s(t) = 0$ can be obtained in the following theorem.

Theorem 2. Consider the uncertain stochastic time-delay systems (1)–(2) with sliding surface as in (7) where $P > 0$, $Q > 0$, and scalars $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$ satisfying LMI (14) and equality constraint (38). Then, the SMC law (8)–(10) will guarantee that the sliding surface (7) is attained (with probability 1) for all $v(t) \in L_2[0,\infty) \cap L_\infty[0,\infty)$.

Proof: To analyze the reachability of sliding surface $s(t) = 0$, we choose the Lyapunov function candidate as:

$$V(t) = \frac{1}{2}s(t)^T \left(B^T PB\right)^{-1} s(t).$$

Utilizing (8), (9) and (40) yields:

$$\begin{aligned}
\dot{V}(t) &= s(t)^T \left(B^T PB\right)^{-1} B^T P \left((A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-d)\right) \\
&\quad + s(t)^T (I + \delta(t))u(t) + s^T(t) \left(B^T PB\right)^{-1} B^T P D_1 v(t) \\
&\leq s^T(t) \left(B^T PB\right)^{-1} B^T P \left(A - BK + \Delta A(t)\right) x(t) - s^T(t)\delta(t)Kx(t) \\
&\quad + s^T(t) \left(B^T PB\right)^{-1} B^T P(A_d + \Delta A_d(t))x(t-d) \\
&\quad - s(t)^T (I + \delta(t))B^T P \left(Ax(t) + A_d x(t-d)\right) \\
&\quad - s^T(t) \left(B^T PB\right)^{-1} B^T P D_1 v(t) - s(t)^T (I + \delta(t))\rho(x,t)\frac{s(t)}{||s(t)||}. \quad (41)
\end{aligned}$$

By (23), we have:

$$\begin{aligned}
\dot{V}(t) &\leq ||s(t)|| \left(||\Phi(A - BK)|| + ||\Phi M_1|| ||N_a|| + \rho_B ||K||\right) ||x(t)|| \\
&\quad + ||s(t)|| \left(||\Phi A_d|| + ||\Phi M_1|| ||N_{ad}||\right) ||x(t-d)|| \\
&\quad + ||s(t)|| \left(1 + \rho_B\right) \left(||B^T PA|| ||x(t)|| + ||B^T PA_d|| ||x(t-d)||\right) \\
&\quad + ||s(t)|| ||\Phi D_1|| ||v(t)|| - \frac{1}{2}\rho(x,t)(1 - \rho_B^2)||s(t)||. \quad (42)
\end{aligned}$$

Hence, it follows from (10) and (42) that:

$$\dot{V}(t) \leq -\mu||s(t)|| < 0 \text{ for } ||s(t)|| \neq 0. \quad (43)$$

This means that the trajectories of the uncertain stochastic system (1) and (3) will be globally driven onto (with probability 1) the specified switching surface $s(t) = 0$ for all $v(t) \in L_2[0,\infty) \cap L_\infty[0,\infty)$. □

Remark 2. It is shown from Theorems 1 and 2 that if there exist matrices $P > 0$, $Q > 0$, and scalars $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$ satisfying LMI (14) and equality constraint (38), the SMC law (8)–(10) can guarantee that the state trajectories of uncertain stochastic systems (1)–(3) are driven onto (with probability 1) the sliding surface $s(t) = 0$ in finite time, and then, asymptotically tend to zero (in mean-square sense) with disturbance attenuation γ.

Remark 3. According to Remark 2, the design of the desired SMC system is presented as the feasibility problem of linear matrix inequality (14) with equality constraint (38). In recent years, there are a number of numerical approaches proposed to solve the problem of LMIs with a nonconvex constraint, among which the LMI-based approaches are promising, such as, the alternating projections method [14], the min-max algorithm [15], XY-centering algorithm [16], and cone complementarity linearization (CCL) algorithm [17] (also referred to as product reduction (PR) algorithm in [18]. The advantage of LMI-based algorithms is that in each iteration only a set of LMIs is needed to be solved. which can be easily implemented with polynomial running time.

Consider the linear equality condition $B^TPG = 0$, where $P > 0$ satisfies LMI (14), which can be equivalently converted to:

$$\mathrm{tr}\left[\left(B^TPG\right)^T B^TPG\right] = 0.$$

Introduce the condition:

$$\left(B^TPG\right)^T B^TPG \leq \beta I, \tag{44}$$

by Schur's complement, the matrix inequality (44) is equivalent to:

$$\begin{pmatrix} -\beta I & G^TPB \\ B^TPG & -I \end{pmatrix} \leq 0. \tag{45}$$

Hence, the designs of both sliding mode controller and sliding surface are now changed to a problem of finding a global solution of the following minimization problem:

$$\min \beta \quad \text{subject to (14),and (45).} \tag{46}$$

The problem is a minimization problem involving linear objective and LMI constraints, which can be solved by means of LMI toolbox in Matlab. It admits a global infinum. If this infinum equals zero, the solutions will satisfy the LMI (14) and the equality $B^TPG = 0$. Thus, the sliding mode control problem is solvable.

4. Simulation Example

Consider the uncertain stochastic state-delay system (1)–(3) with

$$\begin{aligned}
A &= \begin{bmatrix} -1 & 0.8 & -2.4 \\ 3.8 & 0.1 & -0.8 \\ 0.1 & 2.5 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.3 & 0.1 & 0.2 \\ 0.2 & 0.3 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2.2 \\ 2.7 & 3.5 \\ 2 & -0.2 \end{bmatrix} \\
E &= \begin{bmatrix} 0.3 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}, \quad E_d = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \; \delta(t) = \begin{bmatrix} 0.5 & 0.6 \\ -0.4 & 0.2 \end{bmatrix}, \\
G &= \begin{bmatrix} -0.2 & -0.1 \\ 0.35 & 0.2 \\ -0.15 & -0.1 \end{bmatrix}, \; N_e = \begin{bmatrix} -0.1 & 0.2 & 0.2 \\ 0.4 & 0.1 & -0.2 \\ 0.3 & -0.4 & 0.1 \end{bmatrix}, \quad N_{ed} = \begin{bmatrix} -0.2 & 0.1 & 0.4 \\ 0.1 & 0.2 & -0.1 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}, \\
M_1 &= \begin{bmatrix} 0.2 & 0.2 & 0.1 \end{bmatrix}^T, \quad M_2 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}^T, \quad N_a = \begin{bmatrix} 0.2 & 0.1 & 0.1 \end{bmatrix}, \\
N_{ad} &= \begin{bmatrix} 0.1 & 0.3 & 0.1 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0.2 & 0.1 & 0.3 \end{bmatrix}^T, \; D_2 = \begin{bmatrix} 0.3 & 0.5 \end{bmatrix}^T, v(t) = 1/(1+t^2) \\
C &= \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}, \; F_1(t) = 0.2\sin(t), \; F_2(t) = \begin{bmatrix} 0.3\sin(t) & 0.3\cos(t) & 0.2\sin(t) \end{bmatrix}
\end{aligned}$$

It is easily obtained that $\rho_B = 0.7908$ with $||\delta(t)|| \leq \rho_B$. It is assumed that the time delay $d = 2$, and the initial state $x(t) = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix}^T$, $t \in [-2, 0]$. The objective is to design a sliding mode controller such that the state trajectories can be driven (with probability 1) onto the switching surface, and the sliding motion in the specified switching surface is robustly stochastically stable with disturbance attenuation γ .

For $\gamma = 0.2$, and the matrix $K = \begin{bmatrix} 1.5 & 4.3 & 4 \ ; & 3 & 3.5 & -1.2 \end{bmatrix}$, solving LMIs (14) and (38) yields:

$$\begin{aligned} P &= \begin{bmatrix} 0.0594 & -0.0208 & 0.0144 \\ -0.0208 & 0.0244 & -0.0020 \\ 0.0144 & -0.0020 & 0.0204 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0664 & -0.0574 & 0.0165 \\ -0.0574 & 0.0765 & -0.0083 \\ 0.0165 & -0.0083 & 0.0116 \end{bmatrix}, \\ \varepsilon_1 &= 0.0321, \quad \varepsilon_2 = 0.0250, \quad \varepsilon_3 = 0.0153, \quad \varepsilon_4 = 0.0033, \end{aligned}$$

and $\beta \approx 2.3264 \times 10^{-7}$ (hence the linear constraint $B^T PG = 0$ is satisfied). Hence, the sliding surface in (7) can be obtained as:

$$s(t) = \begin{bmatrix} 0.0323 & 0.0411 & 0.0498 \\ 0.0552 & 0.0403 & 0.0205 \end{bmatrix} x(t) = 0$$

and the desired SMC law is obtained as in (8)–(10) with

$$\rho(x,t) = 41.4210||x(t)|| + 1.2145||x(t-d)|| + 0.108||s(t)|| + 0.5.$$

In the work, the simulations are undergone by using the discretization approach as in [5, 19] with initial parameters: the simulation time $t \in [0, T]$ with $T = 5$, the variance of a normal distribution is $\delta t = T/N$ with $N = 2^{15}$, step size $\Delta t = R \cdot \delta t$ with $R = 2$, the number of *discretized Brownian paths* $M = 10$.

The simulation results are given in Figs. 1–6. Among them, Figs. 1–3 show the simulation results along an individual *discretized Brownian path*. Figs. 4–6 show, respectively, the simulation results on $x(t)$ along 10 individual paths (dotted lines) and the average over 10 paths (solid line). It is seen from simulation results that the present method effectively attenuates the effect of both parameter uncertainties and external disturbance.

5. Conclusions

In this work, a novel robust SMC method, i.e., integrating SMC with H_∞ technique, has been provided for uncertain stochastic systems with *unmatched* external disturbances, such that the sliding motion is robustly stochastic stability with a prescribed H_∞ performance level, irrespective of parameter uncertainties and *unmatched* external disturbance. However, it is also seen that the condition $SG = 0$ is stronger and, to some extent, will limits the application of the present method. Hence, we also hope that a more effective method can be found to overcome the above difficulty in future research.

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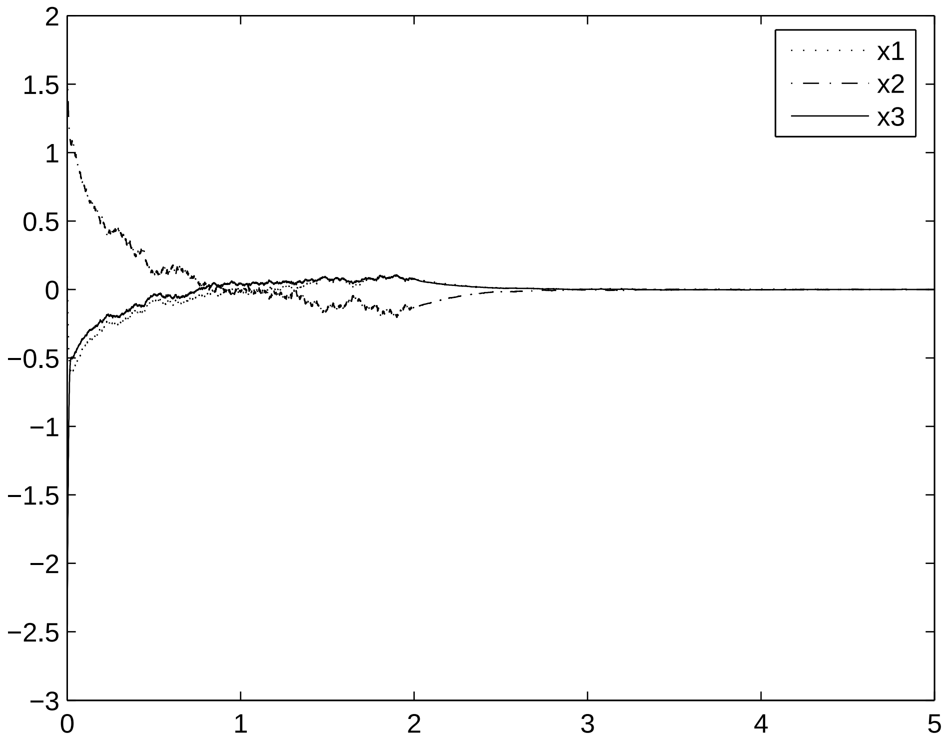

Figure 1. Trajectories of state $x(t)$

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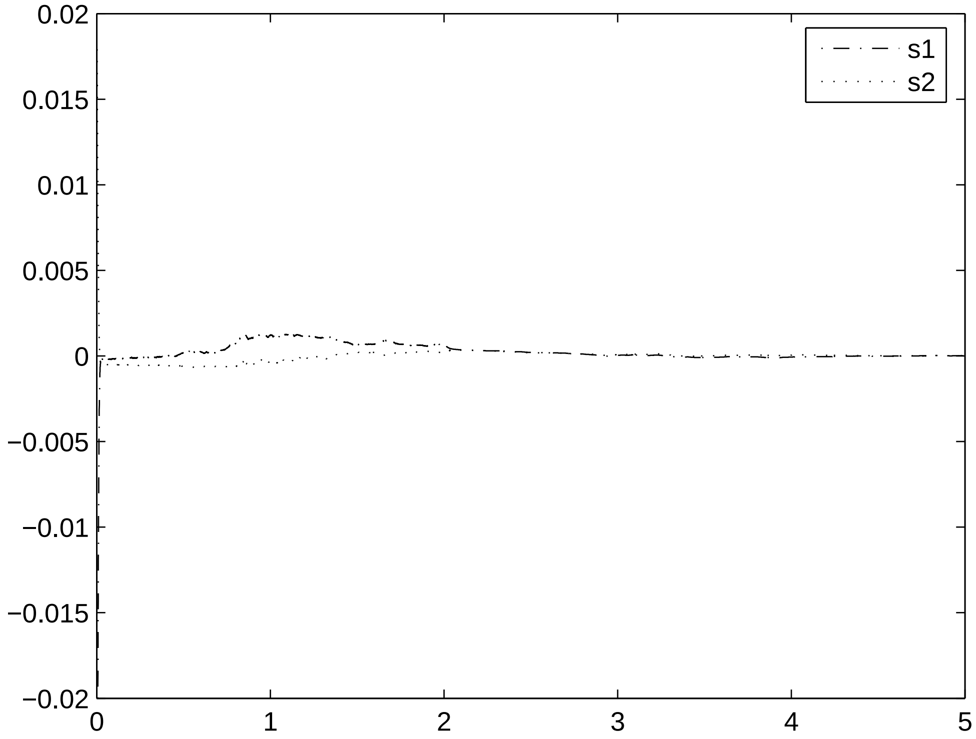

Figure 2. Sliding mode variable $s(t)$

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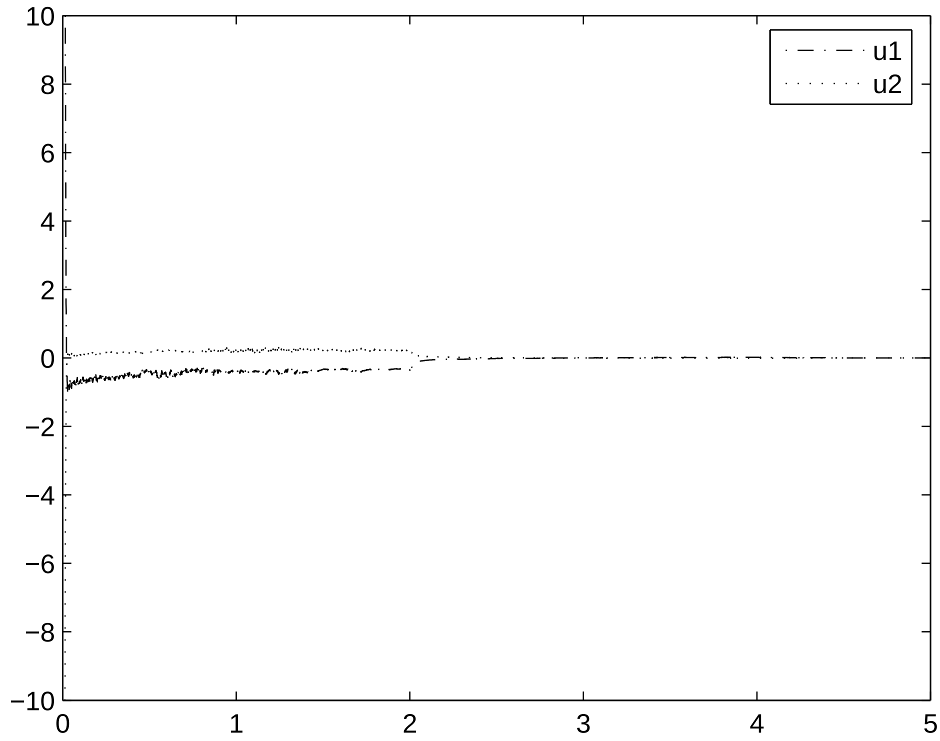

Figure 3. Control signals $u(t)$

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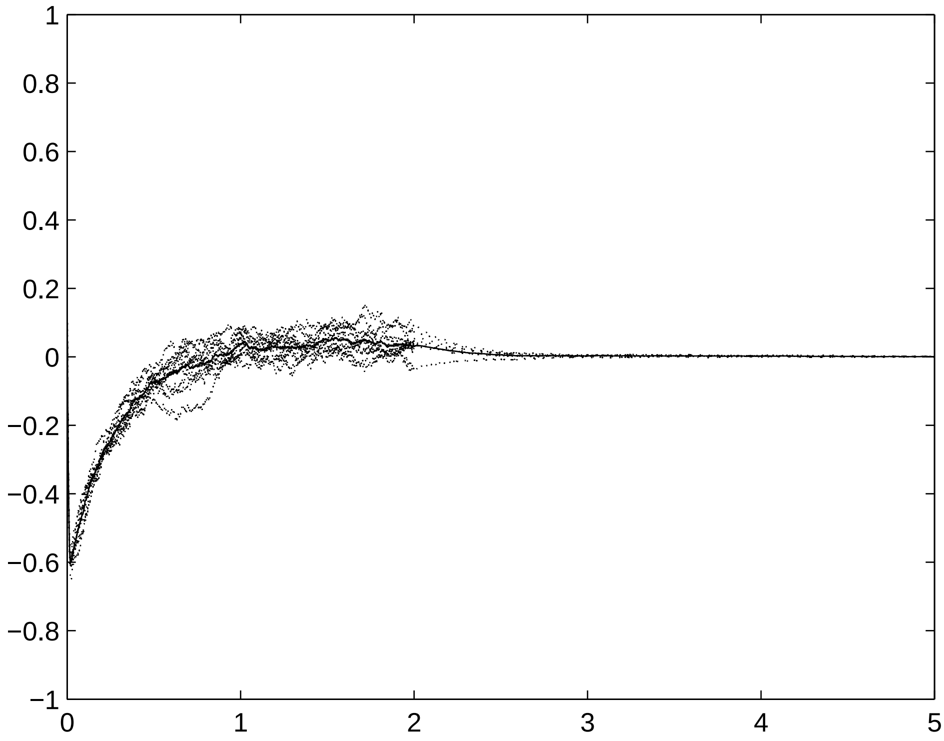

Figure 4. Individual paths and the average of $x_1(t)$

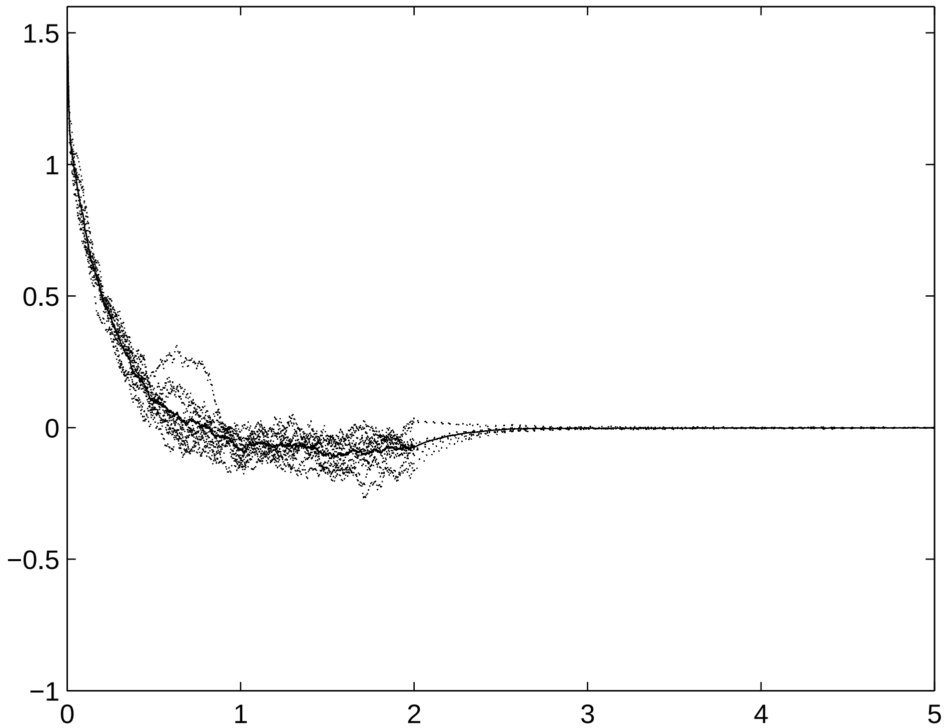

Figure 5. Individual paths and the average of $x_2(t)$

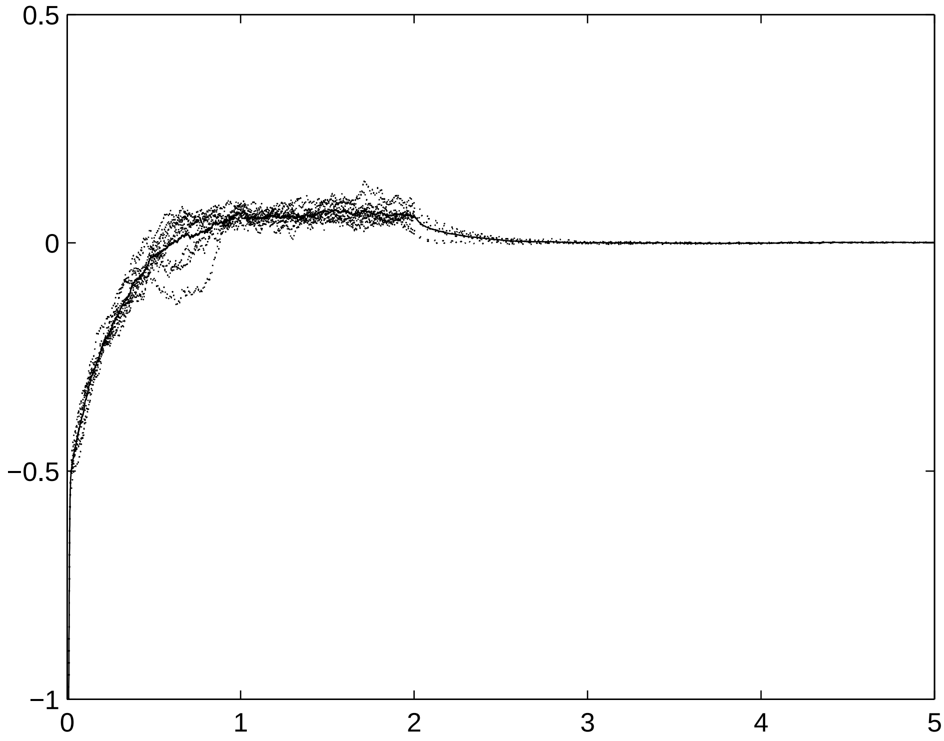

Figure 6. Individual paths and the average of $x_3(t)$