Moving load on elastic structures: passage through the wave speed barriers

A thesis submitted for the degree of
Doctor of Philosophy

by

Vitaly Voloshin

School of Information Systems,
Computing and Mathematics
Brunel University

September 2010

Declaration

I certify that this thesis submitted for the degree of Doctor of Philosophy in Applied Mathematics is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: ______________________

Date: ______________________

"Make things as simple as possible, but not simpler."

Albert Einstein

Abstract

The asymptotic behaviour of an elastically supported infinite string and an elastic isotropic half plane (in frames of specific asymptotic model) under a moving point load are studied. The main results of this work are uniform asymptotic formulae and the asymptotic profile for the string and the exact solution and uniform asymptotic formulae for a half plane. The crucial assumption for both structures is that the acceleration is sufficiently small.

In order to describe asymptotically the oscillations of an infinite string auxiliary canonical functions are introduced, asymptotically analyzed and tabulated. Using these functions uniform asymptotic formulae for the string under constant accelerating and decelerating point loads are obtained. Approximate formulae for the displacement in the vicinity of the point load and the singularity area behind the shock wave using the steady speed asymptotic expansion with additional contributions from stationary points where appropriate are derived. It is shown how to generalise uniform asymptotic results to the arbitrary acceleration case. As an example these results are applied for the case of sinusoidal load speed. It is shown that the canonical functions can successfully be used in the arbitrary acceleration case as well. The graphical comparative analysis of numerical solution and approximations is provided for different moving load speed intervals and values of the parameters.

Vibrations of an elastic half plane are studied within the framework of the asymptotic model suggested by J. Kaplunov et al. in 2006. Boundary conditions for the main problem are obtained as a solution for the problem of a string on the surface of a half plane subject to uniformly accelerated moving load. The exact solution over the interior of the half plane is derived with respect to boundary conditions. Steady speed and Rayleigh wave speed asymptotic expansions are obtained. In the neighborhood of the Rayleigh speed the uniform asymptotic formulae are derived. Some of their interesting properties are discovered and briefly studied. The graphical comparative analysis of the exact solution and approximations is provided for different moving load speed intervals and values of the parameters.

Acknowledgements

I am grateful to Brunel University for supporting my research with a PhD studentship and for the opportunity to taste and work in different country. I have enjoyed every single day of my three years of student life here.

I thank my supervisor Professor Julius Kaplunov for constant inspiration and wise guidance through the toughest moments of my research. I express my sincere gratitude to my second supervisor Dr Evgeniya Nolde for always being there for me, for the great advices and ideas and for moral support. I really appreciate all the help that Prof. Julius Kaplunov, his amazing wife Dr Olga Kaplunov and Dr Evgeniya Nolde provided to me and my family at the beginning of our stay in the UK.

I would like to thank Professor Anthony D. Rawlins, the co-author of our paper [1], for great ideas and nice communication. I also grateful to Dr Igor Krasovsky for stimulating discussions and interesting remarks concerning results published in [1].

I express my gratitude to Department of Mathematical Sciences staff, especially to Ms Deborah Banyard, for making my student life easier and more interesting.

I would like to express my respect and acknowledgement to the Scilab Consortium for creating Scilab software and making it available for free. I enjoyed using it for all my numerical calculations.

I really appreciate all the help and support received from my beloved parents and wife. Thanks to them I had an opportunity to spend as much time doing research as I needed to.

I am grateful to my examiners Dr Matthias Winter and Prof. Gerhard Müller for productive discussion and important suggestions concerning this work.

At the end I would like to say thanks in advance to all readers of my thesis.

Contents

List of Figures

List of Tables

Dedicated to my dear teachers from Saratov State University: $\boxed{\text{N.M. Maslov}}$, P.F. Nedorezov, Yu.P. Gulyaev, Yu.N. Chelnokov, Ya.G. Sapunkov, A.G. Markushin, D.V. Prokhorov, V.I. Shevtsov, I.V. Elistratov and N.I. Igolkina.

Chapter 1

Introductory notes

1.1 Introduction

1.1.1 Waves: history and development

Waves can be defined as *a phenomenon of physical quantity disturbances propagation in space and time.* The mathematical description of waves is based on viewing them as distributed in space oscillations and can be written as:

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t),$$

where $\mathbf{u}$ is a deviation from mean state at point $\mathbf{r}$ at a moment of time t. The exact equation depends on the wave nature.

Waves can be classified in accordance with different characteristics, for example by propagation media, types of wave front, directions of oscillations, etc. By propagation media waves can be divided into elastic, electromagnetic, gravitational and others. In this work we deal with the elastic waves only.

The history of the wave theory is more than 300 years old. The main reason of the interest in studying waves was connected to music in general (sound waves) and to string musical instruments in particular (string vibration). So, it began with experiments with string oscillations and sound wave propagation, carried out by such great scientists as, for example, Galilei, Descartes and Huygens. Especially, here it is worth mentioning the "father of acoustics", the French mathematician, philosopher and music theorist Marin Mersenne, who was the first to discover that the vibrating string frequency is proportional to the square root of the tension, and inversely proportional to the length, to the diameter and to the square root of the unit weight of the string [2]. He found this relation in 1625, almost a century before it was obtained using the mathematical point of view by Taylor in 1713.

The real opportunity for the development of the mechanical explanation of the wave phenomenon appeared after Newton's laws of motion (1687), analysis of infinitesimal, differential and integral calculi were discovered [3]. Taylor considered a string profile at any fixed moment of time as a function $f = f(x)$ and assumed that it should have sinusoidal "main" form $f = A\sin(k\pi x/l)$ and a string tends to this form for any initial condition. It turned out that only the first assumption was right. Taylor's approach is now known as the stationary wave method, it was developed later by Bernoulli but mathematically proven by Fourier. Joseph Fourier was the first, who applied trigonometrical and transcendent functions series expansion to integrate PDEs.

The next big step in string oscillation research belongs to d'Alembert. He considered string point displacements as a function of two variables: coordinate x and time t. It allowed to apply Newton's second law and, finally, obtain a partial differential equation for the string behaviour (1747) in the form it is known today:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

D'Alembert was the first mathematician who found a general solution for this equation.

Cauchy's problem for a string with a given initial profile and zero initial speed conditions was formulated by Euler. In 1766 he discovered a new method of solving this problem which is now called the method of characteristics. For more information about the string equation see, for example, [4].

Further development of the wave equations was made mostly in PDE theory by Euler, Lagrange, Monge, Fourier, Laplace and others.

At the turn of XVIII and XIX centuries the appearance of the industrial revolution caused new problems for specialists in mechanics, connected with the behaviour of structures and media under a moving load. Among the pioneers were the British engineers Robert Willis and Sir George Stokes. A. Krylov and S. Timoshenko should also be added to the list of first scientists who were interested in this area. It is worth mentioning that Stokes was involved in investigations into several railway bridge accidents, which happened because the bridge constructions were not properly proved to withstand the loads of moving trains, Willis was famous for his studies in biomechanics and for several widely popular inventions, Krylov obtained several results which better explained the behaviour of a moving ship, Timoshenko is now called the "father of solid mechanics" for his incredible achievements in numerous areas of mechanics, mathematics and engineering. He was an author of several fundamental books (see, for instance, [5] and [6]). Many different methods, theorems and objects in mechanics and applied mathematics are named after these great scholars.

In late XIX century another famous British mathematician Lord Rayleigh theoretically predicted the existence of surface acoustic waves, which appear on a free boundary of solid bodies and in 1885 he presented his famous paper [7]. In this work Rayleigh wrote: *"It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wavelength. ... It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."* This great paper was the start for the whole new surface waves school in elastic theory. After that the significance of surface waves in some areas was identified and therefore scientific interest began growing.

One of the biggest scientific search engines Google.Scholar returns more than a million links for request *"Rayleigh wave"* and almost 3 millions for *"surface wave"*. At the same time the patent search engine Google.Patent gives approximately 500 results for each of those requests. This data is really amazing! It shows a tremendous scale of scientific and industrial interests (see, for example, [8]) in this area. Later in this section we mention some works in selected areas.

The famous British mathematician Sir Horace Lamb described special waves in thin solid layers and stated a new problem, involving both recently discovered Rayleigh waves and bulk waves. This problem is now known as the Lamb problem and those special waves are called Lamb, or Rayleigh-Lamb waves. The paper, which contains the formulation and solution of this problem in terms of integral transforms, was published in 1904 [9]. The theory of Lamb waves was developed by Cagniard [10] and de Hoop [11], the inventors of Cagniard-de Hoop method. Numerical methods applied to the integral transforms solution of the Lamb problem can be found in plenty of papers, see, for example, [12],[13], [14], etc.

Lord Rayleigh is also an author of the secular equation for the surface wave speed in elastic isotropic media. One of the first (but incorrect) attempts to prove the existence of the real solution for the complex secular equation without further reduction for general anisotropic elastic media was made by Synge in 1956 [15]. Later, in 1958 Stroh in [16] introduced his famous formalism and in [17] and [18] it was shown that the complex secular equation can be reduced to a purely real expression. In 1974 and 1976 Barnett in collaboration with Lothe gave two different proofs of the existence of the real solution (see [19] and [20]). One more proof was given by Chadwick and Smith in [21]. Proof of the uniqueness was performed under the Stroh formalism in [22] or using another method in [23].

Although existence and uniqueness theorems for this equation were proved, it remained unsolved for more than 100 years because of its complicated and transcendent nature. Before the exact solution was derived some approximations were obtained, see, for instance, Achenbach's book [24]. The first attempt to find the exact solution was made by Rahman and Barber in [25]. Their result is valid only for a limited range of parameters. Later, in 1997, Nkemzi obtained a general formula [26], which was disproved in 2000 by Malischewsky in [27]. Another way to find the exact expression for the solution was used by Vihn and Ogden in [28], published in 2004. All these solutions turned out to be too complicated for engineering applications, but there is a really good high precision approximate formula that was suggested by Rahman and Michelitsch in 2006 [29].

Now we revert to the question about the waves caused by the moving loads on lengthy linear elastic systems. This subject can be divided into three groups: stationary problems for uniformly moving load, non-stationary problems for uniformly moving load and problems for load moving with varying speed.

Among the papers dedicated to the stationary problem for uniformly moving load first of all fundamental work made by Cole and Huth [30] in 1958 can be mentioned. They found the quasi-steady solutions for the elastic waves generated by concentrated loads moving over the surface of the half space with a uniform speed. Their solution was corrected by Georgiadis and Barber in their paper [31] published in 1993. A very interesting problem was considered by Singh and Kuo in 1970. They dealt with a half plane with an unusual load, namely, the circular surface load and considered the three dimensional case. According to their result (see [32]) the effect of the circular shaped load appears only in its sufficiently small (approximately 5 radii of the load) vicinity. Outside this neighborhood the circular load can be successfully approximated by an equivalent point load. Müller in 1990 considered a stripe moving load on a half space (see [33]). He is also an author of paper [34], published in 1991, where an expanding circular load on a layered and non-layered half space was considered. Another interesting case of the moving load problem was considered by Belotserkovskiy, who worked with a concentrated harmonic force moving on an infinite string, supported by equidistantly spaced identical visco-elastic suspensions (see [35]). Kennedy and Herrmann derived the result for a load moving on the fluid-surface interface and compared it to the "usual" vacuum-surface interface [36], which can be applied to, for example, modeling geophysical activity on the floor of the ocean. An infinite moving system of equivalent forces and the possibility of loss of the contact zone between beam and its support were considered in [37] by Muravski.

The transient solutions of the moving load problem for a constant speed are of a big interest as well. In the paper [38] by Frýba the non-stationary behaviour of a beam subject to moving random force is described. The moment of moving load application was investigated in details in [39] by Craggs, who considered transient effects caused by different types of loads. Gakenheimer and Miklowitz, the authors of [40], were the first to derive a dynamical solution for the interior of a half space subject to a surface moving load. Moreover, they introduced a new solution technique that allows to find transient solutions not only on the surface, but also over the interior of elastic solids. In paper [41] Kanninen and Florence dealt with a string under two loads initially applied at the same point, but moving in the opposite directions. They mention that their model can possibly be used for the description of the behaviour of lengthy structures under loads caused by the explosion shock waves. An important example of the numerical approach to a dynamical problem is the finite element/finite difference method, which was described, for example, in [42], where Cifuentes considered both the uniform speed and the constant acceleration cases for a moving load on beam. Duplyakin in paper [43] considered a deformable carriage of rigid bodies with viscoelastic connections between each other and with the viscoelastic interface with a beam on which a carriage is moving with constant speed. This model actually describes moving railway vehicles well. Because of the rapid growth of modern railway transport speed and the possibility of overcoming the critical speeds in the near future, this and similar works are of great interest for engineering mechanics.

Considering results for the third type of the wave propagation problems, which involve moving loads with non-uniform velocity, Flaherty, who was the pioneer in this sphere, should be mentioned. He was the first, who derived the result for string behaviour under accelerating and decelerating moving forces in his work [44], published in 1968. Another paper dedicated to string vibrations is [45] in which Stronge considered the passage through the critical speed with fast acceleration in order to stay inside the scope of small deformation theory. A year earlier in his paper [46] the case of a load represented as a step function with the front moving along an acoustic half space was discussed. The uniform deceleration case for a load moving along a half space was also given by Beitin in [47]. Myers in [48] considered a two dimensional surface expanding load on a liquid half space, when the fronts of the load are decelerating from the initial supersonic speed. Singularities of the displacements of a half space caused by a load passing through and moving exactly with the critical speed were investigated in [49] by Freund. A reciprocating anti-plane shear load for the homogeneous and layered half space were described in Watanabe's papers [50] and [51] respectively. In these papers the author adapted Cagniard's technique for non-uniformly moving loads. Later he generalized this technique for an arbitrary moving load (see [52] and [53]). The passage through the critical speed by a point load moving on a beam with a damping support was considered by Muravski and Krasikova in their work [54]. Among the recent papers there are Gavrilov's results [55] and [56], where he dealt with a problem of a string under a moving load passing through the critical speed in both directions. His paper [57] is of particular interest because it contains an approvement and necessary conditions of the possibility of the passage through the critical speed under the non-linear theory of elasticity.

A good insight into the dynamical problem for a moving load can be found in book [58] written by Ladislav Frýba in 1972 (another edition of this book was published in 1999). This monograph covers the vibrations of virtually all elements involved in the study of engineering mechanics and the theory of elasticity and plasticity (e.g. beams, strings, elastic space, etc.). In this book the author deals with all basic cases of the moving load. All chapters of this fundamental work provide not only theoretical formulation and mathematical solution for each problem, but also possible applications in the various engineering fields.

At the end of this section it is worth telling briefly about the papers which inspired this research. Among the authors of these results are my supervisors Professor Julius Kaplunov and Dr Evgeniya Nolde.

J. Kaplunov and G.B. Muravski in [59], published in 1986, investigated the non-uniform asymptotic behaviour of the integrals of the Bessel functions with a large parameter which arises from a problem of a uniformly accelerated moving load on an elastically supported string. The paper [1], written in collaboration with Prof. J. Kaplunov and Prof. A.D. Rawlins, based on an approach similar to [59]. We introduced special functions and using them derived the uniform asymptotic formulae for the vicinity of the sonic speed. This result is expanded and discussed in this thesis.

In [60] J. Kaplunov gave the "classical" approach to Rayleigh wave motion for the problem of a moving load on a half space. Later, in collaboration with A. Zakharov and D.A. Prikazchikov, he created an asymptotic model which allows to derive explicitly the Rayleigh waves on a surface of an elastic half space (see [61]). This model was applied to the case of constant velocity, described in [62] by J. Kaplunov, E. Nolde and D.A. Prikazchikov. In the thesis we adopt this model to the case of a uniformly accelerated load.

1.1.2 Main objectives of the thesis

There are two different problems considered in this thesis: the asymptotic behaviour of a string and a half plane subject to a moving load.

The main aim of the work is to create and analyze the uniform asymptotic solutions for a small magnitude of the load acceleration, which cover all the values of the load speed, in particular, the vicinity of the wave speed. To construct approximations which describe the behaviour of elastic solids not only for points under the load but also for other points of a string and the interior of the half plane, especially for the vicinities of the load and shock wave, and to compare them numerically (and graphically) with the exact solutions and with each other are also among the common objectives for both structures.

For the problem of string behaviour the aims also are to consider a load moving with an arbitrary acceleration and to improve the asymptotic analysis technique for integrals with Bessel functions (which usually arise in similar string vibration problems).

Apart from all mentioned above the important objective of the problem about the half plane vibrations is to obtain the exact solution over the interior of a half plane in frames of the existing asymptotic model analytically in a case of uniformly accelerated load.

1.1.3 Structure of the thesis

The thesis consists of three chapters, concluding remarks and bibliography. Chapter 1 is an auxiliary part of the work that provides introductory and necessary technical data, in Chapters 2 and 3 we describe two different problems which arose in the research and their solutions.

Chapter 1 contains a brief insight into the history of wave related problems research and publications (Section 1.1) and some theoretical information that is required for clear understanding throughout the thesis (Section 1.2).

In Chapter 2 we study the asymptotic behaviour of an elastically supported infinite string under a moving point load. In Section 2.1 we state the classic non-homogeneous string equation and give the general integral solution for homogeneous initial data. This equation and its solution were described in detail in [59]. In Section 2.2 three auxiliary canonical integral functions are introduced. The asymptotic expansions for the limit values of these functions' argument are derived and numerically analyzed. The next two sections are dedicated to the calculations and numerical analysis of the uniform asymptotic behaviour of a string for constant acceleration and deceleration cases respectively. These asymptotic formulae are based on the introduced in Section 2.2 canonical functions, which also appear in Section 2.5, where we obtain the uniform asymptotic expansion for the case when the path (and, consequently, speed and acceleration) function is arbitrary. As an example we show there how to deal with the sinusoidal speed changing. In Section 2.6 we find the steady speed asymptotic expansions for the vicinity of the load. By adding contributions from the stationary points (one before the shock wave and two behind) of a phase function to the steady speed asymptotic formulae, we managed to a find good approximation for the singularity area near the shock wave. Brief conclusions are given in Section 2.7. The results given in Chapter 2 were published in The Quarterly Journal of Mechanics and Applied Mathematics (see [1]) and presented at British Applied Mathematics Colloquium 2008 in Manchester, UK and International Conference On Vibration Problems 2009 in Kolkata, India.

In Chapter 3 we investigate Rayleigh waves which appear in an elastic isotropic half plane subject to uniformly accelerated moving point load using the asymptotic model described in [61]. This model which contains hyperbolic equations on the surface along with elliptic equations over the interior was extracted using the perturbation method from the general linear elasticity theory. In frames of this model we state the problem for a point load moving with a constant acceleration in Section 3.1. Calculations of boundary conditions are given in Section 3.2. The problem stated in Section 3.1 can be solved analytically over the interior of a half plane, the process of the solution is provided in Section 3.3. In the next section there are two parts. In the first one, Section 3.4.1, we obtain the asymptotic expansions in the cases of zero acceleration and Rayleigh speed and compare them with the exact solution from Section 3.3 using graphical representation. In Section 3.4.2 we find the uniform asymptotic solution for the vicinity of the Rayleigh wave speed and again provide a graphical comparison with the accurate solution. At the end of this section we briefly study some remarkable properties of the obtained uniform approximate formulae. The conclusions given in Section 3.5 finish this chapter.

1.1.4 Ideas for the future

Both problems and the ways of their solution, presented in this thesis, give a wide range of the ideas for the development of these scientific areas. Varying the problem statements in different directions one can extend the results of this work.

In Chapter 2 the behaviour of an elastically supported infinite string under uniformly accelerated moving point load is considered. Our conjecture is that solutions for the different types of load, its speed and acceleration, support of the string, its geometrical properties, etc. can be found using the special functions $\mathcal{F}_i$, $i = 1, 2, 3$ introduced in Section 2.2 (probably, with some modifications) and similar approaches. For example, instead of a point load one can consider a finite distributed or step function load, a system of connected point (or not) forces or loads moving in different directions simultaneously. An elastically supported string can be changed by a string on a damping or elasto-plastic support, or, for instance, with (non)equidistant fixation points, etc. The problem can be considered for semi-infinite or finite string.

In Chapter 3 we deal with the vibration of an elastic homogeneous half plane under uniformly accelerated moving point load. Using a similar approach one can derive a solution for the problems with different types of load, its speed and acceleration, physical properties of a half plane, etc. Finite surface or expanding surface load or a system of point or surface loads can be considered instead of a point load. It is worth trying to find solutions for an anisotropic (or, in particular, orthotropic), pre-stressed (or not), layered (or not) half space or one with the cracks or local inhomogeneities on (or near) the surface. Another idea is to consider the liquid above a solid half plane instead of vacuum. Uniform acceleration can be changed to the arbitrary accelerated case. However, in comparison with the arbitrary accelerating case described in Chapter 2 this is not easy at all, since it is impossible to calculate the boundary conditions for an arbitrary path (and, consequently, speed and acceleration) function.

I believe that some of the ideas mentioned above can find response not only from applied mathematicians but also from mechanical engineers.

1.2 Theoretical notes

1.2.1 Gamma function

The Gamma function is an extension of the factorial function to the complex numbers. It was originally introduced by Leonhard Euler. The Gamma function is usually denoted as $\Gamma(z)$, this notation belongs to Legendre.

If the real part of a complex number z is positive then one can define the Gamma function via the integral:

$$\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-t} dt.$$

To extend the function to the whole complex plane one can use the identity:

$$\Gamma(z+1) = z\Gamma(z).$$

An alternative definition is:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)...(z+n)} = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}},$$

it is valid for all complex z except 0 and the negative integers.

Main properties:

1. $\overline{\Gamma(z)} = \Gamma(\bar{z})$;
2. $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$;
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$;
4. Gamma function has a pole at $z = -n$ for $n \in \mathbb{N} \bigcup \{0\}$ and the residue is: $\operatorname{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}$.

1.2.2 Bessel function

The Bessel functions are the canonical solutions of the Bessel's differential equation:

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \alpha^2) y = 0, \tag{1.1}$$

where α is an arbitrary real or complex number, α is called the order of the Bessel function. The most common special case is where α is an integer.

The Bessel functions are named after the famous German mathematician Friedrich Wilhelm Bessel.

Since (1.1) is a second-order differential equation, it has two linear independent solutions. However, different formulations of the solutions are convenient under the different circumstances. Some of them, namely, Bessel functions of the first and second kind and Hankel functions, are described below.

Bessel functions of the first kind, usually denoted as $J_\alpha(x)$, are solutions of the Bessel's differential equation (1.1) that are finite at $x = 0$ for non-negative integer α and infinite for $x \to 0$ for negative or non-integer α.

Bessel functions of the second kind, usually denoted as $Y_\alpha(x)$, are solutions of (1.1) that have a singularity at $x = 0$. $J_\alpha(x)$ and $Y_\alpha(x)$ are related for non-integer α via the following formula:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha(x)}}{\sin(\alpha\pi)},$$

in case of integer α, i.e. $\alpha = n \in \mathbb{Z}$, one should take the limit as $\alpha \to n$.

The Bessel functions of the third kind, also known as the Hankel functions (named after German mathematician Hermann Hankel), usually denoted as $H_\alpha^{(1)}(x)$ and $H_\alpha^{(2)}(x)$, are linearly independent solutions of (1.1) defined by:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x),$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x),$$

where i is the imaginary unit.

There are also Bessel functions of a complex argument. Important special case is that of a purely imaginary argument. In this case, the solutions to the (1.1) are called the modified Bessel functions. MacDonald function $K_\alpha(x)$ is an example of the modified Bessel functions, it can be defined as:

$$K_\alpha(x) = \frac{\pi}{2} i^{\alpha+1} H_\alpha^{(1)}(ix).$$

The Bessel functions have the following asymptotic forms for non-negative α:

$$J_\alpha(x) \approx \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha, \quad \text{for} \quad 0 < x \ll \sqrt{\alpha+1}, \tag{1.2}$$

$$J_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right), \quad \text{for} \quad x \gg \left|\alpha^2 - \frac{1}{4}\right|, \tag{1.3}$$

$$H_\alpha^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)}, \quad \text{for} \quad x \gg \left|\alpha^2 - \frac{1}{4}\right|. \tag{1.4}$$

More information on special functions mentioned in Sections 1.2.1 – 1.2.2 and their asymptotic expansions can be found in [63, 64, 65, 66].

1.2.3 Saddle-point method

The saddle-point approximation or steepest descent method (sometimes it is also called generalized Laplace method), is a method used to approximate integrals of the form:

$$\int\limits_\gamma \Phi(z) e^{\lambda\phi(z)} \mathrm{d}z, \tag{1.5}$$

where $\Phi(z)$ and $\phi(z)$ are some meromorphic functions, λ is an arbitrary sufficiently large number, contour $\gamma \in \mathbb{C}$ can be infinite.

Algorithm:

1. Transform the given integral to the form:

$$I(\lambda) = \int\limits_\gamma \Phi(z) e^{\lambda\phi(z)} \mathrm{d}z.$$

2. Since $\lambda \to \infty$ then the behavior of $I(\lambda)$ is defined by the exponential part. So, the following analysis of $\phi(z)$ required:
   - Find the saddle points, i.e. such points that $\phi'(z) = 0$;
   - Plot the steepest descent lines.
3. Transform the contour γ using the steepest descent lines.
4. Using Laplace's method find an asymptotic form.

The matter of the Laplace's method: assume that the function $f(x)$ has a unique global maximum at x_0. Then, the value $f(x_0)$ will be larger than other values $f(x)$. If one multiplies this function by a large number M, the gap between $Mf(x_0)$ and $Mf(x)$ will only increase, and then it will grow exponentially for the function $e^{Mf(x)}$. So, significant contributions to the integral of this function will come only from points x in a neighborhood of x_0.

$$\int_a^b e^{Mf(x)}\mathrm{d}x \approx \sqrt{\frac{2\pi}{M|f''(x_0)|}}e^{Mf(x_0)}, \quad \text{as} \quad M \to \infty,$$

where x_0 is not an endpoint of the interval of integration, second derivative $f''(x_0) < 0$.

Theory, background and other information about saddle point method can be found in [66, 67].

1.2.4 Stationary phase method for one-dimensional integrals

The method of stationary phase was developed by Lord Kelvin in the 1887 to solve integrals encountered in the study of hydrodynamics.

Using the stationary phase method one can evaluate integrals of the form:

$$I = \int_{-\infty}^{\infty} F(x)e^{i\nu\phi(x)}\mathrm{d}x,$$

where $\phi(x)$ is a rapidly varying function of x over most of the range of integration, $F(x)$ is by comparison slowly varying, ν is a large positive parameter. The major contribution to the value of the integral I arises from the neighborhood of the end points of the domain of integration and from the neighborhood of stationary points, i.e. where $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0$. Stationary phase points can be denoted as x_s and defined by $\phi'(x_s) = 0$. In the neighborhood of stationary points $F(x) \approx F(x_s)$ since $F(x)$ is assumed to be slowly varying function. Hence, this term can be moved outside the integral. First two non-zero terms of a Taylor expansion of

$\phi(x)$ near the point x_s are:

$$\phi(x) \approx \phi(x_s) + \frac{1}{2}\phi''(x_s)(x - x_s)^2.$$

Substituting this into the initial integral gives

$$I \approx F(x_s)e^{i\nu\phi(x_s)} \int\limits_{-\infty}^{\infty} e^{i\nu\phi''(x_s)(x-x_s)^2/2}\mathrm{d}x.$$

Further integration and contributions from the end points lead to the formula:

$$I \approx \sqrt{\frac{2\pi}{\nu\phi''(x_s)}}F(x_s)e^{i(\nu\phi(x_s)+\pi/4)}.$$

For more detailed information see, for example, [66, 67, 68].

1.2.5 Numerical integration

The main idea of the numerical integration is to compute an approximate solution to a definite integral:

$$I = \int\limits_a^b f(x)\mathrm{d}x.$$

There are many methods of approximating the integral with arbitrary precision, especially for smooth well-behaved functions $f(x)$, integrated over a small number of dimensions and if the limits of integration are bounded. A good example of those methods is Trapezium Method (for other methods see, for example, [69, 70]).

Trapezium Method:

To calculate the value of the integral over the given segment $[a, b]$ one should consider a partition $\{x_0 = a,\ x_1 = a + \frac{b-a}{n},\ \ldots,\ x_{n-1} = a + (n-1)\frac{(b-a)}{n},\ x_n = b\}$ of the $[a, b]$. Hence,

$$I = \int\limits_a^b f(x)\mathrm{d}x = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} f(x)\mathrm{d}x,$$

$$I_i = \int\limits_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1}).$$

The last formula means that the value of I_i can be approximated by the area of the corresponding trapezium with the precision

$$|R_i| \leqslant \frac{(b-a)^3}{12n^2} M_i, \text{ where } M_i = \max_{x \in [x_{i-1}, x_i]} |f''(x)|.$$

So, the approximation for the initial integral is:

$$I \approx h \left(\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i) \right), \text{ where } h = \frac{b-a}{n},$$

with the precision

$$|R| \leqslant \frac{(b-a)^3}{12n^2} M, \text{ where } M = \max_{x \in [a,b]} |f''(x)|.$$

Chapter 2

Behaviour of elastically supported infinite string under accelerated moving point load

2.1 Statement of the problem

Consider an infinite string lying on an elastic support and subject to a point force uniformly accelerating from the rest (see Figure 2.1). Transverse vibrations of a string are described by the equation

$$-T\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} + ky = P\delta\left[x - s(t)\right], \quad s(t) = \frac{\alpha t^2}{2}, \tag{2.1}$$

where T - a string tension, m - a linear mass, k - a support stiffness coefficient, α - an acceleration of a point where a load P is applied. We introduce the non-dimensional variables:

$$\xi = x\sqrt{\frac{k}{T}}, \quad \tau = t\sqrt{\frac{k}{m}}, \quad a = \alpha\frac{m}{\sqrt{kT}},$$
$$s_1(\tau) = \frac{\alpha\tau^2}{2}, \quad y = \frac{P}{\sqrt{kT}}w,$$
$$\delta[\xi - s_1(\tau)] = \sqrt{\frac{T}{k}}\delta[x - s(t)].$$

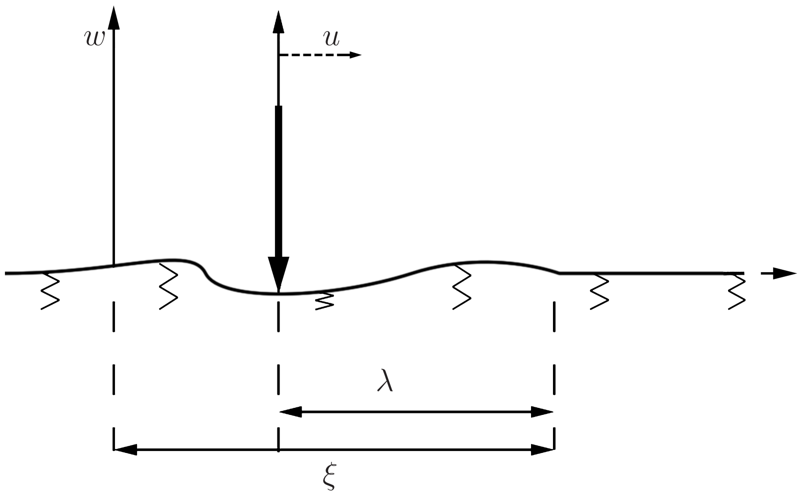


FIGURE 2.1: Elastically supported string under moving load

This system of parameters were selected as the most suitable. It was previously used in [59]. We have got 7 dimensional parameters and 4 non-dimensional parameters and three global dimensions (length, time, mass) and by the Buckingham π theorem, it is a complete system of parameters.

Using these parameters we can rewrite (2.1) as a non-dimensional equation of motion (e.g. see [59, 55] for more details)

$$-\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + w = \delta\left(\xi - s_1(\tau)\right), \quad s_1(\tau) = \frac{1}{2}a\tau^2, \tag{2.2}$$

where τ is a time, ξ is a coordinate, a is an acceleration, w is transverse displacement and δ denotes Dirac delta function; in doing so, dimensionless sound wave speed, stiffness of the elastic support and magnitude of the moving load all take the unit values.

The solution of the equation (2.2) with homogeneous initial data can be expressed as

$$w(\nu,\ \lambda,\ u) = \nu I, \tag{2.3}$$

where

$$I = \int_0^u J_0(\nu\phi(u,\lambda,t))H(t^2 - (\lambda + ut - t^2/2)^2)\mathrm{d}t$$

with

$$\phi(u, \lambda, t) = \sqrt{t^2 - (\lambda + ut - t^2/2)^2},$$

where H denotes the Heaviside step function. The last integral depends on three problem parameters including the load speed $u = a\tau$, the moving coordinate $\lambda = \xi - \frac{1}{2}a\tau^2$, and the parameter $\nu = 1/a$. The large values of the parameter ν ($\nu \gg 1$) are associated with the dynamic phenomena observed when the load speed passes through the critical value $u = 1$, i.e. the sound wave speed in a string, with a small acceleration a. Non-uniform asymptotic forms of the function (2.3) were derived in [59].

2.2 Canonical integrals introduction and their asymptotic behaviour

Consider the integral

$$\mathcal{F}(\gamma) = \int_a^b J_0(\gamma f(p))\mathrm{d}p, \ b > a, \tag{2.4}$$

where γ is a large real parameter and J_0 denotes the zero-order Bessel function of the first kind. Away from the zeros of the argument $f(p)$, the Bessel function of the integrand (2.4) behaves as (see [71])

$$J_0(\gamma f(p)) \sim \sqrt{\frac{2}{\pi\gamma f(p)}} \cos\left(\gamma f(p) - \frac{\pi}{4}\right), \ \gamma \gg 1. \tag{2.5}$$

As a result, the integral (2.4) can be evaluated using the standard method of stationary phase.

Let now $f(a) = 0$, $f'(a) > 0$ and $f(p) = f'(a)(p - a) + \ldots$ ($|p - a| \ll 1$). Assume for the sake of simplicity that the function $f(p)$ has no stationary points and zeros over the domain of integration in (2.4) (see [68, 72] for further details). So, substitution $s = \gamma f'(a)(p - a)$ gives:

$$\mathcal{F}(\gamma) \sim \frac{1}{\gamma f'(a)} \int_0^{\gamma f'(a)(b-a)} J_0(s)\mathrm{d}s.$$

Finally, for $\gamma \gg 1$ [73] it appears that

$$\mathcal{F}(\gamma) \sim \frac{1}{\gamma f'(a)} \int_0^\infty J_0(s)\mathrm{d}s = \frac{1}{\gamma f'(a)}.$$

Thus, the contribution of the zeros of the J_0 argument is of the same asymptotic order $O(\gamma^{-1})$ as that of ordinary stationary phase points (the additional factor $\gamma^{-1/2}$ comes from the asymptotic formula (2.5)).

In this thesis integrals of the following type

$$\mathcal{F}(\gamma, \beta) = \int_a^b J_0(\gamma f(p, \beta))\mathrm{d}p, \tag{2.6}$$

are investigated with an extra real parameter β. The main focus is on the uniform asymptotic analysis in terms of the parameters γ and β, dealing in particular with the dominant contributions of the J_0 zeros, which cannot be reduced to the well known uniform generalizations of the stationary phase method including, for example, the Airy function (e.g. see [68] and reference therein). To this end, canonical integrals are introduced. They play the same role as the above mentioned Airy function (and some others) do in the well established case of the oscillating sinusoidal functions. If, for example, in (2.6)

$$f(p, \beta) = p\sqrt{p + \beta}, \ \ \beta \geq 0, \tag{2.7}$$

and the limits of integration are $a = 0$ and $b = \infty$, then the substitution $p = \gamma^{-2/3}s$ implies

$$\mathcal{F}(\gamma, \beta) = \gamma^{-2/3}\mathcal{F}_1(\vartheta),$$

where

$$\mathcal{F}_1(\vartheta) = \int_0^\infty J_0\left(s\sqrt{\vartheta + s}\right)\mathrm{d}s. \tag{2.8}$$

Here and below $\vartheta = \beta\gamma^{2/3}$ is a real non-negative parameter.

For the same limits of integration in (2.6) and with

$$f(p, \beta) = (p + \beta)\sqrt{p}, \ \ \beta \geq 0, \tag{2.9}$$

it appears that

$$\mathcal{F}(\gamma,\beta) = \gamma^{-2/3}\mathcal{F}_2(\vartheta),$$

where

$$\mathcal{F}_2(\vartheta) = \int\limits_0^\infty J_0(\sqrt{s}(\vartheta + s))\mathrm{d}s. \tag{2.10}$$

The last canonical integral arises from letting

$$f(p,\beta) = p\sqrt{\beta - p}, \ \beta \geq 0 \tag{2.11}$$

with the limits $a = 0$ and $b = \beta$. In this case

$$\mathcal{F}(\gamma,\beta) = \gamma^{-2/3}\mathcal{F}_3(\vartheta),$$

where

$$\mathcal{F}_3(\vartheta) = \int\limits_0^\vartheta J_0\left(s\sqrt{\vartheta - s}\right)\mathrm{d}s. \tag{2.12}$$

In more general situations when the formulae (2.7), (2.9) and (2.11) correspond to the local approximations of the Bessel function argument near its zeros and for the arbitrary limits of integration one may expect that the canonical integrals (2.8), (2.10) and (2.12) will appear as the leading order terms in related asymptotic expansions. All of these integrals naturally arise in the moving load problem for a string (this will be considered in Section 2.3).

The behaviour of an argument of the Bessel function in all the canonical integrals (2.8), (2.10) and (2.12) is strongly affected by the parameter ϑ. In particular, in (2.8) and (2.10) it has, respectively, the limiting forms $\vartheta^{1/2}s$ and $\vartheta s^{1/2}$ for $\vartheta \gg 1$ and tends to $s^{3/2}$ for $\vartheta \ll 1$ in both integrals. In (2.12) the argument of J_0 is uniformly small for $\vartheta \ll 1$, whereas it takes large values outside the vicinities of the end points, in this integral for $\vartheta \gg 1$.

The asymptotic behaviour of the functions $\mathcal{F}_i$ $(i = 1, 2, 3)$ in the domain of small and large values of the parameter ϑ plays a very important role in the present thesis and definitely should be considered in detail. It is clear that (see e.g. [73])

$$\lim_{\vartheta\to 0}\mathcal{F}_j(\vartheta) = \int\limits_0^\infty J_0(s^{3/2})\mathrm{d}s = \frac{2\sqrt{\pi}}{3\Gamma\left(5/6\right)}, \quad j = 1, 2. \tag{2.13}$$

It is also evident that

$$\mathcal{F}_3(\vartheta) \sim \vartheta \quad \text{as} \quad \vartheta \ll 1, \tag{2.14}$$

since $J_0(s\sqrt{\vartheta - s}) \sim 1$.

The asymptotic analysis for $\vartheta \gg 1$ requires more delicate calculations. Namely, the functions $\mathcal{F}_i$ $(i = 1, 2, 3)$ should be expressed in terms of the integrals of the Hankel function $H_0^{(1)}$. Changing variables in (2.8), (2.10) and (2.12) by the formulae $s = -\vartheta(z^2 + 1)$, $s = \vartheta z^2$ and $s = \vartheta(z^2 + 1)$, respectively gives

$$\mathcal{F}_1(\nu) = -2\nu^{2/3}\mathrm{Re}\int_{i}^{i\infty} H_0^{(1)}(i\nu h) z \mathrm{d}z, \tag{2.15}$$

$$\mathcal{F}_2(\nu) = 2\nu^{2/3}\mathrm{Re}\int_{0}^{\infty} H_0^{(1)}(\nu h) z \mathrm{d}z, \tag{2.16}$$

and

$$\mathcal{F}_3(\nu) = 2\nu^{2/3}\mathrm{Re}\int_{-i}^{0} H_0^{(1)}(i\nu h) z \mathrm{d}z, \tag{2.17}$$

where $\nu = \vartheta^{3/2} \gg 1$ and $h(z) = z(z^2 + 1)$.

The asymptotic behaviour of the Hankel function in (2.15) - (2.17) for $\nu|h| \gg 1$ is given by (see [71])

$$H_0^{(1)}(i\nu h) \sim -i\sqrt{\frac{2}{\pi\nu h}} e^{-\nu h} \tag{2.18}$$

and

$$H_0^{(1)}(\nu h) \sim e^{-\frac{\pi i}{4}} \sqrt{\frac{2}{\pi\nu h}} e^{i\nu h}. \tag{2.19}$$

The exponentials in the right-hand sides of the formulae (2.18) and (2.19) motivate making use of the steepest descent method (e.g. see [72, 74]) when evaluating the original integrals (2.15)–(2.17). The introduction of a complex variable $z = x + iy$ gives the following representation for $h(z)$

$$h(x + iy) = h_r(x, y) + ih_i(x, y),$$

where

$$\begin{aligned} h_r(x, y) &= \mathrm{Re}h(x + iy) = x(x^2 - 3y^2 + 1), \\ h_i(x, y) &= \mathrm{Im}h(x + iy) = y(3x^2 - y^2 + 1). \end{aligned} \tag{2.20}$$

In case of the function $\mathcal{F}_1$, the integral (2.15) can be presented as (see Figure 2.2)

$$\int\limits_{C_1} = \int\limits_{C_{11}} + \int\limits_{C_{12}}, \tag{2.21}$$

where C_1 is the original path of integration in (2.15), C_{11} is the steepest descent path through the point $z = i$ corresponding to the exponential in (2.18) and C_{12} is the path along the circle of an infinitely large radius. Here and below the integrands in the all symbolic formulae are omitted.

Along C_{11}, which is the steepest descent path, $\mathrm{Im}h(z) = \mathrm{Im}h(i) = 0$ (see (2.18)). Therefore, from (2.20)

$$y = \sqrt{3x^2 + 1}.$$

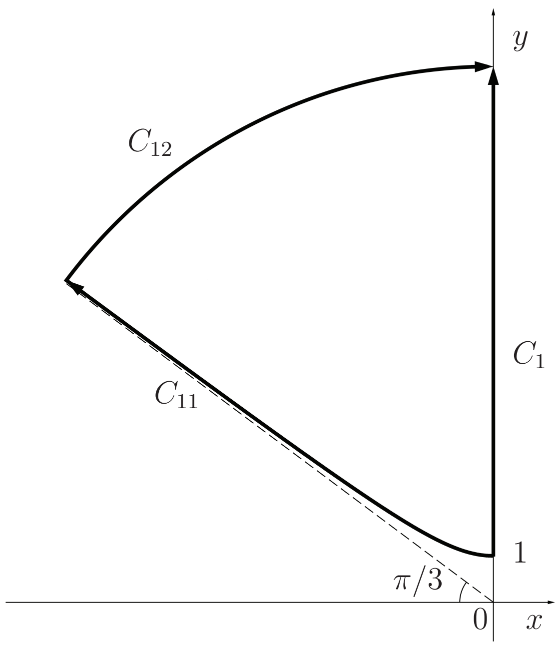


FIGURE 2.2: Contour integration in (2.15)

Start with the first integral in (2.21). Near the end point $z = i$ on C_{11} one has $h(z) \approx -2x$, $z \approx i$ and $\mathrm{d}z \approx \mathrm{d}x$. Thus,

$$\int\limits_{C_{11}} \sim i \int\limits_{0}^{-\infty} H_0^{(1)}(-2i\nu x)\mathrm{d}x. \tag{2.22}$$

It is clear that the contribution of the integral along C_{12} vanishes. Then, by substituting $x_1 = -2\nu x$ in (2.22) from (2.21) follows

$$\int\limits_{C_1} \sim -\frac{i}{2\nu}\int\limits_0^\infty H_0^{(1)}(ix_1)\mathrm{d}x_1 = -\frac{1}{\pi\nu}\int\limits_0^\infty K_0(x_1)\mathrm{d}x_1 = -\frac{1}{2\nu},$$

where K_0 denotes the Macdonald function. Finally, from (2.15)

$$\mathcal{F}_1(\vartheta) \sim \frac{1}{\sqrt{\vartheta}}. \tag{2.23}$$

To establish the asymptotic behaviour of the functions $\mathcal{F}_2$ and $\mathcal{F}_3$ the calculation of the saddle points required for the function $h(z)$. Setting $h'(z) = 0$ gives $3z^2 + 1 = 0$. The saddle points become $z_{1,2} = \pm\frac{i}{\sqrt{3}}$.

Next, consider the integral (2.16). The steepest descent path through the saddle point $z_1 = \frac{i}{\sqrt{3}}$ is determined by the condition $\mathrm{Re}h(z) = \mathrm{Re}h(\frac{i}{\sqrt{3}}) = 0$ resulting in (see (2.19) and (2.20))

$$y = \frac{1}{\sqrt{3}}\sqrt{x^2+1}. \tag{2.24}$$

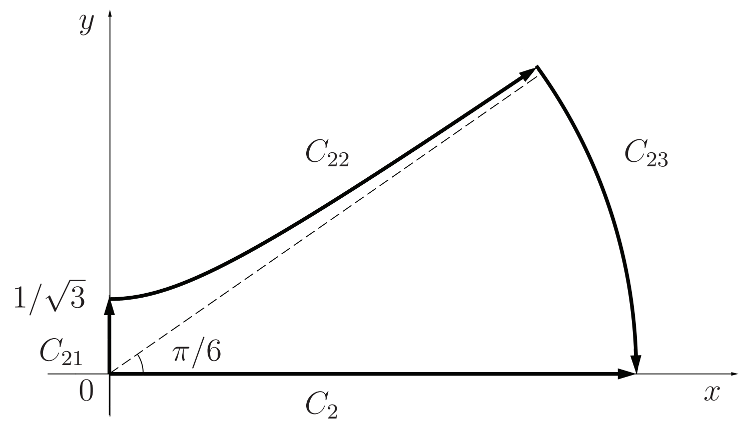


FIGURE 2.3: Contour integration in (2.16)

Similarly to (2.21) the integral (2.16) can be presented as (see Figure 2.3)

$$\int\limits_{C_2} = \int\limits_{C_{21}} + \int\limits_{C_{22}} + \int\limits_{C_{23}},$$

where C_{21} is the part of the imaginary axis between the points $z = 0$ and $z = i/\sqrt{3}$, C_{22} is the steepest descent path and C_{23} is the path along the circle of an infinitely large radius.

Along the path C_{21} $z = iy$ and $h(z) = i(1-y)$. Then

$$\int\limits_{C_{21}} = -\int\limits_{0}^{\frac{1}{\sqrt{3}}} H_0^{(1)}\left[i(1-y^2)\right] \mathrm{d}y.$$

The real part of the last integral is equal to zero and it does not affect the asymptotic behaviour of $\mathcal{F}_2$ (see (2.16)). Use of the formula (2.19) gives

$$\int\limits_{C_{22}} \sim -i\sqrt{\frac{2}{\pi\nu}} \int\limits_{0}^{\infty} \frac{1}{h_i} \exp\left(-\nu h_i\right)(x+iy)\left(1+i\frac{\mathrm{d}y}{\mathrm{d}x}\right) \mathrm{d}x, \tag{2.25}$$

where the steepest descent path $y(x)$ is given by (2.24) whereas

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sqrt{3(x^2+1)}}, \quad h_i = \frac{2}{3\sqrt{3}}\sqrt{x^2+1}(4x^2+1).$$

Laplace's method (e.g. see [72, 74]) in (2.25) provides the asymptotic formula. It is

$$\int\limits_{C_2} \sim \exp\left(-\frac{2\nu}{3\sqrt{3}}\right) \frac{1}{\sqrt{\pi\nu}} \int\limits_{0}^{\infty} \exp\left(-\sqrt{3}\nu x^2\right) \mathrm{d}x = \frac{1}{2\nu} \exp\left(-\frac{2\nu}{3\sqrt{3}}\right). \tag{2.26}$$

Now, inserting (2.26) into (2.16) gives

$$\mathcal{F}_2(\vartheta) = \frac{1}{\vartheta^{1/2}} \exp\left(-\frac{2\vartheta^{3/2}}{3\sqrt{3}}\right). \tag{2.27}$$

The path of integration for the function $\mathcal{F}_3$ is shown in Figure 2.4. Here the path C_{31} goes along the real axis, the path C_{32} goes along the steepest descent paths associated with the saddle point $z_2 = -\frac{i}{\sqrt{3}}$ and C_{33} is the steepest descent path through the point $z = -i$.

Along the path C_{32} $\mathrm{Im}h(z) = \mathrm{Im}h(-\frac{i}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$ and on this path

$$x = \left(y + \frac{1}{\sqrt{3}}\right)\sqrt{\frac{y - 2/\sqrt{3}}{3y}}. \tag{2.28}$$

The equation of the path C_{33} follows from the condition $\mathrm{Im}h(z) = \mathrm{Im}h(-i) = 0$. The result is

$$y = -\sqrt{3x^2 + 1}.$$

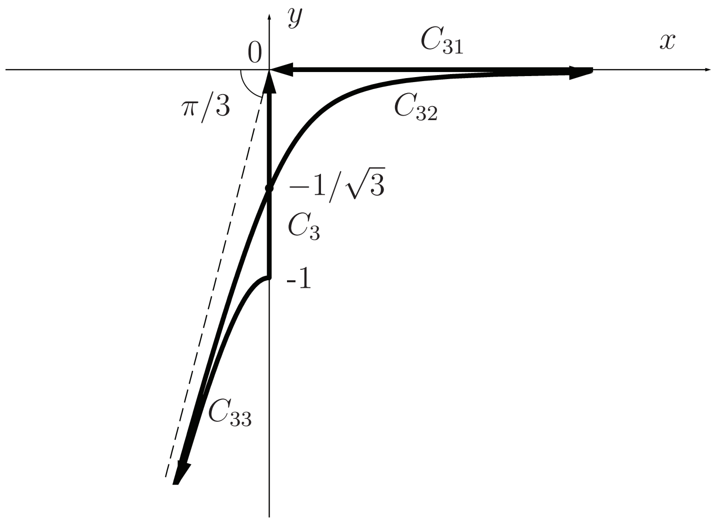


FIGURE 2.4: Contour integration in (2.17)

As above, the studied integral can be presented as a sum, i.e.

$$\int\limits_{C_3} = \int\limits_{C_{31}} + \int\limits_{C_{32}} + \int\limits_{C_{33}} .$$

Along the path C_{31} $y = 0$ and $h(z) = x(x^2 + 1)$. The integral

$$\int\limits_{C_{31}} = -\int\limits_0^\infty H_0^{(1)} \left[ix(x^2 + 1)\right] x\mathrm{d}x,$$

takes an imaginary value and does not contribute to the function $\mathcal{F}_3$.

The integral along the path C_{33} is similar to that along C_{11} (see (2.22)). In this case

$$\int\limits_{C_{31}} \sim -i \int\limits_0^{-\infty} H_0^{(1)}(-2i\nu x)\mathrm{d}x = \frac{1}{2\nu}.$$

Near the saddle point $z_2 = -\frac{i}{\sqrt{3}}$, one can derive from (2.28) $y \approx -\frac{1}{\sqrt{3}} + x$, $z \approx -\frac{i}{\sqrt{3}}$, $\mathrm{d}z \approx (1+i)\mathrm{d}x$ and $h(z) \approx 2\sqrt{3}x^2 - \frac{2}{3\sqrt{3}}i$. Hence,

$$\int\limits_{C_{32}} \sim i\sqrt{\frac{2\sqrt{3}}{\pi\nu}} \exp\left(-i\frac{2\nu}{3\sqrt{3}}\right) \int\limits_{-\infty}^{+\infty} e^{-2\sqrt{3}\nu x^2} \mathrm{d}x,$$

and

$$\int\limits_{C_3} \sim \frac{1}{\nu}\left[\frac{1}{2} + i\exp\left(-i\frac{2\nu}{3\sqrt{3}}\right)\right]. \tag{2.29}$$

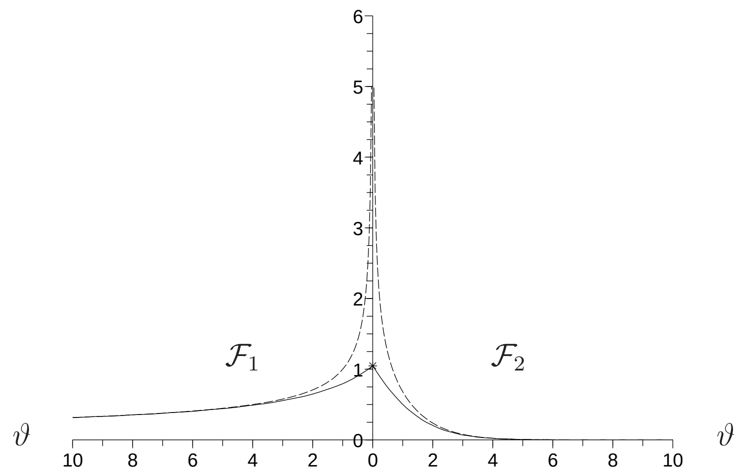


FIGURE 2.5: The functions $\mathcal{F}_1$ and $\mathcal{F}_2$. Asymptotic functions (dashed line and asterisk) and numerics (solid line).

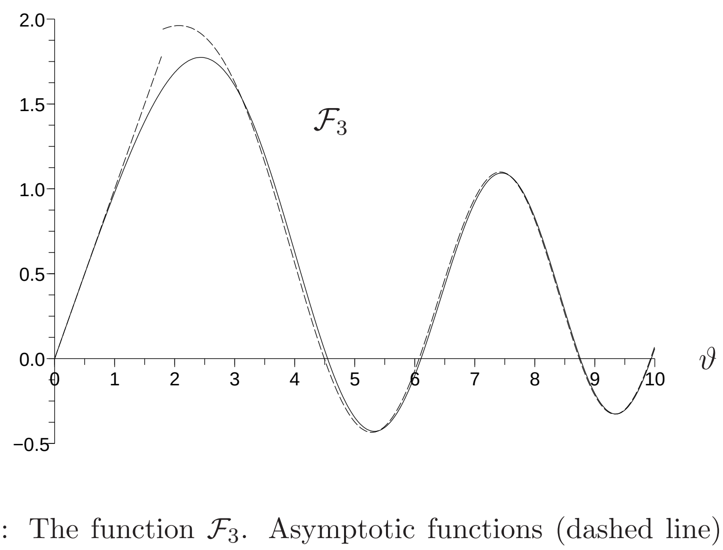


FIGURE 2.6: The function $\mathcal{F}_3$. Asymptotic functions (dashed line) and numerics (solid line).

ϑ	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	ϑ	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
0.0	1.04713	1.04713	0.00000	2.6	0.58912	0.11115	1.76045
0.1	1.01559	0.98280	0.10000	2.7	0.58088	0.09995	1.73827
0.2	0.98573	0.91931	0.19997	2.8	0.57263	0.08843	1.70562
0.3	0.95731	0.85697	0.29983	2.9	0.56431	0.07774	1.66250
0.4	0.93014	0.79721	0.39947	3.0	0.55592	0.06911	1.60906
0.5	0.90419	0.74135	0.49870	3.1	0.54759	0.06270	1.54554
0.6	0.87948	0.68948	0.59730	3.2	0.53947	0.05738	1.47232
0.7	0.85611	0.64039	0.69501	3.3	0.53174	0.05167	1.38993
0.8	0.83418	0.59273	0.79150	3.4	0.52454	0.04511	1.29901
0.9	0.81374	0.54625	0.88640	3.5	0.51796	0.03857	1.20037
1.0	0.79476	0.50200	0.97932	3.6	0.51197	0.03343	1.09492
1.1	0.77713	0.46135	1.06979	3.7	0.50648	0.03026	0.98372
1.2	0.76065	0.42469	1.15733	3.8	0.50131	0.02824	0.86796
1.3	0.74507	0.39101	1.24142	3.9	0.49626	0.02589	0.74891
1.4	0.73014	0.35877	1.32152	4.0	0.49116	0.02241	0.62798
1.5	0.71564	0.32725	1.39704	4.1	0.48587	0.01832	0.50663
1.6	0.70144	0.29713	1.46740	4.2	0.48036	0.01503	0.38642
1.7	0.68751	0.26977	1.53200	4.3	0.47467	0.01345	0.26893
1.8	0.67391	0.24590	1.59023	4.4	0.46893	0.01316	0.15576
1.9	0.66077	0.22483	1.64149	4.5	0.46332	0.01276	0.04852
2.0	0.64826	0.20502	1.68519	4.6	0.45802	0.01116	-0.05122
2.1	0.63651	0.18539	1.72079	4.7	0.45315	0.00852	-0.14195
2.2	0.62562	0.16623	1.74774	4.8	0.44880	0.00612	-0.22225
2.3	0.61556	0.14884	1.76558	4.9	0.44492	0.00515	-0.29082
2.4	0.60625	0.13423	1.77387	5.0	0.44142	0.00561	-0.34651
2.5	0.59751	0.12214	1.77225				

TABLE 2.1: Tabulated values of canonical integrals

Substitution (2.29) into (2.17) leads to the formula

$$\mathcal{F}_3(\vartheta) \sim \frac{1}{\vartheta^{1/2}} \left(1 + 2\sin\left(\frac{2\vartheta^{3/2}}{3\sqrt{3}}\right)\right). \tag{2.30}$$

Further, we may expect that the canonical integrals (2.8), (2.10) and (2.12) describe the uniform asymptotic behaviour of more complicated integrals of this type for the case in which the intermediate range $\vartheta \sim 1$ is also of interest.

The comparison of the asymptotic forms of the functions $\mathcal{F}_i$ ($i = 1, 2, 3$) with the results of numerical computations for the integrals (2.8), (2.10) and (2.12) is presented in Figures 2.5 and 2.6. Here and below the trapezium method with 10000-30000 points was used for numerical integration. The solid line in the first and second quadrants of Figure 2.5 corresponds to the computed values of the integrals (2.10) and (2.8), respectively. The asymptotic representations (2.23) and (2.27) are plotted in this figure by the dashed line. In addition, the limiting value (2.13) is denoted by an asterisk. In Figure 2.6 the results of the numerical evaluation of the integral (2.12) (solid line) are shown with its asymptotic forms (2.14) and (2.30) (dashed line). The tabulated values of the functions $\mathcal{F}_i$ are also displayed in Table 2.1. Here and below all the numerical calculations were performed in SciLab.

2.3 Constant acceleration case

Below we investigate the string behaviour at a moving point $\lambda = 0$, where a force is applied, and also at a moving singularity $\lambda = -\frac{(u-1)^2}{2}$, which is actually a shock wave which appeared after the passage through the sound wave speed. This value can be easily found (see details in [59]). The latter arises when passing through the sound speed and coincides with a point $\lambda = 0$ at $u = 1$. The aforementioned moving points are of most interest when investigating the passage through the sound wave barrier. Here three combinations of the problem parameters are studied.

2.3.1 The displacement under the load before the passage ($u \leq 1$ and $\lambda = 0$)

In this case the original integral in (2.3) becomes

$$I = \int_0^u J_0(\nu\phi(u,0,t))\mathrm{d}t, \tag{2.31}$$

with

$$\phi(u,0,t) = t\sqrt{1 - u^2 + t - t\left[(1-u) + \frac{t}{4}\right]}. \tag{2.32}$$

In the vicinity of the end point $t = 0$ in the integral (2.31) one has $\phi(u,0,t) \approx t\sqrt{1-u^2}$, if $t \ll 1-u$. Otherwise, for $1 - u \ll t \ll 1$ it appears that $\phi(u,0,t) \approx$

$t^{3/2}$. The formula

$$\phi(u,0,t) \approx t\sqrt{1-u^2+t}, \quad t \ll 1, \tag{2.33}$$

contains both of the limiting forms.

As above, to the leading order we can substitute infinity at the upper limit of the last integral. Finally, the following simpler integral is obtained

$$I \sim \int_0^\infty J_0\left(\nu t\sqrt{1-u^2+t}\right) \mathrm{d}z. \tag{2.34}$$

Next, changing the independent variable by $t_1 = t\nu^{2/3}$ we establish the sought for uniform asymptotic behaviour in the parameters ν and u

$$I \sim \nu^{-2/3} \int_0^\infty J_0\left(t_1\sqrt{\nu^{2/3}(1-u^2)+t_1}\right) \mathrm{d}t_1, \tag{2.35}$$

or

$$I \sim \nu^{-2/3}\mathcal{F}_1(\eta), \tag{2.36}$$

where the fundamental parameter η determines the scaling of the problem. It is given by

$$\eta = \nu^{2/3}(1-u^2). \tag{2.37}$$

Outside the characteristic zone $\eta \sim 1$ $(1-u^2 \sim \nu^{-2/3})$ the function $\mathcal{F}_1$ in (2.36) can be reduced to the local forms (2.13) for $\eta \ll 1$ $(1-u^2 \ll \nu^{-2/3})$ and (2.23) for $\eta \gg 1$ $(1-u^2 \gg \nu^{-2/3})$. Such an observation is relevant for other integrals considered below in this section.

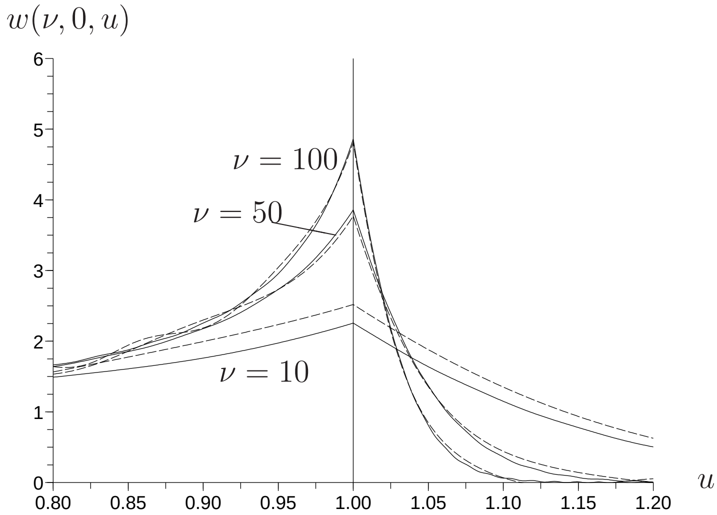
2.3.2 The displacement under the load after the passage ($u \geq 1$ and $\lambda = 0$)

Here

$$I = \int_{2(u-1)}^{u} J_0(\nu\phi(u,0,t))\mathrm{d}t, \tag{2.38}$$

with

$$\phi(u,0,t) = \frac{t}{2}\sqrt{(t-2(u-1))(2(u+1)-t)}.$$



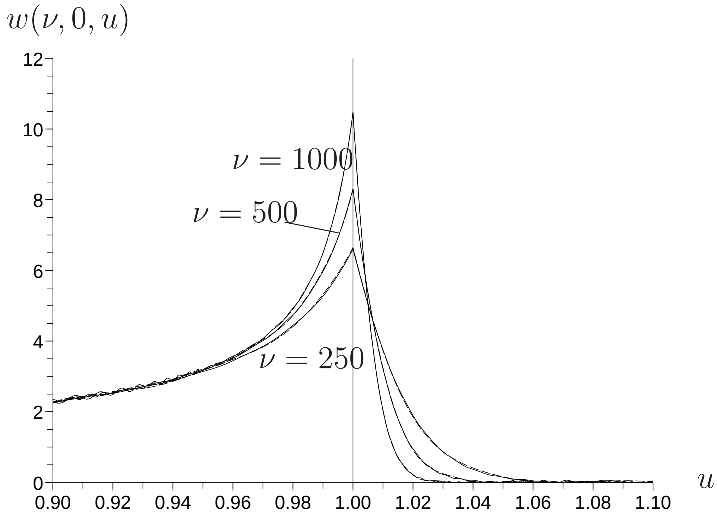


FIGURE 2.7: Uniform asymptotic behaviour (solid line) and numerics (dashed line) of the function (2.3) using integrals (2.31) ($u \leq 1$) and (2.36) ($u \geq 1$).

In this case the function $\phi(u, 0, t)$ near the end point $t = 2(u-1)$ is presented as

$$\phi(u, 0, t) \approx t\sqrt{t - 2(u-1)}. \tag{2.39}$$

After changing the independent variable by

$$t_1 = \nu^{2/3}(t - 2(u-1)), \tag{2.40}$$

it appears that

$$I \sim \nu^{-2/3} \int_0^\infty J_0\left((t_1 + 2\nu^{2/3}(u-1))\sqrt{t_1}\right) \mathrm{d}t_1 = \nu^{-2/3}\mathcal{F}_2(2\nu^{2/3}(u-1)). \quad (2.41)$$

This asymptotic result is of interest only over the narrow vicinity of the sound wave speed ($u-1 \sim \nu^{-2/3}$) due to the exponential decay of the function $\mathcal{F}_2$. In this case, the parameter η may be introduced in the last formula setting $2-u \approx u^2-1$.

2.3.3 The displacement at the moving singularity ($u \geq 1$ and $\lambda = -\frac{1}{2}(u-1)^2$)

We have in (2.3)

$$I = I_1 + I_2, \quad (2.42)$$

with

$$I_1 = \int_{u-1}^{u} J_0\left(\nu\phi\left(u, -\frac{1}{2}(u-1)^2, t\right)\right) \mathrm{d}t, \quad (2.43)$$

and

$$I_2 = \int_{(\sqrt{u}-1)^2}^{u-1} J_0\left(\nu\phi\left(u, -\frac{1}{2}(u-1)^2, t\right)\right) \mathrm{d}t, \quad (2.44)$$

where

$$\phi\left(u, -\frac{1}{2}(u-1)^2, t\right) = \\ = \pm(t-(u-1))\sqrt{-\frac{1}{4}(t-(\sqrt{u}-1)^2)(t-(\sqrt{u}+1)^2)}. \quad (2.45)$$

Here and below the signs "+" and "−" correspond to the integrals I_1 and I_2, respectively.

The parameter range can be restricted by the values of u that are close enough to the value $u = 1$. In this case $\sqrt{u}+1 \approx 2$ may be set. Next, the function (2.45) might be expanded near the left end point in the integral (2.43) and over the whole integration domain in the integral (2.44). It becomes

$$\phi\left(u, -\frac{1}{2}(u-1)^2, t\right) \approx \pm(t-(u-1))\sqrt{t-(\sqrt{u}-1)^2}.$$

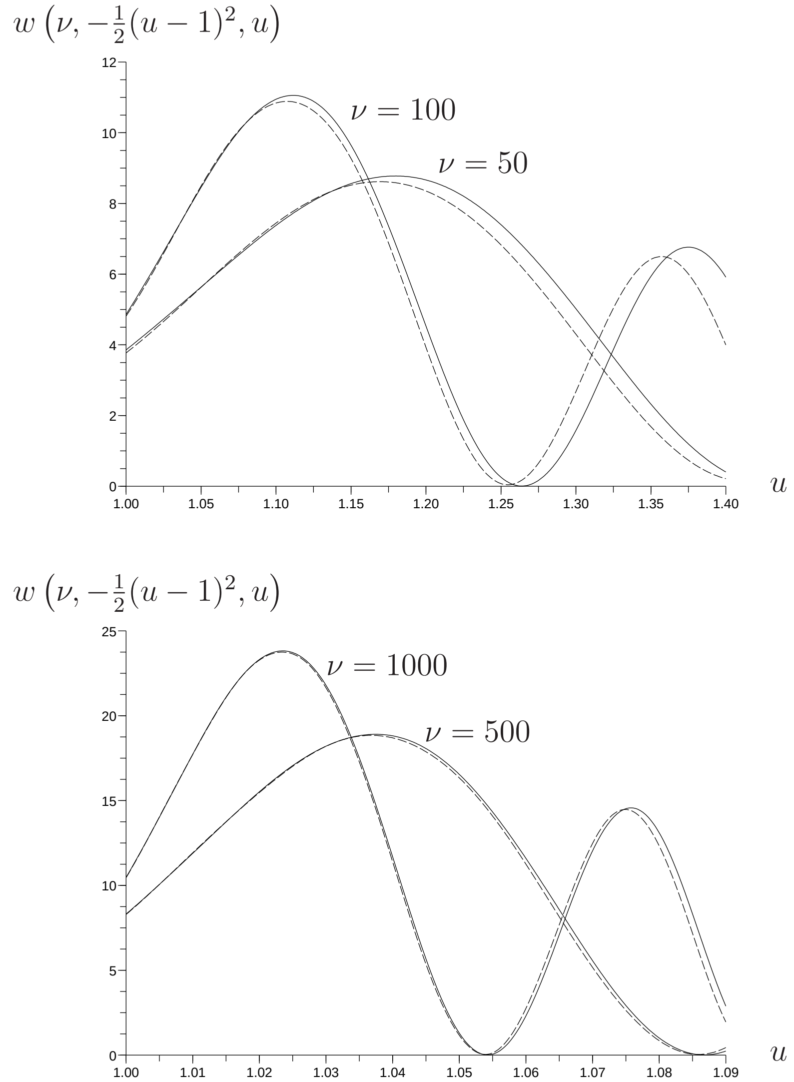


FIGURE 2.8: Uniform asymptotic behaviour (dashed line) and numerics (solid line) of the function (2.3) using integrals (2.42).

Now, change of the variables in the integrals (2.43) and (2.44) by the formula

$$t_1 = \pm\nu^{2/3}(t - (u-1)) \tag{2.46}$$

gives the result

$$I_1 \sim \nu^{-2/3} \int\limits_0^\infty J_0\left(t_1\sqrt{t_1 + 2(\sqrt{u}-1)\nu^{2/3}}\right) \mathrm{d}t_1 =$$
$$= \nu^{-2/3}\mathcal{F}_1(2(\sqrt{u}-1)\nu^{2/3}),$$

and

$$I_2 \sim \nu^{-2/3} \int\limits_0^{2(\sqrt{u}-1)\nu^{2/3}} J_0\left(t_1\sqrt{2(\sqrt{u}-1)\nu^{2/3}-t_1}\right)\mathrm{d}t_1 =$$

$$= \nu^{-2/3}\mathcal{F}_3(2(\sqrt{u}-1)\nu^{2/3}).$$

Finally,

$$I \sim \nu^{-2/3}\left[\mathcal{F}_1(2\nu^{2/3}(\sqrt{u}-1)) + \mathcal{F}_3(2\nu^{2/3}(\sqrt{u}-1))\right]. \tag{2.47}$$

Similar to the previous case, we may operate with the functions $\mathcal{F}_j\left(-\frac{1}{2}\eta\right)$ $(j = 1,3)$ in the vicinity of the sound wave speed where $\sqrt{u}-1 \approx \frac{1}{4}(u^2-1)$.

2.3.4 Numerical results

The numerical examples are presented in Figures 2.7 and 2.8. In Figure 2.7 the computed values of the function (2.3) are displayed using the formulae (2.31) and (2.38) (dashed line) along with its uniform asymptotic behaviour given by the formulae (2.36) and (2.41) (solid line). The graphs of the function (2.3) in case of the integral (2.42) (dashed line) and the uniform asymptotic formula (2.47) (solid line) are plotted in Figure 2.8.

These figures illustrate the uniform validity of the derived asymptotic formulae in case of the moving load problem for a string. The striking difference between the asymptotic behaviour of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ for $\vartheta \gg 1$ leads to a strong asymmetry of the transition curves in Figure 2.7. In Figure 2.8 the function $\mathcal{F}_3$ reproduces the oscillatory patterns associated with the passage through the sound wave barrier.

2.4 Constant deceleration case

Here we briefly describe an inverse transition through the sonic speed. In particular we would like to pay attention on a string behaviour under a moving load just after the passage. In this case $u \leq 1$ and $\lambda = 0$. Similar to the previous section we obtain an integral:

$$w = \nu \int\limits_{-2(1-u)}^{0} J_0 \left(\nu \sqrt{-\frac{1}{4} z^2 (z + 2(1-u))(z - 2(1+u))} \right) \mathrm{d}z. \tag{2.48}$$

Since $z \approx 0$ and u is in the vicinity of 1 then $z - 2(1 + v1) \approx -4$. Thus

$$w = \nu \int\limits_{-2(1-u)}^{0} J_0 \left(\nu z \sqrt{z + 2(1-u)} \right) \mathrm{d}z.$$

Changing variables as $z = \nu^{-2/3} y$ gives

$$w = \nu^{1/3} \int\limits_{-\nu^{2/3}(1-u)}^{0} J_0 \left(y \sqrt{y + 2\nu^{2/3}(1-u)} \right) \mathrm{d}y.$$

Introducing a new notation $a = 2\nu^{2/3}(1-u)$ we obtain

$$w = \nu^{1/3} \int\limits_{-a}^{0} J_0 \left(y \sqrt{y + a} \right) \mathrm{d}y.$$

Since J_0 is an even function then substitution $y = -t$ leads to

$$w = \nu^{1/3} \int\limits_{0}^{a} J_0 \left(t \sqrt{a - t} \right) \mathrm{d}t = \nu^{1/3} \mathcal{F}_3(a) = \nu^{1/3} \mathcal{F}_3(2\nu^{2/3}(1-u)). \tag{2.49}$$

So, the derived asymptotic solution (2.49) is based on the function $\mathcal{F}_3$ introduced above. As one can see in Figure 2.9 it provides a very close approximation (solid line) to the exact integral solution (2.48) (dashed line) in the vicinity of sound speed.

2.5 Arbitrary acceleration case

In this section we provide a draft of how to deal with the arbitrary acceleration using the results for the constant one, which were obtained above. As an example

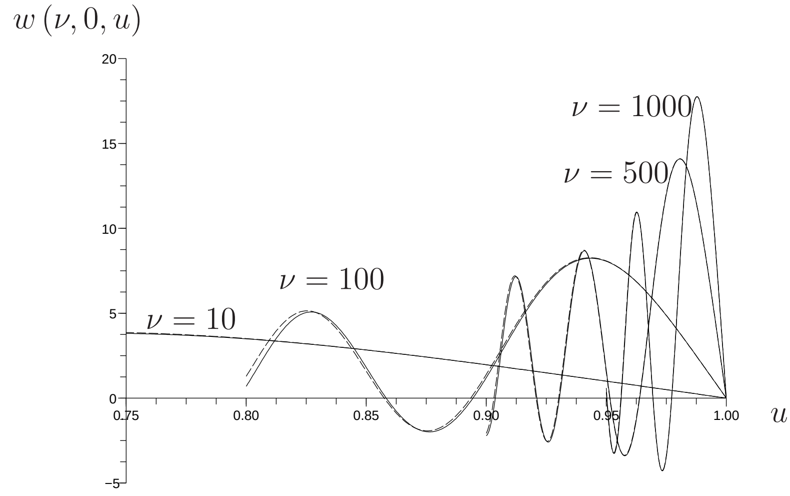


FIGURE 2.9: Asymptotic and numerical results for the integral (2.48).

we consider two possible paths: $s(\tau) = u_0\tau + \frac{\varepsilon\tau^2}{2}$ and $s(\tau) = -A\cos(\varepsilon\tau)$ (respective speeds are $u(\tau) = u_0 + \varepsilon\tau$ and $u(\tau) = A\sin(\varepsilon\tau)$) (see Figure 2.10). Note that the first path presents inverse transition from arbitrary to uniform acceleration case.

Consider the general integral solution for the main problem (2.2) with a general path function $s_1(\tau)$ (not necessary equal to $\frac{a\tau^2}{2}$) in the right hand side of the equation.

$$w = \int_0^\tau J_0\left(\sqrt{\zeta^2 - (\lambda + s_1(\tau) - s_1(\tau-\zeta))^2}\right)\sigma \mathrm{d}\zeta. \tag{2.50}$$

Here and below σ has the following behaviour:

$$\sigma = \begin{cases} 1, & \text{if } \zeta^2 - (\lambda + s_1(\tau) - s_1(\tau-\zeta))^2 > 0; \\ 0, & \text{if } \zeta^2 - (\lambda + s_1(\tau) - s_1(\tau-\zeta))^2 < 0. \end{cases}$$

In the vicinity of $\zeta = 0$, one can expand the path function s_1 as

$$s_1(\tau-\zeta) \approx s_1(\tau) - u(\tau)\zeta + \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\zeta^2}{2}, \zeta \ll 1. \tag{2.51}$$

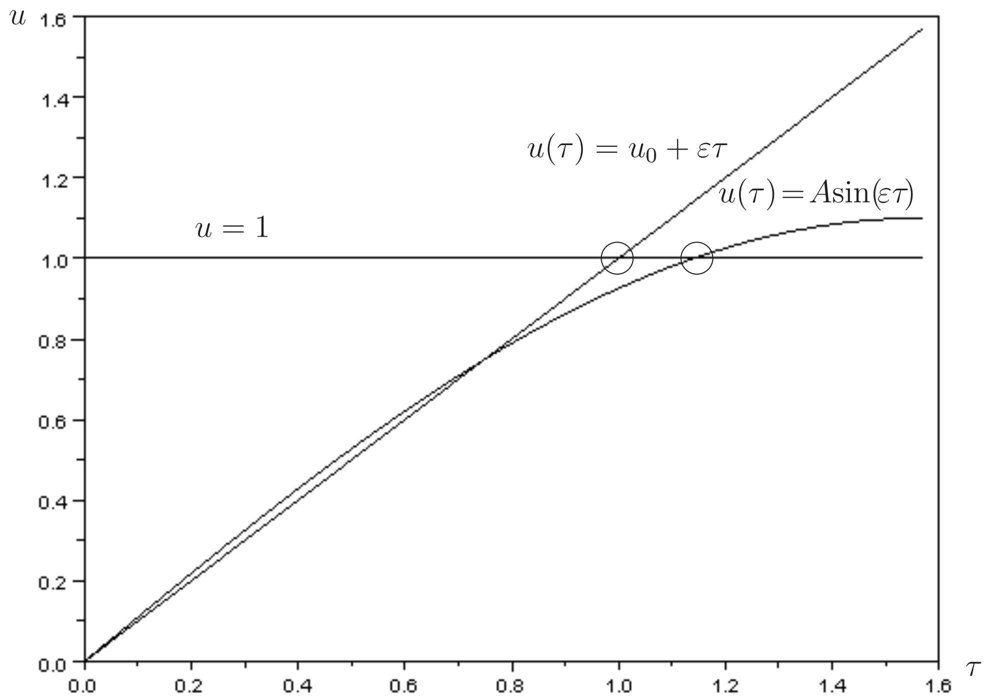


FIGURE 2.10: Transition through the sound wave barrier with constant acceleration and sinusoidal speed function. The transition moment for each case is shown by a circle

Substitution (2.51) into (2.50) leads to

$$w = \int_0^\delta J_0\left(\sqrt{\zeta^2 - \left(\lambda + u(\tau)\zeta - \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\zeta^2}{2}\right)^2}\right)\sigma\mathrm{d}\zeta. \tag{2.52}$$

Assumption $u = k(\varepsilon\tau)$ gives

$$w = \int_0^\delta J_0\left(\sqrt{\zeta^2 - \left(\lambda + k\zeta - \varepsilon\frac{\mathrm{d}k}{\mathrm{d}\tau}\frac{\zeta^2}{2}\right)^2}\right)\sigma\mathrm{d}\zeta.$$

After the substitutions $\zeta = \dfrac{z}{\varepsilon}$ it appears

$$w = \frac{1}{\varepsilon}\int_0^{\varepsilon\delta} J_0\left(\frac{1}{\varepsilon}\sqrt{z^2 - \left(\varepsilon\lambda + kz - \frac{\mathrm{d}k}{\mathrm{d}\tau}\frac{z^2}{2}\right)^2}\right)\sigma\mathrm{d}z.$$

Now, assume that $\nu = \dfrac{1}{\varepsilon}$ and $\lambda_1 = \varepsilon\lambda = \dfrac{\lambda}{\nu}$ and get

$$w = \nu \int_0^{\delta/\nu} J_0 \left(\nu \sqrt{z^2 - \left(\lambda_1 + kz - \frac{\mathrm{d}k}{\mathrm{d}\tau}\frac{z^2}{2} \right)^2} \right) \sigma \mathrm{d}z.$$

Consider the case when $\lambda_1 = 0$:

$$w = \nu \int_0^{\delta/\nu} J_0 \left(\nu \sqrt{z^2 - k^2 z^2 + k\frac{\mathrm{d}k}{\mathrm{d}\tau} z^3 - \left(\frac{\mathrm{d}k}{\mathrm{d}\tau} \right)^2 \frac{z^4}{4}} \right) \sigma \mathrm{d}z.$$

Since the last term is a small value of the fourth order, it can be neglected and we obtain the following formula

$$w = \nu \int_0^{\delta/\nu} J_0 \left(\nu \sqrt{z^2(1-k^2) + k\frac{\mathrm{d}k}{\mathrm{d}\tau} z^3} \right) \sigma \mathrm{d}z. \tag{2.53}$$

Two cases with respect to values of k, namely, $k < 1$ and $k > 1$ (here k is assumed to be in the vicinity of unity), should be considered:

(i) $k < 1$

Consider the (2.53) and assume that in the second term under a radical sign $k \approx 1$. In this case $z^2 \dfrac{\mathrm{d}k}{\mathrm{d}\tau}$ can be moved from the square root sign and

$$w = \nu \int_0^{\delta/\nu} J_0 \left(\nu z \sqrt{\frac{\mathrm{d}k}{\mathrm{d}\tau}} \sqrt{(1-k^2)/\frac{\mathrm{d}k}{\mathrm{d}\tau} + z} \right) \mathrm{d}z.$$

Making a substitution $z = y \left(\nu \sqrt{\dfrac{\mathrm{d}k}{\mathrm{d}\tau}} \right)^{-2/3}$ and also assuming that the upper limit in integral is going towards ∞, one may get

$$w = \nu^{1/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau} \right)^{-1/3} \int_0^{\infty} J_0 \left(y \sqrt{(1-k^2)\nu^{2/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau} \right)^{-2/3} + y} \right) \mathrm{d}y,$$

or

$$w = \nu^{1/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau} \right)^{-1/3} \mathcal{F}_1 \left((1-k^2)\nu^{2/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau} \right) \right).$$

(ii) $k > 1$

By analogy with the previous case, we consider the integral (2.53), but here the lower limit is changed because the phase function should be non-negative on the entire interval of integration.

$$w = \nu \int\limits_{\frac{2(k-1)}{\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{4/3}}}^{\delta/\nu} J_0\left(\nu z\sqrt{\frac{\mathrm{d}k}{\mathrm{d}\tau}}\sqrt{z-\frac{k^2-1}{\frac{\mathrm{d}k}{\mathrm{d}\tau}}}\right)\mathrm{d}z.$$

Substitution $z = y\left(\nu\sqrt{\frac{\mathrm{d}k}{\mathrm{d}\tau}}\right)^{-2/3}$ and the assumption that the upper integration limit is going towards ∞ imply that

$$\begin{aligned} w &= \nu^{1/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-1/3}\times \\ &\times \int\limits_{\frac{2(k-1)\nu^{2/3}}{\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{4/3}}}^{\infty} J_0\left(y\sqrt{y-(k^2-1)\nu^{2/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-2/3}}\right)\mathrm{d}y. \end{aligned}$$

Now, assume that $k^2-1 \approx 2(k-1)$ and make the substitution $x = y - (k^2-1)\nu^{2/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-2/3}$. The result is

$$w = \nu^{1/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-1/3}\int\limits_0^{\infty} J_0\left(\sqrt{x}\left(x+2(k-1)\nu^{2/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-2/3}\right)\right)\mathrm{d}x,$$

or

$$w = \nu^{1/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-1/3}\mathcal{F}_2\left(2(k-1)\nu^{2/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-2/3}\right).$$

So, the general case asymptotic forms are:

$$w = \nu^{1/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-1/3}\mathcal{F}_1\left((1-k^2)\nu^{2/3}\left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)\right), (k<1)$$

and

$$w = \nu^{1/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-1/3} \mathcal{F}_2 \left(2(k-1)\nu^{2/3} \left(\frac{\mathrm{d}k}{\mathrm{d}\tau}\right)^{-2/3}\right), (k > 1).$$

It remains to substitute the example paths into the latter formulae.

(i) $k = \varepsilon\tau = v$

Thus, $\frac{\partial k}{\partial \tau} = 1$ and it leads to

$$w = \nu^{1/3}\mathcal{F}_1\left((1-v^2)\nu^{2/3}\right), (k < 1)$$

and

$$w = \nu^{1/3}\mathcal{F}_2\left(2(v-1)\nu^{2/3}\right), (k > 1).$$

Here we note that these results are exactly the same as the ones given in Section 2.3, but here they are obtained for a more general case.

(ii) $k = A\sin(\varepsilon\tau)$

Assume that $\nu = \frac{1}{\varepsilon}$ and $k \approx 1$ (so $\frac{\partial k}{\partial \tau} = A\cos(\varepsilon\tau) = A\sqrt{1-\sin^2(\varepsilon\tau)} = \sqrt{A^2-1}$). In this case

$$w = \left(\varepsilon\sqrt{A^2-1}\right)^{-1/3} \mathcal{F}_1 \left(\frac{1 - A^2\sin^2(\varepsilon\tau)}{\left(\varepsilon\sqrt{A^2-1}\right)^{2/3}}\right), (k < 1) \tag{2.54}$$

and

$$w = \left(\varepsilon\sqrt{A^2-1}\right)^{-1/3} \mathcal{F}_2 \left(\frac{2\left(A\sin(\varepsilon\tau) - 1\right)}{\left(\varepsilon\sqrt{A^2-1}\right)^{2/3}}\right), (k > 1). \tag{2.55}$$

The graphs given in Figures 2.11–2.13 geometrically describe the exact and asymptotic solutions for the different values of the parameters A and ν. Comparing these figures one can notice that the derived formulae (2.54) – (2.55) provide a better approximation for integral solution (2.50) when parameter ν is sufficiently large and parameter A is close to the unity (note that A can not be less or equal to 1, otherwise there is no passage through the critical speed), i.e. when the transition is quite slow. As you can see, there are some local corners and roughness (especially sufficiently far from critical speed) on the lines on the figures. These features appear because of some local errors in numerical integration.

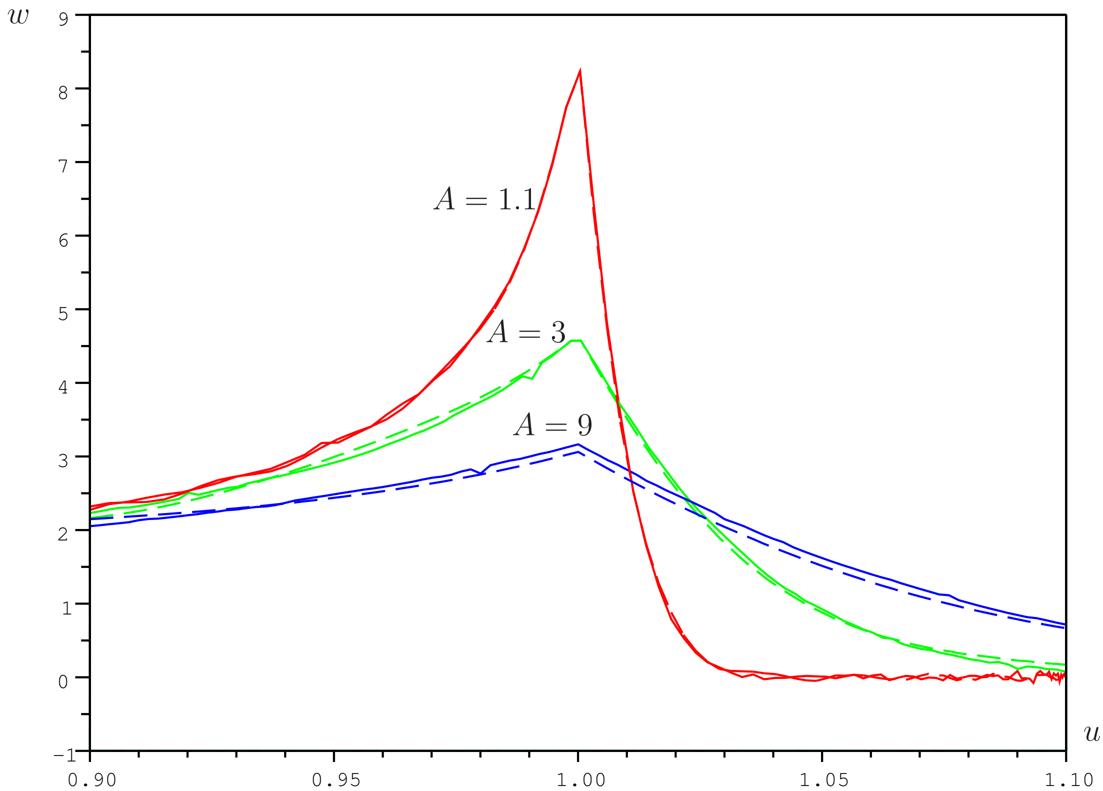


FIGURE 2.11: The exact solution (2.50) (dashed line) along with the asymptotic solution (2.54)–(2.55) (solid line) under the load for $\nu = 250$

2.6 Special asymptotic forms

2.6.1 Steady speed asymptotic behaviour near the load case

Again, consider (2.50) and the series (2.51), but now only the first two terms of the series are the object of the attention, due to the fact that the speed is steady, or zero acceleration. So,

$$w = \int\limits_0^\delta J_0\left(\sqrt{\zeta^2 - (\lambda + u\zeta)^2}\right)\sigma \mathrm{d}\zeta,$$

or after some transformations of the phase function

$$w = \int\limits_0^\delta J_0\left(\sqrt{(1-u^2)\left(\zeta - \frac{\lambda}{1-u}\right)\left(\zeta + \frac{\lambda}{1+u}\right)}\right)\sigma \mathrm{d}\zeta.$$

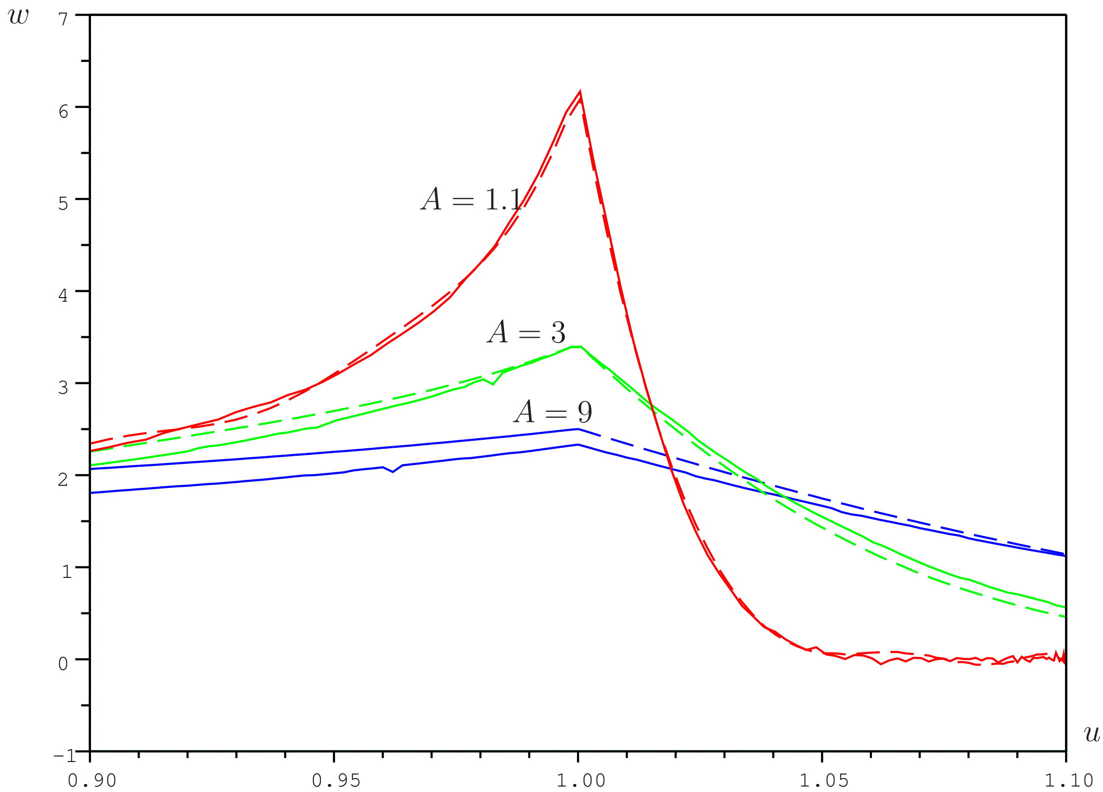


FIGURE 2.12: The exact solution (2.50) (dashed line) along with the asymptotic solution (2.54)–(2.55) (solid line) under the load for $\nu = 100$

The roots of the phase function are

$$T_0 = \frac{\lambda}{1-u}; \; T_1 = \frac{\lambda}{1+u}, \quad (T_0 > T_1).$$

Now one can make a substitution

$$p = \zeta - T_i, \; (i = 0 \text{ if } \lambda > 0; \; i = 1 \text{ if } \lambda < 0).$$

So now there are two cases of λ values for the case of $u < 1$:

(i) $\lambda > 0$

In this case

$$w = \int_0^{\delta - T_0} J_0 \left(\sqrt{(1-u^2)\, p \left(p + \frac{2\lambda}{1-u^2} \right)} \right) \mathrm{d}p.$$

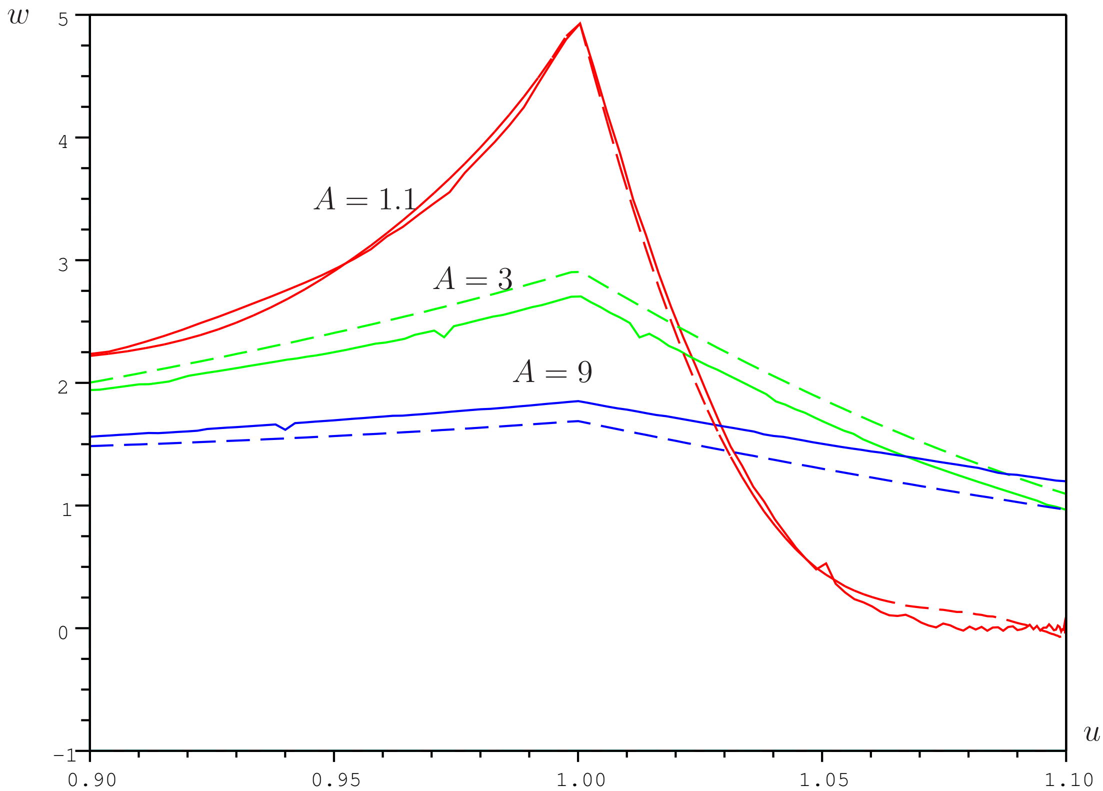


FIGURE 2.13: The exact solution (2.50) (dashed line) along with the asymptotic solution (2.54)–(2.55) (solid line) under the load for $\nu = 50$

(ii) $\lambda < 0$ It brings:

$$w = \int_0^{\delta - T_1} J_0 \left(\sqrt{(1-u^2)\, p \left(p + \frac{2\lambda}{1-u^2} \right)} \right) \mathrm{d}p.$$

So, the limit case will be if $d = \delta - T_i \to \infty$, and the result for a steady speed is:

$$w_{st} = \int_0^{\infty} J_0 \left(\sqrt{(1-u^2) \left(p^2 + \left| \frac{2\lambda}{1-u^2} \right| p \right)} \right) \sigma \mathrm{d}p. \tag{2.56}$$

The integral (2.56) is a well-known integral, that can be calculated analytically

$$w_{st} = \frac{1}{\sqrt{1-u^2}} \exp \left(\frac{|\lambda|}{\sqrt{1-u^2}} \right).$$

For the case of $u > 1$ there is another substitution $q = \zeta - T_1$.

Again there are two cases:

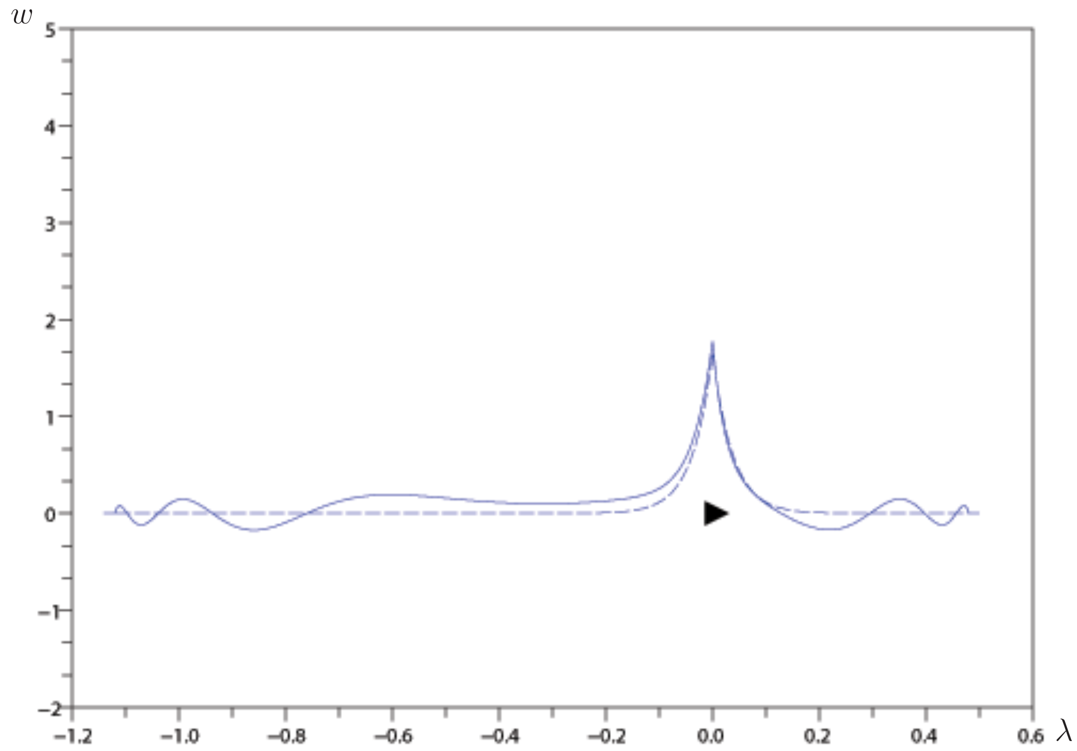


FIGURE 2.14: The exact solution (solid line) along with the static speed solution (dashed line) for $u < 1$ ($u = 0.8$).

(i) $T_1 < \tau < T_0$

In other words, the above mentioned substitution gives $0 < d < \frac{2\lambda}{1-u^2} = 2\gamma$. And the integral transforms into

$$w = \int_{T_1}^{\tau} J_0\left(\sqrt{(1-u^2)q\,(q+T_1-T_0)}\right) \mathrm{d}q =$$

$$= \int_{0}^{a} J_0\left(\sqrt{(u^2-1)\,(-q^2+2\gamma q)}\right) \mathrm{d}q.$$

(ii) $\tau > T_0$

Here $d > 2\gamma$ and thus

$$w = \int_{T_1}^{T_0} J_0\left(\sqrt{(1-u^2)q\,(q+T_1-T_0)}\right) \mathrm{d}q =$$

$$= \int_0^{2\gamma} J_0\left(\sqrt{(u^2-1)\,(-q^2+2\gamma q)}\right) \mathrm{d}q.$$

The limit case is when $d = 2\gamma$ ($\tau = T_0$) and hence

$$w_{st} = \int_0^{d} J_0\left(\sqrt{(u^2-1)\,(-q^2+2\gamma q)}\right) \mathrm{d}q.$$

This is a well-known integral that can be calculated analytically

$$w_{st} = \frac{2\sin\left[\frac{|\lambda|}{\sqrt{u^2-1}}\right]}{\sqrt{u^2-1}}.$$

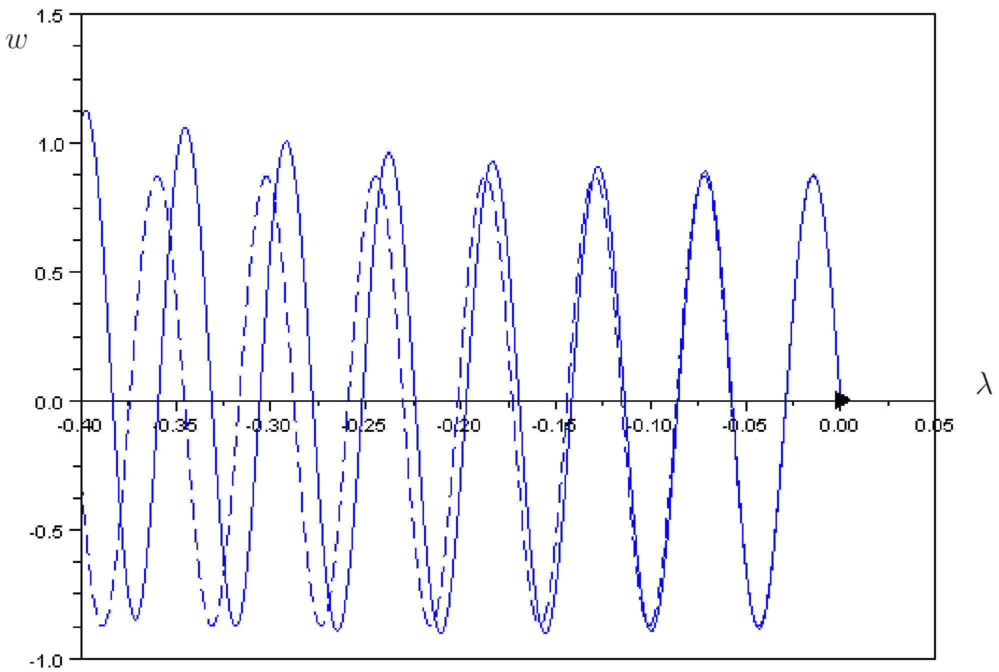


FIGURE 2.15: The exact solution (solid line) along with the static speed solution (dashed line near the load point, black triangle) for $u > 1$ ($u = 2.5$).

2.6.2 Asymptotic behaviour near singularity area

Here the stationary points are considered, one for the area in front of the shock wave and two of them beyond the shock wave. This section contains only the constant positive acceleration case, because it is too hard to find such an asymptotic representations for more complicated cases.

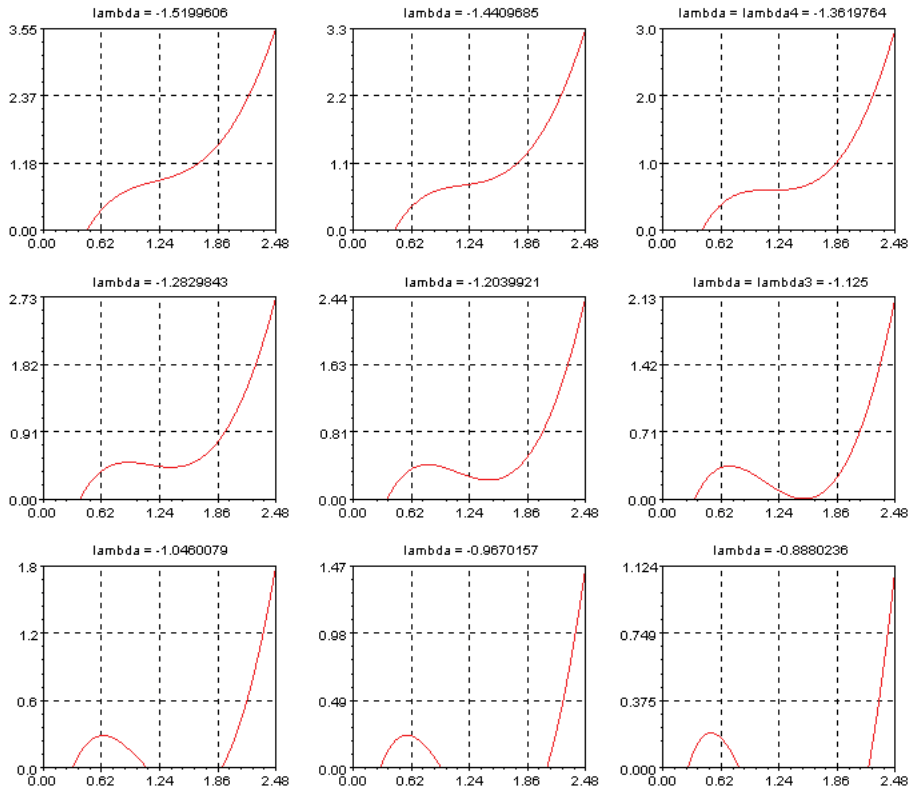


FIGURE 2.16: Evolution of the phase function (expression under square roots in (2.57)) against z for the different values of λ for $u = 2.5$

So, consider the following integral:

$$I = \int_0^u J_0 \left(\nu \sqrt{z^2 - \left(\lambda_1 + uz - \frac{1}{2} z^2 \right)^2} \right) \mathrm{d}z. \tag{2.57}$$

We need to find the contribution of all stationary points by the classic stationary phase method. For this, first we have to transfer the Bessel function in the integral (2.57) to its asymptotic formula with an exponential function, as follows

$$J_0(\nu \sqrt{\Phi(z)}) \approx \sqrt{\frac{2}{\pi \Phi(z)}} \mathrm{Re} \exp \left(\Phi(z) - \frac{\pi}{4} \right). \tag{2.58}$$

Now one may use the stationary phase method. To do so one or two (depending on the value of λ, see Figure 2.16) stationary points should be found. It is impossible to make it analytically, due to the complicated nature of the phase function. So, it was done by numerical methods. After finding these phase points, it remains to substitute the results into the following formula

$$I_{1,2} = \sqrt{\frac{2\pi}{\nu|\Phi''(z_0)|}} f(z_0) \cos\left(\nu\sqrt{\Phi(z_0)} - \frac{\pi}{4} + \frac{\pi}{4}\mathrm{sgn}\left(\Phi''(z_0)\right)\right),$$

where Φ is a phase function, $f(z)$ - is a term, staying before exp in the (2.58), z_0 is a stationary point, calculated numerically.

The final step is to find a sum of contributions of all the stationary points. This will tend to the asymptotic result (see Figures 2.15 and 2.17), that works near the shock wave (λ_3, red diamond) and at the end of singularity area (λ_4, green diamond).

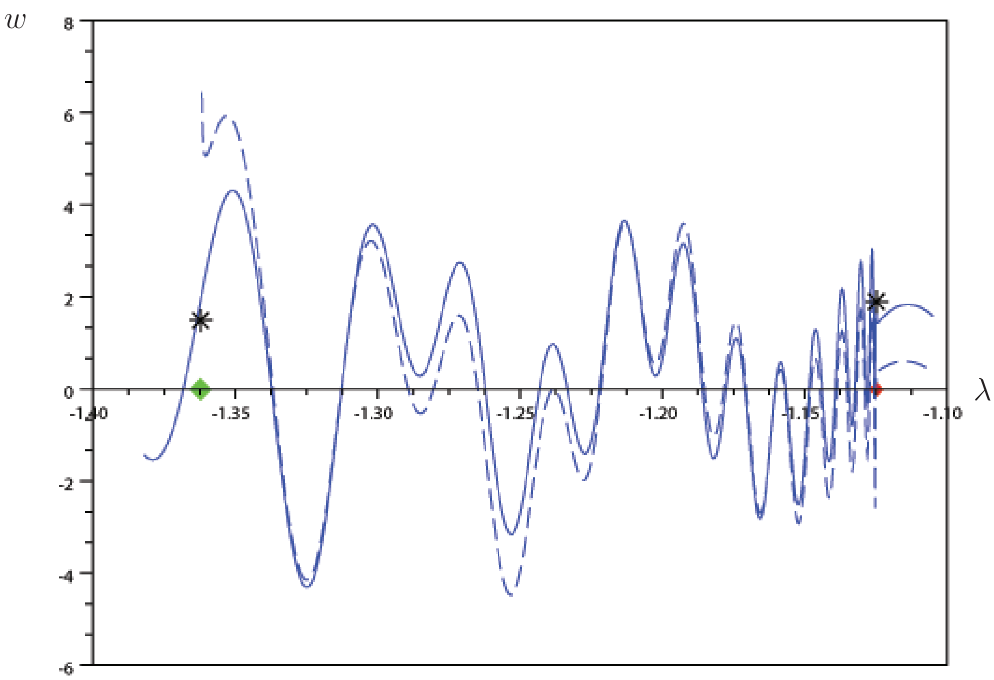


FIGURE 2.17: The exact solution (solid line) and static speed solution with contributions from stationary points of phase function (dashed line) for $u > 1$ ($u = 2.5$).

2.7 Summary

In this chapter we investigated the asymptotic behaviour of an elastically supported infinite string under a moving point load.

The auxiliary canonical functions $\mathcal{F}_i,\ i = 1, 2, 3$, were introduced, asymptotically analyzed and tabulated in Section 2.2. Using these functions the uniform asymptotic formulae for a string under the constant accelerating and decelerating point loads were obtained in Sections 2.3 and 2.4 respectively.

The asymptotic formulae for an arbitrary acceleration case were presented and then applied for the case of sinusoidal load speed $u(\tau) = A\sin(\varepsilon\tau)$ in Section 2.5. It was shown that the canonical functions $\mathcal{F}_i,\ i = 1, 2, 3$, can successfully be used in an arbitrary acceleration case as well.

In Section 2.6 we obtained the approximate formulae for the vicinity of point load and singularity area behind the shock wave using the steady speed asymptotic expansion with the additional contributions from the stationary points where appropriate.

Chapter 3

Vibrations of an infinite half plane under moving point load

3.1 Statement of the problem

The main object of consideration is an elastic isotropic infinite half plane ($-\infty < x < \infty$, $0 \leq y < \infty$) subject to a point load (line load for three dimensional case) of an amplitude P_0 moving along the surface $y = 0$ with a constant acceleration a (see Figure 3.1).

We start with the asymptotic model which was suggested in [61] and describes the half plane behaviour for the load speeds close to the Rayleigh wave speed. In doing so, the model ignores the bulk compression and shear waves, which appear in general Lamé equation for the half plane. The horizontal and vertical displacements u_1 and u_2 can be expressed in terms of the Lamé potentials ϕ and ψ (in frames of general elastic theory):

$$u_1 = \phi_x - \psi_y, \quad u_2 = \phi_y + \psi_x. \tag{3.1}$$

Here and below subscripts x and y denote the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Further on we will use both of these notations.

The elliptic equations for the interior field in the above mentioned model can be expressed in the form

$$\phi_{yy} + k_1^2 \phi_{xx} = 0, \quad \psi_{yy} + k_2^2 \psi_{xx} = 0, \tag{3.2}$$

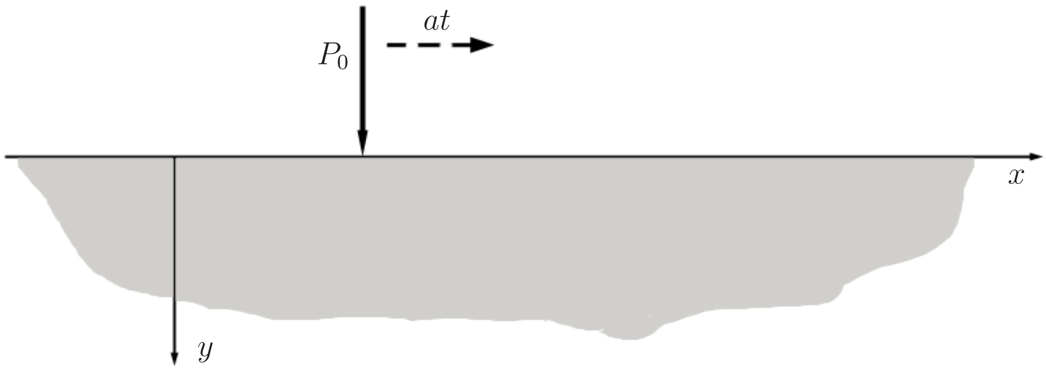


FIGURE 3.1: Elastic isotropic infinite half plane subject to a moving point load

where

$$k_i^2 = 1 - \frac{c_R^2}{c_i^2}, \ (i = 1,\ 2), \tag{3.3}$$

and

$$(1 + k_2^2)^2 = 4k_1k_2. \tag{3.4}$$

The formula (3.4), using expression (3.3), can be written as the classic Rayleigh equation:

$$4\sqrt{1 - \frac{c_R^2}{c_1^2}}\sqrt{1 - \frac{c_R^2}{c_2^2}} - \left(2 - \frac{c_R^2}{c_2^2}\right)^2 = 0$$

and, thus, gives the dependence between c_R and the Poisson ratio, which is coming from the bulk wave speeds c_1 and c_2.

Here c_R denotes the Rayleigh wave speed, c_1 and c_2 are the longitudinal and shear wave speeds, respectively.

At the surface $y = 0$ we have the following boundary conditions

$$\phi_{xx}(x, 0, t) - c_R^{-2}\phi_{tt}(x, 0, t) = AP_0\delta\left(x - \frac{1}{2}at^2\right) \tag{3.5}$$

and

$$-\frac{2}{1 + k_2^2}\phi_y(x, 0, t) = \psi_x(x, 0, t). \tag{3.6}$$

Here t is a time, δ is the Dirac function, A is a material constant

$$A = -\frac{k_1 k_2 (1 + k_2^2)}{2\mu[k_2(1 - k_1^2) + k_1(1 - k_2^2) - k_1 k_2 (1 - k_2^4)]}, \tag{3.7}$$

where μ is the Lamé elastic modulus. The equation (3.5) is a hyperbolic equation for an elastic potential ϕ for $y = 0$, equation (3.6) gives a differential relation between both potentials. Later we assume the equation (3.5) subject to homogeneous initial conditions.

So, in this chapter we consider the problem (3.2) with the boundary conditions (3.5) and (3.6), find an explicit analytic solution and provide the further analysis of the solution by creating the simpler asymptotic forms.

3.2 Boundary conditions. An infinite string under the moving load.

In this section we analyze the hyperbolic problem on the surface. In order to do it we need to solve the equation:

$$f_{xx} - c_R^{-2} f_{tt} = A P_0 \delta\left(x - \frac{1}{2} a t^2\right),$$

or, equivalently,

$$f_{tt} = c_R^2 f_{xx} - A P_0 c_R^2 \delta\left(x - \frac{1}{2} a t^2\right) \tag{3.8}$$

with homogeneous initial conditions. This is an inhomogeneous second order partial differential equation. It describes the behaviour of an infinite string under the moving point load. All the parameters are given in Section 3.1.

To solve the equation (3.8) consider the corresponding homogeneous one (details can be found in [75])

$$f_{tt} = c_R^2 f_{xx}$$

with the following conditions

$$f\mid_{t=\tau} = 0, \quad f_t\mid_{t=\tau} = -A P_0 c_R^2 \delta\left(x - \frac{a\tau^2}{2}\right).$$

The solution to the homogeneous equation has a form

$$f(x,t) = -\frac{AP_0c_R}{2}\int\limits_0^t\left[\int\limits_{x-c_R(t-\tau)}^{x+c_R(t-\tau)}\delta\left(\xi-\frac{a\tau^2}{2}\right)\mathrm{d}\xi\right]\mathrm{d}\tau.$$

The result of the inner integration is the Heaviside function denoted as H. Hence,

$$f(x,t) = -\frac{AP_0c_R}{2}\int\limits_0^t\left[H\left(x+c_R(t-\tau)-\frac{a\tau^2}{2}\right)-H\left(x-c_R(t-\tau)-\frac{a\tau^2}{2}\right)\right]\mathrm{d}\tau.$$

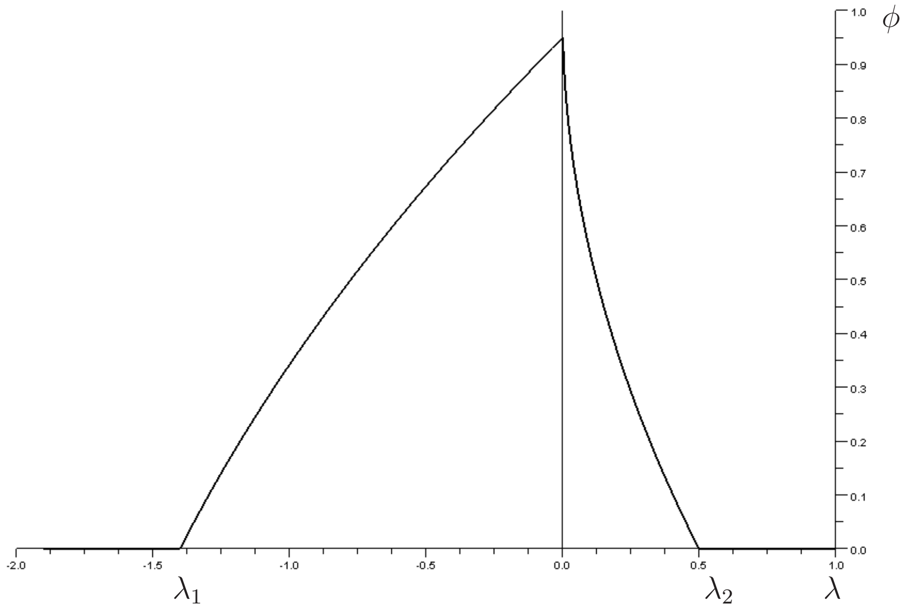


FIGURE 3.2: Boundary condition for $v = 0.95$ ($\lambda_1 = -1.40125$, $\lambda_2 = 0.49875$)

Introducing a moving coordinate system: $(s,t) = \left(x - \frac{at^2}{2}, t\right)$ and changing variable $r = t - \tau$ give

$$\begin{aligned} f(s,t) = \frac{AP_0c_R}{2}\int\limits_0^t\bigg[&H\left(s+r(at+c_R)-\frac{ar^2}{2}\right)-\\ &-H\left(s+r(at-c_R)-\frac{ar^2}{2}\right)\bigg]\mathrm{d}r. \end{aligned} \tag{3.9}$$

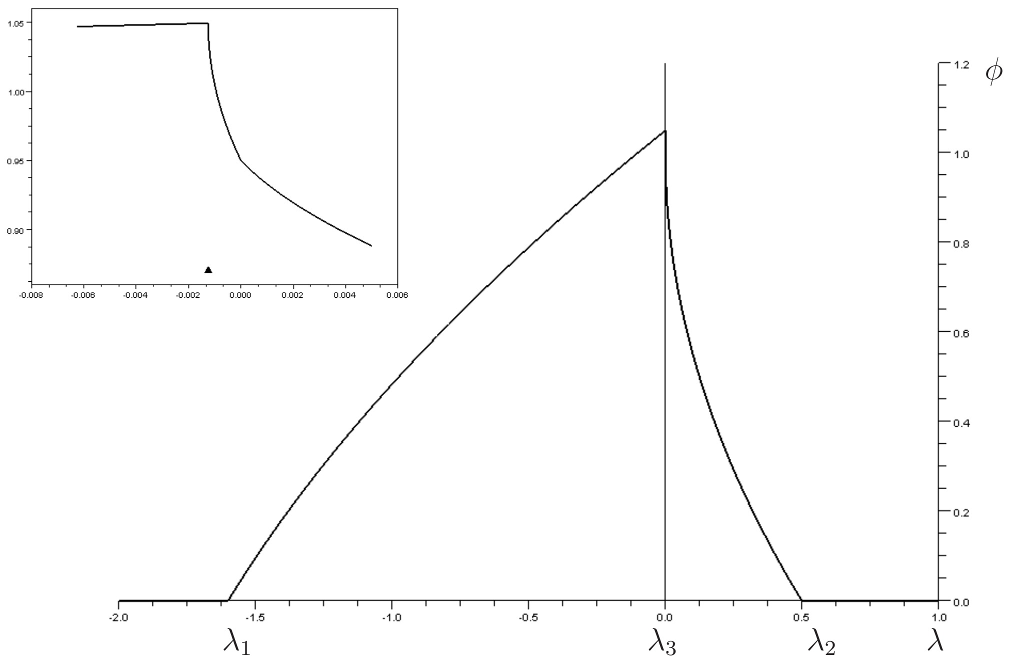


FIGURE 3.3: Boundary condition for $v = 1.05$ ($\lambda_1 = -1.60125$, $\lambda_3 = -0.00125$, $\lambda_2 = 0.49875$). On the insert you can see magnified vicinity of the shock wave, shown as black triangle

The arguments of the Heaviside functions in the integrand are the quadratic polynomials of r. Their roots are

$$\begin{aligned} r_1^{1,2} &= \frac{(at \pm c_R) + \sqrt{(at \pm c_R)^2 + 2as}}{a}, \\ r_2^{1,2} &= \frac{(at \pm c_R) - \sqrt{(at \pm c_R)^2 + 2as}}{a}, \end{aligned} \tag{3.10}$$

where numbers 1 and 2 in superscripts correspond to the first and second Heaviside functions in (3.9) and, consequently, plus and minus signs respectively.

In order to resolve the integral one should define the intervals inside $[0, t]$, where the Heaviside functions are not equal to zero. Only these intervals bring a non-trivial contribution to the result.

To reduce the number of parameters and simplify the further calculations it is useful to make (3.10) non-dimensional by introducing new variables

$$v = t\frac{a}{c_R} \text{ and } \lambda = \frac{as}{c_R^2}.$$

In this case the roots (3.10) can be written as

$$\rho_1^{1,2} = (v \pm 1) + \sqrt{(v \pm 1)^2 + 2\lambda},$$
$$\rho_2^{1,2} = (v \pm 1) - \sqrt{(v \pm 1)^2 + 2\lambda},$$

where upper indices 1 and 2 correspond to the plus and minus signs respectively. Note that $\rho_2^{1,2} < \rho_1^{1,2}$.

To define the intervals mentioned above it is sufficient to consider the following cases

(i) $0 < \rho_2^{1,2} < v$; $0 < \rho_1^{1,2} < v$: the length of the interval is $\rho_1^{1,2} - \rho_2^{1,2}$.

(ii) $\rho_2^{1,2} < 0$; $0 < \rho_1^{1,2} < v$: the length of the interval is $\rho_1^{1,2}$.

(iii) $0 < \rho_2^{1,2} < v$; $\rho_1^{1,2} > v$: the length of the interval is $v - \rho_2^{1,2}$.

(iv) $\rho_2^{1,2} < 0$; $\rho_1^{1,2} > v$: the length of the interval is v.

These cases should be considered for $v \leq 1$, $1 \leq v \leq 2$ and $v \geq 2$.

To classify the roots we use the notation

$$\lambda_1 = \frac{1-(1+v)^2}{2}, \quad \lambda_2 = \frac{1-(1-v)^2}{2}, \quad \lambda_3 = -\frac{(1-v)^2}{2},$$

$$s_1 = \frac{c_R^2 - (c_R + at)^2}{2a}, \quad s_2 = \frac{c_R^2 - (c_R - at)^2}{2a}, \quad s_3 = -\frac{(c_R - at)^2}{2a}.$$

The results (for non-dimensional variables along with dimensional ones) obtained via straightforward calculations are presented in the Table 3.1.

Remark: Further on in this thesis we consider only the data displayed in the first and second columns of Table 3.1 (see Figures 3.2 and 3.3 respectively), because our main object of interest is the vicinity of the transition through the Rayleigh speed. Note also that the model, described in this work, does not involve the passage through the sonic speed. So, the results obtained within the framework of this model do not describe adequately the real behaviour of a half plane in case when the velocity of the load is greater than the sound wave speed.

$t \le \frac{c_R}{a}$ $(v \le 1)$	$\frac{c_R}{a} \le t \le \frac{2c_R}{a}$ $(1 \le v \le 2)$	$t \ge \frac{2c_R}{a}$ $(v \ge 2)$
For $s_1 \le s \le 0$ $(\lambda_1 \le \lambda \le 0)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_2^1) = \frac{AP_0c_R}{2}\left(t-r_2^1\right)$	For $0 \le s \le s_2$ $(0 \le \lambda \le \lambda_2)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_1^2) = \frac{AP_0c_R}{2}(t-r_1^2)$	For $s_2 \le s \le 0$ $(\lambda_1 \le \lambda \le \lambda_3)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_2^1) = \frac{AP_0c_R}{2}(r_2^2-r_2^1)$
For $0 \le s \le s_2$ $(0 \le \lambda \le \lambda_2)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_1^2) = \frac{AP_0c_R}{2}\left(t-r_1^2\right)$	For $s_3 \le s \le 0$ $(\lambda_3 \le \lambda \le 0)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_1^2-(\rho_2^1-\rho_2^2)) = \frac{AP_0c_R}{2}(t-r_1^2+r_2^2-r_2^1)$	For $s_3 \le s \le s_2$ $(\lambda_3 \le \lambda \le \lambda_2)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_2^1-(\rho_1^2-\rho_2^2)) = \frac{AP_0c_R}{2}(t-r_1^2+r_2^2-r_2^1)$
	For $s_1 \le s \le s_3$ $(\lambda_1 \le \lambda \le \lambda_3)$: $f = \frac{AP_0c_R^2}{2a}(v-\rho_2^1) = \frac{AP_0c_R}{2}(t-r_2^1)$	For $s_1 \le s \le s_3$ $(\lambda_2 \le \lambda \le 0)$: $f = \frac{AP_0c_R^2}{2a}(\rho_2^2-\rho_2^1) = \frac{AP_0c_R}{2}(r_2^2-r_2^1)$

TABLE 3.1: Classification of the roots

3.3 The exact solution over the interior

First of all, it is necessary to state again, that the term "exact solution" does not mean exact solution within the framework of the general theory of elasticity, but only the solution within the framework of the asymptotic model (see [61]) without any additional assumptions and approximations.

This section is dedicated to the calculation of the displacements (3.1). In order to do that we need to find the Lamé potentials ϕ and ψ by resolving the equations (3.2) with the boundary conditions (3.5) and (3.6).

Two speed intervals $at \leq c_R$ and $at \geq c_R$ are considered separately and the corresponding results are given below.

3.3.1 Before the passage

Start with the equation (3.2) for the potential ϕ, rewritten in the variables y and s:

$$\phi_{yy} + k_1^2 \phi_{ss} = 0 \tag{3.11}$$

with the boundary condition, obtained in Section 3.2

$$\phi(s,0,t) = f(s,t) = \begin{cases} \dfrac{AP_0 c_R}{2} \left(t - r_2^1\right), & \text{for } s_1 \leq s \leq 0; \\ \dfrac{AP_0 c_R}{2} \left(t - r_1^2\right), & \text{for } 0 \leq s \leq s_2; \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

The associated fundamental solution $\Phi(x,y)$ of the boundary-value problem

$$\Phi_{yy} + k_1^2 \Phi_{xx} = 0, \quad \Phi(x,0) = \delta(x)$$

is

$$\Phi(x,y) = \frac{k_1 y}{\pi(x^2 + (k_1 y)^2)}. \tag{3.13}$$

Then the solution for the potential $\phi(s,y,t)$ can be found as a convolution of (3.13) and (3.12) (details can be found in [75]):

$$\phi(s,y,t) = \int_{-\infty}^{\infty} \phi(r,0,t)\Phi(s-r,y)\mathrm{d}r = \frac{1}{\pi}\int_{s_1}^{s_2} \frac{k_1 y}{(r-s)^2 + k_1^2 y^2}\phi(r,0,t)\mathrm{d}r$$

Integration by parts gives

$$\phi(s,y,t) = -\frac{1}{\pi}\int_{s_1}^{s_2} \arctan\left(\frac{r-s}{k_1 y}\right)\frac{\partial\phi(r,0,t)}{\partial r}\mathrm{d}r. \tag{3.14}$$

Partial derivative of the potential $\phi(r,y,t)$ with respect to r at $y=0$ is

$$\frac{\partial\phi}{\partial r}(r,0,t) = \begin{cases} \frac{AP_0 c_R}{2}\frac{1}{\sqrt{(c_R+at)^2+2ar}}, & \text{for } s_1 < r < 0; \\ -\frac{AP_0 c_R}{2}\frac{1}{\sqrt{(c_R-at)^2+2ar}}, & \text{for } 0 < r < s_2. \end{cases} \tag{3.15}$$

Since the boundary condition (3.12) and, consequently, the partial derivative (3.15) are different for the intervals $s_1 \le s \le 0$ and $0 \le s \le s_2$, then the integral (3.14) can be written as:

$$\phi(s,y,t) = -\frac{AP_0 c_R}{2\pi}\left(I_1 - I_2\right),$$

where

$$I_1 = \int_{s_1}^{0} \frac{\arctan\left(\frac{r-s}{k_1 y}\right)}{\sqrt{(c_R+at)^2+2ar}}\mathrm{d}r; \; I_2 = \int_{0}^{s_2} \frac{\arctan\left(\frac{r-s}{k_1 y}\right)}{\sqrt{(c_R-at)^2+2ar}}\mathrm{d}r. \tag{3.16}$$

First, we calculate the integral I_1. After an elementary transformation it can be written in the form

$$I_1 = \frac{1}{\sqrt{2a}}\int_{s_1}^{0} \frac{\arctan\left(\frac{r-s}{k_1 y}\right)}{\sqrt{r + \left(\frac{c_R+at}{\sqrt{2a}}\right)^2}}\mathrm{d}r.$$

Applying integration by parts with

$$u = \arctan\left(\frac{r-s}{k_1 y}\right), \qquad \mathrm{d}u = \frac{1}{k_1 y}\,\frac{1}{1+\left(\frac{r-s}{k_1 y}\right)^2}\mathrm{d}r;$$

$$v = 2\sqrt{r+\left(\frac{c_R+at}{\sqrt{2a}}\right)^2}, \qquad \mathrm{d}v = \frac{1}{\sqrt{r+\left(\frac{c_R+at}{\sqrt{2a}}\right)^2}}\mathrm{d}r,$$

one can obtain

$$I_1 = \frac{1}{\sqrt{2a}}\left[2\sqrt{r+\left(\frac{c_R+at}{\sqrt{2a}}\right)^2}\arctan\left(\frac{r-s}{k_1 y}\right)\right]_{s_1}^{0} - \tilde{I},$$

where

$$\tilde{I} = \frac{1}{k_1 y}\int_{s_1}^{0}\frac{2\sqrt{r+\left(\frac{c_R+at}{\sqrt{2a}}\right)^2}}{1+\left(\frac{r-s}{k_1 y}\right)^2}\mathrm{d}r.$$

To simplify the calculation of $\tilde{I}$ let us introduce the change of variables

$$z = \sqrt{r+\left(\frac{c_R+at}{\sqrt{2a}}\right)^2}.$$

Then,

$$r = z^2 - \left(\frac{c_R+at}{\sqrt{2a}}\right)^2, \ \mathrm{d}r = 2z\mathrm{d}z,$$

and the limits of integration are:

$$\frac{c_R}{\sqrt{2a}}, \quad \frac{c_R+at}{\sqrt{2a}}.$$

Hence,

$$\tilde{I} = k_1 y \int_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}}\frac{4z^2\mathrm{d}z}{z^4-2z^2b_1+b_1^2+(k_1y)^2},$$

where $b_1 = \dfrac{(c_R+at)^2}{2a} + s.$

Factorizing the denominator and applying partial fractions to the integrand, one can get:

$$\tilde{I} = k_1 y \int\limits_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} \left[\frac{\frac{2z}{\sqrt{2b_1+2\alpha_1}}}{z^2 - \sqrt{2b_1+2\alpha_1} z + \alpha_1} - \frac{\frac{2z}{\sqrt{2b_1+2\alpha_1}}}{z^2 + \sqrt{2b_1+2\alpha_1} z + \alpha_1} \right] \mathrm{d}z,$$

where $\alpha_1 = \sqrt{b_1^2 + (k_1 y)^2}$.

The further transformations of $\tilde{I}$ are

$$\begin{aligned}
\tilde{I} &= \frac{k_1 y}{\sqrt{2b_1+2\alpha_1}} \left[\int\limits_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} \frac{2z - \sqrt{2b_1+2\alpha_1} + \sqrt{2b_1+2\alpha_1}}{z^2 - \sqrt{2b_1+2\alpha_1} z + \alpha_1} \mathrm{d}z - \right. \\
&\left. - \int\limits_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} \frac{2z + \sqrt{2b_1+2\alpha_1} - \sqrt{2b_1+2\alpha_1}}{z^2 + \sqrt{2b_1+2\alpha_1} z + \alpha_1} \mathrm{d}z \right] = \\
&= \frac{k_1 y}{\sqrt{2b_1+2\alpha_1}} \ln \frac{z^2 - \sqrt{2b_1+2\alpha_1} z + \alpha_1}{z^2 + \sqrt{2b_1+2\alpha_1} z + \alpha_1} \Bigg|_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} + \\
&+ k_1 y \int\limits_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} \frac{\mathrm{d}z}{\left(z - \frac{\sqrt{2b_1+2\alpha_1}}{2}\right)^2 + \left(\alpha_1 - \frac{2b_1+2\alpha_1}{4}\right)} + \\
&+ k_1 y \int\limits_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} \frac{\mathrm{d}z}{\left(z + \frac{\sqrt{2b_1+2\alpha_1}}{2}\right)^2 + \left(\alpha_1 - \frac{2b_1+2\alpha_1}{4}\right)} =
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{k_1 y}{\sqrt{2b_1+2\alpha_1}} \ln \frac{z^2 - \sqrt{2b_1+2\alpha_1} z + \alpha_1}{z^2 + \sqrt{2b_1+2\alpha_1} z + \alpha_1} + \right. \\
&+ \left. \frac{2k_1 y}{\sqrt{2\alpha_1 - 2b_1}} \left(\arctan \frac{z - \frac{\sqrt{2b_1+2\alpha_1}}{2}}{\frac{\sqrt{2\alpha_1-2b_1}}{2}} + \arctan \frac{z + \frac{\sqrt{2b_1+2\alpha_1}}{2}}{\frac{\sqrt{2\alpha_1-2b_1}}{2}} \right) \right] \Bigg|_{\frac{c_R}{\sqrt{2a}}}^{\frac{c_R+at}{\sqrt{2a}}} .
\end{aligned}$$

Finally,

$$I_1 = \frac{1}{\sqrt{2a}} [\xi_1(s, y, z_2) - \xi_1(s, y, z_1)],$$

where

$$\xi_1(s,y,z) = 2z \arctan \frac{z^2 - b_1}{k_1 y} - \frac{k_1 y}{\beta_1} \ln \frac{z^2 - \beta_1 z + \alpha_1}{z^2 + \beta_1 z + \alpha_1} - \\ - \frac{2k_1 y}{\gamma_1} \left[\arctan \frac{2z - \beta_1}{\gamma_1} + \arctan \frac{2z + \beta_1}{\gamma_1} \right], \tag{3.17}$$

$$z_1 = \frac{c_R}{\sqrt{2a}}, \quad z_2 = \frac{c_R + at}{\sqrt{2a}}$$

and

$$b_1 = \frac{(at + c_R)^2}{2a} + s, \quad \alpha_1 = \sqrt{b_1^2 + (k_1 y)^2}, \\ \beta_1 = \sqrt{2\alpha_1 + 2b_1}, \quad \gamma_1 = \sqrt{2\alpha_1 - 2b_1}. \tag{3.18}$$

The integral I_2 can be calculated in exactly the same way. The result is:

$$I_2 = \frac{1}{\sqrt{2a}} [\xi_2(s,y,z_4) - \xi_2(s,y,z_3)],$$

where

$$\xi_2(s,y,z) = 2z \arctan \frac{z^2 - b_2}{k_1 y} - \frac{k_1 y}{\beta_2} \ln \frac{z^2 - \beta_2 z + \alpha_2}{z^2 + \beta_2 z + \alpha_2} - \\ - \frac{2k_1 y}{\gamma_2} \left[\arctan \frac{2z - \beta_2}{\gamma_2} + \arctan \frac{2z + \beta_2}{\gamma_2} \right], \tag{3.19}$$

$$z_3 = \frac{c_R - at}{\sqrt{2a}}, \quad z_4 = \frac{c_R}{\sqrt{2a}},$$

and

$$b_2 = \frac{(c_R - at)^2}{2a} + s, \quad \alpha_2 = \sqrt{b_2^2 + (k_1 y)^2}, \\ \beta_2 = \sqrt{2b_2 + 2\alpha_2}, \quad \gamma_2 = \sqrt{2\alpha_2 - 2b_2}. \tag{3.20}$$

Hence, the potential $\phi(s,y,t)$ for $c_R \geq at$ is

$$\phi(s,y,t) = \frac{AP_0 c_R}{2\pi\sqrt{2a}} [\xi_1(s,y,z_2) - \xi_1(s,y,z_1) + \xi_2(s,y,z_3) - \xi_2(s,y,z_4)]. \tag{3.21}$$

To calculate the potential ψ one needs to solve the second equation in (3.2) with the boundary condition (3.6). In order to use this boundary condition, the partial derivative $\dfrac{\partial \phi}{\partial y}$ should be calculated. To simplify this operation, one can use the notation:

$$b_i = \frac{1}{2a} b_i', \quad \alpha_i = \frac{1}{2a}\alpha_i', \quad \beta_i = \frac{1}{\sqrt{2a}}\beta_i', \quad \gamma_i = \frac{1}{\sqrt{2a}}\gamma_i', \quad i = 1, 2,$$

$$z_j = \frac{1}{\sqrt{2a}} z_j', \quad j = 1, 2, 3, 4,$$

$$\xi_i(s, y, z) = \sqrt{2a}\,\xi_i'(s, y, z), \quad i = 1, 2,$$

where

$$b_i' = (c_R \pm at)^2 + 2as, \quad \alpha_i' = \sqrt{(b_1')^2 + (2ak_1 y)^2}, \quad i = 1, 2,$$

$$\beta_i' = \sqrt{2(b_1' + \alpha_1')}, \quad \gamma_i' = \sqrt{2(\alpha_1' - b_1')}, \quad i = 1, 2,$$

$$z_1' = z_4' = c_R, \quad z_2' = (c_R + at), \quad z_3' = (c_R - at)$$

and

$$\begin{aligned}\xi_i'(s, y, z') = {} & \frac{1}{a} z' \arctan \frac{(z')^2 - b_i'}{2ak_1 y} - \frac{k_1 y}{\beta_i'} \ln \frac{(z')^2 - \beta_i' z' + \alpha_i'}{(z')^2 + \beta_i' z' + \alpha_i'} - \\ & - \frac{2k_1 y}{\gamma_i'} \left[\arctan \frac{2z' - \beta_i'}{\gamma_i'} + \arctan \frac{2z' + \beta_i'}{\gamma_i'} \right], \quad i = 1, 2.\end{aligned}$$

Here and below we use the notation without dashes but keep in mind that all the parameters are defined as the "dashed" ones above.

Partial derivative of the potential ϕ can be obtained straightforward. Performing the massive (but relatively simple) calculations one can get:

$$\frac{\partial \phi}{\partial y}(s, y, t) = \frac{\vartheta}{\pi} \left[\frac{\partial \xi_1}{\partial y}(z = c_R + at) - \frac{\partial \xi_1}{\partial y}(z = c_R) + \frac{\partial \xi_2}{\partial y}(z = c_R - at) - \frac{\partial \xi_2}{\partial y}(z = c_R) \right].$$

where

$$\begin{aligned}\frac{\partial \xi_i}{\partial y} &= -k_1 \left[\left(\frac{1}{\beta_i} - \frac{f^2 y^2}{\alpha_i \beta_i^3} \right) \ln \frac{z^2 - \beta_i z + \alpha_i}{z^2 + \beta_i z + \alpha_i} + \right. \\ &+ \left. 2 \left(\frac{1}{\gamma_i} - \frac{f^2 y^2}{\alpha_i \gamma_i^3} \right) \left(\arctan \frac{2z - \beta_i}{\gamma_i} + \arctan \frac{2z + \beta_i}{\gamma_i} \right) \right]. \end{aligned} \tag{3.22}$$

Here and below $f = 2ak_1$, $\vartheta = \frac{AP_0 c_R}{2}$.

For the sake of simplicity one can rewrite (3.22) as

$$\frac{\partial \xi_i}{\partial y} = -k_1 \left[\frac{\beta_i}{4\alpha_i} \ln \frac{z^2 - \beta_i z + \alpha_i}{z^2 + \beta_i z + \alpha_i} + \frac{\gamma_i}{2\alpha_i} \left(\arctan \frac{2z - \beta_i}{\gamma_i} + \arctan \frac{2z + \beta_i}{\gamma_i} \right) \right],$$

using the equalities

$$\frac{1}{\beta_i} - \frac{f^2 y^2}{\alpha_i \beta_i^3} = \frac{\beta_i}{4\alpha_i}, \quad \frac{1}{\gamma_i} - \frac{f^2 y^2}{\alpha_i \gamma_i^3} = \frac{\gamma_i}{4\alpha_i}.$$

Note that both equations in (3.2) have the same form, the only difference is in the coefficient for $\frac{\partial^2}{\partial s^2}$. Hence, using the boundary condition (3.6), we can state that the partial derivatives $\frac{\partial \psi}{\partial s}(s, y, t)$ and $\frac{\partial \phi}{\partial y}(s, y, t)$ are different only in a constant coefficient $-\frac{2}{1+k_2^2}$. So, partial derivative of the potential ψ is

$$\begin{aligned} \frac{\partial \psi}{\partial s}(s, y, t) &= \frac{2\vartheta k_1}{(1 + k_2^2)\pi} \left[\frac{\partial \zeta_1}{\partial s}(z = c_R + at) - \frac{\partial \zeta_1}{\partial s}(z = c_R) + \right. \\ &+ \left. \frac{\partial \zeta_2}{\partial s}(z = c_R - at) - \frac{\partial \zeta_2}{\partial s}(z = c_R) \right], \end{aligned}$$

where

$$\frac{\partial \zeta_i}{\partial s} = \frac{\tilde{\beta}_i}{4\tilde{\alpha}_i} \ln \frac{z^2 - \tilde{\beta}_i z + \tilde{\alpha}_i}{z^2 + \tilde{\beta}_i z + \tilde{\alpha}_i} + \frac{\tilde{\gamma}_i}{2\alpha_i} \left[\arctan \frac{2z - \tilde{\beta}_i}{\tilde{\gamma}_i} + \arctan \frac{2z + \tilde{\beta}_i}{\tilde{\gamma}_i} \right]. \quad (3.23)$$

Here and below $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\gamma}_i$ are the same as α_i, β_i, γ_i, but with k_2 instead of k_1 in their definitions. This distinction comes from the different coefficients in the equations (3.2).

Note that

$$\frac{\partial}{\partial s} \left(\frac{\tilde{\beta}_i}{4a} \right) = \frac{\tilde{\beta}_i}{4\tilde{\alpha}_i}, \quad \frac{\partial}{\partial s} \left(\frac{\tilde{\gamma}_i}{2a} \right) = -\frac{\tilde{\gamma}_i}{2\tilde{\alpha}_i},$$

hence, by integrating (3.23), one can obtain:

$$\zeta_i = \frac{\tilde{\beta}_i}{4a} \ln \frac{z^2 - \tilde{\beta}_i z + \tilde{\alpha}_i}{z^2 + \tilde{\beta}_i z + \tilde{\alpha}_i} - \frac{\tilde{\gamma}_i}{2a} \left[\arctan \frac{2z - \tilde{\beta}_i}{\tilde{\gamma}_i} + \arctan \frac{2z + \tilde{\beta}_i}{\tilde{\gamma}_i} \right] + F. \quad (3.24)$$

To complete the calculation of the potential ψ, it remains to find explicitly the function F in (3.24). If one compares the formulae (3.23) and (3.24) it becomes clear that the partial derivative of F has a form:

$$\frac{\partial F}{\partial s} = -\frac{2z(b_i - z^2)}{(b_i - z^2)^2 + f^2y^2},$$

consequently,

$$F = -\frac{z}{2a} \ln \frac{(b_i - z^2)^2 + f^2y^2}{B}.$$

The quantity B is an arbitrary element and does not depend on s. Since the argument of the logarithm should be non-dimensional, one can put $B = c_R^4$. Then,

$$F = -\frac{z}{2a} \ln \frac{(b_i - z^2)^2 + f^2y^2}{c_R^4}$$

and the formula (3.24) transforms to

$$\zeta_i = \frac{\tilde{\beta}_i}{4a} \ln \frac{z^2 - \tilde{\beta}_i z + \tilde{\alpha}_i}{z^2 + \tilde{\beta}_i z + \tilde{\alpha}_i} - \frac{\tilde{\gamma}_i}{2a} \left[\arctan \frac{2z - \tilde{\beta}_i}{\tilde{\gamma}_i} + \arctan \frac{2z + \tilde{\beta}_i}{\tilde{\gamma}_i} \right] - \\ - \frac{z}{2a} \ln \frac{(b_i - z^2)^2 + f^2y^2}{c_R^4}. \quad (3.25)$$

With respect to (3.25), the potential ψ has a form

$$\begin{aligned} \psi(s, y, t) &= \frac{2AP_0c_R}{2\pi} \frac{k_1}{1 + k_2^2} \left[\zeta_1(z = c_R + at) - \zeta_1(z = c_R) + \right. \\ &+ \left. \zeta_2(z = c_R - at) - \zeta_2(z = c_R) \right]. \end{aligned} \quad (3.26)$$

It remains to calculate explicitly the displacements (3.1). We remind that $(s, t) = (x - \frac{1}{2}at^2, t)$, it means that the partial derivative with respect to s can be used instead of $\dfrac{\partial}{\partial x}$.

So, the derivatives of the potentials ϕ and ψ are

$$\begin{aligned} \frac{\partial \phi}{\partial s} &= \frac{AP_0c_R}{2\pi} \left[\frac{\partial \xi_1}{\partial s}(z = c_R + at) - \frac{\partial \xi_1}{\partial s}(z = c_R) + \right. \\ &+ \left. \frac{\partial \xi_2}{\partial s}(z = c_R - at) - \frac{\partial \xi_2}{\partial s}(z = c_R) \right], \end{aligned} \quad (3.27)$$

where

$$\frac{\partial \xi_i}{\partial s} = \frac{ak_1y}{\alpha_i\beta_i} \ln \frac{z^2 - \beta_i z + \alpha_i}{z^2 + \beta_i z + \alpha_i} - \frac{2ak_1y}{\alpha_i\gamma_i} \left[\arctan \frac{2z - \beta_i}{\gamma_i} + \arctan \frac{2z + \beta_i}{\gamma_i} \right];$$

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{AP_0c_R}{2\pi}\left[\frac{\partial\xi_1}{\partial y}(z=c_R+at)-\frac{\partial\xi_1}{\partial y}(z=c_R)+\right.\\ &+ \left.\frac{\partial\xi_2}{\partial y}(z=c_R-at)-\frac{\partial\xi_2}{\partial y}(z=c_R)\right], \end{aligned}\tag{3.28}$$

where

$$\frac{\partial\xi_i}{\partial y}=-k_1\left[\frac{\beta_i}{4\alpha_i}\ln\frac{z^2-\beta_i z+\alpha_i}{z^2+\beta_i z+\alpha_i}+\frac{\gamma_i}{2\alpha_i}\left(\arctan\frac{2z-\beta_i}{\gamma_i}+\arctan\frac{2z+\beta_i}{\gamma_i}\right)\right];$$

$$\begin{aligned}\frac{\partial\psi}{\partial s} &= \frac{2AP_0c_R}{2\pi}\frac{k_1}{1+k_2^2}\left[\frac{\partial\zeta_1}{\partial s}(z=c_R+at)-\frac{\partial\zeta_1}{\partial s}(z=c_R)+\right.\\ &+ \left.\frac{\partial\zeta_2}{\partial s}(z=c_R-at)-\frac{\partial\zeta_2}{\partial y}(z=c_R)\right], \end{aligned}\tag{3.29}$$

where

$$\frac{\partial\zeta_i}{\partial s}=\frac{\tilde\beta_i}{4\tilde\alpha_i}\ln\frac{z^2-\tilde\beta_i z+\tilde\alpha_i}{z^2+\tilde\beta_i z+\tilde\alpha_i}+\frac{\tilde\gamma_i}{2\tilde\alpha_i}\left[\arctan\frac{2z-\tilde\beta_i}{\tilde\gamma_i}+\arctan\frac{2z+\tilde\beta_i}{\tilde\gamma_i}\right];$$

$$\begin{aligned}\frac{\partial\psi}{\partial y} &= \frac{AP_0c_R}{2\pi}\frac{k_1}{1+k_2^2}\left[\frac{\partial\zeta_1}{\partial y}(z=c_R+at)-\frac{\partial\zeta_1}{\partial y}(z=c_R)+\right.\\ &+ \left.\frac{\partial\zeta_2}{\partial y}(z=c_R-at)-\frac{\partial\zeta_2}{\partial y}(z=c_R)\right], \end{aligned}\tag{3.30}$$

where

$$\frac{\partial\zeta_i}{\partial y}=\frac{f^2y}{4a\tilde\alpha_i\tilde\beta_i}\ln\frac{z^2-\tilde\beta_i z+\tilde\alpha_i}{z^2+\tilde\beta_i z+\tilde\alpha_i}-\frac{f^2y}{2a\tilde\alpha_i\tilde\gamma_i}\left[\arctan\frac{2z-\tilde\beta_i}{\tilde\gamma_i}+\arctan\frac{2z+\tilde\beta_i}{\tilde\gamma_i}\right].$$

The last step is to substitute the formulae (3.27)–(3.30) into (3.1). The result is:

$$\begin{aligned}u_1=\frac{\partial\phi}{\partial s}-\frac{\partial\psi}{\partial y} &= \frac{AP_0c_R}{2\pi}\left[\frac{\partial\xi_1}{\partial s}(z=c_R+at)-\frac{\partial\xi_1}{\partial s}(z=c_R)+\right.\\ &+ \left.\frac{\partial\xi_2}{\partial s}(z=c_R-at)-\frac{\partial\xi_2}{\partial s}(z=c_R)\right]-\\ &- \frac{2k_1}{1+k_2^2}\left[\frac{\partial\zeta_1}{\partial y}(z=c_R+at)-\frac{\partial\zeta_1}{\partial y}(z=c_R)+\right.\\ &+ \left.\frac{\partial\zeta_2}{\partial y}(z=c_R-at)-\frac{\partial\zeta_2}{\partial y}(z=c_R)\right], \end{aligned}\tag{3.31}$$

$$\begin{aligned} u_2 &= \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial s} = \frac{AP_0 c_R}{2\pi}\left[\frac{\partial \xi_1}{\partial y}(z = c_R + at) - \frac{\partial \xi_1}{\partial y}(z = c_R) + \right.\\ &+ \left. \frac{\partial \xi_2}{\partial y}(z = c_R - at) - \frac{\partial \xi_2}{\partial y}(z = c_R)\right] + \\ &+ \frac{2k_1}{1+k_2^2}\left[\frac{\partial \zeta_1}{\partial s}(z = c_R + at) - \frac{\partial \zeta_1}{\partial s}(z = c_R) + \right.\\ &+ \left. \frac{\partial \zeta_2}{\partial s}(z = c_R - at) - \frac{\partial \zeta_2}{\partial s}(z = c_R)\right]. \end{aligned} \tag{3.32}$$

3.3.2 After the passage

This part of the thesis describes how to calculate the displacements (3.1) for the load speed greater than the Rayleigh speed. The procedure is very similar to the one given in the previous subsection. First, we calculate the potential ϕ, then the potential ψ and, finally, the displacements u_1 and u_2.

To calculate the potential ϕ we start with the same equation (3.11) as before but the boundary condition is different, namely,

$$\phi(s,0,t) = \begin{cases} \frac{AP_0 c_R}{2}(t - r_1^2), & \text{for } 0 \le s \le s_2; \\ \frac{AP_0 c_R}{2}(t - r_2^1 + r_2^2 - r_1^2), & \text{for } s_3 \le s \le 0; \\ \frac{AP_0 c_R}{2}(t - r_2^1), & \text{for } s_1 \le s \le s_3; \\ 0, & \text{otherwise.} \end{cases} \tag{3.33}$$

One can notice that the difference between the boundary conditions (3.12) and (3.33) is an additional contribution that appears in the interval $s_3 \le s \le 0$. So, it is clear that the potential ϕ is

$$\phi(s,y,t) = -\frac{AP_0 c_R}{2\pi}\left(I_1 - I_2 + I_3\right), \tag{3.34}$$

where I_1 and I_2 are the same as in formulae (3.16) in Section 3.3.1. Integral I_3 is the "additional contribution" mentioned above. It can be obtained in the same way as the integrals I_1 and I_2 in previous section and has the following form:

$$I_3 = -2\int_{s_3}^{0} \frac{1}{\sqrt{(at - c_R)^2 + 2ar}} \arctan\frac{r - s}{k_1 y}\mathrm{d}r, \tag{3.35}$$

where

$$s_3 = -\frac{(at - c_R)^2}{2a}.$$

Similar to the previous section, to simplify the calculations we introduce a change of variables:

$$z = \sqrt{r + \frac{(at - c_R)^2}{2a}}.$$

Hence, the integral (3.35) can be rewritten as

$$I_3 = -2 \int\limits_0^{\frac{at-c_R}{\sqrt{2a}}} \frac{1}{\sqrt{(at - c_R)^2 + 2ar}} \arctan \frac{r - s}{k_1 y} \mathrm{d}r.$$

Repeating the same steps as in Section 3.3.1 for the integral (3.35) one can obtain

$$I_3 = -\frac{1}{\sqrt{2a}}[2\xi_2(z = z_3) - \underbrace{2\xi_2(z = 0)}_{=0}] = -\frac{2}{\sqrt{2a}}\xi_2(z = z_3),$$

where ξ_2 is defined by the formula (3.19) and $z_3 = \dfrac{at - c_R}{\sqrt{2a}}$. To simplify the expression for I_3 the "dashed" parameters from the previous section can be used. As it was mentioned before dashes themselves are omitted. It makes the notation shorter and easier to read. After this operation I_3 becomes

$$I_3 = -2\xi_2(z = at - c_R). \tag{3.36}$$

The potential ϕ can be found by substituting the formulae (3.16) and (3.36) into (3.34). The result is

$$\begin{aligned} \phi(s, y, t) &= \frac{AP_0 c_R}{2\pi} \left[\xi_1(z = at + c_R) - \xi_1(z = c_R) - \right. \\ &- \left. \xi_2(z = at - c_R) - \xi_2(z = c_R)\right]. \end{aligned} \tag{3.37}$$

By analogy with Section 3.3.1 the potential ψ can be calculated by using the result for ϕ and the second equation in (3.2) with the boundary condition (3.6). Note that in the formulae (3.21) (before the Rayleigh speed) and (3.37) (after the Rayleigh speed) the expressions for the potential ϕ have the same structure.

Using this fact and the calculations for ψ from the previous section, we get

$$\begin{aligned}\psi(s,y,t) &= \frac{AP_0c_R}{2\pi}\frac{k_1}{1+k_2^2}\left[\zeta_1(z=at+c_R)-\zeta_1(z=c_R)-\right.\\ &- \left.\zeta_2(z=at-c_R)-\zeta_2(z=c_R)\right]. \end{aligned} \tag{3.38}$$

It only remains to find the displacements u_1 and u_2. To do it the derivatives ϕ_s, ϕ_y, ψ_s and ψ_y should be calculated and substituted into formulae (3.1). The result is

$$\begin{aligned} u_1=\frac{\partial\phi}{\partial s}-\frac{\partial\psi}{\partial y} &= \frac{AP_0c_R}{2\pi}\left[\frac{\partial\xi_1}{\partial s}(z=at+c_R)-\frac{\partial\xi_1}{\partial s}(z=c_R)-\right.\\ &- \left.\frac{\partial\xi_2}{\partial s}(z=at-c_R)-\frac{\partial\xi_2}{\partial s}(z=c_R)\right]-\\ &- \frac{2k_1}{1+k_2^2}\left[\frac{\partial\zeta_1}{\partial y}(z=at+c_R)-\frac{\partial\zeta_1}{\partial y}(z=c_R)-\right.\\ &- \left.\frac{\partial\zeta_2}{\partial y}(z=at-c_R)-\frac{\partial\zeta_2}{\partial y}(z=c_R)\right], \end{aligned} \tag{3.39}$$

$$\begin{aligned} u_2=\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial s} &= \frac{AP_0c_R}{2\pi}\left[\frac{\partial\xi_1}{\partial y}(z=at+c_R)-\frac{\partial\xi_1}{\partial y}(z=c_R)-\right.\\ &- \left.\frac{\partial\xi_2}{\partial y}(z=at-c_R)-\frac{\partial\xi_2}{\partial y}(z=c_R)\right]+\\ &+ \frac{2k_1}{1+k_2^2}\left[\frac{\partial\zeta_1}{\partial s}(z=at+c_R)-\frac{\partial\zeta_1}{\partial s}(z=c_R)-\right.\\ &- \left.\frac{\partial\zeta_2}{\partial s}(z=at-c_R)-\frac{\partial\zeta_2}{\partial s}(z=c_R)\right]. \end{aligned} \tag{3.40}$$

Let us mention a remarkable property of the expressions for the displacements u_1 and u_2. Since the derivatives of ξ_2 and ζ_2 are odd functions, we can state that the pairs of formulae (3.31), (3.32) and (3.39), (3.40) are equivalent. Moreover, they provide the same results at a critical point which is the Rayleigh speed. So, the solution of the main problem is smooth for all the considered speed intervals, including the passage through the critical speed.

The graphs given in Figures 3.4 and 3.5 display the horizontal and vertical displacements u_1 and u_2 under a moving point load for the different values of a dimensionless parameter $\hat{a}$ that represents a combination of an acceleration a and a depth y as $\hat{a}=\frac{ay}{c_R^2}$. As one can see the most dramatic transition effects

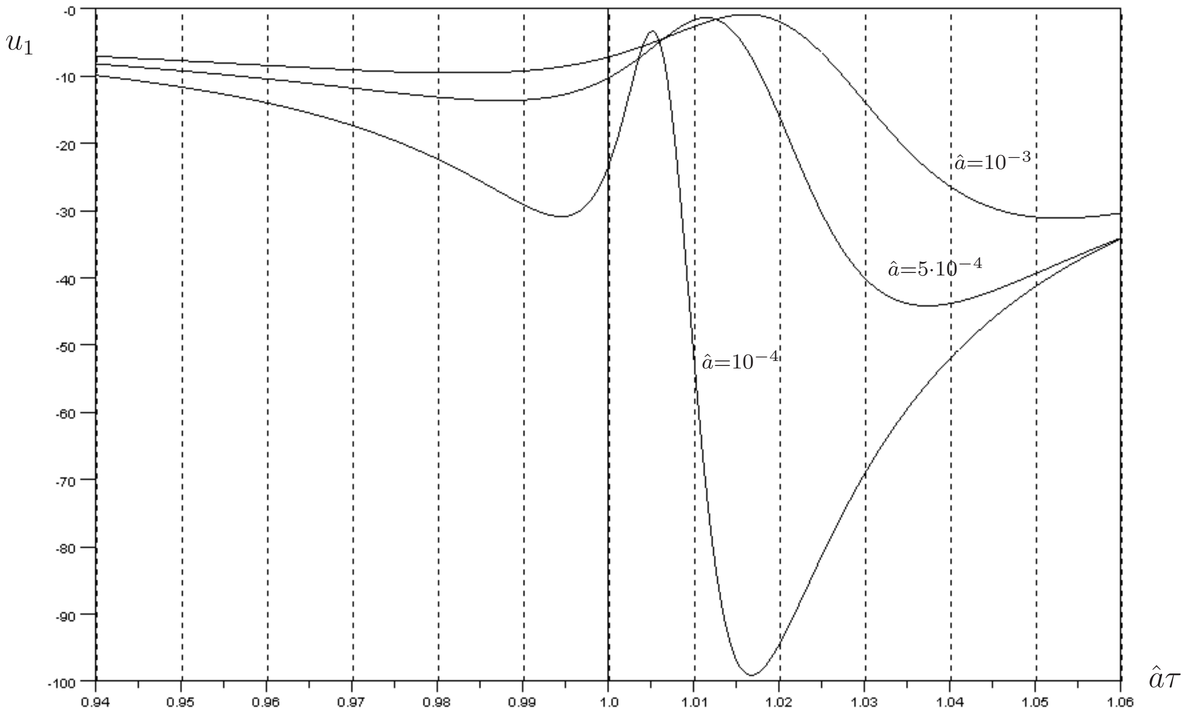


FIGURE 3.4: Dimensionless exact solution for horizontal displacement u_1 for various values of parameter $\hat{a}$

appear for the smallest value of the parameter. Clearly, the regions of the maximum effect (in particular, the peaks right after the critical speed) become more local as the parameter $\hat{a}$ decreases.

3.4 Asymptotic forms for the solution

In Section 3.3 the exact solution for the main problem was obtained. The explicit formulae for the displacements u_1 and u_2 contain many parameters and look massive. This section is dedicated to the asymptotic analysis of formulae (3.31)–(3.32) and (3.39)–(3.40).

To make the analysis of the expressions for the displacements u_1 and u_2 a bit simpler sometimes it is useful to introduce the following new parameters:

$$\sigma = \frac{s}{y}, \ \tau = \frac{tc_R}{y}, \ \hat{a} = \frac{ay}{c_R^2}.$$

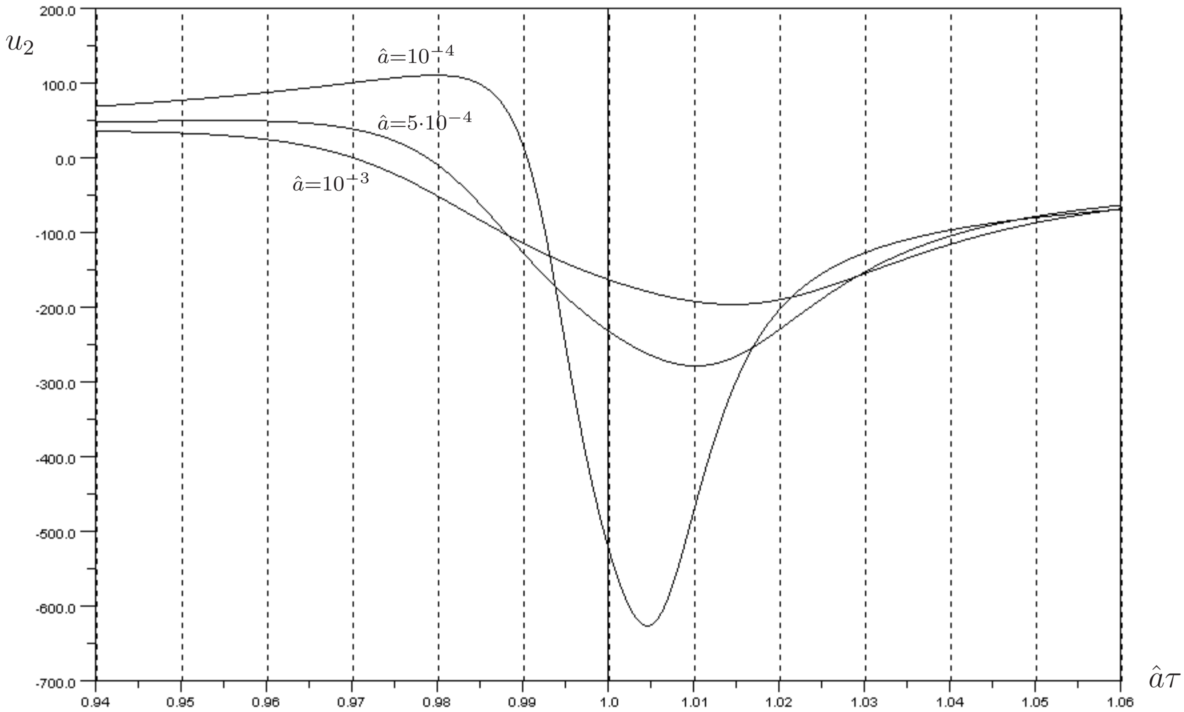


FIGURE 3.5: Dimensionless exact solution for vertical displacement u_2 for various values of parameter $\hat{a}$

Further on we assume that $y \neq 0$, so we consider the solution over the interior only.

The main reason for introducing these parameters is to make the solution non-dimensional. Let us note that the parameters a, s, t and y appear in the explicit formulae for u_1 and u_2 in the following "pairs": as, at and ay, which can be expressed in the language of the new parameters σ, τ and $\hat{a}$ as

$$as = \hat{a}\sigma c_R^2, \ at = \hat{a}\tau c_R, \ ay = \hat{a}c_R^2. \tag{3.41}$$

It is clear that there is no depth variable y in the new notation but it is contained implicitly in all three new parameters. Further on one should keep in mind that the main focus is to analyze the solution near the surface, i.e. for $y \ll 1$.

In this section we use the non-dimensional quantities $\hat{b}_i$, $\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{\gamma}_i$ (and, respectively, $\hat{\hat{\alpha}}_i$, $\hat{\hat{\beta}}_i$ and $\hat{\hat{\gamma}}_i$) which can be expressed as

$$\hat{b}_i = (1 \pm \hat{a}\tau)^2 + 2\hat{a}\sigma, \quad \hat{\alpha}_i = \sqrt{\hat{b}_i^2 + (2k_1\hat{a})^2}, \tag{3.42}$$

$$\hat{\beta}_i = \sqrt{2\hat{\alpha}_i + 2\hat{b}_i}, \quad \hat{\gamma}_i = \sqrt{2\hat{\alpha}_i - 2\hat{b}_i}, \tag{3.43}$$

instead of b_i, α_i, β_i and γ_i (and, respectively, $\tilde{\alpha}_i$, $\tilde{\beta}_i$ and $\tilde{\gamma}_i$). Note that the "old" quantities are directly proportional to the corresponding "new" ones (with some constant coefficients). Note also, that here and below for $\hat{b}_i$ plus and minus signs refer to $i = 1$ and $i = 2$ respectively. In this view the partial derivatives of the potentials ϕ and ψ are

$$\frac{\partial \hat{\phi}}{\partial \sigma} = \frac{\partial \hat{\xi}_1}{\partial \sigma}(\hat{z} = 1 + \hat{a}\tau) - \frac{\partial \hat{\xi}_1}{\partial \sigma}(\hat{z} = 1) + \frac{\partial \hat{\xi}_2}{\partial \sigma}(\hat{z} = 1 - \hat{a}\tau) - \frac{\partial \hat{\xi}_2}{\partial \sigma}(\hat{z} = 1), \tag{3.44}$$

where

$$\frac{\partial \hat{\xi}_i}{\partial \sigma} = \frac{k_1 \hat{a}}{\hat{\alpha}_i \hat{\beta}_i} \ln \frac{\hat{z}^2 - \hat{\beta}_i \hat{z} + \hat{\alpha}_i}{\hat{z}^2 + \hat{\beta}_i \hat{z} + \hat{\alpha}_i} - \frac{2k_1 \hat{a}}{\hat{\alpha}_i \hat{\gamma}_i} \left[\arctan \frac{2\hat{z} - \hat{\beta}_i}{\hat{\gamma}_i} + \arctan \frac{2\hat{z} + \hat{\beta}_i}{\hat{\gamma}_i} \right];$$

$$\frac{\partial \hat{\phi}}{\partial y} = \frac{\partial \hat{\xi}_1}{\partial y}(\hat{z} = 1 + \hat{a}\tau) - \frac{\partial \hat{\xi}_1}{\partial y}(\hat{z} = 1) + \frac{\partial \hat{\xi}_2}{\partial y}(\hat{z} = 1 - \hat{a}\tau) - \frac{\partial \hat{\xi}_2}{\partial y}(\hat{z} = 1), \tag{3.45}$$

where

$$\frac{\partial \hat{\xi}_i}{\partial y} = -k_1 \left[\frac{\hat{\beta}_i}{4\hat{\alpha}_i} \ln \frac{\hat{z}^2 - \hat{\beta}_i \hat{z} + \hat{\alpha}_i}{\hat{z}^2 + \hat{\beta}_i \hat{z} + \hat{\alpha}_i} + \frac{\hat{\gamma}_i}{2\hat{\alpha}_i} \left[\arctan \frac{2\hat{z} - \hat{\beta}_i}{\hat{\gamma}_i} + \arctan \frac{2\hat{z} + \hat{\beta}_i}{\hat{\gamma}_i} \right] \right];$$

$$\frac{\partial \hat{\psi}}{\partial \sigma} = \frac{2k_1}{1 + k_2^2} \left[\frac{\partial \hat{\zeta}_1}{\partial \sigma}(\hat{z} = 1 + \hat{a}\tau) - \frac{\partial \hat{\zeta}_1}{\partial \sigma}(\hat{z} = 1) + \frac{\partial \hat{\zeta}_2}{\partial \sigma}(\hat{z} = 1 - \hat{a}\tau) - \frac{\partial \hat{\zeta}_2}{\partial \sigma}(\hat{z} = 1) \right], \tag{3.46}$$

where

$$\frac{\partial \hat{\zeta}_i}{\partial \sigma} = \frac{\hat{\tilde{\beta}}_i}{4\hat{\tilde{\alpha}}_i} \ln \frac{\hat{z}^2 - \hat{\tilde{\beta}}_i \hat{z} + \hat{\tilde{\alpha}}_i}{\hat{z}^2 + \hat{\tilde{\beta}}_i \hat{z} + \hat{\tilde{\alpha}}_i} - \frac{\hat{\tilde{\gamma}}_i}{2\hat{\tilde{\alpha}}_i} \left[\arctan \frac{2\hat{z} - \hat{\beta}_i}{\hat{\gamma}_i} + \arctan \frac{2\hat{z} + \hat{\beta}_i}{\hat{\gamma}_i} \right];$$

$$\frac{\partial \hat{\psi}}{\partial y} = \frac{2k_1}{1 + k_2^2} \left[\frac{\partial \hat{\zeta}_1}{\partial y}(\hat{z} = 1 + \hat{a}\tau) - \frac{\partial \hat{\zeta}_1}{\partial y}(\hat{z} = 1) + \frac{\partial \hat{\zeta}_2}{\partial y}(\hat{z} = 1 - \hat{a}\tau) - \frac{\partial \hat{\zeta}_2}{\partial y}(\hat{z} = 1) \right], \tag{3.47}$$

where

$$\frac{\partial \hat{\zeta}_i}{\partial y} = \frac{2\hat{a}k_2^2}{2\hat{\tilde{\alpha}}_i \hat{\tilde{\beta}}_i} \ln \frac{\hat{z}^2 - \hat{\tilde{\beta}}_i \hat{z} + \hat{\tilde{\alpha}}_i}{\hat{z}^2 + \hat{\tilde{\beta}}_i \hat{z} + \hat{\tilde{\alpha}}_i} - \frac{4\hat{a}k_2^2}{2\hat{\tilde{\alpha}}_i \hat{\tilde{\beta}}_i} \left[\arctan \frac{2\hat{z} - \hat{\beta}_i}{\hat{\gamma}_i} + \arctan \frac{2\hat{z} + \hat{\beta}_i}{\hat{\gamma}_i} \right].$$

As a result, the expressions for the displacements u_1 and u_2 can be written as

$$\begin{aligned}\hat{u}_1 &= \frac{\partial\hat{\phi}}{\partial\sigma}-\frac{\partial\hat{\psi}}{\partial y}=\frac{\partial\hat{\xi}_1}{\partial\sigma}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\xi}_1}{\partial\sigma}(\hat{z}=1)+\frac{\partial\hat{\xi}_2}{\partial\sigma}(\hat{z}=1-\hat{a}\tau)-\frac{\partial\hat{\xi}_2}{\partial\sigma}(\hat{z}=1)-\\ &- \frac{2k_1}{1+k_2^2}\left[\frac{\partial\hat{\zeta}_1}{\partial y}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\zeta}_1}{\partial y}(\hat{z}=1)+\frac{\partial\hat{\zeta}_2}{\partial y}(\hat{z}=1-\hat{a}\tau)-\frac{\partial\hat{\zeta}_2}{\partial y}(\hat{z}=1)\right], \quad (3.48)\end{aligned}$$

$$\begin{aligned}\hat{u}_2 &= \frac{\partial\hat{\phi}}{\partial y}+\frac{\partial\hat{\psi}}{\partial\sigma}=\frac{\partial\hat{\xi}_1}{\partial y}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\xi}_1}{\partial y}(\hat{z}=1)+\frac{\partial\hat{\xi}_2}{\partial y}(\hat{z}=1-\hat{a}\tau)-\frac{\partial\hat{\xi}_2}{\partial y}(\hat{z}=1)+\\ &+ \frac{2k_1}{1+k_2^2}\left[\frac{\partial\hat{\zeta}_1}{\partial\sigma}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\zeta}_1}{\partial\sigma}(\hat{z}=1)+\frac{\partial\hat{\zeta}_2}{\partial\sigma}(\hat{z}=1-\hat{a}\tau)-\frac{\partial\hat{\zeta}_2}{\partial\sigma}(\hat{z}=1)\right]. \quad (3.49)\end{aligned}$$

The partial derivatives of ϕ and ψ and, thus, the displacements u_1 and u_2 above are given for the speed less than c_R. To obtain the solution after the transition through the critical speed one should change the sign of the third summand in the formulae for partial derivatives of the potentials and use $z = \hat{a}\tau - 1$ instead of the argument $z = 1 - \hat{a}\tau$. In this case the displacements are:

$$\begin{aligned}\hat{u}_1 &= \frac{\partial\hat{\phi}}{\partial\sigma}-\frac{\partial\hat{\psi}}{\partial y}=\frac{\partial\hat{\xi}_1}{\partial\sigma}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\xi}_1}{\partial\sigma}(\hat{z}=1)-\frac{\partial\hat{\xi}_2}{\partial\sigma}(\hat{z}=\hat{a}\tau-1)-\frac{\partial\hat{\xi}_2}{\partial\sigma}(\hat{z}=1)-\\ &- \frac{2k_1}{1+k_2^2}\left[\frac{\partial\hat{\zeta}_1}{\partial y}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\zeta}_1}{\partial y}(\hat{z}=1)-\frac{\partial\hat{\zeta}_2}{\partial y}(\hat{z}=\hat{a}\tau-1)-\frac{\partial\hat{\zeta}_2}{\partial y}(\hat{z}=1)\right], \quad (3.50)\end{aligned}$$

$$\begin{aligned}\hat{u}_2 &= \frac{\partial\hat{\phi}}{\partial y}+\frac{\partial\hat{\psi}}{\partial\sigma}=\frac{\partial\hat{\xi}_1}{\partial y}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\xi}_1}{\partial y}(\hat{z}=1)-\frac{\partial\hat{\xi}_2}{\partial y}(\hat{z}=\hat{a}\tau-1)-\frac{\partial\hat{\xi}_2}{\partial y}(\hat{z}=1)+\\ &+ \frac{2k_1}{1+k_2^2}\left[\frac{\partial\hat{\zeta}_1}{\partial\sigma}(\hat{z}=1+\hat{a}\tau)-\frac{\partial\hat{\zeta}_1}{\partial\sigma}(\hat{z}=1)-\frac{\partial\hat{\zeta}_2}{\partial\sigma}(\hat{z}=\hat{a}\tau-1)-\frac{\partial\hat{\zeta}_2}{\partial\sigma}(\hat{z}=1)\right]. \quad (3.51)\end{aligned}$$

3.4.1 Non-uniform asymptotic formulae

This part of the thesis is divided into two sections. The first one is dedicated to the steady speed case, i.e. when the load speed is constant. The result is an asymptotic expansion, which can well describe the behaviour of a half plane while the moving load speed is sufficiently far from the critical one. The other section contains the approximation of displacements exactly in the case $at = c_R$.

The acceleration there is assumed to be quite small, but non-zero. In the second section all the calculations are provided in dimensional quantities.

3.4.1.1 Steady speed asymptotic expansion

Consider the displacements $\hat{u}_1$ and $\hat{u}_2$ in case when $\hat{a} \to 0$ and $\hat{a}\tau \to v =$ const and, consequently, $\frac{1}{2}\hat{a}\tau^2 \to v\tau$. This assumption allows to approximate the parameters $\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{\gamma}_i$ by expanding a square roots using the formula:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots \text{ for } |x| < 1. \tag{3.52}$$

In view of this expansion the approximate formulae for the mentioned quantities are

$$\begin{aligned}
\hat{\alpha}_i &= \sqrt{((1 \pm \hat{a}\tau)^2 + 2\hat{a}\sigma)^2 + (2\hat{a}k_1)^2} = \\
&= (1 \pm \hat{a}\tau)^2 \sqrt{1 + \frac{4\hat{a}\sigma}{(1 \pm \hat{a}\tau)^2} + \frac{(2\hat{a}\sigma)^2 + (2\hat{a}k_1)^2}{(1 \pm \hat{a}\tau)^4}} \approx \\
&\approx (1 \pm \hat{a}\tau)^2 \left(1 + \frac{2\hat{a}\sigma}{(1 \pm \hat{a}\tau)^2} + \frac{2(\hat{a}\sigma)^2 + 2(\hat{a}k_1)^2}{(1 \pm \hat{a}\tau)^4} - \frac{2(\hat{a}\sigma)^2}{(1 \pm \hat{a}\tau)^4}\right) = \\
&= b_i + \frac{2(\hat{a}k_1)^2}{(1 \pm \hat{a}\tau)^2},
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
\hat{\beta}_i &\approx \sqrt{2b_i + \frac{(2\hat{a}k_1)^2}{(1 \pm \hat{a}\tau)^2} + 2b_i} = 2(1 \pm \hat{a}\tau)\sqrt{1 + \frac{2\hat{a}\sigma}{(1 \pm \hat{a}\tau)^2} + \frac{\hat{a}^2k_1^2}{(1 \pm \hat{a}\tau)^4}} \approx \\
&\approx 2(1 \pm \hat{a}\tau)\left(1 + \frac{\hat{a}\sigma}{(1 \pm \hat{a}\tau)^2} + \frac{\hat{a}^2k_1^2}{2(1 \pm \hat{a}\tau)^4} - \frac{\hat{a}^2\sigma^2}{2(1 \pm \hat{a}\tau)^4}\right),
\end{aligned} \tag{3.54}$$

$$\hat{\gamma}_i \approx \sqrt{2b_i + \frac{(2\hat{a}k_1)^2}{(1 \pm \hat{a}\tau)^2} - 2b_i} = \frac{2\hat{a}k_1}{1 \pm \hat{a}\tau}. \tag{3.55}$$

Let us note that the displacements $\hat{u}_i$, $i = 1, 2$, are expressed via a sum of the partial derivatives of the potentials, see the formulae (3.48)–(3.51). Each partial derivative is a sum of the logarithms and the pairs of arctangents with some coefficients. In every case one can note that there are two different coefficients for logarithms and the same for pairs of arctangents. All the further calculations start with the consideration of these coefficients.

Consider the case $v < 1$. To calculate the displacement $\hat{u}_1$ we deal with the partial derivatives of the potentials separately. Start with $\dfrac{\partial\hat{\phi}}{\partial\sigma}$. First we calculate the coefficients in front of the logarithms using formulae (3.53)–(3.55).

$$\frac{\hat{a}k_1}{\hat{\alpha}_i\hat{\beta}_i} \approx \frac{\hat{a}k_1}{2(1\pm\hat{a}\tau)^3} \xrightarrow[\hat{a}\to 0]{} 0,\ \ i=1,2. \tag{3.56}$$

The coefficients for the arctangents are

$$\frac{2\hat{a}k_1}{\hat{\alpha}_i\hat{\gamma}_i} \approx \frac{1}{1\pm\hat{a}\tau} \xrightarrow[\hat{a}\tau\to v]{} \frac{1}{1\pm v},\ \ i=1,2. \tag{3.57}$$

It is clear that the terms with the logarithms in $\dfrac{\partial\phi}{\partial\sigma}$ vanish. So, to approximate $\dfrac{\partial\hat{\phi}}{\partial\sigma}$ it remains only to calculate the arctangents. To do it one needs to substitute the expressions (3.53)–(3.55) and (3.57) into the formula (3.44). After some simple transformations one can get

$$\begin{aligned}
\frac{\partial\hat{\phi}}{\partial\sigma} &\approx \frac{1}{1+v}\left[-\arctan\frac{\sigma+\frac{\hat{a}(k_1^2-\sigma^2)}{2(1+v)^2}}{k_1}+\arctan\frac{4(1+v)^2+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^2}}{2\hat{a}k_1}+\right.\\
&+\left.\arctan\frac{2v(1+v)+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^2}}{2\hat{a}k_1}-\arctan\frac{2(1+v)(2+v)+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^2}}{2\hat{a}k_1}\right]+\\
&+\frac{1}{1-v}\left[-\arctan\frac{\sigma+\frac{\hat{a}(k_1^2-\sigma^2)}{2(1-v)^2}}{k_1}+\arctan\frac{4(1-v)^2+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^2}}{2\hat{a}k_1}-\right.\\
&-\left.\arctan\frac{2v(1-v)+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^2}}{2\hat{a}k_1}-\arctan\frac{2(1-v)(2-v)+2\hat{a}\sigma+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^2}}{2\hat{a}k_1}\right].
\end{aligned}$$

Taking a limit when $\hat{a}\to 0$ in the last expression gives the asymptotic formula for the $\dfrac{\partial\hat{\phi}}{\partial\sigma}$:

$$\begin{aligned}
\frac{\partial\hat{\phi}}{\partial\sigma} &\approx \frac{1}{1+v}\left[-\arctan\frac{\sigma}{k_1}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{2}\right]+\frac{1}{1-v}\left[-\arctan\frac{\sigma}{k_1}+\frac{\pi}{2}-\frac{\pi}{2}-\frac{\pi}{2}\right]=\\
&= -\frac{\pi v}{1-v^2}-\frac{2}{1-v^2}\arctan\frac{\sigma}{k_1}.
\end{aligned} \tag{3.58}$$

The calculations for $\dfrac{\partial\hat{\psi}}{\partial y}$ are very similar. The coefficients in front of the logarithms in the expressions for $\dfrac{\partial\hat{\psi}}{\partial y}$ differ from the corresponding ones in $\dfrac{\partial\hat{\phi}}{\partial\sigma}$ only by a constant factor. So, they are equal to zero. By analogy with a partial derivative for the potential $\hat{\phi}$ the coefficients in front of arctangents are

$$\frac{2\hat{a}k_2^2}{\hat{\hat{\alpha}}_i\hat{\hat{\gamma}}_i} \approx \frac{k_2}{1 \pm v}, \ \ i = 1, 2,$$

and, thus, the asymptotic formula for the partial derivative of ψ is:

$$\frac{\partial\hat{\psi}}{\partial y} \approx -\frac{2k_1k_2}{1+k_2^2}\frac{\pi v}{1-v^2} - \frac{2k_1k_2}{1+k_2^2}\frac{2}{1-v^2}\arctan\frac{\sigma}{k_2}. \tag{3.59}$$

So, substituting the formulae (3.58) and (3.59) into (3.48) and using the property (3.4) the asymptotic formula for the displacement u_1 in case of $v < 1$ is

$$\hat{u}_1 \approx \frac{2}{1-v^2}\left[\arctan\frac{\sigma}{k_1} - \frac{2k_1k_2}{1+k_2^2}\arctan\frac{\sigma}{k_2}\right] + \frac{\pi v}{2(1-v^2)}(1-k_2^2). \tag{3.60}$$

To find an approximation for the displacement $\hat{u}_2$ in case of $v < 1$ one can follow the steps performed for $\hat{u}_1$. Start with a derivative $\dfrac{\partial\hat{\phi}}{\partial y}$. The coefficients in front of arctangents are

$$k_1\frac{\hat{\gamma}_i}{2\hat{\alpha}_i} \approx \frac{\hat{a}k_1^2}{(1 \pm v)^3} \xrightarrow[\hat{a}\to 0]{} 0, \ \ i = 1, 2, \tag{3.61}$$

in front of the logarithms are

$$-k_1\frac{\hat{\beta}_i}{4\hat{\alpha}_i} \approx -\frac{k_1}{2(1 \pm v)}, \ \ i = 1, 2. \tag{3.62}$$

So, the terms with the arctangents vanish and to calculate the derivative of the potential $\hat{\phi}$ it only remains to substitute the expressions (3.53)–(3.55) and (3.62) into the formula (3.45). After some algebraic transformations it becomes

$$
\begin{aligned}
\frac{\partial\hat{\phi}}{\partial y} \approx -\frac{k_1}{2(1+v)} & \left[\ln\left(\frac{\frac{2\hat{a}^2k_1^2-\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^2}}{4(1+v)^2+4\hat{a}\sigma+\frac{2\hat{a}^2k_1^2+\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^2}}\right)-\right. \\
- \quad & \left.\ln\left(\frac{1-2(1+v)+(1+v)^2-\frac{2\hat{a}\sigma}{(1+v)}-\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^3}+2\hat{a}\sigma+\frac{2\hat{a}^2k_1^2}{(1+v)^2}}{1+2(1+v)+(1+v)^2+\frac{2\hat{a}\sigma}{(1+v)}+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1+v)^3}+2\hat{a}\sigma+\frac{2\hat{a}^2k_1^2}{(1+v)^2}}\right)\right] \\
- \quad & \frac{k_1}{2(1-v)}\left[\ln\left(\frac{\frac{2\hat{a}^2k_1^2-\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^2}}{4(1-v)^2+4\hat{a}\sigma+\frac{2\hat{a}^2k_1^2+\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^2}}\right)-\right. \\
- \quad & \left.\ln\left(\frac{1-2(1-v)+(1-v)^2-\frac{2\hat{a}\sigma}{(1-v)}-\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^3}+2\hat{a}\sigma+\frac{2\hat{a}^2k_1^2}{(1-v)^2}}{1+2(1-v)+(1-v)^2+\frac{2\hat{a}\sigma}{(1-v)}+\frac{\hat{a}^2(k_1^2-\sigma^2)}{(1-v)^3}+2\hat{a}\sigma+\frac{2\hat{a}^2k_1^2}{(1-v)^2}}\right)\right].
\end{aligned}
$$

Taking a limit when $\hat{a} \to 0$ or, equivalently, $\hat{a}\tau \to v$ one can get the asymptotic formula for $\dfrac{\partial\hat{\phi}}{\partial y}$:

$$
\begin{aligned}
\frac{\partial\hat{\phi}}{\partial y} \approx & -k_1\left[\frac{1}{1-v^2}\ln(k_1^2+\sigma^2)-\frac{1}{1-v^2}\ln(4\tau^2)-\right. \\
& - \left.\frac{\ln(1-v)}{1-v}-\frac{\ln(1+v)}{1+v}+\frac{\ln(2-v)}{1-v}+\frac{\ln(2+v)}{1+v}\right]. \qquad (3.63)
\end{aligned}
$$

To obtain an approximation for the partial derivative of the potential $\hat{\psi}$ let us note that the coefficients in front of arctangents tend to zero as well. The form of coefficients in front of the logarithms is very similar to one in $\dfrac{\partial\hat{\phi}}{\partial y}$, namely,

$$
\frac{\hat{\beta}_i}{4\hat{\alpha}_i} \approx \frac{1}{2(1\pm v)} \qquad (3.64)
$$

and, thus, the asymptotic formula for $\dfrac{\partial\hat{\psi}}{\partial\sigma}$ is:

$$
\begin{aligned}
\frac{\partial\hat{\psi}}{\partial\sigma} \approx & \frac{2k_1}{1+k_2^2}\left[\frac{1}{1-v^2}\ln(k_2^2+\sigma^2)-\frac{1}{1-v^2}\ln(4\tau^2)-\right. \\
& - \left.\frac{\ln(1-v)}{1-v}-\frac{\ln(1+v)}{1+v}+\frac{\ln(2-v)}{1-v}+\frac{\ln(2+v)}{1+v}\right]. \qquad (3.65)
\end{aligned}
$$

Finally, the substitution of the formulae (3.63) and (3.65) into (3.49) gives the approximation for the displacement $\hat{u}_2$:

$$\begin{aligned}\hat{u}_2 \approx & -\frac{k_1}{1-v^2}\left[\ln(k_1^2+\sigma^2)-\frac{2}{1+k_2^2}\ln(k_2^2+\sigma^2)\right]- \\ & -\frac{k_1(1-k_2^2)}{1+k_2^2}\left[\frac{\ln(1-v)}{1-v}+\frac{\ln(1+v)}{1+v}\right]- \\ & -\frac{2k_1(1-k_2^2)}{(1-v^2)(1+k_2^2)}\ln\tau+ \\ & +\frac{k_1(1-k_2^2)}{1+k_2^2}\left[\frac{\ln(2-v)-\ln 2}{1-v}+\frac{\ln(2+v)-\ln 2}{1+v}\right]. \end{aligned} \tag{3.66}$$

Consider now the case $v>1$. By analogy with the previous case we need to find the asymptotic formulae for the displacements $\hat{u}_1$ and $\hat{u}_2$. But instead of formulae (3.48) and (3.49), one should use (3.50) and (3.51). The asymptotic forms for the derivatives of the potentials $\hat{\phi}$ and $\hat{\psi}$ and, thus, the displacements can be obtained using exactly the same technique as above. They are

$$\frac{\partial\hat{\phi}}{\partial\sigma}\approx\frac{2}{v^2-1}\arctan\left(\frac{\sigma}{k_1}\right)+\frac{\pi v}{v^2-1}-\frac{2\pi}{v-1},$$

$$\frac{\partial\hat{\psi}}{\partial y}\approx\frac{2}{v^2-1}\arctan\left(\frac{\sigma}{k_2}\right)+\frac{\pi v}{v^2-1}-\frac{2\pi}{v-1},$$

$$\begin{aligned}\hat{u}_1 \approx & \frac{2}{v^2-1}\left[\arctan\left(\frac{\sigma}{k_1}\right)-\frac{2k_1k_2}{1+k_2^2}\arctan\left(\frac{\sigma}{k_2}\right)\right]+ \\ & +\frac{\pi v}{2(v^2-1)}(1-k_2^2)-\frac{2\pi}{v-1}(1-k_2^2); \end{aligned} \tag{3.67}$$

$$\begin{aligned}\frac{\partial\hat{\phi}}{\partial y} \approx & -k_1\left[-\frac{1}{v^2-1}\ln(k_1^2+\sigma^2)+\frac{1}{v^2-1}\ln(4\tau^2)-\right. \\ & \left.-\frac{\ln(v+1)}{v+1}+\frac{\ln(2+v)}{v+1}+\frac{\ln(v-1)}{v-1}+\frac{\ln(2-v)}{v-1}-2\frac{\ln v}{v-1}\right], \end{aligned}$$

$$\begin{aligned}\frac{\partial\hat{\psi}}{\partial\sigma} \approx & \frac{2k_1}{1+k_2^2}\left[-\frac{1}{v^2-1}\ln(k_2^2+\sigma^2)+\frac{1}{v^2-1}\ln(4\tau^2)-\right. \\ & \left.-\frac{\ln(v+1)}{v+1}+\frac{\ln(2+v)}{v+1}+\frac{\ln(v-1)}{v-1}+\frac{\ln(2-v)}{v-1}-2\frac{\ln v}{v-1}\right], \end{aligned}$$

$$\begin{aligned}\hat{u}_2 &\approx \frac{k_1}{v^2-1}\left[\ln(k_1^2+\sigma^2) - \frac{2}{1+k_2^2}\ln(k_2^2+\sigma^2)\right] - \\ &- \frac{k_1(1-k_2^2)}{1+k_2^2}\left[\frac{\ln(v+1)}{v+1} - \frac{\ln(v-1)}{v-1}\right] + \\ &+ \frac{k_1(1-k_2^2)}{1+k_2^2}\left[\frac{\ln(2+v)-\ln 2}{v+1} + \frac{\ln(2-v)+\ln 2}{v-1}\right] + \\ &+ \frac{2vk_1(1-k_2^2)}{(1+k_2^2)(v^2-1)}\ln\tau - \frac{4k_1\ln v}{(1+k_2^2)(v-1)}. \end{aligned} \tag{3.68}$$

Although the exact solutions (3.48)–(3.49) and (3.50)–(3.51) are, generally speaking, the same pairs of functions (see the end of Section 3.3), pairs of the asymptotic formulae (3.60), (3.66) and (3.67), (3.68) are different. The difference is expressed as an extra term in the second pair of the asymptotic forms. This "addition" is generated by dynamical effects, caused by the passage through the critical speed.

3.4.1.2 Asymptotic expansion on the Rayleigh speed

In this section we deal with the moment of passage through the Rayleigh speed using the same dimensional notation that was used for calculating the exact solution in Section 3.3. To construct the approximations for the displacements on the critical speed we should put $at = c_R$. In this case the parameters b_1 and b_2 from (3.18) and (3.20) can be rewritten as

$$b_1 = 4c_R^2 + 2as \text{ and } b_2 = 2as.$$

The arguments of the derivatives of functions ξ_i and ζ_i, $i = 1, 2$ become

$$z_1 = c_R, \quad z_2 = 2c_R, \quad z_3 = 0, \quad z_4 = c_R.$$

By analogy with the formulae (3.53)–(3.55), since $\dfrac{ay}{c_R^2} = \hat{a} \ll 1$ and $\dfrac{s}{y} = O(1)$, one can use Taylor's expansion (3.52) to obtain the expressions:

$$\alpha_1 = 4c_R^2\sqrt{1 + \frac{as}{c_R^2} + \frac{a^2(s^2+(k_1y)^2)}{4c_R^4}} \approx 4c_R^2\left(1 + \frac{as}{2c_R^2} + \frac{a^2(k_1y)^2}{8c_R^4}\right),$$

$$\beta_1 \approx 4c_R\sqrt{1 + \frac{as}{2c_R^2} + \frac{a^2(k_1y)^2}{16c_R^4}} \approx 4c_R\left(1 + \frac{as}{4c_R^2} + \frac{a^2((k_1y)^2-s^2)}{32c_R^4}\right),$$

$$\gamma_1 \approx \frac{ak_1y}{c_R}.$$

Note that the parameters α_2, β_2 and γ_2 can not be expanded in the same way. The direct substitution $at = c_R$ into the formulae (3.20) leads to

$$\alpha_2 = 2a\sqrt{s^2+(k_1y)^2},\ \beta_2 = 2\sqrt{a}\sqrt{\sqrt{s^2+(k_1y)^2}+s},\ \gamma_2 = 2\sqrt{a}\sqrt{\sqrt{s^2+(k_1y)^2}-s}.$$

Similar to Section 3.4.1.1 we consider the partial derivatives of the potentials separately for each displacement. But here it is easier to deal with the derivatives of the functions ξ_i and ζ_i individually. As before the expressions for the displacements contain logarithms and arctangents with some coefficients.

Start with the first displacement. Consider the expression for $\dfrac{\partial\phi}{\partial s}$ (see formula (3.27)). It consists of a sum of the four partial derivatives: $\dfrac{\partial\xi_1}{\partial s}(z = 2c_R)$, $\dfrac{\partial\xi_1}{\partial s}(z = c_R)$, $\dfrac{\partial\xi_2}{\partial s}(z = c_R)$ and $\dfrac{\partial\xi_2}{\partial s}(z = 0)$. Clearly, $\dfrac{\partial\xi_2}{\partial s}(z = 0)$ is equal to zero. All other terms are considered separately.

For $\dfrac{\partial\xi_1}{\partial s}(z = 2c_R)$ the coefficients in front of logarithm and pair of arctangents are

$$\frac{ak_1y}{\alpha_1\beta_1} \approx \frac{ak_1y}{16c_R^3} = \frac{1}{16c_R}O(\hat{a}) \text{ and } \frac{2ak_1y}{\alpha_1\gamma_1} \approx \frac{1}{2c_R}O(1).$$

Here and below all the approximations are obtained by substituting the parameters α_i, β_i and γ_i, $i = 1, 2$ into the corresponding expressions and keeping only terms of the main order. Also, I would like to mention that nevertheless, in general, in this section we use dimensional variables, we also use non-dimensional variable $\hat{a}$ to describe and compare orders of terms in the expressions for derivative of the potentials ϕ and ψ and, thus, for the displacements u_1 and u_2. Now we estimate a logarithm and each arctangent:

$$\ln\left[\frac{4c_R^2 - 2c_R\beta_1 + \alpha_1}{4c_R^2 + 2c_R\beta_1 + \alpha_1}\right] \approx$$
$$\approx \ln\left[\frac{\frac{a^2((k_1y)^2+s^2)}{4c_R^2}}{16c_R^2 + 4as + \frac{a^2(3(k_1y)^2-s^2)}{8c_R^2}}\right] \approx \ln\left[\frac{a^2((k_1y)^2+s^2)}{64c_R^4}\right],$$

$$\arctan\left[\frac{4c_R - \beta_1}{\gamma_1}\right] \approx -\arctan\left[\frac{s + \frac{a^2((k_1y)^2-s^2)}{8c_R^2}}{k_1y}\right] \approx -\arctan\frac{s}{k_1y},$$

$$\arctan\left[\frac{4c_R+\beta_1}{\gamma_1}\right]\approx\arctan\left[\frac{8c_R^2+as+\frac{a^2((k_1y)^2-s^2)}{8c_R^2}}{ak_1y}\right]\approx-\arctan\frac{8c_R^2}{ak_1y}\approx\frac{\pi}{2}.$$

Hence, the asymptotic formula for the first term of the derivative $\dfrac{\partial\phi}{\partial s}$ is

$$\frac{\partial\xi_1}{\partial s}(z=2c_R)\approx\frac{ak_1y}{16c_R^3}\ln\left[\frac{a^2((k_1y)^2+s^2)}{64c_R^4}\right]+\frac{1}{2c_R}\left[\arctan\frac{s}{k_1y}-\frac{\pi}{2}\right]. \qquad (3.69)$$

For $\dfrac{\partial\xi_1}{\partial s}(z=c_R)$ the coefficients in front of logarithm and pair of arctangents are the same as for $\dfrac{\partial\xi_1}{\partial s}(z=2c_R)$. Now we can calculate a logarithm and each arctangent:

$$\ln\left[\frac{c_R^2-c_R\beta_1+\alpha_1}{c_R^2+c_R\beta_1+\alpha_1}\right]\approx\ln\left[\frac{c_R^2}{9c_R^2}\right]=-2\ln 3,$$

$$\arctan\left[\frac{2c_R-\beta_1}{\gamma_1}\right]\approx-\arctan\left[\frac{2c_R^2+as+\frac{a^2((k_1y)^2-s^2)}{8c_R^2}}{ak_1y}\right]\approx-\arctan\frac{2c_R^2}{ak_1y}\approx-\frac{\pi}{2},$$

$$\arctan\left[\frac{2c_R+\beta_1}{\gamma_1}\right]\approx\arctan\left[\frac{6c_R+\frac{as}{c_R}+\frac{a^2((k_1y)^2-s^2)}{8c_R^3}}{\frac{ak_1y}{c_R}}\right]\approx\arctan\frac{6c_R^2}{ak_1y}\approx\frac{\pi}{2}.$$

Hence, the asymptotic formula for the second term of the derivative $\dfrac{\partial\phi}{\partial s}$ is

$$\frac{\partial\xi_1}{\partial s}(z=c_R)=-\frac{ak_1y}{8c_R^3}\ln 3. \qquad (3.70)$$

For $\dfrac{\partial\xi_2}{\partial s}(z=c_R)$ the coefficients in front of logarithm and pair of arctangents are

$$\frac{ak_1y}{\alpha_2\beta_2}=\frac{k_1y}{4\sqrt{a}\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}+s}}\sim O\left(\frac{1}{\sqrt{\hat{a}}}\right)$$

and

$$\frac{2ak_1y}{\alpha_1\gamma_1}=\frac{k_1y}{2\sqrt{a}\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}-s}}\sim O\left(\frac{1}{\sqrt{\hat{a}}}\right).$$

Now we estimate a logarithm and pair of arctangents:

$$\ln\left[\frac{c_R^2-c_R\beta_2+\alpha_2}{c_R^2+c_R\beta_2+\alpha_2}\right]\approx\ln\frac{c_R^2}{c_R^2}=0,$$

$$\arctan\left[\frac{2c_R-\beta_2}{\gamma_2}\right]+\arctan\left[\frac{2c_R+\beta_2}{\gamma_2}\right]\approx\pi.$$

Hence, the asymptotic formula for the third term of the derivative $\dfrac{\partial\phi}{\partial s}$ is

$$\frac{\partial\xi_2}{\partial s}(z=c_R)=-\frac{k_1y\pi}{2\sqrt{a}\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}-s}}\sim O\left(\frac{1}{\sqrt{\hat{a}}}\right). \qquad (3.71)$$

Using the formulae (3.69), (3.70) and (3.71) and also the fact that $\dfrac{\partial\xi_2}{\partial s}(z=0)=0$ and keeping only the leading order term one can obtain the asymptotic formula for the derivative of the potential ϕ:

$$\frac{\partial\phi}{\partial s}\approx\frac{k_1y\pi}{2\sqrt{a}\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}+s}}. \qquad (3.72)$$

To find an approximation for the derivative of the potential ψ one should use the same technique as for ϕ. Consider the expression for $\dfrac{\partial\psi}{\partial y}$ (see formula (3.30)). It consists of a sum of the four partial derivatives: $\dfrac{\partial\zeta_1}{\partial y}(z=2c_R)$, $\dfrac{\partial\zeta_1}{\partial y}(z=c_R)$, $\dfrac{\partial\zeta_2}{\partial y}(z=c_R)$ and $\dfrac{\partial\zeta_2}{\partial y}(z=0)$.

For $\dfrac{\partial\zeta_1}{\partial y}(z=2c_R)$ the coefficients in front of logarithm and pair of arctangents are

$$\frac{2k_2^2}{1+k_2^2}\,\frac{ak_1y}{\tilde{\alpha}_1\tilde{\beta}_1}\approx\frac{2k_2^2}{1+k_2^2}\,\frac{ak_1y}{16c_R^3}\sim O(\hat{a})$$

and

$$\frac{2k_2^2}{1+k_2^2}\,\frac{2ak_1y}{\tilde{\alpha}_1\tilde{\gamma}_1}\approx\frac{2k_2^2}{1+k_2^2}\,\frac{1}{2c_R}\sim O(1).$$

Now we estimate a logarithm and pair of arctangents:

$$\ln\left[\frac{4c_R^2-2c_R\tilde{\beta}_1+\tilde{\alpha}_1}{4c_R^2+2c_R\tilde{\beta}_1+\tilde{\alpha}_1}\right]\approx\ln\left[\frac{a^2((k_2y)^2+s^2}{64c_R^4}\right],$$

$$\arctan\left(\frac{4c_R-\tilde{\beta}_1}{\tilde{\gamma}_1}\right)\approx-\arctan\left(\frac{s}{k_2y}\right),\quad \arctan\left(\frac{4c_R+\tilde{\beta}_1}{\tilde{\gamma}_1}\right)\approx\frac{\pi}{2}.$$

Hence, the asymptotic formula for the first term of the derivative $\dfrac{\partial\psi}{\partial y}$ is

$$\frac{\partial\zeta_1}{\partial y}(z=2c_R) = \frac{2k_2^2}{1+k_2^2}\left(\frac{ak_1y}{16c_R^3}\ln\left[\frac{a^2((k_1y)^2+s^2)}{64c_R^4}\right]+ \right.$$
$$\left. +\frac{1}{2c_R}\left[\arctan\left(\frac{s}{k_1y}\right)-\frac{\pi}{2}\right]\right). \tag{3.73}$$

For $\dfrac{\partial\zeta_1}{\partial y}(z=c_R)$ the coefficients in front of a logarithm and pair of arctangents are the same as ones for $\dfrac{\partial\zeta_1}{\partial y}(z=2c_R)$. Now we estimate logarithm and pair of arctangents:

$$\ln\left[\frac{c_R^2-c_R\tilde{\beta}_1+\tilde{\alpha}_1}{c_R^2+c_R\tilde{\beta}_1+\tilde{\alpha}_1}\right]\approx -2\ln 3$$

$$\arctan\left(\frac{2c_R-\tilde{\beta}_1}{\tilde{\gamma}_1}\right)\approx -\frac{\pi}{2},\quad \arctan\left(\frac{2c_R+\tilde{\beta}_1}{\tilde{\gamma}_1}\right)\approx \frac{\pi}{2};$$

Hence, the asymptotic formula for the second term of the derivative $\dfrac{\partial\psi}{\partial y}$ is

$$\frac{\partial\zeta_1}{\partial y}(z=c_R)\approx -\frac{ak_1y}{8c_R^3}\frac{2k_2^2}{1+k_2^2}\ln 3. \tag{3.74}$$

For $\dfrac{\partial\zeta_2}{\partial y}(z=c_R)$ the coefficients in front of logarithm and pair of arctangents are

$$\frac{2k_1}{1+k_2^2}\,\frac{ak_2^2y}{\tilde{\alpha}_2\tilde{\beta}_2}=\frac{2k_2^2}{1+k_2^2}\,\frac{k_1y}{4\sqrt{a}\sqrt{s^2+(k_2y)^2}\sqrt{\sqrt{s^2+(k_2y)^2}+s}}\sim O\left(\frac{1}{\sqrt{\hat{a}}}\right),$$

and

$$\frac{2k_1}{1+k_2^2}\,\frac{2ak_2^2y}{\tilde{\alpha}_2\tilde{\gamma}_2}=\frac{2k_2^2}{1+k_2^2}\,\frac{k_1y}{2\sqrt{a}\sqrt{s^2+(k_2y)^2}\sqrt{\sqrt{s^2+(k_2y)^2}-s}}\sim O\left(\frac{1}{\sqrt{\hat{a}}}\right).$$

Now we estimate a logarithm and pair of arctangents:

$$\ln\left[\frac{c_R^2-c_R\tilde{\beta}_2+\tilde{\alpha}_2}{c_R^2+c_R\tilde{\beta}_2+\tilde{\alpha}_2}\right]\approx \ln\frac{c_R^2}{c_R^2}=0,$$

$$\arctan\left[\frac{2c_R - \tilde{\beta}_2}{\tilde{\gamma}_2}\right] + \arctan\left[\frac{2c_R + \tilde{\beta}_2}{\tilde{\gamma}_2}\right] \approx \pi.$$

Hence, the asymptotic formula for the third term of the derivative $\dfrac{\partial \phi}{\partial s}$ is

$$\frac{\partial \zeta_2}{\partial y}(z = c_R) \approx -\frac{2k_2^2}{1+k_2^2}\frac{k_1 y \pi}{2\sqrt{a}\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}-s}}. \tag{3.75}$$

Using the formulae (3.73), (3.74) and (3.75) and also the fact that $\dfrac{\partial \zeta_2}{\partial y}(z = 0) = 0$ and keeping only the leading order term one can obtain the asymptotic formula for the derivative of the potential ϕ:

$$\frac{\partial \psi}{\partial y} \approx \frac{2k_2^2}{1+k_2^2}\frac{k_1 y \pi}{2\sqrt{a}\sqrt{s^2+(k_2y)^2}\sqrt{\sqrt{s^2+(k_2y)^2}-s}}. \tag{3.76}$$

Finally, the asymptotic form of the first displacement can be written using formulae (3.72) and (3.76):

$$\begin{aligned} u_1 = \frac{\partial \phi}{\partial s} - \frac{\partial \psi}{\partial y} \;&=\; \frac{k_1 y \pi}{2\sqrt{a}}\left[\frac{1}{\sqrt{s^2+(k_1y)^2}\sqrt{\sqrt{s^2+(k_1y)^2}-s}} - \right. \\ &- \left.\frac{2k_2^2}{1+k_2^2}\,\frac{1}{\sqrt{s^2+(k_2y)^2}\sqrt{\sqrt{s^2+(k_2y)^2}-s}}\right]. \end{aligned} \tag{3.77}$$

All the auxiliary calculations for the second displacement are quite similar to the ones for u_1. The final asymptotic representations for the derivatives of the potentials ϕ and ψ and also for u_2 are given below.

$$\frac{\partial \phi}{\partial y} \approx \frac{k_1 \pi \sqrt{\sqrt{s^2+(k_1y)^2}-s}}{2\sqrt{a}\sqrt{s^2+(k_1y)^2}}, \tag{3.78}$$

$$\frac{\partial \psi}{\partial s} \approx \frac{k_1}{1+k_2^2}\,\frac{\pi\sqrt{\sqrt{s^2+(k_2y)^2}-s}}{\sqrt{a}\sqrt{s^2+(k_2y)^2}}, \tag{3.79}$$

$$u_2 \approx \frac{k_1\pi}{\sqrt{a}}\left[\frac{\sqrt{\sqrt{s^2+(k_1y)^2}-s}}{2\sqrt{s^2+(k_1y)^2}} + \frac{\sqrt{\sqrt{s^2+(k_2y)^2}-s}}{(1+k_2^2)\sqrt{s^2+(k_2y)^2}}\right]. \tag{3.80}$$

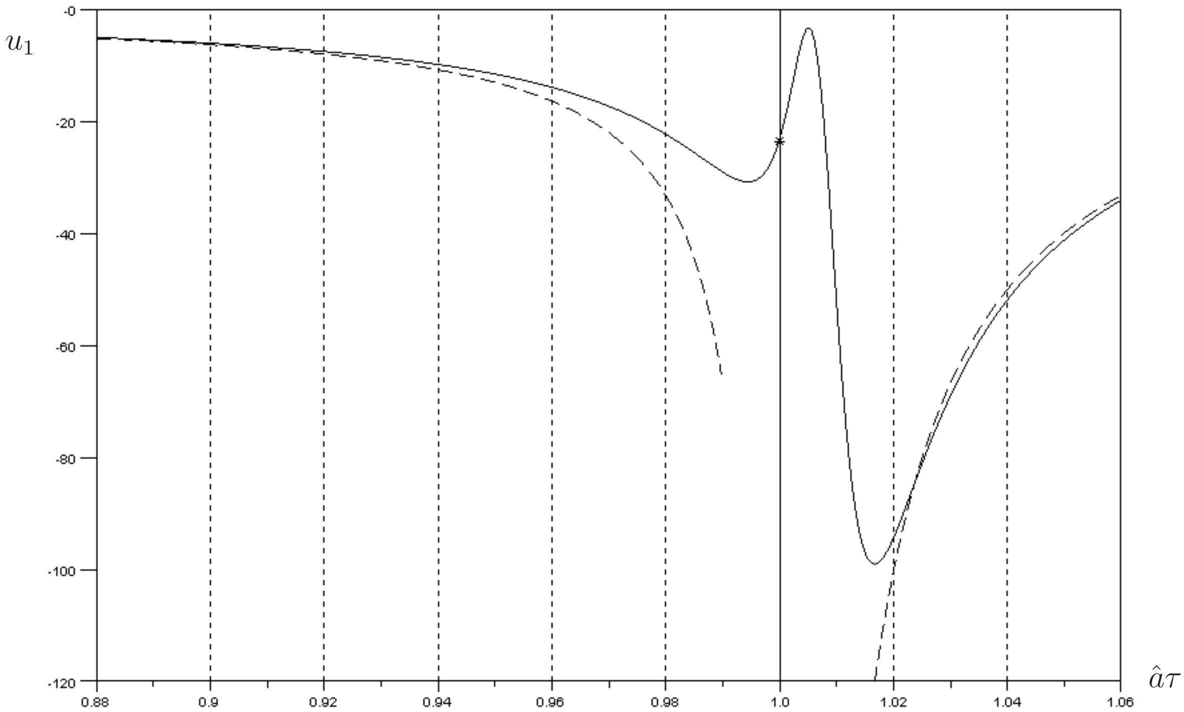


FIGURE 3.6: Asymptotic behaviour for uniform speed case (dashed line) with exact solution (solid line) and asymptotic result on Rayleigh speed (star) with $\hat{a} = 0.0001$ for horizontal displacement u_1

So, formulae (3.77) and (3.80) can be used to approximate the formulae (3.31)–(3.32) on the Rayleigh speed.

Figures 3.6–3.7 and 3.8–3.9 show the graphical representation of the exact solution (solid line), steady speed (dashed line) and Rayleigh speed (star) approximations for the displacements u_1 and u_2 respectively for different values of the parameter $\hat{a}$. As one can see the steady speed asymptotic formulae provide a good approximation out of the transient effect region. As mentioned above the region of transition effects decreases while the parameter $\hat{a}$ becomes smaller, so, the smaller the parameter $\hat{a}$, the better steady speed approximation works closer to the Rayleigh speed. It is also clear that our quite simple asymptotic formulae derived for the Rayleigh speed give a very accurate approximation.

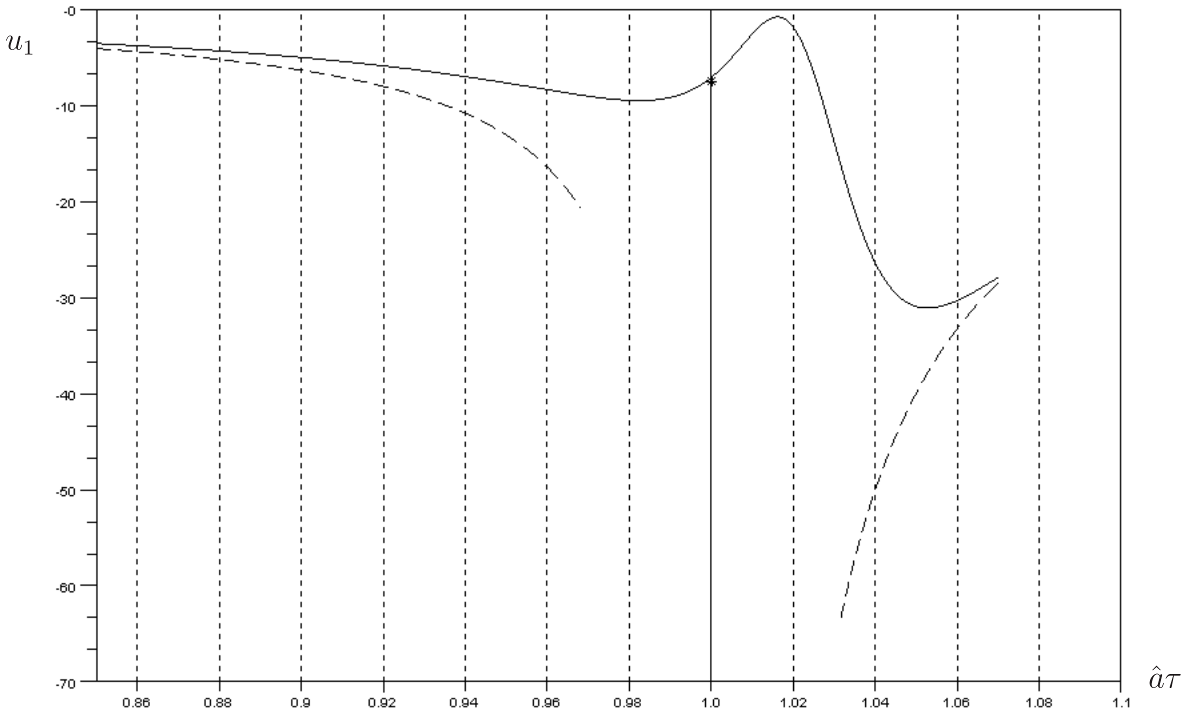


FIGURE 3.7: Asymptotic behaviour for uniform speed case (dashed line) with exact solution (solid line) and asymptotic result on Rayleigh speed (star) with $\hat{a} = 0.001$ for horizontal displacement u_1

3.4.2 Uniform asymptotic solution

In Section 3.4.1 we obtain several asymptotic formulae, which approximate the exact solution (3.31)–(3.32) (or, equivalently, (3.39)–(3.40)) either exactly on the Rayleigh speed (see (3.77) and (3.80)) or far from it (see (3.60) and (3.66) or (3.67) and (3.68)). So, the local vicinity of the critical speed is not covered by them. Note that in the previous section we have the only one small parameter, namely, acceleration $\hat{a}$. To describe the asymptotic behaviour of the exact solution near the Rayleigh speed we should consider one more parameter $(1 - \hat{a}\tau)^2$ of the same order as an acceleration $\hat{a}$, i.e. $\hat{a} \sim (1 - \hat{a}\tau)^2$.

We introduce an auxiliary small quantity ε that has the same order as a parameter $\hat{a}$ (and, consequently, $(1 - \hat{a}\tau)^2$). So,

$$O(\hat{a}) = O((1 - \hat{a}\tau)^2) = O(\varepsilon), \quad O(1 - \hat{a}\tau) = O(\sqrt{\varepsilon}).$$

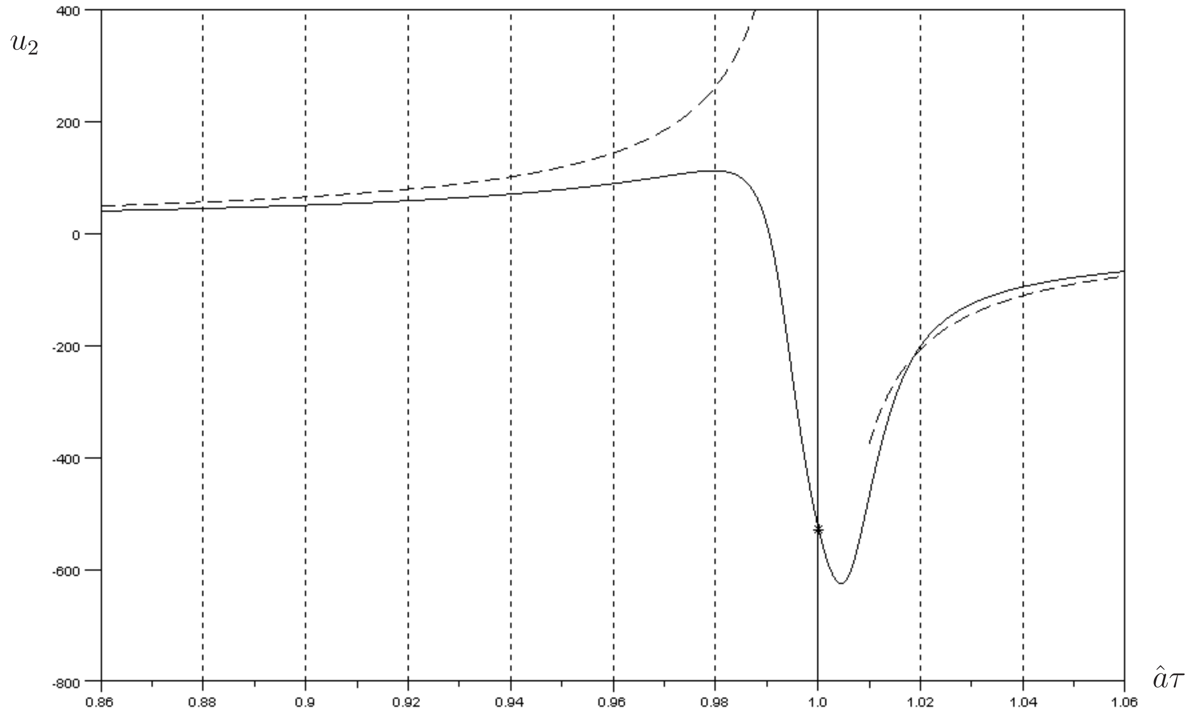


FIGURE 3.8: Asymptotic behaviour for uniform speed case (dashed line) with exact solution (solid line) and asymptotic result on Rayleigh speed (star) with $\hat{a} = 0.0001$ for vertical displacements $\hat{u}_2$

To find an approximation we use a procedure similar to the one for the non-uniform asymptotic representations and use the non-dimensional parameters. So, for each displacement the partial derivatives of the potentials are considered separately and as before we are interested only in terms of the leading order. Here to make the description less massive we start with an identification of the orders of all the auxiliary quantities. For the parameters (3.42)–(3.43) we get

$$\hat{b}_1 = (1 + \hat{a}\tau)^2 + 2\hat{a}\sigma \approx (1 + \hat{a}\tau)^2 = O(1),$$

$$\hat{\alpha}_1 = \sqrt{(1 + \hat{a}\tau)^4 + (2\hat{a}k_1)^2} \approx (1 + \hat{a}\tau)^2 = O(1),$$

$$\hat{\beta}_1 \approx \sqrt{2(1 + \hat{a}\tau)^2 + 2(1 + \hat{a}\tau)^2} = 2(1 + \hat{a}\tau) = O(1),$$

$$\hat{\gamma}_1 = \sqrt{2(\sqrt{((1 + \hat{a}\tau)^2 + 2\hat{a}\sigma)^2 + (2\hat{a}k_1)^2} - (1 + \hat{a}\tau)^2 - 2\hat{a}\sigma)} \approx \frac{2\hat{a}k_1}{1 + \hat{a}\tau} = O(\varepsilon),$$

$$\hat{b}_2 = (1 - \hat{a}\tau)^2 + 2\hat{a}\sigma = O(\varepsilon),$$

$$\hat{\alpha}_2 = \sqrt{((1 - \hat{a}\tau)^2 + 2\hat{a}\sigma)^2 + (2\hat{a}k_1)^2} = O(\varepsilon),$$

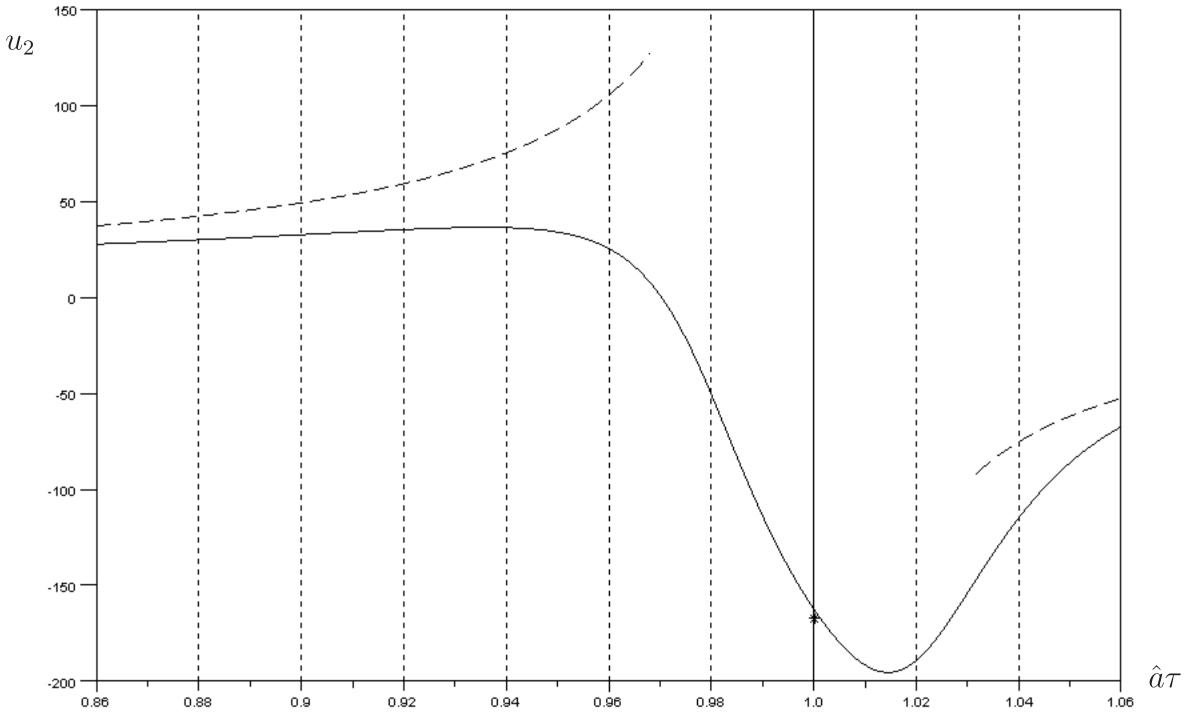


FIGURE 3.9: Asymptotic behaviour for uniform speed case (dashed line) with exact solution (solid line) and asymptotic result on Rayleigh speed (star) with $\hat{a} = 0.001$ for vertical displacements $\hat{u}_2$

$$\hat{\beta}_2 = \sqrt{2(\alpha_2 + b_2)} = O(\sqrt{\varepsilon}),$$

$$\hat{\gamma}_2 = \sqrt{2(\alpha_2 - b_2)} = O(\sqrt{\varepsilon}).$$

Consider the first displacement. The coefficients in front of the logarithm and pair of arctangents for the derivative of $\dfrac{\partial\hat{\phi}}{\partial\sigma}$ are

$$\frac{\hat{a}k_1}{\hat{\alpha}_1\hat{\beta}_1} = O(\varepsilon), \quad \frac{\hat{a}k_1}{\hat{\alpha}_2\hat{\beta}_2} = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

and

$$\frac{2\hat{a}k_1}{\hat{\alpha}_1\hat{\gamma}_1} = O(1), \quad \frac{2\hat{a}k_1}{\hat{\alpha}_2\hat{\gamma}_2} = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

respectively. Let us note that the leading order here is $O\left(\frac{1}{\varepsilon}\right)$. Hence, to find an approximation for $\dfrac{\partial\hat{\phi}}{\partial\sigma}$ we can neglect the terms of smaller order in advance. Now we calculate the logarithm and arctangents:

$$\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}{(1-\hat{a}\tau)^2+\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}\right]-\ln\left[\frac{1-\hat{\beta}_2+\hat{\alpha}_2}{1+\hat{\beta}_2+\hat{\alpha}_2}\right]\approx$$
$$\approx\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}{(1-\hat{a}\tau)^2+\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}\right].$$

Since the leading order term of the second logarithm's argument is unity, it vanishes as $\ln 1 = 0$, and

$$\arctan\left[\frac{2(1-\hat{a}\tau)-\hat{\beta}_2}{\hat{\gamma}_2}\right]+\arctan\left[\frac{2(1-\hat{a}\tau)+\hat{\beta}_2}{\hat{\gamma}_2}\right]-$$
$$-\arctan\left[\frac{2-\hat{\beta}_2}{\hat{\gamma}_2}\right]-\arctan\left[\frac{2+\hat{\beta}_2}{\hat{\gamma}_2}\right]\approx$$
$$\approx\arctan\left[\frac{2(1-\hat{a}\tau)-\hat{\beta}_2}{\hat{\gamma}_2}\right]+\arctan\left[\frac{2(1-\hat{a}\tau)+\hat{\beta}_2}{\hat{\gamma}_2}\right]-\pi,$$

because the arguments of the last two arctangents are of order $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$, which is a sufficiently large number.

So, combining the last two results, one can write the approximation for the derivative of the potential $\hat{\phi}$:

$$\frac{\partial\hat{\phi}}{\partial s}\approx\frac{\hat{a}k_1}{\hat{\alpha}_2\hat{\beta}_2}\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}{(1-\hat{a}\tau)^2+\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}\right]-$$
$$-\frac{2\hat{a}k_1}{\hat{\alpha}_2\hat{\gamma}_2}\left[\arctan\left[\frac{2(1-\hat{a}\tau)-\hat{\beta}_2}{\hat{\gamma}_2}\right]+\arctan\left[\frac{2(1-\hat{a}\tau)+\hat{\beta}_2}{\hat{\gamma}_2}\right]\right]+\frac{2\hat{a}k_1\pi}{\hat{\alpha}_2\hat{\gamma}_2}. \quad (3.81)$$

The coefficients in front of the logarithms and pairs of arctangents for $\dfrac{\partial\hat{\psi}}{\partial y}$ are:

$$\frac{2\hat{a}k_2^2}{2\hat{\hat{\alpha}}_1\hat{\hat{\beta}}_1}=O(\varepsilon),\quad\frac{\hat{a}k_2^2}{\hat{\hat{\alpha}}_2\hat{\hat{\beta}}_2}=O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

and

$$\frac{4\hat{a}k_2^2}{2\hat{\hat{\alpha}}_1\hat{\hat{\gamma}}_1}=O(1),\quad\frac{2\hat{a}k_2^2}{\hat{\hat{\alpha}}_2\hat{\hat{\gamma}}_2}=O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Note that all these coefficients have the same order as the respective ones for $\dfrac{\partial\hat{\phi}}{\partial\sigma}$. So, in this case one should consider the same logarithms and arctangents as before. Hence, an approximation for the derivative of the potential $\hat{\psi}$ has the following form:

$$\frac{\partial\hat{\psi}}{\partial y} \approx \frac{2k_2^2}{1+k_2^2}\,\frac{\hat{a}k_1}{\hat{\hat{\alpha}}_2\hat{\hat{\beta}}_2}\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\hat{\beta}}_2(1-\hat{a}\tau)+\hat{\hat{\alpha}}_2}{(1-\hat{a}\tau)^2+\hat{\hat{\beta}}_2(1-\hat{a}\tau)+\hat{\hat{\alpha}}_2}\right]-$$
$$-\frac{2k_2^2}{1+k_2^2}\,\frac{2\hat{a}k_1}{\hat{\hat{\alpha}}_2\hat{\hat{\gamma}}_2}\left[\arctan\left(\frac{2(1-\hat{a}\tau)-\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right)+\arctan\left(\frac{2(1-\hat{a}\tau)+\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right)\right]+$$
$$+\frac{2k_2^2}{1+k_2^2}\,\frac{2\hat{a}k_1\pi}{\hat{\hat{\alpha}}_2\hat{\hat{\gamma}}_2}. \quad (3.82)$$

Combining the formulae (3.81) and (3.82), we can write down a uniform asymptotic representation for the first displacement:

$$\hat{u}_1 \approx \frac{\hat{a}k_1}{\hat{\alpha}_2\hat{\beta}_2}\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}{(1-\hat{a}\tau)^2+\hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}\right]-$$
$$-\frac{2k_2^2}{1+k_2^2}\,\frac{\hat{a}k_1}{\hat{\hat{\alpha}}_2\hat{\hat{\beta}}_2}\ln\left[\frac{(1-\hat{a}\tau)^2-\hat{\hat{\beta}}_2(1-\hat{a}\tau)+\hat{\hat{\alpha}}_2}{(1-\hat{a}\tau)^2+\hat{\hat{\beta}}_2(1-\hat{a}\tau)+\hat{\hat{\alpha}}_2}\right]-$$
$$-\frac{2\hat{a}k_1}{\hat{\alpha}_2\hat{\gamma}_2}\left[\arctan\left(\frac{2(1-\hat{a}\tau)-\hat{\beta}_2}{\hat{\gamma}_2}\right)+\arctan\left(\frac{2(1-\hat{a}\tau)+\hat{\beta}_2}{\hat{\gamma}_2}\right)\right]+$$
$$+\frac{2k_2^2}{1+k_2^2}\,\frac{2\hat{a}k_1}{\hat{\hat{\alpha}}_2\hat{\hat{\gamma}}_2}\left[\arctan\left(\frac{2(1-\hat{a}\tau)-\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right)+\arctan\left(\frac{2(1-\hat{a}\tau)+\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right)\right]+$$
$$+2\hat{a}k_1\pi\left[\frac{1}{\hat{\alpha}_2\hat{\gamma}_2}-\frac{2k_2^2}{1+k_2^2}\,\frac{1}{\hat{\hat{\alpha}}_2\hat{\hat{\gamma}}_2}\right]. \quad (3.83)$$

To obtain an asymptotic formula for the second displacement let us find the orders of the coefficients in front of the logarithms and arctangents in the expressions for $\dfrac{\partial\hat{\phi}}{\partial y}$ and $\dfrac{\partial\hat{\psi}}{\partial\sigma}$. For the derivative of the potential $\hat{\phi}$ they are

$$-k_1\frac{\beta_1}{4\alpha_1}=O(1), \quad -k_1\frac{\beta_2}{4\alpha_2}=O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

$$-k_1\frac{\gamma_1}{2\alpha_1}=O(\varepsilon), \quad -k_1\frac{\gamma_2}{2\alpha_2}=O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

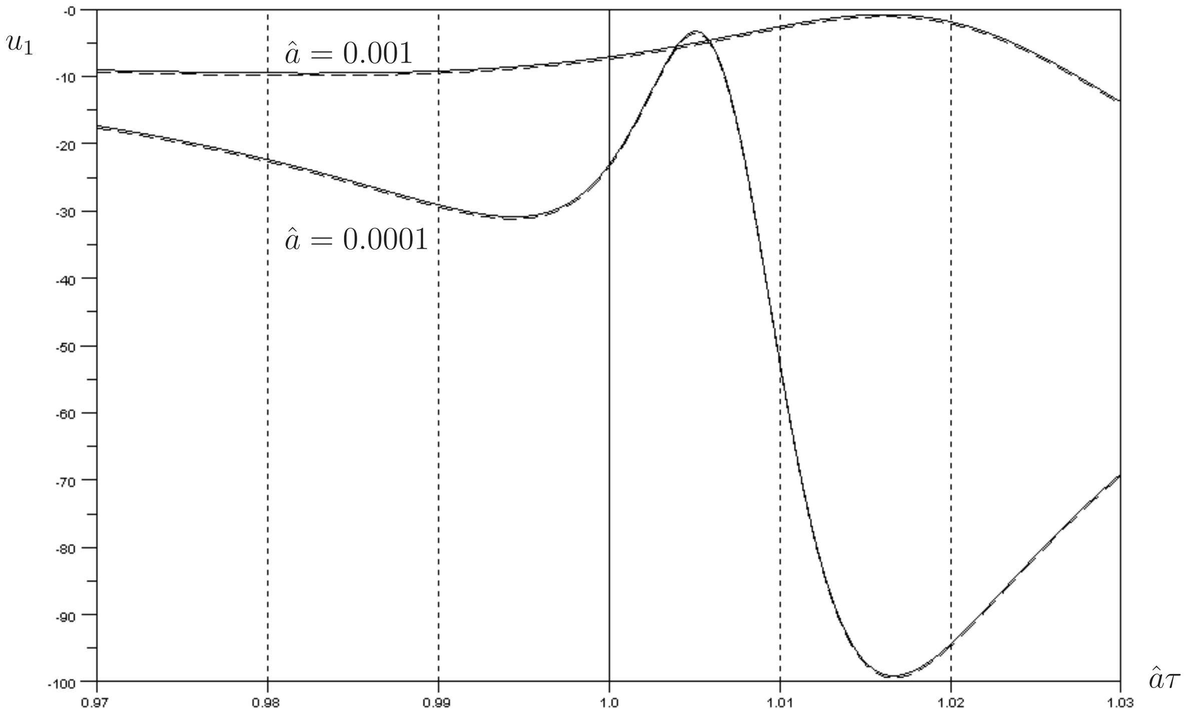


FIGURE 3.10: Uniform asymptotic behaviour (dashed line) and exact solution (solid line) of horizontal displacement u_1 under moving point load

For the derivative of the potential $\hat{\psi}$ the coefficients are

$$\frac{\beta_1}{4\alpha_1} = O(1), \quad \frac{\beta_2}{4\alpha_2} = O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

$$\frac{\gamma_1}{2\alpha_1} = O(\varepsilon), \quad \frac{\gamma_2}{2\alpha_2} = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Let us note that again we are interested only in the coefficients in front of exactly the same logarithms and arctangents as those considered in case for $\hat{u}_1$. So, by analogy with the first displacement, we can immediately write down the asymptotic formulae for $\dfrac{\partial\hat{\phi}}{\partial y}$:

$$\frac{\partial\hat{\phi}}{\partial y} \approx -k_1\left(\frac{\hat{\beta}_2}{4\hat{\alpha}_2}\ln\left(\frac{(1-\hat{a}\tau)^2 - \hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}{(1-\hat{a}\tau)^2 + \hat{\beta}_2(1-\hat{a}\tau)+\hat{\alpha}_2}\right)+\right.$$
$$\left.+\frac{\hat{\gamma}_2}{2\hat{\alpha}_2}\left[\arctan\left(\frac{2(1-\hat{a}\tau)-\hat{\beta}_2}{\hat{\gamma}_2}\right)+\arctan\left(\frac{2(1-\hat{a}\tau)+\hat{\beta}_2}{\hat{\gamma}_2}\right)-\pi\right]\right), \quad (3.84)$$

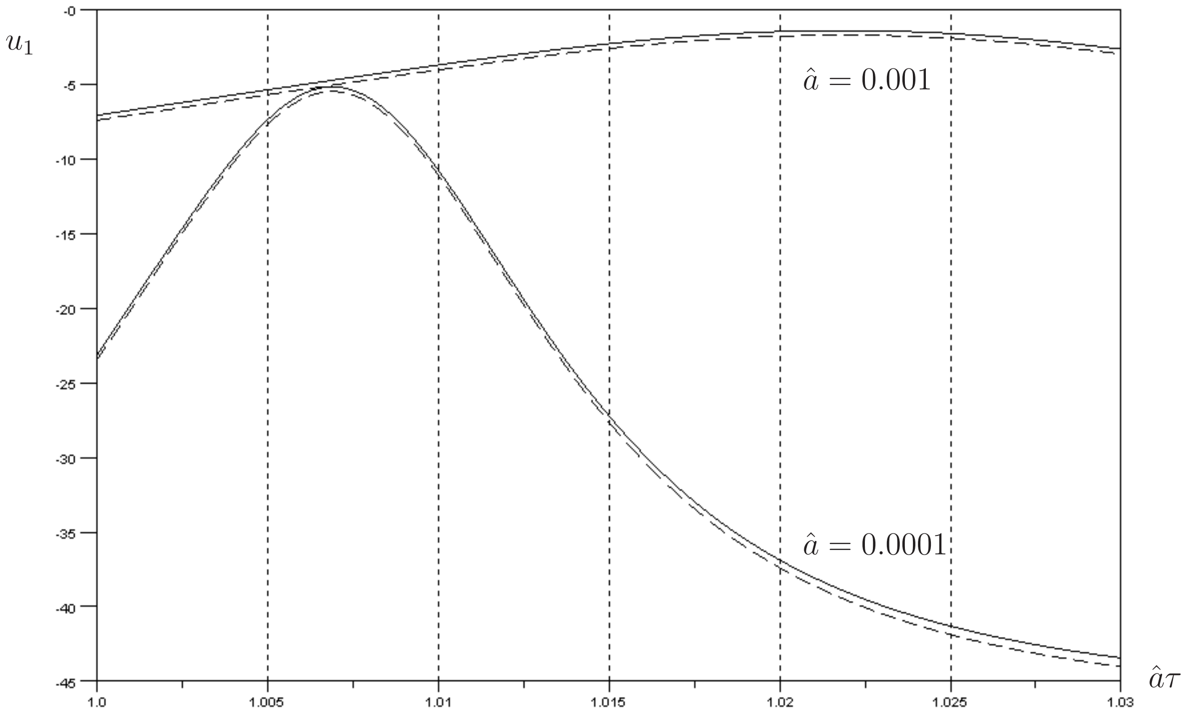


FIGURE 3.11: Uniform asymptotic behaviour (dashed line) and exact solution (solid line) of horizontal displacement u_1 under shock wave

and for $\dfrac{\partial\hat{\psi}}{\partial\sigma}$:

$$\frac{\partial\hat{\psi}}{\partial\sigma} \approx \frac{2k_1}{1+k_2^2}\frac{\hat{\hat{\beta}}_2}{4\hat{\hat{\alpha}}_2}\ln\left(\frac{(1-\hat{a}\tau)^2 - \hat{\hat{\beta}}_2(1-\hat{a}\tau) + \hat{\hat{\alpha}}_2}{(1-\hat{a}\tau)^2 + \hat{\hat{\beta}}_2(1-\hat{a}\tau) + \hat{\hat{\alpha}}_2}\right) -$$
$$- \frac{2k_1}{1+k_2^2}\frac{\hat{\hat{\gamma}}_2}{2\hat{\hat{\alpha}}_2}\left[\arctan\left(\frac{2(1-\hat{a}\tau)-\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right) + \arctan\left(\frac{2(1-\hat{a}\tau)+\hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2}\right) - \pi\right]. \quad (3.85)$$

Finally, using the formulae (3.84)–(3.85) we arrive at the asymptotic representation for the second displacement:

$$\hat{u}_2 \approx k_1 \left[\frac{2}{1+k_2^2} \frac{\hat{\hat{\beta}}_2}{4\hat{\hat{\alpha}}_2} \ln \left(\frac{(1-\hat{a}\tau)^2 - \hat{\hat{\beta}}_2(1-\hat{a}\tau) + \hat{\hat{\alpha}}_2}{(1-\hat{a}\tau)^2 + \hat{\hat{\beta}}_2(1-\hat{a}\tau) + \hat{\hat{\alpha}}_2} \right) - \right.$$
$$\left. - \frac{\hat{\beta}_2}{4\hat{\alpha}_2} \ln \left(\frac{(1-\hat{a}\tau)^2 - \hat{\beta}_2(1-\hat{a}\tau) + \hat{\alpha}_2}{(1-\hat{a}\tau)^2 + \hat{\beta}_2(1-\hat{a}\tau) + \hat{\alpha}_2} \right) \right] -$$
$$-k_1 \left[\frac{2}{1+k_2^2} \frac{\hat{\hat{\gamma}}_2}{2\hat{\hat{\alpha}}_2} \left[\arctan \left(\frac{2(1-\hat{a}\tau) - \hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2} \right) + \arctan \left(\frac{2(1-\hat{a}\tau) + \hat{\hat{\beta}}_2}{\hat{\hat{\gamma}}_2} \right) - \pi \right] + \right.$$
$$\left. + \frac{\hat{\gamma}_2}{2\hat{\alpha}_2} \left[\arctan \left(\frac{2(1-\hat{a}\tau) - \hat{\beta}_2}{\hat{\gamma}_2} \right) + \arctan \left(\frac{2(1-\hat{a}\tau) + \hat{\beta}_2}{\hat{\gamma}_2} \right) - \pi \right] \right]. \quad (3.86)$$

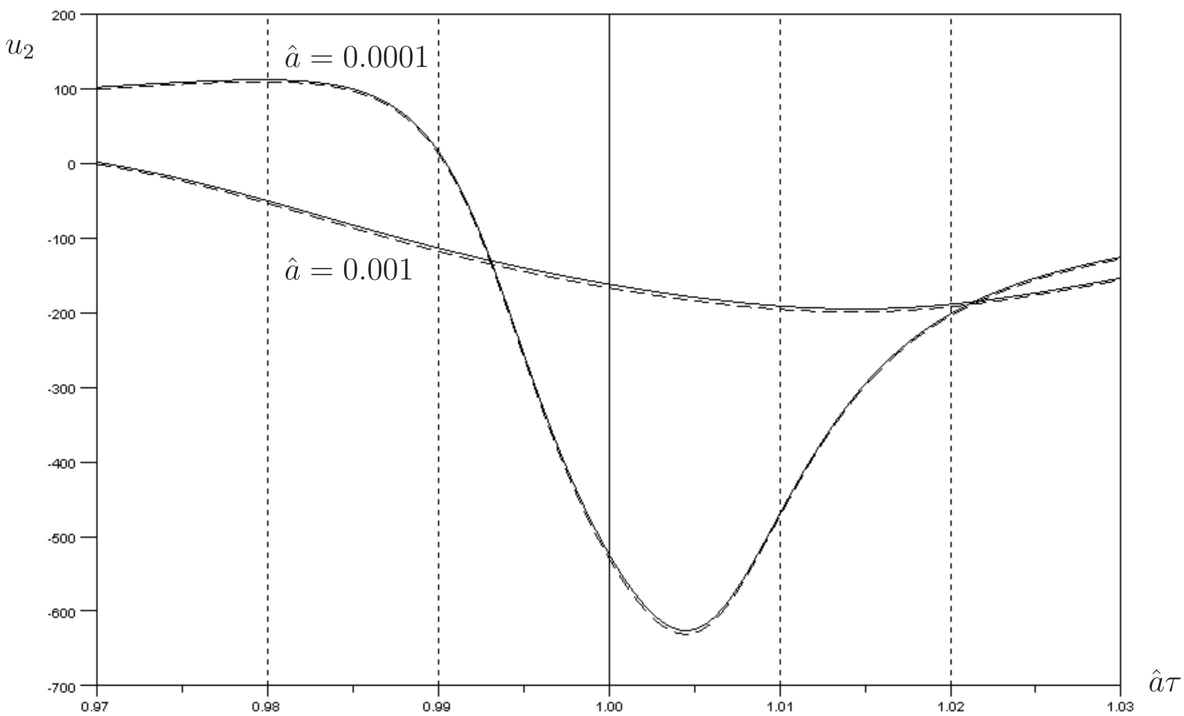


FIGURE 3.12: Uniform asymptotic behaviour (dashed line) and exact solution (solid line) of vertical displacement u_2 under moving point load

Formulae (3.83) and (3.86) are the same for the cases when $\hat{a}\tau \leq 1$ and $\hat{a}\tau \geq 1$, i.e. valid for all values of the load speed that are mentioned at the end of Section 3.2. Let us remind that this is so because all the terms with the logarithms and the arctangents in the expressions for the uniform asymptotic forms of $\hat{u}_1$ and $\hat{u}_2$ are odd.

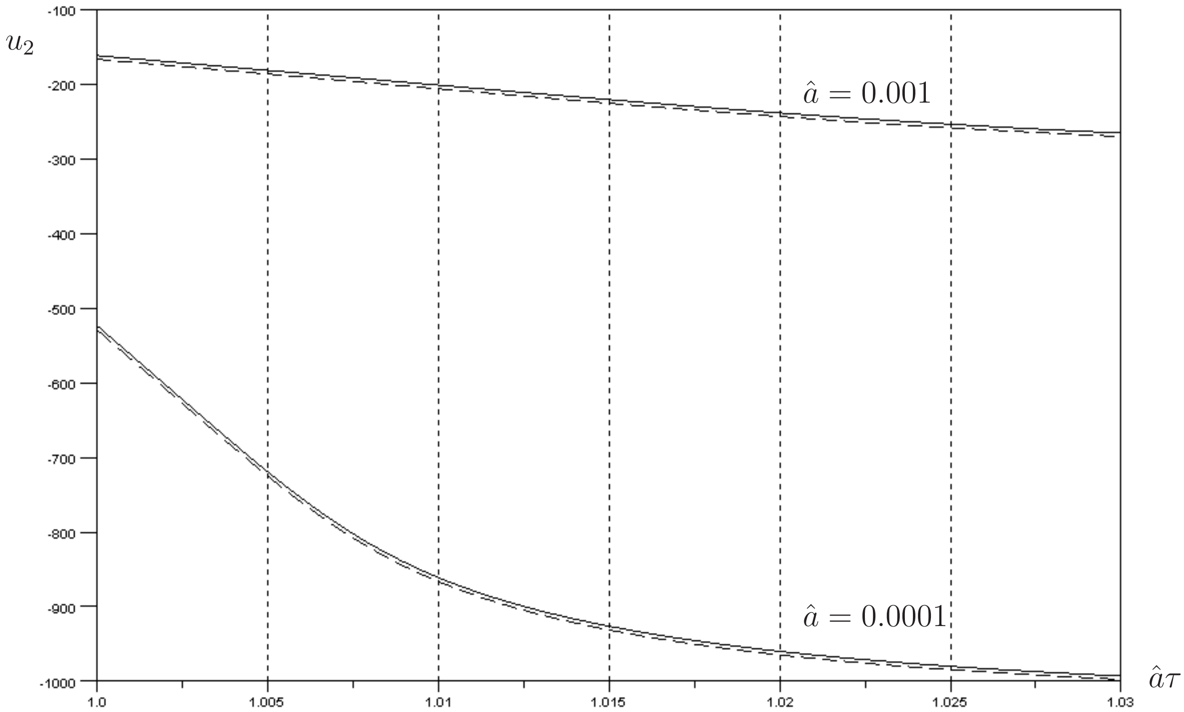


FIGURE 3.13: Uniform asymptotic behaviour (dashed line) and exact solution (solid line) of vertical displacement u_2 under shock wave

Now we will check how the obtained uniform asymptotic forms (3.83) and (3.86) are connected to the respective formulae for the stationary case (see (3.60) and (3.66)) and for the Rayleigh speed case (see (3.77) and (3.80)).

Start with considering (3.83) and (3.86) for $\hat{a}\tau = 1$, i.e. for the moment of passage through the critical speed. In this case the auxiliary quantities $\hat{b}_2$, $\hat{\alpha}_2$, $\hat{\beta}_2$ and $\hat{\gamma}_2$ can be rewritten as

$$\hat{b}_2 = 2\hat{a}\sigma, \quad \hat{\alpha}_2 = 2\hat{a}\sqrt{\sigma^2 + k_1^2},$$

$$\hat{\beta}_2 = 2\sqrt{\hat{a}}\sqrt{\sqrt{\sigma^2 + k_1^2} + \sigma}, \quad \hat{\gamma}_2 = 2\sqrt{\hat{a}}\sqrt{\sqrt{\sigma^2 + k_1^2} - \sigma}.$$

The formulae for the respective parameters with tildes are the same apart k_2 instead of k_1.

The direct substitution of $\hat{a}\tau = 1$, $\hat{b}_2$, $\hat{\alpha}_2$, $\hat{\beta}_2$, $\hat{\gamma}_2$ (and quantities with tildes) into (3.83) and (3.86) and some obvious transformations lead to the following results:

$$\hat{u}_1 \approx \frac{k_1\pi}{2\sqrt{\hat{a}}}\left[\frac{1}{\sqrt{\sigma^2+k_1^2}\sqrt{\sqrt{\sigma^2+k_1^2}-\sigma}} - \right.$$
$$\left. - \frac{2k_2^2}{1+k_2^2}\,\frac{1}{\sqrt{\sigma^2+k_2^2}\sqrt{\sqrt{\sigma^2+k_2^2}-\sigma^2}}\right], \tag{3.87}$$

$$\hat{u}_2 \approx \frac{\pi k_1}{\sqrt{\hat{a}}}\left[\frac{\sqrt{\sqrt{\sigma^2+k_1^2}-\sigma}}{2\sqrt{\sigma^2+k_1^2}} + \frac{1}{1+k_2^2}\frac{\sqrt{\sqrt{\sigma^2+k_2^2}-\sigma}}{\sqrt{\sigma^2+k_2^2}}\right]. \tag{3.88}$$

One can see that these expressions are a non-dimensional analogue to the asymptotic formulae, obtained for the case of the Rayleigh speed in Section 3.4.1.2. So, we can state that our uniform approximations (3.83) and (3.86) transform to the formulae (3.77) and (3.80) exactly on the critical speed.

Now in order to check how the uniform asymptotic formulae for $\hat{u}_1$ and $\hat{u}_2$ deal with the stationary case we consider (3.83) and (3.86) for the load speed sufficiently far from the critical one. So, the same assumptions as in Section 3.4.1.1, i.e. when $\hat{a} \to 0$, $\hat{a}\tau \to v = \text{const}$ and $\frac{1}{2}\hat{a}\tau^2 \to v\tau$, can be used. All the calculations and arguments below are given for the case $\hat{a}\tau < 1$, but can be equally applied for $\hat{a}\tau > 1$. Using the expansion (3.52) one can find the following representation for the auxiliary parameters $\hat{\alpha}_2$, $\hat{\beta}_2$ and $\hat{\gamma}_2$:

$$\hat{\alpha}_2 \approx (1-\hat{a}\tau)^2\left(1+\frac{2\hat{a}\sigma}{(1-\hat{a}\tau)^2}+\frac{(2\hat{a}k_1)^2}{(1-\hat{a}\tau)^4}\right), \tag{3.89}$$

$$\hat{\beta}_2 \approx 2(1-\hat{a}\tau)\left(1+\frac{\hat{a}\sigma}{(1-\hat{a}\tau)^2}+\frac{\hat{a}^2(k_1^2-\sigma^2)}{2(1-\hat{a}\tau)^4}\right), \tag{3.90}$$

$$\gamma_2 \approx \frac{2\hat{a}k_1}{1-\hat{a}\tau}. \tag{3.91}$$

Substituting (3.89)–(3.91) into (3.83) and (3.86) and taking a limit while $\hat{a} \to 0$, $\hat{a}\tau \to v = \text{const}$ and $\frac{1}{2}\hat{a}\tau^2 \to v\tau$ we get

$$\hat{u}_1 \approx \frac{1}{1-v}\left[\arctan\left(\frac{\sigma}{k_1}\right) - \frac{2k_1k_2}{1+k_2^2}\arctan\left(\frac{\sigma}{k_2}\right)\right] + \frac{\pi(1-k_2^2)}{1-v} \tag{3.92}$$

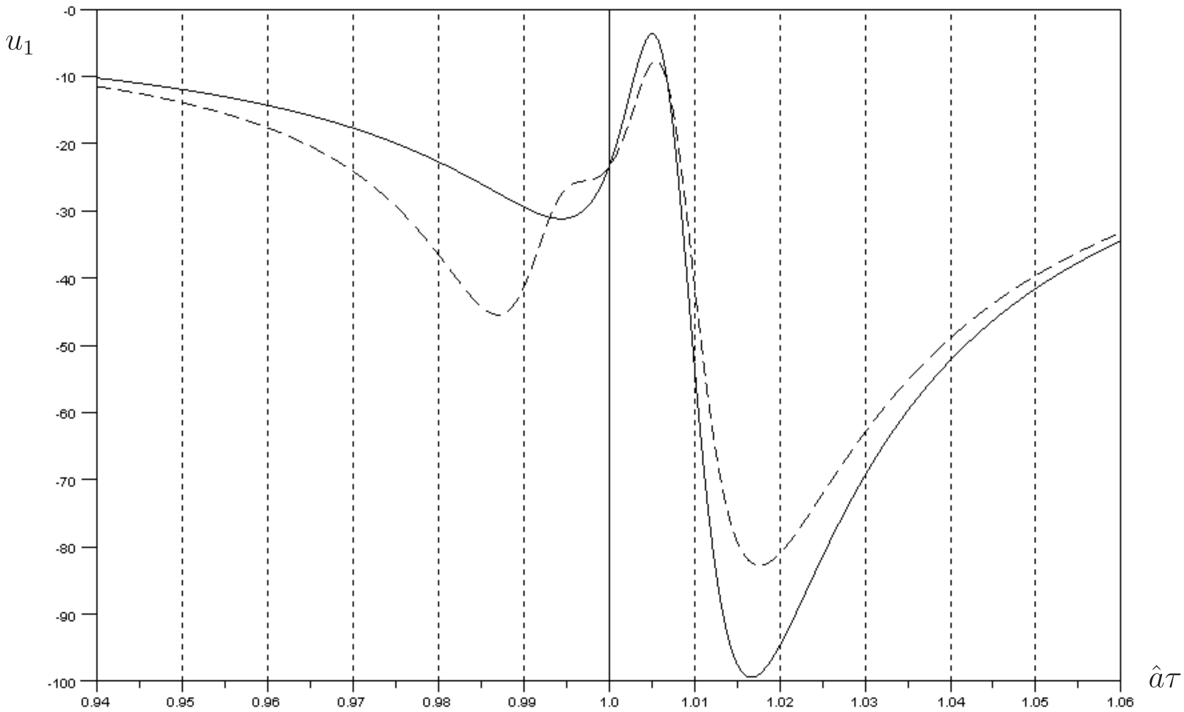


FIGURE 3.14: Uniform asymptotic behaviour with (solid line) and without (dashed line) terms with logarithms of horizontal displacement u_1 under moving point load

and

$$\hat{u}_2 \approx -k_1 \left[\frac{1}{2(1-v)} \left(\ln\left(k_1^2+\sigma^2\right) - \frac{2}{1+k_2^2} \ln\left(k_2^2+\sigma^2\right) \right) \right] + \\ + \frac{\ln v}{1-v}\, \frac{k_2^2-1}{k_2^2+1} - \frac{\ln 2\tau}{1-v}\, \frac{k_2^2-1}{k_2^2+1} - \frac{2\ln(1-v)}{1-v}\, \frac{k_2^2-1}{k_2^2+1}. \tag{3.93}$$

As one can see, the latter expressions are different from those obtained in Section 3.4.1.1 (see (3.60) and (3.66)). However, to find the asymptotic formulae (3.83) and (3.86) we assumed that $1 - \hat{a}\tau$ is a small parameter, thus, found that the leading order terms are of order $O\left(\frac{1}{1-\hat{a}\tau}\right)$ and neglected all the others. This can explain the difference between the formulae (3.60), (3.66) and (3.92), (3.93).

The graphs on Figures 3.10–3.13 represent the exact solution (solid line) and the uniform asymptotic solution (dashed line) obtained in this section for the displacements under a moving point load (Figure 3.10 and Figure 3.12) and under a shock wave (Figure 3.11 and Figure 3.13). Clearly, our asymptotic solution

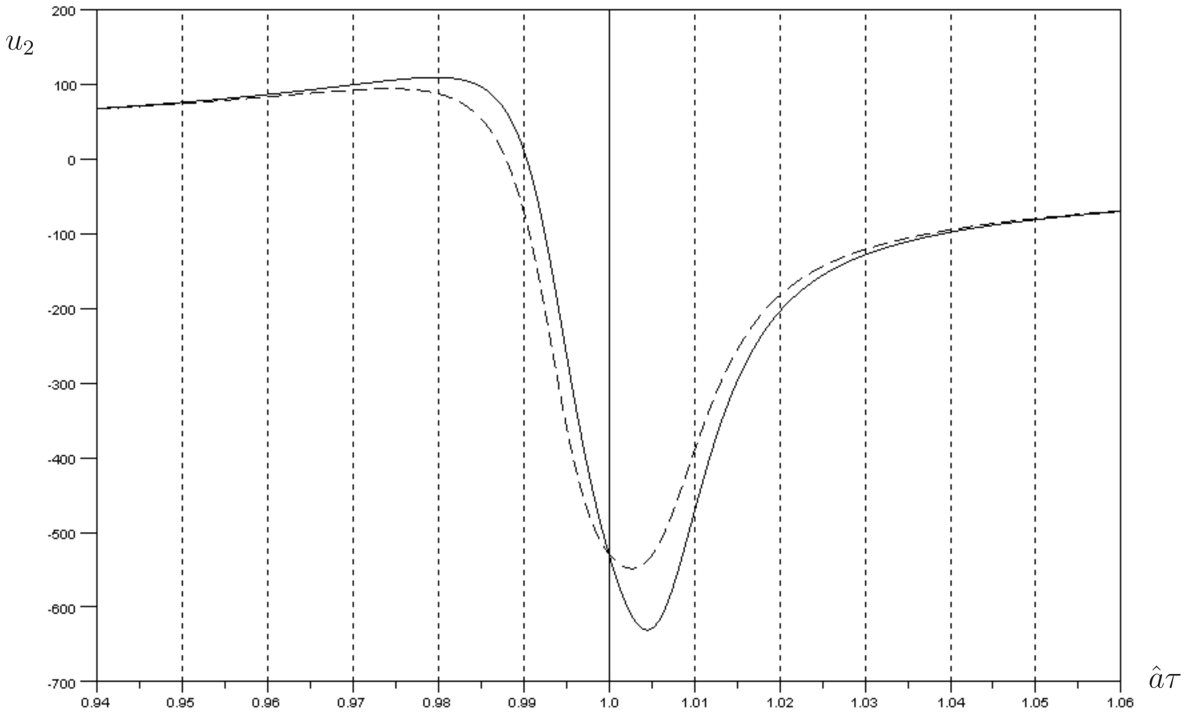


FIGURE 3.15: Uniform asymptotic behaviour with (solid line) and without (dashed line) terms with arctangents of vertical displacement u_2 under moving point load

provide a very accurate approximation for the vicinity of the Rayleigh speed, i.e. for the transient effect region. As one can see the derived uniform asymptotic formulae work well for the different sufficiently small values of the parameter $\hat{a}$.

Note also that the formulae (3.92) and (3.87) do not contain terms with the logarithms, as well as there are no terms with the arctangents in the formulae (3.93) and (3.88). So, it is necessary to define their contribution to the main uniform asymptotic expressions (3.83) and (3.86). As one can see from Figure 3.14 and Figure 3.15 these terms play a very important role in the region where the uniform asymptotic formulae can be applied.

3.5 Summary

In this chapter we dealt with the asymptotic model given in [61] which describes the Rayleigh waves which appear in an elastic half plane subject to a moving point load.

The boundary conditions for the main problem were obtained in Section 3.2 as a solution for the problem of a string on the surface of a half plane subject to a moving load.

The exact solution over the interior of a half plane was derived for a uniformly accelerated load within the framework of the above mentioned model. The asymptotic analysis and graphical representation for this solution was given.

In Section 3.4.1 the steady speed and the Rayleigh wave speed asymptotic expansions were obtained. The graphical comparative analysis of the exact solution and the approximations were provided for different moving load speed intervals and values of the key small parameter $\hat{a}$.

In the neighborhood of the Rayleigh speed we derived uniform asymptotic formulae which can be applied for those values of moving load speed where $\hat{a} \sim (1-v)^2$. The obtained asymptotic forms were compared graphically with the exact solution. Some interesting properties of the uniform asymptotic formulae were discovered and briefly studied.

So, our set of sufficiently simple asymptotic formulae obtained in this chapter gives quite an accurate approximation for the very complicated exact solution within the framework of the model.

Concluding remarks

In this thesis the asymptotic behaviour of an elastically supported string and an elastic homogeneous half plane, both subject to a moving load, were considered.

For both structures uniform approximations which describe their behaviour for the wide range of the load speed and in particular in the vicinity of the wave speeds were derived; small magnitude of the load acceleration was assumed. The asymptotic formulae obtained in this work were compared numerically (and graphically) with the exact solutions and, in some cases, with each other.

To describe the string behaviour subject to a moving load the auxiliary canonical functions $\mathcal{F}_i, \quad i = 1, 2, 3$, were introduced, asymptotically analyzed and tabulated. Some asymptotic analysis techniques for integrals with Bessel functions were improved. Uniform asymptotic formulae for a string under the constant accelerating and decelerating point loads as well as for an arbitrary acceleration case were obtained using the introduced canonical functions $\mathcal{F}_i, \quad i = 1, 2, 3$. As an example of the approximation for an arbitrary accelerated load, the sinusoidal load speed case was considered. Approximate formulae for the displacements in the vicinity of a point load and a shock wave using the steady speed asymptotic expansion with additional contributions from the stationary points where appropriate were derived.

Within the framework of the asymptotic model given in [61] the boundary conditions and the exact solution (in form of elementary functions) for the main problem over the interior of a half plane subject to a point load moving with a small constant acceleration were obtained. The steady speed, the Rayleigh wave speed and uniform asymptotic expansions were derived and compared with the exact solution for different values of the parameters. All those formulae can be applied for all the interior points of a half plane under the certain assumptions.

Some interesting properties of the uniform asymptotic forms were discovered and briefly studied.

Bibliography

[1] J. Kaplunov, V. Voloshin, and A. D. Rawlins. Uniform asymptotic behaviour of integrals of bessel functions with a large parameter in the argument. *The Quarterly Journal of Mechanics and Applied Mathematics*, 63(1):57–72, 2010.

[2] O P Miridonova. The first steps in physical acoustics in the works of galileo and mersenne (in Russian). *Voprosy Istor. Estest. i Tehn.*, 16:85–90, 1964.

[3] J. Stillwell. *Mathematics and its history.* Springer, New York, 2002.

[4] R. Knobel. *An introduction to the mathematical theory of waves.* American Mathematical Society, Providence, R.I., 2000.

[5] W. Weaver, S. Timoshenko and D.H. Young. *Vibration problems in engineering.* John Wiley and Sons, New York, 1990.

[6] S.P. Timoshenko and J.M. Gere. *Theory of elastic stability.* McGraw-Hill, New York, Toronto, London, 1961.

[7] Lord Rayleigh. On waves propagated along the plane surface of an elastic solid. *Proc. London Math. Soc.*, s1-17(1):4–11, 1885.

[8] A. Mal, S. Banerjee and F. Ricci. An automated damage identification technique based on vibration and wave propagation data. *Phil. Trans. R. Soc. A*, 365:479–491, 2007.

[9] H. Lamb. On the propagation of tremors over the surface of an elastic solid. *Phil. Trans. R. Soc. Lond. A*, 203:1–42, 1904.

[10] L. Cagniard. *Reflection and refraction of progressive seismic waves.* McGraw-Hill, New York, 1962.

[11] A. T. De Hoop. A modification of cagniards method for solving seismic pulse problems. *Applied Scientific Research, Section B*, 8(1):349–356, 1960.

[12] H.M. Mooney. Some numerical solutions for lamb's problem. *Bulletin of the Seismological Society of America*, 64(2):473–491, 1974.

[13] S. W. Liu, S. K. Datta and T. H. Ju. Transient scattering of rayleigh-lamb waves by a surface-breaking crack: Comparison of numerical simulation and experiment. *Journal of Nondestructive Evaluation*, 10(3):111–126, 1991.

[14] J.M. Galán and R. Abascal. Numerical simulation of lamb wave scattering in semi-infinite plates. *International Journal for Numerical Methods in Engineering*, 53(5):1145–1173, 2001.

[15] J.L. Synge. Elastic waves in anisotropic media. *J. Math. and Phys.*, 35:323–334, 1956.

[16] A.N. Stroh. A note on the formula for the rayleigh wave speed. *Phil. Mag.*, 3:625–646, 1958.

[17] A.N. Stroh. Steady state problems in anisotropic elasticity. *J. Math. and Phys.*, 41:77–103, 1962.

[18] P.K. Currie. Rayleigh waves on elastic crystals. *The Quarterly Journal of Mechanics and Applied Mathematics*, 27(4):489–496, 1974.

[19] D.M. Barnett and J. Lothe. Consideration of the existence of surface wave (rayleigh wave) solutions in anisotropic elastic crystals. *J. Phys. F: Metal Phys.*, 4(5):671–686, 1974.

[20] J. Lothe and D.M. Barnett. On the existence of surface-wave solutions for anisotropic elastic half-spaces with free surface. *J. Appl. Phys.*, 47:428–433, 1976.

[21] P. Chadwick and G.D. Smith. Foundations of the theory of surface waves in anisotropic elastic materials. In C.-S. Yih, editor, *Advances in Applied Mechanics*, pages 303–376. Academic Press, New York, 1977.

[22] D.M. Barnett, J. Lothe, K. Nishioka and R.J. Asaro. Elastic surface waves in anisotropic crystals : a simplified method for calculating rayleigh velocities using dislocation theory. *J. Phys. F: Metal Phys.*, 3(6):1083–1096, 1973.

[23] A. Mielke and Y.B. Fu. Uniqueness of the surface-wave speed: A proof that is independent of the stroh formalism. *Mathematics and Mechanics of Solids*, 9(1):5–15, 2004.

[24] J.D. Achenbach. *Wave propagation in elastic solids*. North-Holland, Amsterdam, 1993.

[25] M. Rahman and J.R. Barber. Exact expressions for the roots of the secular equation for rayleigh waves. *Journal of Applied Mechanics*, 62(1):250–252, 1995.

[26] D. Nkemzi. A new formula for the velocity of rayleigh waves. *Wave Motion*, 26(2):199–205, 1997.

[27] P.G. Malischewsky. Comment to a new formula for the velocity of rayleigh waves by d. nkemzi [wave motion 26 (1997) 199205]. *Wave Motion*, 31(1):93–96, 2000.

[28] P.C. Vinh and R.W. Ogden. On formulas for the rayleigh wave speed. *Wave Motion*, 39:191–197, 2004.

[29] M. Rahman and T. Michelitsch. A note on the formula for the rayleigh wave speed. *Wave Motion*, 43(3):272–276, 2006.

[30] J. Cole and J. Huth. Stresses produced in half plane by moving loads. *Journal of applied mechanics*, 25:433–436, 1958.

[31] H.G. Georgiadis and J.R. Barber. Steady-state transonic motion of a line load over an elastic half-space: the corrected cole/huth solution. *Transaction of the ASME*, 60:772–774, 1993.

[32] S.K. Singh and J.T. Kuo. Response of an elastic half space to uniformly moving circular surface load. *Journal of applied mechanics*, 37:109–115, 1970.

[33] G. Müller. Moving load on an elastic isotropic half space. *Acustica*, 72:47–53, 1990.

[34] G. Müller. Circulary expanding load on the surface of a half-space. *Archive of Applied Mechanics*, 61:276–283, 1991.

[35] P.M. Belotserkovskiy. Forced oscilations and resonance of infinite periodic strings. *Journal of Sound and Vibration*, 204(1):41–57, 1997.

[36] T.C. Kennedy and G. Herrmann. Moving load on a fluid-solid interface:subsonic and intersonic regimes. *Journal of applied mechanics*, 40:885–890, 1973.

[37] G.B. Muravski. Effect of moving force system on an elastically supported beam (in Russian). *Izvestia AN SSSR. Mechanika tverdogo tela*, (3):190–195, 1975.

[38] L. Frýba. Non-stationary response of a beam to a moving random force. *Journal of Sound and Vibration*, 46(3):323–338, 1976.

[39] J.W. Craggs. On two dimentional waves in an elastic half space. *Proceedings of the Cambridge Philosophical Society*, 56(3):269–285, 1960.

[40] D.C. Gakenheimer and J. Miklowitz. Transient excitation of an elastic half space by a point load traveling on the surface. *Journal of applied mechanics*, 30:505–515, 1969.

[41] M.F. Kanninen and A.L. Florence. Traveling forces on strings and membranes. *Int. J. Solids Structures*, 3(5b):143–154, 1967.

[42] A.O. Cifuentes. Dynamic response of a beam excited by a moving mass. *Finite Elements in Analysis and Design*, 5:237–246, 1989.

[43] I.A. Duplyakin. The motion of a carriage with constant velocity along a beam of infinite length resting on a base with two elastic characteristics. *J. Appl. Maths Mechs*, 55(3):376–384, 1991.

[44] F.T. Flaherty, Jr. Transient resonance of an ideal string under a load moving with varying speed. *Int. J. Solids Structures*, 4(12):1221–1231, 1968.

[45] W.J. Stronge. An accelerating force on a string. *The journal of the acoustical society of America*, 50(5):1382–1383, 1971.

[46] W.J. Stronge. A load accelerating on the surface of an acoustic half space. *Journal of applied mechanics*, 37:1077–1082, 1970.

[47] K.I. Beitin. Response of an elastic half space to a decelerating surface point load. *Journal of applied mechanics*, 30:819–826, 1969.

[48] M.K. Myers. Transient excitation of a liquid half space by a decelerating load. *Journal of applied mechanics*, 37:475–479, 1970.

[49] L.B. Freund. The response of an elastic solid to nonuniformly moving surface loads. *Journal of applied mechanics*, 40:699–704, 1973.

[50] K. Watanabe. Transient response of an elastic half space subjected to a reciprocating antiplane shear load. *Journal of applied mechanics*, 98(4):625–629, 1976.

[51] K. Watanabe. Transient response of a layered elastic half space subjected to a reciprocating anti-plane shear load. *Int. J. Solids Structures*, 13:63–74, 1977.

[52] K. Watanabe. Transient response of an elastic half space to moving load. *Bulletin of the JSME*, 24:913–920, 1981.

[53] K. Watanabe. Transient response of an infinite elastic solid to a moving point load. *Bulletin of the JSME*, 24:1115–1122, 1981.

[54] G.B. Muravski and N.P. Krasikova. Vibrations of beam on deforming base with uniformly varying motion of point load (in Russian). *Stroitelnaya mechanika i raschet sooruzheniy*, (3):46–50, 1984.

[55] S. Gavrilov. Non-stationary problems in dynamics of a string on an elastic foundation subjected to a moving load. *Journal of Sound and Vibration*, 222:345–361, 1999.

[56] S. Gavrilov. On transition of moving load through the critical speed in elastic waveguide (in Russian). *Journal of technical physics (Zhurnal tehnicheskoi fiziki)*, 70:138–140, 2000.

[57] S. Gavrilov. Nonlinear investigation of the possibility to exceed the critical speed by a load on a string. *Acta Mechanica*, 154:47–60, 2002.

[58] L. Frýba. *Vibration of solids and structures under moving loads.* Noordhoff international publishing, Groningen, Prague, 1972.

[59] J. Kaplunov and G. Muravski. Vibrations of an infinite string on a deformable base under a uniformly accelerated load: passage through the critical velocity. *Mechanics of Solids (Izvestiya AN SSSR, Seriya Mekhanika Tverdogo Tela)*, 21(1):155–160, 1986.

[60] J. Kaplunov. *Transient dynamics of an elastic half-plane subject to a moving load (in Russian, Preprint No. 277).* Institute for Problems in Mechanics, Moscow, 1986.

[61] J. Kaplunov, A. Zakharov and D.A. Prikazchikov. Explicit models for elastic and piezoelastic surface waves. *IMA J. Appl. Math.*, 71:768–782, 2006.

[62] J. Kaplunov, E. Nolde and D.A. Prikazchikov. A revisit to the moving load problem using an asymptotic model for the rayleigh wave. *Wave motion*, 2010. Accepted, in press.

[63] B.C. Carlson. *Special functions of applied mathematics.* Academic press, New York, San Francisco, London, 1977.

[64] N.N. Lebedev. *Special functions and their applications.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1965.

[65] B.G. Korenev. *Bessel functions and their applications.* Taylor and Fracis, London, New York, 2002.

[66] A. Erdelyi. *Asymptotic Expansions.* Dover, New York, 1955.

[67] E.T. Copson. *Asymptotic Expansions.* Cambridge University Press, Cambridge, 1965.

[68] V.A. Borovikov. *Uniform stationary phase method.* The Institute of Electrical Engineers, London, 1994.

[69] P.J. Davis and P. Rabinowitz. *Methods of numerical integration.* Academic press, New York, London, 1984.

[70] J.D. Hoffman. *Numerical Methods for Engineers and Scientists.* M. Dekker, New York, 2001.

[71] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, Washington, D.C., 1964.

[72] N. Bleistein and R.A. Handelsman. *Asymptotic expansions of integrals.* Courier Dover Publications, New York, 1986.

[73] I.S. Gradshteyn and I.M. Ryzhik. *Tables of Integrals, Series and Products.* Academic Press, New York, 1994.

[74] R. Wong. *Asymptotic approximations of integrals.* Computer Science and Scientific Computing, Academic Press, London, 1989.

[75] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 2. John Wiley and Sons, New York, 1989.