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Hartley Sets and Injectors of a Finite Group

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Abstract—By a Fitting set of a group G one means a nonempty set of subgroups $\mathscr{F}$ of a finite group G which is closed under taking normal subgroups, their products, and conjugations of subgroups. In the present paper, the existence and conjugacy of $\mathscr{F}$-injectors of a partially π-solvable group G is proved and the structure of $\mathscr{F}$-injectors is described for the case in which $\mathscr{F}$ is a Hartley set of G.

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1. INTRODUCTION

In this paper, we consider only finite groups. We use the notation of [1]–[3].

A fundamental result in the theory of classes of finite solvable groups is a generalization of the fundamental theorems of Sylow and Hall which was obtained by Gaschütz, Fischer, and Hartley in [4], where it was established that, for any Fitting class $\mathfrak{F}$, every solvable group G admits $\mathfrak{F}$-injectors, and any two of them are conjugate in G.

Let G be a group, and let $\mathfrak{F}$ be a class of groups. If H is a subgroup of G and $H \in \mathfrak{F}$, then H is called an $\mathfrak{F}$-*subgroup*.

Recall that by a *Fitting class* one means a class $\mathfrak{F}$ of groups which is closed under taking normal subgroups and products of normal $\mathfrak{F}$-subgroups. It follows from the definition that, for any nonempty Fitting class $\mathfrak{F}$, every group G has a unique maximal normal $\mathfrak{F}$-subgroup, which is called the $\mathfrak{F}$-*radical* of G and denoted by $G_{\mathfrak{F}}$.

If $\mathfrak{F}$ is a nonempty class of groups, then a subgroup V of a group G is said to be

(a) $\mathfrak{F}$-*maximal* if $V \in \mathfrak{F}$ and $U{=}V$ provided that $V \le U \le G$ and $U \in \mathfrak{F}$;

(b) an $\mathfrak{F}$-*injector* if $V \cap N$ is an $\mathfrak{F}$-maximal subgroup N for every subnormal subgroup N of G.

Localizing the notion of a Fitting class, Shemetkov [5] (and, in the solvable case, Anderson [6]) introduced the notion of a Fitting set of a group G. By a *Fitting set* of G one means a nonempty set of subgroups of G which is closed under taking normal subgroups, their products, and conjugations of subgroups. The definition of an $\mathscr{F}$-*injector* of a group G for its Fitting set is similar to the corresponding definition for a Fitting class.

Developing and generalizing the result of Fischer, Gaschütz, and Hartley [4], Shemetkov proved that if $\mathscr{F}$ is a Fitting set of a group G, π is the set of all prime divisors of all $\mathscr{F}$-subgroups of G, and the group G is π-solvable, then G has a unique class of conjugate $\mathscr{F}$-injectors.

Note that to every nonempty Fitting class $\mathfrak{F}$ there corresponds a Fitting set

$$\mathrm{Tr}_{\mathfrak{F}}(G) = \{H \le G : H \in \mathfrak{F}\}$$

in the group G, the so-called *trace* of the Fitting class $\mathfrak{F}$ in the group G, although the converse fails to hold in the general case (see [1, Example VIII.2.2(c)]).

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If $\mathscr{F} = \mathrm{Tr}_{\mathfrak{F}}(G)$, then the sets of $\mathfrak{F}$-injectors and $\mathscr{F}$-injectors of the group G coincide, and the Fischer–Gaschütz–Hartley theorem mentioned above is a consequence of Shemetkov's theorem [5].

Let $\mathbb{P}$ be the set of all primes, and let $\pi \subseteq \mathbb{P}$. We denote the complement to π in the set $\mathbb{P}$ by π'; thus, $\pi' = \mathbb{P} \setminus \pi$. For the case in which $\pi = \{p\}$, we denote the set $\{p\}'$ by p'. If $\varnothing \neq \pi \subseteq \mathbb{P}$, then a Fitting set $\mathscr{F}$ of a group G is said to be π-*saturated* if $\mathscr{F} = \{H \le G : H/H_{\mathscr{F}} \in \mathfrak{E}_{\pi'}\}$.

Vorob'ev and Semenov [7] proved that, in every π-solvable group G, there are $\mathscr{F}$-injectors for any π-saturated Fitting set $\mathscr{F}$ of G, and every two of these $\mathscr{F}$-injectors are conjugate in G.

The problem of describing the structure of injectors for local Fitting classes was first considered by Hartley [8] and D'Arcy [9]. In the present paper, we develop a local method for studying the Fitting sets $\mathscr{F}$ of a group G (nonsolvable in the general case) and use this method to describe the structure of $\mathscr{F}$-injectors of G.

Recall that by the *product of Fitting classes* $\mathfrak{F}$ *and* $\mathfrak{H}$ one means the class of groups

$$\mathfrak{F} \diamond \mathfrak{H} = (G : G/G_{\mathfrak{F}} \in \mathfrak{H}).$$

As is well known, the product of any two Fitting classes is also a Fitting class, and the multiplication operation for Fitting classes is associative (see [1, Theorem IX.(1.12)(a),(c)]). Note that if $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes, then $\mathfrak{F} \diamond \mathfrak{H} \subseteq \mathfrak{F}\mathfrak{H}$, where

$$\mathfrak{F}\mathfrak{H} = (G : \exists\, K \trianglelefteq G,\ K \in \mathfrak{F} \text{ and } G/K \in \mathfrak{H}).$$

If $\mathfrak{H}$ is a homomorph, i.e., if it is closed under homomorphic images, then $\mathfrak{F} \diamond \mathfrak{H} = \mathfrak{F}\mathfrak{H}$ (see [1, Remark II.1.11]). Therefore, we denote the product $\mathfrak{F} \diamond \mathfrak{H}$ of Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ by the symbol $\mathfrak{F}\mathfrak{H}$ for the case in which $\mathfrak{H}$ is a homomorph.

We use the following customary notation for classes of groups:

- $\mathfrak{E}$ is the class of all groups;
- $\mathfrak{N}$ is the class of all nilpotent groups;
- $\mathfrak{E}_\pi$ is the class of all π-groups;
- $\mathfrak{S}_\pi$ is the class of all solvable π-groups;
- $\mathfrak{N}_\pi$ is the class of all nilpotent π-groups;
- $\mathfrak{S}^\pi$ is the class of all π-solvable groups;
- $\mathfrak{N}^\pi$ is the class of all π-nilpotent groups.

Hartley [8] described the injectors for the local Fitting class $\mathfrak{X}\mathfrak{N}$ in a solvable group G in terms of radicals, and, later, Guo and Vorob'ev [10] described the $\mathfrak{H}$-injectors G for every local Fitting class $\mathfrak{H}$ of the form $\bigcap_{p\in\mathbb{P}} h(p)\mathfrak{E}_{p'}\mathfrak{N}_p$ (Hartley class), where h is a mapping $h\colon \mathbb{P} \to \{\textit{Fitting classes}\}$.

Following [10], we refer to a mapping

$$h\colon \mathbb{P} \to \{\text{Fitting sets of a group } G\}$$

as a *Hartley function* (or, briefly, an H-*function*) of the group G. The *product of a Fitting set* $\mathscr{F}$ *of a group G and a Fitting class* $\mathfrak{X}$ (see [11, Definition 2.2]) is defined as the set $\{H \le G : H/H_{\mathscr{F}} \in \mathfrak{X}\}$ of subgroups, which we denote by $\mathscr{F} \diamond \mathfrak{X}$, following [1, Definition IX.(1.10)].

Let $\mathscr{H}$ be a nonempty set of subgroups of a group G. For $\mathscr{H}$ and a nonempty class $\mathfrak{F}$ of groups, we refer to the set $\{H \le G : L \trianglelefteq H$, where $L \in \mathscr{H},\ H/L \in \mathfrak{F}\}$ of subgroups of G as the *product of the set* $\mathscr{H}$ *and the class* $\mathfrak{F}$ *of groups* and denote this set by $\mathscr{H}\mathfrak{F}$, by analogy with the definition of the product of classes of groups (see [1, Definition II.(1.3)]).

Remark 1.1. If $\mathscr{H}$ is a Fitting set of a group G and $\mathfrak{F}$ is a Fitting class, then, clearly, $\mathscr{H} \diamond \mathfrak{F} \subseteq \mathscr{H}\mathfrak{F}$. Suppose that the Fitting class $\mathfrak{F}$ is a homomorph. In this case, if $H \le G$ and $H \in \mathscr{H}\mathfrak{F}$, then $H/L \in \mathfrak{F}$ for some normal $\mathscr{H}$-subgroup L of H. Since $L \le H_{\mathscr{H}}$ and $H/L/H_{\mathscr{H}}/L \cong H/H_{\mathscr{H}}$, it follows that $H \in \mathscr{H} \diamond \mathfrak{F}$. Therefore, $\mathscr{H} \diamond \mathfrak{F} = \mathscr{H}\mathfrak{F}$ in this case, and we denote $\mathscr{H} \diamond \mathfrak{F}$ by $\mathscr{H}\mathfrak{F}$.

Let $\varnothing \neq \pi \subseteq \mathbb{P}$, and let h be a Hartley function of G. We set $HS(h) = \bigcap_{p\in\pi} h(p)(\mathfrak{E}_{p'}\mathfrak{N}_p)$. A Fitting set $\mathscr{H}$ of G is called a *Hartley set of* G if $\mathscr{H} = HS(h)$ for some H-function h.

Let us classify the H-functions of a group G.

Let $\varnothing \neq \pi \subseteq \mathbb{P}$, and let h be an H-function of a Hartley set $\mathscr{H}$ of the group G. Then h is said to be

(1) *integrated* if $h(p) \subseteq \mathscr{H}$ for all $p \in \pi$;

(2) *stable* if $h(p) \subseteq h(q)\mathfrak{E}_{q'}$ for all distinct $p, q \in \pi$;

(3) *stable integrated* if h is simultaneously stable and integrated;

(4) *invariable* if $h(p) = h(q)$ for all distinct $p, q \in \pi$.

By analogy with [10, Lemma 2.2], we shall show that every Hartley set of a group G is defined by a stable integrated H-function G.

Note that every Hartley set of a group G can be defined by using an integrated H-function (see Lemma 3.2 below). It can readily be noticed that an invariable H-function is stable integrated. Indeed, since $h(p) = h(q)$ for all distinct $p, q \in \pi$, we have $h(p) \subseteq h(p)\mathfrak{E}_{q'} = h(q)\mathfrak{E}_{q'}$ and, therefore,

$$h(p) \subseteq \bigcap_{q\in\pi} h(q)(\mathfrak{E}_{q'}\mathfrak{N}_q) = \mathscr{H}.$$

Note that Theorem 7.1.3 in [3] implies the existence of Fitting sets of nonsolvable groups G that are not Hartley sets of G and contain no $\mathscr{F}$-injectors. For example, if $S = A_7$ is the symmetric group of degree 7, $T = PSL(2, 11)$, the Fitting class is $\mathfrak{F} = D_0(S, T, 1)$, and the group G is defined as in [3, Theorem 7.1.3], then the group G has no $\mathscr{F}$-injectors in the Fitting set $\mathscr{F} = \mathrm{Tr}_{\mathfrak{F}}(G)$. Moreover, there are known examples of sets π of primes and non-Abelian groups G such that G has a Hall π-subgroup and the Hall π-subgroups are not conjugate (see [12, Theorem 1.1]). Recall that a group G is called an E_π*-group* if G has a Hall π-subgroup. As is known, the Hall π-subgroups of an E_π-group G are $\mathfrak{E}_\pi$-injectors of G (see [3, p. 328]). Thus, there are E_π-groups whose $\mathfrak{E}_\pi$-injectors are not conjugate.

In this connection, the following questions arise.

Question 1. For what (generally nonsolvable) groups G and Hartley sets $\mathscr{H}$ of G does there exist a unique class of conjugate $\mathscr{H}$-injectors in G?

Question 2. What is the structure of $\mathscr{H}$-injectors for G, provided that they exist?

In the present paper, we give positive answers to the above questions for a partially π-solvable group G and a Hartley set defined by an invariable H-function.

The main result of the paper is the following theorem.

Theorem 1.2. *Let $\mathscr{X}$ be a nonempty Fitting set of a group G, let $\varnothing \neq \pi \subseteq \mathbb{P}$, and let $\mathscr{H} = HS(h)$ be a Hartley set of G defined by an H-function h such that $h(p) = \mathscr{X}$ for every $p \in \pi$. If $G \in \mathscr{X}\mathfrak{S}^\pi$, then the following assertions hold*:

1) *G has an $\mathscr{H}$-injector, and every two $\mathscr{H}$-injectors are conjugate*;

2) *every $\mathscr{H}$-injector V of G is a subgroup of the form $G_{\mathscr{X}\mathfrak{E}_{\pi'}}L$, where L is a subgroup of G such that $L/G_{\mathscr{X}}$ is an $\mathfrak{N}_\pi$-injector of some Hall π-subgroup $G/G_{\mathscr{X}}$.*