Numerical Solution and Stability for Model of Extensible Beam

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ABSTRACT- In this paper, numerical methods (finite differences methods for explicit and implicit) has been applied, to solve nonlinear partial differential equations. In methodology, the beam was divided into very smaller squares, then the study discussed three partial differential equations generating from model. The first equation called longitudinal vibrations of a beam, second equation known as transverse vibrations of a beam and then the third equation considered the extensible beam. The equation of extensible beam was defined by Woiniwsky- Krieger as a model for transverse deflection of an extensible beam of natural length. The study discussed the stability of these models (longitudinal vibrations, transverse vibrations and extensible beams). The stability solution has been counted and considered unconditionally for implicit method, but it's conditional for an explicit method. Obtaining the stability and convergent solution for longitudinal vibrations of a beam if width divisions is less than length divisions ($\mathrm{R} < 2$), and for transverse vibrations of a beam if width divisions less than the square length divisions ($\mathrm{R} < 0.25$), as well as for extensible beam if width divisions less than the square length divisions, the study recommended to use an implicit method. But in case of using an explicit method, the divisions must be adhered for a stable and convergent solution.

Keywords*:* Partial Differential Equations, Finite Differences, Beam, MATLAB Programming.

المستخلص - في هذه الورقة طبقنا طرق عددية "طريقة الفروقات المحددة الصريحة والضمنية" لحل المعادلات التفاضلية الجزئية، في هذه الطريقة يتم تقسيم العارضة الي مربعات صغيرة جدا، حيث ناقشنا ثلاثة معادلات تفاضلية جزئية مولدة من نموذج للعارضة، المعادلة الاولي تسمي الاهتزازات الطولية للعارضة، المعادلة الثانية تسمي الاهتزازات المستعرضة للعارضة والمعادلة الثالثة تسمي العارضة الممتدة. معادلة العارضة الممتدة افترضت بواسطة وينوسكي– كريجر لنموذج الانحناء وامتداد العارضة من الطول الطبيعي. حيث ناقشنا الاستقرار للنماذج (الاهتزازات الطولية للعارضة، الاهتزازات المستعرضة للعارضة والعارضة الممتدة) وجدنا ان استقرار الحل للنماذج عند تطبيق الطريقة الضمنية غير مشروط، لكنها مشروطة (مقيدة) عند تطبيق الطريقة الصريحة، فالعارضة الطولية يكون الحل مستقر ومتقارب إذا كان عدد تقسيمات العرض اقل من عدد تقسيمات الطول (قيمة الباروميتر اقل من 2) اما العارضة المستعرضة يكون الحل مستقر ومتقارب إذا كان عدد تقسيمات العرض اقل من عدد تقسيمات مربع الطول (قيمة الباروميتر اقل من 0.25)، والعارضة الممتدة يكون الحل مستقر ومتقارب اذا كان عدد تقسيمات العرض اقل من عدد تقسيمات مربع الطول، نوصي باستخدام الطريقة الضمنية اما في حالة استخدام الطريقة الصريحة يجب التقيد بالتقسيمات للطول والعرض للحصول على حل مستقر ومتقارب.

INTRODUCTION

Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering [1]. A beam is a bar-like structural member whose primary function is to support transverse loading and carry it to the supports, this equation describes the motion of a beam initially located on the x-axis which is vibrating transversely "perpendicular to the x-direction", in this case $u(x,t)$ is the transverse displacement or deflection at any time t of any points x [2].

In the recent literature the behavior of a clamped free non-linear inextensible Euler Elastic introduced in Euler [9] , see Luongo and Zulli [10], Eugster[11], Steigmann and Faulkner [12] for general reference works, has been mathematically investigated under distributed load (2016) [3]. In particular, the set of stable equilibrium configurations has been completely characterized in Della Corte et al (2019) [4].

Today we get the numerical solution is very important especial for nonlinear models, because the traditional methods for solving nonlinear models is very difficult and sometime is impossible to applicable.

In this paper, we discuss two methods, explicit finite difference method and implicit finite difference method.

The objective of this research is to estimation of stability of longitudinal vibrations of a beam equation, transverse vibrations of a beam equation and extensible beam equation.

The stability of problems is very important, because give optimal option to choose the parameters to obtain best approximate solution.

The Model

Consider an extensible beam whose ends are held at $x-0$ and $x-L$, let H be the axial force set up in the beam when it is constrained to lie along the x −axis. The model of deflection $u(x,t)$, which we discuss, is

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial^4 u}{\partial x^4}-\left(\beta+\gamma\left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2}=0 \quad (1)$$

where $\alpha = EI/\rho$, $\beta = EA/\rho L$ and $\gamma = EA/2\rho L$, where E is young's modulus, I is the cross sectional of second moment area, ρ is density and A is the cross sectional area. Consider the boundary conditions at both ends [1].

Equation (1) was proposed by Woiniwsky-Krieger as a model for transverse deflection of an extensible beam of natural length whose ends are held a fixed distance apart. If we assume

i. $\alpha = 0$ & $\gamma = 0$, equation (1), is called longitudinal vibrations of a beam.

ii. $\beta = 0$ & $\gamma = 0$, equation (1), is called transverse vibrations of a beam.

Consider the initial- boundary value problem at both ends. Consider the boundary conditions:

$$u(0,t)=A,\quad u(L,t)=B,$$
$$u_x(0,t)=C \;\&\; u_x(L,t)=D$$

Initial conditions:

$$u(x,0)=f(x) \quad \& \quad u_t(x,0)=\mathrm{g}(x)$$

TABLE 1: IMPORTANT PARAMETERS

Parameters	Mining
α	Is ratio between young's modulus multiply by cross sectional of second moment area and density
β	Is ratio between young's modulus multiply by cross sectional area and density multiply by length
γ	Is ratio between young's modulus multiply by cross sectional area and density multiply by tow length

MATERIALS AND METHODS

In this paper, we use finite differences methods (Explicit Method & Implicit Method), the finite differences approximations for derivatives are one of the simplest and of the oldest methods to solve differential equations. L. Euler knew it, in this paper, we using the explicit method to approximate the derivatives for central operator difference [5].

$$\frac{\partial u(x_i,t_j)}{\partial x}=\frac{u(x_{i+1},t_j)-u(x_{i-1},t_j)}{2h}+O(h^2) \quad (2)$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial x^2}=\frac{u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2}+O(h^2) \quad (3)$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial t^2}=\frac{u(x_i,t_{j+1})-2u(x_i,t_j)+u(x_i,t_{j-1})}{k^2}+O(k^2) \quad (4)$$

$$\frac{\partial^4 u(x_i,t_j)}{\partial x^4}=\frac{u(x_{i+2},t_j)-4u(x_{i+1},t_j)+6u(x_i,t_j)-4u(x_{i-1},t_j)+u(x_{i-2},t_j)}{h^4}+O(h^2) \quad (5)$$

Note: the implicit method we defined the derivative of x-axis at point (x_i, t_{j+1})

STABILITY ANALYSIS

The stability analysis is giving optimal option to choose the parameters to obtain best approximate solution.

Stability of explicit method for longitudinal vibrations of a beam equation.

$$\frac{\partial^2 u}{\partial t^2}-\beta\frac{\partial^2 u}{\partial x^2}=0 \quad (6)$$

$$\left(\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}\right)-\beta\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^4}\right)=O(k^2)+O(h^2) \quad (7)$$

The local truncation error for this equation is

$$t_{ij}=\frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4}-\beta\frac{h^2}{12}\frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}=O(k^2)+O(h^2) \quad (8)$$

$$u_{i,j+1}=ru_{i+1,j}+(2-2r)u_{i,j}+ru_{i-1,j}-u_{i,j-1} \quad (9)$$

where $r = \frac{\beta k^2}{h^4}$, R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by β.

Let us: $u_{i,j} = (-1)^i \lambda^j$ or $u_{i,j} = \lambda^j e^{in\Delta x\theta}$

$$(-1)^i \lambda^{j+1} = r(-1)^{i+1}\lambda^j + (2-2r)(-1)^i\lambda^j + r(-1)^{i-1}\lambda^j - (-1)^i\lambda^{j-1} \quad (10)$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$\lambda = -r + (2-2r) - r - r - \lambda^{-1} \quad (11)$$

$$\lambda + \lambda^{-1} = 2 - 2r \quad (12)$$

$$\Rightarrow \frac{\lambda^2+1}{\lambda} = 2 - 2r \quad (13)$$

Suppose $w = 1 - r \Rightarrow \lambda^2 - 2w\lambda + 1 = 0$

$$\Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2-1} \quad (14)$$

Now since λ_1 and λ_2 are roots of this quadratic equation, we may conclude that $\lambda_1\lambda_2 = 1$. However, for stability of solutions we require

$|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. Given the constraint $\lambda_1\lambda_2 = 1$, the only possibility if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies [6].

$|w| = |1-r| < 1 \Rightarrow |r-1| < 1$, $\Rightarrow r < 2$, $r = \frac{\beta k^2}{h^2} < 2 \Rightarrow k^2 < \frac{2h^2}{\beta}$

The stability of explicit method is conditionally.

Stability of implicit method for longitudinal vibrations of a beam equation.

$$\frac{\partial^2 u}{\partial t^2} - \beta\frac{\partial^2 u}{\partial x^2} = 0 \quad (15)$$

$$\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right) - \beta\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^4}\right) = O(k^2) + O(h^2) \quad (16)$$

The local truncation error for this equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4} - \beta\frac{h^2}{12}\frac{\partial^4 u(\xi_i, t_{j+1})}{\partial x^4} = O(k^2) + O(h^2) \quad (17)$$

$$2u_{i,j} = -ru_{i+1,j+1} + (1+2r)u_{i,j+1} - ru_{i-1,j+1} + u_{i,j-1} \quad (18)$$

where $r = \frac{\beta k^2}{h^4}$. R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by β.

Let us: $u_{i,j} = (-1)^i \lambda^j$ or $u_{i,j} = \lambda^j e^{in\Delta x\theta}$

$$2(-1)^i\lambda^j = -r(-1)^{i+1}\lambda^{j+1} + (1+2r)(-1)^i\lambda^{j+1} - r(-1)^{i-1}\lambda^{j+1} + (-1)^i\lambda^{j-1} \quad (19)$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$2 = r\lambda + (1+2r)\lambda + r\lambda + \lambda^{-1} \quad (20)$$

$$(1+4r)\lambda + \lambda^{-1} = 2 \quad (21)$$

$$\Rightarrow \frac{(1+4r)\lambda^2+1}{\lambda} = 2 \quad (22)$$

Suppose $w = \frac{1}{1+4r} \Rightarrow \lambda^2 - 2w\lambda + w = 0, \Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - w}$ (23)

Now since λ_1 and λ_2 are roots of this quadratic equation. However, for stability of solutions we require $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. The only possibility, if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies [6].

$|w| = \left|\frac{1}{1+4r}\right| < 1 \Rightarrow |1+4r| > 1$, $\Rightarrow r > 0$, $r = \frac{\beta k^2}{h^2} > 0 \Rightarrow k > 0$

The stability of implicit method is unconditionally.

Stability of explicit method for transvers vibrations of a beam equation.

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} = 0 \quad (24)$$

$$\left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right) + c^2\left(\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}\right) = O(k^2) + O(h^2) \quad (25)$$

The local truncation error for this equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4} + \alpha\frac{2h^2}{8!}\frac{\partial^8 u(\xi_i,t_j)}{\partial x^8} = O(k^2) + O(h^2) \tag{26}$$

$$u_{i,j+1} = -ru_{i+2,j} + 4ru_{i+1,j} + 2(1-3r)u_{i,j} + 4ru_{i-1,j} - ru_{i-2,j} - u_{i,j-1} \tag{27}$$

where $r = \frac{\alpha k^2}{h^4}$. R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by α.

Let us: $u_{i,j} = (-1)^i\lambda^j$ or $u_{i,j} = \lambda^j e^{in\Delta x\theta}$

$$(-1)^i\lambda^{j+1} = -r(-1)^{i+2}\lambda^j + 4r(-1)^{i+1}\lambda^j + 2(1-3r)(-1)^i\lambda^j + 4r(-1)^{i-1}\lambda^j - r(-1)^{i-2}\lambda^j - (-1)^i\lambda^{j-1} \tag{28}$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$\lambda = -r - 4r + 2(1-3r) - 4r - r - \lambda^{-1} \tag{29}$$

$$\lambda + \lambda^{-1} = 2 - 16r \tag{30}$$

$$\frac{\lambda^2 + 1}{\lambda} = 2 - 16r \tag{31}$$

Suppose $w = 1 - 8r \implies \lambda^2 - 2w\lambda + 1 = 0, \implies \lambda_{1,2} = w \pm \sqrt{w^2 - 1}$ (32)

Now since λ_1 and λ_2 are roots of this quadratic equation, we may conclude that $\lambda_1\lambda_2 = 1$. However, for stability of solutions we require $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$. Given the constraint $\lambda_1\lambda_2 = 1$, the only possibility if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies [6].

$$|w| = |1 - 8r| < 1 \implies |8r - 1| < 1 \ , \implies r < \tfrac{1}{4},\ r = \tfrac{\alpha k^2}{h^4} < \tfrac{1}{4} \implies k^2 < \tfrac{h^4}{4\alpha}$$

The stability of explicit method is conditionally.

Stability of implicit method for transvers vibrations of a beam equation.

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} = 0 \tag{33}$$

$$\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right) + c^2\left(\frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4}\right) = O(k^2) + O(h^2) \tag{34}$$

The local truncation error for this equation is

$$t_{ij} = \frac{k^2}{12}\frac{\partial^4 u(x_i,\eta_j)}{\partial t^4} + \alpha\frac{2h^2}{8!}\frac{\partial^8 u(\xi_i,t_{j+1})}{\partial x^8} = O(k^2) + O(h^2) \tag{35}$$

$$2u_{i,j} = ru_{i+2,j+1} - 4ru_{i+1,j+1} + (1+6r)u_{i,j+1} - 4ru_{i-1,j+1} + ru_{i-2,j+1} + u_{i,j-1} \tag{36}$$

where $r = \frac{\alpha k^2}{h^4}$. R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by α.

Let us: $u_{i,j} = (-1)^i\lambda^j$ or $u_{i,j} = \lambda^j e^{in\Delta x\theta}$

$$2(-1)^i\lambda^j = r(-1)^{i+2}\lambda^{j+1} - 4r(-1)^{i+1}\lambda^{j+1} + (1+6r)(-1)^i\lambda^{j+1} - 4r(-1)^{i-1}\lambda^{j+1} + r(-1)^{i-2}\lambda^{j+1} + (-1)^i\lambda^{j-1} \tag{37}$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$2 = r\lambda + 4r + (1+6r)\lambda + 4r\lambda + r\lambda + \lambda^{-1} \tag{38}$$

$$(1+16r)\lambda + \lambda^{-1} = 2 \tag{39}$$

$$\implies \frac{(1+16r)\lambda^2 + 1}{\lambda} = 2 \tag{40}$$

Suppose $w = \frac{1}{1+16r} \implies \lambda^2 - 2w\lambda + w = 0, \implies \lambda_{1,2} = w \pm \sqrt{w^2 - w}$ (41)

Now since λ_1 and λ_2 are roots of this quadratic equation. However, for stability of solutions we require $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$. The only possibility, if the solution to be stable is $|\lambda_1| = |\lambda_2| = 1$, thus λ must fall on the unit disk, which implies [6].

$$|w| = \left|\tfrac{1}{1+16r}\right| < 1 \implies |1 + 16r| > 1 \ , \implies r > 0, r = \tfrac{\alpha k^2}{h^4} > 0 \implies k > 0$$

The stability of implicit method is unconditionally.

Stability of explicit method for extensible beam equation.

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \beta\frac{\partial^2 u}{\partial x^2} - \gamma\left|\frac{\partial u}{\partial x}\right|^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{42}$$

$$\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}\right)+\alpha\left(\frac{u_{i+2,j}-4u_{i+1,j}+6u_{i,j}-4u_{i-1,j}+u_{i-2,j}}{h^4}\right) -\beta\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right)-\gamma\left|\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)\right|^2\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right) = 0 \quad (43)$$

$$u_{i,j+1}=-ru_{i+2,j}+4ru_{i+1,j}+(2-6r)u_{i,j}+4ru_{i-1,j}-ru_{i-2,j} +\frac{\beta h^2}{\alpha}\left(ru_{i+1,j}-2ru_{i,j}+ru_{i-1,j}\right) +\frac{\gamma}{4\alpha}\left(ru_{i+1,j}-ru_{i-1,j}\right)^2\left(ru_{i+1,j}-2ru_{i,j}+ru_{i-1,j}\right)-u_{i,j-1} \quad (44)$$

where $r=\frac{\alpha k^2}{h^4}$. R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by α.

Let us: $u_{i,j}=(-1)^i\lambda^j$ or $u_{i,j}=\lambda^j e^{in\Delta x\theta}$

$$(-1)^i\lambda^{j+1}=-r(-1)^{i+2}\lambda^j+4r(-1)^{i+1}\lambda^j+(2-6r)(-1)^i\lambda^j+4r(-1)^{i-1}\lambda^j-r(-1)^{i-2}\lambda^j +\frac{\beta h^2}{\alpha}\left(r(-1)^{i+1}\lambda^j-2r(-1)^i\lambda^j+r(-1)^{i-1}\lambda^j\right) +\frac{\gamma}{4\alpha}\left(r(-1)^{i+1}\lambda^j-r(-1)^{i-1}\lambda^j\right)^2\left(r(-1)^{i+1}\lambda^j-2r(-1)^i\lambda^j+r(-1)^{i-1}\lambda^j\right) -(-1)^i\lambda^{j-1} \quad (45)$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$\lambda=-r-4r+(2-6r)-4r-r-\frac{4\beta h^2}{\alpha}r-\lambda^{-1} \quad (46)$$

$$\lambda+\lambda^{-1}=2-16r-\frac{4\beta h^2}{\alpha}r \quad (47)$$

$$\frac{\lambda^2+1}{\lambda}=2-16r-\frac{4\beta h^2}{\alpha}r \quad (48)$$

Suppose $w=1-8r-\frac{2\beta h^2}{\alpha}r$, $\Rightarrow$ $\lambda^2-2w\lambda+1=0, \Rightarrow$ $\lambda_{1,2}=w\pm\sqrt{w^2-1}$ (49)

Now since λ_1 and λ_2 are roots of this quadratic equation, we may conclude that $\lambda_1\lambda_2=1$. However, for stability of solutions we require $|\lambda_1|\le 1$ and $|\lambda_2|\le 1$. Given the constraint $\lambda_1\lambda_2=1$, the only possibility if the solution to be stable is $|\lambda_1|=|\lambda_2|=1$, thus λ must fall on the unit disk, which implies [6].

$$|w|=\left|1-8r-\frac{2\beta h^2}{\alpha}r\right|<1 \quad\Rightarrow\quad \left|(8+\frac{2\beta h^2}{\alpha})r-1\right|<1 \quad\Rightarrow\quad r<\frac{1}{(4+\frac{\beta h^2}{\alpha})}$$

$$r=\frac{\alpha k^2}{h^4}<\frac{1}{(4+\frac{\beta h^2}{\alpha})} \quad\Rightarrow k^2<\frac{h^4}{(4\alpha+\beta h^2)}$$

The stability of explicit method is conditionally.

Stability of implicit method for extensible beam equation.

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial^4 u}{\partial x^4}-\beta\frac{\partial^2 u}{\partial x^2}-\gamma\left|\frac{\partial u}{\partial x}\right|^2\frac{\partial^2 u}{\partial x^2}=0 \quad (50)$$

$$\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}\right)+\alpha\left(\frac{u_{i+2,j+1}-4u_{i+1,j+1}+6u_{i,j+1}-4u_{i-1,j+1}+u_{i-2,j+1}}{h^4}\right) -\beta\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right) -\gamma\left|\left(\frac{u_{i+1,j+1}-u_{i-1,j+1}}{2h}\right)\right|^2\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right)=0 \quad (51)$$

$$2u_{i,j}=ru_{i+2,j+1}-4ru_{i+1,j+1}+(1+6r)u_{i,j+1}-4ru_{i-1,j+1}+ru_{i-2,j+1} -\frac{\beta h^2}{\alpha}\left(ru_{i+1,j+1}-2ru_{i,j+1}+ru_{i-1,j+1}\right) -\frac{\gamma}{4\alpha}\left(ru_{i+1,j+1}-ru_{i-1,j+1}\right)^2\left(ru_{i+1,j+1}-2ru_{i,j+1}+ru_{i-1,j+1}\right)+u_{i,j-1} \quad (52)$$

where $r=\frac{\alpha k^2}{h^4}$. R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by α. Let us: $u_{i,j}=(-1)^i\lambda^j$ or $u_{i,j}=\lambda^j e^{in\Delta x\theta}$

$$2(-1)^i\lambda^j=r(-1)^{i+2}\lambda^{j+1}-4r(-1)^{i+1}\lambda^{j+1}+(1+6r)(-1)^i\lambda^{j+1}-4r(-1)^{i-1}\lambda^{j+1}+r(-1)^{i-2}\lambda^{j+1}-\frac{\beta h^2}{\alpha}\left(r(-1)^{i+1}\lambda^{j+1}-2r(-1)^i\lambda^{j+1}+r(-1)^{i-1}\lambda^{j+1}\right)-\frac{\gamma}{4\alpha}\left(r(-1)^{i+1}\lambda^{j+1}-r(-1)^{i-1}\lambda^{j+1}\right)^2\left(r(-1)^{i+1}\lambda^{j+1}-2r(-1)^i\lambda^{j+1}+r(-1)^{i-1}\lambda^{j+1}\right)+(-1)^i\lambda^{j-1} \tag{53}$$

Multiply both sides by, $(-1)^{-i}\lambda^{-j}$, we obtain

$$2=r\lambda+4r\lambda+(1+6r)\lambda+4r\lambda+r\lambda+\frac{4\beta h^2}{\alpha}r\lambda+\lambda^{-1} \tag{54}$$

$$\left(1+16r+\frac{4\beta h^2}{\alpha}r\right)\lambda+\lambda^{-1}=2 \tag{55}$$

$$(1+16r+\frac{4\beta h^2}{\alpha}r)\lambda^2+1=2\lambda \tag{56}$$

Suppose $w=\frac{1}{1+16r+\frac{4\beta h^2}{\alpha}r} \Rightarrow \lambda^2-2w\lambda+w=0, \Rightarrow \lambda_{1,2}=w\pm\sqrt{w^2-w}$ (57)

Now since λ_1 and λ_2 are roots of this quadratic equation. However, for stability of solutions we require $|\lambda_1|\le 1$ and $|\lambda_2|\le 1$. The only possibility, if the solution to be stable is $|\lambda_1|=|\lambda_2|=1$, thus λ must fall on the unit disk, which implies [6].

$$|w|=\left|\frac{1}{1+16r+\frac{4\beta h^2}{\alpha}r}\right|<1 \Rightarrow \left|1+\left(16+\frac{4\beta h^2}{\alpha}\right)r\right|>1 \Rightarrow r>0, r=\frac{\alpha k^2}{h^4}>0 \Rightarrow k>0$$

The stability of implicit method is unconditionally.

Algorithm and Numerical Results

Algorithm of equation (1) for applied explicit method

To obtain the numerical solution of equation 1.

Input: *endpoint L; maximum time T; constants α, β, γ; integers n and m*

Output: *approximations $u(x_i,t_j)$, for each i=0,1,….,m and j=0,1,….,n*

Step 1: *h=L/n*
k=T/m
*r= α *k^2/h^4*
*p= γ *k^2/(2*h)^2*

Step 2: *for i=0,1,…..,m*
for j=0,1,……,n
Do step 3 and step 4

Step 3: $u(x_0,t_j)=A$
$u(x_n,t_j)=B$

Step 4: $u(x_i,t_0)=f(x_i)$

Step 5: *for i=1,…..,n-1*
for j=1,……,m-1

$u(x_i,t_{j+1})=-ru(x_{i+2},t_j)+4ru(x_{i+1},t_j)+2(1-3r)u(x_i,t_j)+4ru(x_{i-1},t_j)-ru(x_{i-2},t_j)-u(x_i,t_{j-1})+\left(\beta k^2+p\left(u(x_{i+1},t_j)-u(x_{i-1},t_j)\right)^2\right)*(u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j))/h^2$

Step 6: *output $u_{00},u_{01},u_{02},\dots.u_{mn}$*

Step 7: *Stop (the producer is complete)*

Algorithm of equation (1) for applied implicit method

To obtain the numerical solution of equation 1.

Input: *endpoint L; maximum time T; constants* α, β, γ *; integers n and m*

Output: approximations $u(x_i, t_j)$, *for each i=0,1,....,m and j=0,1,....,n*

Step 1: *h=L/n*

k=T/m

*r=α *k^2/h^4*

*p=γ *k^2/(2*h)^2*

Step 2: *for i=0,1,.....,m*

for j=0,1,......,n

Do step 3 and step 4

Step 3: $u(x_0, t_j) = A$

$u(x_n, t_j) = B$

Step 4: $u(x_i, t_0) = f(x_i)$

Step 5: *for i=1,.....,n-1*

for j=1,......,m-1

$$ru(x_{i+2}, t_{j+1}) - 4ru(x_{i+1}, t_{j+1}) + (1 + 6r)u(x_i, t_{j+1}) - 4ru(x_{i-1}, t_{j+1}) + ru(x_{i-2}, t_{j+1}) + \left(\beta k^2 + p\left(u(x_{i+1}, t_{j+1}) - u(x_{i-1}, t_{j+1})\right)^2\right) * \frac{u(x_{i+1},t_{j+1}) - 2u(x_i,t_{j+1}) + u(x_{i-1},t_{j+1})}{h^2} = 2u(x_i, t_j) + u(x_i, t_{j-1})$$

Step 6: *output* $u_{00}, u_{01}, u_{02}, \ldots\ldots u_{mn}$

Step 7: *Stop (the producer is complete)*

Example 1: Consider length of x-axis equal 10 and width equal 5. n=6, m=6, $(x) = \sin(x)$, $g(x) = x, A = 1, B = 3, C = 0, D = 0$ & $\beta = 0.5$. In longitudinal vibrations of a beam.

TABLE 2: APPROXIMATE SOLUTION BY USING EXPLICIT METHOD

x	t=0.000	t=0.8333	t=1.6667	t=2.5000	t=3.3333	t=4.1667	t=5.000
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.6667	0.9954	4.2549	7.1801	9.6746	11.7530	13.5285	15.1707
3.3333	-0.1906	4.8355	9.9145	15.0203	20.0831	24.9907	29.6056
5.0000	-0.9589	5.8391	12.8638	20.0216	27.1717	34.1115	40.5378
6.6667	0.3742	8.6562	16.8776	24.9621	32.5768	39.1247	43.8851
8.3333	0.8873	10.9873	19.7975	26.1430	29.4480	29.8381	28.0343
10.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

TABLE 3: APPROXIMATE SOLUTION BY USING IMPLICIT METHOD

x	t=0.000	t=0.8333	t=1.6667	t=2.5000	t=3.3333	t=4.1667	t=5.000
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.6667	0.9954	7.2035	12.0082	15.3947	17.6556	19.2424	20.6113
3.3333	-0.1906	5.5527	11.7895	18.4439	25.3062	32.0917	38.4798
5.0000	-0.9589	6.9284	15.5191	24.8334	34.6200	44.3271	53.1692
6.6667	0.3742	12.4075	24.8759	37.0110	47.7129	55.9261	60.9392
8.3333	0.8873	22.2715	37.7122	46.5234	49.3390	47.6162	43.1087
10.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

TABLE 4: ERROR ESTIMATION FOR LONGITUDINAL VIBRATIONS OF A BEAM

Explicit Method		Implicit Method	
R=2.000	R=0.125	R=2.000	R=0.125
1.0e+003*0.0000	0.0000	0.0000	0.0000
1.6477	1.6422	1.7672	1.3689
3.8304	4.6149	3.0251	6.3881
6.0236	6.4263	3.3392	8.8422
6.6218	4.7604	2.6466	5.0132
4.4476	1.8038	1.3533	4.5075
0.0000	0.0000	0.0000	0.0000

Error=$|\underline{u}^{k+1} - \underline{u}^{k}|$

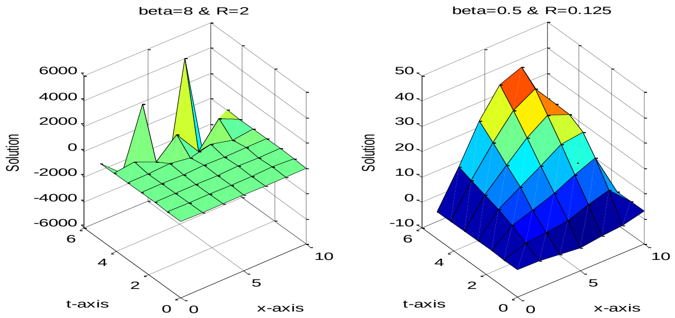

Figure 1: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method is unstable when R ≥ 2 and is stable at R < 2

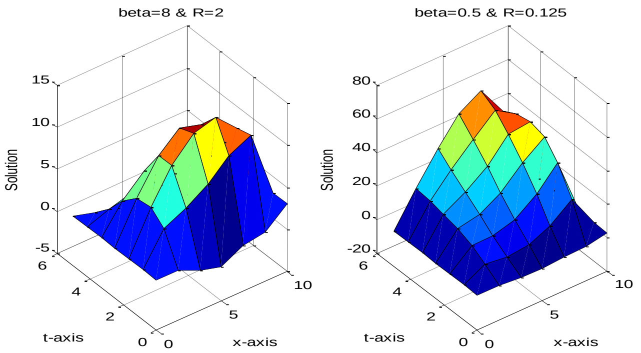

Figure 2: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method

TABLE 5: APPROXIMATE SOLUTION BY USING EXPLICIT METHOD

x	t=0.000	t=0.8333	t=1.6667	t=2.5000	t=3.3333	t=4.1667	t=5.000
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.6667	0.9954	2.3218	3.4727	4.4396	5.2584	5.9882	6.6826
3.3333	-0.1906	2.5853	5.3707	8.1630	10.9290	13.6497	16.3972
5.0000	-0.9589	3.3114	7.7355	12.3045	17.1430	22.4569	28.3601
6.6667	0.3742	5.8095	11.4556	17.5807	24.0409	30.2844	35.5885
8.3333	0.8873	7.9697	14.2020	18.7187	21.1971	21.9133	21.5585
10.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

TABLE 6: APPROXIMATION SOLUTION BY USING IMPLICIT METHOD

x	t=0.000	t=0.8333	t=1.6667	t=2.5000	t=3.3333	t=4.1667	t=5.000
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.6667	0.9954	2.3157	3.5509	4.7631	6.0049	7.3019	8.6418
3.3333	-0.1906	2.6153	5.4710	8.3565	11.2285	14.0129	16.6015
5.0000	-0.9589	3.3305	7.7607	12.1701	16.3523	20.0756	23.1096
6.6667	0.3742	5.8642	11.0936	15.8294	19.7933	22.7383	24.5131
8.3333	0.8873	7.4274	12.4961	15.8322	17.5160	17.8539	17.2489
10.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

TABLE 7: ERROR ESTIMATE FOR TRANSVERSE OF VIBRATIONS OF A BEAM

Explicit Method		Implicit Method	
R=2.000	**R=0.125**	**R=2.000**	**R=0.125**
0.0000	0.0000	0.0000	0.0000
99.8310	0.6944	1.1442	1.3399
161.1570	2.7476	2.0841	2.5885
178.0735	5.9032	2.4935	3.0341
153.6377	5.3041	2.1296	1.7749
97.9247	0.3548	1.1387	0.6050
0.0000	0.0000	0.0000	0.0000

Error=$|\underline{u}^{k+1} - \underline{u}^{k}|$

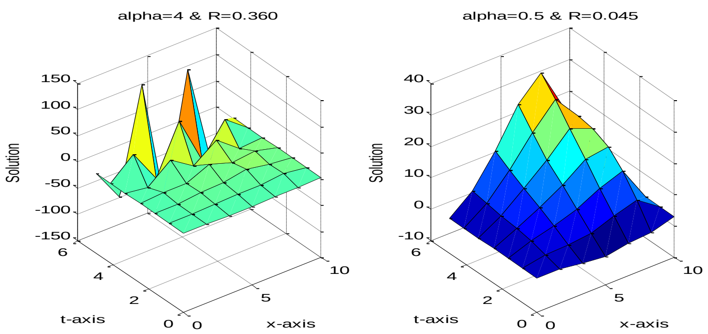

Figure 3: Graphical Representation of transverse vibrations of a beam equation is unstable when R ≥ 0.25 and is stable at R < 0.25

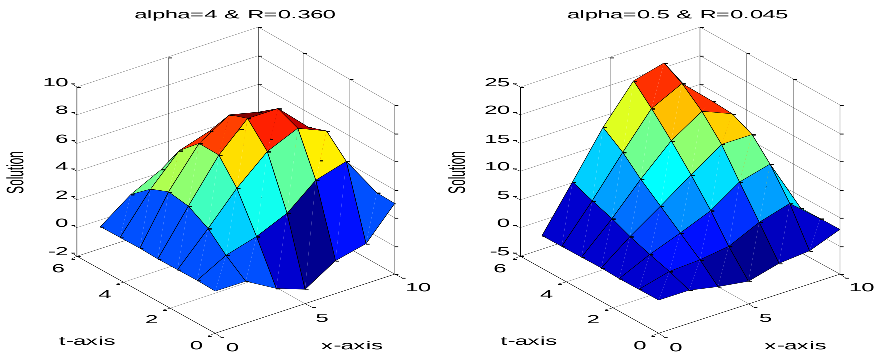

Figure 4: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method

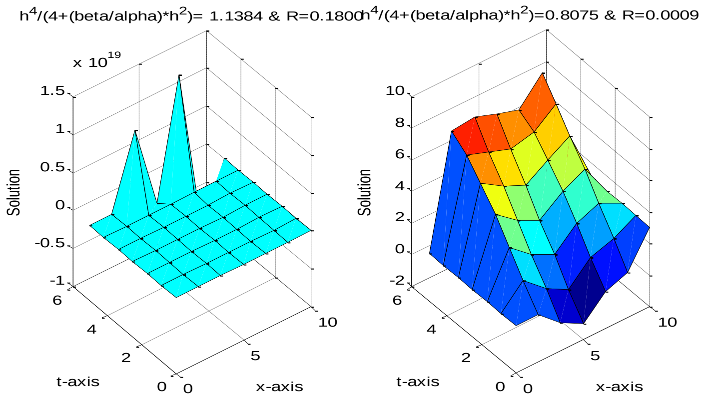

Figure 5: Graphical Representation of extensible beam equation is unstable when R = 0.1800 and is stable at R =0.0009.

TABLE 8: APPROXIMATE SOLUTION BY USING EXPLICIT METHOD

x	t=0.000	t=0.8333	t=1.6667	t=2.5000	t=3.3333	t=4.1667	t=5.000
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.6667	0.9954	2.3791	3.7381	5.0561	6.3092	7.4624	8.4678
3.3333	-0.1906	1.2005	2.5950	3.9928	5.3937	6.7967	8.2005
5.0000	-0.9589	0.4396	1.8585	3.3003	4.7645	6.2495	7.7518
6.6667	0.3742	1.7589	3.1342	4.4583	5.7042	6.8328	7.7915
8.3333	0.8873	2.2867	3.6892	5.0805	6.4443	7.7575	8.9880
10.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000

Table 4, shown the error is smaller in stability case. Figure 1, shown the solution is convergent in stability case, but is divergent in instability case, from figure 2 we get a solution is convergent at all case.

Example 2: Consider length of x-axis equal 10 and width equal 5. n=6, m=6, $f(x) = \sin(x), g(x) = x, A = 1, B = 3, C = 0, D = 0$ & $\alpha = 0.5$. In Transverse of vibrations of a beam.

Table 7, shown the error is smaller in stability case. Figure 3, shown the solution is convergent in stability case, but is divergent in instability case, from figure 4 we get a solution is convergent at all case.

Example 3: Consider length of x-axis equal 10 and width equal 5. n=6, m=6, $f(x) = \sin(x), g(x) = x, A = 1, B = 3, C = 0, D = 0, \alpha = 0.01,\ \beta = 0.02\ \ \&\ \ \gamma = 0.03.$ In extensible beam.

Table 9, shown the error is smaller in stability case. Figure 5, shown the solution is convergent in stability case, but in instability case, the solution is divergent.

TABLE 9: ERROR ESTIMATION FOR EXTENSIBLE BEAM

R=0.1800	R=0.0009
1.0e+019* 0.0000	0.0000
0.1533	1.0055
4.6940	1.4038
0.9630	1.5023
8.1515	0.9586
7.1266	1.2305
0.0000	0.0000

Error=$|\underline{u}^{k+1} - \underline{u}^{k}|$

CONCLUSIONS

In this paper, the study discussed solutions of extensible beam linear or/and nonlinear partial differential equation dependent for parameterγ, by using finite difference methods also we discuss the stability we get i) The stability of implicit method unconditionally but the stability of explicit method is conditionally, ii) The explicit method of longitudinal vibrations of a beam equation is stable if R < 2 and unstable when R ≥ 2, iii) The explicit method of Transverse vibrations of a beam equation is stable if R < 0.25 and unstable when R ≥ 0.25, iv) The explicit method of extensible beam equation is stable if $R < \frac{\alpha}{(4\alpha+\beta h^2)}$ and unstable when $R \geq \frac{\alpha}{(4\alpha+\beta h^2)}$, v) The implicit method of longitudinal vibrations of a beam equation, Transverse vibrations of a beam equation and extensible beam equation are stable for any value of R, vi) From tables 4, 7 & 9 we get the error is very small when we applied implicit method, but in explicit method to make small error use stability case, vii) From figures 1, 2 & 3, at stability case for explicit method and implicit method the figures is similar and uniform, but in instability case the figures in not similar and differences, viii) Future work, we hop the research applied the implicit method for solving, but sometime the implicit method for nonlinear model is very difficult to compute solution in this case applied the explicit method and choose the parameters to give stability.

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