Adaptive prediction of Ornstein-Uhlenbeck process by observations with additive noise

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Abstract

This paper considers the Ornstein-Uhlenbeck process by observations with additive noise that also satisfies Ornstein-Uhlenbeck equation. The truncated parameter estimation problem of non-observable process with guaranteed accuracy is solved. On the basis of these estimators adaptive predictors of observable process are constructed. Asymptotic property of predictors is established. The presented algorithm works for predictors of any depth.

Keywords: Truncated estimation method, fixed sample size, Ornstein-Uhlenbeck process, guaranteed accuracy, adaptive prediction

Intrduction

One of the important problems of modern applied mathematics is the construction of mathematical models and development of the identification and prediction algorithms with guaranteed accuracy for discrete and continuous time stochastic dynamic systems. Such systems are widely used for the description of databases, for information processing, as well as for mathematical model construction of random processes in economics, financial mathematics, physics, sociology, biology, medicine etc.

The most frequently used for these purposes continuous-time models are the diffusion-type models and the Ito processes. The structure of the abovementioned models implies essential dependence of observations which corresponds to demands for real stochastic processes.

According to Ljung's concept the prediction is a crucial part in constructing complete probabilistic models of dynamical systems (see [1, 2]). A model is considered to be useful if it allows to make predictions of high statistical quality.

Models of dynamical systems often have unknown parameters, which requires estimation in order to build adaptive predictors. The quality of adaptive prediction explicitly depends on the chosen estimators of model parameters. Possible estimation methods include the classic stochastic approximation, maximum likelihood, least squares and sequential estimation methods among others. The first three methods provide estimators with given statistical properties under asymptotic assumptions, when the duration of observations tends to infinity (see, e.g., [3, 4]).

The sequential estimation method makes it possible to obtain estimators with guaranteed accuracy by samples of finite but random and unbounded size (see, e.g., [4]–[11] among others).

Both approaches do not guarantee prescribed estimation accuracy when using samples of non-random finite size and lead up to complicated analytical problems in adaptive procedures.

However, the more recent truncated sequential estimation method yields estimators with prescribed accuracy by samples of random but bounded size, see [7], [8] among others.

Then the truncated estimation method was introduced in [12]. Truncated estimators were constructed for ratio type multivariate functionals by samples of fixed size and have guaranteed accuracy in the sense of the L_{2m}-norm, $m \geq 1$ (see also [11]). The truncated estimation method is simpler in implementation then the truncated sequential estimation one. At the same time, both methods are very effective in problems of parameter estimation of dynamical systems.

The main aim of the paper is the construction and investigation of adaptive predictors' properties of observable process wich is a sum of two unobservable Ornstein-Uhlenbeck processes. The presented algorithm based on the usage of truncated estimators and works for making predictions of any depth. Similar problems for continuous-time systems were solved in, e.g., [13, 14]. Properties of adaptive optimal control of continuous-time processes constructed on the basis of sequential parameters were considered in [15]. Adaptive optimal predictors for discre-time multivariate system were constructed in [16].

1 Problem statement. Guaranteed parameter estimation of Ornstein-Uhlenbeck process

Consider the estimation problem of the parameter a of the first order stable autoregressive process

$$dx_t = ax_tdt + dw_t, \quad t \geq 0 \tag{1}$$

with the initial value x_0 by oservation of the process y_t with the known parameter λ of the noise θ

$$y_t = x_t + \theta_t, \qquad \theta_t = \lambda\theta_tdt + dv_t, \tag{2}$$

where w_t and v_t are independent standard Wiener processes,θ_0 - initional value for θ, $a < 0, \lambda < 0,\ \lambda^2 \neq a^2$.

Let's substitute an unobservable process x_t in differential equation (1) by the difference $y_t - \theta_t$ and get the eqation

$$dy_t = ay_tdt + dw_t + d\theta_t - a\theta_tdt. \tag{3}$$

Since the parameter λ is known, we have a possibility to exclude the dependent noise θ_t from the equation (3). To this end we integrate it from 0 to t and multiply the result by dt

$$y_tdt = y_0dt + a\int_0^t y_sdsdt + w_tdt + d\theta_tdt - a\int_0^t \theta_sdsdt.$$

Multiply the obtained equation by λ and subtract it from the equation (3)

$$dy_t - \lambda y_tdt = -\lambda y_0dt + a\left[y_t - \lambda\int_0^t y_sds\right]dt + dw_t + \lambda d\theta_tdt$$

$$-\lambda w_t dt - a\left[\theta_t - \lambda\int_0^t \theta_s ds\right]dt.$$

Define $z_t = y_t - \lambda\int_0^t y_s ds$ and then $dz_t = dy_t - \lambda y_t dt$. From the equation $d\theta_t - \lambda\theta_t dt = dv_t$ it follows, that $\theta_t - \lambda\int_0^t \theta_s ds = \theta_0 + v_t$. Last equation can be written in a form

$$dz_t = az_t dt + d(w_t + v_t) - (\lambda w_t + av_t)dt - (\lambda y_0 + a\theta_0)dt.$$

Let us define the difference operator $\delta_h z_t = z_t - z_{t-h}$ with a step h, $h > 0$ and apply it to the previous equation

$$d\delta_h z_t = a\delta_h z_t dt + d(\delta_h w_t + \delta_h v_t) - (\lambda\delta_h w_t + a\delta_h v_t)dt. \tag{4}$$

Note that $\delta_h z_t$ is an observable process as well. In view of the fact that $\delta_h z_t$ and model noises are correlated, we construct the correlation (or Yule-Walker) type estimator with the shift h

$$\hat{a}_T = \frac{\int_{2h}^T \delta_h z_{t-h} d\delta_h z_t}{\int_{2h}^T \delta_h z_{t-h}\delta_h z_t dt}. \tag{5}$$

We rewrite the deviation of estimator (5), having replaced $d\delta_h z_t$ by the right hand side of (4)

$$\hat{a}_T - a = \frac{1}{\int_{2h}^T \delta_h z_{t-h}\delta_h z_t dt}\Big[\int_{2h}^T \delta_h z_{t-h} d(\delta_h w_t + \delta_h v_t)$$
$$- \int_{2h}^T \delta_h z_{t-h}(\lambda\delta_h w_t + a\delta_h v_t)dt\Big].$$

Analogously to [17, 18],

$$\frac{1}{T}\left[\int_{2h}^T \delta_h z_{t-h} d(\delta_h w_t + \delta_h v_t) - \int_{2h}^T \delta_h z_{t-h}(\lambda\delta_h w_t + a\delta_h v_t)dt\right] \to 0 \quad \text{a.s.}$$

Taking into account the independence $\delta_h z_{t-h}$ from $\delta_h w_t$ and $\delta_h v_t$,

$$E\left[\int_{2h}^T \delta_h z_{t-h} d(\delta_h w_t + \delta_h v_t) - \int_{2h}^T \delta_h z_{t-h}(\lambda\delta_h w_t + a\delta_h v_t)dt\right]^2$$
$$\leq C \cdot E\int_{2h}^T (\delta_h z_{t-h})^2 dt \leq C \tag{6}$$

and there exists the limit

$$\sigma_h^2 = \lim_{T\to\infty}\frac{1}{T}\int_{2h}^T \delta_h z_{t-h}\delta_h z_t dt \quad \text{a.s.},$$

where $\sigma_h^2 = \left(1 - \frac{\lambda^2}{a^2}\right)\frac{ea^h - 1}{2a} \neq 0$.

It is easy to verify that

$$\lim_{n\to\infty}\hat{a}_T = a \quad \text{a.s.}$$

and for every $T > 0$ the following inequality holds

$$E\left[\frac{1}{T}\int_{2h}^{T}\delta_h z_{t-h}\delta_h z_t dt - \sigma^2\right]^4 \le \frac{C}{T^2}. \tag{7}$$

The truncated estimator $\tilde{a}_T$ of the parameter a can be defined similar to [12] for some $T_0 > 0$ as

$$\tilde{a}_T = \frac{\int_{2h}^{T}\delta_h z_{t-h} d\delta_h z_t}{\int_{2h}^{T}\delta_h z_{t-h}\delta_h z_t dt}\cdot\chi\left(\left|\int_{2h}^{T}\delta_h z_{t-h}\delta_h z_t dt\right| \ge T\cdot\log^{-1}T\right). \tag{8}$$

Using (13), (7) and similar to the scheme of the proof for truncated estimators in [12], we get

$$E(\tilde{a}_T - a)^2 \le \frac{C}{T}, \qquad T \ge T_0. \tag{9}$$

By the condition $a < -r$, $r > 0$ the estimator σ_h^2 has the form

$$\sigma_h^2 = \left(1 - \frac{\lambda^2}{\overline{a}^2}\right)\frac{e\overline{a}^h - 1}{2\overline{a}},$$

where $\overline{a} = \text{proj}_{(-\infty,-r]}\tilde{a}_s$ and satisfy the condition

$$E(\sigma_h^2 - \sigma^2)^2 \le \frac{C}{T}, \qquad T \ge T_0.$$

Without a priory information about a, the truncated estimation method can be applied for estimation σ_h^2 .

2 Adaptive prediction

Consider the model (1), (2). The purpose is to construct an adaptive predictor for y_t by observations $y^{t-u} = (y_s)_{0\le s\le t-u}$. Here $u > 0$ - is a fixed time delay.

Using the solution of the equation (1), we get

$$x_t = \mu x_{t-u} + \xi_{t,t-u}, \quad t \ge u, \tag{10}$$

where $\xi_{t,t-u} = \int\limits_{t-u}^{t} e^{a(t-s)}dw_s$, $\mu = e^{au}$.

Define

$$\mu_s = e^{\hat{a}_s u}, \quad s \ge 0. \tag{11}$$

Here

$$\hat{a}_s = \text{proj}_{(-\infty,0]}\tilde{a}_s,$$

$\hat{a}_s$ is a projection of the truncated estimator $\tilde{a}_s$ of the parameter a, defined in (7).

It can be shown that

$$E(\mu_t - \mu)^{2p} \le \frac{C}{t^p}, \quad p \ge 1. \tag{12}$$

Replacing x_t in the formula (9) using (2) we get

$$y_t = \mu y_{t-u} + \xi_{t-u,t} + \theta_t - \mu\theta_{t-u},$$

Introduce the notation

$$\eta_{t-u,t} = \int_{t-u}^{t} e^{\lambda(t-s)} dw_s, \quad \overline{\xi}_{t-u,t} - e^{\lambda u}\xi_{t-2u,t-u}, \quad \overline{\eta}_{t-u,t} - e^{\lambda u}\xi_{t-2u,t-u}.$$

and

$$z_t = y_t - e^{\lambda u} y_{t-u}.$$

The function z_t satisfies the equation

$$z_t = \mu z_{t-u} + \overline{\xi}_{t-u,t} + \overline{\eta}_{t-u,t} - \mu\overline{\eta}_{t-2u,t-u}. \qquad (13)$$

Applying operator of conditional mathematical expectation $E(\cdot|y^{t-3u})$ to the last equation we get

$$E(z_t|y^{t-3u}) = \mu E(z_{t-u}|y^{t-3u}).$$

By the definition of z_t we have

$$E(z_t|y^{t-3u}) = E(y_t|y^{t-3u}) - e^{\lambda u} E(y_{t-u}|y^{t-3u}).$$

Let us define $s_i(t) = E(y_t|y^{t-iu})$, $i = \overline{1,3}$.

The equation for optimal predictions $s_i(t)$, $i = \overline{1,3}$, has the form

$$s_3(t) = (e^{au} + e^{\lambda u})s_2(t) + e^{(a+\lambda)u} s_1(t).$$

Define adaptive predictors $\hat{s}_i(t)$, $i = \overline{1,3}$. The equation for $\hat{s}_i(t)$, is constructed with truncated estimators instead of unknown parameters

$$\hat{s}_3(t) = (e^{\hat{a}_{t-3u}u} + e^{\lambda u})\hat{s}_2(t) + e^{(\hat{a}_{t-3u}+\lambda)u}\hat{s}_1(t).$$

Prediction errors can be written as

$$e_i(t) = s_i(t) - \hat{s}_i(t), \qquad i = \overline{1,3}.$$

It can be shown that

$$\overline{\lim}_{t\to\infty} Ee_i^2(t) < \infty, \qquad i = \overline{1,3}.$$

In the conclusion we note that obtained property for this model probably can not be improved in view of complicated structure of noise dependence. At the same time this property reflects proximity of adaptive and optimal predictors in L_2 - metric, which is important in analytical investigations and practical applications.

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