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Simulation of the Measurer of the Time of Appearance and the Average Power of the Random Pulse Signal

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***Abstract*—The maximum likelihood measurer is considered of the time of appearance and the average power of the fast fluctuating Gaussian band pulse against Gaussian white noise. The possibilities of its practical implementation are demonstrated and its accuracy characteristics are determined. By statistical simulation methods, the experimental values of biases and variances of the resulting estimates are found. The error ranges of the theoretical formulas describing the measurer performance are established. There have been determined the conditions of high a posteriori accuracy for the measurer operation, that is, such signal-to-noise ratios above which the anomalous errors in estimating the pulse time parameter are practically non-existent.**

Keywords-maximum likelihood estimate; random pulse; unknown time of apperance, average power; bias; variance; probability of anomalous error; statistical simulation

I. INTRODUCTION

The problem of estimating the parameters of the random pulse signals has a wide application in analyzing the operation of the various radio engineering devices [1-3]. One of the adequate models of a random pulse is the mathematical model of the form

$$s(t) = \xi(t) I[(t-\lambda_0)/\tau], \qquad (1)$$

where $I(x)=1$, if $|x| \le 1/2$, and $I(x)=0$, if $|x|>1/2$; $\xi(t)$ is the stationary centered Gaussian random process; λ_0 is the time of appearance and τ is the duration of the pulse. The spectral density of the process $\xi(t)$ is described by the expression [1-3]

$$G(\omega) = (\pi D_0/\Omega)\{I[(\vartheta-\omega)/\Omega] + I[(\vartheta+\omega)/\Omega]\}.$$

here the designations are: ϑ is the band center, Ω is the bandwidth of the spectral density, and D_0 is the average power (dispersion) of the process $\xi(t)$. It is presupposed that the pulse (1) duration τ is much longer than the correlation time $2\pi/\Omega$ of the random substructure $\xi(t)$ (the process $\xi(t)$ fluctuations are "fast"), so that the following condition is satisfied: $\mu = \tau\Omega/2\pi >> 1$.

Examples of the signal (1) include the reflected radar signals, the signals in noise carrier communication systems, the pulses describing the optical noise flash, explosive noise in transistors, etc. [1-3].

Let the signal (1) be observed against Gaussian white noise $n(t)$ with the one-sided spectral density N_0. By the observable realization

$$x(t) = s(t, \lambda_0, D_0) + n(t),$$

the parameters λ_0 and D_0 have to be measured taking the values from the prior intervals $[\Lambda_1, \Lambda_2]$ and $[0, \infty)$, respectively. Thus, the boundaries of the observation interval $[T_1, T_2]$ are chosen according to the conditions $T_1 \le \Lambda_1 - \tau/2 < \Lambda_2 + \tau/2 \le T_2$, i.e. the pulse (1) is always located within the observation interval.

II. THE ESTIMATION ALGORITHM

In order to estimate the parameters of the random pulse (1), the maximum likelihood method is applied. According to [4, 5], the logarithm of the functional of the likelihood

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ratio (FLR) $L(\lambda, D)$, as the function of the current values λ, D of the unknown parameters λ_0, D_0, can be represented in the form of

$$L(\lambda, D) = \left[DM(\lambda)/(D+E_N) - \tau_0 E_N \ln(1+D/E_N)\right]/N_0, \tag{2}$$

$$M(\lambda) = \int_{\lambda-\tau/2}^{\lambda+\tau/2} \left[\int_{-\infty}^{+\infty} x(t')h(t-t')\mathrm{d}t'\right]^2 \mathrm{d}t.$$

here $h(t)$ is the pulse response of the filter whose transfer function $H(\omega)$ satisfies the condition $|H(\omega)|^2 = I[(\vartheta-\omega)/\Omega] + I[(\vartheta+\omega)/\Omega]$, and $E_N = N_0\Omega/2\pi$ is the average power of the noise $n(t)$ within the bandwidth Ω of the process $\xi(t)$.

Then, the joint maximum likelihood estimates (MLEs) λ_m and D_m of the time of appearance λ_0 and the dispersion D_0 are written down as follows [4]

$$(\lambda_m, D_m) = \underset{\lambda\in[\Lambda_1,\Lambda_2],\, D>0}{\arg\sup}\; L(\lambda, D),$$

or

$$\lambda_m = \underset{\lambda\in[\Lambda_1,\Lambda_2]}{\arg\sup}\; L(\lambda, D_m) = \underset{\lambda\in[\Lambda_1,\Lambda_2]}{\arg\sup}\; \mathrm{M}(\lambda),$$
$$D_m = \underset{D\geq 0}{\arg\sup}\; \mathrm{L}(\lambda_\mathrm{m}, D) = \Gamma(\lambda_m), \tag{3}$$

where $\Gamma(\lambda) = \max[0; M(\lambda)/\tau - E_N]$, while $M(\lambda)$ is determined from (2).

According to (3), the maximum likelihood measurer of the time of appearance and the average power can be implemented in the form shown in Fig. 1. Here the designations are: 1 is the switch that is open for the time $[\Lambda_1 - \tau/2, \Lambda_2 + \tau/2]$; 2 is the filter with the transfer function $H(\omega)/\sqrt{\tau}$ (2); 3 is the squarer; 4 is the delay line for the time τ; 5 is the integrator; 6 is the extremator that fixes the location of the greatest maximum of the signal as the estimate λ_m (3) of the time of appearance; 7 is the nonlinear element with the characteristic $f(x) = \max(0, x)$; 8 is the gating unit generating the signal sample at the point of time λ_m. The sample magnitude at the output of the gating unit 8 is the estimate D_m (3).

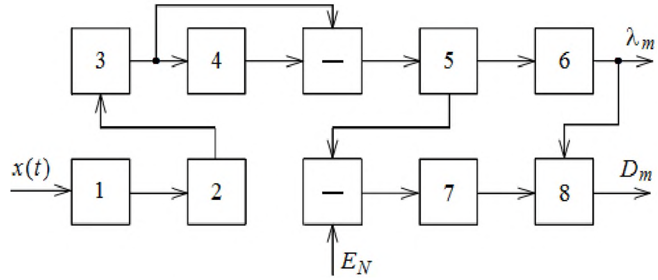

Figure 1. The block diagram of the measurer of the time of appearance and the average power of the random pulse

III. The Characteristics of the Maximum Likelihood Estimates

Considering the characteristics of the joint MLEs of the time of appearance λ_m and the average power D_m is our next task. From (3), it follows that the structure of the algorithm of the estimate λ_m is invariant with respect to the unknown average power D_0. Therefore, by applying the results of [6], for the conditional bias (systematic error) and variance (mean square error) of MLE λ_m, one gets:

$$b(\lambda_m|\lambda_0) = P_0 b_0(\lambda_m|\lambda_0) + (1-P_0)[(\Lambda_2+\Lambda_1)/2 - \lambda_0],$$
$$V(\lambda_m|\lambda_0) = P_0 V_0(\lambda_m|\lambda_0) + (1-P_0)\times$$
$$\times\left[(\Lambda_2^2 + \Lambda_1\Lambda_2 + \Lambda_1^2)/3 - (\Lambda_2+\Lambda_1)\lambda_0 + \lambda_0^2\right]. \tag{4}$$

here P_0 is the probability of a reliable estimate, while $b_0(\lambda_m|\lambda_0)$ and $V_0(\lambda_m|\lambda_0)$ denote the conditional bias and variance of a reliable estimate, respectively. As a reliable estimate [6, 7], the estimate found under the assumption that $|\lambda_m - \lambda_0| < \tau$ is considered.

From [6], it follows that

$$P_0 \approx 2\psi z \exp\left(\frac{\psi^2 z^2}{2} + \psi z^2\right) \int_{1/(1+q_0)}^{\infty} \exp\left[-\frac{m(1+q_0)x}{\sqrt{2\pi}} \times \right.$$
$$\left.\times \exp\left(-\frac{(1+q_0)^2 x^2}{2}\right)\right]\left\{\exp(-\psi z x)\,\Phi[x - z(\psi+1)] - \right.$$
$$\left. - \exp\left[\frac{3\psi^2 z^2}{2} + \psi z(z-2x)\right]\Phi[x - z(2\psi+1)]\right\} dx, \tag{5}$$

$$b_0(\lambda_m|\lambda_0) \approx 0,\; V_0(\lambda_m|\lambda_0) \approx 13\tau^2\left[1 + (1+q_0)^2\right]^2 / 8\mu^2 q_0^4,$$

where

$$\psi = 2(1+q_0)^2 \Big/ \left[1+(1+q_0)^2\right], \qquad q_0 = D_0 / E_N ,$$
$$m = (\Lambda_2 - \Lambda_1)/\tau, \qquad z^2 = \mu q_0^2 / (1+q_0)^2 \qquad (6)$$

is the power signal-to-noise ratio (SNR), and $\Phi(x) = \int_{-\infty}^{x} \exp(-t^2/2) \mathrm{d}t / \sqrt{2\pi}$ is the probability integral. The accuracy of the formulas (4), (5) increases with μ, z, m.

The characteristics of the estimate D_m (3), while the time of appearance λ_0 is unknown, are found in [4]. Under $m > 1$, the expressions for the conditional bias and variance of MLE D_m, while taking into account the possible anomalous errors [7] in estimate λ_m of the parameter λ_0, can be written as

$$b(D_m | D_0) = \langle D_m \rangle - D_0 , \; V(D_m | D_0) = \langle D_m^2 \rangle - 2D_0 \langle D_m \rangle + D_0^2 , \qquad (7)$$

$$\langle D_m \rangle = \frac{D_0}{z} \left\{ \int_0^\infty [1 - F(x)] \mathrm{d}x \right\}, \; \langle D_m^2 \rangle = \frac{2D_0^2}{z^2} \left\{ \int_0^\infty x [1 - F(x)] \mathrm{d}x \right\},$$

where

$$F(x) = F_S(x) F_N[x(1+q_0)],$$

$$\begin{aligned} F_S(x) = {} & \Phi(x - z) - \\ & - 2\exp\left[\psi^2 z^2/2 - \psi z (x - z)\right] \Phi[x - z(\psi + 1)] + \\ & + \exp\left[2\psi^2 z^2 + 2\psi z (z - x)\right] \Phi[x - z(2\psi + 1)], \end{aligned} \qquad (8)$$

$$F_N(x) = \begin{cases} \exp\left[-\left(mx/\sqrt{2\pi}\right)\exp\left(-x^2/2\right)\right], & x \ge 1, \\ 0, & x < 1, \end{cases}$$

and z, ψ, m are determined from (6).

Under $m \le 1$, when the estimate of the time of appearance λ_m are the reliable one, the formulas (7) are simplified and take the form of [4]

$$\begin{aligned} b(D_m | D_0) = D_0 \Bigg\{ & \left(1 + \frac{3}{2\psi z^2}\right)\Phi(z) + \frac{1}{\sqrt{2\pi} z} \exp\left(-\frac{z^2}{2}\right) - \\ & -1 + \frac{2}{\psi z^2} \exp\left(\frac{\psi^2 z^2}{2} + \psi z^2\right) \left[1 - \Phi(z(\psi + 1))\right] - \\ & - \frac{1}{2\psi z^2} \exp\left(2\psi^2 z^2 + 2\psi z^2\right)\left[1 - \Phi(z(2\psi + 1))\right] \Bigg\}, \end{aligned} \qquad (9)$$

$$\begin{aligned} V(D_m | D_0) = D_0^2 \Bigg\{ & 1 - \left(1 - \frac{1}{z^2} - \frac{7}{2\psi^2 z^4}\right)\Phi(z) - \\ & - \frac{1}{\sqrt{2\pi} z}\left(1 - \frac{3}{\psi z^2}\right)\exp\left(-\frac{z^2}{2}\right) - \frac{4}{\psi z^2}\left(1 - \frac{1}{\psi z^2}\right) \times \\ & \times \exp\left(\frac{\psi^2 z^2}{2} + \psi z^2\right)\left[1 - \Phi(z(\psi + 1))\right] + \frac{1}{2\psi z^2} \times \\ & \times \left(2 - \frac{1}{\psi z^2}\right)\exp\left(2\psi^2 z^2 + 2\psi z^2\right)\left[1 - \Phi(z(2\psi + 1))\right] \Bigg\}. \end{aligned}$$

The accuracy of the expressions (7), (8) increases with μ, z, m, while the accuracy of the expressions (9) – with μ, z [4].

IV. The Results of the Statistical Simulation

Analytical calculation of the error ranges of the formulas specified above is very difficult. Therefore, it is of interest to study the noise immunity of the maximum likelihood measurer and the limits of applicability of the approximate expressions (4), (5) and (7)-(9) for the characteristics of the joint MLEs λ_m and D_m by the methods of the statistical computer simulation. To reduce the amount of computer time required for the simulation, the representation is used of the response of the narrowband filter $h(t)$ (2) through its low-frequency quadratures [6]. This allows forming the decision statistics (2) as the sum of the two independent random processes as follows

$$M(\lambda) = \frac{1}{2}\left[M_1(\lambda) + M_2(\lambda)\right],$$
$$M_i(\lambda) = \int_{\lambda - \tau/2}^{\lambda + \tau/2} y_i^2(t) \mathrm{d}t , \; y_i(t) = \int_{-\infty}^{+\infty} x_i(t') h_0(t - t') \mathrm{d}t' , \qquad (10)$$
$$x_i(t) = \xi_i(t) I[(t - \lambda_0)/\tau] + n_i(t), \; i = 1,2 ,$$

where $\xi_i(t)$ and $n_i(t)$ are the statistically independent centered Gaussian random processes with the spectral densities $G_\xi(\omega) = (2\pi D_0 / \Omega) I(\omega / \Omega)$ and $G_n(\omega) = N_0$, respectively, while $h_0(t)$ is the function whose spectrum $H_0(\omega)$ satisfies the condition $|H_0(\omega)|^2 = I(\omega/\Omega)$.

During the simulation within the interval $\left[\widetilde{\Lambda}_1, \widetilde{\Lambda}_2\right]$, $\widetilde{\Lambda}_i = \Lambda_i / \tau$, $i = 1, 2$ with the discretization step Δ, the samples were formed of the realizations of the random processes $y_i(t)$ (10). It allowed us to obtain the stepwise approximation of the decision statistics of the form of

$$M(l) = \frac{1}{2} \sum_{k = k_{\min}}^{k_{\max}} \left(y_{1k}^2 + y_{2k}^2\right)\Delta , \qquad (11)$$

here $k_{\min} = \mathrm{int}\{(l - 1/2)/\Delta\}$, $k_{\max} = \mathrm{int}\{(l + 1/2)/\Delta\}$, $l = \lambda/\tau$ is the normalized current value of the time of appearance, $\mathrm{int}\{\cdot\}$ is an integer. In case when $\Delta = 0.05/\mu$ and $\Delta l = 0.01$ (Δl is the discretization step along the variable l), the mean square error of the step approximation (11) of the continuous realization (10) does not exceed 10 %.

The samples of the processes y_{ik}, $i = 1, 2$ are generated in terms of the sequence of independent Gaussian random numbers by a moving summation method [6] as follows:

$$y_{ik} = \frac{1}{\sqrt{\Delta}} \sum_{m=k-p}^{k+p} \alpha_{im} H_{k,m} + \sum_{m=\max(m_{\min}, k-p)}^{\min(m_{\max}, k+p)} \xi_{im} H_{k,m}, \qquad (12)$$
$$\xi_{iim} = \frac{1}{\pi}\sqrt{\frac{q_0}{\Delta}} \sum_{n=0}^{2p} H_{n,p} \beta_{in+m+1}.$$

here $m_{\min} = \mathrm{int}\{(l_0 - 1/2)/\Delta\}$, $m_{\max} = \mathrm{int}\{(l_0 + 1/2)/\Delta\}$, $l_0 = \lambda_0/\tau$, $H_{k,m} = \sin[2\pi\mu\Delta(k-m)]/[\pi(k-m)]$, and α_{im}, β_{im} are independent Gaussian random numbers with zero mathematical expectations and unit dispersions.

In the sums (12), the number of summands corresponds to the value $p = 50$ providing a relative deviation of the generated sample dispersion from the simulated process dispersion to be no more than 5 %. Formation of the Gaussian numbers α_{im}, β_{im} with the parameters (0,1) has been implemented based on the sequences of the independent random numbers ϕ_n, φ_n uniformly distributed within the interval [0,1] by the Cornish-Fisher method [6, 8]:

$$\zeta_i = Z_i + \frac{Z_i^3 - 3Z_i}{20N}, \quad Z_i = \sqrt{\frac{12}{N}} \sum_{n=1}^{N} \left[\theta_{N(i-1)+n} - 0.5\right], \qquad (13)$$

where ζ_i is one of the sequences α_{im}, β_{im}, and θ_n is sequence ϕ_n, φ_n corresponding to it. The number of summands N in the sum (13), following [6, 8], has been chosen as equal to 5.

By the realization of the process $M(l)$ obtained with the help of the formulas (11), (12), according to (3), the normalized estimates $l_m = \lambda_m/\tau$, $q_m = D_m/E_N$ are determined and the variances of these estimates are found. Some results of the statistical simulation are presented in Figs. 2-6 where the corresponding theoretical dependences are also shown. Each experimental value has been obtained as a result of processing of no less than 10^4 realizations of $M(l)$ under $\tilde{\Lambda}_1 = 0$, $\tilde{\Lambda}_2 = m$, $l_0 = (\tilde{\Lambda}_2 + \tilde{\Lambda}_1)/2$. Thus, with the probability of 0.9, the confidence intervals boundaries deviate from the experimental values no more than by 10...15 %.

In Fig. 2, there is presented the theoretical dependence (5) of the normalized variance $V_{0l} = V_0(\lambda_m|\lambda_0)/\tau^2$ of the reliable estimate λ_m under $m = 1$.

In Fig. 3 one can see the analogous dependence (4) of the normalized variance $V_l = V(\lambda_m|\lambda_0)/\tau^2$ of the estimate λ_m with the anomalous errors taken into account under $m = 20$. The solid lines depict the results of the calculations when $\mu = 50$, while the dashed lines demonstrate them under μ=100 and the dash-dotted lines – under μ=200. The corresponding experimental values of the variances V_{0l} and V_l are designated by rectangles, crosses and diamonds under $\mu = 50$, μ=100 and μ=200, respectively.

In Fig. 4, there are shown the theoretical and experimental dependences of the probability of the anomalous error $P_a = P[|\lambda_m - \lambda_0| > \tau_0] = 1 - P_0$ (5). In Figs. 5, 6, one can see the theoretical and experimental dependences (9) and (7), (8) of the normalized variances $V_q = V(D_m|D_0)/E_N^2$ of the estimate D_m under $m = 1$ (when MLE λ_m is reliable) and $m = 20$ (when the anomalous errors are possible in estimating the pulse (1) time of appearance), respectively. The designations in Figs. 4-6 correspond to those given in Fig. 3.

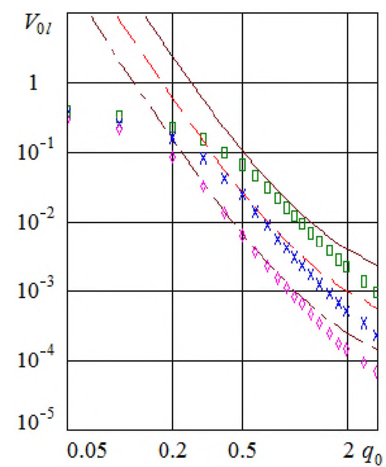

Figure 2. The variance of the reliable estimate of the time of appearance of the random pulse

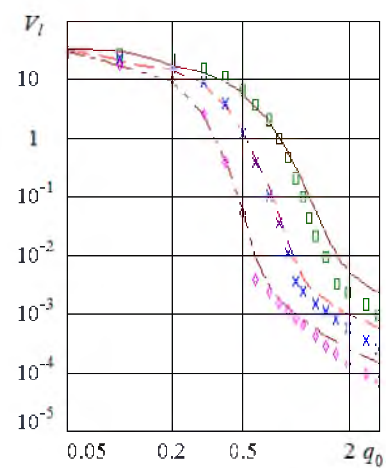

Figure 3. The variance of the estimate of the time of appearance of the random pulse when there are the anomalous errors

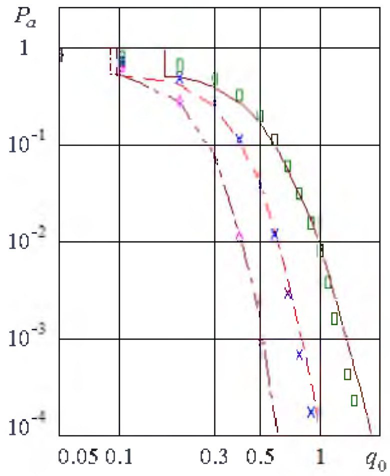

Figure 4. The the probability of the anomalous error when estimating the time of apperance of the random pulse

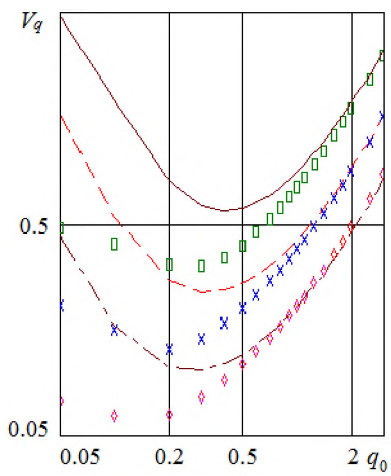

Figure 5. The variance of the estimate of the average power of the random pulse in case of the reliable estimate of the time of apperance

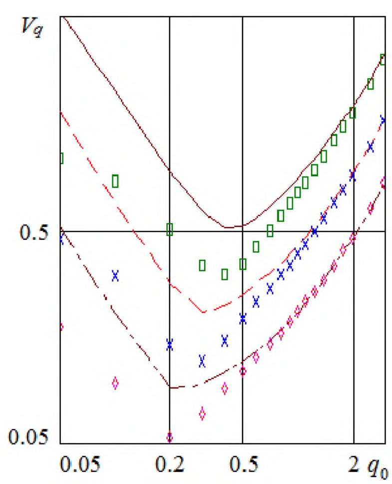

Figure 6. The variance of the estimate of the average power of the random pulse if the anomalous errors occur when estimating the time of apperance

V. Conclusion

Based on the results obtained, the following conclusions can be drawn. As follows from Figs. 2, 3, the theoretical dependences (5) for the variance of the reliable estimate λ_m well approximate the experimental data under SNR $z > 1.5\ldots2$, while the theoretical dependences (4) for the variance of the estimate λ_m with the anomalous errors taken into account agree generally with the experimental data under $\mu \geq 50$ and $z \geq 0.5$. If $z < 1.5$, then the theoretical dependences (5) deviate from the experimental values, as the formula (5) for the variance of the reliable estimate of the time of appearance does not take into account the finite length of the prior interval $[\Lambda_1, \Lambda_2]$ of the possible values of the parameter λ_0. As a result, when the variance becomes comparable with or greater than the value $(\Lambda_2 - \Lambda_1)^2/12$, the accuracy of the formulas (5) deteriorates significantly. The deviation of the theoretical dependences $V_0(\lambda_m|\lambda_0)$ (5), $V(\lambda_m|\lambda_0)$ (4) from the experimental values is also observed in case of the large SNRs, when $q_0 > 2\ldots3$. This is due to the fact that the formula (5) for the variance of the reliable estimate of the time of appearance has been obtained in neglecting the estimation errors of the order of the correlation time of the random process $\xi(t)$ [6]. Therefore, when the normalized variance decreases up to the value of the order of μ^{-2}, the error of the formulas (4), (5) becomes significant.

If the SNR is not large enough ($z < 4\ldots5$) and the reduced length of the prior interval is $m >> 1$, then it is necessary to take into account the anomalous errors in estimating the time of appearance. In this case, the accuracy of the MLE λ_m can significantly deteriorate. Under $z > 5$ and $m \leq 10\ldots20$, when the probability of the anomalous errors can be neglected, the values of the variances of the estimate λ_m obtained by the formulas (4) and (5) almost coincide.

According to Figs. 5, 6, the formulas (7), (8) and (9) are consistent satisfactorily with the experimental values of the variance of the estimated dispersion, if $z > 3\ldots4$. Under $z > 5$, when the probability of anomalous errors in estimating the parameter λ_0 is sufficiently small (the estimate of the time of appearance is reliable), the variances $V(D_m|D_0)$ (7), (8) and (9) of the estimated dispersion coincide.

It may be noted that, as it is stated in [9, 10], the accuracy of the discontinuous parameter (time of appearance) can be increased by 20 percent (under big SNRs), approximately, by applying the Bayesian method to obtain the estimates. However, in this case the structure of the measurer becomes more complex.

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