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Zafar Rakhmonov
National University of Uzbekistan, zraxonov@inbox.ru

Jasur Urunbaev
Samarkand State University, jasururunbayev@gmail.com

Bobur Allaberdiyev
National University of Uzbekistan

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ON THE PROPERTIES OF SOLUTIONS OF A CROSS-DIFFUSION SYSTEM WITH NONLINEAR BOUNDARY FLUX

Z.R.Rakhmonov[1], J.E.Urunbaev[2], B.Allaberdiyev[1]
[1]*National university of Uzbekistan*
[2]*Samarkand state university*

Abstract: In this paper, based on a self-similar analysis and the method of standard equations, the properties of a nonlinear cross-diffusion system coupled via nonlocal boundary conditions are studied. We are investigated the qualitative properties of solutions of a nonlinear system of parabolic equations of cross-diffusion in a medium coupled with nonlinear boundary conditions. It is proved that for certain values of the numerical parameters of the nonlinear cross-diffusion system of parabolic equations coupled via nonlinear boundary conditions, they may not have global solutions in time. Based on a self-similar analysis and the principle of comparing solutions, a critical exponent of the Fujita type and a critical value of global solvability are established. Using the comparison theorem, upper bounds for global solutions and lower bounds for blow-up solutions are obtained.

Keywords: asymptotic, cross-diffusion, nonlinear system, self-similar solution

Nolokal chegaraviy oqim bilan berilgan kross-diffuziya sistemalarining yechimi xususiyati haqida

Annotatsiya. Ushbu ishda avtomodel tahlil va etalon tenglamalar usuli asosida nochiziqli chegaraviy shartlar bilan bog'langan nochiziqli kross-diffuziya sistemalarining xususiyatlari o'rganilgan. Chiziqsiz chegaraviy shartlari bilan bog'langan muhitda kross-diffuziya sistemasi yechimlarining sifat xossalari o'rganilgan. Nochiziqli chegaraviy shartlar bilan bog'langan kross-diffuziya sistemasining sonli parametrlarining ba'zi bir qiymatlari uchun vaqt bo'yicha global yechimlari mavjud bo'lmasligi isbotlangan. Avtomodel tahlil va yechimlarni taqqoslash prinsipi asosida Fujita tipidagi kritik eksponenta va yechimlarning global mavjudlik kritik qiymati topilgan. Taqqoslash teoremasidan foydalanib, global yechimlar uchun yuqori va blow-up yechimlar uchun quyi baholar olingan.

Kalit soʻzlar: asimptotika, kross-diffuziya, nochiziqli sistema, avtomodel yechim.

О свойствах решений системы кросс-диффузии с нелинейным граничным потоком

Аннотация. В данной работе на основе автомодельного анализа и метода стандартных уравнений изучаются свойства нелинейной кросс-диффузионной системы, связанной через нелокальные граничные условия. Исследованы качественные свойства решений нелинейной системы параболических уравнений кросс-диффузии в среде, связанной с нелинейными граничными условиями. Доказано, что при некоторых значениях числовых параметров нелинейной кросс-диффузионной системы параболических уравнений, связанных нелинейными граничными условиями, они могут не иметь глобальных решений во времени. На основе автомодельного анализа и принципа сравнения решений установлены критический показатель типа Fujita и критическое значение глобальной разрешимости. Используя теорему сравнения, получены верхние оценки для глобальных решений и нижние оценки для blow-up решений.

Ключевые слова: асимптотика, кросс-диффузия, нелинейная система, автомодельное решение.

Qualitative properties of solutions of a nonlinear cross-diffusion system coupled with nonlocal boundary conditions are studied in the paper

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\upsilon^{m_1-1} \left| \dfrac{\partial u^k}{\partial x} \right|^{p-2} \dfrac{\partial u}{\partial x} \right), \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{m_2-1} \left| \dfrac{\partial \upsilon^k}{\partial x} \right|^{p-2} \dfrac{\partial \upsilon}{\partial x} \right), \quad x \in R_+, \ t > 0, \end{cases} \tag{1}$$

$$\begin{cases} -\upsilon^{m_1-1}\left|\dfrac{\partial u^k}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}(0,t)=u^{q_1}(0,t), \\ -u^{m_2-1}\left|\dfrac{\partial \upsilon^k}{\partial x}\right|^{p-2}\dfrac{\partial \upsilon}{\partial x}(0,t)=\upsilon^{q_2}(0,t),\ t>0, \end{cases} \tag{2}$$

$$u(x,0)=u_0(x),\quad \upsilon(x,0)=\upsilon_0(x),\quad x\in R_+, \tag{3}$$

where $p>\max\{m_1,m_2\}+1$, $k>0,\ \ m_i>1,\ \ q_i>0(i=1,2)$, u_0 and $\upsilon_0(x)$ are the non-negative continuous functions with compact carrier in R_+.

Recently, they began to intensively engage in the analysis of mathematical models of the reaction-diffusion type in the presence of the so-called cross (cross) diffusion. Cross-diffusion is a process in which a concentration or density gradient of one chemical or biological species induces a flow (linear or non-linear) of another species. Accordingly, applications of the reaction-cross-diffusion of a system are widespread in the literature and include the formation of pattern development in biology [16], electrochemistry [22], cancer motility [5, 8, 11] and biofilms [10]. The introduction of cross-diffusion in standard reaction-diffusion models has been shown to prevent cross-diffusion in order to prevent exacerbation phenomena associated with such systems [4]. Explicit analytical solutions of these complex and often nonlinearly connected systems of partial differential equations are rarely existed, and thus, numerical methods are used to obtain approximate solutions.

Cross-diffusion models are also found in various fields of science. For example, in physical systems (plasma physics) [1, 12, 23], in chemical systems (dynamics of electrolytic solutions), in biological systems (cross-diffusion transport, dynamics of population systems), in ecology (dynamics of the age structure of the forest), in seismology - Burridge-Knopoff model describing the interaction of tectonic plates [13-15, 21]. In recent years, in the study of the biological population and the movement of tectonic plates, mathematical models with cross-diffusion have been actively used [14, 15].

It is known that a system of degenerate equations may not have a classical solution in the region, where $u,\ \upsilon\equiv 0$. In this case, the generalized solution of system (1) is studied in a class having a physical meaning

$$u(x,t),\ \upsilon(x,t)\ge 0,\ \upsilon^{m_1-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x},\ u^{m_2-1}\left|\frac{\partial \upsilon}{\partial x}\right|^{p-2}\frac{\partial \upsilon}{\partial x}\in C\big(R\times(0,+\infty)\big).$$

and satisfying system (1) from the point of distribution [1, 3].

In recent years, the condition for global existence of solutions and the condition for the emergence of a blow-up regime have been intensively studied (see . [2-5,8-10, 13, 6,7,11,15,17-22, 16, 24-26]). In [24,25] the conditions for global solvability and unsolvability in terms of time have been studied and the estimation of the solution near exacerbations time of a nonlocal problem of diffusion has been stated

$$u_t=u_{xx},\quad \upsilon_t=\upsilon_{xx},\quad x>0,\ 0<T<0, \tag{4}$$

$$-u_x(0,t)=u^{\alpha}\upsilon^{p},\quad -\upsilon_x(0,t)=u^{q}\upsilon^{\beta},\quad 0<t<T, \tag{5}$$

$$u(x,0)=u_0(x),\ \upsilon(x,0)=\upsilon_0(x),\ x>0. \tag{6}$$

It is proved that if $pq\le(1-\alpha)(1-\beta)$, then each solution of problem (4) - (6) is global.

In [6], systems of cross-diffusion equations on a stationary surface of the following form are investigated

$$\begin{cases} \dfrac{\partial u_m}{\partial t}-\sum_{k=1}^{r}d_{mk}\Delta_\Gamma u_k = f_m(u_1,...,u_r),\ in\,\Gamma\times(0,T), \\ u_m(x,0)=u_{0,m}(x),\ \forall_x\in\Gamma,\ m=1,...,r \end{cases}$$

where r≥1.*They providea fully-discrete scheme by applying the Implicit–Explicit Euler method. In addition, they provide sufficient conditions for the existence of polytopal invariant regions for the numerical solution after spatial and full discretization's. Furthermore, they prove* optimal error bounds for the semi- and *fully-discrete methods, that is the convergence rates are quadratic in the mesh size and linear in the time step.*

In [17] the authors have investigated the following problem

$$u_t = \left(u^n\right)_{xx}, \quad \upsilon_t = \left(\upsilon^k\right)_{xx}, \quad x \in R_+, \ t > 0, \tag{7}$$

$$-\left(u^n\right)_x (0,t) = \upsilon^p (0,t), \quad -\left(\upsilon^k\right)_x (0,t) = u^q (0,t), \quad t > 0, \tag{8}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \tag{9}$$

They have shown that the solution of problem (7) - (8) is global if $pq \le (n+1)(k+1)/4$. Conditions were obtained for the numerical parameters of systems (7) - (9) under which the solution of the problem grows infinitely in a finite time.

The paper [21] should be also mentioned; here system (7) has been investigated with the following boundary conditions

$$-\left(u^n\right)_x (0,t) = u^\alpha \upsilon^p (0,t), \quad -\left(\upsilon^k\right)_x (0,t) = u^q \upsilon^\beta (0,t), \quad t > 0.$$

It is shown, that $\min\{y_1 - r_1, y_2 - r_2\} = 0$, where

$$r_1 = \frac{2p + k + 1 - 2\beta}{4pq - (k + 1 - 2\alpha)(n + 1 - 2\beta)}, \ r_2 = \frac{2p + n + 1 - 2\beta}{4pq - (k + 1 - 2\alpha)(n + 1 - 2\beta)},$$

$$y_1 = \frac{1 - r_1(n-1)}{2}, \ y_2 = \frac{1 - r_2(k-1)}{2}.$$

are the critical exponents of Fujita type.

Introduce the notation

$$\beta = \frac{(q_1-1)(q_2-1) - k(p-2)(q_2-1) - (m_1-1)(q_1-1)}{\big(k(p-2)+2\big)(q_1-1)(q_2-1) - k(p-2)(q_2-1) - (m_1-1)(q_1-1)} =$$

$$\frac{(q_1-1)(q_2-1) - k(p-2)(q_1-1) - (m_2-1)(q_2-1)}{\big(k(p-2)+2\big)(q_1-1)(q_2-1) - k(p-2)(q_1-1) - (m_2-1)(q_2-1)},$$

$$\alpha_1 = \frac{\big(k(p-2)+1\big)(q_2-1)}{l_1}, \alpha_2 = \frac{\big(k(p-2)+1\big)(q_1-1)}{l_2},$$

$$l_1 = \big(k(p-2)+2\big)(q_1-1)(q_2-1) - k(p-2)(q_2-1) - (m_1-1)(q_1-1),$$

$$l_2 = \big(k(p-2)+2\big)(q_1-1)(q_2-1) - k(p-2)(q_1-1) - (m_2-1)(q_2-1).$$

Theorem 1. Let $\min\{l_1, l_2\} > 0$, then any solution to problem (1)-(3) is unbounded for sufficiently large initial data

Proof. Introduce new functions $\hat{u}$ and $\hat{\upsilon}$ of the types:

$$\begin{cases} \hat{u}(x,t) = (T-t)^{-\alpha_1} \varphi(\xi), \\ \hat{\upsilon}(x,t) = (T-t)^{-\alpha_2} \phi(\xi), \ \xi = x(T-t)^{-\beta}, \quad x \ge 0, \ t \ge 0, \end{cases} \tag{10}$$

which are self-similar ones. Then theorem 1 can be proved at $T > 0$. Substituting these functions in (1) - (3), we obtain a self-similar problem consisting of the following systems of equations relative to $\varphi(\xi)$ and $\phi(\xi)$:

$$\begin{cases} \dfrac{d}{d\xi}\left(\phi^{m_1-1}\left|\dfrac{d\varphi^k}{d\xi}\right|^{p-2}\dfrac{d\varphi}{d\xi}\right)-\beta\xi\dfrac{d\varphi}{d\xi}-\alpha_1\varphi=0, \\ \dfrac{d}{d\xi}\left(\varphi^{m_2-1}\left|\dfrac{d\phi^k}{d\xi}\right|^{p-2}\dfrac{d\phi}{d\xi}\right)-\beta\xi\dfrac{d\phi}{d\xi}-\alpha_2\phi=0, \end{cases} \tag{11}$$

$$\begin{cases} -\phi^{m_1-1}\left|\dfrac{d\varphi^k}{d\xi}\right|^{p-2}\dfrac{d\varphi}{d\xi}(0)=\varphi^{q_1}(0), \\ -\varphi^{m_2-1}\left|\dfrac{d\phi^k}{d\xi}\right|^{p-2}\dfrac{d\phi}{d\xi}(0)=\phi^{q_2}(0). \end{cases} \tag{12}$$

obtained by substituting (10) into (1) - (3) and some simplifications. Define the conditions under which (10) is an unbounded lower solution to problem (1) - (3). As the compared functions, we choose the following ones:

$$\begin{cases} \tilde{\varphi}(\xi)=A_1(a-\xi)^k, \\ \tilde{\phi}(\xi)=A_2(a-\xi)^z, \end{cases} \tag{13}$$

where $A_i>0\ (i=1,2)$, $y=\dfrac{(p-1)(k(p-2)-m_1-1)}{k^2(p-1)^2-(m_1-1)(m_2-1)}$, $z=\dfrac{(p-1)\left(k(p-2)-m_2-1\right)}{k^2(p-2)^2-(m_1-1)(m_2-1)}$.

To apply the comparison theorem, the following inequalities are required:

$$\begin{cases} \dfrac{d}{d\xi}\left(\tilde{\phi}^{m_1-1}\left|\dfrac{d\tilde{\varphi}^k}{d\xi}\right|^{p-2}\dfrac{d\tilde{\varphi}}{d\xi}\right)-\beta\xi\dfrac{d\tilde{\varphi}}{d\xi}-\alpha_1\tilde{\varphi}\geq 0, \\ \dfrac{d}{d\xi}\left(\tilde{\varphi}^{m_2-1}\left|\dfrac{d\tilde{\phi}^k}{d\xi}\right|^{p-2}\dfrac{d\tilde{\phi}}{d\xi}\right)-\beta\xi\dfrac{d\tilde{\phi}}{d\xi}-\alpha_2\tilde{\phi}\geq 0, \end{cases}$$

$$\begin{cases} -\tilde{\phi}^{m_1-1}\left|\dfrac{d\tilde{\varphi}^k}{d\xi}\right|^{p-2}\dfrac{d\tilde{\varphi}}{d\xi}(0)\leq\tilde{\varphi}^{q_1}(0), \\ -\tilde{\varphi}^{m_2-1}\left|\dfrac{d\tilde{\phi}^k}{d\xi}\right|^{p-2}\dfrac{d\tilde{\phi}}{d\xi}(0)\leq\tilde{\phi}^{q_2}(0). \end{cases}$$

Two systems of inequalities with respect to A_1 and A_2 are obtained

$$\begin{cases} A_1^{k(p-2)}A_2^{m_1-1}y^pk^{p-2}-\alpha_1a+\left(\alpha_1+y\beta\right)\xi\geq 0, \\ A_1^{m_2-1}A_2^{k(p-2)}z^pk^{p-2}-\alpha_2a+\left(\alpha_2+z\beta\right)\xi\geq 0, \end{cases} \tag{I}$$

$$\begin{cases} A_1^{k(p-2)+1}A_2^{m_1-1}k^{p-2}y^{p-1}a^y\leq A_1^{q_1}a^{yq_1}, \\ A_2^{k(p-2)+1}A_1^{m_2-1}k^{p-2}z^{p-1}a^z\leq A_2^{q_2}a^{zq_2}. \end{cases} \tag{II}$$

$a\leq\min\left\{\dfrac{A_1^{p-2}A_2^{m_1-1}y^pk^{p-2}}{\alpha_1},\dfrac{A_1^{m_2-1}A_2^{p-2}z^pk^{p-2}}{\alpha_2}\right\},\min\{l_1,l_2\}>0$. Similarly, we can find the condition for systems (II); the upper bound for parameter a is:

$$a \ge \max\left\{ \left(\frac{A_2^{m_1-1} k^{p-2} y^{p-1}}{A_1^{q_1-k(p-2)-1}} \right)^{\frac{1}{y(q_1-1)}}, \left(\frac{A_1^{m_2-1} k^{p-2} z^{p-1}}{A_2^{q_2-k(p-2)-1}} \right)^{\frac{1}{z(q_2-1)}} \right\}.$$

Thus, choosing the parameters $A_1,\ A_2,\ a$, we can obtain the system of inequalities (I, II) under condition $\min\{l_1, l_2\} > 0$. By the principle of comparing solutions, we have estimates for the initial data, with respect to the lower self-similar solutions (10), (13):

$$\begin{cases} u_0(x) \ge T^{-\alpha_1} A_1 \left(a - xT^{-\beta}\right)_+^y, \\ \upsilon_0(x) \ge T^{-\alpha_2} A_2 \left(a - xT^{-\beta}\right)_+^z. \end{cases}$$

It follows that the solution to problem (1) - (3) is unbounded

$$\begin{cases} u(x,t) \ge (T-t)^{-\alpha_1} \tilde{\varphi}(0) \to \infty,\ t \to T, \\ \upsilon(x,t) \ge (T-t)^{-\alpha_2} \tilde{\phi}(0) \to \infty,\ t \to T. \end{cases}$$

at $\min\{l_1, l_2\} > 0$. The theorem is proved.

Theorem 2. Let $\max\{\alpha_1 - \beta, \alpha_2 - \beta\} < 0$ and the initial data are sufficiently small, then any solution to problem (1)-(3) is global.

Proof. Constructing bounded upper solutions, we can determine the conditions of solvability in time in the following way:

$$\begin{cases} u_+(x,t) = (T+t)^{-\alpha_1} f(\xi), \\ \upsilon_+(x,t) = (T+t)^{-\alpha_2} g(\xi),\ \xi = x(T+t)^{-\beta}, \end{cases} \tag{14}$$

where $T > 0$, $f(\xi)$ and $g(\xi)$ are the sought for functions, which, by the solution comparison theorem, must satisfy the system of inequalities:

$$\begin{cases} \frac{d}{d\xi}\left(g^{m_1-1} \left| \frac{df^k}{d\xi} \right|^{p-2} \frac{df}{d\xi} \right) + \beta\xi \frac{df}{d\xi} + \alpha_1 f \le 0, \\ \frac{d}{d\xi}\left(f^{m_2-1} \left| \frac{dg^k}{d\xi} \right|^{p-2} \frac{dg}{d\xi} \right) + \beta\xi \frac{dg}{d\xi} + \alpha_2 g \le 0, \end{cases} \tag{15}$$

$$\begin{cases} -g^{m_1-1} \left| \frac{df^k}{d\xi} \right|^{p-2} \frac{df}{d\xi}(0) \ge f^{q_1}(0), \\ -f^{m_2-1} \left| \frac{dg^k}{d\xi} \right|^{p-2} \frac{dg}{d\xi}(0) \ge g^{q_2}(0). \end{cases} \tag{16}$$

Along with this, consider the following

$$\begin{cases} \bar{f}(\xi) = A_1 \left(a - (\xi + h)^{\frac{p}{p-1}} \right)^y, \\ \bar{g}(\xi) = A_2 \left(a - (\xi + h)^{\frac{p}{p-1}} \right)^z, \end{cases} \tag{17}$$

Where $h \in \left(0, a^{\frac{p-1}{p}}\right)$, $a > 0$, $A_2^{m_1-1} A_1^{k(p-2)} k^{p-2} \left(\frac{yp}{p-1}\right)^{p-1} = \beta$, $A_1^{m_2-1} A_2^{k(p-2)} k^{p-2} \left(\frac{zp}{p-1}\right)^{p-1} = \beta$. The solvability of systems of inequalities (15)-(16) with respect to unknown parameters a, h, and under conditions $q_1 > m_2 + 1$, $q_2 > m_1 + 1$ is shown as follows. Substituting functions (17) into (15) and (16), we obtain

$$\begin{cases} (\alpha_1 - \beta)\left(a - (\xi + h_1)^{\frac{p}{p-1}}\right) \le 0, \\ (\alpha_2 - \beta)\left(a - (\xi + h_2)^{\frac{p}{p-1}}\right) \le 0. \end{cases}$$

hence the condition for the restrictions on $\max\{\alpha_1 - \beta, \alpha_2 - \beta\} < 0$ and conditions for further calculations of a, h are given in the form:

$$a \le h^{\frac{p}{p-1}} + \min\left\{ \left(A_1^{1-q_1} h\beta\right)^{\frac{1}{y(q_1-1)}}, \left(A_2^{1-q_2} h\beta\right)^{\frac{1}{z(q_2-1)}} \right\} \qquad (18)$$

Given this, we can conclude that if $\max\{\alpha_1 - \beta, \alpha_2 - \beta\} < 0$ and the initial functions $u_0(x)$ and $\upsilon_0(x)$ satisfy the following inequalities:

$$\begin{cases} u_0(x) \le T^{-\alpha_1} A_1 \left(a - \left(xT^{-\beta} + h\right)^{\frac{p}{p-1}}\right)^y, \\ \upsilon_0(x) \le T^{-\alpha_2} A_2 \left(a - \left(xT^{-\beta} + h\right)^{\frac{p}{p-1}}\right)^z, \end{cases}$$

then the solution to problem (1) - (3) is global. The values of parameters a, h are selected from condition (18).

Theorem 3. If $q_1 \le 1$, $q_2 \le 1$ then every solution to problem (1)-(3) is global.

Theorem 3 is proved by the method described in [20].

Note 1. Theorem 3 shows that the critical exponents of the global existence of a solution are $q_{10} = 1$, $q_{20} = 1$.

Note 2. Theorem 1 shows that the critical Fujita exponents are $\min\{\alpha_1 - \beta, \alpha_2 - \beta\} = 0$.

Theorem 4. Assume that $p > \max\{m_1, m_2\} + 1$ and $m_i > 1$, then the solution with compact carrier of the system of equations (15) at $\xi \to a^{(p-1)/p}$ has an asymptotic

$$\begin{cases} \varphi(\xi) = \tilde{\varphi}(\xi)(1 + o(1)), \\ \phi(\xi) = \tilde{\phi}(\xi)(1 + o(1)). \end{cases} \qquad (18)$$

A numerical scheme is constructed based on the finite difference method. For this, system (1) is approximated with the second order of accuracy in spatial coordinates and with the first order in t, by the integro-interpolation method (balance method). Let's build an iterative process. In the internal nodes, the values are calculated by the sweep method.

It is known that the choice of an appropriate initial approximation for the iterative process of solving the nonlinear problem (1)-(3) in a general case is the main difficulty in numerical solution of the problem.

When solving specific tasks, the functions are used that reflect some properties of the sought for solutions; these functions are obtained on the basis of a qualitative analysis of the problem. This difficulty, depending on the value of the numerical parameters of equations, is overcome by successful selecting the initial approximations, for which the established asymptotic formula is taken in the calculations. Based on the above results, numerical calculations were made. Below are the numerical schemes and some results of computational experiments.

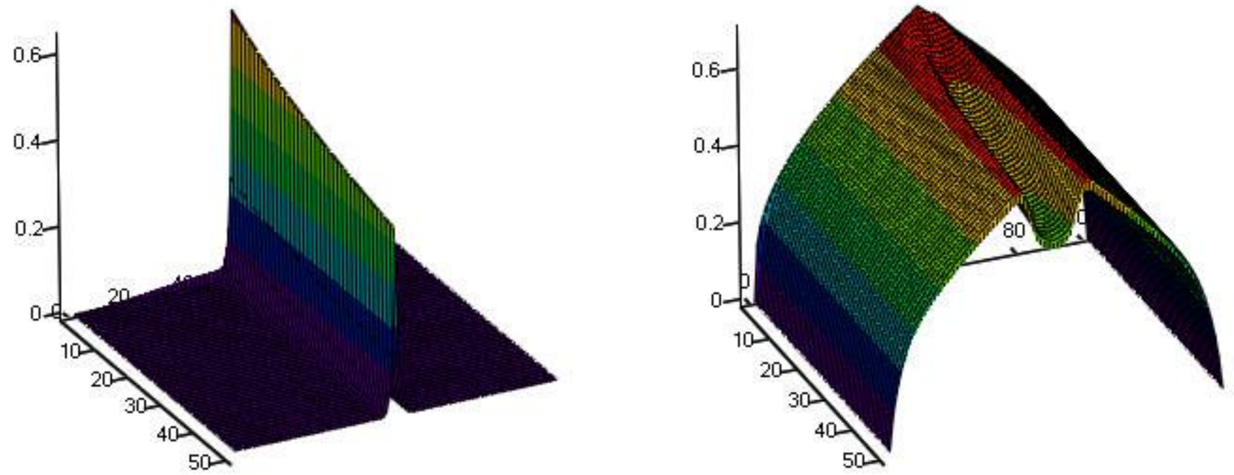

Fig. 1.Numerical solution of the problem (1)-(3) $q_1 = 3,\ q_2 = 2.7\ \ p = 1.3, p = 3.3,\ m_1 = 1.5,\ m_2 = 1.8$

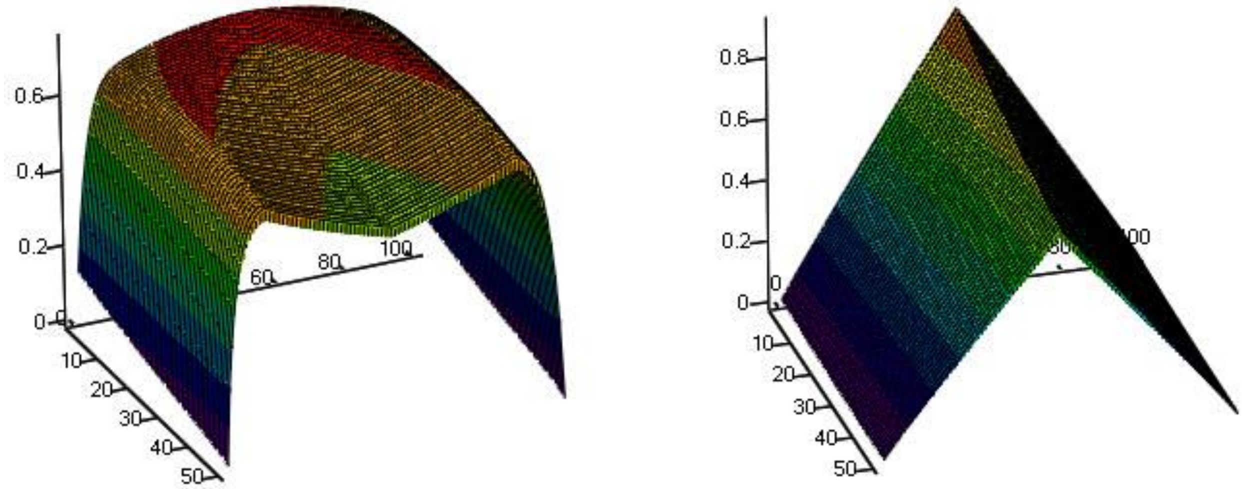

Fig. 2.Numerical solution of the problem (1)-(3) at

$q_1 = 3.1,\ q_2 = 3.7, k = 1.25, p = 3.8,\ m_1 = 2.1,\ m_2 = 1.2$

Figs. 1-2, show the results of numerical solution of problem (1) - (3) at $m_i > 1\ (i = 1, 2)$ corresponding to the case of slow diffusion. At $m_i > 1\ (i = 1, 2)$ as follows from the asymptotic formulas (14), (18) and graphs, the object moves with a finite velocity. The depth of penetration of a diffusion wave depends on time and the wave front (the point at which $\underline{u}(x,t)$, $\underline{\upsilon}(x,t)$ for each medium located at the end point: $x_\phi = a^{(p-1)/p}(T+t)^\beta < \infty$

References

1. Aripov M.M. Methods of reference equations for solving nonlinear boundary value problems. - Tashkent, Fan, 1988M.

2. Aripov M.M., Matyakubov A.S. Self-similar solutions of a cross-diffusion parabolic system with variable density: explicit estimates and asymptotic behavior. Nanosystems: Physics, Chemistry, Mathematics, 2017, 8(1), 5-12. https://doi.org/10.17586/2220-8054-2017-8-1-5-12

3. Aripov M.M., Matyakubov A.S. To the qualitative properties of solution of system equations not in divergence form of polytrophic filtration in variable density. Nanosystems: Physics, Chemistry, Mathematics, 2017, 8(3), 317-322. doi: https://doi.org/10.17586/2220-8054-2017-8-3-317-322

4. Dziuk G., Elliott C.M., Finite element methods for surface PDEs, Acta Numer. 22 (2013) 289–396. http://dx.doi.org/10.1017/s0962492913000056.

5. Elliott C.M., Stuart A.M., The global dynamics of discrete semilinear parabolic equations, SIAM J Num. Anal. 30 (1993) 1622–1663. http://dx.doi.org/ 10.1137/0730084.

6. Frittelli Massimo, Madzvamuse Anotida, Sgura Ivonne, Venkataraman Chandrasekhar. Lumped finite elements for reaction–cross-diffusion systems on stationary surfaces. Computers and Mathematics with Applications 74 (2017) 3008–3023 doi: https://doi.org/10.1016/j.camwa.2017.07.044

7. Francesco M. Di, Esposito A., Fagioli S.. Nonlinear degenerate cross-diffusion systems with nonlocal interaction. Nonlinear Analysis, 169, 2018, 94-117. doi: http://dx.doi.org/10.1016/j.na.2017.12.003.

8. Gerisch A., Chaplain M.A.J., Robust numerical methods for taxis–diffusion–reaction systems: applications to biomedical problems, Math. Comp. Mod. 43(2006)49–75.http://dx.doi.org/10.1016/j.mcm.2004.05.016.

9. Gerstenmayer Anita, Jüngel Ansgar. Analysis of a degenerate parabolic crossdiffusion system for ion transport. Journal of Mathematical Analysis and Applications, Nonlinear Analysis, 461(1), 2018, 523-543. doi: https://doi.org/10.1016/j.jmaa.2018.01.024

10. Hoff D. Stability and convergence of finite difference methods for systems of nonlinear reaction-diffusion equations, SIAM J Num. Anal. 15 (1978) 1161–1177. http://dx.doi.org/10.1137/0715077.

11. Hittmeir S., Jüngel A., Cross diffusion preventing blow-up in the two-dimensional Keller–Segel Model, SIAM J Math. Anal. 43 (2011) 997–1022. http://dx.doi.org/10.1137/100813191.

12. Kalashnikov A.S. Some questions of the qualitative theory of nonlinear second order degenerate parabolic equations // UMN, V.42, Iss. 2 (254), 1987, 135–176.

13. Levine, H., The role of critical exponents in blowup theorems, *SIAM Rev*., 32(2), 1990, 262-288.

14. Murray J.D. *Mathematical Biology,* 3rd ed., Berlin: Springer, 2002-2003.

15. Malchow H, Petrovskii SV, Venturino E. Spatiotemporal patterns in ecology and epidemiology: theory, models, and simulations. London: Chapman & Hall/CRC Press; 2008.

16. Nie Y-Y, Thomée V., A lumped mass finite-element method with quadrature for a non-linear parabolic problem, IMA J. Numer. Anal. 5 (1985) 371–396.http://dx.doi.org/10.1093/imanum/5.4.371.

17. Quiros F, Rossi J D. Blow-up set and Fujita-type curves for a degenerate parabolic system with nonlinear conditions. Indiana Univ Math J, 2001, 50: 629–654.

18. Rakhmonov Z. Estimates for solutions of a nonlinear system of heat conduction equations with variable density and with a non-local boundary condition // Vestnik NUUz, №1 (2), 2016, 145-154.

19. Rakhmonov Z. On the properties of solutions of multidimensional nonlinear filtration problem with variable density and nonlocal boundary condition in the case of fast diffusion // Journal of Siberian Federal University. Mathematics & Physics 2016, 9(2), 236–245.doi: https://doi.org/10.17516/1997-1397-2016-9-2-225-234

20. Rakhmonov Z. R., Urunbayev J. E., On a Problem of Cross-Diffusion with Nonlocal Boundary Conditions. Journal of Siberian Federal University. Mathematics & Physics, №5, 2019, 614-620. doi: https://doi.org/10.17516/1997-1397-2019-12-5-614-620.

21. Tsyganov M.A., Biktashev V.N., Brindley J., Holden A.V., Ivanitsky G.R., Waves in cross-diffusion systems – a special class of nonlinear waves, UFN, 2007, vol. 177, issue 3, 275-300.doi:http://dx.doi.org/10.1070/PU2007v050n03ABEH006114

22. Vanag V.K., Epstein I.R., Cross-diffusion and pattern formation in reaction-diffusion systems, Phys. Chem. Chem. Phys. 11 (2009) 897–912. http://dx.doi.org/10.1039/B813825G.

23. Wu, Z.Q., Zhao, J.N., Yin, J.X. and Li, H.L., Nonlinear Diffusion Equations, Singapore: World Scientific, 2001.

24. Wang S, Xie C H, Wang M X. Note on critical exponents for a system of heat equations coupled in the boundary conditions. J Math Analysis Applic, 1998, 218: 313–324.

25. Wang S, Xie C H, Wang M X, The blow-up rate for a system of heat equations completely coupled in the boundary conditions. Nonlinear Anal, 1999, 35: 389–398.

26. Zheng S N, Song X F, Jiang Z X. Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux. J Math Anal Appl, 2004, 298: 308–324. https://doi.org/10.1016/j.jmaa.2004.05.017

27. Sadulayeva Sh. A., Khaydarov A.N., Kabiljanova F.A. Modeling of the multidimensional problem of nonlinear heat condictivity in non-divergent case. “3 rd International Symposium on Multidisciplinary Studies and Innovative Technologies”. Ankara, Turkey. Oct 11-13, 2019, 299-303. https://ieeexplore.ieee.org/xpl/ conhome/8928468/proceeding.